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Two-dimensional wave-number spectral analysis techniques for phase contrast imaging turbulence imaging data on large helical device

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An analysis method for unfolding the spatially resolved wave-number spectrum and phase velocity from the 2D CO₂ laser phase contrast imaging system on the large helical device is described. This is based on the magnetic shear technique which identifies propagation direction from 2D spatial Fourier analysis of images detected by a 6 × 8 detector array. Because the strongest modes have wave-number at the lower end of the instrumental k range, high resolution spectral techniques are necessary to clearly resolve the propagation direction and hence the spatial distribution of fluctuations along the probing laser beam. Multiple-spatial point cross-correlation averaging is applied before calculating the spatial power spectrum. Different methods are compared, and it is found that the maximum entropy method (MEM) gives best results. The possible generation of artifacts from the over-narrowing of spectra are investigated and found not to be a significant problem. The spatial resolution $\Delta\rho$ (normalized radius) around the peak wave-number, for conventional Fourier analysis, is ~ 0.5 , making physical interpretation difficult, while for MEM, $\Delta\rho \sim 0.1$. © 2015 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4928668>]

I. INTRODUCTION

Turbulence plays an important role in particle and energy transport in fusion-relevant plasmas. In both tokamaks and heliotron/stellarator devices, measured particle and heat fluxes can be significantly larger than neoclassical predictions. The deficit, or so called “anomalous” transport, is thought to be driven by drift-wave scale micro-turbulence for which the wave-number $k \approx \rho_{ci}^{-1}$ (where ρ_{ci} is the ion Larmor radius, typically of order of millimeters). Therefore, understanding the nature of plasma micro-turbulence is essential for understanding confinement characteristics. A range of diagnostics have been developed, for measuring turbulence in the core of high temperature plasmas, each technique having different features (such as localization, signal to noise ratio, and sensitivity scale lengths).

The CO₂ laser ($\lambda = 10.6 \mu\text{m}$) phase contrast imaging (PCI) technique is a non-perturbing method to measure the wave-number resolved fluctuation spectrum, integrated along a line of sight.¹ One of the difficulties of the PCI technique is the lack of spatial resolution along the beam because scattering volume is larger than plasma size since the scattering angle is small for such a short wavelength. The “magnetic shear technique” is one such approach that has been used to diagnose this variation,^{2–4} that is based around the assumption that fluctuations propagate perpendicular to field lines, and that the field line pitch changes spatially throughout the plasma. Other techniques include the phase-amplitude technique,⁵ cross-beam correlation techniques,⁶ and hybrid schemes.⁷ Previous

systems (e.g., Refs. 2 and 3) using the magnetic shear localization technique using only 1d detectors can change the measurement spatial location on a shot-to-shot basis or through time multiplexing within a single shot.⁸ On large helical device (LHD), we apply the first use of a 6 × 8 2D detector array to simultaneously measure fluctuations from different positions within the plasma. This is able to work well because of the high shear of magnetic field direction along the line of sight (around 80°).

The PCI technique using a 2D detector was first presented in Ref. 4. Hardware improvements were also published (using the techniques in this paper) in Refs. 9 and 10. An overview of the system and results, including a brief description of the analysis technique, was given in Ref. 11. However, in that paper, the image processing technique was not discussed, and it is the focus of this paper.

The use of a 2D array is a novel application in PCI, however is not unique to plasma fluctuation diagnostics, for example, microwave imaging reflectometry (MIR)¹² and beam emission spectroscopy (BES).¹³ The use of advanced methods for 2D spectral analysis of plasma turbulence data has not been considered to our knowledge, despite the fact that most diagnostics view a limited region of the plasma and have only a modest number of channels (similar to the 48 channels here). This may be because for direct imaging diagnostics, most analyses have been carried out in real space, for example, comparison of the cross-correlation function with a “synthetic diagnostic” simulation.¹⁴ However, in the presence of a large low k components (for example, the contamination of ion-gyro scale drift-wave turbulence by large-scale MHD activity), it is difficult to filter these out using conventional techniques. For 2D PCI, however, the signal interpretation

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from the magnetic shear technique is considerably different to direct imaging diagnostics, because of unfolding the line integration effect. Because of a peculiarity of this technique, the need for Cartesian to polar conversion in k space, unresolved low k components result in a large spatial smearing of the analyzed fluctuation spatial profile. It was therefore necessary to investigate methods for providing higher k resolution. In this search, data analysis techniques have used other areas of science, such as radio interferometry and seismic wave detection. The maximum entropy method (MEM) (and computer code) described in this paper was first implemented in 2005 (and was used in earlier papers on LHD PCI). However, since then, it has been applied to MIR data on TPRX reversed field pinch¹⁵ and has also been useful in analyzing BES data on MAST¹⁶ (in separating large coherent modes from the drift wave turbulence and in terms of the consideration of geometrical line integration effects). In parallel to this work, Matsuo *et al.*¹⁷ worked on a maximum entropy algorithm in polar coordinates for deriving the localization of fluctuations from a 3×15 array with a YAG laser ($\lambda = 1064$ nm), which has considerably lower scattering angle than here.

This paper is organized as follows. In Sec. II, the geometric properties of the diagnostic and plasma are introduced. In Sec. III, the mathematical details of localization principle are derived. In Sec. IV, a complete cross-correlation matrix from all pairs of channels is calculated, then in Sec. V, based on the Wiener-Khinchine theorem, the Fourier transform is taken to get the (real, positive) power spectral density. Since the wavelength/correlation length are similar to or larger than the aperture of the array, high resolution spectral analysis is helpful to distinguish such waves. Different high resolution techniques have been described and tested on experimental data, and the maximum entropy method is considered most suitable. Sec. VI is devoted to consideration of reliability and generation of artifacts including spurious peaks, and the leakage of signals from stronger peaks into regions with no signal. Finally, the paper is summarized in Sec. VII. The same data set is used throughout this paper, for consistency (#71160@ $t = 0.615 - 0.620$ s; $f = 0-500$ kHz). This was an interesting shot exhibiting “non-local” core temperature rise in response to edge cooling.¹⁸

II. GEOMETRY OF THE 2D PCI SYSTEM

Both the transmission and detection optics are mounted on the vibration isolation stand of the FIR interferometer.¹⁹⁻²¹ The PCI system consists of a laser beam passing vertically through the plasma region as shown in Fig. 1(a). The transmission optics, situated below the plasma chamber, are shared with an imaging heterodyne interferometer,²²⁻²⁴ while the detection optics are on top of the chamber. The projection of the magnetic field lines from different points along the line of sight into the frame of the detector array is shown in Fig. 1(b), with crosses denoting the location of the detector pixels. A discrete Fourier transform of the image would yield points in k space as shown in Fig. 1(c). These provide a characteristic type of resolution. However, additional constraints in the optical system including focusing and phase groove width determine the low k limit, while the depth of focus determines the high

FIG. 1. (a) Geometry of sight line through poloidal cross section. (b) Imaged region of plasma, along the beam axis. Triangles denote the position of detector elements and the lines indicate the direction of field lines for given positions. (c) “Sampling” in Fourier space (up to Nyquist limit, and down to lower limit set by finite image size) and lines corresponding to positions in the plasma.

k limit.¹¹ The first report on the hardware description of the system was given in Ref. 4, and raw fluctuation images are demonstrated in Ref. 25. This system, which used a narrow injected beam size of $\varnothing 32$ mm could measure fluctuations in the range $0.5 \text{ mm}^{-1} < k < 3 \text{ mm}^{-1}$. However, because the peak wave-number was less than 0.5 mm^{-1} , spatial resolution was poor and so, the system was upgraded in the 2005-2006 experimental campaign to incorporate a wider beam and flexible imaging optics⁹ to have a measurable range $0.1 \text{ mm}^{-1} < k < 0.9 \text{ mm}^{-1}$ (this is what is shown in Fig. 1). However, the targeted k can be changed to higher values by changing the optical configuration, and asymmetric compression and rotation can be used to target high k fluctuations with better spatial resolution.

III. LOCALIZATION PRINCIPLE AND INVERSE PROBLEM

The magnetic shear principle, described here, is the means to localize fluctuation signals to different parts of the plasma.

The purpose of this section is to illustrate mathematically how this technique works; however, some assumptions are made. For more complete description, see Ref. 26. The basis of this method is the assumption that fluctuations propagate perpendicular to the field lines and have a very long wavelength parallel to B . For the following notation, x, y are coordinate axes perpendicular to injected laser beam (direction z). The y direction is chosen oriented along the field magnetic field in the center. The time resolved line-integrated image $\tilde{N}(x, y, t)$ is first Fourier transformed and ensemble averaged to give the power spectral density function $\partial\langle\tilde{N}^2\rangle/\partial(k_x k_y \omega)(k_x, k_y, \omega)$ (denoted conventionally in this paper by $S(k_x, k_y)$ according to the procedure described in Sec. IV, in order to determine the propagation direction $\theta = \tan^{-1}(k_y/k_x)$ of fluctuations which can be localized to the coordinate z , along the beam, $\theta(z) = \chi(z) + \pi/2$, with $\chi(z) = \tan^{-1}(B_y(z)/B_x(z))$. This is shown by lines from various values of $z(\rho)$ in Fig. 1(c). In this way, the angle θ can be associated with z which can then be mapped to a flux coordinate ρ , where we conventionally denote positive (negative) ρ for locations above (below) the mid-plane. The innermost accessible flux surface is determined by the perpendicular distance between the beam and the magnetic axis, which can change depending on magnetic configuration.

Although the fluctuation $k = \sqrt{k_x^2 + k_y^2}$ spectrum should be relatively broad, fluctuations from one particular z will lie on a line $\theta(z) = \text{const}$, and so, a high $(\Delta k_x, \Delta k_y)$ resolution is required to attain best spatial resolution. The instrumental broadening $(\Delta k_x, \Delta k_y)$ arising from the finite width of the beam is inverse to the size imaged region, as shown in Fig. 1(c), where the points are inverse to the image in Fig. 1(b), and the broadening is given by the spacing between these points. The spatial resolution is considered in more detail in Ref. 9. The main limitation for the resolution of low k components is due to the finite beam width, rather than the limited number of detectors. The high resolution spectral analysis methods effectively extrapolate the correlation function outside the imaged region (as illustrated later in Fig. 5).

The line integral density fluctuations are represented in the time domain by $\tilde{N}$. The space/time cross correlation functions are evaluated and Fourier transformed with respect to the space/time difference parameters to yield the spectral density function $\partial\langle\tilde{N}^2\rangle/\partial(k_x k_y \omega)$, which, strictly, should be a function of space and time. The assumption is made; however, that this is independent over the size of the probing beam (which is much smaller than the plasma). The notation with brackets and partial derivatives is used for power spectral density to make clear which variables need to be integrated over to reconstruct the total fluctuation intensity. Change of variables from Cartesian to polar coordinates requires multiplication of the spectral density function by the Jacobean,

$$\frac{\partial\langle\tilde{N}^2\rangle}{\partial(k\theta\omega)} = \frac{\partial\langle\tilde{N}^2\rangle}{\partial(k_x k_y \omega)} k. \quad (1)$$

It can be shown^{26,27} that line integrated density fluctuation signals $\tilde{N}$ can be related to local density fluctuation signals $\tilde{n}$ by

$$\frac{\partial\langle\tilde{N}^2\rangle}{\partial(k_x k_y \omega)} = \int \frac{\partial\langle\tilde{n}^2(k_z=0)\rangle(z)}{\partial(k_x k_y k_z \omega)} dz. \quad (2)$$

Note here that the above formula is only valid when spectral density envelope function $\partial\langle\tilde{n}^2\rangle/\partial(k_x k_y k_z \omega)$ varies over a slower scale (in x, y , and z directions) than the true turbulence correlation length turbulence. Therefore, it is invalid for large scale modes like global MHD. However, even for data presented here, for example, with the MEM in Fig. 8, the scale length for density fluctuation changes is around $\Delta\rho = 0.1$, which corresponds to a distance of 5 cm, whilst the wavelength corresponding to peak wavenumber is ≈ 2 cm. On the other hand, it was shown^{9,28} that the instrumental resolution with conventional Fourier techniques is $l_{instr} \approx a\lambda/B_w$, where a is the plasma minor radius and B_w is the probing beam and can be evaluated to be $\sim 10\lambda$, indicating that the scale separation is valid. Therefore, this scale separation issue pertains only to high resolution spectral analysis.

Introducing l_z as the ratio of the power density propagating perpendicular to the beam ($k_z = 0$) to the total power density, Eq. (2) can be simplified to

$$\frac{\partial\langle\tilde{N}^2\rangle}{\partial(k_x k_y \omega)} = \int \frac{\partial\langle\tilde{n}^2\rangle(z)}{\partial(k_x k_y \omega)} l_z(z) dz, \quad (3)$$

or in polar coordinates in x - y plane,

$$\frac{\partial\langle\tilde{N}^2\rangle}{\partial(k\theta\omega)} = \int \frac{\partial\langle\tilde{n}^2\rangle(z, \theta)}{\partial(k\theta\omega)} l_z(z) dz. \quad (4)$$

Because there is a unique mapping $\chi(z)$, $\partial\langle\tilde{n}^2\rangle(z, \theta)/\partial(k\theta\omega) = \partial\langle\tilde{n}^2\rangle(z, \theta)/\partial(k\theta\omega)\delta(\theta - \chi(z) - \pi/2)$, integration over z is equivalent to an integration over θ , and the fluctuations can be localized along z by

$$\frac{\partial\langle\tilde{N}^2\rangle(\theta = \chi(z) + \pi/2)}{\partial(k\theta\omega)} = \frac{\partial\langle\tilde{n}^2\rangle(z)}{\partial(k\omega)} l_z(z) \frac{dz}{d\theta}(z), \quad (5)$$

so that

$$\langle\tilde{n}^2\rangle(z) = \int \frac{\partial\langle\tilde{N}^2\rangle(\theta = \chi(z) + \pi/2)}{\partial(k\theta\omega)} l_z^{-1}(z) \frac{d\theta}{dz}(z) d\omega dk. \quad (6)$$

The main uncertainty in predicting the local density fluctuation level is the l_z parameter. This is because line integration is only sensitive to components propagating perpendicular to the beam, since all other components cancel out. While l_z is unknown, it should have the ordering of the correlation length, or hence k^{-1} . Therefore, expressing $l_z^{-1} = \alpha(k, z)k$ (which may not necessarily be spatially constant) and substituting the Cartesian spectral density into Eq. (6), we can write

$$\langle\tilde{n}^2\rangle(z) = \int \frac{\partial\langle\tilde{N}^2\rangle(\theta = \chi(z))}{\partial(k_x k_y \omega)} \alpha(k, z) k^2 \frac{d\theta}{dz} d\omega dk. \quad (7)$$

The value of α was assessed experimentally on the TCA (Tokamak Chauffage Alfvén) tokamak,²⁷ and it was deduced that $l_z \sim \lambda/4$, giving a value for $\alpha \sim 2/\pi$, rather close to unity. It was also shown in Ref. 26 how l_z could be computed assuming a Gaussian correlation envelope $\Gamma(\Delta z) = \exp(-\Delta z^2/\mathcal{L}^2) \exp(ik_{z0}\Delta z)$ (with $\mathcal{L}$ the correlation length and k_{z0} the peak wavenumber), $l_z = \mathcal{L} \exp(k_{z0}^2 \mathcal{L}^2/4)/\sqrt{\pi}$. If the ratio $\mathcal{L}/\lambda = 0.45$, then $l_z = \lambda/4$. This implies that the correlation length is less than the wavelength. In situations where the correlation length is larger, l_z decreases rapidly on account of destructive interference.

Aside from the assumption of scale separation of the turbulence and envelope made in Eq. (2), the lack of knowledge about the parameter $\alpha(k, z)$ is the main cause of uncertainty in determining the local fluctuation level. In Ref. 26, a model was made where the flux-surface oriented spectral density fluctuation was transformed to the lab frame, and it was shown that an asymmetry in the in-out propagation of turbulence (equivalently to an “eddy tilting” effect) could lead to dramatically different values of α for the upper and lower positions near the last-closed flux surface, thereby leading to an apparent signal asymmetry.

Equation (7) also shows that noise or broadband background leakage occurring at higher k values can obscure signals at lower k when the total fluctuation amplitude is computed. In this paper, the spectrum in Cartesian coordinates $\partial\langle\tilde{N}^2\rangle/\partial(k_x k_y \omega)$ is shown in Fig. 6, whilst the conversion to polar coordinates $\partial\langle\tilde{N}^2\rangle/\partial(k\theta\omega)$ is shown in Fig. 7, and the fluctuation amplitude profile $\langle\tilde{n}^2\rangle$ is shown in Fig. 8. Further analysis of the phase velocity of fluctuations can be carried out by transforming variables from ω to v , and the moment can be found to obtain the mean phase velocity for each position in the plasma. Details of this procedure and the results can be found in Ref. 11, and clarification of the type of fluctuation based on the velocity relative to the $E \times B$ velocity has been topics of other publications.^{18,29,30}

IV. CORRELATION AVERAGING

The first step of the procedure is to produce the cross-correlation function, which is related to the power spectral density via the Fourier transform according to the Wiener-Khinchine theorem. The cross-correlation³¹ inherently couples a pair of signals together. One single channel may be chosen as a “reference” channel; however, no such obvious choice exists, and not all information is utilized. In this procedure, we average over all possible pairs of channels, first averaging in time (for which we use a subscript t) then averaging in space (subscript s).

Signals are represented on a grid x, y such that y is parallel to the magnetic field in the middle of the plasma. First, channels are ordered in a 1D array having a single subscript i , $0 < i < N_{ch} - 1$, where N_{ch} is number of channels $N_x N_y$ for an array of size number of channels $N_x \times N_y$. For the present array, $N_x = 6$, $N_y = 8$. Raw line-integrated signals are represented as $\tilde{N}(\mathbf{x}_i, t)$, where t is time and $\mathbf{x}_i = (i_x(i)\Delta_x, i_y(i)\Delta_y)$, where $i_x(i) = i \bmod N_x$ and $i_y(i) = i/N_x$, and the inter-channel spacing (in the object plane in the plasma) is (Δ_x, Δ_y) . The set of channel positions $\mathbf{x}_i$ is referred to the array denoted by $\mathcal{A}$. We first perform the time domain Fourier transform,

$$\tilde{N}(\mathbf{x}_i, \omega) = \int w_t(t) N(\mathbf{x}_i, t) \exp(i\omega t) dt, \quad (8)$$

where $w_t(t)$ is a time domain windowing function, which defines the averaging time τ_{av} . Typically, a Hanning window is used. (Though, through the use of wavelet transforms, a frequency dependent averaging time is chosen for a fixed number of waves, this issue is important for tracking fast changes in fluctuation characteristics, though is not dealt with here.) The

spatial cross-correlation function is calculated by averaging over a frequency interval $\Delta\omega \gg 1/\tau_{av}$,

$$\Gamma_t(\mathbf{x}_i, \delta_j, \omega) = \int_{\omega - \Delta\omega/2}^{\omega + \Delta\omega/2} N(\mathbf{x}_i, \omega') N^*(\mathbf{x}_i + \delta_j, \omega') d\omega', \quad (9)$$

where δ_j is the channel separation. This is defined on the grid $\delta_j = (j_x(j)\Delta_x, j_y(j)\Delta_y)$, with $-(N_x - 1) < j_x < (N_x - 1)$ and $-(N_y - 1) < j_y < (N_y - 1)$ and is indexed by a single variable j : $0 < j < N_{co} - 1$, where $N_{co} = (2N_x - 1)(2N_y - 1)$ such that $j_x = j \bmod (2N_x - 1) - N_x + 1$, $j_y = j/(2N_x - 1) - N_y + 1$. The set of δ_j is referred to as the co-array denoted $\mathcal{C}$ and represents positions where it is possible to compute cross-correlations. In Eq. (9), $\Gamma_t(\mathbf{x}_i, \delta_j)$ is defined for values of j such that $\{\mathbf{x}_i + \delta_j \in \mathcal{A}\}$.

The cross correlation function $\Gamma_t(\mathbf{x}, \delta, \omega)$ should be independent of $\mathbf{x}$, since the image size is small in comparison with the plasma, and the line of sight passes close to the plasma center. The magnitude of the auto-power spectra differs between channels, on account of the characteristic profile shape of the probing laser beam and detector sensitivity variations. However, in addition, the power spectra of different channels exhibit a slightly different frequency dependence, possibly on account of non-ideal effects of the system. This is rectified by adopting a frequency dependent sensitivity, by introducing the total average power spectrum $P(\omega)$,

$$P(\omega) = (N_{ch})^{-1} \sum_{i=0}^{N_{ch}-1} \Gamma_t(\mathbf{x}_i, \mathbf{0}, \omega), \quad (10)$$

and the normalized correlation function (coherence), γ ,

$$\gamma_t(\mathbf{x}_i, \delta_j, \omega) = \Gamma_t(\mathbf{x}_i, \delta_j, \omega) / \Gamma_t(\mathbf{x}_i, \mathbf{0}, \omega), \quad (11)$$

so that the correlation function can be reconstructed by

$$\Gamma'_t(\mathbf{x}_i, \delta_j, \omega) = \gamma_t(\mathbf{x}_i, \delta_j, \omega) P(\omega). \quad (12)$$

Further ensemble averaging over the spatial domain is performed to produce an “averaged” Γ_{ts} , with a corresponding error $\Delta\Gamma_{ts}$,

$$\Gamma_{ts}(\delta_j) = N_{av}^{-1}(j) \sum_{i=0}^{N_{ch}-1} \Gamma'_t(\mathbf{x}_i, \delta_j), \quad (13)$$

$$\Delta\Gamma_{ts}(\delta_j) = \left[N_{av}^{-1}(j) \sum_{i=0}^{N_{ch}-1} (\Gamma_{ts}(\delta_j) - \Gamma'_t(\mathbf{x}_i, \delta_j))^2 \right]^{1/2}, \quad (14)$$

where $N_{av}(j)$ is the number of values of i such that $\mathbf{x}_i + \delta_j \in \mathcal{A}$. For the regular grid considered here, this is equal to

$$N_{av}(j) = (N_x - |j_x(j)|)(N_y - |j_y(j)|). \quad (15)$$

There are several reasons for using the co-array rather than selecting a particular reference channel: Hermitian symmetry is guaranteed by the interchangeability of terms in Eq. (9), the dimensions are twice as large, improving the resolution, and information in all signals is incorporated, relieving the bias of selecting a particular reference channel.³² A depiction of the array sampling and co-array sampling N_{av} is given in Fig. 2, where each panel depicts the array location within the co-array. When all of these panels are “averaged” (with appropriate weighting), the co-array is generated, and images of the real

FIG. 2. Real part of the correlation function $\Gamma'_r(\mathbf{x}_i, \delta_j)$ for all reference channels $\mathbf{x}_i \in \mathcal{A}$. In each plot, x and y axes represent δ_j . The reference channel $\mathbf{x}_i$ is moved systematically in each plot such that the column number indicates $i_x(i)$ and the row number indicates $i_y(i)$. Certain cells are not plotted because 3 channels were not acquired. Points where the correlation function can be computed are indicated with a + symbol. The domain of each plot is $\mathcal{C}$, while the crosses in each plot are on the domain of $\mathcal{A}$ shifted such that the reference channel is in the middle of the plot. The average correlation function is plotted in Fig. 3.

FIG. 3. Real (left) and imaginary (right) parts of $\Gamma_{ts}(\delta_j)$, averaged from individual components plotted in Fig. 2. The x and y axes indicate $j_x(j)$ and $j_y(j)$, and the center of the images correspond to $j_x = 0, j_y = 0$.

and imaginary parts of that are shown in Fig. 3. Line slices through the correlation function $\Gamma_{ts}(\delta_j)$ are plotted as shown in Fig. 4. The error bars represent $\Delta\Gamma_{ts}$. The statistical uncertainty should scale with N_{av} giving high confidence in the center and lower confidence towards the edge. This error estimate can be incorporated in the power spectrum calculation. The standard deviation at delay zero is necessarily zero by the normalization given in Eq. (14). It can be seen that the error is larger for lower frequencies. This may be because at lower frequency, the average k is lower and may be more prone to diffraction phenomena which render the system response different for different channels.

V. 2D POWER SPECTRUM ESTIMATION

The k_x, k_y spectra generated from each of these techniques are compared in Fig. 6 and discussed in Sec. V C.

FIG. 4. Comparison of slices through the correlation function $\Gamma((\delta_x, \delta_y))$ through fixed values of δ_y before spatial averaging (grey lines), after spatial averaging (red error bars), and reconstructed from the inverse Fourier transform of the MEM power spectrum (green crosses). Top row: Real part. Bottom row: Imaginary part. Horizontal axis: j_x from $-5 \dots 5$. Successive plots are for different j_y , from $j_y = 0 \dots 7$. Gray lines indicate $\Gamma'_r(\mathbf{x}_i, \delta_j)$ for all reference channels $\mathbf{x}_i \in \mathcal{A}$. Red error bars are centered on $\Gamma_{ts}(\delta_j)$ and have a magnitude $\Delta\Gamma_{ts}(\delta_j)$ as computed by Eq. (14). The small error bars and narrow spread in gray lines indicate that averaging over different reference channels is well justified. The slight mismatch between red error bars and green crosses is a consequence of tolerating a non-zero χ^2 in the MEM spectral estimate.

A. Fourier methods

There are two main types of “standard” Fourier methods. The periodogram utilizes the Wiener-Khinchine theorem and is based on the Fourier transform of the windowed cross correlation,

$$S_{\text{periodogram}}(\mathbf{k}, \omega) = \sum_{j=0}^{N_{co}-1} W(\delta_j) \Gamma''_{ts}(\delta_j, \omega) e^{i\mathbf{k} \cdot \delta_j}, \quad (16)$$

where $W(\delta_j)$ is a windowing function (Hanning or top-hat). This spectrum can be computed on a fine grid in $\mathbf{k}$ using the FFT by zero-padding the correlation function. In Welch's method, each frequency is spatially Fourier transformed then the averaging is performed in the frequency domain,

$$S_{\text{welch}}(\mathbf{k}, \omega) = \int_{\omega-\Delta\omega/2}^{\omega+\Delta\omega/2} \sum_{i=0}^{N_{ch}-1} \mathcal{N}''(\mathbf{x}_i, \omega') e^{i\mathbf{k} \cdot \mathbf{x}_i} d\omega'. \quad (17)$$

Both these methods presume the data beyond the measurement region to be zero. The resolution of the periodogram is better than that of Welch's method, since it is based on the co-array which is twice as large than the array, whereas Welch's method is based on measurements in the array. The resolution is ultimately limited by the extent of the detector, since the $\Gamma_{ts}(\delta_j)$ does not go to zero over the scale of the co-array, as in Fig. 3.

The use of a window function reduces side-lobes but broadens the main lobe significantly, so it is not used. Because the data contain a strong component around $k = 0$, side lobes from this can give rise to artificial peaks. To mitigate this, low k components are directly filtered out according to

$$\mathcal{N}''(\mathbf{x}_i) = \mathcal{N}(\mathbf{x}_i) - N_{ch}^{-1} \sum_{k=0}^{N_{ch}-1} \mathcal{N}(\mathbf{x}_k) \quad (18)$$

and

$$\Gamma''_{ts}(\delta_j) = \Gamma_{ts}(\delta_j) - N_{co}^{-1} \sum_{k=0}^{N_{co}-1} \Gamma_{ts}(\delta_k). \quad (19)$$

B. High resolution methods

Since the correlation length and peak wavelength are generally larger than the size of the image, Fourier methods become inadequate. The truncation may also cause the power spectrum to be negative which is a nonphysical situation. An ideal analysis method should be highly resolved in θ though not necessarily in k . One could achieve this by representing the power spectrum using non-Cartesian basis functions, where the number of basis functions (parameters describing the power spectrum) is chosen according to the number of measurements. A solution could then be obtained using a least-squares fit (Singular Value Decomposition). Tests of this technique showed that the broadening in the θ direction was similar to conventional Fourier analysis. Though the use of a non-Cartesian basis may be helpful, linear solution techniques are inadequate to provide higher resolution.

We consider spectral analysis techniques which give higher resolution by extending $\Gamma_{ts}(\delta_j)$ beyond the measured image by using some parametric form or *a priori* information.

Methods developed for 2d spectrum estimation are generally more complicated than their 1d counterparts.

One of the simplest high resolution techniques due to Ref. 33 is the maximum likelihood method (MLM). A filter is constructed for arbitrary values of $\mathbf{k}$ within the Nyquist band which maximizes the output power at the desired value of $\mathbf{k}$ and suppresses power from other values of $\mathbf{k}$. This technique was developed for seismic array processing, and it is simple to incorporate non-uniform arrays. The spectral estimate as a function of $\mathbf{k}$ is given by

$$S(\mathbf{k}) = \frac{1}{E^H(\mathbf{k}) R^{-1} E(\mathbf{k})}, \quad (20)$$

where R is the complete $N_{ch} \times N_{ch}$ correlation matrix $R_{ij}(\omega) = \Gamma'_{ts}(\mathbf{x}_i, \mathbf{x}_j - \mathbf{x}_i, \omega)$, and the N_{ch} length vector $E(\mathbf{k})$ has elements $E_i(\mathbf{k}) = \exp(i\mathbf{k} \cdot \mathbf{x}_i)$. The superscript H refers to the conjugate transpose.

The MEM is another well-known high resolution technique based on *a priori* information, defined as the power spectrum $S(\mathbf{k})$ which maximizes the Burg entropy,

$$H = \int \log S(\mathbf{k}) d\mathbf{k} \quad (21)$$

and satisfies the “correlation matching” constraints $\Gamma_{ts}(\delta_j) = \hat{\Gamma}(\delta_j)$, for all $\delta_j \in C$, where

$$\hat{\Gamma}(\delta_j) = \int S(\mathbf{k}) \exp(i\mathbf{k} \cdot \delta_j) d\mathbf{k}. \quad (22)$$

Upon maximization, it can be shown that the MEM estimate is parameterized by N_{ch} Lagrange multipliers a_j for all $\delta_j \in C$,³²

$$S(\mathbf{k}) = \left(\sum_{i=1}^{N_{ch}} a_j \exp(-i\mathbf{k} \cdot \delta_j) \right)^{-1}. \quad (23)$$

If Eq. (23) can be factorized in the following form as

$$S(\mathbf{k}) = \frac{\sigma^2}{|\sum b_j \exp(-i\mathbf{k} \cdot \delta_j)|^2} \quad (24)$$

for $\delta_j \in C \setminus \mathbf{0}$ (excluding zero), then the coefficients b_j can be computed by the auto-regressive (AR) linear prediction and can be obtained simply from the (linear) Yule-walker equations.³⁴ Such a factorization is possible in 1d; however, in 2d, no such factorization is possible.³⁵ Another explanation is that in 2D, linear prediction is not so well defined as in 1D because there is no natural ordering in the 2D plane (in 1D, for example, there is a concept of “past” and “future”). Therefore, to obtain the MEM spectrum in 2d, it is necessary to maximize nonlinear equation (21) subject to the linear constraints in Eq. (22). We treat this problem later.

The AR model can still deliver high resolution estimates in 2d and is attractive because of the ease of solution. For improving the condition of the solution, the set of delays δ_j used in the solution is chosen to be a subset of the co-array C . One common technique is to use “quarter plane” masks, which effectively linearly predict each quadrant of the 2D plane based on separately computed filter coefficients.³⁴

High resolution techniques tend to give very sharp peaks. One of the problems in calculating MEM spectra Eq. (23) is

the ability to represent these peaks. The generalized Pisarenko method³⁶ is parametric model which gets around the problem of representing sharp peaks by expressing the power spectrum as a sum of N_{co} delta functions in a background of colored or white noise σ^2 , the offsets $\mathbf{k}_j$, and amplitudes a_j of which are determined from the data,

$$S(\mathbf{k}) = \sigma^2 + \sum_{j=0}^{N_{co}-1} a_j \delta(\mathbf{k} - \mathbf{k}_j). \quad (25)$$

An exact mathematical formalism can be derived for discrete regions of support. The parameters are determined by solving a linear programming problem which maximizes σ^2 subject to Eq. (22). This method is attractive because of the ease of solution and the ability to provide high resolution.

For high resolution techniques which extrapolate correlation measurements beyond the co-array, such as AR, Pisarenko, and MEM, the existence of a positive definite spectral estimate $S(\mathbf{k})$ is not guaranteed. This is a question of *extendability*. For the AR and Pisarenko methods, if the measurements $\Gamma(\delta)$ are not extendable, then the noise component σ^2 is negative. A result due to Ref. 36 states that if the term σ^2 in Pisarenko's method is positive, then the measurements are extendable. It was found in practice with the PCI data that often the data set was not extendable.

Several methods obtaining the MEM spectral estimate were considered.^{35,37,38} When searching for a MEM solution, often convergence in the iterative solution method could not be obtained because the data are not extendable. This problem was overcome by relaxing the correlation matching constraints to allow for some experimental error. We achieve this by allowing for some non-zero value of χ^2 in the solution,

$$\chi^2 = \sum_{j=0}^{N_{co}-1} \left(\frac{\Gamma_{ts}(\delta_j) - \hat{\Gamma}(\delta_j)}{\Delta \Gamma_{ts}(\delta_j)} \right)^2. \quad (26)$$

With exact matching of all $\Gamma_{ts}(\delta_j) = \hat{\Gamma}(\delta_j)$ ($\chi^2 = 0$), the spectral estimate can be parameterized by the Lagrange multipliers a_j as in Eq. (23), from which calculation of each a_j gives the solution to the MEM problem. However, for solving with a constraint such that $\chi^2 = \chi_{aim}^2$, no such parameterization exists, and it is easier to use a primal space method, similar to that of Wernecke and D'Addario³⁸ where the power spectrum is described on a Cartesian grid with a specified resolution (routinely we used 35×35). We adapted this method to use optimization techniques described in the work of Skilling,³⁹ including the use of the entropy metric $\nabla \nabla H$ and using four search directions at every iteration. Though, we adapted this method to use the Burg entropy defined in Eq. (21) rather than the Skilling entropy $H = - \int S(\mathbf{k}) \log S(\mathbf{k}) d\mathbf{k}$ used in Ref. 39. The advantage of the Skilling method over the Wernecke method is that the target value of $\chi^2 \leq \chi_{aim}^2$ could be specified, and the method converges more reliably in less time.

While the measurement error Γ_{ts} can be taken from the data as in Eq. (14), it was found that the error was sometimes too large, and the measurement error was not so reliable particularly towards the edges of C , where N_{av} is small. Therefore, we modified the error to a prescribed function (not related to

FIG. 5. Real (left) and imaginary (right) parts of the correlation function calculated from the Fourier transform of $S_{MEM}(\mathbf{k})$ at points δ inside and outside C . Points with + are members of the set C where the correlation function can be measured. This demonstrates how high resolution spectral estimation extrapolates outside the domain of C , reducing the side lobe broadening, whereas Fourier periodogram estimation only uses points within C and therefore has much stronger side-lobe broadening since Γ does not go to zero over C . The x and y axes indicate $j_x(j)$ and $j_y(j)$, and the center of the images correspond to $j_x = 0, j_y = 0$.

the data),

$$\Delta \Gamma_{ts}(\delta_j) = 0.01 \times \Gamma_{ts}(\delta_{j_0}) \left(\frac{N_{av}(j)}{N_{av}(j_0)} \right)^{-0.5}, \quad (27)$$

where j_0 is in the middle of the co-array, $(j_x(j_0), j_y(j_0)) = (0, 0)$.

Using this type of tapered bias in the error results in a reduction of the overall background noise level when exponent of N_{av} is closer to zero, all the mismatches tend to be concentrated around $\delta = 0$ effectively increasing the broadband noise floor. The solution proceeds iteratively. Initially, $\chi_{aim}^2 = N_{co}$. If convergence to the solution slows too much, the data may not be extendable within the given error bars, and so, the value of χ_{aim}^2 is increased appropriately.

The problem of MEM spectral analysis is identical to MEM image reconstruction either in tomography or for image sharpening,⁴⁰ since Fourier transformation is just another integral equation, where the “response” matrix has elements given by Eq. (22). The MEM estimate is described has very desirable properties, enforcing positivity and smoothness. The higher the values of χ^2 that are tolerated, the “flatter” is the spectral estimate. It has also been shown that the MEM estimate maximizes the ratio of the peak to the mean.³⁴ For this method, the reduction of χ^2 means that over the domain of the co-array, the correlation function is matched closely to the data. However, outside the co-array, i.e., for larger separations δ which are “inaccessible” since they are larger than the array size, the MEM predicts a non-zero value of the correlation function at those points and is illustrated in Fig. 5. On the other hand, for the conventional Fourier periodogram method, the correlation function at these points is enforced to be zero.

C. Comparison and discussion

When dealing with mathematical inverse problems, it is common to use synthetic data to test different inversion

FIG. 6. Power spectra $S(k_x, k_y)$ (over the Nyquist domain with the origin in the center of the images, for all frequencies) for the different analysis methods in the text. The color is on a log scale, and the text indicates the dynamic range of each individual image.

methods. A simple test using synthetic data was conducted for the MEM as explained below; however, for the purposes of comparing different methods, it was chosen to use real plasma data for the following reasons. First, if one were to construct a synthetic correlation function, it would be most realistic to make one which matched closely to the experimental data. So, why not just use the experimental data itself. Second, for high resolution methods which are intrinsically non-linear, features of the k spectrum such as the broadening of peaks and broadband background level are not the same when the number and location of peaks changed, implying that the best test can be done on real data. Furthermore, owing to instrument effects, the spatial correlation function may be distorted in such a way that its Fourier transform, the power spectrum, is not positive definite which is a nonphysical situation. This is very important since side-lobe features tend to try to make the spectrum estimate go negative, and this produces problems

FIG. 7. Fluctuation amplitude $\partial\langle\tilde{N}^2\rangle/\partial(k\theta\omega)$ as a function of k and ρ (mapped from θ) transformed from the MEM panel in Fig. 6. The dashed lines correspond to a transformation of the “characteristic grid” k_x, k_y as in Fig. 1(c).

for the maximum entropy method (which we assume that it is positive).

Images of the 2D power spectra for each of the different methods are shown in Fig. 6. For clarifying the comparison, the axes labels are removed and the color scales are adjusted, and the dynamic range in each image (logarithmic scale) is indicated. Each of these images exhibit different artifacts but all depict the same broad types of features: lines extending from the origin in a “v” shape, which correspond to fluctuations spread over a range of k on both the upper and lower sides of the plasma, propagating in opposite directions within the image, corresponding poloidal propagation in the ion diamagnetic direction. These data sets are then converted to polar coordinates, and the one for the MEM is shown in Fig. 7. Fluctuation amplitude profiles using Eq. (7) are compared in Fig. 8. Given that the fluctuations are strongest near the edge of the plasma, and that it is nonphysical to have any finite fluctuation amplitude outside the plasma boundary where the density and temperature are zero. For the present shot, the Thomson scattering system shows this to be the case at around $\rho = 1.1$. The fluctuation amplitude outside $|\rho| = 1.1$ can therefore be interpreted as being due to artifacts, primarily the broadband background level, due to leakage from the primary peak. The signal profile is also observed to be asymmetric, independent of which method is being used. This is a common feature of the PCI data and may be interpreted as being due to line integration effects of tilted eddies,²⁶ resulting a change of the longitudinal integration length l_z . However, we cannot dismiss the possibility that it might also be an inherent asymmetry of the turbulence. Another noteworthy feature of the data is that the MEM shows an additional peak near the core of the plasma. Most other methods cannot detect this, and in fact, the AR method shows a big spike here; this is likely to be an artifact.

The difference in spatial resolution of each method can be easily observed in Fig. 8, based on the full-width at half-maximum (FWHM) of the main peak at $\rho = +1.0$. The maximum entropy method has a broadening of $\Delta\rho = 0.1$, the Fourier periodogram with no window has $\Delta\rho = 0.3$, and

FIG. 8. Fluctuation amplitude $\langle \tilde{n}^2 \rangle^{1/2}$ as a function of ρ (or equivalently polar angle θ , upper axis), integrated over ω and k , for the different methods described in the text. The sign convention of ρ indicates above or below the mid-plane.

other Fourier methods (Hanning periodogram and Welch) have $\Delta\rho > 0.5$. The maximum likelihood and auto-regressive methods have resolutions in between MEM and the Fourier methods, whilst Pisarenko's method also has resolution similar to that of MEM. For the experimental data analyzed here, the entire (positive) frequency domain was summed together to illustrate the improvement of the resolution of low k components; however, if only high frequency components are selected, then obviously better resolution can be obtained for these components. Note however, according to the discussion around Eq. (2), that the steep gradient near $\rho = 1$, shown by the maximum entropy method, has a scale length approaching the fluctuation wavelength, calling into question the validity of that equation and therefore the relative intensity in that region.

A simple test to determine instrument resolution inherent to the MEM was conducted as follows. A correlation function was synthesized based the analytic Fourier transform of an infinite correlation length plane wave, with a k value of 0.4 mm^{-1} (corresponding to the k value in Fig. 7), located at an angle of 40° , corresponding to the edge of the plasma. The MEM analysis was applied to this correlation function, and it was found that the angular FWHM corresponded to an instrumental spatial resolution of $\Delta\rho = 0.1$ (whilst this could be improved, it reduces the reliability of convergence).

The Fourier periodogram method with Hanning window shows broad lobes corresponding to fluctuations at the edge of the plasma. However, due to their broadness, this implies considerable fluctuation amplitude at high angles which are not perpendicular to any field lines along the line of sight. Furthermore, the bleeding of this amplitude around the region $k = 0$ results in a very broad fluctuation profile from the core to the edge, with no strong localization. Given that it is known the instrumental function wings contribute towards this broadness, this profile clearly does not correspond to the real fluctuation

profile. The Fourier periodogram method without any window clearly shows more localization towards the edge of the plasma and has less low k leakage, but, according to Fourier theory, contains significant side-lobes, essentially vertically and horizontally displaced “phantoms” of the v -structure. Whilst these side lobes have lower amplitude, they would obscure any real fluctuations existing in that part of k space. It is fortunate that in this case, these power spectra are actually positive definite, with about 20-21 dB between the peak and the minimum, as this is not guaranteed, and in other cases, the power spectra can be negative. The Fourier Welch method has similar features to the Hanning window version of the periodogram, as it is derived from the array rather than the coarray.

The maximum likelihood method gives better resolution than Fourier periodogram and Fourier Welch methods; however, since correlation matching is not specifically enforced, the total fluctuation power in the spatial Fourier transform is not equal to the power auto-correlation function of each signal (Parseval's theorem does not hold). This problem arises when there are several different waves with strongly different intensities; the response to each component can be quite different.⁴¹ For this reason, this method is considered inappropriate.

The AR model spectrum is seen to give strong peaks, though they tend to blend into one another, and when integrated (for example, over k for fixed angle), the spatial resolution is not good. For both MLM and the AR spectra, it is necessary to perform matrix inversion on a highly ill-conditioned matrix ($\sim 10^{10}$).

The spectrum from Pisarenko's method clearly has higher resolution than other methods and shows clear regions where the fluctuation power is localized, and the peaks do not bleed into one another. However, it is evident that the noise sensitivity is high since many spurious peaks are produced. Furthermore, since turbulence should be broadband in nature, the parametric form of a sum of discrete peaks is not theoretically justified; however, the discrete peaks are clumped together in such a way as to create a main broader peak. To relieve the noise sensitivity, it may be possible to perform boxcar smoothing or to tolerate some mismatch between the measured correlations and the correlations based on the power spectrum.

The MEM estimate is clearly the best trade-off between resolution and stability. Maximum entropy is global *a priori* information and tends to make the power spectrum smooth. Since the method allows for some error in the measurements, the smoothest possible solution is found. The computation time for MEM is larger than other methods considered but the quality of the images is clearly better. All analyses presented from here use the MEM.

VI. ACCURACY OF MEM EXTENDED CORRELATION FUNCTION

One problem with high resolution analysis methods such as MEM is the tendency to make peaks narrower than they are in reality. The width of the individual peaks of the reconstruction depends on the ratio of the peak power to the noise background power, since MEM estimate in Eq. (23) has the form of an all-pole model. Further tests of the response in the presence of multiple peaks become complicated because

FIG. 9. Test to determine the “narrowing” effect of the MEM. Left: The Fourier periodogram method with Hanning window is applied to the plasma data to generate a k spectrum which is the most conservative in terms peaks broadening. Right: Reconstructed MEM spectrum, based on the correlation function generated from the data on the left.

algorithm is inherently non-linear. However, the properties of MEM estimates in 2d have been studied before.^{40,42} In regions of steep gradient, an over-shoot may occur leading to a spurious peak. In order to test whether the peaks in the MEM spectrum derived in Fig. 6 were likely more narrow than in reality, a test was conducted, as illustrated in Fig. 9. For this test, the power spectrum derived using the Fourier periodogram with Hanning window was taken as reference data, for this is in some sense the “least” narrow that the spectrum could be. The Fourier transform is taken to generate the correlation function on the coarray, then the MEM is implemented to re-derive the power spectrum. It is clear that although there is a slight perturbation to the spectrum, the width of the peaks remains largely unchanged, and so, one can conclude that broad peaks can be reconstructed by the MEM.

The broadband noise floor in the power spectrum can also generate artifacts in the spatial profile of the fluctuation amplitude, as in Fig. 8. Even for the MEM, there is a finite amplitude outside the plasma. Furthermore, it is questionable how much of the fluctuation amplitude in the core is due to this noise floor. The broadband noise floor is related to the relative error criterion assigned in Eq. (27). The MEM derived spectral power, for two values of the relative error criterion is plotted in Fig. 10. When the relative error is increased by an order of magnitude, the minimum value of the image increases by about 5-10 dB and the dynamic range decreases accordingly. Additionally, some features of the image are less well resolved. Consequently, it is important to assign the relative error to be

FIG. 10. MEM spectral estimate obtained over the Nyquist domain (logarithmic color scale) when using relative errors (at $\delta = 0$) of 0.1% (left) and 1% (right).

FIG. 11. Comparison of fluctuation amplitude (normalized to maximum) vs angle (vertical lines denoting the edges of the plasma) for the original MEM reconstruction (black thick line), a modification to zero out the power over all k within $|\theta| < 15^\circ$ and $|\theta| > 55^\circ$ (dotted red line), and the reconstructed profile based on the forward computed correlation function and MEM derived power spectrum (dashed blue line).

as small a value as possible; however, when it gets too small, it can become difficult for the MEM to converge because it is trying to make the power spectrum negative at a point in the image. A value of 1% is conventionally used, despite not giving as a good resolution, in order to improve the convergence properties in general.

The influence of the noise floor on the fluctuation profile is assessed in Fig. 11. The aim of this test is to determine to what extent the background level (outside the plasma) is related to the amplitude of the peak. For this test, the MEM spectral function is taken as a source test k spectrum (plotted in black solid line). The 2D k_x, k_y spectrum is then converted to k, θ coordinates, the power is “zeroed out” in the domain $|\theta| < 15^\circ$ and $|\theta| > 55^\circ$ (the profile for which is denoted by dotted red line, it does not go to zero precisely because of the grid-size representation in Cartesian coordinates), and then converted back to Cartesian coordinates. The correlation function is then computed based on this modified k spectrum, then the MEM estimate is taken again, and it is converted into a profile again via a polar transform (dashed blue line). It can be seen that in those regions which were zeroed, the analysis technique produces a signal of about 5%-7% of the maximum peak near the edge. Correspondingly, we can conclude that we cannot distinguish fluctuations in the core which are weaker than this value. However, in this case, there is a peak towards the core which is about 30% of the value at the edge.

VII. CONCLUSIONS

The data analysis method for 2D phase contrast imaging on LHD has been described. By performing a particular averaging scheme of all cross-correlation pairs, the coarray is derived which has Hermitian symmetry (so, it would correspond to a real power spectrum) and has a larger size than the array. Such a scheme is appropriate to any turbulence image data.

Different 2D spectral analysis methods were compared: conventional Fourier, maximum likelihood, auto-regressive

sequence, Pisarenko's method, and the maximum entropy method implemented using a technique developed by Skilling. These techniques were found to be commonly used in other areas of science such as astronomy and earthquake detection. The maximum entropy method was found to be the most appropriate method here, and for the shot analyzed, a spatial broadening of $\Delta\rho = 0.1$ was found. However, as the resolution is inverse to the value of k , this is variable depending on things such as the nature of turbulence which can vary with the axis position of the plasma and also the configuration of the optical system (for example, prior to the upgrade described in Ref. 9, the peak k value was much lower compared to the Nyquist k because of the image size being smaller). Implementing the maximum entropy technique was the most challenging of these methods, as it is based on complicated numerical scheme and the entropy functional itself has such a "weak" maximum. Since this work was carried out, suggestions for other maximum entropy algorithms have been put forward: Iwama and colleagues⁴³ have used the Sherman-Morrison-Woodbury technique to improve the speed of convergence, by searching in a subspace with dimension greater than the four used here, whilst smaller than the total subspace. However, there are many other implementations of the maximum entropy method that could be used. The numerical precision is satisfactory here for the 35×35 image of the power spectrum, but for more implementations on a finer grid (for example, in cases where there are more channels available), certainly more advanced algorithms might prove useful. The technique described has been the basis for physics analysis in other publications:^{11,30} for example, resolving the velocity of fluctuations as a function of plasma radius and comparing with $E \times B$ rotation profile help to resolve the type of fluctuation mode. Also, a comparison of the k spectrum with non-linear gyro-kinetic simulation has also been carried out in Ref. 29. Comparison of the (branch-resolved) fluctuation amplitude (analyzed using techniques described in this paper) with the particle flux and diffusion coefficient has been published in Refs. 44 and 45.

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