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An improved explicit bound on $|\zeta(\frac{1}{2}+it)|$

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ABSTRACT

This article proves the bound $|\zeta(\frac{1}{2}+it)| \le 0.732t^{\frac{1}{6}} \log t$ for $t \ge 2$, which improves on a result by Cheng and Graham. We also show that $|\zeta(\frac{1}{2}+it)| \le 0.732|4.678+it|^{\frac{1}{6}} \log |4.678+it|$ for all t.

1. Introduction

The Riemann zeta-function $\zeta(s)$ is known [4] to satisfy $\zeta(\frac{1}{2}+it) \ll_\epsilon t^{\frac{32}{205}+\epsilon}$ for all $t \gg 1$ and for every $\epsilon > 0$. Explicit estimates of the sort

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \le k_1 t^{k_2} (\log t)^{k_3}, \quad (t \ge t_0)$$

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are difficult to produce since attempts at small values of k_2 lead to complicated arguments in the calculation of k_1. Using the approximate functional equation and the Riemann–Siegel formula one may show that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \le \frac{4}{(2\pi)^{\frac{1}{4}}} t^{\frac{1}{4}}, \quad (t \ge 0.2). \tag{1}$$

Lehman [7, Lem. 2] proved this for $t \ge 128\pi$ — see also [10, Thm. 2] and [14, Thm. 1] — one may verify that (1) holds in the range $0.2 \le t < 128\pi$ by a direct computation. The only other result of which we are aware is due to Cheng and Graham [1], viz.

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \le 3t^{\frac{1}{6}} \log t, \quad (t \ge e). \tag{2}$$

The upper bound in (2) is smaller than that in (1) when $t \ge 1.4\times 10^{21}$. This is unfortunate since for some problems one seeks information for $t \ge T_0$, where T_0 is at most the height to which the Riemann hypothesis has been verified. The first author [8] has confirmed that for $0 \le t \le 3.06 \times 10^{10}$ all non-trivial zeroes of $\zeta(\sigma + it)$ lie on the critical line.

In [13, (5.4)] the second author showed that one could combine Theorem 3 of [1] with (1) to show that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \le 2.38t^{\frac{1}{6}} \log t, \quad (t \ge e),$$

which is better than the bound in (1) only when $t \ge 10^{19}$. The purpose of this article is to revisit the paper by Cheng and Graham and to prove

Theorem 1.

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \le 0.732t^{\frac{1}{6}} \log t, \quad (t \ge 2).$$

The bound in Theorem 1 improves on that in (1) whenever $t \ge 5.868 \times 10^9$. Three applications are apparent: [9,12,13] which respectively relate to explicit estimates for zero-density theorems, bounding $\int_0^T S(t)\,dt$, and bounding $S(t)$, where $\pi S(t)$ is the argument of the zeta-function on the critical line. More precisely, when t does not coincide with an ordinate of a zero of $\zeta(\sigma + it)$, $S(t)$ is defined as

$$S(t) = \pi^{-1} \arg \zeta\left(\frac{1}{2}+it\right),$$

where the argument is determined via continuous variation along the straight lines connecting $2, 2+it$ and $\frac{1}{2}+it$, with $S(0) = 0$. If t is such that $\zeta(\sigma + it) = 0$ then define $S(t)$ to be $\frac{1}{2}\lim_{\epsilon\to 0}\{S(t-\epsilon) + S(t+\epsilon)\}$.

The estimate for $S(t)$ can be improved immediately to give

Corollary 1. *If $T \geq e$, then*

$$\left|S(T)\right| \leq 0.110 \log T + 0.290 \log\log T + 2.290.$$

Proof. Using Theorem 1 one may take $(k_1, k_2, k_3) = (0.732, 1/6, 1)$ in [13, (4.8)]. Instead of choosing $Q_0 = 2$ on page 291 of [13], we choose $Q_0 = 5$. The choice of $\eta = 0.064$, $r = 2.032$ on the same page establishes Corollary 1. □

This improves the constant term in Theorem 1 of [13] from 2.510 to 2.290.

The improvement of Theorem 1 over the result in [1] comes from two ideas. First, an explicit form of the 'standard' approximate functional equation is used (cf. Lemma 3), in which one needs to estimate sums of the form $\sum_{n \leq Y} n^{it}$, where $t^{\frac{1}{2}} \ll Y \ll t^{\frac{1}{2}}$. This requires only one round of applying estimates for exponential sums. Cheng and Graham considered an approximation to $\zeta(\frac{1}{2} + it)$ in which one needs to estimate a longer sum with $t \ll Y \ll t$. They require two different estimates for exponential sums to cover this range. Second, some minor adjustments are made to some of the results in [1], and more variables are optimised.

We prove some necessary lemmas in Section 2. We prove Theorem 1 for large t in Section 3 and for small t in Section 4. We conclude with some computational remarks in Section 5.

2. Preparatory lemmas

It is necessary to record some estimates for exponential sums. Versions of the following lemmas without explicit constants can be found in [11, Thm. 5.9 and Lemma 5.10]. Slightly coarser explicit versions can be found in [5, p. 36] and [3, Lemma 2.2]

Lemma 1. *Assume that $f(x)$ is a real-valued function with two continuous derivatives when $x \in [N+1, N+L]$. If there exist two real numbers $V < W$ with $W > 1$ such that*

$$\frac{1}{W} \leq \left|f''(x)\right| \leq \frac{1}{V}$$

for $x \in [N+1, N+L]$, then

$$\left|\sum_{n=N+1}^{N+L} e^{2\pi i f(n)}\right| \leq \left(\frac{L-1}{V} + 1\right)\left(2\sqrt{\frac{2}{\pi}} W^{1/2} + 2\right) + 1.$$

Proof. This is Lemma 3 in [1] with three slight adjustments. First, when applying the mean-value theorem on the first line of page 1268 of [1] one obtains $k \leq (L-1)/V + 2$ instead of $k \leq L/V + 2$. Second, when estimating the $2(k-1)$ intervals trivially, one may note that there are two intervals of length $W\Delta + 1$, namely those intervals from $(C_k - \Delta, C_k)$ and $(C_1, C_1 + \Delta)$, whereas there are $k - 2$ intervals of length $2W\Delta + 1$. Third, we retain the constant $2\sqrt{2/\pi}$ as opposed to (the only slightly larger) $8/5$. □

Lemma 2. *Let $f(n)$ be a real-valued function and let M be a positive integer. Then*

$$\left|\sum_{n=N+1}^{N+L} e^{2\pi i f(n)}\right|^2 \le \frac{L(L+M-1)}{M} + \frac{2(L+M-1)}{M}\sum_{m=1}^{M-1}\left(1-\frac{m}{M}\right)\max_{K\le L}\left|\sum_{m,K}\right|, \tag{3}$$

where

$$\sum_{m,K} = \sum_{n=N+1}^{N+K} e^{2\pi i(f(n+m)-f(n))}.$$

Proof. This is Lemma 5 in [1] with $L+M$ changed to $L+M-1$, a substitution that is clearly permitted as per the displayed equation at the bottom of [1, p. 1272]. This differs from Lemma 5.10 in [11] in three respects: there is no upper restriction on M, the coefficients are smaller (in [11] both terms in (3) have 4 as their leading coefficients), and the factor $(1-m/M)$ is present. □

Lemma 3. *For $t \ge 100$,*

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \le 2\left|\sum_{n\le\sqrt{\frac{t}{2\pi}}} n^{-\frac{1}{2}-it}\right| + 1.53t_0^{-\frac{1}{4}} + 3.23t_0^{-\frac{3}{4}}. \tag{4}$$

Proof. We use Theorem 1 of [10], from which it follows that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \le 2\left|\sum_{n\le\sqrt{\frac{t}{2\pi}}} n^{-\frac{1}{2}-it}\right| + \frac{|\Gamma(\frac{1}{2}+it)|}{2\pi}e^{\frac{1}{2}\pi t}(2\pi)^{\frac{1}{2}}\left|g\left(\frac{t}{2\pi}\right)\right| + \left|R(s)\right|, \tag{5}$$

where, in Titchmarsh's expression for $R(s)$, there appears to be a blemish on the page: the ^{8} ought to be $\frac{8}{3}$, as per Eq. (4.1) of [10]. By the last line on p. 235 of [10] we have

$$\left|g\left(\frac{t}{2\pi}\right)\right| \le (2\pi)^{\frac{1}{4}}t^{-\frac{1}{4}}\left|\frac{\cos 2\pi(x^2-x-\frac{1}{16})}{\cos 2\pi x}\right|, \tag{6}$$

where $0 \le x \le 1$. By (2.6) and (2.7) of [6] we have $|g(\frac{t}{2\pi})| \le (\cos\frac{\pi}{8})(2\pi)^{\frac{1}{4}}t^{-\frac{1}{4}}$. With the version of Stirling's theorem given in Lemma ϵ in [10] we can now bound the second term in (5). Finally, using Titchmarsh's expression for $R(s)$, we note that $R(s)t^{-\frac{3}{4}}$ is decreasing in t provided that $t > (5/2)^3$. A computation of the constants involved proves the lemma. □

3. Proof of Theorem 1 for large t

Write the sum in (4) as

$$\sum_{n\le A_0 t^{\frac{1}{3}}} n^{-\frac{1}{2}-it} + \sum_{A_0 t^{\frac{1}{3}}<n\le\sqrt{\frac{t}{2\pi}}} n^{-\frac{1}{2}-it}$$

provided that the interval of summation in the second sum is non-empty, that is, provided that

$$t_0 > A_0^6(2\pi)^3. \tag{7}$$

The trivial estimate gives

$$\left|\sum_{n\le A_0 t^{\frac{1}{3}}} n^{-\frac{1}{2}-it}\right| \le \sum_{n\le A_0 t^{\frac{1}{3}}} n^{-\frac{1}{2}} \le 2A_0^{\frac{1}{2}} t^{\frac{1}{6}} - 1.$$

Now consider

$$X_j = A_0 k^j t^{\frac{1}{3}},$$

where $k > 1$ is a parameter to be determined later, and $j = 0, 1, 2, \ldots, J$, where

$$J \le \frac{\frac{1}{6}\log t - \log(A_0(2\pi)^{\frac{1}{2}})}{\log k} + 1.$$

Also, let $N_j = [X_j]$ be the integer part of X_j. It follows that

$$\sum_{A_0 t^{\frac{1}{3}}<n\le\sqrt{\frac{t}{2\pi}}} n^{-\frac{1}{2}-it} = \sum_{j=1}^{J} \sum_{n=N_{j-1}+1}^{\min\{N_j,\sqrt{\frac{t}{2\pi}}\}} n^{-\frac{1}{2}-it},$$

whence, by partial summation we have

$$\left|\sum_{A_0 t^{\frac{1}{3}}<n\le\sqrt{\frac{t}{2\pi}}} n^{-\frac{1}{2}-it}\right| \le \sum_{j=1}^{J} \frac{1}{X_{j-1}^{\frac{1}{2}}} \max_{L\le N_j-N_{j-1}} \left|\sum_{n=N_{j-1}+1}^{N_{j-1}+L} e^{-it\log n}\right|. \tag{8}$$

Denote the sum over n in (8) by S_j. We may estimate S_j using Lemmas 1 and 2. First apply Lemma 2 to S_j and thence apply Lemma 1 to the resulting

$$\sum_{m,K} = \sum_{n=N_{j-1}+1}^{N_{j-1}+K} e^{-it(\log(n+m)-\log n)}.$$

Choose $M = [k^j\theta] + 1$, for some θ to be determined later. We impose the additional restriction that $M \geq 2$ so as to use the bounds in (10) without worry.

We need to determine V and W in Lemma 1. We have

$$f(x) = -\frac{t}{2\pi}\big(\log(x+m) - \log x\big), \qquad \left|f''(x)\right| = \frac{tm}{2\pi}\left(\frac{m+2x}{x^2(x+m)^2}\right).$$

Since $(m+2x)/(x(x+m))^2$ is decreasing in both x and m we take $m = 0$, $x = A_0k^{j-1}t^{\frac{1}{3}}$, and $m = M - 1 \leq k^j\theta$, $x = A_0k^jt^{\frac{1}{3}}$ to find that $1/W \leq |f''(x)| \leq 1/V$, where

$$V = \frac{\pi A_0^3 k^{3j}}{k^3 m}, \qquad W = \frac{\pi k^{3j} A_0^3}{m}\left(1 + \frac{\theta}{A_0 t_0^{\frac{1}{3}}}\right)^2.$$

In order to apply Lemma 1 it remains only to note that

$$L \leq (k-1)X_{j-1} + 1 \leq (k-1)A_0k^{j-1}t^{\frac{1}{3}} + 1. \tag{9}$$

One may now apply Lemma 1 to find that

$$\left|\sum_{m,K}\right| \leq A_1 t^{\frac{1}{3}} m^{\frac{1}{2}} k^{-\frac{1}{2}j} + A_2 t^{\frac{1}{3}} m k^{-2j} + A_3 m^{-\frac{1}{2}} k^{\frac{3}{2}j} + 3,$$

where

$$A_1 = \frac{2\sqrt{2}(k-1)k^2Y_0}{\pi A_0^{\frac{1}{2}}}, \qquad A_2 = \frac{2(k-1)k^2}{\pi A_0^2}, \qquad A_3 = 2\sqrt{2}A_0^{\frac{3}{2}}Y_0, \qquad Y_0 = 1 + \frac{\theta}{A_0 t_0^{\frac{1}{3}}}.$$

One of the advantages of using Lemma 1 over Lemma 3 in [1] is that, according to (9), $L - 1$ generates only one term.

The displayed formulae on page 1277 of [1] show that

$$\sum_{1\leq m\leq M-1}\left(1 - \frac{m}{M}\right)m^{\frac{1}{2}} \leq \frac{4}{15}M^{\frac{3}{2}}, \qquad \sum_{1\leq m\leq M-1}\left(1 - \frac{m}{M}\right)m^{-\frac{1}{2}} \leq \frac{4}{3}M^{\frac{1}{2}}. \tag{10}$$

Applying this gives

$$\frac{1}{M}\sum_{m=1}^{M-1}\left(1 - \frac{m}{M}\right)\left|\sum_{m,K}\right| \leq \frac{4}{15}A_1 t^{\frac{1}{3}} M^{\frac{1}{2}} k^{-\frac{1}{2}j} + \frac{1}{6}A_2 t^{\frac{1}{3}} M k^{-2j} + \frac{4}{3}A_3 M^{-\frac{1}{2}} k^{\frac{3}{2}j} + \frac{3}{2}.$$

Return now to Lemma 2

$$|S_j|^2 \leq \frac{L(L+M-1)}{M} + 2(L+M-1)\left(\frac{1}{M}\sum_{m=1}^{M-1}\left(1 - \frac{m}{M}\right)\left|\sum_{m,K}\right|\right).$$

For $\alpha > 0$, $(L+M-1)M^{\alpha}$ is an increasing function of M; $(L+M-1)/M$ is decreasing. We use an upper bound for the numerator and a lower bound for the denominator in $(L+M-1)/M^{1/2}$. With $M = [k^j\theta] + 1$ we have,

$$|S_j|^2 \le B_1 k^j t^{\frac{2}{3}} + B_2 t^{\frac{2}{3}} + B_3 k^j t^{\frac{1}{3}} + B_4 k^{2j} t^{\frac{1}{3}},$$

where

$$\begin{aligned}
A_4 &= \frac{(k-1)^2 A_0^2}{k^2\theta}\left(1+\frac{1}{(k-1)A_0 t_0^{\frac{1}{3}}}\right)\left(1+\frac{\theta k}{(k-1)A_0 t_0^{\frac{1}{3}}}\right)\\
A_5 &= \frac{2(k-1)A_0}{k}\left(1+\frac{1}{(k-1)A_0 t_0^{\frac{1}{3}}}+\frac{\theta k}{(k-1)A_0 t_0^{\frac{1}{3}}}\right)\\
A_6 &= \frac{4}{15}A_1\theta^{\frac{1}{2}}\left(1+\frac{1}{k\theta}\right)^{\frac{1}{2}}, \qquad A_7 = \frac{A_2\theta}{6}\left(1+\frac{1}{k\theta}\right)\\
A_8 &= \frac{4A_3}{3\theta^{\frac{1}{2}}}, \qquad B_1 = A_4 + A_5A_6, \qquad B_2 = A_5A_7, \qquad B_3 = \frac{3}{2}A_5, \qquad B_4 = A_5A_8.
\end{aligned} \tag{11}$$

Using the inequality $\sqrt{(x+y+\cdots)} \le \sqrt{x} + \sqrt{y} + \cdots$ we have

$$\sum_{j=1}^{J}\frac{1}{X_{j-1}^{\frac{1}{2}}}|S_j| \le \frac{k^{\frac{1}{2}}}{A_0^{\frac{1}{2}}}\left(\left(\sqrt{B_1}t^{\frac{1}{6}}+\sqrt{B_3}\right)\sum_{j=1}^{J}1+\sqrt{B_2}t^{\frac{1}{6}}\sum_{j=1}^{J}k^{-\frac{1}{2}j}+\sqrt{B_4}\sum_{j=1}^{J}k^{\frac{1}{2}j}\right).$$

Since

$$\sum_{j=1}^{J}k^{-\frac{1}{2}j} = k^{-\frac{1}{2}}\left(\frac{1-k^{-\frac{1}{2}J}}{1-k^{-\frac{1}{2}}}\right),$$

this gives

$$\sum_{j=1}^{J}\frac{1}{X_{j-1}^{\frac{1}{2}}}|S_j| \le \left(\frac{k}{A_0}\right)^{\frac{1}{2}}\left(C_1 t^{\frac{1}{6}}\log t + C_2 t^{\frac{1}{6}} + C_3 t^{\frac{1}{12}} + C_4\log t + C_5\right),$$

where

$$\begin{aligned}
C_1 &= \frac{\sqrt{B_1}}{6\log k}, \qquad C_2 = \sqrt{B_1}\left(1-\frac{\log(A_0(2\pi)^{\frac{1}{2}})}{\log k}\right)+\frac{\sqrt{B_2}k^{-\frac{1}{2}}}{1-k^{-\frac{1}{2}}}\\
C_3 &= \frac{\sqrt{B_4}k}{A_0^{\frac{1}{2}}(2\pi)^{\frac{1}{4}}(k^{\frac{1}{2}}-1)} - \frac{\sqrt{B_2}A_0^{\frac{1}{2}}(2\pi)^{\frac{1}{4}}}{k(1-k^{-\frac{1}{2}})}, \qquad C_4 = \frac{\sqrt{B_3}}{6\log k}\\
C_5 &= \sqrt{B_3}\left(1-\frac{(\log A_0(2\pi)^{\frac{1}{2}})}{\log k}\right) - \frac{\sqrt{B_4}k^{\frac{1}{2}}}{k^{\frac{1}{2}}-1}.
\end{aligned}$$

This means that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \le D_1 t^{\frac{1}{6}} \log t + D_2 t^{\frac{1}{6}} + D_3 t^{\frac{1}{12}} + D_4 \log t + D_5,$$

where

$$D_1 = 2C_1\left(\frac{k}{A_0}\right)^{\frac{1}{2}}, \qquad D_2 = 2\left(2A_0^{\frac{1}{2}} + C_2\left(\frac{k}{A_0}\right)^{\frac{1}{2}}\right), \qquad D_3 = 2C_3\left(\frac{k}{A_0}\right)^{\frac{1}{2}},$$
$$D_4 = 2C_4\left(\frac{k}{A_0}\right)^{\frac{1}{2}}, \qquad D_5 = 2\left(C_5\left(\frac{k}{A_0}\right)^{\frac{1}{2}} - 1 + \frac{0.77}{t_0^{\frac{1}{4}}} + \frac{1.62}{t_0^{\frac{3}{4}}}\right). \tag{12}$$

To reduce the right side of (12) as much as possible it is desirable to choose a large value of t_0. We shall, in the next section, use (1) to handle smaller values of t. With this in mind, the choice

$$k = 1.16, \qquad \theta = 7.5, \qquad A_0 = 3.37, \qquad t_0 = 5.867 \times 10^9$$

means that $|\zeta(\frac{1}{2}+it)| \le 0.732 t^{\frac{1}{6}} \log t$ for $t \ge t_0$, (7) is satisfied, and that $M \ge 2$. We now turn our attention to $t < 5.867 \times 10^9$.

4. Proof of Theorem 1 for small t

Lemma 4. *For $t \in [2, 5.867 \times 10^9]$ we have*

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| < 0.732 t^{\frac{1}{6}} \log t.$$

Proof. The bound (1) is tighter than our new bound at $t = 5.867 \times 10^9$ and remains so for t all the way down to $t = 226.7088\ldots$. We checked the range $[2, 230]$ rigorously by computer as follows.

We implemented an interval arithmetic version of the Euler–MacLaurin summation formula that, given an interval $\underline{t}$ returns an interval that includes $|\zeta(\frac{1}{2}+it)|$ for all $t \in \underline{t}$. We divided the line segment $[2, 230]$ into pieces of length $1/1024$ and for each piece, checked that $|\zeta(\frac{1}{2}+it)|$ did not exceed our bound. Specifically, if we are considering $\underline{t} = [a, a+1/1024]$ and we know that for $t \in \underline{t}$ we have $|\zeta(\frac{1}{2}+it)| \in [x, y]$, then we check $y < 0.732 a^{\frac{1}{6}} \log a$. No counterexamples exist for $t \in [2, 230]$ and this establishes the lemma. □

Corollary 2. *For t real and $Q \ge 4.678$ we have*

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| < 0.732 |Q+it|^{\frac{1}{6}} \log |Q+it|.$$

Table 1
Bounds on $|\zeta(\frac{1}{2}+it)| \leq At^{\frac{1}{6}}\log t$ for ranges of t.

t	A
$[2, 200]$	0.7090
$[200, 10^3]$	0.4873
$[10^3, 10^4]$	0.4682
$[10^4, 10^5]$	0.4217
$[10^5, 10^6]$	0.3765
$[10^6, 10^7]$	0.3238
$[10^7, 10^8]$	0.2854

Proof. For $|t| \geq 2$ we use Lemma 4. For $t \in (-2, 2)$ we know that $|\zeta(\frac{1}{2}+it)|$ attains a maximum at $t = 0$ so we determine a Q such that

$$\left|\zeta\left(\frac{1}{2}\right)\right| < 0.732Q^{\frac{1}{6}}\log Q$$

and we are done. □

5. Conclusion

Since an Euler–MacLaurin computation of $\zeta(\frac{1}{2}+it)$ becomes inefficient as t increases, we also implemented an interval version of the Riemann–Siegel formula (R-S) for $t \geq 200$. Above this height we have explicit error bounds due to Gabcke [2]. The only nuance is that the main sum of R-S runs from 1 to $\lfloor\sqrt{t/2\pi}\rfloor$ and we must be careful not to compute with intervals $\underline{t} = [a, b]$ such that $\lfloor\sqrt{a/2\pi}\rfloor \neq \lfloor\sqrt{b/2\pi}\rfloor$. We get around this by using Euler–MacLaurin for such intervals.

So armed, we can continue to compute $|\zeta(\frac{1}{2}+it)|$ for $t \in [a, b]$ and each time we come across an interval where (possibly) $|\zeta(\frac{1}{2}+i[a, b])|$ sets a new record $[x, y]$, we store a and y. Running through the data files produced, it is a trivial matter to find an A such that $|\zeta(\frac{1}{2}+it)| < At^{\frac{1}{6}}\log t$ throughout the range. Our results are summarised in Table 1.

It seems that the bound in Theorem 1 is still very far from optimal.

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