Nonlinearly $\mathcal{PT}$-symmetric systems: Spontaneous symmetry breaking and transmission resonances

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We consider a class of $\mathcal{PT}$-symmetric systems which include mutually matched nonlinear loss and gain (in other words, a class of $\mathcal{PT}$-invariant Hamiltonians in which both the harmonic and anharmonic parts are non-Hermitian). For a basic system in the form of a dimer, symmetric and asymmetric eigenstates, including multistable ones, are found analytically. We demonstrate that, if coupled to a linear chain, such a nonlinear $\mathcal{PT}$-symmetric dimer generates previously unexplored types of nonlinear Fano resonances, with completely suppressed or greatly amplified transmission, as well as a regime similar to the electromagnetically induced transparency. The implementation of the systems is possible in various media admitting controllable linear and nonlinear amplification of waves.

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I. INTRODUCTION

In the past few years, the study of systems exhibiting the parity-time ($\mathcal{PT}$) symmetry has drawn a great deal of attention. The underlying idea is to extend canonical quantum mechanics by introducing a class of non-Hermitian Hamiltonians which may exhibit entirely real eigenvalue spectra below a certain phase-transition point [1]. A necessary condition for the Hamiltonian to be $\mathcal{PT}$ symmetric is that its linear-potential part $\mathcal{V}(x)$, being complex, is subject to the spatial-symmetry constraint $\mathcal{V}(x) = \mathcal{V}^*(-x)$. The complex $\mathcal{PT}$-symmetric potentials can be realized in the most straightforward way in optics, by combining the spatial modulation of the refractive index with properly placed gain and loss [2]. This possibility has stimulated extensive theoretical [3,4] and experimental [5] studies.

In the $\mathcal{PT}$-symmetric Hamiltonians introduced in the context of the field theory and optics, the harmonic part features matched gain and loss, while the anharmonic part, if any, is usually Hermitian, giving rise to nonlinear dynamical models in which only the linear part features balanced dissipation and amplification [6]. In this work, we consider an extension of the $\mathcal{PT}$ symmetry, in the form of Hamiltonians whose anharmonic part also includes mutually matched loss and gain. Very recently, a similar setting was proposed in Ref. [4], with the $\mathcal{PT}$-symmetric part of the system represented *solely* by the nonlinear terms, while the linear ones were conservative. We consider a more general situation, with *both* the linear and nonlinear terms in the dynamical equations carrying $\mathcal{PT}$-matched loss and gain.

Solving the corresponding dynamical equations, in Sec. II, we demonstrate that such nonlinear systems also give rise to eigenstates with real frequencies. Among our findings, which are specific to the systems with matched nonlinear gain and loss, are eigenstates with a spontaneously broken spatial symmetry (as mentioned above, it is the spatial symmetry which implements the $\mathcal{PT}$ symmetry of the system), and multistability of eigenstates.

Straightforward physical applications of these states are realized, in Sec. III, by coupling the $\mathcal{PT}$ system to linear chains: We demonstrate that it gives rise to previously unexplored types of multistable nonlinear Fano resonances, transmission regimes with a very strong amplification, and those similar to electromagnetically induced transparency (EIT). Finally, in Sec. IV we demonstrate that solutions for nonpropagating modes in the chain with the inserted $\mathcal{PT}$ system can be reduced to those for the isolated $\mathcal{PT}$ system.

Such systems can be implemented in optics, using saturable absorbers [7] and two-photon losses to realize the nonlinear $\mathcal{PT}$ symmetry, as well as in any medium which allows nonlinear amplification of waves, including cavity polaritons [8], surface plasmons [9], and magnons [10].

II. $\mathcal{PT}$-SYMMETRIC DIMER

We start by introducing a solvable system, in the form of a dimer, in which the symmetric linear gain and loss terms come along with their nonlinear mutually conjugate counterparts,

$$\begin{aligned} i\dot{\psi}_A &= (E + i\gamma_0 - i\gamma_2|\psi_A|^2 + \chi|\psi_A|^2)\psi_A + V\psi_B, \\ i\dot{\psi}_B &= (E - i\gamma_0 + i\gamma_2|\psi_A|^2 + \chi|\psi_B|^2)\psi_B + V\psi_A. \end{aligned} \tag{1}$$

Here the overdot stands for the time derivative, $\gamma_0 > 0$ accounts for the linear gain and loss acting on complex variables ψ_A and ψ_B, respectively, E is a frequency shift, γ_2 accounts for the $\mathcal{PT}$-symmetric nonlinear loss and gain (as shown below, stable eigenstates are obtained with $\gamma_2 > 0$, i.e., if the nonlinear loss competes with the linear gain and vice versa), χ determines the nonlinear frequency shift, and V is a coupling coefficient.

A. Symmetric modes

Symmetric eigenstates, with $|\psi_A| = |\psi_B|$, are sought for as $\psi_{A,B}(t) = A\exp(-i\omega t \pm i\delta/2)$, with the amplitude and phase shift determined by the following equations:

$$[\chi A^2 - (\omega - E)]^2 + (\gamma_0 - \gamma_2 A^2)^2 = V^2, \tag{2}$$

$$\tan\delta = (\gamma_0 - \gamma_2 A^2)(\omega - E - \chi A^2)^{-1}. \tag{3}$$

Depending on the parameters, Eq. (2) may yield no physical solutions with $A^2 > 0$, a single solution (monostability), and bistability, with two physical roots. The bistability occurs under conditions

$$\begin{aligned} &(\omega - E)^2 > V^2 - \gamma_0^2, \\ &\gamma_0\gamma_2\chi^{-1} + (\omega - E) > \sqrt{(\omega - E)^2 + \gamma_0^2 - V^2}, \end{aligned} \tag{4}$$

while the monostability condition is $(\omega - E)^2 < V^2 - \gamma_0^2$. Although the system is dissipative, the symmetric eigenstates with the real frequencies form a continuous family parameterized by arbitrary frequency ω, which is a manifestation of the $\mathcal{PT}$ symmetry.

B. Asymmetric states and multistability

The system admits solutions with broken symmetry too, $(A \neq B)$: $\psi_{A,B}(t) = \{Ae^{i\delta/2}, Be^{-i\delta/2}\}e^{-i\omega t}$, with δ determined by the same equation (3) as above. Unlike the symmetric eigenstates, the asymmetric ones exist at the single frequency, which is typical to dissipative systems:

$$\omega_{\rm AS} = E + (\gamma_0/\gamma_2)\,\chi. \tag{5}$$

The amplitudes are determined by equations

$$(A^2)^2 - (\gamma_0/\gamma_2)A^2 + V^2\left(\chi^2 + \gamma_2^2\right)^{-1} = 0 \tag{6}$$

[cf. Eq. (2)] and $B^2 = (\gamma_0/\gamma_2) - A^2$, which shows that the asymmetric eigenmode exists only for $\gamma_2 > 0$, that is, when the nonlinear $\mathcal{PT}$-symmetric loss or gain terms compete with their linear counterparts. We stress that the asymmetric solutions are supported by the balance of the nonlinear and linear gain and loss, as they do not exist for $\gamma_2 = 0$ and for $\gamma_0 = 0$.

The above relations yield two physical solutions (i.e., the bistability) for the asymmetric modes, with $A^2, B^2 > 0$, at $\chi^2/\gamma_2^2 > 4(V^2/\gamma_0^2) - 1$, and no solutions in the opposite case. Under this condition, inequalities (4) hold too for $\omega = \omega_{\rm AS}$; that is, the system gives rise to the *multistability*, with four coexisting eigenstates, two symmetric and two asymmetric.

III. SCATTERING PROBLEM

The next step is to couple the dimer to a chain transmitting linear discrete waves $\psi_n(t)$, as shown in Fig. 1 (note that the entire system remains $\mathcal{PT}$ symmetric). Here we focus on the most fundamental version of the system, with $\chi = 0$, the nonlinearity being represented by the matched cubic loss and gain, the respective coupled system being

$$i\dot{\psi}_A = E\psi_A + i(\gamma_0 - \gamma_2|\psi_A|^2)\psi_A + V\psi_0, \tag{7}$$

$$i\dot{\psi}_n = C(\psi_{n-1} + \psi_{n+1}) + V\delta_{n,0}\,(\psi_A + \psi_B)\,, \tag{8}$$

$$i\dot{\psi}_B = E\psi_B - i(\gamma_0 - \gamma_2|\psi_B|^2)\psi_B + V\psi_0, \tag{9}$$

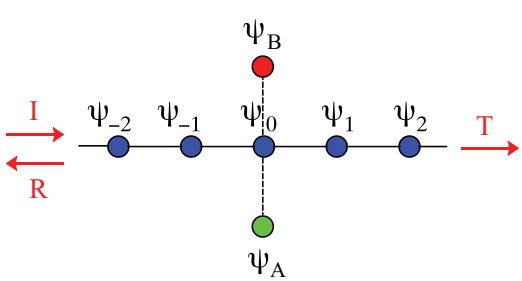

FIG. 1. (Color online) The linear chain with the side-coupled elements featuring the nonlinear $\mathcal{PT}$ symmetry. The arrows indicate incident, reflected, and transmitted waves.

where C is the coupling constant in the linear chain. The solution corresponding to the scattering of incident waves with amplitude I on the $\mathcal{PT}$ complex is looked for as

$$\psi_n = \begin{cases} Ie^{i(kn-\omega t)} + Re^{-i(kn+\omega t)} & (n \leqslant 0)\,, \\ Te^{i(kn-\omega t)} & (n \geqslant 0)\,, \end{cases} \tag{10}$$

where wave number $k > 0$ is determined by the dispersion equation for the linear chain, $k = \cos^{-1}(\omega/2C)$, while R and T are the amplitudes of reflected and transmitted waves. A straightforward analysis of Eqs. (8) and (10) at $n = 0$ yields $R = \psi_0 - I$, $T = \psi_0$, and the expression for ψ_0 in terms I and $\psi_{A,B}^{(0)}$:

$$\psi_0 = I + iV\,(2C\sin k)^{-1}\left(\psi_A^{(0)} + \psi_B^{(0)}\right). \tag{11}$$

The substitution of expression (11) into the stationary version of Eqs. (7) and (9) leads to a system of complex cubic equations:

$$(E - \omega)\,\psi_{A,B}^{(0)} + iV^2\,(2C\sin k)^{-1}\left(\psi_A^{(0)} + \psi_B^{(0)}\right) \pm i\left(\gamma_0 - \gamma_2\left|\psi_{A,B}^{(0)}\right|^2\right)\psi_{A,B}^{(0)} = -VI, \tag{12}$$

which should be solved for $\psi_{A,B}^{(0)}$ at given I and ω. Then, ψ_0 can be found from Eq. (11), and, eventually, the reflection and transmission coefficients can be found.

A. Scattering in the symmetric regime

In the linear system ($\gamma_2 = 0$), Eq. (12) yields only symmetric solutions, with $|\psi_A| = |\psi_B|$. The corresponding transmission spectrum, displayed in Fig. 2, demonstrates two noteworthy effects. One is the suppression of the transmission by the degenerate side-coupled elements without the gain and loss, $\gamma_0 = 0$. In this case, the eigenfrequencies of both elements are identical, and their excitation results in the resonant reflection at $\omega = E$, which can be explained in terms of the Fano resonance [11]. The presence of the weak linear gain and loss, with $\gamma_0 \ll 1$, lifts the degeneracy between the attached sites, leading to a response resembling the EIT effect [12], with the total transmissivity ($|T/I|^2 = 1$) at $\omega = E$, between resonant reflections on the pair of slightly detuned linear $\mathcal{PT}$ elements.

In the system combining the linear chain and the nonlinear $\mathcal{PT}$ scatterer with $\gamma_2 > 0$, one can find symmetric solutions with $\psi_A^{(0)} = -\psi_B^{(0)} = -i\phi$, where ϕ is real. First, we consider

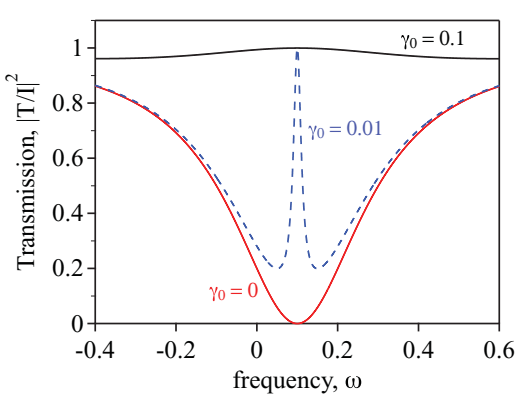

FIG. 2. (Color online) The normalized transmission coefficient for several values of the gain/loss factor γ_0 in the linear system ($\gamma_2 = 0$). Other parameters are $E = 0.1$, $V = 0.2$, and $C = 1$.

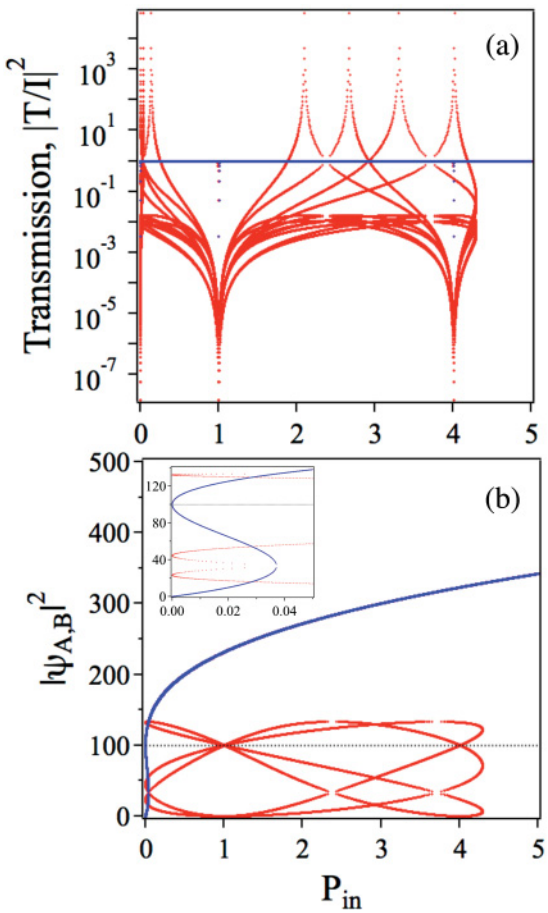

FIG. 3. (Color online) (a) Normalized transmission coefficient and (b) excitation intensity of the side-coupled $\mathcal{PT}$ elements for $\gamma_0 = 0.01$, $\gamma_2 = 0.0001$, $V = 0.2$, $C = 1$, and $\omega = E = 0.1$. The nonlinear Fano resonances correspond to $T = 0$. The red and blue curves depict, respectively, the full multitude of asymmetric scattering regimes, produced by Eq. (12), and the symmetric one corresponding to Eq. (13). The inset in (b) shows the tristability region in the latter case. The horizontal dotted line in (b) corresponds to Eq. (15).

the case of $\phi \neq \sqrt{\gamma_0/\gamma_2}$ (this value plays a special role, as shown below). Then, the symmetric mode exists at $\omega = E$, with ϕ determined by the equation

$$\gamma_2\phi^3 - \gamma_0\phi + VI = 0, \tag{13}$$

which yields a single real root for $P_{\rm in} \equiv |I|^2 > (4/27)\gamma_0^3/(V^2\gamma_2)$, and three real solutions (*tristability*) in the opposite case. According to Eq. (11), all these solutions realize the perfect EIT-like transmissivity, with $T \equiv 1$ [the horizontal blue line in Fig. 3(a)]. The family of the symmetric states is displayed by the blue curves in Fig. 3(b), where the tristability occurs at $P_{\rm in} < 1/27$.

B. Nonlinear fano resonances

In contrast to its linear counterpart, the nonlinear system may support complete suppression of the transmission ($T = 0$), that is, nonlinear Fano resonances [11]. From Eq. (11) it follows that $\psi_A^{(0)} + \psi_B^{(0)} = 2iICV^{-1}\sin k$ for $\psi_0 = T = 0$. The substitution of this into Eq. (12) leads to the system

$$(E-\omega)\,\psi_{A,B}^{(0)} \pm i\left(\gamma_0 - \gamma_2\left|\psi_{A,B}^{(0)}\right|^2\right)\psi_{A,B}^{(0)} = 0, \tag{14}$$

which gives rise to a *continuous family* of symmetric nonlinear Fano resonances, with $\omega = E$:

$$\psi_{A,B}^{(0)} = i\sqrt{\gamma_0/\gamma_2}\exp(\pm i\delta), \tag{15}$$

$$\cos\delta = (2V)^{-1}\sqrt{(4C^2 - E^2)(\gamma_2/\gamma_0)\,P_{\rm in}}. \tag{16}$$

(Recall the above EIT-like symmetric family, corresponding to $|T/I|^2 \equiv 1$, had $|\psi_{A,B}^{(0)}| \neq \sqrt{\gamma_0/\gamma_2}$ and $\cos\delta = 0$.) Equation (16) imposes condition $\cos^2\delta \leqslant 1$, that is,

$$P_{\rm in} \leqslant 4V^2(\gamma_0/\gamma_2)(4C^2 - E^2)^{-1}. \tag{17}$$

Thus, at $\omega = E$, *a family of the nonlinear Fano resonances* with the symmetrically excited side-coupled $\mathcal{PT}$ elements exists in this interval of the intensity of the incident wave. The novelty of the result is that the Fano resonance is usually obtained as an isolated solution.

C. Asymmetric scattering regimes

Equation (14) for the nonlinear Fano resonances also admits two *ultimate asymmetric states*, with the vanishing excitation at one of the $\mathcal{PT}$ elements: $\omega = E$ and

$$\psi_A^{(0)} = \sqrt{\gamma_0/\gamma_2}, \psi_B^{(0)} = 0, \tag{18}$$

or vice versa, with $A \rightleftarrows B$. In either case, this solution exists at $P_{\rm in} = V^2(\gamma_0/\gamma_2)(4C^2 - E^2)^{-1}$. This point falls into the range (17) of the existence of the symmetric Fano-resonance solutions, which implies an intrinsic bistability of the nonlinear Fano resonances.

Equation (12) gives rise to other asymmetric scattering regimes, which, in particular, may produce a strong resonant amplification of the transmitted wave. The complete set of the asymmetric scattering states is depicted by the red curves in Fig. 3.

D. Stability

The stability of the above analytical solutions was checked in direct simulations of Eqs. (7)–(9). The results demonstrate that the ultimate asymmetric state (18) with the nonzero excitation at the linear-loss element is stable, while its counterpart with the excitation at the linear-gain element is unstable, transforming itself into an oscillatory mode (apparently, a limit cycle); see the left panels in Fig. 4. These results can be understood following the similarity to previously studied systems composed of coupled cores with the linear gain and loss acting separately in them, which also give rise to a pair of stable and unstable modes [13].

Symmetric Fano-resonance modes (15) are unstable and also develop oscillatory states, with a very low transmissivity and spontaneously broken symmetry between the $\mathcal{PT}$ elements, $|\psi_A| \neq |\psi_B|$, as shown in the right panels in Fig. 4. In fact, such asymmetric states correspond to the nearly perfect Fano resonance in Fig. 3 at $P_{\rm in} = 4$. The resonantly amplified transmission regimes are also unstable, due to enhancement of the field on the linear-gain element.

IV. NONPROPAGATING MODES

The dispersion relation for Eq. (8) demonstrates that frequencies of the propagating waves belong to the respective

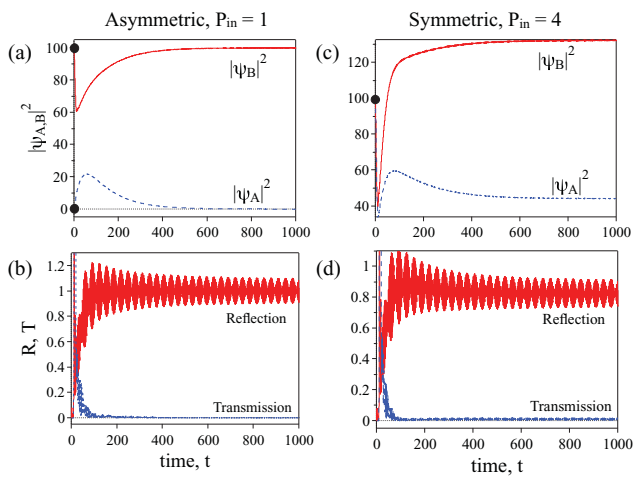

FIG. 4. (Color online) The perturbed evolution of those asymmetric (left) and symmetric (right) Fano-resonance modes from Fig. 3 which are dynamically unstable. Filled circles indicate the initial states.

phonon band, $|\omega| < 2C$. Above the band, at $\omega > 2C$, it is possible to find exact solutions for localized modes pinned to the $\mathcal{PT}$ complex:

$$\psi_n = \frac{V(\tilde{\psi}_A + \tilde{\psi}_B)}{2\omega - \sqrt{\omega^2 + 4C^2}} \left(\frac{\sqrt{\omega^2 + 4C^2} + \omega}{2C} \right)^{-|n|}, \quad (19)$$

where $\tilde{\psi}_A$ and $\tilde{\psi}_B$ are given by the above solutions for the symmetric and asymmetric modes, that is, Eqs. (2), (3) and Eqs. (5), (6), with V and E replaced by $\tilde{V} \equiv V^2/(2\omega - \sqrt{\omega^2 + 4C^2})$ and $\tilde{E} \equiv E + \tilde{V}$. This means that the solution for the nonpropagating symmetric modes remains explicit, while Eq. (5) for the asymmetric mode takes the form of a quartic equation for ω_{AS}.

V. CONCLUSIONS

We have introduced the $\mathcal{PT}$-symmetric systems that feature balanced gain and loss at both linear and nonlinear levels. For the basic dimer system, we have produced a complete set of analytical solutions, which feature the spontaneous symmetry breaking and multistability. We have demonstrated, also in the analytical form, that the symmetric and asymmetric excitations in the dimer, if it is coupled to the linear chain, give rise to a variety of nonlinear Fano resonances, including the bistability between them, as well as perfect-transmission regimes resembling EIT, and the resonantly amplified transmission. The coexistence of these scattering channels suggests applications to the design of data-processing schemes. Nonpropagating modes in the chain, pinned to the $\mathcal{PT}$ scatterer, were found too.

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