Competitive Equilibriums and Social Shaping for Multi-Agent Systems ⋆

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Abstract

In this paper, we study multi-agent systems with decentralized resource allocations. Agents have local demand and resource supply, and are interconnected through a network designed to support sharing of the local resource; and the network has no external resource supply. It is known from classical welfare economics theory that by pricing the flow of resource, balance between the demand and supply is possible. Agents decide on the consumed resource, and perhaps further the traded resource as well, to maximize their payoffs considering both the utility of the consumption, and the income from the trading. When the network supply and demand are balanced, a competitive equilibrium is achieved if all agents maximize their individual payoffs, and a social welfare equilibrium is achieved if the total agent utilities are maximized. First, we consider multi-agent systems with static local allocations, and prove from duality theory that under general convexity assumptions, the competitive equilibrium and the social welfare equilibrium exist and agree. Compared to similar results in the literature based on KKT arguments, duality theory provides a direct way for connecting the two notions and for a more general (e.g. nonsmooth) class of utility functions. Next, we show that the agent utility functions can be prescribed in a family of socially admissible functions, under which the resource price at the competitive equilibrium is kept below a threshold. Finally, we extend the study to dynamical multi-agent systems where agents are associated with dynamical states from linear processes, and prove that the dynamic the competitive equilibrium and social welfare equilibrium continue to exist and coincide with each other. In addition, we also present a recursive representation of the competitive equilibriums using dynamic programming, and a receding horizon approach for smoothing the dynamic pricing as a dynamic competitive equilibrium social shaping method.

Key words:

1 Introduction

Next generation technologies are leveraging the internet of things (IoT) to support critical infrastructure systems including energy distribution and automotive transportation, and are being organized as interconnected multi-agent systems [7]. Such systems involve data collection, resource allocation, and control coordination between geographically distributed subsystems. Each subsystem, termed an 'agent', is an intelligent functioning unit with its own decisions, objectives and preferences, and remarkably, network-level goals such as consensus, formation, and optimality can be achieved by agents interacting with others over a *network* e.g., [2–6]. The underlying network for multi-agent systems can be physical such as transmission lines in a power grid, non-physical such as wireless communication channels, or a combination of the two. The key promise of organizing subsystems into networked multi-agent systems is a radical improvement in scalability, efficiency, and sustainability through shared inputs and outputs, and coordinated decisions and controls.

One important problem for multi-agent system operation is efficient resource allocation, where demand and supply must be balanced for efficient and secure operations at the system level. In a typical resource allocation problem, agents have local demand and internal

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and external resource suppliers, interconnected through a network that allows for transmission of the resource. In light of classical welfare economics theory [11, 12], careful pricing of the transmission flow potentially balances the demand against the supply across the entire system. Agents decide on the resource consumed, and perhaps further the resource traded, to maximize their payoffs considering both the utility from production/consumption, and income from the trading. When network supply and demand is balanced, a competitive equilibrium is achieved if all agents maximize their individual payoffs; a social welfare equilibrium is achieved if the total agent utilities are maximized [10].

The concept of resource allocation via a competitive equilibrium has been widely applied in the smart grid literature, where consumers and suppliers of electrical energy make trading decisions over an energy market to achieve a competitive equilibrium [14–20]. The dynamics of power networks can even be coupled with the market pricing dynamics [21–25], where energy price becomes an effective controller for quasi steady-state grid frequency stability. We refer to [13] for a comprehensive survey on transactive energy systems. Moreover, in climate-economy frameworks, a market approach was also proposed as a principled way for mitigating global carbon emissions while balancing regional interests [27,28]. The price for carbon emissions is calculated under the competitive equilibrium of the carbon market, becoming a benchmark for the social cost of carbon [29]. We refer to [30] for an excellent introduction to the dynamic integration of climate and economy models from a feedback system perspective.

Despite the aforementioned successes in coordinating multi-agent systems via market pricing in critical engineering and societal problems, the resilience of the pricing mechanism is potentially a serious challenge even for theoretically optimal equilibrium conditions. In fact, it is well known in welfare economics that an efficient market equilibrium may imply Pareto optimality, while individual equity and fairness of the agents may be discarded entirely [31]. In the context of power grids, market-driven prices have exceeded affordable thresholds for an individual household. Specifically, in February 2021, Texas encountered extreme cold weather conditions resulting in a power outage disaster throughout the state, with customers on rolling blackouts. In Texas, competitive pricing did not curtail non-critical loads to balance the grid under the power shortage conditions [32]. Residential customers in Texas paying wholesale electricity prices during the power shortage event reported electricity bill shock with their bills exceeding previous invoices by a factor of over one hundred. Moreover, in the context of climate change mitigation, it was only possible for the Paris agreement to be reached in 2016 after a decade-long negotiation, as the estimated social cost of carbon was perceived as unfair among different nations [33]. It follows that competitive equilibriums for multi-agent systems are in need of social shaping at the network level: only equilibriums within a prescribed range of fairness can be accepted; only individual agent utility functions within a prescribed range of social responsibilities can be admissible. More importantly, the two directions of social shaping should be consistent: socially admissible agent utility functions should always lead to socially acceptable equilibriums.

In this paper, multi-agent systems with decentralized resource allocations are entirely self-sustained, i.e. there is no external resource supply. First, we consider multi-agent systems with static local allocations, and we prove that under general convexity assumptions, the competitive equilibrium and the social welfare equilibrium exist and agree. Our proof is based on duality theory, as opposed to the commonly attributed KKT arguments (e.g. [18]) that support similar results. Here duality theory provides a more direct way of connecting the competitive equilibrium to the social welfare equilibrium, and supports a more general (e.g. nonsmooth) class of utility functions. Next, we investigate the case when the pricing under a competitive equilibrium is associated with an upper bound for social acceptance. By means of constructive analysis, we show that the agent utility function is prescribed by a family of socially admissible quadratic functions, under which the pricing at the competitive equilibrium is always below a threshold. Finally, we extend the study to dynamical multi-agent systems where agents are associated with dynamical states from linear processes, and prove that the dynamic competitive equilibrium and the social welfare equilibrium continue to exist and coincide in an optimal control context. We also present a recursive way of representing and computing the competitive equilibriums in view of the dynamic programming principle. In order to shape the dynamic pricing in the sense that the pricing trajectory would be stationary, we propose a receding horizon approach for smoothing the dynamic pricing.

Some preliminary results of the work were submitted for possible presentation at the IEEE Conference on Decision and Control in 2021[1]. The remainder of the paper is organized as follows. In Section 2, we introduce the multi-agent system with static decisions. In Section 3, we formulate social shaping of competitive equilibriums. In Section 4, we formulate dynamic pricing for resource allocation of multi-agent systems with an underlying dynamical process. Some concluding remarks are presented in Section 5.

2 Static Multi-Agent Systems

In this section, we study multi-agent systems with static resource allocation and load decisions.

2.1 Competitive Equilibrium for Static Multi-Agent Systems

We consider a multi-agent system (MAS) with n agents. The agents are indexed in $\mathrm{V} = \{1, \dots, n\}$. We consider a basic MAS setup with static agent decisions on load allocations.

MAS with Static Agent Load Decisions (MAS-SALD). Each agent i holds a local resource of a_i units, and makes a (static) decision to allocate $x_i \in \mathbb{R}^{\geq 0}$ units of load for itself. The utility function related to agent i allocating x_i amount of load is $f_i(x_i) : \mathbb{R}^{\geq 0} \mapsto \mathbb{R}$. Consequently, agent i would incur an $a_i - x_i$ amount of surplus ($a_i > x_i$), or a shortcoming ($a_i < x_i$). We assume that there is a connected network among the n agents so that they can balance the surplus and shortcomings through a pricing mechanism. To be precise, each unit of resource moved across the network is priced at $\lambda \in \mathbb{R}$. Therefore, agent i will yield $(a_i - x_i)\lambda$ in income or expenditure.

Denoting $\mathbf{x} = (x_1 \dots x_n)^\top \in (\mathbb{R}^{\geq 0})^n$ as the network resource allocation profile, we introduce the following definitions.

Definition 1 *A pair of price-allocation decisions (λ^*, $\mathbf{x}^*$) is a competitive equilibrium for the MAS-SALD if the following conditions hold:*

(i) each agent i maximizes her combined payoff at x_i^*, i.e., x_i^* is an optimizer for the following constrained optimization problem:

$$\begin{aligned} \max_{x_i} \quad & f_i(x_i) + \lambda^*(a_i - x_i) \\ \text{s.t.} \quad & x_i \in \mathbb{R}^{\geq 0}. \end{aligned} \tag{1}$$

(ii) the total demand and supply are balanced across the network:

$$\sum_{i=1}^{n} x_i^* = \sum_{i=1}^{n} a_i. \tag{2}$$

Definition 2 *A resource allocation profile $\mathbf{x}^\star$ is a social welfare equilibrium for the MAS-SALD if it is a solution to the following optimization problem:*

$$\begin{aligned} \max_{\mathbf{x}} \quad & \sum_{i=1}^{n} f_i(x_i) \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} a_i, \\ & x_i \in \mathbb{R}^{\geq 0};\ i \in \mathrm{V}. \end{aligned} \tag{3}$$

We present the following result which establishes the equivalence between a competitive equilibrium and a social welfare equilibrium. The result is based only on a concavity assumption for the utility functions f_i.

Theorem 1 *Consider the MAS-SALD. Suppose each $f_i(\cdot)$ is concave over the domain $\mathbb{R}^{\geq 0}$. Then the social welfare equilibrium(s) and the competitive equilibrium(s) coincide. To be precise, the following statements hold.*

(i) If (λ^*, $\mathbf{x}^*$) is a competitive equilibrium, then $\mathbf{x}^*$ is a social welfare equilibrium.

(ii) If $\mathbf{x}^\star$ is a social welfare equilibrium, then there exists $\lambda^* \in \mathbb{R}$ such that (λ^*, $\mathbf{x}^\star$) is a competitive equilibrium.

Proof. (i) Let $(\lambda^*, \mathbf{x}^*)$ be a competitive equilibrium. The proof proceeds by contradiction. Suppose that $\mathbf{x}^*$ is not a social welfare equilibrium. Then there must exist $\bar{\mathbf{x}}^*$ such that $\sum_{i=1}^n \bar{x}_i^* = \sum_{i=1}^n x_i^* = \sum_{i=1}^n a_i$, and $\sum_{i=1}^n f_i(x_i^*) < \sum_{i=1}^n f_i(\bar{x}_i^*)$. Consequently, there holds

$$\sum_{i=1}^{n} \Big(f_i(x_i^*) + \lambda^*(a_i - x_i^*) \Big) < \sum_{i=1}^{n} \Big(f_i(\bar{x}_i^*) + \lambda^*(a_i - \bar{x}_i^*) \Big). \tag{4}$$

This implies that there is at least one $m \in \mathrm{V}$ such that

$$f_m(x_m^*) + \lambda^*(a_m - x_m^*) < f_m(\bar{x}_m^*) + \lambda^*(a_m - \bar{x}_m^*),$$

which contradicts the fact that $(\lambda^*, \mathbf{x}^*)$ is a competitive equilibrium.

(ii) We propose a proof using duality. To be consistent with the literature on duality theory for continuous optimization, we denote $g_i = -f_i$, and rewrite (3) as

$$\begin{aligned} \min_{\mathbf{x}} \quad & \sum_{i=1}^{n} g_i(x_i) \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} a_i; x_i \in \mathbb{R}^{\geq 0}, i \in \mathrm{V}. \end{aligned} \tag{5}$$

Let $\mathbf{x}^\star$ be a social welfare equilibrium [1]. Then from its definition there holds $\sum_{i=1}^n x_i^\star = \sum_{i=1}^n a_i$. Since (5) is a convex optimization problem with a linear equality constraint, strong duality holds [9] and we denote the optimal primal and dual costs of (5) as p_* and d^*, respectively.

The Lagrangian function of (5) is

$$L(\mathbf{x}, \lambda) = \sum_{i=1}^{n} g_i(x_i) + \lambda \Big(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} a_i \Big) : (\mathbb{R}^{\geq 0})^n \times \mathbb{R} \mapsto \mathbb{R}.$$

[1] Note that $\mathbf{x}^\star$ must be finite as the feasible set of $\mathbf{x}$ is compact.

Then we introduce

$$L^*(\lambda) = \min_{\mathbf{x}\in(\mathbb{R}^{\geq 0})^n} L(\mathbf{x},\lambda).$$

If λ^* is dual optimal (i.e., $\lambda^* \in \arg\max_{\lambda\in\mathbb{R}} L^*(\lambda)$), there holds from strong duality [9] that

$$\begin{aligned} d^* = L^*(\lambda^*) &= \min_{\mathbf{x}\in(\mathbb{R}^{\geq 0})^n} L(\mathbf{x},\lambda^*) && (6)\\ &\leq L(\mathbf{x}^\star,\lambda^*) && (7)\\ &= \sum_{i=1}^n g_i(x_i^\star) && (8)\\ &= p^*. && (9)\end{aligned}$$

This implies the inequality from the above equation actually holds at equality:

$$\mathbf{x}^\star \in \arg\min_{\mathbf{x}\in(\mathbb{R}^{\geq 0})^n} L(\mathbf{x},\lambda^*). \quad (10)$$

Note that $L(\mathbf{x},\lambda^*) = \sum_{i=1}^n \big(g_i(x_i)+\lambda^*(x_i-a_i)\big)$ implies

$$x_i^\star \in \arg\max_{x_i\in\mathbb{R}^{\geq 0}} \big(f_i(x_i)+\lambda^*(a_i-x_i)\big). \quad (11)$$

Thus, we have proved that $(\lambda^*, \mathbf{x}_i^\star)$ is a competitive equilibrium. □

Clearly, in this basic multi-agent system setup, the price λ^* associated with a competitive equilibrium could take negative values. From an economic point of view, the resource at every agent must either be consumed or traded, and in cases of an oversupply of resource a negative price for load balancing would occur. From an optimization point of view, the price λ^* is the Lagrangian multiplier associated with an equality constraint for a constrained optimization problem, which can take positive or negative values. Proposition 1 indicates that as long as one agent is associated with a non-decreasing utility function, oversupply will not happen.

Proposition 1 *Consider the MAS-SALD. Suppose each $f_i(\cdot)$ is concave over the domain $\mathbb{R}^{\geq 0}$. Let (λ^*, $\mathbf{x}^*$) be a competitive equilibrium. Then $\lambda^* \geq 0$ if there exists at least one agent $m \in \mathrm{V}$ such that $f_m(\cdot)$ is non-decreasing.*

Proof. Let $f_m(\cdot)$ be non-decreasing. Assume $\lambda^* < 0$. Then $f_m(x_m)+\lambda^*(a_m-x_m)$ is a strictly increasing function with respect to x_m. Therefore, there can not be a finite x_m^* such that $x_m^* \in \arg\max_{x_m\in\mathbb{R}^{\geq 0}} \big(f_m(x_m) + \lambda^*(a_m - x_m)\big)$, contradicting the definition of the competitive equilibrium. □

2.2 MAS with Trading Decisions

In our standing multi-agent system model, agent i only decides on its allocated load x_i with the surplus/shortcoming $a_i - x_i$ returned to the network. Next we relax the network restriction, and introduce the following extended MAS.

MAS with Static Agent Load and Trading Decisions (MAS-SALTD) Here we extend the MAS-SALD. Each agent i further makes a decision on the traded amount of resource, denoted e_i. Specifically, e_i is physically constrained by x_i and a_i in the following way:

(i) if $x_i < a_i$, then agent i can sell, in which case $e_i \geq 0$ and $e_i \leq a_i - x_i$;

(ii) if $x_i \geq a_i$, then agent i can only buy, in which case $e_i \leq 0$ and $e_i = a_i - x_i$.

Let λ^* continue to represent the price for a unit of shared resource. Denote $\mathbf{e} = (e_1 \dots e_n)^\top$ as the vector representing the traded resource profile across the network.

Definition 3 *A triplet of price-allocation-trade profile (λ^*, $\mathbf{x}^*$, $\mathbf{e}^*$) is a competitive equilibrium for the MAS-SALTD if the following conditions hold:*

(i) Each agent i maximizes her combined payoff at ($\mathbf{x}^*$, $\mathbf{e}^*$) while meeting the physical constraint, i.e., (x_i^*, e_i^*) is an optimizer for the following constrained optimization problem:

$$\begin{aligned} \max_{x_i,e_i}\quad & f_i(x_i)+\lambda^* e_i \\ \text{s.t.}\quad & x_i + e_i \leq a_i, && (12)\\ & x_i\in\mathbb{R}^{\geq 0}, e_i\in\mathbb{R}. \end{aligned}$$

(ii) The total demand and supply are balanced across the network:

$$\sum_{i=1}^n e_i^* = 0. \quad (13)$$

Definition 4 *A pair of resource allocation-trade profile ($\mathbf{x}^\star, \mathbf{e}^\star$) is a social welfare equilibrium for the MAS-SALTD if it is an optimizer to the following optimization problem:*

$$\begin{aligned} \max_{\mathbf{x},\mathbf{e}}\quad & \sum_{i=1}^n f_i(x_i) && (14)\\ \text{s.t.}\quad & \sum_{i=1}^n e_i = 0, && (15)\\ & x_i + e_i \leq a_i; i\in\mathrm{V}, && (16)\\ & x_i\in\mathbb{R}^{\geq 0}, e_i\in\mathbb{R}; i\in\mathrm{V}. && (17)\end{aligned}$$

Theorem 2 *Consider the MAS-SALTD. Suppose each $f_i(\cdot)$ is concave over the domain $\mathbb{R}^{\geq 0}$. Then the social welfare equilibrium(s) and the competitive equilibrium(s) continue to coincide under the shared load decisions for the agents. To be precise, the following statements hold.*

(i) If $(\lambda^*, \mathbf{x}^*, \mathbf{e}^*)$ is a competitive equilibrium, then $(\mathbf{x}^*, \mathbf{e}^*)$ is a social welfare equilibrium.

(ii) If $(\mathbf{x}^\star, \mathbf{e}^\star)$ is a social welfare equilibrium, then there exists $\lambda^* \in \mathbb{R}$ such that $(\lambda^*, \mathbf{x}^\star, \mathbf{e}^\star)$ is a competitive equilibrium.

Proof. (i) The proof follows the same analysis as the proof of Theorem 1(i), where the desired connection is in place with the definitions of the optimization goals, respectively, for the competitive equilibrium and the social welfare equilibrium.

(ii) The key idea of the proof continues to be based on strong duality applied to the definition of the social welfare equilibrium, as the proof of Theorem 1. However, now the social welfare equilibrium contains additional inequality constraints; that is $x_i + e_i \leq a_i$, for all $i \in \mathrm{V}$. Inevitably, such constraints will lead to auxiliary dual variables, in addition to the dual variable related to the equality constraint $\sum e_i = 0$ if we simply repeat the proof of Theorem 1. In order to highlight the role of the dual variable corresponding to the equality constraint, and establish it as the price in the competitive equilibrium, we need a refined treatment. To this end, we define a set $\mathbb{X}_i$ for all $i \in \mathrm{V}$ in terms of the inequality constraint in (16) as $\mathbb{X}_i = \{(x_i, e_i) | x_i + e_i \leq a_i; x_i \in \mathbb{R}^{\geq 0}; e_i \in \mathbb{R}\}$. Clearly, $\mathbb{X}_i$ is a polyhedral set (see Chapter 3.4.2, Duality Theory in [8]). Denoting again $f_i = -g_i$, the problem (14)-(17) can be written as:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} g_i(x_i) \\ \text{s.t.} \quad & (x_i, e_i) \in \mathbb{X}_i, i \in \mathrm{V} \\ & \sum_{i=1}^{n} e_i = 0. \end{aligned} \tag{18}$$

Let τ be the Lagrange multiplier associated with constraint $\sum_{i=1}^{n} e_i = 0$. Subsequently, we can define the dual function where the primal variables are in a polyhedral set as ([8], section 5.1.6):

$$L^*(\tau) = \sum_{i=1}^{n} L_i^*(\tau), \tag{19}$$

where

$$L_i^*(\tau) = \inf_{(x_i, e_i) \in \mathbb{X}_i} \Big(g_i(x_i) + \tau e_i \Big), \qquad i \in \mathrm{V}. \tag{20}$$

Let $(\mathbf{x}^\star, \mathbf{e}^\star)$ be a social welfare equilibrium and τ^* be the dual optimal i.e. $\tau^* \in \arg\max_{\tau \in \mathbb{R}} L^*(\tau)$. Since the problem (18) is feasible and its optimal value is finite, strong duality holds ([8], Proposition 5.2.1). This means that

$$\begin{aligned} \sum_{i=1}^{n} g_i(x_i^\star) &= L^*(\tau^*) && (21) \\ &= \sum_{i=1}^{n} \left(\inf_{(x_i, e_i) \in \mathbb{X}_i} \Big(g_i(x_i) + \tau^* e_i \Big) \right) && (22) \\ &\leq \sum_{i=1}^{n} g_i(x_i^\star) + \tau^* \sum_{i=1}^{n} e_i^\star && (23) \\ &\leq \sum_{i=1}^{n} g_i(x_i^\star). && (24) \end{aligned}$$

Equation (21) states that the duality gap is zero, (22) comes from the definition of the dual function, (23) follows since the minimization of $\sum_{i=1}^{n} g_i(x_i) + \tau^* \sum_{i=1}^{n} e_i$ over $(x_i, e_i) \in \mathbb{X}_i$ is always less than or equal to the value at $\sum_{i=1}^{n} g_i(x_i^\star) + \tau^* \sum_{i=1}^{n} e_i^\star$, (24) follows from $\sum_{i=1}^{n} e_i^\star = 0$. We conclude that the two inequalities hold with equality which implies $(\mathbf{x}^\star, \mathbf{e}^\star)$ minimizes $\sum_{i=1}^{n} g_i(x_i) + \tau^* \sum_{i=1}^{n} e_i$ over $(x_i, e_i) \in \mathbb{X}_i$. Therefore, there holds

$$(\mathbf{x}^\star, \mathbf{e}^\star) \in \underset{\substack{(x_i, e_i) \in \mathbb{X}_i, \\ i \in \mathrm{V}}}{\arg\min} \sum_{i=1}^{n} g_i(x_i) + \tau^* \sum_{i=1}^{n} e_i. \tag{25}$$

Since (25) is separable in all $i \in \mathrm{V}$, an equivalent formulation is

$$(x_i^\star, e_i^\star) \in \underset{(x_i, e_i) \in \mathbb{X}_i}{\arg\min}\ g_i(x_i) + \tau^* e_i, \quad i \in \mathrm{V}. \tag{26}$$

Let us define the equilibrium price λ^* as $\lambda^* = -\tau^*$. It follows from (26) that $(x_i^\star, e_i^\star)$ is the solution of the following optimization problem:

$$\begin{aligned} \max \quad & f_i(x_i) + \lambda^* e_i \\ \text{s.t.} \quad & x_i + e_i \leq a_i \\ & x_i \in \mathbb{R}^{\geq 0}, e_i \in \mathbb{R}. \end{aligned} \tag{27}$$

Hence, we conclude that the triplet $(\lambda^*, \mathbf{x}^\star, \mathbf{e}^\star)$ is a competitive equilibrium. □

In the presence of agent trading decisions, the price λ^* under any competitive equilibrium must be nonnegative, as shown in Proposition 2.

Proposition 2 *Consider the MAS-SALTD. Suppose each $f_i(\cdot)$ is concave over the domain $\mathbb{R}^{\geq 0}$. Let $(\lambda^*, \mathbf{x}^*, \mathbf{e}^*)$ be a competitive equilibrium under the agent trading decisions. Then there always holds that $\lambda^* \geq 0$.*

Proof. Assume $\lambda^* < 0$. Then $f_i(x_i) + \lambda^* e_i$ is a strictly decreasing function with respect to e_i. Since e_i is unbounded below and upper bounded by $e_i \leq$

$a_i - x_i$, there can not be a finite e_i^* such that $e_i^* \in \arg\max_{e_i \in \mathbb{R}} \left(f_i(x_i) + \lambda^* e_i\right)$, contradicting the definition of the competitive equilibrium. This completes the proof. □

2.3 Numerical Examples

Example 1. Consider a multi-agent system with four agents who have local resource $(a_1, a_2, a_3, a_4) = (13, 14, 4, 7)$. Each agent i is associated with a utility function f_i which is represented by $f_i(x_i) = \min(k_i x_i, \beta_i)$ with $(k_1, k_2, k_3, k_4) = (21, 20, 23, 32)$ and $(\beta_1, \beta_2, \beta_3, \beta_4) = (135, 600, 130, 150)$.

(i) Let the multi-agent system be MAS-SALD. The social welfare equilibrium can be computed by numerically solving the optimization problem (3) as $\mathbf{x}^\star = (6.429, 21.23, 5.652, 4.688)^\top$, and the corresponding optimal dual variable is also obtained as $\lambda^* = 20$. Letting $\lambda^* = 20$, we then compute a competitive equilibrium that satisfies (1)-(2) as $\mathbf{x}^* = (6.429, 21.23, 5.652, 4.688)^\top$. In particular, we obtain $x_1^* = 6.429$, $x_3^* = 5.652$, and $x_4^* = 4.688$ by solving (1), and further establish $x_2^* = 21.232$ from (2). Clearly there holds $\mathbf{x}^\star = \mathbf{x}^*$, which is consistent with Theorem 1.

(ii) Let the multi-agent system be MAS-SALTD. We compute the social welfare equilibrium $(\mathbf{x}^\star, \mathbf{e}^\star)$ by solving the optimization problem (14)-(17) as

$$\mathbf{x}^\star = (6.429, 21.23, 5.652, 4.688)^\top,$$

$$\mathbf{e}^\star = (6.571, -7.23, -1.652, 2.313)^\top.$$

The optimal dual variable τ^* corresponding to the equity constraint (15) can be obtained as $\tau^* = -20$. We take $\lambda^* = -\tau^* = 20$ and establish a competitive equilibrium that satisfies (12)-(13) as

$$\mathbf{x}^* = (6.429, 21.23, 5.652, 4.688)^\top,$$

$$\mathbf{e}^* = (6.571, -7.23, -1.652, 2.313)^\top.$$

In particular, we compute $(x_1^*, e_1^*) = (6.429, 6.571)$, $(x_3^*, e_3^*) = (5.652, -1.652)$ and $(x_4^*, e_4^*) = (4.688, 2.313)$ by solving (12), and further obtain $(x_2^*, e_2^*) = (21.23, -7.23)$ from (13). Again there holds $(\mathbf{x}^\star, \mathbf{e}^\star) = (\mathbf{x}^*, \mathbf{e}^*)$, which validates Theorem 2. □

Example 2. Consider a multi-agent system with four agents. The utility function for agent i is in the quadratic form $f_i = -\frac{1}{2} b_i x_i^2 + k_i x_i$ for $i = 1, 2, 3, 4$. We consider two pairs of system parameters

$$\mathbf{b} = (2, 5, 3, 4)^\top \quad \mathbf{k} = (21, 17, 23, 13)^\top; \qquad \text{(PM.1)}$$

$$\mathbf{b}' = (2, 5, 3, 4)^\top \quad \mathbf{k}' = (25, 22, 24, 14)^\top. \qquad \text{(PM.2)}$$

Let the network resource capacity $C = \sum_{i=1}^4 a_i$ take values in an interval $(0, 40)$. We sample the interval $(0, 40)$ uniformly with a step-size 0.8 to obtain 50 different values for C. For each C, we compute the optimal prices of the system under MAS-SALD and MAS-SALTD.

For MAS-SALD, the optimal dual variables $\lambda_{\text{SALD}}^{*\text{(PM.1)}}$ and $\lambda_{\text{SALD}}^{*\text{(PM.2)}}$ are computed for 50 times corresponding to each value of C by solving (3), respectively, under the parameter setting (PM.1) and (PM.2). For MAS-SALTD, the optimal dual variables $\tau_{\text{SALTD}}^{*\text{(PM.1)}}$ and $\tau_{\text{SALTD}}^{*\text{(PM.2)}}$ related to the equity constraint (15) are also computed for 50 times corresponding to each value of C by solving (14)-(17), respectively, under the parameter setting (PM.1) and (PM.2), and then we take $\lambda_{\text{SALTD}}^{*\text{(PM.1)}} = -\tau_{\text{SALTD}}^{*\text{(PM.1)}}$ and $\lambda_{\text{SALTD}}^{*\text{(PM.2)}} = -\tau_{\text{SALTD}}^{*\text{(PM.2)}}$. In Fig. 1, we plot the 50 points of optimal prices versus C, to obtain an approximate trajectory of the optimal price as a function of C.

From Fig. 1 we observe that the optimal price λ_{SALD}^* in MAS-SALD can indeed take negative values; while the optimal price λ_{SALTD}^* in MAS-SALTD is always nonnegative. These observations are consistent with Proposition 1 and Proposition 2. Moreover, for both MAS-SALD and MAS-SALTD, we observe in Fig. 1 that the optimal prices $\lambda_{\text{SALD}}^*, \lambda_{\text{SALTD}}^*$ are decreasing as the network resource capacity C increases. □

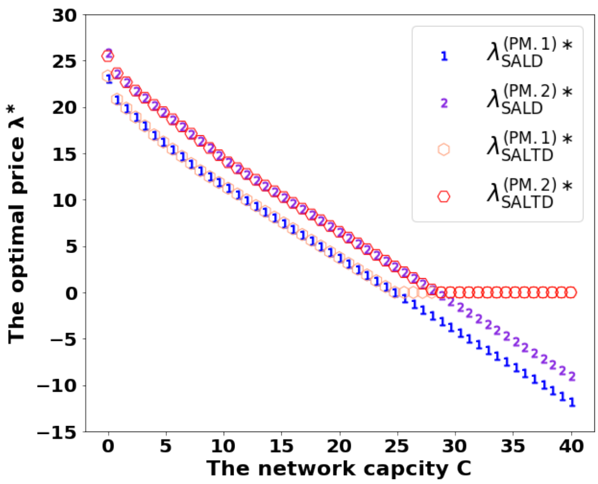

Figure 1. The curves of the optimal prices as functions of the network resource capacity in Example 2.

3 Social Shaping for Competitive Equilibrium

Consistent with classical welfare economics theory, a competitive equilibrium, despite being a social welfare equilibrium as well, indicates nothing about fairness or sustainability. If the optimal pricing λ^* is too high, agents would opt out of the system, instead of participating in the self-sustained multi-agent system. When members leave the system, the achievable payoff for the remaining agents would go down. Therefore, the agents share a social responsibility in shaping their utility functions so that λ^* is within a socially acceptable range.

3.1 Shaping the Competitive Equilibrium

Here we present an approach to achieve a socially acceptable competitive equilibrium, by synthesizing a class of utility functions from which agents can select. We make the following assumption.

Assumption 1. Each f_i is represented by $f_i(x_i) = -\frac{1}{2}b_i x_i^2 + k_i x_i$, where $b_i \in \mathbb{R}^{>0}$ and $k_i \in \mathbb{R}^{\geq 0}$. A utility function f_i is socially admissible if there hold $k_i \in [k_{\min}, k_{\max}]$ and $b_i \in [b_{\min}, b_{\max}]$.

Let $\lambda^\dagger > 0$ represent the highest pricing for λ^* that agents can accept, and we term such a competitive equilibrium $\lambda^* \leq \lambda^\dagger$ a *socially resilient equilibrium.* Let $\mathbf{a} = (a_1 \dots a_n)^\top$ represent the network resource allocation profile, and let $C := \sum_{i=1}^n a_i$ represent the network resource capacity. Assuming C and $\mathbf{a}$ are given network characteristics, we consider the following problem of shaping the competitive equilibrium.

Problem. (Social Competitive Equilibrium Shaping) Consider the MAS-SALD. Find the range for $k_{\min}$, $k_{\max}$, $b_{\min}$, $b_{\max}$ under which there always exists a competitive equilibrium that leads to $0 \leq \lambda^* \leq \lambda^\dagger$, for all socially admissible utility functions.

3.2 Socially Admissible Utility Functions

Denote $\mathbf{k} = (k_1, \dots, k_n)$ and $\mathbf{b} = (b_1, \dots, b_n)^\top$. For two vectors $\mathbf{l} = (l_1, \dots, l_n)$ and $\mathbf{l}' = (l'_1, \dots, l'_n)$, we write $\mathbf{l} \preceq \mathbf{l}'$ if there holds $l_i \leq l'_i$ for all $i \in \mathrm{V}$. In other words, $\preceq$ defines a partial order for all vectors in $\mathbb{R}^n$.

Define

$$\mathscr{S}_* := \Big\{ \big(k_{\min}, k_{\max}, b_{\min}, b_{\max}\big) \in \mathbb{R}^4_{\geq 0} : \frac{nk_{\min}}{b_{\max}} \geq C; \; -\frac{nk_{\min}}{b_{\max}} + \frac{nk_{\max}}{b_{\min}} \leq C; -\frac{n\lambda^\dagger}{b_{\max}} + \frac{nk_{\max}}{b_{\min}} \leq C \Big\}. \tag{28}$$

We present the following theorem.

Theorem 3 *Consider the MAS-SALD. Let Assumption 1 hold. The following statements hold.*

(i) The competitive equilibrium is unique, and therefore, there exists a well-defined mapping, denoted by $\mathcal{F}(\cdot, \cdot)$, that maps $(\mathbf{k}, \mathbf{b})$ to $\lambda^* := \mathcal{F}(\mathbf{k}, \mathbf{b})$ where λ^* belongs to the competitive equilibrium.

(ii) The competitive equilibrium is always socially resilient (ie $\lambda^* \leq \lambda^\dagger$) for all socially admissible utility functions as long as $\big(k_{\min}, k_{\max}, b_{\min}, b_{\max}\big) \in \mathscr{S}_*$.

(iii) Let $\big(k_{\min}, k_{\max}, b_{\min}, b_{\max}\big) \in \mathscr{S}_*$ be given. The mapping $\mathcal{F}(\cdot, \cdot)$ is monotone under the partial order $\preceq$ over $\mathbf{k}$ in the sense that

$$\mathcal{F}(\mathbf{k}, \mathbf{b}) \leq \mathcal{F}(\mathbf{k}', \mathbf{b})$$

for all socially admissible $\mathbf{k} \preceq \mathbf{k}'$.

Proof. (i) Under Assumption 1, all f_i are strictly concave. Thus, the optimization problem (3) is strictly convex, leading to a unique optimal solution. According to Theorem 1, in a competitive equilibrium $(\lambda^*, \mathbf{x}^*)$, $\mathbf{x}^*$ must be the unique primal optimal solution for (3). While from the definition of equilibrium, λ^* is by definition the optimal dual variable for (3), which is also unique.

(ii) Let $(\lambda^*, \mathbf{x}^*)$ be a competitive equilibrium. Then based on the definition of competitive equilibrium, there holds x_i^* is the optimal solution to

$$\max_{x_i \in \mathbb{R}_{\geq 0}} -\frac{1}{2}b_i x_i^2 + k_i x_i + \lambda^*(a_i - x_i), \quad i = 1, 2, \cdots, n.$$

This implies that under the condition $\lambda^* \leq k_i$, there holds

$$x_i^* = \frac{k_i - \lambda^*}{b_i}. \tag{29}$$

We substitute the above form of x_i^* into the constraint $\sum_{i=1}^n x_i = C$ to obtain

$$\Big(\sum_{i=1}^n \frac{1}{b_i}\Big)\lambda^* = \sum_{i=1}^n \frac{k_i}{b_i} - C. \tag{30}$$

We thus confirm that

$$\begin{aligned} 0 \leq \lambda^* &= \Big(\sum_{i=1}^n \frac{k_i}{b_i} - C\Big) / \Big(\sum_{i=1}^n \frac{1}{b_i}\Big) \\ &\leq \Big(\frac{nk_{\max}}{b_{\min}} - C\Big) / \Big(\frac{n}{b_{\max}}\Big) \end{aligned} \tag{31}$$

if there holds

$$\frac{nk_{\min}}{b_{\max}} \geq C.$$

From (31) we also know

$$-\frac{nk_{\min}}{b_{\max}} + \frac{nk_{\max}}{b_{\min}} \leq C \tag{32}$$

guarantees $\lambda^* \leq k_i$. Collecting all conditions for $(k_{\min}, k_{\max}, b_{\min}, b_{\max})$, we obtain that $\lambda^* \leq \lambda^\dagger$ for all socially admissible utility functions with

$$(k_{\min}, k_{\max}, b_{\min}, b_{\max}) \in \mathscr{S}_*.$$

(iii) We have established that if $(k_{\min}, k_{\max}, b_{\min}, b_{\max}) \in \mathscr{S}_*$, then

$$\lambda^* = \mathcal{F}(\mathbf{k}, \mathbf{b}) = \Big(\sum_{i=1}^{n} \frac{k_i}{b_i} - C\Big) / \Big(\sum_{i=1}^{n} \frac{1}{b_i}\Big). \tag{33}$$

It is straightforward to verify that

$$\mathcal{F}(\mathbf{k}, \mathbf{b}) \leq \mathcal{F}(\mathbf{k}', \mathbf{b})$$

for all socially admissible $\mathbf{k} \preceq \mathbf{k}'$. □

3.3 *Numerical Examples*

Example 3. Consider a MAS-SALD with three agents and network capacity $C = 18$. Each agent's utility function is set as the quadratic form $f_i = -\frac{1}{2} b_i x_i^2 + k_i x_i$ for $i = 1, 2, 3$. The system's highest pricing for λ^* that agents can accept socially is assumed to be $\lambda^\dagger = 39$. Take $b_{\min} = 4$, $b_{\max} = 6$, $k_{\min} = 40$, and $k_{\max} = 50$. We can verify such a configuration of $(b_{\min}, b_{\max}, k_{\min}, k_{\max})$ is a point in $\mathscr{S}_*$ defined in (28).

(i) Let $\mathbf{b}$ be fixed to be $\mathbf{b} = (4, 5, 6)^\top$. Take $k_3 \in \{44, 48\}$. We sample the space for $(k_1, k_2) \in [40, 50]^2$ and compute the optimal pricing λ^* by solving the optimal dual variable of (3). Then we plot the contour maps for the optimal price as a function of k_1 and k_2 in the first row of Fig. 2.

(ii) Let $\mathbf{k}$ be fixed to be $\mathbf{k} = (44, 46, 48)^\top$. Take $b_3 \in \{4.8, 5.2\}$. We sample the space for $(b_1, b_2) \in [4, 6]^2$ and compute the optimal pricing λ^* by solving the optimal dual variable of (3). Then we plot the contour maps for the optimal price as a function of b_1 and b_2 in the second row of Fig. 2.

In Fig. 2 we observe that the maximum value for the price λ^* is 20, which is lower than $\lambda^\dagger = 39$. This illustrates all socially admissible utility functions for parameters in the set $\mathscr{S}_*$ lead to socially acceptable prices, providing a validation for Theorem 3(ii). From the first row of Fig. 2, the optimal price is monotone under the partial order $\preceq$ with respect to $\mathbf{k}$, which is consistent with Theorem 3(iii). □

4 Dynamic Multi-Agent Systems

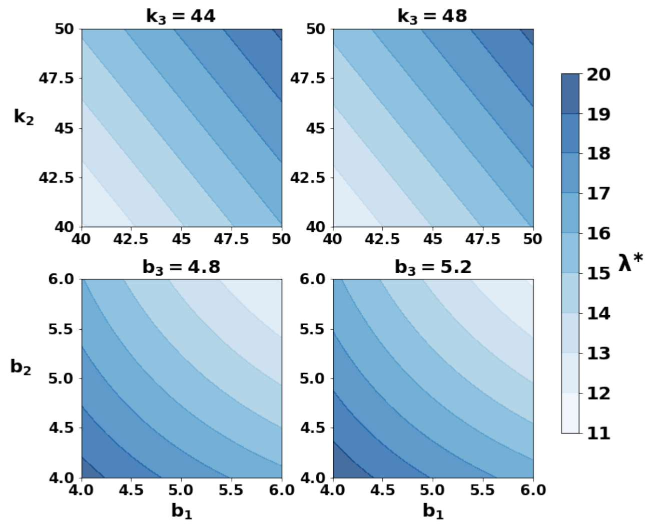

Figure 2. Contour maps for the optimal prices in Example 3.

4.1 *MAS with Dynamic Agent Load/Trading Decisions*

Here we consider the load balancing problem for dynamical multi-agent systems.

MAS with Dynamic Agent Load/Trading Decisions (MAS-DALTD). Each agent $i \in \mathrm{V}$ is associated with a dynamical state $\mathbf{y}_i(t) \in \mathbb{R}^m$, described by

$$\mathbf{y}_i(t+1) = \mathbf{A}_i \mathbf{y}_i(t) + \mathbf{B}_i \mathbf{u}_i(t), \quad t \in \mathcal{T}, \tag{34}$$

where $\mathbf{u}_i(t) \in \mathbb{R}^d$ is the control input, and $\mathbf{A}_i$ and $\mathbf{B}_i$ are real matrices with proper dimensions. The time steps are indexed in $\mathcal{T} = \{0, \dots, T-1\}$. Associated with $t \in \mathcal{T}$, agent i incurs a utility function $f_i(\mathbf{y}_i(t), \mathbf{u}_i(t))$; the terminal utility for agent i is $\Phi_i(\mathbf{y}_i(T))$. Upon taking the control action $\mathbf{u}_i(t)$, the required resource is defined by the function $h_i(\mathbf{u}_i(t))$. Each agent can produce an $a_i(t)$ units of resource at time t, and also makes a trading decision $e_i(t)$ units of resource over the network at time t. Similarly,

(i) if $h_i(\mathbf{u}_i(t)) < a_i(t)$, then agent i can sell, in which case $e_i(t) \geq 0$ and $e_i(t) \leq a_i(t) - h_i(\mathbf{u}_i(t))$;
(ii) if $h_i(\mathbf{u}_i(t)) \geq a_i(t)$, then agent i will buy, in which case $e_i(t) \leq 0$ and $e_i(t) = a_i(t) - h_i(\mathbf{u}_i(t))$.

We denote $\boldsymbol{\lambda} = (\lambda_0 \dots \lambda_{T-1})^\top$ as the pricing vector through the time horizon, where λ_t is the unit price for traded energy at step t. Consequently, the payoff of agent i throughout $[0, T]$ is described by

$$\sum_{t=0}^{T-1} \Big(f_i(\mathbf{y}_i(t), \mathbf{u}_i(t)) + \lambda_t e_i(t) \Big) + \Phi_i(\mathbf{y}_i(T)).$$

We denote $\mathbf{y}(t) = (\mathbf{y}_1(t)^\top \dots \mathbf{y}_n(t)^\top)^\top$, $\mathbf{u}(t) =$

$(\mathbf{u}_1(t)^\top \dots \mathbf{u}_n(t)^\top)^\top$, and $\mathbf{e}(t) = (\mathbf{e}_1(t)^\top \dots \mathbf{e}_n(t)^\top)^\top$. We further define $\mathbf{Y} = (\mathbf{y}(0)^\top \dots \mathbf{y}(T)^\top)^\top$, $\mathbf{U} = (\mathbf{u}(0)^\top \dots \mathbf{u}(T-1)^\top)^\top$ and $\mathbf{E} = (\mathbf{e}(0)^\top \dots \mathbf{e}(T-1)^\top)^\top$. Also introduce $\mathbf{U}_i = (\mathbf{u}_i(0)^\top \dots \mathbf{u}_i(T-1)^\top)^\top$, $\mathbf{E}_i = (\mathbf{e}_i(0)^\top \dots \mathbf{e}_i(T-1)^\top)^\top$ and $\mathbf{a}_i = (a_i(0)^\top \dots a_i(T-1)^\top)^\top$.

Definition 5 *Let $\mathbf{y}(0) = \mathbf{y}_0 \in \mathbb{R}^{mn}$ be given. A triple of price-control-trading profiles ($\boldsymbol{\lambda}^*, \mathbf{U}^*, \mathbf{E}^*$) is a dynamic competitive equilibrium if the following conditions hold:*

(i) each agent i maximizes its combined payoff under $\mathbf{U}_i^*$ and $\mathbf{E}_i^*$:

$$\begin{aligned} \max_{\mathbf{U}_i, \mathbf{E}_i} \quad & \sum_{t=0}^{T-1} \Big(f_i(\mathbf{y}_i(t), \mathbf{u}_i(t)) + \lambda_t^* e_i(t) \Big) + \Phi_i(\mathbf{y}_i(T)) \\ \text{s.t.} \quad & \mathbf{y}_i(t+1) = \mathbf{A}_i \mathbf{y}_i(t) + \mathbf{B}_i \mathbf{u}_i(t), \quad t \in \mathcal{T}, \\ & e_i(t) \le a_i(t) - h_i(\mathbf{u}_i(t)), \quad t \in \mathcal{T}; \end{aligned} \tag{35}$$

(ii) the total demand and supply are balanced across the network for all time, i.e., there holds

$$\sum_{i=1}^{n} e_i(t) = 0, \quad t \in \mathcal{T}. \tag{36}$$

Definition 6 *Let $\mathbf{y}(0) = \mathbf{y}_0 \in \mathbb{R}^{mn}$ be given. A pair of control-trading profiles $(\mathbf{U}^\star, \mathbf{E}^\star)$ is a dynamic social welfare equilibrium if it is a solution to the following optimal control problem:*

$$\max_{\mathbf{U}, \mathbf{E}} \quad \sum_{i=1}^{n} \Big(\sum_{t=0}^{T-1} f_i(\mathbf{y}_i(t), \mathbf{u}_i(t)) + \Phi_i(\mathbf{y}_i(T)) \Big) \tag{37}$$

$$\text{s.t.} \quad \mathbf{y}_i(t+1) = \mathbf{A}_i \mathbf{y}_i(t) + \mathbf{B}_i \mathbf{u}_i(t), \quad t \in \mathcal{T}, \ i \in \mathrm{V}, \tag{38}$$

$$e_i(t) \le a_i(t) - h_i(\mathbf{u}_i(t)), \quad t \in \mathcal{T}, \ i \in \mathrm{V}, \tag{39}$$

$$\sum_{i=1}^{n} e_i(t) = 0, \quad t \in \mathcal{T}. \tag{40}$$

4.2 Dynamic Competitive Equilibrium

We impose the following assumption.

Assumption 2. (i) the Φ_i are concave functions for $i \in \mathrm{V}$; (ii) the f_i are concave functions for $i \in \mathrm{V}$; (iii) the h_i are nonnegative convex functions for $i \in \mathrm{V}$, and $h_i(\mathbf{z}) < b$ defines a bounded open set of $\mathbf{z}$ in $\mathbb{R}^d$ for $b > 0$; (iv) $\sum_{i=1}^n a_i(t) > 0$ for all $t \in \mathcal{T}$.

We present the following result which establishes a similar connection between the competitive equilibrium and social welfare equilibrium under this dynamic setting.

Theorem 4 *Consider the MAS-DALTD with $\mathbf{y}(0) = \mathbf{y}_0 \in \mathbb{R}^{mn}$ be given. Let Assumption 2 hold. The dynamic social welfare equilibrium(s) and the dynamic competitive equilibrium(s) coincide and the following statements hold.*

(i) If $(\boldsymbol{\lambda}^*, \mathbf{U}^*, \mathbf{E}^*)$ is a dynamic competitive equilibrium, then $(\mathbf{U}^*, \mathbf{E}^*)$ is a dynamic social welfare equilibrium.

(ii) If $(\mathbf{U}^\star, \mathbf{E}^\star)$ is a dynamic social welfare equilibrium, then there exists $\boldsymbol{\lambda}^* \in \mathbb{R}^T$ such that $(\boldsymbol{\lambda}^*, \mathbf{U}^\star, \mathbf{E}^\star)$ is a competitive equilibrium.

Proof: (i) The proof of sufficiency follows from a similar analysis as in the proof of Theorem 1. The transition from competitive equilibrium to social welfare equilibrium under this dynamical setting continues to be a direct consequence of the formulations of the two underlying optimization problems.

(ii) First of all, the dynamics $\mathbf{y}_i(t+1) = \mathbf{A}_i \mathbf{y}_i(t) + \mathbf{B}_i \mathbf{u}_i(t)$ with given $\mathbf{y}(0)$ ensures that any $\mathbf{y}_i(t)$ for $t = 1, \dots, T$ is a linear combination of $\mathbf{y}(0)$ and $\mathbf{u}_i(0), \dots, \mathbf{u}_i(t-1)$. Therefore, we can always write for any $i \in \mathrm{V}$ that

$$\mathbf{y}_i(t) = p_{i,t} \mathbf{y}_0 + q_{i,t} \mathbf{U}_i, \ t = 0, \dots, T \tag{41}$$

with $p_{i,t}$ and $q_{i,t}$ being matrices with proper dimensions. As a result, we have

$$f_i(\mathbf{y}_i(t), \mathbf{u}_i(t)) = f_i(p_t \mathbf{y}_0 + q_t \mathbf{U}_i, \mathbf{u}_i(t)) := \tilde{f}_{i,t}(\mathbf{U}_i). \tag{42}$$

In view of Assumption 2, and the fact that the composition of a concave function and an affine function continues to be concave, we conclude that $g_{i,t}(\cdot)$ is a concave function. Similarly,

$$\Phi(\mathbf{y}_i(t)) = \Phi(p_{i,T} \mathbf{y}_0 + q_{i,T} \mathbf{U}_i) := \Phi_i(\mathbf{U}_i)$$

where $\Phi_i(\cdot)$ is a concave function.

The optimization problem (37)-(40) can be equivalently rewritten as the following convex programming:

$$\begin{aligned} \min_{\mathbf{U}, \mathbf{E}} \quad & -\sum_{i=1}^{n} \Big(\sum_{t=0}^{T-1} \tilde{f}_{i,t}(\mathbf{U}_i) + \Phi_i(\mathbf{U}_i) \Big) \\ \text{s.t.} \quad & h_i(\mathbf{u}_i(t)) + e_i(t) \le a_i(t), \quad t = 0, \dots, T-1, \ i \in \mathrm{V} \\ & \sum_{i=1}^{n} e_i(t) = 0, \quad t = 0, \dots, T-1. \end{aligned} \tag{43}$$

Similarly, (35) can be equivalently written as convex programming

$$\begin{aligned} \min_{\mathbf{U}_i,\mathbf{E}_i} \quad & -\sum_{t=0}^{T-1}\Big(\tilde{f}_{i,t}(\mathbf{U}_i)+\lambda_t^* e_i(t)\Big)+\Phi_i(\mathbf{U}_i) \\ \text{s.t.} \quad & e_i(t)\le a_i(t)-h_i(\mathbf{u}_i(t)), t=0,\dots,T-1. \end{aligned} \tag{44}$$

Next, with Assumption 2.(iii)-(iv), we can verify that Slater's condition holds for (43) and (44), which guarantees strong duality in both problems [9]. Also, noting

$$\sum_{i=1}^n h_i(\mathbf{u}_i(t)) \le \sum_{i=1}^n a_i(t) \tag{45}$$

and the assumption that $h_i(\mathbf{z})<b$ defines a bounded open set of $\mathbf{z}$ in $\mathbb{R}^m$ for $b>0$, $\mathbf{u}_i(t)$ takes values in a compact set for all i and t. Moreover, since $h_i(\mathbf{u}_i(t))\ge 0$ holds for all i and for all t, there holds $e_i(t)\le a_i(t)$. The constraint $\sum_{i=1}^n e_i(t)=0$ further ensures $e_i(t)\ge -\sum_{i=1}^n a_i(t)$ for all i and t. Thus $e_i(t)$ also takes values in a compact set for all i and t. The convex programming problem (43) leads to finite primal solution.

The Lagrangian dual function of (43) can be written as

$$\begin{aligned} L(\mathbf{U},\mathbf{E},\boldsymbol{\lambda},\boldsymbol{\mu}) &= -\sum_{i=1}^n\Big(\sum_{t=0}^{T-1}\tilde{f}_{i,t}(\mathbf{U}_i)+\Phi_i(\mathbf{U}_i)\Big) \\ &\quad+\sum_{t=0}^{T-1}\sum_{i=1}^n \lambda_t e_i(t) \\ &\quad+\sum_{t=0}^{T-1}\sum_{i=1}^n \mu_{i,t}\big(h_i(\mathbf{u}_i(t))+e_i(t)-a_i(t)\big) \\ &= \sum_{i=1}^n L_i(\mathbf{U}_i,\mathbf{E}_i,\boldsymbol{\lambda},\boldsymbol{\mu}_i) \end{aligned} \tag{46}$$

where

$$\begin{aligned} L_i(\mathbf{U}_i,\mathbf{E}_i,\boldsymbol{\lambda},\boldsymbol{\mu}_i) &= -\sum_{t=0}^{T-1}\tilde{f}_{i,t}(\mathbf{U}_i)+\Phi_i(\mathbf{U}_i)+\sum_{t=0}^{T-1}\lambda_t e_i(t) \\ &\quad+\sum_{t=0}^{T-1}\mu_{i,t}\big(h_i(\mathbf{u}_i(t))+e_i(t)-a_i(t)\big). \end{aligned} \tag{47}$$

Here $\mu_{i,t}\ge 0$ since they correspond to the inequality constraints. We have used the conventional notation $\boldsymbol{\mu}_i=(\mu_{i,0},\dots,\mu_{i,T-1})^\top$ and $\boldsymbol{\mu}=(\boldsymbol{\mu}_1^\top,\dots,\boldsymbol{\mu}_n^\top)^\top$.

Finally, letting an optimal dual solution of (43) be $(\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*)$, there holds from strong duality

$$(\mathbf{U}^\star,\mathbf{E}^\star)\in\arg\min\ L(\mathbf{U},\mathbf{E},\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*) \tag{48}$$

if $(\mathbf{U}^\star,\mathbf{E}^\star)$ is a dynamic social welfare equilibrium. This implies from (46) that

$$(\mathbf{U}_i^\star,\mathbf{E}_i^\star)\in\arg\min\ L_i(\mathbf{U},\mathbf{E},\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*). \tag{49}$$

Now, $\boldsymbol{\mu}^*$ is obtained by solving

$$\max_{\boldsymbol{\lambda},\boldsymbol{\mu}}\ \min_{\mathbf{U},\mathbf{E}}\ L(\mathbf{U},\mathbf{E},\boldsymbol{\lambda},\boldsymbol{\mu})=\max_{\boldsymbol{\lambda},\boldsymbol{\mu}}\ \min_{\mathbf{U},\mathbf{E}}\ \sum_{i=1}^n L_i(\mathbf{U}_i,\mathbf{E}_i,\boldsymbol{\lambda},\boldsymbol{\mu}_i) \tag{50}$$

where the maximization and minimization are taken in their respective domains for $\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{U},\mathbf{E}$. As a result, there must hold

$$\boldsymbol{\mu}_i^*\in\arg\max_{\boldsymbol{\mu}_i}\ \min_{\mathbf{U}_i,\mathbf{E}_i}\ L_i(\mathbf{U}_i,\mathbf{E}_i,\boldsymbol{\lambda}^*,\boldsymbol{\mu}_i). \tag{51}$$

It is worth emphasizing that $L_i(\mathbf{U}_i,\mathbf{E}_i,\boldsymbol{\lambda}^*,\boldsymbol{\mu}_i)$ is, precisely, the Lagrangian of (44). Therefore, (51) ensures that $\boldsymbol{\mu}_i^*$ is an optimal dual solution of (44), and then from strong duality (49) ensures that $(\mathbf{U}^\star,\mathbf{E}^\star)$ is an optimal primal solution of (44). In other words, we have proven $(\boldsymbol{\lambda}^*,\mathbf{U}^\star,\mathbf{E}^\star)$ is a competitive equilibrium.

The proof of the theorem is now complete. □

Example 4. Consider a MAS-DALTD with three agents who have initial states

$$\mathbf{y}_1(0)=\begin{bmatrix}1\\4\end{bmatrix},\mathbf{y}_2(0)=\begin{bmatrix}2\\5\end{bmatrix},\mathbf{y}_3(0)=\begin{bmatrix}3\\3\end{bmatrix},$$

and local resource

$$\mathbf{a}_1=[50\cdots 50],\quad \mathbf{a}_2=[50\cdots 50],\quad \mathbf{a}_3=[30\cdots 30].$$

The dynamical state $\mathbf{y}_i(t)\in\mathbb{R}^2$ of agent i is described by

$$\mathbf{y}_i(t+1)=\mathbf{A}_i\mathbf{y}_i(t)+\mathbf{B}_i\mathbf{u}_i(t),\quad t\in\mathcal{T},$$

where

$$\mathbf{A}_1=\begin{bmatrix}\frac{-3}{5}&0\\0&\frac{-7}{10}\end{bmatrix},\mathbf{A}_2=\begin{bmatrix}\frac{-1}{2}&0\\0&\frac{-1}{5}\end{bmatrix},\mathbf{A}_3=\begin{bmatrix}\frac{-2}{5}&0\\0&\frac{-4}{5}\end{bmatrix},$$

$$\mathbf{B}_1=\begin{bmatrix}2&0\\0&7\end{bmatrix},\mathbf{B}_2=\begin{bmatrix}4&0\\0&6\end{bmatrix},\mathbf{B}_3=\begin{bmatrix}9&0\\0&3\end{bmatrix}.$$

The utility function of agent i is in the quadratic form

$$f_i=\mathbf{y}_i^\top(t)\mathbf{R}_i\mathbf{y}_i(t)+\mathbf{W}_i\mathbf{y}_i(t)+\mathbf{u}_i^\top(t)\mathbf{Q}_i\mathbf{u}_i(t)+\mathbf{K}_i\mathbf{u}_i(t),$$

where

$$\mathbf{R}_1 = \begin{bmatrix} -5 & 0 \\ 0 & -8 \end{bmatrix}, \mathbf{R}_2 = \begin{bmatrix} -3 & 0 \\ 0 & -7 \end{bmatrix}, \mathbf{R}_3 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\mathbf{Q}_1 = \begin{bmatrix} -5 & 0 \\ 0 & -4 \end{bmatrix}, \mathbf{Q}_2 = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}, \mathbf{Q}_3 = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix},$$

$$\mathbf{W}_1 = \begin{bmatrix} 200 & 300 \end{bmatrix}, \mathbf{W}_2 = \begin{bmatrix} 200 & 400 \end{bmatrix}, \mathbf{W}_3 = \begin{bmatrix} 450 & 300 \end{bmatrix},$$

$$\mathbf{K}_1 = \begin{bmatrix} 50 & 60 \end{bmatrix}, \mathbf{K}_2 = \begin{bmatrix} 50 & 20 \end{bmatrix}, \mathbf{K}_3 = \begin{bmatrix} 80 & 20 \end{bmatrix}.$$

The terminal utility of agent i is also set as the quadratic form $\Phi_i(\mathbf{y}_i(T)) = \mathbf{y}_i^\top(T)\mathbf{R}_i\mathbf{y}_i(T) + \mathbf{W}_i\mathbf{y}_i(T)$. Upon taking $\mathbf{u}_i(t)$, the required resource is determined by $h_i(\mathbf{u}_i(t)) = \mathbf{u}_i^\top(t)\mathbf{H}_i\mathbf{u}_i(t)$, where

$$\mathbf{H}_1 = \begin{bmatrix} 5 & 0 \\ 0 & 8 \end{bmatrix}, \mathbf{H}_2 = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}, \mathbf{H}_3 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

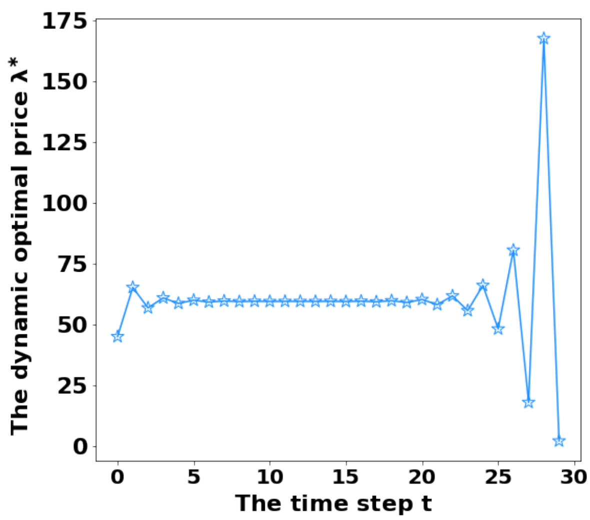

Figure 3. The dynamic optimal price versus time steps in Example 4.

Let the time horizon take the value of $T = 30$. We compute the dynamic social welfare equilibrium $(\mathbf{U}^\star, \mathbf{E}^\star)$ by solving the optimization problem (37)-(40) and the optimal dual variables $-\boldsymbol{\lambda}^*$ corresponding to (40). Given $\boldsymbol{\lambda}^*$, we further compute the dynamic competitive equilibrium $(\mathbf{U}^*, \mathbf{E}^*)$ by solving (35). In Fig. 4, we plot the dynamic social welfare equilibrium and the dynamic competitive equilibrium. In Fig. 4 we observe that the dynamic social welfare equilibrium and the dynamic competitive equilibrium agree, which is consistent with Theorem 4. The dynamic optimal price for traded resource versus time steps is also shown in Fig. 3, where the price experiences oscillations both at the beginning and in the end of the time horizon, and holds a steady value in between. □

4.3 Recursive Computation of Social Welfare Equilibrium

One of the most widely used methods in optimal control problems is dynamic programming [8]. Here we investigate the possibility of using a dynamic programming approach to represent and compute the social welfare equilibriums described in (37)-(40) as optimal feedback decisions.

Denote

$$\mathscr{E}_t := \{\mathbf{e}(t) : \sum_{i=1}^{n} e_i(t) = 0; e_i(t) \leq a_i(t), \forall i \in V\} \quad (52)$$

and

$$\mathscr{U}_t := \{\mathbf{u}(t) : e_i(t) \leq a_i(t) - h_i(\mathbf{u}_i(t), \forall i \in V\}. \quad (53)$$

Also define

$$\mathbf{f}_t(\mathbf{y}(t), \mathbf{u}(t)) = -\sum_{i=1}^{n} f_i(\mathbf{y}_i(t), \mathbf{u}_i(t))$$

and

$$\boldsymbol{\Phi}(\mathbf{y}(T)) = -\sum_{i=1}^{n} \Phi_i(\mathbf{y}_i(T)).$$

Definition 7 *Consider the MAS-DALTD with* $\mathbf{y}(0) = \mathbf{y}_0$. *Let* $\mathbf{y}(t) \in \mathscr{Y}_t, \mathbf{u}(t) \in \mathscr{U}_t$, *and* $\mathbf{e}(t) \in \mathscr{E}_t$. *We say that* $(\mathbf{u}(t), \mathbf{e}(t))$ *follows the feedback policy* $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \cdots, \boldsymbol{\pi}_{T-1})$ *if there holds*

$$(\mathbf{u}(t), \mathbf{e}(t)) = \boldsymbol{\pi}_t(\mathbf{y}(t)),$$

for all $t \in \mathcal{T}$.

Further introduce the cost-to-go function associated with any feedback policy $\boldsymbol{\pi}$ by

$$V^{\boldsymbol{\pi}}(k, \mathbf{y}_k) := \sum_{t=k}^{T-1} \mathbf{f}_t(\mathbf{y}(t), \mathbf{u}(t)) + \boldsymbol{\Phi}(\mathbf{y}_T) \quad (54)$$

with $(\mathbf{u}(t), \mathbf{e}(t)) = \boldsymbol{\pi}_t(\mathbf{y}(t))$, for all $t = k, \cdots, T-1$, $\mathbf{y}(k) = \mathbf{y}_k$, and

$$\mathbf{y}(s+1) = \mathbf{A}\mathbf{y}(s) + \mathbf{B}\mathbf{u}(s), s = k, \cdots, T-1. \quad (55)$$

Here $\mathbf{A}$ and $\mathbf{B}$ are block diagonal matrices

$$\text{diag}(\mathbf{A}_0, \cdots, \mathbf{A}_n) \text{ and } \text{diag}(\mathbf{B}_0, \cdots, \mathbf{B}_n)$$

respectively.

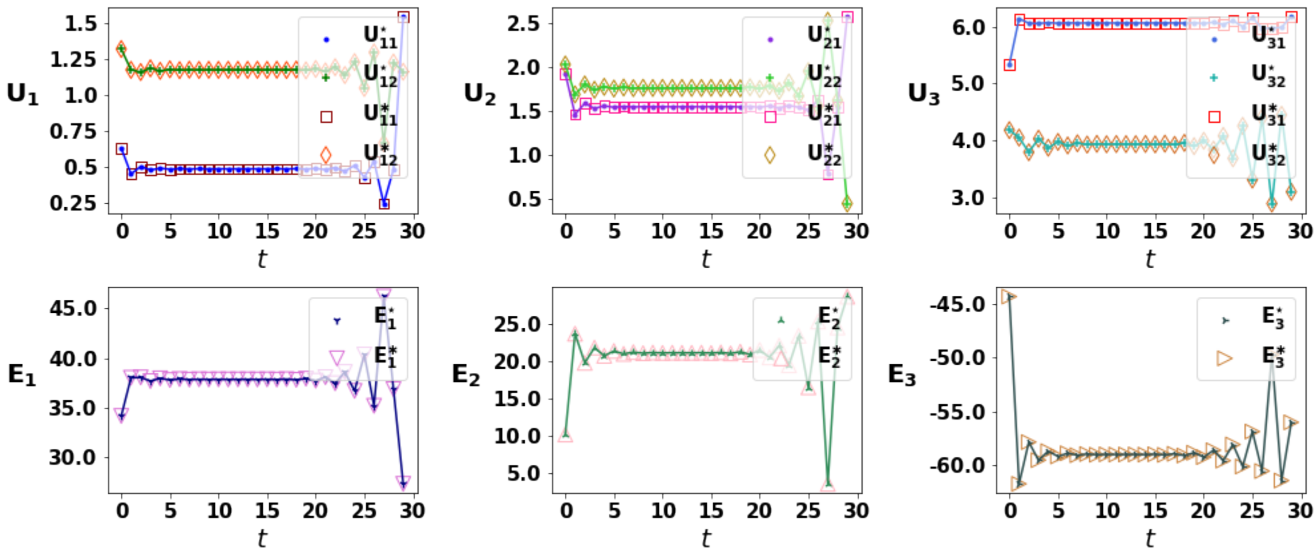

Figure 4. The dynamic social welfare equilibrium and competitive equilibrium in Example 4.

It is clear that the cost-to-go function must satisfy the boundary condition that

$$V^{\boldsymbol{\pi}}(T, \mathbf{y}(T)) = \boldsymbol{\Phi}(\mathbf{y}_T). \tag{56}$$

Theorem 5 *Consider the MAS-DALTD with* $\mathbf{y}(0) = \mathbf{y}_0$. *Let* $(\mathbf{U}^\star, \mathbf{E}^\star)$ *be a dynamic social welfare equilibrium from* (37)-(40). *Then there exists* $\boldsymbol{\pi}^\star$ *such that* $(\mathbf{u}^\star(t), \mathbf{e}^\star(t)) = \boldsymbol{\pi}^\star_t(\mathbf{y}(t))$, *for all* $t \in \mathcal{T}$, *and the cost-to-go function* $V^{\boldsymbol{\pi}^\star}$ *satisfies the following recurrance equation*

$$V^{\boldsymbol{\pi}^\star}(k, \mathbf{y}_k) = \min_{\substack{\mathbf{u}(k)\in\mathscr{U}_k;\\ \mathbf{e}(k)\in\mathscr{E}_k}} \Big[\mathbf{f}_k(\mathbf{y}(k), \mathbf{u}(k)) + V^{\boldsymbol{\pi}^\star}(k+1, \mathbf{y}_{k+1})\Big]. \tag{57}$$

Proof: First of all, the optimization problem (37)-(40) can be rewritten as:

$$\min_{\mathbf{U},\mathbf{E}} \quad \sum_{t=0}^{T-1} \mathbf{f}_t(\mathbf{y}(t), \mathbf{u}(t)) + \boldsymbol{\Phi}(\mathbf{y}_T) \tag{58}$$

$$\text{s.t.} \quad \mathbf{y}(t+1) = \mathbf{A}\mathbf{y}(t) + \mathbf{B}\mathbf{u}(t), \quad t \in \mathcal{T} \tag{59}$$

$$\mathbf{e}(t) \in \mathscr{E}_t, \quad t \in \mathcal{T} \tag{60}$$

$$\mathbf{u}(t) \in \mathscr{U}_t, \quad t \in \mathcal{T}. \tag{61}$$

Given the form of cost-to-go function (54) and the boundary condition of the terminal cost (56), we next have

$$V^{\boldsymbol{\pi}}(T-1, \mathbf{y}_{T-1}) = \mathbf{f}_{T-1}(\mathbf{y}_{T-1}, \mathbf{u}(T-1)) + V^{\boldsymbol{\pi}}(T, \mathbf{y}_T) \tag{62}$$

where $V^{\boldsymbol{\pi}}(T-1, \mathbf{y}_{T-1})$ is a one-step process with initial state $\mathbf{y}_{T-1}$. The value of $V^{\boldsymbol{\pi}}(T-1, \mathbf{y}_{T-1})$ depends only on $\mathbf{y}_{T-1}$ and $\mathbf{u}(T-1)$, since $\mathbf{y}_T$ is related to $\mathbf{y}_{T-1}$ and $\mathbf{u}(T-1)$ through the system dynamics (59). The optimal cost is then

$$V^{\boldsymbol{\pi}^\star}(T-1, \mathbf{y}_{T-1}) \triangleq \min_{\substack{\mathbf{u}(T-1)\in\mathscr{U}_{T-1};\\ \mathbf{e}(T-1)\in\mathscr{E}_{T-1}}} \Big[\mathbf{f}_{T-1}(\mathbf{y}_{T-1}, \mathbf{u}(T-1)) + V^{\boldsymbol{\pi}}(T, \mathbf{y}_T)\Big]. \tag{63}$$

where the optimal choice of $(\mathbf{u}(T-1), \mathbf{e}(T-1))$ only depends on $\mathbf{y}_{T-1}$. The cost over the last two intervals is given by

$$V^{\boldsymbol{\pi}}(T-2, \mathbf{y}_{T-2}) = \mathbf{f}_{T-2}(\mathbf{y}_{T-2}, \mathbf{u}(T-2)) + V^{\boldsymbol{\pi}}(T-1, \mathbf{y}_{T-1}) \tag{64}$$

where $V^{\boldsymbol{\pi}}(T-2, \mathbf{y}_{T-2})$ is a two-step process with initial state $\mathbf{y}_{T-2}$. The optimal policy during these two steps is found from

$$V^{\boldsymbol{\pi}^\star}(T-2, \mathbf{y}_{T-2}) \triangleq \min_{\substack{\mathbf{u}(T-2)\in\mathscr{U}_{T-2};\\ \mathbf{e}(T-2)\in\mathscr{E}_{T-2}}} \Big[V^{\boldsymbol{\pi}}(T-1, \mathbf{y}_{T-1}) + \mathbf{f}_{T-2}(\mathbf{y}_{T-2}, \mathbf{u}(T-2))\Big]. \tag{65}$$

The optimality principle states that for this two-step process, whatever the initial state $\mathbf{y}_{T-2}$, initial control action and trading decison $(\mathbf{u}(T-2), \mathbf{e}(T-2))$, the remaining $(\mathbf{u}(T-1), \mathbf{e}(T-1))$ must be optimal with respect to $\mathbf{y}_{T-1}$ resulted by applying $(\mathbf{u}(T-2), \mathbf{e}(T-2))$

to the system; that is,

$$V^{\boldsymbol{\pi}^\star}(T-2,\mathbf{y}_{T-2}) \triangleq \min_{\substack{\mathbf{u}(T-2)\in\mathscr{U}_{T-2};\\ \mathbf{e}(T-2)\in\mathscr{E}_{T-2}}} \Big[V^{\boldsymbol{\pi}^\star}(T-1,\mathbf{y}_{T-1}) + \mathbf{f}_{T-2}(\mathbf{y}_{T-2},\mathbf{u}(T-2))\Big]. \quad (66)$$

Applying the same analysis recursively, we obtain

$$V^{\boldsymbol{\pi}^\star}(k,\mathbf{y}_k) = \min_{\substack{\mathbf{u}(k)\in\mathscr{U}_k;\\ \mathbf{e}(k)\in\mathscr{E}_k}} \Big[\mathbf{f}_k(\mathbf{y}_k,\mathbf{u}(k)) + V^{\boldsymbol{\pi}^\star}(k+1,\mathbf{y}_{k+1})\Big]. \quad (67)$$

Now let $(\mathbf{U}^\star,\mathbf{E}^\star)$ be a dynamic social welfare equilibrium. Since the dynamics of $\mathbf{y}(t)$ is Markovian, the open-loop optimal solution $(\mathbf{U}^\star,\mathbf{E}^\star)$ of (37)-(40) coincides with the feedback optimal policies $\boldsymbol{\pi}^\star$.

The proof of the theorem is now complete. □

4.4 Social Smoothing via Receding Horizon Pricing

From Example 4, it is evident that dynamic multi-agent systems operating under competitive equilibriums for a fixed horizon may encounter significant pricing oscillations, especially at the beginning and towards the end of the time period. In practice, this means users are experiencing market shocks, which is not desirable from a social point of view. Therefore, for dynamic multi-agent systems, socially resilient competitive equilibriums should have pricing trajectories that are as smooth as possible. In addition, computing the dynamic competitive equilibriums over a long period of time is also a challenging task, and may even be infeasible for large-scale multi-agent systems.

The receding-horizon approach [35, 36] is a proven method for delivering robust and computationally efficient controllers for dynamical systems, with successful applications in a wide range of areas ranging from emergency vehicle scheduling[37] to dynamic hedging of options [38]. The control input trajectories derived from a receding-horizon approach may even be good approximations of the optimal control solution under suitable conditions [39]. With this view, we next propose a receding horizon pricing procedure for the considered dynamic multi-agent systems.

Consider the MAS-DALTD with $\mathbf{y}(0)=\mathbf{y}_0$. We fix a prediction horizon N and denote $\mathcal{K}=\{0,1,\dots,N-1\}$. The receding horizon approach approximates the solution to the optimization problem of (37)-(40) as follows. Assume a full measurement of the estimate of the state $\mathbf{y}_i(t), i\in\mathrm{V}$, is available at each time step $t, t\in\mathcal{T}$. We then propose a new optimization problem over the horizon $[t,t+N]$ at each time step $t, t\in\mathcal{T}$:

$$\min_{\substack{\mathbf{U}_{t\to t+N|t}\\ \mathbf{E}_{t\to t+N|t}}} \boldsymbol{\Phi}(\mathbf{y}_{t+N|t}) + \sum_{k=0}^{N-1}\mathbf{f}_k(\mathbf{y}_{t+k|t},\mathbf{u}_{t+k|t}) \quad (68)$$

$$\text{s.t.}\quad \mathbf{y}_{i,t+k+1|t} = \mathbf{A}_i\mathbf{y}_{i,t+k|t} + \mathbf{B}_i\mathbf{u}_{i,t+k|t}, k\in\mathcal{K}; i\in\mathrm{V}, \quad (69)$$

$$e_{i,t+k|t} \le a_{i,t+k|t} - h_i(\mathbf{u}_{i,t+k|t}), k\in\mathcal{K}; i\in\mathrm{V}, \quad (70)$$

$$\sum_{i=1}^{n} e_{i,t+k|t} = 0, k\in\mathcal{K}, \quad (71)$$

where $\mathbf{U}_{t\to t+N|t} = \{\mathbf{u}_{t|t},\dots,\mathbf{u}_{t+N-1|t}\}$, $\mathbf{E}_{t\to t+N|t} = \{\mathbf{e}_{t|t},\dots,\mathbf{e}_{t+N-1|t}\}$. Here $\mathbf{y}_{i,t+k|t}$ is the state of agent i at time step $t+k$ predicted at time step t. Similarly, $\mathbf{u}_{i,t+k|t}$ and $\mathbf{e}_{i,t+k|t}$ are the control action and trading decision of agent i at time step $t+k$ predicted at time step t obtained by starting from the current state $\mathbf{y}_{i,t|t}=\mathbf{y}_i(t)$ and applying to (69).

Let $\mathbf{U}^*_{t\to t+N|t} = \{\mathbf{u}^*_{t|t},\dots,\mathbf{u}^*_{t+N-1|t}\}$, $\mathbf{E}^*_{t\to t+N|t} = \{\mathbf{e}^*_{t|t},\dots,\mathbf{e}^*_{t+N-1|t}\}$ be the optimal solution of (68)-(71) and $\boldsymbol{\lambda}^*_{t\to t+N|t} = \{\lambda^*_{t|t},\dots,\lambda^*_{t+N-1|t}\}$ be the optimal dual variables for constraints (71). The first element of $\mathbf{U}^*_{t\to t+N|t}$, $\mathbf{E}^*_{t\to t+N|t}$ and $\boldsymbol{\lambda}^*_{t\to t+N|t}$ is applied to the MAS-DALTD at time step t:

$$\mathbf{u}(t) = \mathbf{u}^*_{t|t}(x(t)), \quad (72)$$

$$\mathbf{e}(t) = \mathbf{e}^*_{t|t}(x(t)), \quad (73)$$

$$\lambda(t) = \lambda^*_{t|t}(x(t)). \quad (74)$$

Based on the new state $\mathbf{y}_{i,t+1|t+1}=\mathbf{y}_i(t+1), i\in\mathrm{V}$, the optimization problem (68)-(71) is solved repeatedly at time step $t+1$ and it yields a receding horizon control and pricing. The procedure of receding horizon control and pricing is summarized in Algorithm 1:

Algorithm 1 Receding horizon control and pricing

while $t<T$ **do**
  measure the state $x(t)$ at time step t;
  obtain $\mathbf{U}^*_{t\to t+N|t}, \mathbf{E}^*_{t\to t+N|t}, \boldsymbol{\lambda}^*_{t\to t+N|t}$ by solving (68)-(71);
  apply the first element $\mathbf{u}^*_{t|t}(x(t)), \mathbf{e}^*_{t|t}(x(t)), \lambda^*_{t|t}(x(t))$ to MAS-DALTD;

Note that Algorithm 1 still applies, even if the new optimization problem proposed in a receding horizon fashion over the horizon $[t,t+N]$ exceeds the entire time horizon $[0,T]$ when $t\ge T-N$.

Example 5. Consider a MAS-DALTD with the same setting in Example 4. Let the entire time horizon take the value of 200. We fix a prediction horizon as $N = 40$. First, we follow the procedure in Example 4 and compute the dynamic optimal pricing over the entire time horizon. Then we apply Algorithm 1 to obtain the receding horizon pricing. The resulting trajectories of the two pricing approaches are shown in Figure 5.

From Fig. 5, it is clear that the dynamic optimal pricing and receding horizon pricing coincide over most of the time horizon, which shows that receding horizon pricing is a good approximation of the optimal pricing planned for the entire time horizon. We also note that the receding horizon pricing does not experience the large oscillations found in dynamic optimal pricing during the end periods of the time horizon. □

5 Conclusions

We studied multi-agent systems with decentralized resource allocation without external resource supply. For multi-agent systems with static local allocations, we showed that under general convexity assumptions, the competitive equilibrium and the social welfare equilibrium exist and agree using a duality analysis. We also studied the problem of social shaping for competitive equilibriums, where the pricing under a competitive equilibrium is associated with an upper bound. We presented an explicit family of socially admissible utility functions under which the pricing is always socially acceptable. Finally, a dynamical multi-agent system was studied and generalized in an optimal control context. For future works, efficient numerical algorithms for the computation of class of socially admissible utility functions would be an interesting direction.

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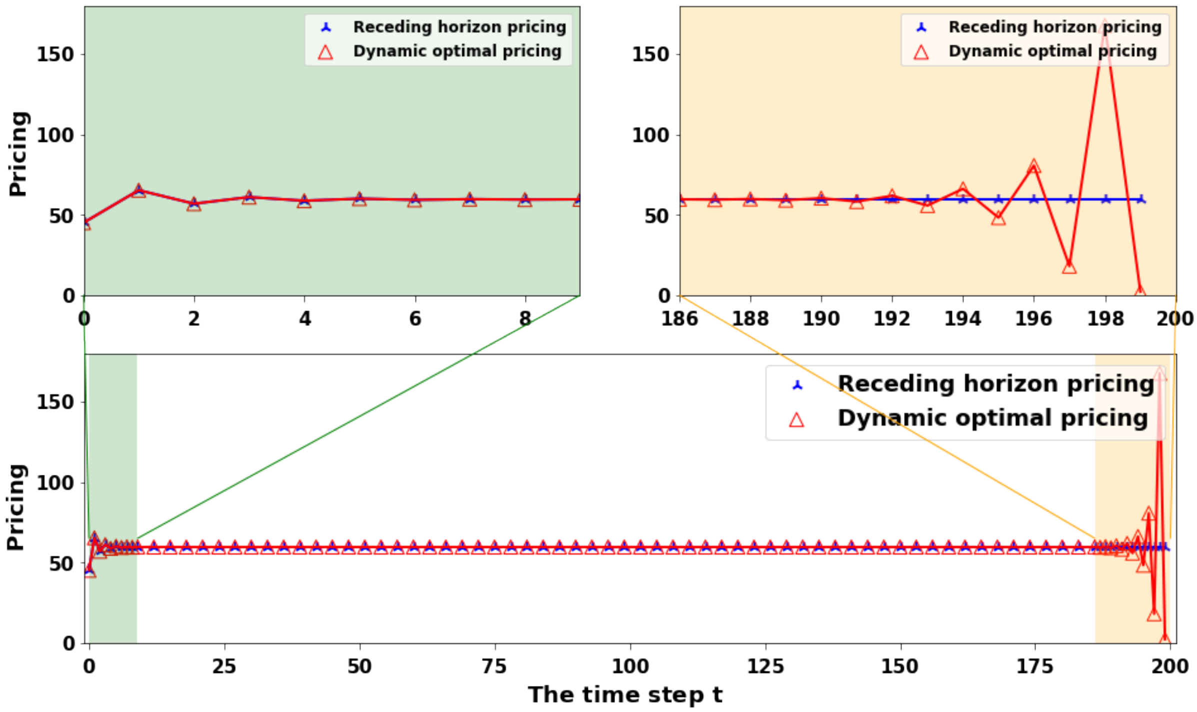

Figure 5. The dynamic optimal pricing v.s. the receding horizon pricing in Example 5.

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