Numerical solution to partial differential equations using fractals

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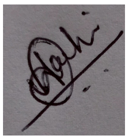
Declaration

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

Signed: 

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Abstract

The goal of this work is to construct and use continuous fractal functions to find a numerical solution of partial differential equations in one dimension. We focus on univariate, $\mathbb{R}-$valued, continuous fractal functions defined on interval $[0,1]$. These fractal functions provide more flexibility for approximation and interpolation. They are used as interpolation functions in the concept of hierarchical basis in the finite element method. Based on a continuous Read-Bajraktarevic operator defined on a space of bounded continuous functions, a fractal bubble is initially constructed on the interval $[0,1]$. Further, linear nodal basis functions defined on the interval $[0,1]$ are expressed as fixed points of an iterated function system. Consequently, they are established as continuous fractal functions. Thus, a hierarchical fractal basis is constructed on the interval $[0,1]$, consisting of three shape functions: two linear nodal basis functions and a hierarchical element namely, the fractal bubble. The nodal basis functions ensure the continuity at connecting nodes while the fractal bubble yields an additional degree of freedom without increasing the numbers of nodes in a mesh.

Subsequently, a finite element mesh is constructed by discretizing the interval $[0,1]$ in 2^n elements. The hierarchical fractal basis on a finite element mesh is then obtained by translating and dilating the three hierarchical basis functions. This basis defined on the finite element mesh spans the space of fractal functions and is used as a finite element space. This space has a direct sum hierarchical decomposition into a space of piecewise linear nodal basis functions and a space of piecewise fractal functions which are zero at the nodes. Subsequently, to set up element matrices in a finite element model, the fractal functions are integrated explicitly using the self-similarity property. However, a trapezoidal rule is used for numerical integration of fractal functions where the explicit integration of fractal functions is unattainable. As an illustration, a numerical solution to the Poisson equation with homogeneous Dirichlet boundary conditions is found. A numerical error is computed in L^2 norm. Bound of an interpolation error in L^2 norm and H^1 seminorm is established theoretically. A python code is used for generating the fractal bubble from an iterated function system and for the computational purpose to find an approximate solution to the Poisson equation in one dimension.

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Introduction

Partial differential equations play an important role in modelling diverse physical phenomena. For many of these equations, only an approximate solution can be determined. The partial differential equations are classified as elliptic, parabolic and hyperbolic. In this work, we aim to find numerical solution of second order elliptic partial differential equation with homogeneous boundary conditions using fractals.

The finite element method is the most popular numerical method used to find an approximate solution of partial differential equations. It uses interpolation functions to find the approximate solution. A nodal or Lagrange basis is a traditional choice for interpolation functions. However, different types of functions namely, polynomial functions, splines, wavelets have been extensively used for interpolation in the finite element method.

The novel algorithms have been developed to establish the viability of wavelet bases to the solution of partial differential equations [1, 2, 3, 4, 5]. G. Donovan, J. Geronimo, D. Hardin and P. Massopust used fractal interpolation functions to construct compactly supported, continuous, orthogonal wavelet basis spanning $L^2(\mathbb{R})$ [6]. These fractal wavelets were used by A. Kudrila, T. Sun, P. Grama and J. Ko to derive a class of finite elements for the finite element approximation schemes [7]. Likewise, S De Marchi and M Morandi Cecchi have derived fractal interpolation functions for a class of finite elements [8]. Moreover, DE. Dutkay and Palle E. T. Jorgensen have constructed wavelet bases in separable Hilbert spaces built on affine fractals and Hausdorff measure [9]. J. Bohnstengel and M. Kessebohmer have constructed wavelets and generalised Fourier bases on fractal sets constructed via iterated function systems (IFS) [10]. Recently, L F. Contreras Hernandez and J. Galvis have presented the finite difference and the finite element methods to compute solutions of partial differential equations posed on fractals [11].

Further, the hierarchical concept for the finite element shape functions was introduced as a convenient tool for mixed order interpolation. The hierarchical elements are equally simple to implement as "standard" fixed order elements in the finite element approximations. The hierarchical finite element approximations with polynomial functions [12, 13, 14], B-splines [15, 16], T- splines [17], wavelets [18, 19] have already been analyzed previously and the list is non-exhaustive. But the utilization of fractal functions in the hierarchical finite element approximation is not been investigated.

The fractal functions were originally introduced in [20] as continuous functions interpolating a set of data. These fractal functions display some sort of self-similarity under magnification and have an integral dimension. Their complicated mathematical structure, specifically their recursive construction has led to applications in several domains namely, structural mechanics, physics and chemistry, applied wavelet theory, signal processing and decoding, economics, among others. One of the main areas in mathematics where fractals are heavily used is for interpolation and smooth approximation [21, 22, 23].

This thesis focuses on the mathematical foundations of continuous fractal functions and their utilization as interpolation functions in the hierarchical finite element method. A space of continuous fractal functions is used as as a finite element space. The utilization of fractal functions for interpolation in the hierarchical finite element approximations begins with the construction of a hierarchical fractal basis. The fractal functions used in the hierarchical fractal basis are continuous fractal functions. They are associated with iterated function systems. Iterated function systems provide a natural framework for a specific type of contractive operators called Read-Bajraktarevic operators. These continuous Read-Bajraktarevic operators generate continuous fractal functions as their fixed points. Some of the advantages of this approach over approaches based on the linear basis as discussed in [24] are:

1. Partial differential equations are solved on very general grids defined by iterated function systems using fractal sets.

2. The theory is based on the theory of fractals and IFS which is well established.

This thesis establishes answers for the following questions:

1. How can continuous fractal functions be used to approximate a solution of the boundary value problem, expressed as follows.

$$-\frac{d^2u}{dx^2} = f(x), \quad x \in \Omega \tag{1}$$

Subject to homogeneous boundary conditions:

$$u(x) = 0, \quad x \in \partial\Omega \tag{2}$$

where $u(x)$ is defined on the domain Ω and its boundary $\partial\Omega$.

2. What are the error estimates in the proposed method?

3. What is the computational efficiency of the proposed method?

To conclude this introduction, we give a brief outline of chapters in this thesis. Chapter I focuses on the basic theory of functional analysis which forms the foundation of the finite element method. Chapter II deals with the fundamentals of finite element analysis. Chapter III discusses the fundamentals of the iterated function system and the construction of continuous fractal function on interval $[0, 1]$. Chapter IV introduces continuous fractal functions in the hierarchical finite element method. Chapter V illustrates the application of fractal functions to find a numerical solution to the Poisson equation with homogeneous Dirichlet boundary conditions.

Overall, current study is viewed as an attempt to stimulate the continuous fractal functions and their properties to use them for interpolation in the concept of hierarchical finite element method to find numerical solution of second order elliptic partial differential equations.

Chapter 1

Preliminaries

This chapter deals with some basic concepts of functional analysis to establish a foundation for the finite element method. Most of the results stated in this chapter are without proof and sourced from [25, 26, 27, 28, 29, 30, 31] where a detailed discussion is available. This chapter is a background and does not contribute to original work.

1.1 Functional spaces and generalized derivatives

This section is devoted to reviewing some suitable function spaces that are used in the variational formulation of partial differential equations.

1.1.1 Banach space

Definition 1 (Normed linear space)**.** *[29]. A normed vector space X is a linear space equipped with a norm $\|.\| : X \to \mathbb{R}$ that satisfies the following properties.*

1. $\|x\| \geq 0$ *and* $\|x\| = 0 \Leftrightarrow x = 0$
2. $\|\alpha x\| = |\alpha| \, \|x\|$, *for any* $\alpha \in \mathbb{R}$
3. $\|x + y\| \leq \|x\| + \|y\|$.

Definition 2 (Cauchy sequence)**.** *[29]. A sequence $\{x^k\}$ in a normed space is called Cauchy* *sequence if for each $\epsilon > 0$ there exists a number N such that*

$$\left\| x^k - x^l \right\| \leq \epsilon \quad \text{for all } k, l \geq N.$$

Definition 3 (Complete normed linear space)**.** *[29]. A normed linear space X is called complete if every Cauchy sequence $x^k \subset X$ converges to a limit $x \in X$. That is, there exists $x \in X$ with*

$$\lim_{k \to \infty} \left\| x^k - x \right\| = 0.$$

Equivalently,

$$x = \lim_{k \to \infty} x^k.$$

Definition 4 (Banach space)**.** *[29]. A normed linear space $(X, \|.\|)$ is called a Banach space if it is complete with respect to the metric induced by the norm $\|.\|$.*

Example 1. *The space of continuous functions from a domain $\Omega \to \mathbb{R}$ equipped with the supremum norm*

$$\|f\|_\infty = \sup\{f(x) : x \in \Omega\}$$

is a Banach space.

Definition 5. *[29]. A map $T : X \to Y$ is called linear if*

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty, \quad \text{for all } x, y \in X, \quad \alpha, \beta \in \mathbb{R}.$$

Theorem 1. *[31] A linear mapping $T : X \to Y$ is continuous if there exist a constant $M \geq 0$ such that*

$$\|Tx\| \leq M \|x\|, \quad \text{for all } x \in X.$$

Definition 6. *[31] Let X and Y be normed spaces with the norms $\|.\|_X$ and $\|.\|_Y$ respectively. The space X is continuously embedded into Y if $X \subset Y$ and there exists a constant $c > 0$ such that*

$$\|x\|_Y \leq c \|x\|_X, \quad \text{for all } x \in X.$$

Theorem 2. *[31] Let X and Y be two normed spaces with norms $\|.\|_X$ and $\|.\|_Y$ respectively. A mapping $T : U \to V$ is continuous at $x \in X$ if for every sequence $\{x^k\} \subset X$ converging to x, one has*

$$\lim_{k\to\infty} Tx^k = Tx.$$

That is,

$$\lim_{k\to\infty} \left\|x^k - x\right\|_X = 0 \implies \lim_{k\to\infty} \left\|Tx^k - Tx\right\|_Y = 0.$$

A mapping is continuous if it is continuous at every point x in X.

1.1.2 Hilbert space

Definition 7 (Inner product space)**.** *[27]. An inner product space X is a vector space equipped with an inner product, $(.,.) : X \times X \to \mathbb{R}$ that satisfies the following properties:*

1. $(u, v) = (v, u)$.
2. $(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w)$ *for any* $\alpha, \beta \in \mathbb{R}$.
3. $(u, u) \geq 0$ *and* $(u, u) = 0 \Leftrightarrow u = 0$.

Definition 8 (Hilbert space)**.** *[27]. A Hilbert space X is a complete inner product space.*

A norm induced by an inner product is defined as

$$\|u\| = \sqrt{(u, u)}.$$

Example 2. *[25] The best-known Hilbert space of functions on Ω is L^2 , the space of square integrable functions $v : \Omega \to R$ with the norm*

$$\|v\|_{L_2(\Omega)} = \sqrt{\int_\Omega v(x)^2 dx}.$$

Example 3. *[25] The space $H_0^1(\Omega)$ of square-integrable functions $u : \Omega \to R$ with square-integrable derivatives which are zero on the boundary γ is a Hilbert space with inner product*

$$(u, v)_{H_0^1(\Omega)} = \int_\Omega \nabla u \nabla v dx.$$

An important fact about a Hilbert space is Cauchy–Schwarz inequality.

Theorem 3 (Cauchy–Schwarz inequality)**.** *[27]. For a Hilbert space X and any $u, v \in X$,*

$$|(u, v)_X| \leq \|u\|_X \|v\|_X .$$

Triangle inequality is an immediate consequence of the Cauchy–Schwarz inequality.

Corollary 1. *[27] Let $u, v \in X$. Then*

$$\|u + v\| \leq \|u\| + \|v\| .$$

Definition 9 (Linear functional on a Hilbert space)**.** *[27]. Given a Hilbert space X, a linear functional j on X is a function $j : X \to \mathbb{R}$ that satisfies*

$$j(\alpha u + \beta v) = \alpha j(u) + \beta j(v).$$

Definition 10 (Dual of a Hilbert space)**.** *[25]. The dual of a Hilbert space X^* is the space of all bounded linear functionals on X. This has a natural norm induced by the norm on the underlying space*

$$\|j\|_{X^*} = \sup_{\|u\|_X = 1} |j(u)|.$$

In fact, X^* itself is a Hilbert space.

Theorem 4 (Riesz representation theorem)**.** *[27]. Let $j : X \to \mathbb{R}$ be a linear functional on a Hilbert space X. Then there exists a unique $g \in X$ such that*

$$(g, x) = j(x), \text{ for all } x \in X.$$

Moreover, one has $\|j\|_{X^*} = \|g\|_X$.

1.1.3 Lebesgue space

The Lebesgue spaces, denoted by $L_p(\Omega)$ are fundamental function spaces that appear throughout the analysis. We restrict our attention to real-valued functions f on a given domain Ω that are Lebesgue measurable.

Definition 11 (Lebesgue p-norm (for p finite))**.** *[25]. Let Ω be an open set in $\mathbb{R}^n$, $n \in \mathbb{N}$. For $1 \leq p < \infty$, L_p norm is defined as follows.*

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}, \text{ for } 1 \leq p < \infty.$$

Definition 12 (Lebesgue p-norm (for p infinite))**.** *[25]. For $p = \infty$, L_p norm is defined as follows.*

$$\|f\|_{L^\infty(\Omega)} = \text{ ess } \sup_{x \in \Omega} |f(x)|.$$

Definition 13 (Lebesgue spaces)**.** *[25]. For $1 \leq p \leq \infty$, Lebesgue spaces denoted by $L^p(\Omega)$ are defined as follows.*

$$L^p(\Omega) = \{f \text{ measurable } : \|f\|_{L^p(\Omega)} < \infty\}.$$

Theorem 5. *[25] For $p \in [1, \infty]$, all Lebesgue spaces are Banach spaces. For $p = 2$, $L_p(\Omega)$ is a Hilbert space.*

Theorem 6 (Holder's inequality)**.** *[25]. Let $p, q \in [1, \infty]$ such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $f \in L_p(\Omega)$ and $g \in L_q(\Omega)$ then $fg \in L_1(\Omega)$ and

$$\|fg\|_{L_1(\Omega)} \leq \|f\|_{L_p(\Omega)} \|g\|_{L_q(\Omega)}.$$

The elements of such a pair are called Holder conjugates.

Theorem 7 (Inclusion of Lebesgue spaces)**.** *[25]. Let Ω be bounded and $1 \leq p \leq q \leq \infty$. If*

$f \in L_q(\Omega)$ then $f \in L_p(\Omega)$.

1.1.4 Sobolev space

Sobolev spaces are fundamental spaces in the study of partial differential equations and their numerical approximations. Sobolev spaces are vector spaces and are obtained by extending Lebesgue spaces $L_2(\Omega)$. These spaces provide a theoretical framework to analyze partial differential equations. In particular, weak formulation of partial differential equation is examined in Sobolev space. They are important because solutions of partial differential equations, when they exist, belong to a Sobolev space.

Definition 14 (Compact support)**.** *[25]. A function $\phi \in C(\Omega)$ has compact support iff*

$$\textit{supp}(\phi) = \textit{ closure } \{x \in \Omega : \phi(x) \neq 0\}.$$

is compact and is a subset of an interior of the domain Ω.

In particular, the function ϕ vanishes on the boundary γ.

Definition 15 (Bump function)**.** *[25]. A bump function is an infinitely differentiable function that has compact support in the domain Ω. A set of bump functions is denoted by $C_0^\infty(\Omega)$.*

Definition 16 (Locally integrable function)**.** *[25]. Given a domain Ω, a set of locally integrable functions is defined by,*

$$L^1_{loc}(\Omega) = \{f : f \in L^1(K) \textit{ for all compact } K \subset \textit{ interior } \Omega\}.$$

To define higher derivatives, we introduce multi-index notation.

Definition 17 (Multi index notation)**.** *[25]. Consider a multi-dimensional function $u(x) = u(x_1, x_2, \dots, x_n)$ and a corresponding multi index notation that simplifies expressions for partial derivatives. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \alpha_i \geq 0$ is an integer vector in R^n then partial derivative of u(x) is represented by*

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_n}^{\alpha_n}},$$

where $|\alpha| = (\alpha_1 + \alpha_2 + ... + \alpha_n), \alpha_i \geq 0.$

This is called as multi index notation.

Definition 18 (Weak derivative)**.** *[25]. Let* $\Omega \subset \mathbb{R}^n$ *and* $f, g \in L^1_{loc}(\Omega)$*. Then a weak derivative of* f *is defined to be a function* g *such that,* $g = D^\alpha f$ *satisfying*

$$\int_\Omega f\, D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_\Omega g\, \phi\, dx, \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

It follows that if partial derivative of f up to order $|\alpha|$ exists in a usual sense then integration by parts implies an expression for weak derivative and the two derivatives coincide.

Example 4. *[25] Any strongly differentiable function has a weak derivative.*

Example 5. *[25] Let* $\Omega = (-1, 1)$ *and* $f(x) = |x|$*. Then the weak derivative of* $f(x)$ *denoted by* $f'(x)$ *is given by*

$$f'(x) = \begin{cases} -1, & \text{if } 1 < x < 0 \\ 1, & \text{if } 0 < x < 1. \end{cases} \tag{1.1}$$

Theorem 8. *[25]. The weak derivatives are unique up to a set of measure zero.*

Definition 19. *[25]. Let* l *be non-negative integer. Let* $f \in L^1_{loc}(\Omega)$*. Suppose that the weak derivatives* $D^\alpha f$ *exist for all* $|\alpha| \leq l$*. For* $p \in [1, \infty)$*, define Sobolev norm as*

$$\|f\|_{W_p^l(\Omega)} = \left[\sum_{|\alpha| \leq l} \|D^\alpha f\|^p_{L_p(\Omega)} \right]^{1/p},$$

and in case for $p = \infty$*,*

$$\|f\|_{W_\infty^l(\Omega)} = \max_{|\alpha| \leq l} \|D^\alpha f\|_{L_\infty(\Omega)}.$$

Definition 20 (Sobolev space)**.** *[25]. For* $1 \leq p \leq \infty$*, the Sobolev space is defined as*

$$W_p^l(\Omega) = \{f \in L^1_{loc}(\Omega) : \|f\|_{W_p^l} < \infty\}.$$

The Sobolev seminorm is introduced as follows.

Definition 21. *[25]. Sobolev seminorm of the function f is defined as follows.*

For $p \in [1, \infty)$,

$$|f|_{W_p^l(\Omega)} = \left[\sum_{|\alpha|=l} \|D^\alpha f\|_{L_p(\Omega)}^p \right]^{1/p}$$

and in the case $p = \infty$

$$|f|_{W_p^l(\Omega)} = \max_{|\alpha|=l} \|D^\alpha f\|_{L_\infty(\Omega)}.$$

It follows that $\|f\|_{W_p^l(\Omega)} \geq |f|_{W_p^l(\Omega)}$.

Theorem 9. *[25]. The Sobolev space $W_p^l(\Omega)$ is a Banach space.*

Definition 22 (Hilbert space $H^l(\Omega)$)**.** *[25].*

For $p = 2$, the spaces $W_p^l(\Omega)$ are Hilbert spaces with respect to inner product

$$(u, v) = \int_\Omega \sum_{|\alpha|\leq 1} D^\alpha u D^\alpha v.$$

These spaces are denoted by $H^l(\Omega) = W_2^l(\Omega)$. Thus, for integer $l \geq 0$,

$$H^l(\Omega) = \{v \in L_2(\Omega), D^\alpha v \in L_2(\Omega) \text{ for all } |\alpha| \leq l\}.$$

Remark.

1. *$H_0^l(\Omega)$ is a closed subspace of the Sobolev space $H^l(\Omega)$ incorporating boundary conditions.*

2. *For $l = 1$, the space H^1 is equipped with the scalar product*

$$(u, v)_{H^1} = (u, v)_{L^2} + (u', v')_{L^2}$$

and with the associated norm,

$$\|u\|_{H^1} = (\|u\|_{L^2}^2 + \left\|u'\right\|_{L^2}^2)^{\frac{1}{2}}.$$

Chapter 2

Fundamentals of the finite element method.

This chapter describes the fundamentals of the finite element method which provides a framework for further explorations. This chapter is a background and does not contribute to original work. The references [32, 33, 13, 34, 35, 25, 26, 27, 36, 37, 38, 12, 39, 40] have been used to source information discussed here. We restrict our attention to the case of second-order linear elliptic partial differential equations in 1-dimension.

2.1 A two-point boundary value problem

Partial differential equation with suitable boundary conditions forms a two-point boundary value problem usually represented by a single equation

$$Lu = f$$

where L is a linear partial differential operator acting on certain class of functions u which satisfy boundary conditions. Partial differential equations are classified as elliptic, parabolic and hyperbolic equations. For example, the wave equation is hyperbolic equation and the heat equation is parabolic equation. A typical example of elliptic partial differential equation is the Poisson equation/Laplace equation.

The Poisson equation and boundary conditions arise from a steady state problem in which a solution varies only with spatial coordinates but not with time. It has applications in the areas of theory of heat conduction, electrodynamics, fluid dynamics among others.
Consider 2^{nd} order Poisson's equation where an unknown function $u(x)$ defined on the domain Ω is characterized by equation

$$-\frac{d^2u}{dx^2} = f(x), \quad x \in \Omega.$$

The Laplace equation is obtained by letting $f(x) = 0$ in the above equation. The unique solution of the Poisson equation is investigated usually with an associated boundary conditions. Thus, the Poisson equation with Dirichlet boundary conditions is stated below:

$$-\frac{d^2u}{dx^2} = f(x), \quad x \in \Omega \tag{2.1}$$

subject to homogeneous boundary conditions:

$$u(x) = 0, \quad x \in \partial\Omega \tag{2.2}$$

where $u(x)$ is defined on the domain Ω and its boundary $\partial\Omega$.

Definition 23 (Strong solution of partial differential equation)**.** *[25] The exact or strong solution of partial differential equation is a solution that satisfies partial differential equation and specified boundary conditions.*

Example 6. *The strong solution of the Poisson equation with $f(x) = 1$ defined on $[0, 1]$ is $u(x) = \frac{x-x^2}{2}$. It follows that the function $u(x)$ is continuously differentiable. Moreover, it satisfies equation (2.1) and boundary conditions (2.2).*

For complicated functions f and complex domains Ω, it is usually not possible to find strong solutions to (2.1). Hence, numerical techniques are used to find approximations of the strong solution. Here we use the Galerkin approximation for numerical simulation which replaces the boundary value problem with a system of equations for finitely many parameters. The system of equations needs to be solved further to get an approximate solution.

2.2 Abstract formulation

2.2.1 Abstract formulation

This section deals with the mathematical theory of the finite element method from a broader approach. The general theory of second-order elliptic partial differential equation is also introduced which allows dealing with a large class of problems with the same analytical technique. The existence and uniqueness results for an abstract variational problem are presented in this section.

Definition 24 (Bi-linear form)**.** *[27]. A bi-linear form on a vector space V over a field F is a function $a : V \times V \to F$ that is linear in each argument such that*

- $a(v_1 + v_2, w) = a(v_1, w) + a(v_2, w)$, *for all* $v_1, v_2, w \in V$
- $a(v, w_1 + w_2) = a(v, w_1) + a(v, w_2)$, *for all* $v, w_1, w_2 \in V$
- $(cv, w) = c\, a(v, w)$, *for all* $v, w \in V$, $c \in F$
- $a(v, cw) = c\, a(v, w)$, *for all* $v, w \in V$, $c \in F$.

Example 7. *An inner product $a(u, v) : H_0^1 \times H_0^1 \to \mathbb{R}$ defined as*

$$a(u, v) = (\nabla u, \nabla v) = \int_\Omega \nabla u \nabla v \, dx, \quad \text{for all } v \in H_0^1(\Omega)$$

is a bi-linear form.

Definition 25 (Continuous bi-linear form)**.** *[38]. A bi-linear form $a : V \times V \to F$ is called continuous if there exists a constant $M < \infty$ such that*

$$|a(u, v)| \leq M \|u\|_V \|v\|_V, \quad \text{for all } u, v \in V.$$

Definition 26 (V-elliptic bi-linear form)**.** *[38]. A bi-linear form $a : V \times V \to F$ is called V-elliptic if there exists a constant $\alpha > 0$ such that*

$$a(u, u) \geq \alpha \|u\|_V^2, \quad \text{for all } u \in V.$$

The following fundamental result guarantees existence and uniqueness of the solution of variational problem stated below: Find $u \in V$ such that

$$a(u, v) = b(v), \text{ for all } v \in V, \tag{2.3}$$

where V is a Hilbert space, $b \in V'$ and $a(.,.)$ is continuous, coercive bi-linear form.

Lemma 1 (Lax-Milgram lemma). *[38] Let V be a Hilbert space, $a(.,.)$ be a continuous V-elliptic bi-linear form of V and $b \in V'$. There exist exactly one $u \in V$ such that $a(u, v) = b(v)$, for all $v \in V$.*

Proof. Please refer [38] for detailed proof. □

Definition 27 (Functional (J)). *[38] Let V be an infinite dimensional Hilbert space of functions defined over Ω. Let $a(v, w)$ be a continuous, symmetric and positive definite bi-linear form defined on $V \times V$. Let $b(v)$ be a continuous linear functional on V. Then J defined by*

$$J(v) = \frac{1}{2}a(v, v) - b(v), \text{ for } v \in V$$

is a continuous quadratic functional on V.

Usually, J represents energy of some physical system. The problem of solving $a(u, v) - b(v)$ for all $v \in V$ consists in finding an element $u \in V$ such that

$$J(u) = \min_{v \in V} J(v), \text{ where } J(v) = \frac{1}{2}a(v, v) - b(v), \text{ for } v \in V.$$

The symmetry and positive definiteness of a bi-linear form $a(u, v)$ are required for characterization and uniqueness of u.

Theorem 10 (Characterization and uniqueness). *[38]. Let $a(u, v)$ be a positive definite, symmetric bi-linear form defined on V and let $b(v)$ be a linear functional defined on V. Then $u \in V$ is a solution of the variational problem*

$$J(u) = \min_{v \in V} J(v), \text{ where } J(v) = \frac{1}{2}a(v, v) - b(v), \text{ for } v \in V \tag{2.4}$$

if and only if u is a solution of the variational equation

$$a(u, v) = b(v), \quad \text{for all } v \in V. \tag{2.5}$$

The solution u if exists, is unique.

Proof. Please refer [38] for detailed proof. □

Example 8. *For a model problem (2.1) with boundary conditions (2.2), choose a space* $V = H_0^1(\Omega)$ *and* $H = L_2(\Omega)$*. The quadratic functional J is then*

$$J(v) = \frac{1}{2}\int_\Omega \left(\frac{dv}{dx}\frac{dv}{dx}\right) - \int_\Omega f(x)\, v(x)\, dx, \qquad \text{for all } v \in H_0^1(\Omega).$$

Since the bi-linear form is symmetric, a weak solution of the model problem (2.1) with boundary conditions (2.2) is also a solution of the variational problem,

$$J(u) = \min_{v \in V} J(v), \text{ where } J(v) = \frac{1}{2}a(v, v) - b(v), \qquad \text{for } v \in V.$$

Definition 28 (Energy norm)**.** *[27] Let* $\|.\|_E$ *be an associated energy norm defined by*

$$\|v\|_E = a(v, v)^{1/2}.$$

Definition 29 (Equivalent norm)**.** *[25] The two norms are said to be equivalent on V if there exist two constants* C_1 *and* C_2 *such that*

$$C_1\|v\|_1 \leq \|v\|_2 \leq C_2\|v\|_1\,.$$

Let $\|.\|_V$ be a norm defined on a Hilbert space V. Then, the equivalence of energy norm and V- norm are necessary for the existence of the minimizer $u \in V$ of J.

Theorem 11 (Existence)**.** *[38] If energy norm is equivalent to the V- norm, then there exists a minimizer* $u \in V$ *of J.*

2.2.2 Ritz-Galerkin approximation schemes

Ritz- Galerkin approximation is a technique for approximately solving variational problems such as in equation (2.5). Instead of seeking a solution in infinite dimensional space V, the solution of equation (2.5) is found over a finite dimensional subspace $V_h \subset V$.

Definition 30 (Ritz-Galerkin Approximation)**.** *[27]. Given $V_h \subset V$, Ritz-Galerkin approximation of equation (2.5) is*
find $u_h \in V_h$ such that, $a(u_h, v_h) = b(v_h)$ for all $v_h \in V_h$.

As subspace V_h is finite dimensional, it is closed subspace of V and therefore a Hilbert space endowed with the same scalar product $(.,.)$ as V. Consequently, bi-linear form $a(.,.)$ has the same properties on V_h as on V. Therefore an equation

$$J(u_h) = \min_{v_h \in V_h} J(v_h) \text{ where } J(v_h) = \frac{1}{2}a(v_h, v_h) - b(v_h), \text{ for } v_h \in V_h \tag{2.6}$$

has a unique solution $u_h \in V_h$ and a unique minimiser u_h is characterised by Galerkin equation

$$a(u_h, v_h) = b(v_h), \quad v_h \in V_h. \tag{2.7}$$

Thus, Ritz introduced the finite dimensional approximation $u_h \in V_h$ to u which minimizes the functional J in V_h such that

$$J(u_h) \leq J(v_h) \quad \forall v_h \in V_h.$$

A minimiser always exists in V_h as the sub-space V_h is finite dimensional.

2.2.3 The error estimate for the Ritz-Galerkin approximation schemes

Let u be a solution to the variational problem $a(u, v) = b(v)$, for all $v \in V$ and u_h be a solution to an approximation problem $a(u_h, v) = b(v)$ for all $v \in V_h$.

Theorem 12 (Galerkin orthogonality)**.** *[27] The approximation u_h satisfies Galerkin orthogonality*

$$a(u - u_h, v) = 0, \text{ for all } v \in V_h.$$

Proof. Please refer [27] for detailed proof. □

Theorem 13. *[27] Assume that u_h is minimiser of J in V_h defined in (2.6) and energy norm is equivalent to the norm in V with*

$$C_1\|v\|_V \leq \|v\|_E \leq C_2\|v\|_V, \ \text{for all } v \in V$$

and some positive constants, C_1 and C_2. Then

- *u_h is the best approximation to u in the energy norm. That is*

$$\|u - u_h\|_E \leq \|u - v_h\|_E \qquad \text{for all } v_h \in V_h.$$

- *The approximation is order optimal in V and*

$$\|u - u_h\| \leq \frac{C_2}{C_1}\left\|u - u_h^{best}\right\|.$$

Proof. Galerkin's Orthogonality states that the error $u - u_h$ is orthogonal to V_h in the energy norm with the scalar product $a(.,.)$. Therefore for all $v_h \in V_h$,

$$\|u - v_h\|_E^2 = \|u - u_h + u_h - v_h\|_E^2$$

$$= \|u - u_h\|_E^2 + \|u_h - v_h\|_E^2$$

$$\geq \|u - u_h\|_E^2.$$

Therefore u_h is the best approximation of u in V_h with respect to the energy norm. Thus, one has

$$\|u - u_h\|_E \leq \frac{1}{C_1}\|u - u_h\|_E$$

$$\leq \frac{1}{C_1}\left\|u - u_h^{best}\right\|_E$$

$$\leq \frac{C_2}{C_1}\left\|u - u_h^{best}\right\|.$$

□

Thus, the best approximation is just the orthogonal projection of u into V_h. With respect to V-norm, the best approximation satisfies

$$\left\|u - u_h^{best}\right\|_V \leq \|u - v_h\|_V \,, \text{ for all } v_h \in V_h.$$

Cea's lemma is a key to the error analysis of the finite element method for elliptic boundary value problems. According to Cea's Lemma, an accuracy of numerical solution depends essentially on the function space V_h used to approximate the solution u well. It shows that the closer u is to V_h, the smaller is the discretization error.

Theorem 14 (Cea's lemma)**.** *[27] If $a(.,.)$ is V-elliptic with $a(v,v) \geq \alpha\|v\|^2$ and bounded with $a(u,v) \leq C\|u\|\|v\|$ and u_h be a solution to the Galerkin approximation problem $a(u_h, v) = b(v)$ for all $v \in V_h$ then*

$$\|u - u_h\|_V \leq \frac{C}{\alpha} \inf_{v \in V_h} \|u - v\|_V$$

where C is continuity constant and α is coercivity constant of $a(.,.)$ on V.

Proof. Please refer [27] for detailed proof. □

Remark.

- *Cea's theorem shows that the discrete solution u_h is quasi-optimal since its error coincides with the best possible approximation up to a factor in the sense that the error $\|u - u_h\|_V$ is proportional to the best it can be using the subspace V_h.*

- *In the symmetric case, we can prove that,*

$$\|u - u_h\|_E = \min_{v \in V_h} \|u - v\|_E \,.$$

Hence,

$$\|u - u_h\|_V \leq \frac{1}{\sqrt{\alpha}} \|u - u_h\|_E = \frac{1}{\sqrt{\alpha}} \min_{v \in V_h} \|u - v\|_E$$

$$\leq \sqrt{\frac{C}{\alpha}} \min_{v \in V_h} \|u - v\|_V \leq \frac{C}{\alpha} \min_{v \in V_h} \|u - v\|_V \,.$$

- *The energy norm and H^1 norm are equivalent. Therefore, it follows that*

$$\|u - u_h\|_{H^1} \leq \frac{C}{\alpha} \inf_{v \in V_h} \|u - v\|_{H^1} .$$

It is more useful to estimate the error in a weaker norm. This requires a duality argument. A bound of an Error in the L^2 norm is obtained using 'Nitsche trick'. To this end, let $V \subset H$ be a Hilbert space with norm $\|.\|_H$ such that the embedding $V \hookrightarrow H$ be continuous, that is there is a constant $c > 0$ such that

$$\|v\|_H \leq c\|v\|_V , \text{ for all } v \in V.$$

It follows that the order of error in the H- norm is at least as good as in the V- norm. i.e

$$\|u - u_h\|_H \leq c\|u - u_h\|_V .$$

An improved error bound is given by Aubin-Nitsche trick.

Lemma 2 (Aubin-Nitsche trick)**.** *[27] Let a be bounded such that $|a(v, w)| \leq \gamma\|v\|\|w\|$, for $v, w \in V$, and $H \supset V$ be a Hilbert space with norm $\|.\|_H$ such that $\|v\|_H \leq c\|v\|$, for all $v \in V$. If u_h satisfies the Galerkin equation $a(u_h, v_h) = b(v_h)$, for $v_h \in V_h$ for some $V_h \subset V$ then*

$$\|u - u_h\|_H \leq \gamma\|u - u_h\|_V \sup_{g \in H} \inf_{v_h \in V_h} \frac{\left\|\phi_g - v_h\right\|_V}{\|g\|_H}$$

where ϕ_g is the weak solution of dual problem

$$a(w, \phi_g) = (g, w)_H, \quad \textit{for all } w \in V.$$

Proof.

$$\|u - u_h\|_H = \sup_{g \in H} \frac{(u - u_h, g)_H}{\|g\|_H} \text{ ... (Dual characterization of the norm)}$$

$$= \sup_{g \in H} \frac{a(u - u_h, \phi_g)}{\|g\|_H} \text{ ... (definition of } \phi_g)$$

$$= \sup_{g \in H} \inf_{v_h \in V_h} \frac{a(u - u_h, \phi_g - v_h)}{\|g\|_H} \text{... (Galerkin orthogonality)}$$

$$\leq \gamma \|u - u_h\|_V \sup_{g \in H} \inf_{v_h \in V_h} \frac{\left\|\phi_g - v_h\right\|_V}{\|g\|_H} \text{... (by boundedness of a).}$$

□

2.3 The variational formulation of elliptic partial differential equation

The variational formulation of partial differential equation allows us to adopt an alternative solution approach, namely the finite element method to construct an approximate solution. To apply the abstract theory derived in the previous section to the Poisson equation with Dirichlet boundary conditions, choose space $V = H_0^1(\Omega)$ and $H = L_2(\Omega)$. The Sobolev space, L_2 space, and their norms are closely connected by the following theorem.

Theorem 15 (Poincare-Friedrichs)**.** *[38]. Assume that the domain Ω is bounded. The Sobolev space $H_0^1(\Omega)$ is continuously embedded in $L_2(\Omega)$, that is, there exists $C > 0$ such that*

$$\|v\|_{L_2(\Omega)} \leq C \left|\frac{dv}{dx}\right|_{L_2(\Omega)}, \quad \textit{for all } v \in H_0^1(\Omega).$$

Example 9. *For the model problem (2.1) with boundary conditions (2.2), choose the space $V = H_0^1(\Omega)$ and $H = L_2(\Omega)$.*
Multiplying given partial differential equation by a test function $v(x) \in H_0^1(\Omega)$,

$$\int_\Omega -\frac{d^2u}{dx^2} v(x)dx = \int_\Omega f(x)v(x)dx$$

Integrating by parts,

$$\int_\Omega \frac{du}{dx}\frac{dv}{dx}dx = \int_\Omega f(x)v(x)dx. \tag{2.8}$$

That is,

$$a(u, v) = b(v), \qquad \textit{for all } v(x) \in H_0^1(\Omega) \tag{2.9}$$

where, $a(u,v) = (\frac{du}{dx}, \frac{dv}{dx}) = \int_\Omega \frac{du}{dx}\frac{dv}{dx}dx$, $b(v) = (f, v) = \int_\Omega f(x)v(x)dx$ *and* $f \in L_2$.
The equation (2.9) is called variational or weak form of the problem. In this form, problem leads to the Galerkin approximation process.

Definition 31 (Weak Solution)**.** *[25]. A weak solution of given partial differential equation is a function,* $u(x) \in H_0^1(\Omega)$ *which satisfies*

$$a(u,v) = b(v), \text{ for all } v(x) \in H_0^1(\Omega).$$

Example 10. *[25]. Every classical solution is always a weak solution since integrating by parts gives*

$$\int_\Omega \frac{du}{dx}\frac{dv}{dx}dx = \int_\Omega f(x)v(x)dx.$$

Let $C_0^2(\Omega)$ be the space of all twice continuously differentiable functions, defined on $\overline{\Omega}$ satisfying boundary conditions and let $u \in C_0^2(\Omega)$ be a classical solution which satisfies (2.1) and (2.2). Using integration by parts, the variational problem defines the solution of the Poisson equation.

Theorem 16. *[25]. Let* $u \in C_0^2(\Omega)$*. Then, u solves the variational problem (2.9) if and only if*

$$-\frac{d^2u}{dx^2} = f(x)$$

almost everywhere in Ω*.*

Proof. Suppose that

$$-\frac{d^2u}{dx^2} = f(x)$$

almost everywhere in Ω.
Multiplying by $v(x) \in H_0^1(\Omega)$ and integrating over Ω,

$$\int_\Omega -\frac{d^2u}{dx^2}v(x)dx = \int_\Omega f(x)v(x)dx.$$

Since $v(x) \in H_0^1(\Omega)$, $v(\partial u) = 0$, integrating by parts gives

$$\int_\Omega \left(\frac{du}{dx}\right)\left(\frac{dv}{dx}\right) dx = \int_\Omega f(x)v(x)dx.$$

Therefore,

$$a(u, v) = b(v), \text{ for all } v(x) \in H_0^1(\Omega).$$

Conversely, let $u \in C_0^2(\Omega)$ and suppose that u solves the variational problem.

$$a(u, v) = b(v), \text{ for all } v(x) \in H_0^1(\Omega).$$

That is

$$\int_\Omega \left(\frac{du}{dx}\right)\left(\frac{dv}{dx}\right) dx = \int_\Omega f(x)v(x)dx.$$

Integration by parts gives

$$\int_\Omega \left(-\frac{d^2u}{dx^2}\right) v(x)dx = \int_\Omega f(x)v(x)dx.$$

It follows that the weak solution satisfies

$$\int_\Omega \left(-\frac{d^2u}{dx^2} - f(x)\right) v(x)dx = 0,$$

for all $v(x) \in H_0^1(\Omega)$.

As $H_0^1(\Omega)$ is dense in $L_2(\Omega)$, it follows that this characterization holds for all $v \in L_2(\Omega)$. Since, $(-\frac{d^2u}{dx^2} \in L_2(\Omega)$, choose $v = (-\frac{d^2u}{dx^2} - f(x))$

$$\left\|(\frac{d^2u}{dx^2} + f(x))\right\|_{L^2} = 0.$$

Consequently, $\left(-\frac{d^2u}{dx^2} - f(x)\right) = 0$ in $L_2(\Omega)$. Hence,

$$-\frac{d^2u}{dx^2} = f(x)$$

almost everywhere in Ω. □

Thus, if the weak solutions are regular enough and if $f \in L_2(\Omega)$ then the weak variational problem or minimization problem in $H_0^1(\Omega)$ have the same solution as the Dirichlet problem. Consequently, numerical techniques to solve the variational problem provide approximations of the solutions of the Poisson problem.

2.4 The finite element method

This section deals with a general framework of the finite element method. The finite element method is a computational technique for obtaining an approximate solutions to partial differential equations that arise in scientific and engineering applications. It is a technique in which given domain is represented as a collection of simple non-overlapping domains called finite elements and solution of a partial differential equation is approximated by a simpler polynomial function on each element.

The definition of the finite elements was first formalized by Ciarlet [1976] and it remains the standard formulation. The formal definition of the finite elements can be stated as follows.

Definition 32 (Ciarlet 1976)**.** *[36]. A finite element element is a triplet $(T, \sum, P)$ with the following properties.*

1. *$T \subset R^n$ be a bounded closed set with nonempty interior and a Lipschitz continuous boundary ∂T (the element domain).*

2. *$\sum$ be a finite dimensional subspace of functions on T (the space of shape functions). The set $\sum$ is said to be the set of degrees of freedom of the finite element.*

3. *P is finite dimensional subspace of real valued functions over T such that $\sum$ is $P-$ unisolvent. That is, if $\sum = \{\phi_i\}_{i=1}^{N}$ and $\alpha_i, 0 \leq i \leq N$ are any scalars, then there exists a unique function $p \in P$ such that $\phi_i(p) = \alpha_i, \quad 1 \leq i \leq N$.*

The finite element method is based on a variational formulation involving a subspace and special subspace of finite elements. The spaces over which variational problems are associated with boundary value problems are called finite element spaces.

A general framework constructing finite element method is discussed further.

2.4.1 Discretization of domain

The process of dividing the domain Ω of the solution u into small subdomains Ω_k is called discretization of the domain. The sub-domains Ω_k, are called finite elements. If all finite elements are congruent, then the partition is called regular. The discretization of the domain into finite elements

- represents geometry of the domain.
- approximates solution over each element of the mesh to better represent the solution over the entire domain. Moreover, an approximation of a solution over an element of the mesh is simpler than its approximation over the entire domain.

For instance, let $\Omega = [0, 1]$, be the domain of a given problem.

Definition 33 (Mesh). *[34] A mesh of $\Omega = [0, 1]$ is an indexed collection of intervals with non-zero measure $\{I_i = [x_i, x_{i+1}]$ for $0 \leq i \leq n\}$ forming a partition of Ω. That is,*

$$\Omega = \cup_{i=0}^{n} I_i \text{ and } I_i^\circ \cap I_j^\circ = \emptyset, \text{ for } i \neq j.$$

The intervals I_i are called elements of the mesh.

The simplest way to construct a mesh is to divide interval $[0, 1]$ into n subintervals such that,

$$0 = x_0 < x_1 < x_2 < ... < x_n = 1.$$

The points $\{x_0, x_1, \ldots, x_n\}$ are called vertices of the mesh. The mesh may have variable step size, $h_i = x_{i+1} - x_i \quad 0 \leq i \leq n$. Set $h = \max_{0\leq i\leq n} h_i$. Figure (2.1) represents the mesh of the interval $[0, 1]$ into 10 equidistant elements.

2.4.2 Derivation of approximation functions over each element

We restrict our discussion of interpolation functions to 1-dimension. The interpolation function is a function that interpolates the solution between discrete values obtained at mesh nodes.

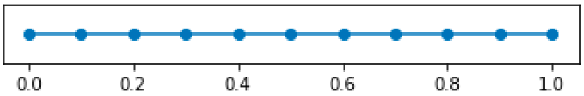


Figure 2.1: Discretization of the interval $[0, 1]$ into subintervals.

Therefore, appropriate functions must be used. The interpolation functions are generally assumed and are required to approximate quantities between the nodes. The continuity of these functions and their derivatives, called regularity plays important role in approximation theory. The regular function when restricted to a small enough domain is assumed to be simple, close to constant, or some lower degree polynomial. Thus, in the finite element method, the solution u is approximated by some u_h which is a polynomial in every finite element.

Piecewise polynomial approximation

For an integer $k \geq 0$, let P_k be a space of polynomials in one variable, with real coefficients and of degree at most k.

P_1 Lagrange finite element

Consider a vector space of continuous, piecewise linear functions

$$P_1^h = \{v_h \in C^0(\Omega); \forall i \in \{0, 1, \dots n\}; v_h |_{I_i} \in P_1\}.$$

This space is used to approximate one dimensional PDE and hence called as approximation space in one dimension. The functions $\{\phi_1, \phi_2, \dots, \phi_n\}$ are defined elementwise as follows. For $i \in \{0, 1, \dots n\}$,

$$\phi_i(x) = \begin{cases} \frac{1}{h_{i-1}}(x - x_{i-1}), \text{ if } x \in I_{i-1} \\ = \frac{1}{h_i}(x - x_i), \text{ if } x \in I_i \\ = 0, \text{ otherwise} \end{cases} \tag{2.10}$$

with obvious modifications if $i = 0$ or n.

Definition 34 (Nodal basis)**.** *[34]. The set $\{\phi_i\}$ is called nodal basis for P_1^h and $\{u_h(x_i)\}$ are the nodal values of the function u_h.*

These functions are called as hat functions in reference to their shape. The nodal basis functions satisfy the following conditions.

- inter-element compatibility condition: The nodal basis functions are such that continuity is achieved at each point on the interface between adjacent elements.

- Completeness condition: The nodal basis functions include complete linear polynomials such that constant derivatives may be obtained in each element.

- Interpolation condition: The nodal basis functions satisfy Kronecker delta property. That is, for all $i = 0, 1, ... n$,

$$\phi_i(x_j) = \begin{cases} 1, \text{ if } i = j \\ = 0, \text{ if } i \neq j. \end{cases} \tag{2.11}$$

- Local support condition: The nodal basis functions are compactly supported.

- Partition of unity condition: Let,$\phi_1, \phi_2, \dots, \phi_n$ be $n$nodal basis functions. Then $\phi_1 + \phi_2 + ... + \phi_n = 1$.

Proposition 1. *[34]. The set $\{\phi_1, \phi_2, \dots, \phi_n\}$ is a basis for P_1^h. Consequently, the space of functions P_h^1 is finite dimensional.*

Proof. Please refer [34] for detailed proof. □

Lemma 3. *[34].* $P_1^h \subset H^1(\Omega)$.

Proof. Please refer [34] for detailed proof. □

Thus, the function $u_h \in P_1^h$ can be represented in terms of basis functions as follows.

$$u_h = \sum_{k=0}^{n} u_k \phi_k, \tag{2.12}$$

where the coefficients $u_k = u_h(x_k)$ if x_k is the k^{th} vertex. The coefficients u_k are uniquely determined by its value at the nodal points x_k for $k = 1, 2, \dots, n$.

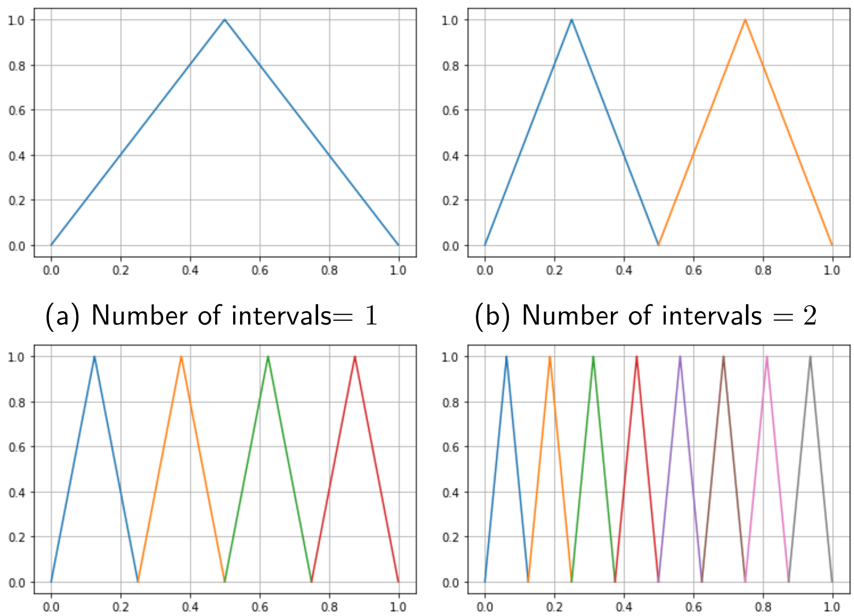
Similarly, for $k > 1$, the space of piecewise polynomials in one variable, with real coefficients and of degree at most k denoted by P_k^h can also be used as an approximation space. The interpolation technique described for P_1 Lagrange finite element can be generalized to higher degree polynomials.

Hierarchical Approximations

The hierarchical approximations are classified as $h-$version and $p-$version hierarchical approximations.

- $h-$version hierarchical approximations:

  The hierarchical approximations for $h-$version are constructed by introducing the additional shape function, maintaining constant polynomial order and reducing the size of the finite element mesh. Let, $i \geq 0$ and let V_i be the finite element space associated with the uniform finite element mesh M_i. The space V_i is generated by the piecewise linear basis functions and an additional shape function, called a hierarchical element which is defined over every finite element of the mesh M_i. Thus, the finite element space is V_i has direct sum hierarchical decomposition $V_i = U_i \bigoplus W_i$. Therefore, for $v \in V_i$ we have the unique decomposition $v_i = u_i + w_i$ where $u_i \in U_i$ and $w_i \in W_i$. Figure (2.2) represents graph of the linear hierarchical elements over the mesh.



(a) Number of intervals$= 1$ (b) Number of intervals $= 2$

(c) Number of intervals $= 4$ (d) Number of intervals $= 8$

Figure 2.2: Hierarchical linear elements over the mesh.

The uniform bisection refinement of the mesh M_i produces another finer mesh M_{i+1}. The set of functions ϕ_k^{i+1} constructed on the mesh M_{i+1} constitutes the basis for a finer space V_{i+1} such that any basis function ϕ_k^i can be expressed as a linear combination of basis functions ϕ_k^{i+1} of V_{i+1}. Thus,

$$\phi_k^i(x) = \sum_k a_k^{i+1} \phi_k^{i+1}(x)$$

where a_k^{i+1} are constants and $\phi_k^{i+1}(x)$ are basis functions of the finer space V_{i+1}. Consequently, the nesting relation $V_i \subset V_{i+1}$ holds. The coarsest space in the hierarchy is V_0 with progressively finer spaces $V_1, V_2, \ldots$ such that

$$V_0 \subset V_1 \subset \ldots \subset V_n \subset \ldots .$$

- $p-$version hierarchical approximations:

  The hierarchical approximations for $p-$version are constructed by adding high degree corrections to lower degree members of the series. A major drawback of the standard interpolation functions is that when element refinement is carried out, new shape functions have to be generated, and hence all calculations have to be repeated. Therefore, it is advantageous to use an additional function with the standard interpolation functions without increasing the number of nodes in the mesh. The additional shape functions represent simply additive refinements of a higher-order. These shape functions are called hierarchical as their contribution to the approximation will be of diminishing importance.

  The hierarchical bases give rise to a hierarchy of finite element spaces namely, $V_0, V_1, V_2 \ldots$. This hierarchy is represented by hierarchical nestedness relation,

$$V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_k.$$

Definition 35 (Hierarchical modal basis)**.** *[27]. A family $\{B_k\}_{k\geq 0}$ where B_k is a set of polynomials, is said to be a hierarchical modal basis if, for all $k \geq 0$*

1. B_k is a basis for P_k.

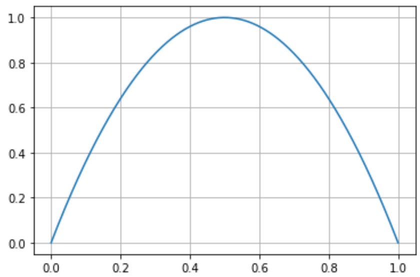
2. $B_k \subset B_{k+1}$.

Example 11. *[27]. The simplest example of hierarchical modal basis is* $B_k = \{1, x, x^2, x^3, \dots, x^k\}$.

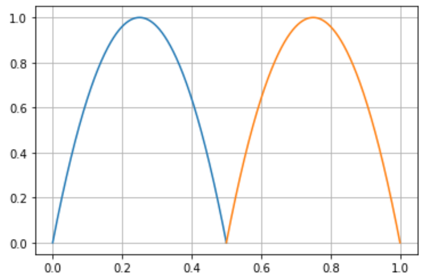
For instance, in case of linear nodal basis functions, a quadratic function is used to form a hierarchical basis over an element. This quadratic function is required to be:

- a quadratic polynomial,
- compactly supported, and
- continuous.

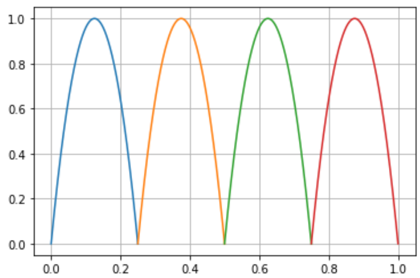
Thus, the hierarchical basis has three functions per element. Two of them are linear functions and one is a hierarchical quadratic element. Figure (2.3) represents graph of the quadratic hierarchical elements over the mesh. Thus, the finite element space is V_i



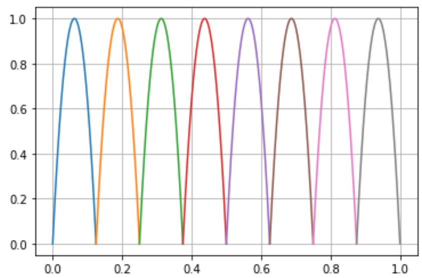
(a) Number of intervals$= 1$



(b) Number of intervals $= 2$



(c) Number of intervals $= 4$



(d) Number of intervals $= 8$

Figure 2.3: Hierarchical quadratic elements over the mesh.

has direct sum hierarchical decomposition $V_i = U_i \bigoplus W_i$. Therefore, for $v \in V_i$, we have the unique decomposition $v_i = u_i + w_i$ where $u_i \in U_i$ and $w_i \in W_i$. The basis of U_i are nodal basis functions and the basis for W_i consists of all the nodal basis functions for the continuous piecewise polynomial of degree 2 except those associated with mesh vertices. Thus, W_i consists of the span of the quadratic functions associated with edge midpoints in the mesh. This is called a hierarchical basis for piecewise quadratic polynomial space. The three basis functions span the space of quadratic polynomials.

Let $\phi_{i+\frac{1}{2}}$ be defined in terms of local element coordinate ξ , as a quadratic function of the form

$$\phi_{i+\frac{1}{2}} = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2$$

with the coefficients $\alpha_0, \alpha_1, \alpha_2$ be chosen such that, $\phi_{i+\frac{1}{2}}(\xi) = 0$ for $\xi = -1, 1$.

Thus, an approximate solution u_h over the typical element $\Omega_e = [x_i, x_{i+1}]$ is expressed as follows.

$$u_h(x) = u_i\phi_i(x) + u_{i+1}\phi_{i+1}(x) + \Delta u_{i+\frac{1}{2}}\phi_{i+\frac{1}{2}}(x), \quad \text{for } x \in \Omega_e \tag{2.13}$$

where, ϕ_i and ϕ_{i+1} are nodal basis functions and $\phi_{i+\frac{1}{2}}$ is hierarchical element.

The parameter $\Delta u_{i+\frac{1}{2}}$ used in equation (2.13) is the magnitude of the departure from linearity of the approximation $u_h(x)$ at the element center since $\phi_{i+\frac{1}{2}}(x_{i+\frac{1}{2}}) = 1$. The value of u at $x_{i+\frac{1}{2}}$ is given by the following expression:

$$u(x_{i+\frac{1}{2}}) = \frac{u_i + u_{i+1}}{2} + \Delta u_{i+\frac{1}{2}}.$$

Subsequently, the finite element solution over the entire domain has the form

$$u_h(x) = \sum_{i=0}^{n} u_i\phi_i(x) + \sum_{i=1}^{n} \Delta u_{i+\frac{1}{2}}\phi_{i+\frac{1}{2}}(x).$$

2.4.3 The finite element model

Let u_h be defined by equation (2.12) be an approximate solution of the given boundary value problem in terms of standard shape functions where $u_i = u_h(x_i)$ are to be determined.

The weak form of differential equation is stated below:

$$a(u_h, \phi_i) = f(\phi_i) \text{ for } i = 0, 1, ... n.$$

Thus,

$$\sum_{j=0}^{n} a(\phi_j, \phi_i)u_j = f(\phi_i) \text{ for } i = 0, 1, \dots n.$$

The above system can be expressed in the matrix form as $AU = F$ where stiffness matrix A is $A = a(i,j)_{n+1\times n+1}$ where $a(i,j) = a(\phi_j, \phi_i) = \int \phi_j'\phi_i'$ and load vector $F = f(\phi_i) = (f, \phi_i)$, for $i = 0, 1, \dots, n$. To calculate the stiffness matrix and the source vector over the entire domain, the coefficient matrix and the source vector for an element are calculated. Since the interpolation functions are compactly supported, there are only two non-zero interpolation functions over each element. Let K^i and F_i be the coefficient matrix and the source vector respectively for the typical element I_i. The element I_i is located between the nodes x_i and x_{i+1}. Let ϕ_i and ϕ_{i+1} be non-zero interpolation functions. Then the entries of the coefficient matrix on the interval I_i are calculated as follows:

$$K^i = \begin{bmatrix} K^i_{11} & K^i_{12} \\ K^i_{21} & K^i_{22} \end{bmatrix} = \begin{bmatrix} a(\phi_i, \phi_i) & a(\phi_i, \phi_{i+1}) \\ a(\phi_{i+1}, \phi_i) & a(\phi_{i+1}, \phi_{i+1}) \end{bmatrix} = \begin{bmatrix} \int_{I_i} \phi_i'\phi_i' dx & \int_{I_i} \phi_i'\phi_{i+1}' dx \\ \int_{I_i} \phi_{i+1}'\phi_i\prime dx & \int_{I_i} \phi_{i+1}'\phi_{i+1}' dx \end{bmatrix}.$$

The entries of the source vector on the interval I_i are calculated as follows.

$$F_i = \begin{bmatrix} \int_{I_i} f\,\phi_i\,dx \\ \int_{I_i} f\,\phi_{i+1}\,dx \end{bmatrix}.$$

The assembly of the finite element matrices is carried out by imposing inter-element continuity of the variables. The natural boundary conditions imposed on the problem reduce the assembled set of equations equal to the number of unknowns. Thus, the assembled coefficient matrix A is represented as follows.

$$A = \begin{bmatrix} K^1_{11} & K^1_{12} & 0 & \cdots & 0 & 0 \\ K^1_{21} & K^1_{22} + K^2_{11} & K^2_{12} & \cdots & 0 & 0 \\ 0 & K^2_{21} & K^1_{22} + K^3_{11} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & K^{n-1}_{22} + K^n_{11} & K^n_{12} \\ 0 & 0 & 0 & \cdots & K^n_{21} & K^n_{22} \end{bmatrix}.$$

The assembled source matrix is represented as follows.

$$F = \begin{bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 \\ \vdots \\ f_2^{n-1} + f_1^n \\ f_2^n \end{bmatrix}.$$

However, there is a disadvantage in using the standard shape functions. The resulting element matrices are completely different as the shape functions for each order of approximation are completely different. Thus, each level of approximation results in a completely new element matrix and an equation set has to be re-evaluated if shape functions of a higher degree are used. On the contrary, in case of hierarchical basis functions, since the shape functions of higher degree are used without increasing the number of nodes, the element matrices produced in the in the previous stage of the approximation reoccur and need not be recomputed. For instance, the linear nodal basis functions and a quadratic polynomial forms hierarchical basis. Let u_h represented by equation (2.12) be an approximate solution in terms of hierarchical basis. Let, $\Omega_e = [x_i, x_{i+1}]$ and let $\{\phi_i, \phi_{i+1}, \phi_{i+\frac{1}{2}}\}$ be hierarchical basis functions. Then, the element matrix corresponding to linear nodal basis functions reoccur and the entries corresponding to quadratic function are added to the element matrix.

Let, $K^i = \begin{bmatrix} K_{11}^i & K_{12}^i & K_{13}^i \\ K_{21}^i & K_{22}^i & K_{23}^i \\ K_{31}^i & K_{32}^i & K_{33}^i \end{bmatrix} = \begin{bmatrix} a(\phi_i, \phi_i) & a(\phi_i, \phi_{i+\frac{1}{2}}) & a(\phi_i, \phi_{i+1}) \\ a(\phi_{i+\frac{1}{2}}, \phi_i) & a(\phi_{i+\frac{1}{2}}, \phi_{i+\frac{1}{2}}) & a(\phi_{i+\frac{1}{2}}, \phi_{i+1}) \\ a(\phi_{i+1}, \phi_i) & a(\phi_{i+1}, \phi_{i+\frac{1}{2}}) & a(\phi_{i+1}, \phi_{i+1}) \end{bmatrix}.$

$$K^i = \begin{bmatrix} \int_{I_i} \phi_i' \phi_i' dx & \int_{I_i} \phi_i' \phi_{i+\frac{1}{2}}' dx & \int_{I_i} \phi_i' \phi_{i+1}' dx \\ \int_{I_i} \phi_{i+\frac{1}{2}}' \phi_i' dx & \int_{I_i} \phi_{i+\frac{1}{2}}' \phi_{i+\frac{1}{2}}' dx & \int_{I_i} \phi_{i+\frac{1}{2}}' \phi_{i+1}' dx \\ \int_{I_i} \phi_{i+1}' \phi_i \prime dx & \int_{I_i} \phi_{i+1}' \phi_{i+\frac{1}{2}}' dx & \int_{I_i} \phi_{i+1}' \phi_{i+1}' dx \end{bmatrix}.$$

The entries of the source vector on the interval I_i are calculated as follows.

$$F_i = \begin{bmatrix} \int_{I_i} f\, \phi_i \, dx \\ \int_{I_i} f\, \phi_{i+\frac{1}{2}} \, dx \\ \int_{I_i} f\, \phi_{i+1} \, dx \end{bmatrix}.$$

The linear nodal basis functions ensure the inter-element continuity of the shape functions.

Thus, the assembled coefficient matrix is represented in the following form.

$$A = \begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ K_{21}^1 & K_{22}^1 & K_{23}^1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ K_{31}^1 & K_{32}^1 & K_{33}^1 + K_{11}^2 & K_{12}^2 & K_{13}^1 2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & K_{21}^2 & K_{22}^2 & K_{23}^2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & K_{31}^2 & K_{32}^2 & K_{33}^2 + K_{11}^3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & K_{22}^{n-1} + K_{11}^n & K_{12}^n & K_{13}^n \\ 0 & 0 & 0 & 0 & 0 & \cdots & K_{21}^n & K_{22}^{2^n} & K_{13}^n \\ 0 & 0 & 0 & 0 & 0 & \cdots & K_{31}^n & K_{32}^{2^n} & K_{33}^n \end{bmatrix}.$$

The assembled source matrix is represented as follows.

$$F = \begin{bmatrix} f_1^1 \\ f_2^1 \\ f_3^1 + f_1^2 \\ f_2^2 \\ f_3^2 + f_1^3 \\ \vdots \\ f_3^{n-1} + f_1^n \\ f_2^n \\ f_3^n \end{bmatrix}.$$

Moreover, if the shape functions are orthogonal then the system matrix shows a heavy diagonal structure which makes better conditioning of the equation system to be solved.

Thus, the linear system of equations has a following matrix form.

$$AU = F$$

where A is called the stiffness matrix and $A = a(i, j)$ with $a(i, j) = a(\phi_j, \phi_i)$ and the load vector $F = (f, \phi_i)$. The unknown vector U can be obtained by solving the linear algebraic system $AU = F$.

Error Analysis

An error estimate denotes a quantity that approximates an actual unknown error. The error estimates in the finite element method can be classified into two major categories, namely:

- " A Priori" error estimates
- "A Posteriori" error estimates

These two types of error estimates serve very different purposes. Priori estimates are obtained before the solution is known. The main feature of the priori estimates is that it tells us the order of convergence of a given finite element method. The goal of these estimates is to give us a reasonable measure of the efficiency of a given method by telling us how fast the error decreases as we decrease the mesh size. The posteriori error estimates give quantitative information about an accuracy of the solution. These estimates use a computed solution u_h to give us an estimate of the form, $\|u - u_h\|_k \leq \epsilon$, where ϵ is simply a number. These estimates give us a much better idea of actual error in the finite element computation than priori error estimates since posteriori error estimates are obtained after the solution is known. These error estimates are computable since they are expressed in terms of the finite element solution. Secondly, in adaptive mesh refinement, the posteriori error estimator is used to indicate where the error is particularly high and more mesh intervals are then placed in those locations. A new finite element solution is computed and the process is repeated until a satisfactory error tolerance is reached.

Interpolation error estimate in Sobolev space

Definition 36 (Interpolant). *[34]. Given $v \in C_0([0,1])$, the interpolant $\pi v \in P_1^h$ of v is the unique, continuous, and piecewise linear function that takes the same value as v at all the mesh vertices.*

Let u be an exact solution and πu denotes its linear interpolant. Then the estimates of interpolation error are stated in the following theorem.

Theorem 17. *[34]. Let h be the mesh size of the element. Then for all $u \in H^1(\Omega)$, there exists a constant C independent of h such that*

$$\|u - \pi u\|_{L_2(\Omega)} \leq Ch|u - \pi u|_{H^1(\Omega)}. \tag{2.14}$$

Moreover, if $u \in H^2(\Omega)$ then

$$\|u - \pi u\|_{L^2} + Ch\|u - \pi u\|_E \leq 2(Ch)^2 \left\|u''\right\|_{L^2}. \tag{2.15}$$

Proof. Let u be an exact solution and πu be its linear interpolant. Let $e = u - \pi u$ be an error. Let I_i be i^{th} element of the mesh composed of $[x_i, x_{i+1}]$.
Recalling the definitions of the two norms, it is sufficient to prove the estimate piecewise. That is,

$$\|u - \pi u\|_{L^2[x_i, x_{i+1}]} \leq (Ch)|u|_{H^1[x_i, x_{i+1}]}$$

as the stated result represented by equation (2.14) follows by summing over i.
Since $u(x_i) = \pi u(x_i)$ and $u(x_{i+1}) = \pi u(x_{i+1})$ it follows that $e(x_i) = e(x_{i+1}) = 0$. Thus, for any $x \in [x_i, x_{i+1}]$ one has

$$e(x) = e(x) - e(x_i) = \int_{x_i}^{x} e'(t)dt.$$

It follows that

$$|e(x)|^2 \leq \int_{x_i}^{x} |e'(t)|^2 dt = \int_{x_i}^{x} |e'(t)\ 1|^2 dt.$$

$$= \int_{x_i}^{x} |1|^2 dx \int_{x_i}^{x} |e'(t)|^2 dt$$

$$= (x - x_i) \int_{x_i}^{x} |e'(t)|^2 dt$$

$$\leq (x - x_i) \int_{x_i}^{x_{i+1}} |e'(t)|^2 dt$$

$$= (x - x_i) \left\| e' \right\|_{L^2[x_i, x_{i+1}]}^2 .$$

Integrating wrt x on the interval $[x_i, x_{i+1}]$ one has

$$\int_{x_i}^{x_{i+1}} |e(x)|^2 dx = \left\| e' \right\|_{L^2[x_i, x_{i+1}]}^2 \int_{x_i}^{x_{i+1}} (x - x_i) dx.$$

$$\|e\|_{L^2[x_i, x_{i+1}]}^2 = \frac{h^2}{2} \left\| e' \right\|_{L^2[x_i, x_{i+1}]}^2 .$$

$$\|e\|_{L^2[x_i, x_{i+1}]} \leq \frac{h}{\sqrt{2}} \left\| e' \right\|_{L^2[x_i, x_{i+1}]} .$$

Thus,

$$\|e\|_{L^2[x_i, x_{i+1}]} \leq Ch \left\| e' \right\|_{L^2[x_i, x_{i+1}]} . \tag{2.16}$$

That is,

$$\|u - u_h\|_{L^2[x_i, x_{i+1}]} \leq Ch |u - u_h|_{H^1[x_i, x_{i+1}]}.$$

Next, let $u \in H^2(\Omega)$.

It is sufficient to prove that

$$\left\| e' \right\|_{L^2[x_i, x_{i+1}]} \leq \frac{h}{\sqrt{2}} \left\| e'' \right\|_{L^2[x_i, x_{i+1}]} .$$

Since $e(x_i) = e(x_{i+1}) = 0$, by Rolle's mean value theorem, there exists at least one $\xi \in [x_i, x_{i+1}]$ such that $e'(\xi) = 0$. Thus, one has

$$e'(x) = e'(x) - e'(\xi) = \int_{\xi}^{x} e''(t) dt.$$

$$|e'(x)|^2 \leq \int_{\xi}^{x} |e''(t)|^2 (t) dt$$

$$= \int_{\xi}^{x} |e''(t)1|^2(t)dt$$

$$= \int_{\xi}^{x} |1|^2 dt \int_{\xi}^{x} |e''(t)|^2 dt$$

$$\leq (x-\xi) \int_{x_i}^{x_{i+1}} |e''(t)|^2 dt$$

$$= (x-\xi) \left\| e''(x) \right\|^2_{L^2[x_i, x_{i+1}]}.$$

Integrating wrt x on the interval $[x_i, x_{i+1}]$ one has

$$\int_{x_i}^{x_{i+1}} |e'(x)|^2 dx = \left\| e''(x) \right\|^2_{L^2[x_i, x_{i+1}]} \int_{x_i}^{x_{i+1}} (x-\xi) dx.$$

$$\left\| e' \right\|^2_{L^2[x_i, x_{i+1}]} = \frac{h^2}{2} \left\| e''(x) \right\|^2_{L^2[x_i, x_{i+1}]}.$$

$$\left\| e' \right\|_{L^2[x_i, x_{i+1}]} \leq \frac{h}{\sqrt{2}} \left\| e'' \right\|_{L^2[x_i, x_{i+1}]}.$$

Thus,

$$\left\| e' \right\|_{L^2[x_i, x_{i+1}]} \leq Ch \left\| e'' \right\|_{L^2[x_i, x_{i+1}]}. \tag{2.17}$$

From equations (2.16) and (2.17) it follows that

$$\|e(x)\|_{L^2[x_i, x_{i+1}]} + Ch \left\| e' \right\|_{L^2[x_i, x_{i+1}]} \leq (Ch)^2 \left\| e'' \right\|_{L^2[x_i, x_{i+1}]}.$$

□

Further, let u be an exact solution and $\pi^k u$ denotes its interpolant of order k. Then for higher-order interpolants, the following estimates of the interpolation error hold.

Theorem 18. *[34]. Let $u \in H^{k+1}(\Omega)$ then the interpolation error is of optimal order. In perticular, for all h and for all $H^{k+1}(\Omega)$*

$$\left\| u - \pi^k u \right\|_{0,\Omega} + ch|u - \pi^k u|_{1,\Omega} \leq ch^{k+1} |u|_{k+1,\Omega}.$$

Quadrature and finite arithmetic error

This error is due to the numerical evaluation of integrals and numerical computation on computers. In most linear problems, these errors are expected to be very small.

Chapter 3

Fundamentals of an iterated function system and construction of fractal functions

This chapter deals with the fundamentals of an iterated function system and the construction of continuous fractal functions which provide a framework for further explorations. The references [20, 21, 31, 41, 24, 42, 43, 44, 45, 46, 47, 22, 48, 23, 49, 32, 50, 6, 8] have been used to source information discussed here. In this chapter, a specific family of fractal functions is constructed and a member of this family is used in hierarchical basis in the finite element method. A validation of mathematical theory which underpins the use of this member as a hierarchical element in the hierarchical basis contributes to original work.

3.1 Iterated function system

Iterated function systems (IFS) are used to generate a broad class of fractal functions. This section describes the mathematical aspects of iterated function systems.

3.1.1 Fundamentals of iterated function system

Let (X, d) be a metric space, $H(X)$ be a space of non-empty compact subsets and d_H denotes a Hausdorff metric on $H(X)$ defined as

$$d_H(A, B) = \max\{\sup_{x\in A}\inf_{y\in B} d(x, y), \sup_{x\in B}\inf_{y\in A} d(x, y)\}$$

where A and B are compact subsets of (X, d).

Definition 37 (Iterated function system)**.** *See [20]. An iterated function system (IFS) is a triplet $(X, d, w_n : n = 1, 2, \dots, N)$ where (X, d) is a complete metric space and $\{w_n : X \to X, n = 1, 2, \dots, N\}$ is a collection of N continuous functions.*

Definition 38 (Contractivity factor)**.** *See [20]. Let $\{w_n : X \to X,\ \ n = 1, 2, \dots, N\}$ be a set of contraction mappings with contractivity factors s_n, for $n \in 1, 2, \dots, N$. Then contractivity factor of an IFS is $s = \max\{s_n,$ for $n \in 1, 2, \dots, N\}$ and $0 \leq s < 1$.*

Definition 39 (Hyperbolic iterated function system)**.** *See [20]. A hyperbolic iterated system consists of a complete metric space (X, d) together with a finite set of contraction mappings on X with respective contractivity factors s_n for $n = 1, 2, \dots N$.*
Thus, if there exists a constant $s = \max\{s_n : n = 1, 2, \dots N\}$ and $0 \leq s < 1$ such that

$$d(w_n(x), w_n(y)) \leq s\, d(x, y), \text{ for } n = \{1, 2, \dots, N\}$$

then IFS is called a hyperbolic iterated function system.

Definition 40 (Hutchinson operator)**.** *[20]. An IFS induces a set valued operator $W : H(X) \to H(X)$ on $(H(X), d_H)$ defined as*

$$W(A) = \cup_n w_n(A), \text{ for } A \in H(X)$$

where $w_n(A) = \{w_n(x) : x \in A\}$. The operator W is called as Hutchinson operator.

The significant property of Hutchinson operator is stated in the following theorem:

Theorem 19. *See [20]. Let* $\{X; w_n : n = 1, 2, \dots, N\}$ *be hyperbolic iterated function system with contractivity factor* s. *Then the operator* $W : H(X) \to H(X)$ *defined by*

$$W(A) = \cup_n w_n(A), \text{ for all } A \in H(X)$$

is a contractive mapping on a complete metric space $(H(X), d_H)$*with contractivity factor* s. *That is,*

$$d_H(W(B), W(C)) \leq s\, d_H(B, C), \text{ for all } B, C \in H(X).$$

Proof. We will prove the result for $n = 2$. The argument then follows by mathematical induction. Let, $B, C \in H(X)$

$$d_H(W(B), W(C)) = d_H(w_1(B) \cup w_2(B), w_1(C) \cup w_2(C))$$

$$\leq d_H(w_1(B), w_1(C)) \vee d_H(w_2(B), w_2(C))$$

$$\leq s_1 d_H(B, C) \vee s_2 d_H(B, C) \leq s d_H(B, C)$$

where $s = \max\{s_1, s_2\}$. □

Definition 41 (Fixed point)**.** *[20]. Any set* $G \in H(X)$ *such that* $W(G) = G$ *is called the fixed point of an iterated function system.*

Definition 42 (Fractal set)**.** *[20]. A fractal set is a fixed point of Hutchinson operator of an IFS.*

The next theorem shows generation of fractal sets using a hyperbolic iterated function system.

Theorem 20. *[20]. Let hyperbolic iterated function system* $\{X; w_n : n = 1, 2, \dots, N\}$ *be given. For any set* $B \in H(X)$, *there exists a unique fixed set* $A \in H(X)$ *such that*

$$A = W(A) = \cup_n w_n(A)$$

and

$$A = \lim_{m\to\infty} W^{om}(B) \text{ for any } B \in H(X).$$

The unique fixed set $A \in H(X)$ is called the attractor of the hyperbolic IFS. Here, the limit is taken in Hausdorff metric

$$W^n(A_0) \overset{n\to\infty}{\longrightarrow} A \iff h(A, W^n(A_0)) \overset{n\to\infty}{\longrightarrow} 0.$$

3.1.2 IFS associated with fractal function

The fractal functions are attractors of the special class of iterated function systems. The construction of the hyperbolic iterated function system associated with fractal functions is stated below. See [31], [43].

Let $B(I)$ be a Banach space of bounded real valued functions $f : I = [x_0, x_N] \to R$ endowed with the supremum norm

$$\|f(x)\|_\infty = \sup_{x\in I}|f(x)| \text{ for all } f(x) \in B(I).$$

Let $x_0 < x_1 < x_2 < ... < x_N$ be real numbers and $I = [x_0, x_N] \subset \mathbb{R}$ be a closed interval.
Set $I_n = [x_{n-1}, x_n)$ and $I_N = [x_{N-1}, x_N]$.
Let $L_n : [x_0, x_N) \to [x_{n-1}, x_n)$, for $n \in 1, 2, \dots, N-1$ be a contractive homeomorphism such that

$$L_n(x_0) = x_{n-1}, \quad L_n(x_N^-) = x_n;$$

and $L_N : [x_0, x_N] \to [x_{N-1}, x_N]$,

$$|L_n(x_1) - L_n(x_2)| \leq l|x_1 - x_2|$$

for all $x_1, x_2 \in I$ and for $0 \leq l < 1$.
Let, $K = I \times R$ and $F_n : K \to R$ be contractive in the second argument satisfying

$$F_n(x_0, y_0) = y_{(n-1)}, \quad F_n(x_N, y_N) = y_n$$

and

$$|F_n(c, d_1) - F_n(c, d_2)| \leq q|d_1 - d_2|$$

for all $c \in I$, and $d_1, d_2 \in R$ and for some $0 \leq q < 1$.

Define functions $w_n : K \to K$ for $n \in 1, 2, \ldots, N$ by

$$w_n(x, y) = (L_n(x), F_n(x, y)).$$

Then $\{K, w_n : n \in 1, 2, \ldots, N\}$ is an iterated function system associated with fractal function.

3.1.3 The Read-Bajraktarevic operator (RB operator)

See [31]. Fractal functions are fixed points of specific contractive operators called Read-Bajraktarevic operators (RB Operators).

Let X be a closed connected subset of R^2 and $X = I \times R$. Further, assume that $W(x, y) = \{w_n(x, y) = (L_n(x), F_n(x, y)) : n = 1, 2, \ldots, N\}$ be a collection of functions on X, where each $L_n : I \to I_n$ is contractive homeomorphism and each $F_n : I \times R \to R$ is contractive in its second argument. Let $f \in B(I)$ and let G be its graph. If G is a fixed point of associated set valued mapping W then

$$W(G) = \cup_{n=1}^{N} w_n(G)$$

$$= \cup_{n=1}^{N} \{w_n(x, f(x)) : x \in I\}$$

$$= \cup_{n=1}^{N} \{w_n(L_n(x), F_n(x, y)) : x \in I\} = G.$$

The above equation suggests that there exists an operator $T : B(I) \to B(I)$ with a property that,

$$(Tf)(x) = F_n(L_n^{-1}(x), f \circ (L_n^{-1}(x)) \quad \text{for } n = 1, 2, \ldots, N \text{ and } x \in [x_{n-1}, x_n).$$

Analogously,

$$(Tf)(x) = \sum_{n=1}^{N} F_n(L_n^{-1}(x), f \circ (L_n^{-1}(x))\chi_{[x_{n-1}, x_n)}.$$

Definition 43. *[Read - Bajraktarevic operator(RB operator)] See [31]. An operator $T : B(I) \to B(I)$ defined by the equation*

$$(Tf)(x) = \sum_{n=1}^{N} F_n(L_n^{-1}(x), f(L_n^{-1}(x))\chi_{[x_{n-1},x_n)} \tag{3.1}$$

is called 'Read -Bajraktarevic operator'(RB operator).

3.2 Fractal function

The fractal functions are the functions whose graphs are fractal sets and are generated by certain classes of iterated function systems. The graphs of these functions usually have non-integral dimensions. Hence, these functions are referred to as fractal functions. These functions display some sort of self-similarity under magnification.

This section is focused on the construction of a specific class of fractal functions generated by iterated function system and their properties.

3.2.1 Construction of fractal function on $[x_0, x_N]$

See [31]. The fractal functions are described naturally in terms of hyperbolic IFS using the Read-Bajraktarevic operator. Let, $I = [x_0, x_N]$ and $\{x_0, x_1, \ldots, x_N\}$ be partition of a closed interval I satisfying $x_0 < x_1 < \ldots < x_N$. Let, $I_n = [x_{n-1}, x_n)$ for $i = 1, 2, \ldots, N-1$ and $I_N = [x_{N-1}, x_N]$.

The general construction of fractal functions based on the Read -Bajraktarevic operator is described in the following theorem:

Theorem 21. *See [31]. Suppose that $L_n : I \to I_n$ is contractive homeomorphism and each $F_n : I \times R \to R$ is contractive in its second argument with Lipschitz constants σ_n, $n = 1, 2 \ldots, N$. Furthermore, assume that $f \in B(I)$.*

Define a Read -Bajraktarevic operator $T : B(I) \to B(I)$ by,

$$T(f)(x) = \sum_{n=1}^{N} F_n(L_n^{-1}(x), f \circ (L_n^{-1}(x))\chi_{[x_{n-1},x_n)} \quad n = 1, 2, \ldots, N \quad x \in [x_{n-1}, x_n).$$

Then if

$$\sigma = \max\{\sigma_n : n = 1, 2, \dots, N\} < 1$$

the operator T is contractive on B(I) and its unique fixed point satisfies

$$T(f)(x) = f(x).$$

Proof. Let $f \in B(I)$. It follows that $T(f) \in B(I)$ and T is well defined.

Next, to prove that the operator T is contractive, let $f, g \in B(I)$. Then,

$$\|T(f)(x) - T(g)(x)\|_\infty = \max|\sigma_n| \left\| f(L_n^{-1}(x)) - g(L_n^{-1}(x)) \right\|_\infty$$

$$\leq |\sigma| \ \|f - g\|_\infty \text{ where } |\sigma|_\infty < 1.$$

Consequently, T is contractive on $B(I)$. Let $f_0(x)$ be linear function. Then the repeated application of the operator T generates a sequence $f_0, f_1 = T(f_0), f_2 = T(f_1), \dots, f_n = T(f_{n-1})$ of bounded functions on I. By contraction mapping theorem, there exists unique $f^* \in B(I)$ such that $T(f^*) = f^*$. □

The function f^* is a fractal function. The graph of f^* is made up of a finite number of copies of itself.

Definition 44 (Fractal function)**.** *See [31]. A function $f^* \in B(I)$ which is a fixed point of the Read-Bajraktarevic operator and which satisfies equation,*

$$f^*(x) = \sum_{n=1}^{N} F_n(L_n^{-1}(x), f^* \circ (L_n^{-1}(x))\chi_{L_n(x)} \tag{3.2}$$

is called fractal function of class B(I).

3.2.2 Continuity of fractal function

Generally, fractal functions denoted by equation (3.2) are not continuous. For a fractal function to be continuous, additional conditions need to be imposed on the function F_n of the IFS represented by the equation, $w_n(x, y) = (L_n(x), F_n(x, y))$.

A condition for continuity of the fractal function is stated below:

Condition for continuity of the fractal function

The function $F_n : I \to R$ is to be continuous and must satisfy the following equation.

$$F_{n+1}(x_0, y_0) = F_n(x_N, y_N), \quad \text{for } n = 0, 1, 2, \dots, (N-1). \tag{3.3}$$

Theorem 22. *See [31]. Let $B_c(I) \subset B(I)$ be a Banach space of continuous real valued functions $f : I = [x_0, x_N] \subset \mathbb{R} \to \mathbb{R}$ such that $f(x_0) = y_0$ and $f(x_n) = y_n$ endowed with supremum norm. If $T : B_c(I) \to B_c(I)$ is an operator defined by equation (3.1) such that*

$$F_{n+1}(x_0, f(x_0)) = F_n(x_N, f(x_N)), \quad \textit{for } n = 0, 1, \dots, N-1$$

then T is the contractive operator and its associated fractal function f^* is continuous.

Proof. Since $F_n : I \times \mathbb{R} \to \mathbb{R}$ is continuous, that is

$$F_{n+1}(x_0, f(x_0)) = F_n(x_N, f(x_N)), \quad \text{for } n = 0, 1, \dots, N-1$$

the Read -Bajraktarevic operator $T(f)$ is well defined.

Let $f(x), g(x) \in B_c(I)$. Then

$$\|T(f)(x) - T(g)(x)\|_\infty = \max |\sigma_n| \left\| f(L_n^{-1}(x)) - g(L_n^{-1}(x)) \right\|_\infty$$

$$\leq |\sigma| \, \|f - g\|_\infty \text{ where } |\sigma|_\infty < 1.$$

Therefore, the operator T is a contractive on $B_c(I)$. Let $f_0(x)$ be linear function. Then the repeated application of the operator T generates a sequence $f_0, f_1 = T(f_0), f_2 = T(f_1), \dots, f_n = T(f_{n-1})$ of bounded, continuous functions on I. The contraction mapping theorem implies that the Read -Bajraktarevic operator T has a unique fixed-point $f^* \in B_c(I)$.

Hence, the fractal function $f^* \in B_c(I)$ and is continuous. □

An important class of fractal function is obtained by letting,

$$L_n(x) = a_n x + b_n, \quad \textit{for } n = 1, 2, \dots, N.$$

$$F_n(x, y) = \lambda_n(x) + \sigma_n y, \quad \textit{for } n = 1, 2, \dots, N.$$

With

$$F_n(x_0, y_0) = y_{(n-1)}, \quad F_n(x_N, y_N) = y_n$$

By choosing the function $\lambda_n(x)$, the overall approximate shape of $f(x)$ can be fixed.
If $\sigma_n = 0$, then $f(x) = \lambda_n(x)$ for $x \in I_n$. Consequently, $\lambda_n(x)$ is called a condensation function.
If $\sigma_n \neq 0$ and $\sigma_n \to 0$ then the graph of $f(x)$ over I_n condenses on that of $\lambda_n(x)$.
Let, $\lambda_n(x) \in P_d(I)$,where $P_d(I)$ is a space of real polynomials of degree at most d on I, and $|s_n| < 1$ are free parameters. The associated operator T is then given by,

$$T(f)(x) = F_n(\lambda_n \circ L_n^{-1}(x), \sigma_n f \circ (L_n^{-1}(x)) \quad n = 1, 2, \dots, N \quad x \in [x_{n-1}, x_n).$$

3.2.3 Construction of fractal function on $[0, 1]$

See [41]. We focus on univariate $\mathbb{R}-$valued fractal functions defined on subset $[0, 1]$ of $\mathbb{R}$. To discuss the existence, construction and properties of fractal functions $f : [0, 1] \to \mathbb{R}$ whose graphs are attractors of IFS $\{L_i(x), F_i(x, y)\}$ we proceed as follows.
Let $I = [0, 1], I_1 = [0, \frac{1}{2}), I_2 = [\frac{1}{2}, 1]$.
Define the functions, $L_1 : [0, 1) \to I_1$, and $L_2 : [0, 1] \to I_2$ as,

$$\begin{aligned} L_1(x) &= \frac{x}{2} \\ L_2(x) &= \frac{x+1}{2}. \end{aligned} \tag{3.4}$$

It follows that for $i = 1, 2$;

$$|L_i(x) - L_i(y)| \leq c|x - y|, \quad \text{for all } x, y \in [0, 1], c = \frac{1}{2} \in (0, 1).$$

Consequently, L_1 and L^2 are contractive homeomorphisms.

Define a continuous maps $F_i : [0,1] \times \mathbb{R} \to \mathbb{R}$ for $i = 1,2$ as follows.

$$\begin{aligned} F_1(x,y) &= \sigma y + a_0 x + b_0 \\ F_2(x,y) &= \sigma y + a_1 x + b_1 \end{aligned} \tag{3.5}$$

where, $a_0, a_1, b_0, b_1 \in \mathbb{R}$ and $\sigma \in [0,1)$.

It follows that

$$|F_i(x,y_1) - F_i(x,y_2)| \leq \sigma |y_1 - y_2| \text{ for all } y_1, y_2 \in \mathbb{R} \text{ and } i = 1,2.$$

Consequently, the functions $F_i(x,y)$ are uniformly Lipschitz in the second argument.

To construct continuous fractal functions, we impose the following condition on maps $F_i(x,y)$ for $i = 1,2$.

$$F_1(1, f(1)) = F_2(0, f(0)).$$

Thus, the constants a_0, a_1, b_0, b_1 are chosen such that:

$$\begin{aligned} F_1(0,y_0) &= y_0, \\ F_2(1,y_1) &= y_1, \\ F_1(1,f(1)) = F_2(0,f(0)) &= y_{\frac{1}{2}}. \end{aligned} \tag{3.6}$$

Theorem 23. *See [41]. The fractal function corresponding to the IFS $\{L_i(x), F_i(x,y)\}$ defined by equations (3.4) and (3.5) satisfying equation (3.6) is continuous.*

Proof. Let $B_c[0,1]$ be a Banach space of bounded and continuous functions f such that $f(0) = y_0, f(1) = y_1$ and with the norm

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|.$$

Thus,

$$B_c[0,1] = \{f : [0,1] \to \mathbb{R} \mid \exists\, c \in \mathbb{R} \text{ such that } \forall\, x \in [0,1],\ |f(x)| \le c,$$
$$f \in C^0[0,1] \text{ with } f(0) = y_0, f(1) = y_1\}.$$

Define an operator $T : B_c[0,1] \to B_c[0,1]$ by

$$T(f)(x) = F_1(L_1^{-1}(x), f(L_1^{-1}(x))\chi_{[0,\frac{1}{2})} + F_2(L_2^{-1}(x), f(L_2^{-1}(x))\chi_{[\frac{1}{2},1]}.$$

From equations (3.4) and (3.5), it follows that

$$T(f)(x) = (\sigma f(2x) + a_0(2x) + b_0)\chi_{[0,\frac{1}{2})} + (\sigma f(2x-1) + a_1(2x-1) + b_1)\chi_{[\frac{1}{2},1]}.$$

Let, $\alpha_0 = 2a_0, \beta_0 = b_0, \alpha_1 = 2a_1, \beta_1 = b_1 - a_1$.

Thus, the operator T has the following form.

$$T(f)(x) = (\sigma f(2x) + \alpha_0 x + \beta_0)\chi_{[0,\frac{1}{2})} + (\sigma f(2x-1) + \alpha_1 x + \beta_1)\chi_{[\frac{1}{2},1]} \tag{3.7}$$

where $\sigma \in (0,1)$.

It follows that $T(f)$ is continuous on $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1]$.

Moreover, from $L_1^{-1}(\frac{1}{2}^-) = 1,\ L_2^{-1}(\frac{1}{2}^+) = 0$ and from equations in (3.5) it implies that

$$T(f)(\frac{1}{2}^-) = F_1(L_1^{-1}(\frac{1}{2}^-), f(L_1^{-1}(\frac{1}{2}^-)) = F_1(1, f(1)) = y_{\frac{1}{2}}.$$

$$T(f)(\frac{1}{2}^+) = F_2(L_2^{-1}(\frac{1}{2}^+), f(L_2^{-1}(\frac{1}{2}^+)) = F_2(0, f(0)) = y_{\frac{1}{2}}.$$

Thus,

$$T(f)(\frac{1}{2}^-) = T(f)(\frac{1}{2}^+).$$

Therefore, $T(f)$ is continuous at $x = \frac{1}{2}$. Hence, $T(f)$ is continuous on $[0,1]$. Moreover, $T(f)$

is bounded on $[0,1]$ since

$$T(f)(0) = F_1((L_1^{-1}(0), f(L_1^{-1}(0))) = F_1(0, y_0) = y_0.$$

$$T(f)(1) = F_1((L_2^{-1}(1), f(L_2^{-1}(1))) = F_2(1, y_1) = y_1.$$

Thus, $T(f)$ is continuous on the interval $[0,1]$ and bounded at the end points of the interval $[0,1]$. Therefore, $T(f)$ is bounded over the interval $[0,1]$. Consequently, T takes $B_c[0,1]$ into itself and is well defined map on $B_c[0,1]$.

Choose $\sigma \in [0, \frac{1}{2})$. We show that T is contraction mapping on metric space $B_c[0,1]$.

To this end, let $f, g \in B_c[0,1]$ then

$$\|T(f) - T(g)\| = \sup_{x\in[0,1]} \{|T(f)(x) - T(g)(x)|\}$$

$$= \sigma \sup_{x\in[0,1]} \{|(f(2x) - g(2x))\chi_{[0,\frac{1}{2})} + (f(2x-1) - g(2x-1))\chi_{[\frac{1}{2},1]}|\}$$

$$\leq \sigma \sup_{x\in[0,\frac{1}{2}]} |(f(2x) - g(2x))\chi_{[0,\frac{1}{2})}| + \sigma \sup_{x\in[\frac{1}{2},1]} |(f(2x-1) - g(2x-1))\chi_{[\frac{1}{2},1]}|$$

$$= \sigma[\sup_{x\in[0,1)} \{|f(x) - g(x)|\} + \sup_{x\in[0,1]} \{|f(x) - g(x)|\}]$$

and thus,

$$\|Tf - Tg\| \leq c \sup_{x\in[0,1]} \{|f(x) - g(x)|\}$$

where,

$$c = \max\{\sigma, \sigma\} = \sigma.$$

Therefore, if $\sigma \in [0, \frac{1}{2})$, the operator $T : B_c[0,1] \to B_c[0,1]$ is a contraction with contractivity factor $c = \sigma < \frac{1}{2}$.

Let $f_0(x)$ be a linear function. Then the repeated application of the operator T generates a sequence $f_0, f_1 = T(f_0), f_2 = T(f_1), \ldots, f_n = T(f_{n-1})$ of bounded, continuous functions on $[0,1]$. The contraction mapping theorem implies that T possesses unique fixed point in

$B_c[0,1]$. Thus, there exists $f^* \in B_c[0,1]$ such that,

$$T(f^*)(x) = f^*(x), \quad \text{for all } x \in [0,1].$$

Moreover,

$$f^*(x_0) = y_0 \text{ and } f^*(x_1) = y_1.$$

The function f^* is identified as a continuous fractal function and the whole image can be interpreted as the attractor of an IFS with condensation, where the condensation set is the graph of initial function $f_0(x) \in B_c[0,1]$.

As $T(f^*)(x) = f^*(x)$, one has,

$$(f^*)(x) = (\sigma f^*(2x) + \alpha_0 x + \beta_0)\chi_{[0,\frac{1}{2})} + (\sigma f^*(2x-1) + \alpha_1 x + \beta_1)\chi_{[\frac{1}{2},1]}.$$

□

Definition 45. *Let $\mathcal{F}$ be a set of continuous fractal functions $f : [0,1] \to \mathbb{R}$ satisfying*

$$f(x) = (\sigma f(2x) + \alpha_0 x + \beta_0)\chi_{[0,\frac{1}{2})} + (\sigma f(2x-1) + \alpha_1 x + \beta_1)\chi_{[\frac{1}{2},1]}, \tag{3.8}$$

for some $\alpha_0, \beta_0, \alpha_1, \beta_1 \in \mathbb{R}$ and

$$f(0) = y_0 \text{ and } f(1) = y_1.$$

We now construct some of the elements of $\mathcal{F}$ as follows.

1. Let $\phi = f$ then for $\alpha_0 = 0, \beta_0 = 1-\sigma, \alpha_1 = 0, \beta_1 = 1-\sigma$

$$\phi(x) = (\sigma\phi(2x) + (1-\sigma))\chi_{[0,\frac{1}{2})} + (\sigma\phi(2x-1) + (1-\sigma))\chi_{[\frac{1}{2},1]}, \text{for } x \in [0,1]. \tag{3.9}$$

Let $\phi(x) = 1$ then $\phi(2x) = \phi(2x-1) = 1$.

Substituting in equation (3.9), one has

$$\phi(x) = \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{if } x \notin [0,1]. \end{cases} \tag{3.10}$$

In particular, $\phi(0) = \phi(1) = 1$. From equation (3.10) it follows that $\phi(x) \in \mathcal{F}$.

2. Similarly, let $\lambda = f$ then for $\alpha_0 = 1 - 2\sigma, \beta_0 = 0, \alpha_1 = 1 - 2\sigma, \beta_1 = \sigma$

$$\lambda(x) = (\sigma\lambda(2x) + (1 - 2\,\sigma)x)\chi_{[0,\frac{1}{2})} + (\sigma\lambda(2x-1) + (1 - 2\,\sigma)x + \sigma)\chi_{[\frac{1}{2},1]}, \text{if } x \in [0,1]. \tag{3.11}$$

Let $\lambda(x) = x$ then $\lambda(2x) = 2x$ and $\lambda(2x - 1) = 2x - 1$.

Substituting in equation (3.11),

$$\lambda(x) = \begin{cases} x, & \text{if } x \in [0,1] \\ 0, & \text{if } x \notin [0,1]. \end{cases} \tag{3.12}$$

In particular, $\lambda(0) = 0$, $\lambda(1) = 1$.

From equation (3.12) it follows that $\lambda(x) \in \mathcal{F}$.

3. Let $\Lambda = f$ then for $\alpha_0 = 2\sigma - 1, \beta_0 = 1 - \sigma, \alpha_1 = 2\sigma - 1, \beta_1 = 1 - 2\sigma$

$$\Lambda(x) = (\sigma\Lambda(2x) + (2\,\sigma - 1)x + (1 - \sigma))\chi_{[0,\frac{1}{2})} + (\sigma\Lambda(2x-1) + (2\,\sigma - 1)x + (1 - 2\sigma))\chi_{[\frac{1}{2},1].} \tag{3.13}$$

Let $\Lambda(x) = 1 - x$ then $\Lambda(2x) = 1 - 2x$ and $\Lambda(2x - 1) = 2 - 2x$.

Substituting in the equation (3.13), one has

$$\Lambda(x) = \begin{cases} 1 - x, & \text{if } x \in [0,1]. \\ 0, & \text{if } x \notin [0,1]. \end{cases} \tag{3.14}$$

In particular, $\Lambda(0) = 1$, $\Lambda(1) = 0$.

From equation (3.14) it follows that $\Lambda(x) \in \mathcal{F}$.

4. For $\Psi = f$ choosing $\alpha_0 = 2, \beta_0 = 0, \alpha_1 = -2, \beta_1 = 2$ leads to recursion:

$$\Psi(x) = (\sigma\Psi(2x) + 2x)\chi_{[0,\frac{1}{2})} + (\sigma\Psi(2x-1) - 2x + 2)\chi_{[\frac{1}{2},1]}. \tag{3.15}$$

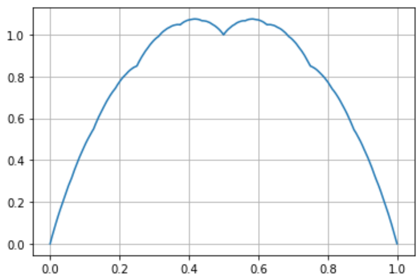
In this case,

- $\Psi(0) = \sigma\Psi(0)$. Thus, $\Psi(0) = 0$.
- $\Psi(1) = \sigma\Psi(1)$. Thus, $\Psi(1) = 0$.
- $\Psi(\frac{1}{2}) = 1$
- $\Psi(\frac{1}{2}^+) = \lim_{x\to\frac{1}{2}^+} \Psi(x) = \lim_{\epsilon\to 0^+} \Psi(\frac{1}{2} + \epsilon) = 1$.
- $\Psi(\frac{1}{2}^-) = \lim_{x\to\frac{1}{2}^-} \Psi(x) = \lim_{\epsilon\to 0^-} \Psi(\frac{1}{2} - \epsilon) = 1$

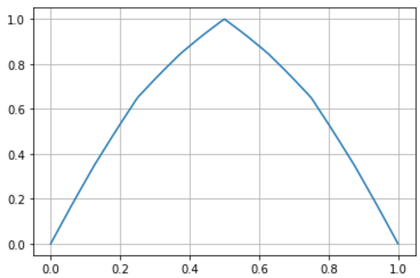
Thus, $\Psi(\frac{1}{2}) = \Psi(\frac{1}{2}^+) = \Psi(\frac{1}{2}^-) = 1$.

Therefore, $\Psi(x)$ is continuous at $x = \frac{1}{2}$.

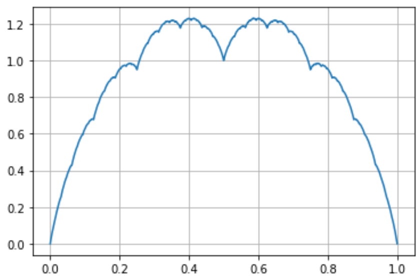
Figure (3.1) represents the graph of fractal function $\Psi(x)$ for different values of $\sigma \in [0, \frac{1}{2})$.



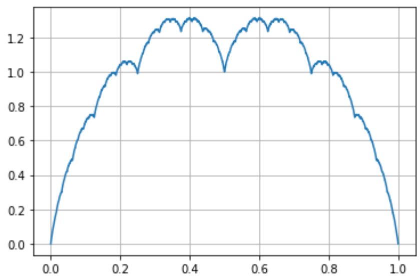
(a) $\sigma = 0.35$



(b) $\sigma = 0.15$



(c) $\sigma = 0.45$



(d) $\sigma = 0.49$

Figure 3.1: Graph of the continuous fractal function $\Psi(x)$ for $\sigma \in [0, \frac{1}{2})$.

Proposition 2. *The fractal function $\Psi(x)$ represented by equation (3.15) defines a function $\Psi(x) = 1 - |2x - 1|$ and a quadratic function $\Psi = 4x(1 - x)$.*

Proof. To find the fixed points of the Read-Bajraktarevic operator denoted by equation (3.15) we proceed as follows.

Put $\sigma = 0$ in equation (3.15). It follows that

$$\Psi(x) = (2x)\chi_{[0,\frac{1}{2})} + (2-2x)\chi_{[\frac{1}{2},1]}.$$

That is,

$$\Psi(x) = \begin{cases} 2x, \text{ if } x \in [0, \frac{1}{2}) \\ 2(1-x), \text{ if } x \in [\frac{1}{2}, 1]. \end{cases} \tag{3.16}$$

Thus,a fixed point of Read-Bajraktarevic operator represented by equation (3.15) for $\sigma = 0$ is a function

$$\Psi(x) = 1 - |2x - 1|, \text{ if } x \in [0, 1].$$

Similarly, let $\sigma = \frac{1}{4}$ and $\Psi(x) = 4x(1-x)$ then one has

$$\Psi(2x) = 4[2x(1-2x)] \text{ and } \Psi(2(1-x)) = 4[(2(1-x))(1-(2(1-x))].$$

From equation (3.15), It follows that

$$\Psi(x) = (4\frac{1}{4}[2x(1-2x)] + 2x)\chi_{[0,\frac{1}{2})} + 4\frac{1}{4}[(2(1-x))(1-(2(1-x))] + 2(1-x)\chi_{[\frac{1}{2},1]}$$

Thus,

$$\Psi(x) = 4x(1-x)\chi_{[0,\frac{1}{2})} + 4x(1-x)\chi_{[\frac{1}{2},1]}.$$

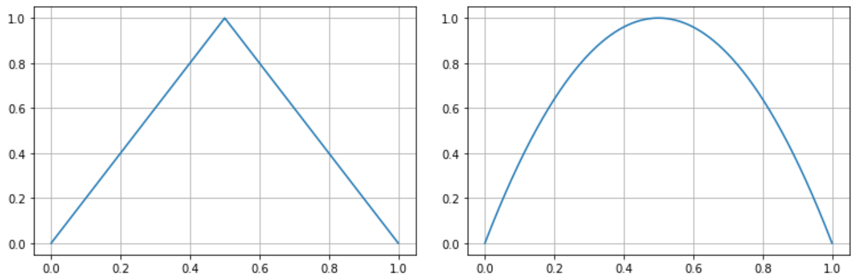
That is,

$$\Psi(x) = \begin{cases} 4x(1-x), \text{ if } x \in [0, \frac{1}{2}) \\ 4x(1-x), \text{ if } x \in [\frac{1}{2}, 1]. \end{cases} \tag{3.17}$$

Hence, $\Psi(x) = 4x(1-x)$ is a fixed point of Read-Bajraktarevic operator represented by equation (3.15) for $\sigma = \frac{1}{4}$.

Consequently, the equation (3.15) defines a function $\Psi(x) = 1-|2x-1|$ and a quadratic function $\Psi = 4x(1-x)$. $\square$

Figure (3.2) represents the graph of fixed points of the Read-Bajraktarevic operator represented by equation (3.15) for $\sigma = 0$ and $\sigma = \frac{1}{4}$.



(a) $\sigma = 0$ (b) $\sigma = \frac{1}{4}$

Figure 3.2: Graph of the fixed points of Read-Bajraktarevic operator represented by equation (3.15)

3.2.4 Properties of class $\mathcal{F}$

The properties of the class $\mathcal{F}$ are stated below:

1. $\mathcal{F} \subset B_c[0,1]$ is a linear space.
2. The values of the function $f(x)$ at boundaries of the interval $[0,1]$ are as follows.
   - $f(0) = \sigma f(0) + \beta_0$. Thus, $f(0) = y_0 = \frac{\beta_0}{1-\sigma}$.
   - $f(1) = \sigma f(1) + \alpha_1 + \beta_1$ Thus, $f(1) = y_1 = \frac{\alpha_1+\beta_1}{1-\sigma}$.
3. $\mathcal{F} \subset L^2[0,1]$.

Lemma 4. *The operator T defined by equation (3.7) is contractive in $L^2[0,1]$ if $\sigma \in [0, \frac{1}{2})$. Moreover, $f(x) \in L^2[0,1]$.*

Proof. Let $f \in \mathcal{F}$. It follows that f is a fixed point of the operator T. That is $Tf = f$.

Therefore, if $f \in L^2[0,1]$ it follows that $Tf \in L^2[0,1]$.

Consequently, T is well defined and T maps $L^2[0,1]$ to $L^2[0,1]$.

To show that T is contractive in $L^2[0,1]$, let $f, g \in L^2[0,1]$. Then

$$\|Tf - Tg\|_{L^2}^2 = \int_0^{\frac{1}{2}} |\sigma\left(f(2x) - g(2x)\right)|^2\, dx + \int_{\frac{1}{2}}^1 |\sigma\left(f(2x-1) - g(2x-1)\right)|^2\, dx$$

$$\leq \int_0^1 |\sigma|^2 |f(x) - g(x)|^2 \, dx + \int_0^1 |\sigma|^2 |f(x) - g(x)|^2 \, dx$$

$$= 2\sigma^2 \int_0^1 |f(x) - g(x)|^2 \, dx = 2\sigma^2 \|f - g\|_{L^2}^2 .$$

Thus,

$$\| Tf - Tg\|_{L^2}^2 \leq 2\sigma^2 \|f - g\|_{L^2}^2 .$$

Since $\sigma \in [0, \frac{1}{2})$ it follows that $\sigma^2 \in [0, \frac{1}{4})$ and

$$2\,\sigma^2 \in [0, \frac{1}{2}) < 1$$

Thus, T is contractive in $L^2[0,1]$.

Let the initial function $f_0(x)$ be linear. Let $f_0(x) = ax + b$. It follows that $f_0(x) \in L^2[0,1]$. Thus, a sequence of iterates $\{T^{on} f_n : n = 0, 1, 2, ...\}$ is obtained in $L^2[0,1]$ by repeated application of contraction mapping T on initial function $f_0(x)$. Therefore,

$$f(x) = \lim_{n \to \infty} f_n(x) \text{ where } f_n(x) = Tf_{n-1}(x).$$

Since $L^2[0,1]$ is a Banach space, by contraction mapping theorem, a unique fixed point of the operator T is an element of $L^2[0,1]$.

Thus, $f(x) \in L^2[0,1]$. □

4. The elements of the class $\mathcal{F}$ are weakly differentiable on the interval $[0,1]$. Let the initial function $f_0(x) \in B_c[0,1]$ be linear.

   Let $T : B_c[0,1] \to B_c[0,1]$ be an operator defined by equation (3.7). The repeated application of contraction mapping T on initial function $f_0(x)$ generates a sequence of iterates $\{T^{on} f_n : n = 0, 1, 2, ...\}$. Thus, the n^{th} iterate of $f_0(x)$ denoted by $f_n(x)$ is given by

$$f_n(x) = T(f_{n-1})(x) = (\sigma f_{n-1}(2x) + \alpha_0 x + \beta_0)\chi_{[0,\frac{1}{2})} + (\sigma f_{n-1}(2x-1) + \alpha_1 x + \beta_1)\chi_{[\frac{1}{2},1]}. \tag{3.18}$$

It follows that $f(x) = \lim_{n\to\infty} f_n(x)$ where $f_n(x) = Tf_{n-1}(x)$.

Proposition 3. *The function $f \in \mathcal{F}$ represented by equation (3.8) is weakly differentiable.*

Proof. We first prove that the weak derivative of functions $f_n(x)$ exist for all $n = \{0, 1, 2, ...\}$ by mathematical induction.
Let, $n = 1$.
Let the initial function $f_0(x)$ be linear. Therefore, $f_0(x)$ is continuous. Let $f_0(0) = y_0, f_0(1) = y_1$. It follows that the classical derivative of $f_0(x)$ exists. Thus, $f_0(x)$ is weakly differentiable and weak derivative coincides with the classical derivative. Denote weak derivative of $f_0(x)$ by $f_0'(x)$.
The first iterate of $f_0(x)$ denoted by $f_1(x)$ is given by

$$f_1(x) = T(f_0)(x) = (\sigma f_0(2x) + \alpha_0 x + \beta_0)\chi_{[0,\frac{1}{2})} + (\sigma f_0(2x-1) + \alpha_1 x + \beta_1)\chi_{[\frac{1}{2},1]}. \quad (3.19)$$

The transformation T is a continuous operator. Hence, it maps a continuous function onto a continuous function. Thus, the continuity of f_0 implies that the function f_1 is continuous.
Let $v(x) \in C_c^\infty[0, 1]$ be a function such that $v(0) = 0$ and $v(1) = 0$. To find the weak derivative of the function $f_1(x)$ we proceed as follows.
Consider

$$\int_0^1 f_1(x)v'(x)\,dx = \int_0^{\frac{1}{2}} (\sigma f_0(2x) + \alpha_0 x + \beta_0)v'(x)\,dx + \int_{\frac{1}{2}}^1 (\sigma f_0(2x-1) + \alpha_1 x + \beta_1)v'(x)dx.$$

Integrating by parts,

$$= [(\sigma f_0(2x) + \alpha_0 x + \beta_0)v(x)]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} (2\,\sigma f_0'(2x) + \alpha_0)v(x)dx$$
$$+ [(\sigma f_0(2x-1) + \alpha_1 x + \beta_1)v(x)]_{\frac{1}{2}}^1 - \int_{\frac{1}{2}}^1 (2\,\sigma f_0'(2x-1) + \alpha_1)v(x)dx.$$
$$= [(\sigma f_0(2x) + \alpha_0 x + \beta_0)_{\frac{1}{2}}\, v\left(\frac{1}{2}\right)] - [(\sigma f_0(2x) + \alpha_0 x + \beta_0)_0\, v(0)]$$
$$- \int_0^{\frac{1}{2}} (2\,\sigma f_0'(2x) + \alpha_0)v(x)dx + [(\sigma f_0(2x-1) + \alpha_1 x + \beta_1)_1\, v(1)]$$
$$- [(\sigma f_0(2x-1) + \alpha_1 x + \beta_1)_{\frac{1}{2}}\, v\left(\frac{1}{2}\right)] - \int_{\frac{1}{2}}^1 (2\,\sigma f_0'(2x-1) + \alpha_1)v(x)dx.$$

Since the function f_1 is continuous at $x = \frac{1}{2}$ from equation (3.19) one has

$$[(\sigma f_0(2x) + \alpha_0 x + \beta_0)]_{\frac{1}{2}} = [(\sigma f_0(2x-1) + \alpha_1 x + \beta_1)]_{\frac{1}{2}}.$$

Hence,

$$[(\sigma f_0(2x) + \alpha_0 x + \beta_0)_{\frac{1}{2}}\, v\left(\frac{1}{2}\right)] = [(\sigma f_0(2x-1) + \alpha_1 x + \beta_1)_{\frac{1}{2}}\, v\left(\frac{1}{2}\right)].$$

Additionally, $v(0) = v(1) = 0$. Thus,

$$\int_0^1 f_1(x)v'(x)\,dx = -\int_0^{\frac{1}{2}} (2\,\sigma f_0'(2x) + \alpha_0)v(x)dx - \int_{\frac{1}{2}}^1 (2\,\sigma f_0'(2x-1) + \alpha_1)v(x)dx$$
$$= -\int_0^1 g_1(x)v(x)dx.$$

Therefore, there exists a function $g_1(x)$ such that

$$\int_0^1 f_1(x)v'(x)dx = -\int_0^1 g_1(x)v(x)dx.$$

Consequently, $g_1(x)$ is a weak derivative of $f_1(x)$ and is represented by equation:

$$g_1(x) = f_1'(x) = (2\,\sigma f_0'(2x) + \alpha_0)\chi_{[0,\frac{1}{2})} + (2\,\sigma f_0'(2x-1) + \alpha_1)\chi_{[\frac{1}{2},1]}.$$

Suppose that the weak derivative of $f_{n-1}(x)$ exists and is denoted by $f'_{n-1}(x)$. To prove that the weak derivative of $f_n(x)$ exists, let $v(x) \in C_c^\infty[0,1]$ be a function such that $v(0) = 0$ and $v(1) = 0$.

To find the weak derivative of the function $f_n(x)$ we proceed as follows.

$$\int_0^1 f_n(x)v'(x)\,dx = \int_0^{\frac{1}{2}} (\sigma f_{n-1}(2x) + \alpha_0 x + \beta_0)v'(x)\,dx \\ + \int_{\frac{1}{2}}^1 (\sigma f_{n-1}(2x-1) + \alpha_1 x + \beta_1)v'(x)dx.$$

Integrating by parts,

$$\begin{aligned}
&= [(\sigma f_{n-1}(2x) + \alpha_0 x + \beta_0)v(x)]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} (2\,\sigma f'_{n-1}(2x) + \alpha_0)v(x)dx \\
&\quad + [(\sigma f_{n-1}(2x-1) + \alpha_1 x + \beta_1)v(x)]_{\frac{1}{2}}^1 - \int_{\frac{1}{2}}^1 (2\,\sigma f'_{n-1}(2x-1) + \alpha_1)v(x)dx \\
&\quad = [(\sigma f_{n-1}(2x) + \alpha_0 x + \beta_0)_{\frac{1}{2}}\, v\left(\frac{1}{2}\right)] - [(\sigma f_{n-1}(2x) + \alpha_0 x + \beta_0)_0\, v(0)] \\
&\quad - \int_0^{\frac{1}{2}} (2\,\sigma f'_{n-1}(2x) + \alpha_0)v(x)dx + [(\sigma f_{n-1}(2x-1) + \alpha_1 x + \beta_1)_1\, v(1)] \\
&\quad - [(\sigma f_{n-1}(2x-1) + \alpha_1 x + \beta_1)_{\frac{1}{2}}\, v\left(\frac{1}{2}\right)] - \int_{\frac{1}{2}}^1 (2\,\sigma f'_{n-1}(2x-1) + \alpha_1)v(x)dx.
\end{aligned}$$

Since the function f_{n-1} is continuous at $x = \frac{1}{2}$ from equation (3.18) one has

$$[(\sigma f_{n-1}(2x) + \alpha_0 x + \beta_0)]_{\frac{1}{2}} = [(\sigma f_{n-1}(2x-1) + \alpha_1 x + \beta_1)]_{\frac{1}{2}}.$$

Hence

$$[(\sigma f_{n-1}(2x) + \alpha_0 x + \beta_0)_{\frac{1}{2}}\, v\left(\frac{1}{2}\right)] = [(\sigma f_{n-1}(2x-1) + \alpha_1 x + \beta_1)_{\frac{1}{2}}\, v\left(\frac{1}{2}\right)].$$

Additionally, $v(0) = v(1) = 0$.

Thus,

$$\int_0^1 f_n(x)v'(x)\,dx = -\int_0^{\frac{1}{2}} (2\,\sigma f'_{n-1}(2x) + \alpha_0)v(x)dx - \int_{\frac{1}{2}}^1 (2\,\sigma f'_{n-1}(2x-1) + \alpha_1)v(x)dx$$

$$= -\int_0^1 g_2(x)v(x)dx.$$

Therefore, there exists a function $g_2(x)$ such that

$$\int_0^1 f_n(x)v'(x)dx = -\int_0^1 g_2(x)v(x)dx.$$

Consequently, $g_2(x)$ is a weak derivative of $f_n(x)$. It is denoted by $f_n'(x)$ and is given by

$$f_n'(x) = (2\sigma f_{n-1}'(2x) + \alpha_0)\chi_{[0,\frac{1}{2})} + (2\sigma f_{n-1}'(2x-1) + \alpha_1)\chi_{[\frac{1}{2},1]}. \tag{3.20}$$

By mathematical induction, the weak derivatives of functions $f_n(x)$ exist for all n. Moreover, the weak derivatives $f_n' \in B[0,1]$ for all n. Further, the above equation (3.20) suggests that, there exists an operator $T' : B[0,1] \to B[0,1]$ such that

$$(T'f)(x) = (2\sigma f(2x) + \alpha_0)\chi_{[0,\frac{1}{2})} + (2\sigma f(2x-1) + \alpha_1)\chi_{[\frac{1}{2},1]}. \tag{3.21}$$

Thus,

$$(T'f_{n-1}')(x) = (2\sigma f_{n-1}'(2x) + \alpha_0)\chi_{[0,\frac{1}{2})} + (2\sigma f_{n-1}'(2x-1) + \alpha_1)\chi_{[\frac{1}{2},1]}. \tag{3.22}$$

From equations (3.20) and (3.22), It follows that

$$f_n'(x) = (Tf_{n-1})'(x) = (T'f_{n-1}')(x). \tag{3.23}$$

For $f, g \in B[0,1]$,

$$\begin{aligned}\left\|T'f - T'g\right\| &= \sup_{x\in[0,\frac{1}{2})} |2\sigma f(2x) - 2\sigma g(2x)| + \sup_{x\in[\frac{1}{2},1]} |2\sigma f(2x-1) - 2\sigma g(2x-1)| \\ &\le |2\sigma| \sup_{x\in[0,1)} |f(x) - g(x)| + |2\sigma| \sup_{x\in[0,1]} |f(x) - g(x)|.\end{aligned}$$

Thus,

$$\left\|T'f - T'g\right\| \le |2\sigma| \|f - g\|$$

with contractivity factor $= \max\{2\sigma, 2\sigma\} = 2\sigma$.

Since $\sigma \in [0, \frac{1}{2})$, $2\sigma \in [0, \frac{1}{4})$, T' is contractive on $B[0,1]$. The space of bounded functions $B[0,1]$ when endowed with supremum norm is a Banach space. Then by Banach fixed point theorem, T' admits a unique fixed point. Thus,

$$T'f'(x) = f'(x).$$

Hence,

$$f'(x) = (2\sigma f'(x) + \alpha_0)\chi_{[0,\frac{1}{2})} + (2\,\sigma f'(2x-1) + \alpha_1)\chi_{[\frac{1}{2},1]}.$$

It follows that a sequence of iterates $f'_n = T'f'_{n-1}$ converges to f' as $n \to \infty$.

Consequently, the fractal function $f(x)$ is weakly differentiable. The weak derivative of f denoted by f' is represented as:

$$f'(x) = (2\sigma f'(x) + \alpha_0)\chi_{[0,\frac{1}{2})} + (2\,\sigma f'(2x-1) + \alpha_1)\chi_{[\frac{1}{2},1]}. \tag{3.24}$$

□

Corollary 2. *The fractal function $\Psi(x)$ denoted by equation (3.15) is weakly differentiable. More precisely, the weak derivative of the function $\Psi(x)$ is:*

$$\Psi'(x) = (\sigma\Psi'(2x) + 2)\chi_{[0,\frac{1}{2})} + (\sigma\Psi'(2x-1) - 2)\chi_{[\frac{1}{2},1]}. \tag{3.25}$$

Proof. The fractal function $\Psi(x)$ denoted by equation (3.15) is an element of $\mathcal{F}$. Hence, by proposition (3), $\Psi(x)$ is weakly differentiable and the weak derivative is represented by (3.25). □

5. The weak derivative of the function f denoted by f' is Riemann integrable.

Proposition 4. *See [41]. The set of points of discontinuities for the function f' denoted by equation (3.24) is a Lebesgue null set. In particular, f' is Riemann integrable.*

Proof. Let $g = f'$.

Since, the fractal function g is obtained by the application of Banach fixed point theorem on the Read-Bajraktarevic operator T' represented by equation (3.21), $g = \lim_{n\to\infty} g_n$, where $g_n = T'g_{n-1}$ and $g_0 = f_0'$. The function f_0 is assumed to be linear and represented by $f_0 = ax + b$. Hence, $g_0 = f_0'$ is a constant function $g_0 = a$ on interval $[0, 1]$. The function $g_1 = T'g_0$ has discontinuity at $x = \frac{1}{2}$, function $g_2 = T'g_1$ has discontinuities at $x = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$. Thus, the n^{th} pre-fractal function g_n has exactly $2^n - 1$ points of discontinuities in $[0, 1]$. Let A_n be a set of discontinuities of g_n, then $|A_n| = 2^n - 1$. Further, $A = \cup_{n=1}^{\infty} A_n$, so that $|A| \leq N_0$. Therefore, A is countable. Next, we prove that g is continuous at $x \in I \backslash A$. Let $x \in I \backslash A$. and choose an $\epsilon > 0$. Since q_i are Lipschitz functions with Lipschitz constants Q_i for $i = 1, 2$ and Q= $\max\{Q_i : i = 1, 2\}$, from the fixed-point equation for $g(L_i(x)) = \sigma g(x) + q_i(x)$ it follows that

$$\omega_g(L_i(I)) = \sup_{x,y\in I} |g(L_i(x)) - g(L_i(y))|$$

$$= \sup_{x,y\in I} |\sigma(g(x) - g(y)) + q_i(x) - q_i(y)|$$

$$\leq \sigma\omega_g(I) + Q|I|$$

where $|I|$ is the length of interval I. Let $S = \{1, 2\}$ and let S^∞ be a set of all infinite sequences $\sigma' = \sigma_1'\sigma_2'\sigma_3' ...$ of elements in S. Then (S, d_S) referred to as code space, is a compact metric space, with the metric d_S defined by $d_S(\sigma', \omega) = 2^k$ where k is the least index for which $\sigma_k' \neq \omega_k$. For any finite code $\sigma' \mid k = \sigma_1'\sigma_2' ... \sigma_k'$, we have

$$\omega_g(L_{\sigma'|k}(I)) = \omega_g(L_{\sigma_1'} \circ L_{\sigma_2'} \circ ... \circ L_{\sigma_k'})$$

$$\leq |\sigma|\omega_g(L_{\sigma_2'} \circ ... \circ L_{\sigma_k'}) + Q|I|(a_{\sigma_2 ... \sigma_k} + sa_{\sigma_3 ... \sigma_k} + s^{k-2}a_k + s^{k-1})$$

$$\leq |\sigma|\omega_g(L_{\sigma_2'} \circ ... \circ L_{\sigma_k'}) + Q|I|(a^{k-1} + sa^{k-2} + ... + s^{k-2}a + s^{k-1})$$

$$\leq |\sigma|\omega_g(L_{\sigma_2'} \circ ... \circ L_{\sigma_k'}) + Qa^{k-1}|I|$$

which on recursion yields

$$\omega_g(L_{\sigma'|k}(I)) + Q|I|\frac{a^k}{|a - s|}.$$

Consider the contractive IFS $\{I, L_i, i = 1, 2\}$ with attractor I. The limit $\lim_{k\to\infty} L_{\sigma'_1} \circ L_{\sigma'_2} \circ ... \circ L_{\sigma'_k}$ is a single point independent of $a \in I$ and the coding map $\pi : S^\infty \to I$

$$\pi(\sigma') = \lim_{k\to\infty} L_{\sigma'_1} \circ L_{\sigma'_2} \circ ... \circ L_{\sigma'_k}(a)$$

is continuous and surjective. Therefore there exists $\sigma' \in S^\infty$ such that

$$x = \pi(\sigma') = \cap_{k=1}^{\infty} L_{\sigma'|_k}(I).$$

For any $k \in \mathbb{N}$, there exists a compact interval I_k such that

$$x \in I_k \subset \cap i = 1^k L'_\sigma \mid_i (I).$$

Set $J = L_{\sigma'|_k(I_k)}$ where $\sigma' \mid_k^{-1} = L^{-1}_{\sigma'_k} \circ L^{-1}_{\sigma'_{k-1}} \circ ... \circ L^{-1}_{\sigma'_1}$. Thus,

$$\omega_g(I_k) = \omega_g(L_{\sigma'|k}(J)) \leq |\sigma|^k \omega_g(I) + Q|J| \frac{a^k}{|a-s|}.$$

Since g is bounded on I, $|J| \leq 1$ and $a^k \to 0$ as $k \to \infty$, we can choose k to be large enough so that each summand in the above inequality is less than $\frac{\epsilon}{2}$. Therefore, $\omega_g(I_k) < \epsilon$. Since ϵ is arbitrary, we deduce that $\omega_g(x) = 0$, and hence g is continuous at $x \in I/A$. A real-valued bounded function f is Riemann integrable if and only if set of points of discontinuities for f has Lebesgue measure zero. Therefore, the fractal function $g = f'$ is Riemann integrable. It follows that $g = f'$ is Lebesgue integrable. □

Corollary 3. *The weak derivative of the function Ψ denoted by Ψ' is Riemann integrable.*

Proof. The result follows from the proposition (4). □

6. $\mathcal{F} \subset H^1[0, 1]$.

Proposition 5. *The fractal function $f \in \mathcal{F}$ represented by equation (3.8) is an element of the Sobolev space $H^1[0, 1]$.*

Proof. We will first show that the operator T defined by equation (3.7) maps the Sobolev space $H^1[0,1]$ onto itself.

To this end, let $f \in H^1[0,1]$ and let $f(0) = y_0$ and $f(1) = y_1$.

The operator T defined by equation (3.7) maps continuous functions onto continuous functions. Hence the function $g = Tf$ is continuous and weakly differentiable. Thus,

$$g'(x) = (Tf)' = (2\sigma f'(2x) + \alpha_0)\chi_{[0,\frac{1}{2})} + (2\sigma f'(2x-1) + \alpha_1)\chi_{[\frac{1}{2},1]}.$$

Next, consider

$$\int_0^1 g'^2(x)dx = \int_0^1 (Tf)'^2(x)dx = \int_0^{\frac{1}{2}} (2\sigma f'(2x) + \alpha_0)^2 dx + \int_{\frac{1}{2}}^1 (2\sigma f'(2x-1) + \alpha_1)^2 dx$$

$$= \int_0^{\frac{1}{2}} 4\sigma^2 f'^2(2x) + 4\sigma\alpha_0 f'(2x) + \alpha_0^2 dx + \int_{\frac{1}{2}}^1 4\sigma^2 f'^2(2x-1) + 4\sigma\alpha_1 f'(2x-1) + \alpha_1^2 dx$$

$$= \int_0^1 4\sigma^2 f'^2(x)dx + 2(\alpha_0 + \alpha_1)\int_0^1 f'(x)dx + \frac{\alpha_0^2 + \alpha_1^2}{2}$$

$$= 4\sigma^2 \int_0^1 f'^2(x)dx + 2(\alpha_0 + \alpha_1)(y_1 - y_0) + \frac{\alpha_0^2 + \alpha_1^2}{2} < \infty.$$

...(since $f \in H^1[0,1]$ implies that $f' \in L^2[0,1]$ and $\int_0^1 f'^2(x)dx < \infty$.)

Thus,

$$\int_0^1 (Tf)'^2(x)dx < \infty.$$

$$(Tf)' \in L^2[0,1].$$

We conclude that, $Tf \in H^1[0,1]$.

Thus, if $f \in H^1[0,1]$ then, $Tf \in H^1[0,1]$ and the operator $T : H^1[0,1] \to H^1[0,1]$ is well defined.

Next, we prove that the operator T is contractive in $H^1[0,1]$. That is

$$\|Tf - Tg\|_{H^1}^2 \le c\|f - g\|_{H^1}^2$$

where $c < 1$. Using the definition of Sobolev norm

$$\| Tf - Tg\|_{H^1}^2 = \| Tf - Tg\|_{L^2}^2 + \left\|(Tf)' - (Tg)'\right\|_{L^2}^2 . \tag{3.26}$$

Therefore, consider

$$\begin{aligned} \| Tf - Tg\|_{L^2}^2 &= \int_0^1 |(Tf)(x) - (Tg)(x)|^2 dx \\ &\leq \int_0^{\frac{1}{2}} |\sigma f(2x) - \sigma g(2x)|^2 dx + \int_{\frac{1}{2}}^1 |\sigma f(2x-1) - \sigma g(2x-1)|^2 dx \\ &= \sigma^2 \int_0^1 |f(x) - g(x)|^2 dx \text{ ... (By changing limits of integration)} \\ &= \sigma^2 \|f - g\|_{L^2}^2 . \end{aligned}$$

Next, consider

$$\begin{aligned} \left\|(Tf)' - (Tg)'\right\|_{L^2}^2 &= \int_0^1 |(Tf)' - (Tg)'|^2 dx \\ &\leq \int_0^{\frac{1}{2}} |2\sigma f'(2x) - 2\sigma g'(2x)|^2 dx + \int_{\frac{1}{2}}^1 |2\sigma f'(2x-1) - 2\sigma g'(2x-1)|^2 dx \\ &= 4\sigma^2 \int_0^1 |f(x) - g(x)|^2 dx \text{ ... (By changing limits of integration)} \\ &= 4\sigma^2 \left\|f' - g'\right\|_{L^2}^2 . \end{aligned}$$

Thus from equation (3.26), one has

$$\| Tf - Tg\|_{H^1}^2 \leq \sigma^2 \int_0^1 |f(x) - g(x)|^2 dx + 4\sigma^2 \int_0^1 |f'(x) - g'(x)|^2 dx$$

$$\begin{aligned} \| Tf - Tg\|_{H^1}^2 &= \sigma^2 \|f - g\|_{L^2}^2 + 4\sigma^2 \left\|(f)' - (g)'\right\|_{L^2}^2 \\ &\leq 4\sigma^2 \|f - g\|_{L^2}^2 + 4\sigma^2 \left\|(f)' - (g)'\right\|_{L^2}^2 \\ &= 4\sigma^2 (\|f - g\|_{L^2}^2 + \left\|(f)' - (g)'\right\|_{L^2}^2). \end{aligned}$$

It follows that

$$\| Tf - Tg\|_{H^1}^2 \leq c\|f - g\|_{H^1}^2$$

where $c = \max\{\sigma^2, 4\sigma^2\} = 4\,\sigma^2$.

Therefore, the operator T is contractive on $H^1[0,1]$.

Assuming initial function $f_0(x)$ linear, the iterates of the function $f_0(x)$ obtained by repeated application of the operator T are elements of $H^1[0,1]$. Since T is contractive on $H^1[0,1]$, by Banach fixed point theorem, it's fixed point $f(x) \in H^1[0,1]$. Thus, we conclude that, the fractal function f is an element of Sobolev space. □

Corollary 4. *The function $\Psi(x)$ denoted by equation (3.15) is an element of the Sobolev space, $H^1[0,1]$.*

Proof. The function $\Psi(x)$ denoted by equation (3.15) is an element of $\mathcal{F}$. Hence, by proposition (5), $\Psi(x)$ is an element of the Sobolev space, $H^1[0,1]$. □

Proposition 6. *The following results hold:*

1.

$$\int_0^1 \Psi'(x)\, dx = 0 \tag{3.27}$$

2.

$$\int_0^1 \Psi'^2(x)\, dx = \frac{4}{1-4\sigma^2}. \tag{3.28}$$

3.

$$\int_0^1 \Psi(x)\, dx = \left[\frac{1}{2(1-\sigma)}\right]. \tag{3.29}$$

4.

$$\int_0^1 \Psi^2(x)\, dx = \left[\frac{2+\sigma}{6(1-\sigma)^2(1+\sigma)}\right]. \tag{3.30}$$

5.

$$\int_0^1 x\,\Psi(x)\, dx = \frac{2-\sigma}{8(1-\sigma)(1-\frac{\sigma}{2})}. \tag{3.31}$$

6.

$$\int_0^1 x^2\,\Psi(x)\, dx = \frac{\sigma(2-\sigma)}{32(1-\sigma)(1-\frac{\sigma}{2})(1-\frac{\sigma}{4})} + \frac{\sigma}{16(1-\sigma)(1-\frac{\sigma}{4})} + \frac{7}{48(1-\frac{\sigma}{4})}. \tag{3.32}$$

Proof. 1.

$$\int_0^1 \Psi'(x)\,dx = \int_0^{\frac{1}{2}} (2\,\sigma\Psi'(2x)+2)dx + \int_{\frac{1}{2}}^1 (2\,\sigma\Psi'(2x-1)-2)dx$$

$$= 2\sigma \int_0^1 \Psi'(x)dx.$$

Therefore,

$$(1-2\sigma)\int_0^1 \Psi'(x)dx = 0.$$

Since $\sigma \in [0, \frac{1}{2})$, $(1-2\sigma) > 0$ it follows that

$$\int_0^1 \Psi'(x)dx = 0.$$

2.

$$\int_0^1 \Psi'^2(x)\,dx = \int_0^{\frac{1}{2}} (2\,\sigma\Psi'(2x)+2)^2 dx + \int_{\frac{1}{2}}^1 (2\,\sigma\Psi'(2x-1)-2)^2 dx$$

$$= \int_0^{\frac{1}{2}} (4\sigma^2\Psi'^2(2x)dx + 8\sigma\Psi'(2x)+4)dx + \int_{\frac{1}{2}}^1 (4\sigma\Psi'^2(2x-1) - 8\sigma\Psi'(2x-1)+4)dx$$

$$= \int_0^1 (2\sigma^2\Psi'^2(x)dx + 4\sigma\Psi'(x)+4)dx + \int_0^1 2\sigma^2\Psi'^2(x)dx - 4\sigma\Psi'(x)+4)dx.$$

$$= 4\sigma^2 \int_0^1 \psi'^2(x)dx + 4.$$

Therefore,

$$(1-4\sigma^2)\int_0^1 \psi'^2(x)dx = 4.$$

Since $\sigma \in [0, \frac{1}{2})$ one has $(1-4\sigma^2) > 0$.

Hence,

$$\int_0^1 \psi'^2(x)dx = \frac{4}{1-4\sigma^2}.$$

3.

$$\int_0^1 \Psi(x)\,dx = \int_0^{\frac{1}{2}} (\sigma\Psi(2x)+2x)dx + \int_{\frac{1}{2}}^1 (\sigma\Psi(2x-1)-2x+2)dx$$

$$= \int_0^1 (\sigma\Psi(x)+x)\frac{dx}{2} + \int_0^1 (\sigma\Psi(x)-(x+1)+2)\frac{dx}{2}$$

$$= \int_0^1 \sigma \Psi(x) dx + \frac{1}{2}.$$

Thus,

$$(1-\sigma) \int_0^1 \Psi(x)\, dx = \frac{1}{2}.$$

$$\int_0^1 \Psi(x)\, dx = \left[\frac{1}{2(1-\sigma)}\right].$$

4.

$$\int_0^1 \Psi^2(x)\, dx = \int_0^{\frac{1}{2}} (\sigma\Psi(2x) + 2x)^2 dx + \int_{\frac{1}{2}}^1 (\sigma\Psi(2x-1) - 2x + 2)^2 dx$$
$$= \int_0^{\frac{1}{2}} (\sigma^2\Psi^2(2x) + 4\sigma x\Psi(2x) + 4x^2) dx + \int_{\frac{1}{2}}^1 (\sigma^2\Psi^2(2x-1)$$
$$- 4\sigma(1-x)\Psi(2x-1) + 4(1-x)^2) dx.$$

Therefore,

$$(1-\sigma^2) \int_0^1 \Psi^2(x) dx = \int_0^1 \sigma x \Psi(x) dx + \int_0^1 \sigma\Psi(x)(1-x) dx + \frac{1}{3}.$$

$$(1-\sigma^2) \int_0^1 \Psi^2(x) dx = \int_0^1 \sigma\Psi(x) dx + \frac{1}{3}.$$

$$(1-\sigma) \int_0^1 \Psi(x)\, dx = \frac{\sigma}{2(1-\sigma)} + \frac{1}{3}.$$

Thus,

$$\int_0^1 \Psi^2(x)\, dx = \left[\frac{2+\sigma}{6(1-\sigma)^2(1+\sigma)}\right].$$

5.

$$\int_0^1 x\,\Psi(x)\, dx = \int_0^{\frac{1}{2}} (\sigma x\,\Psi(2x) + 2x^2) dx + \int_{\frac{1}{2}}^1 (\sigma x\,\Psi(2x-1) - 2x^2 + 2x) dx$$
$$= \int_0^1 \frac{\sigma}{4} x\,\Psi(x)\, dx + \int_0^1 \frac{\sigma}{4}(x+1)\,\Psi(x)\, dx + \left[\frac{2x^3}{3}\right]_0^{\frac{1}{2}} - \left[\frac{2x^3}{3}\right]_{\frac{1}{2}}^1 + \left[x^2\right]_{\frac{1}{2}}^1$$

$$= \int_0^1 \frac{\sigma}{2} x\,\Psi(x)\, dx + \frac{\sigma}{4} \int_0^1 \Psi(x)\, dx + \frac{1}{4}.$$

Therefore,

$$(1 - \frac{\sigma}{2}) \int_0^1 x\, \Psi(x)\, dx \;=\; \frac{\sigma}{4}\left[\frac{1}{2(1-\sigma)}\right] + \frac{1}{4} \;=\; \frac{\sigma}{8(1-\sigma)} + \frac{1}{4} \;=\; \frac{2-\sigma}{8(1-\sigma)}.$$

Thus,

$$\int_0^1 x\Psi(x)\, dx = \frac{2-\sigma}{8(1-\sigma)(1-\frac{\sigma}{2})}.$$

6.

$$\begin{aligned}\int_0^1 x^2\, \Psi(x)\, dx &= \int_0^{\frac{1}{2}} (\sigma x^2\, \Psi(2x) + 2x^3) dx + \int_{\frac{1}{2}}^1 (\sigma x\, \Psi(2x-1) - 2x^3 + 2x^2) dx \\ &= \int_0^1 \frac{\sigma}{8} x^2\, \Psi(x)\, dx + \int_0^1 \frac{\sigma}{8}(x+1)^2\, \Psi(x)\, dx + \left[\frac{2x^4}{4}\right]_0^{\frac{1}{2}} - \left[\frac{2x^4}{4}\right]_{\frac{1}{2}}^1 + \left[\frac{2x^3}{3}\right]_{\frac{1}{2}}^1 \\ &= \frac{\sigma}{4}\int_0^1 x^2\, \Psi(x)\, dx + \frac{\sigma}{4}\int_0^1 x\, \Psi(x)\, dx + \frac{\sigma}{8}\int_0^1 \Psi(x)\, dx + \frac{1}{32} - \frac{15}{32} + \frac{7}{12}.\end{aligned}$$

Therefore,

$$\begin{aligned}(1 - \frac{\sigma}{4}) \int_0^1 x^2\, \Psi(x)\, dx &= \frac{\sigma}{4}\left[\frac{2-\sigma}{8(1-\sigma)(1-\frac{\sigma}{2})}\right] + \frac{\sigma}{8}\left[\frac{1}{2(1-\sigma)}\right] + \frac{7}{48} \\ &= \frac{\sigma(2-\sigma)}{32(1-\sigma)(1-\frac{\sigma}{2})(1-\frac{\sigma}{4})} + \frac{\sigma}{16(1-\sigma)(1-\frac{\sigma}{4})} + \frac{7}{48(1-\frac{\sigma}{4})} \\ &= \frac{\sigma^2 - 9\sigma + 14}{96(1-\sigma)(1-\frac{\sigma}{2})(1-\frac{\sigma}{4})}.\end{aligned}$$

Thus,

$$\int_0^1 x^2\, \Psi(x)\, dx = \frac{(\sigma-2)(\sigma-7)}{96(1-\sigma)(1-\frac{\sigma}{2})(1-\frac{\sigma}{4})}.$$

□

3.2.5 Extension of the fractal function over $\mathbb{R}$

See [43].

Let, $I = [x_0, x_N]$.

A class of fractal functions is obtained by defining $L_i : I \to I_i$ for $i = 1, 2$ as

$$L_i(x) = a_i x + b_i, \text{ for } i = 0, 1, \ldots N-1$$

and $F_i : I \times \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$ as

$$F_i(x, y) = \gamma_i + \sigma y, \text{ for } i = 0, 1, \ldots N-1.$$

where γ_i is a real polynomial with a degree at most 1 on I.

Further, $a_i, b_i \in \mathbb{R}$ and $\sigma \in [0, 1)$. The associated Read-Bajraktarevic operator is then given by

$$T(f) = \sum_{i=0}^{N-1} \gamma_i \circ L_i^{-1} + \sigma f \circ L_i^{-1}. \tag{3.33}$$

The fractal functions are obtained as the fixed points of Read-Bajraktarevic operator represented by equation (3.33). These fractal functions are compactly supported on I. The extension of the fractal function f on $\mathbb{R}$ denoted by $\tilde{f}$ can be defined by setting it to be identically zero for $x \in \mathbb{R} \backslash I$.

Let $\Gamma : \mathbb{R} \to \mathbb{R}$ be defined by

$$\Gamma = \begin{cases} \gamma_i \circ L_i^{-1}, & \text{on } L_i(I) \\ = 0, & \text{on } \mathbb{R} \backslash I. \end{cases}$$

The elements $\tilde{f}$ in $B(\mathbb{R})$ then satisfy fixed point equation of the following form.

$$\tilde{f} = \Gamma + \sum_{i=0}^{1} \tilde{f}\left(\frac{x - b_i}{a_i}\right).$$

Equations of this type are called as inhomogeneous refinement equations since $\tilde{f}$ is expressed in terms of refined version $\tilde{f}(\frac{x-b_i}{a_i})$ of itself.

3.2.6 Extension of the fractal function $\Psi(x)$ over $\mathbb{R}$

Let $B(\mathbb{R})$ be a Banach space of bounded functions $f : \mathbb{R} \to \mathbb{R}$ endowed with the norm

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|.$$

Thus,

$$B(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : \exists C \in \mathbb{R} \text{ such that } |f(x)| \le C\}.$$

Let, $B_c(\mathbb{R})$ denotes the set of bounded functions $f : \mathbb{R} \to \mathbb{R}$ such that the function f restricted to the interval $[0, 1]$, denoted by $f\mid_{[0,1]}$, is continuous and is 0 outside the interval $[0, 1]$. Thus,

$$\begin{aligned} B_c(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid \exists\, C \in \mathbb{R} \text{ such that } |f(x)| \le C, \\ f\mid_{[0,1]} \in C[0,1] \text{ and } f(x) = 0 \text{ if } x \notin [0,1]\}. \end{aligned} \quad (3.34)$$

It follows that $B_c(\mathbb{R})$ is a linear subspace of $B(\mathbb{R})$ endowed with the norm inherited from $B(\mathbb{R})$. Thus, $B_c(\mathbb{R})$ is a normed linear subspace of a Banach space $B(\mathbb{R})$. Define an operator $\tilde{T} : B_c(\mathbb{R}) \to B_c(\mathbb{R})$ as,

$$\tilde{T}(f) = \begin{cases} T(f), & \text{if } x \in [0,1]. \\ 0, & \text{if } x \notin [0,1]. \end{cases} \quad (3.35)$$

where the operator $T : B_c[0, 1] \to B_c[0, 1]$ denoted by equation (3.7) is continuous.

Proposition 7. *Let the operator $\tilde{T} : B_c(\mathbb{R}) \to B_c(\mathbb{R})$ be defined by equation (3.35). Then the fixed point of the operator $\tilde{T}$ is an element of $B_c(\mathbb{R})$.*

Proof. First, we prove that $\tilde{T}$ maps $B_c(\mathbb{R})$ onto itself.
To this end, let $f \in B_c(\mathbb{R})$.
By definition of $B_c(\mathbb{R})$, $f\mid_{[0,1]} \in C[0, 1]$ and $f = 0$ if $x \notin [0, 1]$.
Since $T : B_c[0, 1] \to B_c[0, 1]$ maps smooth functions onto smooth functions, $\tilde{T}(f)\mid_{[0,1]}$ is also a smooth function and $\tilde{T}(f) = 0$ if $x \notin [0, 1]$. Therefore, $\tilde{T}(f) \in B_c(\mathbb{R})$ and $\tilde{T}$ maps $B_c(\mathbb{R})$ onto itself.
A sequence of iterates $\{\tilde{T}^n f : n = 0, 1, 2, ...\}$ is obtained by repeated application of contraction

mapping $\tilde{T}$ on the initial function $f_0(x) \in B_c(\mathbb{R})$. Thus, the sequence $\{\tilde{T}^n f_0 : n = 0, 1, 2, ...\}$ is a sequence of functions in $B_c(\mathbb{R})$.

Moreover, since $B_c(\mathbb{R}) \subset B(\mathbb{R})$, the sequence $\{\tilde{T}^n f : n = 0, 1, 2, ...\}$ lies in Banach space $B(\mathbb{R})$. We conclude that the sequence of iterates $\{\tilde{T}^n f : n = 0, 1, 2, ...\}$ converges to the function $\tilde{f}^* \in B(\mathbb{R})$.

Thus, $\tilde{f}^* = \lim_{n\to\infty} \tilde{T}^n f_0$ and is a bounded function. Further, $\tilde{f}^* = 0$ for $x \notin [0, 1]$.

Since $T : B_c[0, 1] \to B_c[0, 1]$ maps continuous functions onto continuous functions, every element $\tilde{T}^k f$, for all k has a property that $\tilde{f} \mid_{[0,1]}$ is continuous. It follows that, $\tilde{T} f^* \mid_{[0,1]}$ is continuous. Thus, $\tilde{f}^*$ is bounded and continuous on $[0, 1]$.

Hence, $\tilde{f}^* \in B_c[0, 1]$. That is, the fixed point of the operator $\tilde{T}$ is in $B_c[0, 1]$. □

Extension of the fractal function $\Psi(x)$ over $\mathbb{R}$

Let $I = [0, 1]$, $I_1 = [0, \frac{1}{2})$, $I_2 = [\frac{1}{2}, 1]$. A continuous fractal function $\Psi \in \mathcal{F}$ is obtained by defining $L_i : I \to I_i$, for $i = 1, 2$ as,

$$\begin{aligned} L_1(x) &= \frac{x}{2} \\ L_2(x) &= \frac{x+1}{2} \end{aligned} \tag{3.36}$$

and $F_i : I \times \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$ as

$$\begin{aligned} F_1(x, y) &= \sigma y + x \\ F_2(x, y) &= \sigma y - x + 1. \end{aligned} \tag{3.37}$$

The associated Read-Bajraktarevic operator represented by the following equation

$$T(\Psi)(x) = (\sigma\Psi(2x) + 2x)\chi_{[0,\frac{1}{2})} + (\sigma\Psi(2x-1) - 2x + 2)\chi_{[\frac{1}{2},1]} \tag{3.38}$$

is continuous. The fractal function $\Psi(x)$ is fixed point of the Read-Bajraktarevic operator $T(\Psi)$. It follows that the fractal function $\Psi(x)$ is compactly supported on $[0, 1]$. It can be extended on $\mathbb{R}$ by defining it to be identically zero for $x \in \mathbb{R}\backslash[0, 1]$. Let $\Gamma : \mathbb{R} \to \mathbb{R}$ be defined by,

$$\Gamma(x) = \begin{cases} 2x, \text{ on } [0, \frac{1}{2}) \\ 2 - 2x, \text{ on } [\frac{1}{2}, 1] \\ 0, \text{ on } \mathbb{R}\backslash[0, 1]. \end{cases}$$

Let $\tilde{\Psi}$ be an extension of fractal function on $\mathbb{R}$.

The elements $\tilde{\Psi}$ in $B_c(\mathbb{R})$ then satisfy fixed point equation of the form:

$$\tilde{\Psi}(x) = \Gamma + \sum_{i=0}^{1} \tilde{\Psi}(2x - i)$$

Thus, $\tilde{\Psi}$ is expressed in terms of inhomogeneous refinement equations.

Lemma 5. *The fractal function* $\tilde{\Psi} : \mathbb{R} \to \mathbb{R}$ *is continuous over* $\mathbb{R}$.

Proof. By definition, the function $\tilde{\Psi} \mid_{[0,1]}$ is continuous. Therefore, it is sufficient to prove that the function $\tilde{\Psi}(x)$ is continuous at points $x = 0$ and $x = 1$. To this end, consider

- $\tilde{\Psi}(0) = 0$
- $\tilde{\Psi}(0^-) = 0$
- $\tilde{\Psi}(0^+) = \lim_{x\to 0^+} \Psi(x) = \lim_{\epsilon\to 0^+} \Psi(0 + \epsilon) = 0.$

Since $\tilde{\Psi}(0) = \tilde{\Psi}(0^-) = \tilde{\Psi}(0^+)$ the function $\tilde{\Psi}(x)$ is continuous at $x = 0$.

Similarly,

- $\tilde{\Psi}(1) = 0$
- $\tilde{\Psi}(1^+) = 0$
- $\tilde{\Psi}(1^-) = \lim_{x\to 1^-} \Psi(x) = \lim_{\epsilon\to 0^-} \Psi(1 - \epsilon) = 0.$

Since $\tilde{\Psi}(1) = \tilde{\Psi}(1^-) = \tilde{\Psi}(1^+)$ the function $\tilde{\Psi}(x)$ is continuous at $x = 1$.

Hence, the function $\tilde{\Psi} : \mathbb{R} \to \mathbb{R}$ is continuous over $\mathbb{R}$. □

3.3 Construction of continuous fractal function on the compact subintervals of $[0, 1]$

Let, $n \in N \cup \{0\}$.

Let the interval $[0, 1]$ be partitioned into 2^n disjoint subintervals $[x_i, x_{i+1}]$, for $i = \{0, 1, \ldots, 2^n - 1\}$.

Let $I_i = [x_i, x_{i+1}]$ and $x_i = ih$ where $h = \frac{1}{2^n} = x_{i+1} - x_i$.

The functions $\Psi_{i,n}$ are obtained by translating and dilating the fractal function $\Psi(x)$ over 2^n subintervals.

Let $\Psi_{i,n}$ be functions defined on $[\frac{i}{2^n}, \frac{i+1}{2^n}]$ such that

$$\Psi_{i,n}(x) = \Psi(2^n x - i), \text{ for } x \in [0, 1] \text{ and for } i = \{0, 1, \ldots, 2^n - 1\}. \tag{3.39}$$

Figure (3.3) represents the graph of function $\Psi_{i,n}$ for $\sigma \in (0, \frac{1}{2})$.

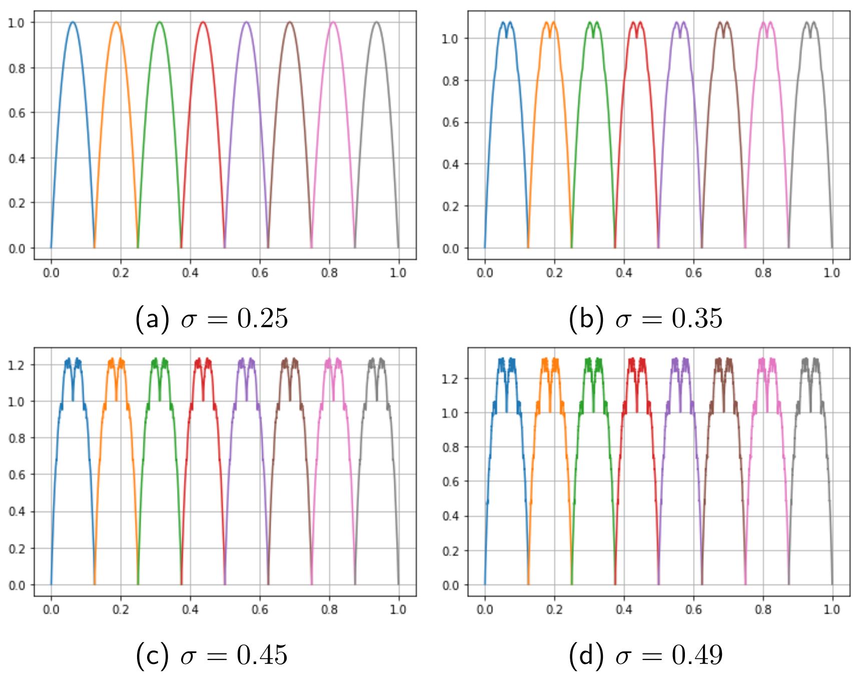


Figure 3.3: graph of the smooth function $\Psi_{i,n}$ for $\sigma \in (0, \frac{1}{2})$.

3.3.1 Properties of the function $\Psi_{i,n}$

1. From the definition of $\Psi_{i,n}$, it follows that the fractal functions $\Psi_{i,n}$ are continuous over the interval $[\frac{i}{2^n}, \frac{i+1}{2^n}]$ for $i = \{0, 1, \dots, 2^n - 1\}$. Furthermore, the compact support of the fractal function $\Psi_{i,n}$ is $[\frac{i}{2^n}, \frac{i+1}{2^n}]$.

2. The values of the function $\Psi_{i,n}$ at boundaries are:

   - $\Psi_{i,n}(\frac{i}{2^n}) = \Psi(2^n x - i)(\frac{i}{2^n}) = \Psi(0) = 0.$
   - $\Psi_{i,n}(\frac{i+1}{2^n}) = \Psi(2^n x - i)(\frac{i+1}{2^n}) = \Psi(1) = 0.$

3. The function $\Psi_{i,n}$ defined by equations (3.39) is an elements of $L^2[\frac{i}{2^n}, \frac{i+1}{2^n}]$.

   Proposition 8. $\Psi_{i,n} \in L^2[\frac{i}{2^n}, \frac{i+1}{2^n}]$, *for all* $i = 0, 1, \dots, 2^n - 1$.

   Proof. From the definition of $\Psi_{i,n}$ it follows that the function $\Psi_{i,n}$ is compactly supported on the interval $[\frac{i}{2^n}, \frac{i+1}{2^n}]$.
   Consider,

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} \Psi_{i,n}^2(x)dx = \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} \Psi^2(2^n x - i)dx.$$

   By changing limits of integration,

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} \Psi_{i,n}^2(x)dx = \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} \Psi^2(2^n x - i)dx = \int_0^1 \frac{1}{2^n}\Psi^2(x)dx.$$

   Since $\Psi(x) \in L^2[0, 1]$ one has $\int_0^1 \Psi^2(x)dx < \infty$. Therefore,

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} \Psi_{i,n}^2(x)dx < \infty.$$

   Hence, $\Psi_{i,n} \in L^2[\frac{i}{2^n}, \frac{i+1}{2^n}]$. □

4. The function $\Psi_{i,n}$ is weakly differentiable.

Corollary 5. *The function $\Psi_{i,n}$ defined by equation (3.39) is weakly differentiable and the weak derivative is as follows.*

$$\Psi'_{i,n}(x) = 2^n\, \Psi'(2^n x - i), \ \ \text{for } i = \{0, 1, \dots, 2^n - 1\}. \tag{3.40}$$

Proof. From a definition of the function $\Psi_{i,n}$, it follows that the function $\Psi_{i,n}$ is continuous and hence weakly differentiable on the interval $[\frac{i}{2^n}, \frac{i+1}{2^n}]$.

Moreover, it is compactly supported on the interval $[\frac{i}{2^n}, \frac{i+1}{2^n}]$.

Let, $v(x) \in C_c^\infty[\frac{i}{2^n}, \frac{i+1}{2^n}]$ be a function such that $v(\frac{i}{2^n}) = 0$ and $v(\frac{i+1}{2^n}) = 0$. To find weak derivative of the function $\Psi_{i,n}(x)$, we proceed as follows.

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} \Psi_{i,n}(x) v'(x)\, dx = \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} \Psi(2^n x - i)\, v'(x)\, dx$$

$$= [\Psi(2^n x - i)\, v(x)]_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} - \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} 2^n \Psi'(2^n x - i) v(x)\, dx$$

$$= 0 - \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} g(x) v(x) dx$$

Thus, there exists a function $g(x)$ such that,

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} \Psi_{i,n}(x) v'(x)\, dx = -\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} g(x) v(x) dx.$$

Therefore, $g(x)$ is the weak derivative of the fractal function $\Psi_{i,n}(x)$.

Thus,the weak derivatives of the function $\Psi_{i,n}(x)$ is

$g(x) = \Psi'_{i,n}(x) = 2^n\, \Psi'(2^n x - i), \ \text{for } i = \{0, 1, \dots, 2^n - 1\}.$ □

5. The function $\Psi_{i,n}$ is an element of the Sobolev space.

Corollary 6. *The function $\Psi_{i,n}$ defined by equation (3.39) is an element of the Sobolev space $H^1[\frac{i}{2^n}, \frac{i+1}{2^n}]$.*

Proof. We will prove that $\Psi'_{i,n} \in L^2[\frac{i}{2^n}, \frac{i+1}{2^n}]$.

Consider,

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} {\Psi'_{i,n}}^2(x)dx = \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} \{2^n\Psi'(2^n x - i)\}^2 dx$$

$$= 2^{2n}\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} \Psi'^2(2^n x - i)dx.$$

By changing limits of integration,

$$= 2^{2n}\int_0^1 \frac{1}{2^n}\Psi'^2(x)\, dx.$$

From equation (3.28),

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} {\Psi'_{i,n}}^2(x)dx = 2^n\left(\frac{4}{1-4\sigma^2}\right) < \infty.$$

Therefore, $\Psi'_{i,n}(x) \in L^2[\frac{i}{2^n}, \frac{i+1}{2^n}]$.

Thus, $\Psi_{i,n}(x) \in H^1[\frac{i}{2^n}, \frac{i+1}{2^n}]$. □

3.4 Python code

```python

import numpy as np
import matplotlib.pylab as plt

n = 20
x = np.linspace(0,1,2**n+1,endpoint=True)
print("x=",x)

z= np.zeros(2**n+1)
psi = np.zeros(2**n+1)
w= np.zeros(2**n+1)

psi[-1] = psi[0]= 0
psi[2**(n-1)]=1
z[0]=1
w[-1]=1
print("psi=",psi)
sigma = 0.35
print("Sigma=",sigma)
z=1-x
w=x
for i in range(n+1):
    psi2 = psi[::2]
    psi[:2**(n-1)+1] = sigma*psi2 + 2*x[:2**(n-1)+1]
    psi[2**(n-1):]   = sigma*psi2 - 2*x[2**(n-1):] + 2
plt.plot(x,psi)
plt.grid()
plt.show()
```

Listing 3.1: Python code to generate fractal function using iterated function system.

Chapter 4

Fractals in the finite element method

A proper selection of basis functions for the finite element space is critical for practical implementation of the finite element method. A hierarchical basis of the finite element space consists of nodal basis functions and hierarchical elements defined over the finite element mesh. Generally, polynomial functions of higher degree are used in the hierarchical basis of the finite element method. Utilization of continuous fractal functions in the hierarchical basis in the finite element method has not been investigated earlier. In this chapter, the mathematical theory which validates the utilization of space of continuous fractal functions as the finite element space and it's practical implementation have been investigated. This chapter contributes to original work.

4.1 Ritz Galerkin approximation

Given a Hilbert space V, a bi-linear form $a : V \times V \to \mathbb{R}$ and a continuous linear functional $b : V \to \mathbb{R}$, an elliptic partial differential equation can be expressed in the variational problem as follows.

Find $u \in V$ satisfying

$$a(u, v) = b(v) \quad \text{for all } v \in V \tag{4.1}$$

where V is an infinite dimensional space.

Let $V = H^1$ be an infinite dimensional space and $V^h_{2^n}$ be a closed finite dimensional subspace of V. The Galerkin approach consists in choosing a finite dimensional subspace $V^h_{2^n} \subset V$ and looking for $u_h \in V^h_{2^n}$ such that

$$a(u_h, v_h) = b(v_h), \text{ for all } v_h \in V^h_{2^n}. \tag{4.2}$$

Thus, instead of seeking a solution in the infinite dimensional space V, we look for an approximation in a finite dimensional space $V^h_{2^n}$. The finite dimensional space $V^h_{2^n}$ can be constructed in many ways. For instance, the finite dimensional space $V^h_{2^n}$ can be chosen to be a space of polynomial functions, splines, wavelets, fractal interpolation functions among others. We introduce a space of piecewise continuous fractal functions as the finite dimensional space $V^h_{2^n}$ in the Galerkin approximation. It is constructed by providing a mesh with piecewise continuous fractal functions generated by iterated function systems. To this end, continuous fractal functions are initially generated on the interval $[0, 1]$. These functions are then translated and dilated over every element of the mesh. Consequently, $V^h_{2^n}$ is constructed as the finite dimensional space of piecewise continuous fractal functions. Thus, the finite dimensional subspace $V^h_{2^n}$ is defined as follows.

$$V^h_{2^n} = \{f \in C[0, 1] \mid f\,|_{[x_i, x_{i+1}]} \in \mathcal{F}\} \tag{4.3}$$

where $\mathcal{F}$ is the class of continuous fractal functions. The elements of class $\mathcal{F}$ are fixed points of iterated function systems and are expressed as follows:
Let $f(x) \in \mathcal{F}$ then

$$f(x) = (\sigma f(2x) + \alpha_0 x + \beta_0)\chi_{[0,\frac{1}{2})} + (\sigma f(2x-1) + \alpha_1 x + \beta_1)\chi_{[\frac{1}{2},1]} \tag{4.4}$$

where $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{R}$ and $\sigma \in [0, \frac{1}{2})$.
Let P_2 be a space of the polynomial functions of degree at most 2. Then the finite dimensional subspace $V^{h^*}_{2^n}$ is defined as follows.

$$V^{h^*}_{2^n} = \{v \in C[0, 1] \mid v\,|_{[x_i, x_{i+1}]} \in P_2\}. \tag{4.5}$$

Proposition 9.

$$P_2 \subset \mathcal{F}.$$

Consequently,

$$V_{2^n}^{h^*} \subset V_{2^n}^{h}.$$

Proof. Let, $v \in P_2$. It is apparent that v is a polynomial function of degree at most 2.

Let $v = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$.

For $f = v$, one has $f(x) = ax^2 + bx + c$.

It follows that

$f(2x) = a(2x)^2 + b(2x) + c = 4ax^2 + 2bx + c$ and

$f(2x-1) = a(2x-1)^2 + b(2x-1) + c = 4ax^2 + (-4a+2b)x + (a-b+c)$.

By choosing $\sigma = \frac{1}{4}$, $\alpha_0 = \frac{b}{2}$, $\beta_0 = \frac{3c}{4}$, $\alpha_1 = a + \frac{b}{2}$, $\beta_1 = \frac{3c-a+b}{4}$ the equation (4.4) leads to,

$$\begin{aligned} f(x) = &\left(\frac{1}{4}(4ax^2 + 2bx + c) + \frac{b}{2}x + \frac{3c}{4}\right)\chi_{[0,\frac{1}{2})} \\ &+ \frac{1}{4}\left(\left[(4ax^2 + (-4a+2b)x + \frac{a-b+c}{4}\right] + \left(a + \frac{b}{2}\right)x + \frac{3c-a+b}{4}\right)\chi_{[\frac{1}{2},1]}. \end{aligned}$$

Solving right hand side,

$$\begin{aligned} f(x) = &\left(ax^2 + \frac{bx}{2} + \frac{c}{4} + \frac{b}{2}x + \frac{3c}{4}\right)\chi_{[0,\frac{1}{2})} \\ &+ \left(\left[ax^2 + (-a + \frac{b}{2})x + (\frac{a-b+c}{4})\right] + \left(a + \frac{b}{2}\right)x + \frac{3c-a+b}{4}\right)\chi_{[\frac{1}{2},1]}. \end{aligned} \quad (4.6)$$

Therefore,

$$f(x) = (ax^2 + bx + c)\chi_{[0,\frac{1}{2})} + (ax^2 + bx + c)\chi_{[\frac{1}{2},1]}.$$

Hence,

$$f(x) = ax^2 + bx + c, \text{ for } x \in [0,1].$$

Thus, v can be expressed as a fixed point of an iterated function system by choosing appropriate values of $\sigma, \alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{R}$ and v is continuous.

Hence, $v \in \mathcal{F}$.

We conclude that $P_2 \subset \mathcal{F}$.

Let $v_h^* \in V_{2^n}^{h^*}$ then $v_h^* \mid_{[x_i,x_{i+1}]} \in P_2 \subset \mathcal{F}$.

Therefore, $v_h^* \mid_{[x_i,x_{i+1}]} \in \mathcal{F}$.

Consequently, $v_h^* \in V_{2^n}^h$.

and; $V_{2^n}^{h^*} \subset V_{2^n}^h$. □

From proposition (9), it is apparent that the space of piecewise continuous fractal functions, $V_{2^n}^h$ is larger than the space of piecewise polynomials of second degree $V_{2^n}^{h^*}$. Hence, approximation is expected to be more accurate in the space $V_{2^n}^h$.

Let $I = [0, 1]$ and $v_1, v_2 \in V_{2^n}^h$. The inner product of v_1 and v_2 denoted by (v_1, v_2) is defined as follows.

$$(v_1, v_2) = \int_I v_1 v_2 dx$$

and

$$a(v_1, v_2) = \int_I v_1' \, v_2' dx.$$

It follows from the definition of $a(.,.)$ that it is continuous, bi-linear and coercive.

4.1.1 Error estimates in the Galerkin approximation

The error e is defined as a difference between exact solution u and approximate solution u_h . Thus, $e = u - u_h$.

The Galerkin orthogonality

Theorem 24 (The Galerkin orthogonality)**.** *Let u and u_h be solutions of equations (4.1) and (4.2) respectively. Then*

$$a(u - u_h, v_h) = 0, \quad \textit{for all } v_h \in V_h.$$

The Galerkin orthogonality states that the error $u - u_h$ is orthogonal to V_h.

Best approximation property

Let $V^h_{2^n}$ be Galerkin approximation space and let $u_h \in V^h_{2^n}$ be a solution of (4.2). Then the following inequality holds.

Corollary 7. *Let u_q be the finite element solution obtained by Lagrange P_2 elements and u_h be the finite element solution of equation (4.2) obtained by fractal functions. Then,*

$$\|u - u_h\|_E \leq \left\|u - u_q\right\|_E. \tag{4.7}$$

Proof. The error in Galerkin approximation is optimal in an energy norm. Therefore, for any $v_h \in V^h_{2^n}$,

$$\|u - u_h\|_E = \min_{v \in V^h_{2^n}} \{\|u - v_h\|\}_E. \tag{4.8}$$

Since $u_q \in V^{h^*}_{2^n} \in V^h_{2^n}$, choosing $v_h = u_q$ in equation (4.8).
Consequently,

$$\|u - u_h\|_E \leq \left\|u - u_q\right\|_E.$$

Thus, u_h is the best approximation to u in the energy norm. □

The Galerkin approximation $u_h \in V^h_{2^n}$ to $u \in V$ is quasi-optimal. Thus, u_h is the best approximation to u using the subspace $V^h_{2^n}$.

Corollary 8. *Let $a(.,.)$ be V-elliptic with $a(v, v) \geq \alpha\|v\|^2$ and bounded with $a(u, v) \leq C\|u\|\|v\|$. Let u be a solution of variational problem denoted by equation (4.1). Let u_h be a fractal solution and u_q be a solution obtained by Lagrange P_2 elements to Galerkin approximation problem denoted by equation (4.2) then*

$$\|u - u_h\|_V \leq \frac{C}{\alpha} \min_{u_q \in V^h_{2^n}} \left\|u - u_q\right\|_V$$

where C is a continuity constant and α is the coercivity constant of $a(.,.)$ on V.

Proof. Ceas Lemma states that, if $a(.,.)$ is V-elliptic with $a(v, v) \geq \alpha\|v\|^2$ and bounded with $a(u, v) \leq C\|u\|\|v\|$ and u_h be a solution to Galerkin approximation problem $a(u_h, v_h) =$

$b(v_h)$ for all $v_h \in V_h$ then

$$\|u - u_h\|_V \leq \frac{C}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V .$$

Choosing the infinite dimensional space $V = H^1$, $V_h = V^h_{2^n}$ and $v_h = u_q$ in Ceas Lemma one has

$$\|u - u_h\|_V \leq \frac{C}{\alpha} \min_{u_q \in V^h_{2^n}} \left\|u - u_q\right\|_V . \tag{4.9}$$

□

4.2 Discretization of the domain

To construct an approximation, the interval $I = [0, 1]$ is divided into 2^n non-overlapping domains called finite elements. We assume that the vertices are equidistant.

Let,

$$0 = x_0 < x_1 < ... < x_{2^n} = 1.$$

Set, $h = x_{i+1} - x_i$, for $i = 0, 1, ... 2^n - 1$.

The mesh of $I = [0, 1]$ is an indexed collection of 2^n disjoint subintervals with non-zero measure $\{I_i = [x_i, x_{i+1}] = [\frac{i}{2^n}, \frac{i+1}{2^n}],$ for $i = 0, 1. ... 2^n - 1\}$ forming a partition of I. That is,

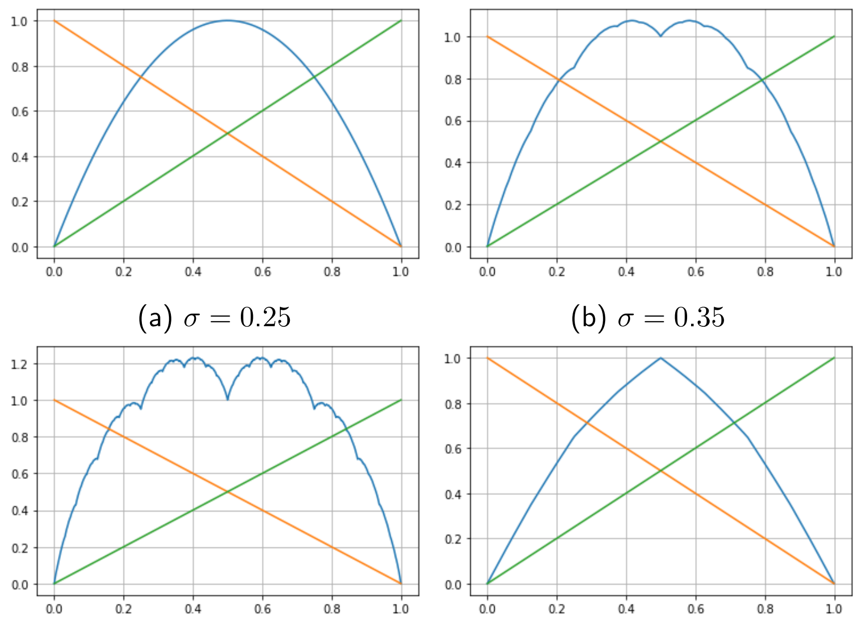
$$I = \cup_{i=0}^{2^n-1} I_i \quad \text{and} \quad I_i^\circ \cap I_j^\circ = \emptyset \text{ for } i \neq j.$$

The subintervals I_i are elements of the mesh and denote the mesh by $\tau_h = \{I_i\}_{i=0}^{2^n-1}$ where $h = \frac{1}{2^n}$ refers to the refinement of the domain I. Thus, the vertices and elements uniquely define the finite element mesh.

4.3 Fractal functions as interpolation functions in the finite element method

The interpolation functions interpolate the solution between the discrete values obtained at the mesh nodes. We propose the use of continuous fractal functions for interpolation in the finite element method. To this end, we introduce a fractal bubble as a hierarchical element in the hierarchical basis. We emphasize that nodal basis functions $\{\lambda(x) = x,\ \Lambda(x) = 1 - x\}$ can be expressed as fixed points of iterated function systems and are elements of class $\mathcal{F}$. Consequently, they are continuous fractal functions. Thus, we establish hierarchical basis on interval $[0, 1]$ consisting of three functions namely, two linear nodal basis functions $\lambda(x), \Lambda(x)$ and a continuous fractal function $\Psi(x)$ denoted by equation (3.15) as hierarchical element. The continuous fractal function $\Psi(x)$ provides additional degree of freedom without increasing the numbers of nodes in the mesh.

Thus, the hierarchical basis on the interval $[0, 1]$ is a set of continuous fractal functions $\{\Lambda(x), \lambda(x), \Psi(x)\}$. Figure (4.1) indicates the graph of the fractal basis functions defined over an interval $[0, 1]$ for different values of $\sigma \in [0, \frac{1}{2})$.



(a) $\sigma = 0.25$ (b) $\sigma = 0.35$

(c) $\sigma = 0.45$ (d) $\sigma = 0.15$

Figure 4.1: Fractal basis functions on the interval $[0, 1]$ for different values of $\sigma \in [0, \frac{1}{2})$

4.3.1 Hierarchical basis functions

Let the interval $[0, 1]$ be divided into 2^n disjoint subintervals $[x_i, x_{i+1}]$, for $i = 0, 1, \ldots, 2^n - 1$. The functions $\Lambda_{i,n}, \Psi_{i,n}, \lambda_{i,n}$ are obtained by translating and dilating the functions Λ, Ψ, λ over 2^n subintervals. Thus,

$$\Lambda_{i,n}(x) = \Lambda(2^n x - i), \text{ for } i = 0, 1, \ldots 2^n - 1.$$

$$\Psi_{i,n}(x) = \Psi(2^n x - i), \text{ for } i = 0, 1, \ldots 2^n - 1.$$

$$\lambda_{i,n}(x) = \lambda(2^n x - i), \text{ for } i = 0, 1, \ldots 2^n - 1.$$

Thus, the hierarchical basis functions over the finish element mesh are $\Lambda_{i,n}, \Psi_{i,n}, \lambda_{i,n}$ for $i = 0, 1, \ldots, 2^n - 1$.

4.3.2 Derivation of element equations

To derive element equations, a typical element is isolated from the mesh and its finite element model is developed. Let $I_i = [x_i, x_{i+1}] = [\frac{i}{2^n}, \frac{i+1}{2^n}]$ be i^{th} element of the mesh.
Then, $h = x_{i+1} - x_i = \frac{1}{2^n}$. The fractal basis functions corresponding to the element I_i of the domain $I = [0, 1]$ are expressed as:

$$\varphi_1^i(x) = \Lambda_{i,n}(x) = \Lambda(2^n x - i) = ((i + 1) - 2^n x). \tag{4.10}$$

$$\varphi_2^i = \Psi_{i,n}(x) = \Psi(2^n x - i) = \begin{cases} \sigma \, \Psi(2^{n+1} x - 2i) + (2^{n+1} x - 2i), & \text{if } x \in [\frac{i}{2^n}, \frac{i+\frac{1}{2}}{2^n}) \\ \sigma \, \Psi(2^{n+1} x - 2i - 1) + 2 - (2^{n+1} x - 2i), & \text{if } x \in [\frac{i+\frac{1}{2}}{2^n}, \frac{i+1}{2^n}]. \end{cases} \tag{4.11}$$

$$\varphi_3^i = \lambda_{i,n}(x) = \lambda(2^n x - i) = (2^n x - i). \tag{4.12}$$

From equations (4.10),(4.11), (4.12) it follows that for $i = \{0, 1, \ldots, 2^n - 1\}$

1. The nodal basis functions φ_1^i, φ_3^i satisfy the Kronecker delta property at nodes x_i and

x_{i+1}. Thus,

- $\varphi_1^i(x_i) = 1$ and $\varphi_1^i(x_{i+1}) = 0$.
- $\varphi_3^i(x_i) = 0$ and $\varphi_3^i(x_{i+1}) = 1$.

However, the hierarchical element φ_2^i is constructed such that

$$\varphi_2^i(x_i) = \varphi_2^i(x_{i+1}) = 0 \text{ and } \varphi_2^i(x_{i+\frac{1}{2}}) = 1.$$

2. $\varphi_1^i(x) + \varphi_3^i(x) = 1$, for all $x \in [x_i, x_{i+1}]$.

   The nodal basis functions φ_1^i, φ_3^i satisfy the partition of unity condition at all points $x \in [x_i, x_{i+1}]$. Moreover, the fractal basis functions $\varphi_1^i, \varphi_2^i, \varphi_3^i$ satisfy the partition of unity condition only at nodes $x = x_i$ and $x = x_{i+1}$. That is

$$\varphi_1^i(x) + \varphi_2^i(x) + \varphi_3^i(x) = 1 \text{ for } x = x_i \text{ and } x = x_{i+1}.$$

3. The compact support of the basis functions $\varphi_1^i, \varphi_2^i, \varphi_3^i$ is the interval $[\frac{i}{2^n}, \frac{i+1}{2^n}]$.

Let u_h^i be the finite element solution over the element I_i. Then one has

$$u_h^i = u_i\varphi_1^i + \Delta u_{i+\frac{1}{2}}\varphi_2^i + u_{i+1}\varphi_3^i \tag{4.13}$$

where $u_i = u_h^i(x_i)$, $u_{i+1} = u_h^i(x_{i+1})$ and $\Delta u_{i+\frac{1}{2}}$ is the magnitude of departure from linearity of the approximation $u_h^i(x)$ at the element center, since $\phi_{i+\frac{1}{2}}(x_{i+\frac{1}{2}}) = 1$. Thus, $\Delta u_{i+\frac{1}{2}}$ represents simply additive refinements. The following lemma provides a formula to find value of $u_h(x_{i+\frac{1}{2}})$.

Lemma 6. *Let u_h^i denoted by equation (4.13) be the finite element solution over subinterval $I_i = [x_i, x_{i+1}]$ in terms of fractal basis functions. The value of $u_h(x)$ at $x = x_{i+\frac{1}{2}}$ is given by*

$$u_h(x_{i+\frac{1}{2}}) = \frac{u_i + u_{i+1}}{2} + \Delta u_{i+\frac{1}{2}}. \tag{4.14}$$

Proof. Consider the interval $[x_i, x_{i+1}]$ with element size $h = x_{i+1} - x_i$. Let $x_{i+\frac{1}{2}}$ be the midpoint of $[x_i, x_{i+1}]$.

Then

$$x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}$$

and

$$x_{i+\frac{1}{2}} = x_i + \frac{h}{2} = x_{i+1} - \frac{h}{2}.$$

The finite element solution over subinterval $[x_i, x_i + 1]$ is expressed as follows.

$$u_h^i = u_i\varphi_1^i + \Delta u_{i+\frac{1}{2}}\varphi_2^i + u_{i+1}\varphi_3^i. \tag{4.15}$$

To find value of u_h at $x = x_{i+\frac{1}{2}}$, substitute $x = x_{i+\frac{1}{2}}$ in the equation (4.13). Thus,

$$\begin{aligned} u_h^i(x_{i+\frac{1}{2}}) &= u_i\varphi_1^i(x_{i+\frac{1}{2}}) + u_{i+1}\varphi_2^i(x_{i+\frac{1}{2}}) + \Delta u_{i+\frac{1}{2}}\varphi_3^i(x_{i+\frac{1}{2}}) \\ = u_i\left(\frac{x_{i+1} - x_{i+\frac{1}{2}}}{h}\right) &+ u_{i+1}\left(\frac{x_{i+\frac{1}{2}} - x_i}{h}\right) + \Delta u_{i+\frac{1}{2}}\Psi(2^n(x_{i+\frac{1}{2}}) - i) \\ &= u_i\left(\frac{\frac{h}{2}}{h}\right) + u_{i+1}\left(\frac{\frac{h}{2}}{h}\right) + \Delta u_{i+\frac{1}{2}}\Psi\left(2^n\left(\frac{i+\frac{1}{2}}{2^n}\right) - i\right) \\ &= \frac{u_i}{2} + \frac{u_{i+1}}{2} + \Delta u_{i+\frac{1}{2}}\Psi\left(\frac{1}{2}\right). \end{aligned}$$

Since $\Psi\left(\frac{1}{2}\right) = 1$, one has

$$u_h^i(x_{i+\frac{1}{2}}) = \frac{u_i + u_{i+1}}{2} + \Delta u_{i+\frac{1}{2}}.$$

□

Global interpolation functions

To develop the finite element model over the entire domain, elements of the domain are connected in a sequence. The node x_i is shared by the two intervals $I_{i-1} = [x_{i-1}, x_i]$ and $I_i = [x_i, x_{i+1}]$, for $i = 0, 1, \ldots, 2^n-1$. Consequently, the interpolation function corresponding to node x_i defined over the element I_{i-1} is $\lambda_{i-1,n} = \frac{x - x_{i-1}}{x_i - x_{i-1}}$ and over the element I_i is $\Lambda_{i,n} = \frac{x_{i+1} - x}{x_{i+1} - x_i}$. The nodal basis functions ensure the inter-element continuity of the interpolation functions at connecting nodes.

The hierarchical basis over the entire domain consists of nodal basis functions and fractal functions defined on every element of the domain as follows.

- Nodal basis functions:

$$\phi_{0,n}(x) = \Lambda_{0,2^n} = 1 - 2^n x \quad \text{for } x \in I_0. \tag{4.16}$$

For $i = 1, 2, \dots 2^n - 2$

$$\phi_{i,n}(x) = \begin{cases} \lambda_{i-1,n} = 2^n x - (i-1), & \text{for } x \in I_{i-1} \\ \Lambda_{i,n} = (i+1) - 2^n x, & \text{for } x \in I_i. \end{cases} \tag{4.17}$$

$$\phi_{2^n,n}(x) = \lambda_{2^n-1,n} = 2^n x - (2^n - 1) \quad \text{for } x \in I_{2^n-1}. \tag{4.18}$$

- Hierarchical element:

$$\phi_{i+\frac{1}{2},n}(x) = \Psi_{i,n}(x) = \Psi(2^n x - i), \quad \text{for } x \in I_i \text{ and for } i = 0, 1, \dots 2^n - 1. \tag{4.19}$$

4.3.3 Construction of the finite element space

We seek an approximate solution as a linear combination of fractal functions. Hence, we propose to use the space of piecewise continuous fractal functions generated by

$$\{\phi_{0,n}, \phi_{\frac{1}{2},n}, \phi_{1,n}, \phi_{\frac{3}{2},n}, \dots, \phi_{2^n,n}\}$$

defined on the interval $[0,1]$ as an approximation space. From the definition of the functions $\phi_{i,n}$ for $i = 1, 2, \dots 2^n - 1$, it follows that they satisfy inter-element continuity property at connecting nodes and hence are continuous on the interval $[0,1]$. Therefore this subspace of piecewise continuous fractal functions is an admissible set of functions for H_0^1 elliptic problems. This space of fractal functions, denoted by $V_{2^n}^h$, is defined as follows.

Definition 46. *[The space $V_{2^n}^h$]*

$$V_{2^n}^h = Span\{\phi_{0,n}, \phi_{\frac{1}{2},n}, \phi_{1,n}, \phi_{\frac{3}{2},n}, \dots, \phi_{2^n,n}\}.$$

Proposition 10. *The set* $\{\phi_{0,n}, \phi_{\frac{1}{2},n}, \phi_{1,n}, \phi_{\frac{3}{2},n}, \dots, \phi_{2^n,n}\}$ *is a basis for* $V^h_{2^n}$*. Consequently, the space* $V^h_{2^n}$ *is finite dimensional.*

Proof. Let $\{c_0, c_{\frac{1}{2}}, c_1, \dots, c_{2^n}\} \in \mathbb{R}^{2^{n+1}+1}$.
Assuming that the continuous function

$$u = \sum_{i=0}^{2^n} c_i \phi_{i,n} + \sum_{i=0}^{2^n-1} c_{i+\frac{1}{2}} \phi_{i+\frac{1}{2},n}$$

vanishes identically on $[0, 1]$,

$$u(x_j) = \sum_{i=0}^{2^n} c_i \phi_{i,n}(x_j) + c_{i+\frac{1}{2}} \phi_{i+\frac{1}{2},n}(x_j) \text{ for } 0 \le j \le 2^n.$$

Using the Kronecker delta property for $0 \le i, j \le 2^n$, since

$$\phi_{i,n}(x_j) = 0 \text{ if } i \neq j \text{ and } \phi_{i+\frac{1}{2},n}(x_j) = 0 \text{ for all } i, j, c_i = 0 \text{ for } 0 \le i \le 2^n.$$

Further for $x_j = x_i + \frac{1}{2}$, $c_{i+\frac{1}{2}} = 0$ for all i.
Therefore, we conclude that the set $\{\phi_{0,n}, \phi_{\frac{1}{2},n}, \phi_{1,n}, \phi_{\frac{3}{2},n}, \dots, \phi_{2^n,n}\}$ is linearly independent. Moreover, for all $u_h \in V_{1,2^n}$, it follows that

$$u_h = \sum_{i=0}^{2^n} u_h(x_i)\phi_{i,n} + u_h(x_{i+\frac{1}{2},n})\phi_{i+\frac{1}{2}},$$

since on each element I_i, u_h and $\sum_{i=0}^{2^n} u_h(x_i)\phi_{i,n} + \sum_{i=0}^{2^n-1} u_h(x_{i+\frac{1}{2}})\phi_{i+\frac{1}{2},n}$ are affine and coincide at two points, namely, x_i and x_{i+1}. Thus,

$$\dim V^h_{2^n} = 2^{n+1} + 1.$$

Consequently, the space $V^h_{2^n}$ is finite dimensional. □

Proposition 11. $V^h_{2^n} \subset H^1[0, 1]$.

Proof. Let $u_h \in V^h_{2^n}$. It follows that $u_h \in L^2[0,1]$.

$$u_h = \sum_{i=0}^{2^n} u_h(x_i)\phi_{i,n} + \sum_{i=0}^{2^n-1} u_h(x_{i+\frac{1}{2}})\phi_{i+\frac{1}{2},n}.$$

Thus u_h can be expressed in terms of the local basis functions as:

$$u_h = \sum_{i=0}^{2^n-1} u_i\Lambda_{i,n} + u_{i+1}\lambda_{i,n} + u_{i+\frac{1}{2}}\Psi_{i,n}.$$

The weak derivative of u_h is:

$$u_h' = \sum_{i=0}^{2^n-1}\left(\frac{-u_i}{h} + \frac{u_{i+1}}{h} + \frac{1}{h}\Psi'_{i,n}\right).$$

To prove $u_h \in H^1[0,1]$ we will prove that $u_h' \in L^2[0,1]$.

To this end, consider

$$\int_0^1 u_h'^2(x)dx = \sum_{i=0}^{2^n-1}\int_{x_i}^{x_{i+1}}\left(\frac{-u_i}{h} + \frac{u_{i+1}}{h} + \frac{1}{h}\Psi'_{i,n}(x)\right)^2 dx$$

$$= \sum_{i=0}^{2^n-1}\int_{x_i}^{x_{i+1}}(\frac{-u_i}{h} + \frac{u_{i+1}}{h})^2 + \frac{1}{h^2}\Psi'^2_{i,n}(x) + \frac{2}{h}(\frac{-u_i}{h} + \frac{u_{i+1}}{h})\Psi'_{i,n}(x)dx$$

$$= \sum_{i=0}^{2^n-1} h(\frac{-u_i}{h} + \frac{u_{i+1}}{h})^2 + \frac{1}{h^2}\int_0^1 \Psi'^2(t)hdt + \frac{2}{h}\int_0^1(\frac{-u_i}{h} + \frac{u_{i+1}}{h})\Psi'(t)hdt.$$

Since $\int_0^1 \Psi'(x)dx = 0$ and $\int_0^1 \Psi'^2(x)dx = \frac{4}{(1-4\sigma^2)}$ one has

$$= \sum_{i=0}^{2^n-1}\frac{1}{h}(u_{i+1}-u_i)^2 + \frac{1}{h}\left(\frac{4}{(1-4\sigma^2)}\right) < \infty$$

Therefore $u_h' \in L^2[0,1]$ and $u_h \in H^1[0,1]$ Thus, $V^h_{2^n} \subset H^1[0,1]$. □

In order to increase an accuracy of the finite element solution, we define an extension process which requires the construction of a sequence of subspaces $V^h_{2^n}$. The hierarchic extensions are such that $V^h_{2^n} \subset V^h_{2^{n+1}}$. This sequence of sub-spaces is constructed by reducing mesh size h or increasing the number of elements in the mesh. The hierarchical extensions assure that the error decreases monotonically.

Proposition 12. *The spaces $V^h_{2^n}$ are nested. That is,*

$$V^h_1 \subset V^h_2 \ldots \subset V^h_{2^n} \subset V^h_{2^{n+1}} \subset \ldots$$

Proof. From equations (4.16) - (4.19), it is apparent that the interpolation functions over the domain are defined in terms of $\Lambda_{i,n}(x), \Psi_{i,n}(x), \lambda_{i,n}(x)$, for $i = \{0, 1, \ldots, 2^n\}$. However, the functions $\Lambda_{i,n}(x), \Psi_{i,n}(x), \lambda_{i,n}(x)$, for $i = \{0, 1, \ldots, 2^n\}$ can be expressed as a linear combination of $\Lambda_{i,n+1}(x), \Psi_{i,n+1}(x), \lambda_{i,n+1}(x) \in V^h_{2^{n+1}}$ as follows.
For $i = \{0, 1, \ldots, (2^n - 1)\}$,

$$\Lambda_{i,n} = \frac{1}{2}\left[\lambda_{2i,n+1} + 2\Lambda_{2i,n+1}\right] + \frac{1}{2}\Lambda_{2i+1,n+1}$$

$$\Psi_{i,n} = \begin{cases} \sigma\Psi_{i,n+1} + \lambda_{i,n+1} \\ \sigma\Psi_{i+1,n+1} + \Lambda_{i+1,n+1} \end{cases} \tag{4.20}$$

$$\lambda_{i,n} = \frac{1}{2}\lambda_{2i,n+1} + \frac{1}{2}\left[2\lambda_{2i+1,n+1} + \Lambda_{2i+1,n+1}\right],$$

Therefore, one has the following identities.

- Nodal basis functions:

$$\begin{aligned}\phi_{0,n}(x) = \Lambda_{0,n} &= \frac{1}{2}\left[\lambda_{0,n+1} + 2\Lambda_{0,n+1}\right] + \frac{1}{2}\Lambda_{1,n+1} \\ &= \frac{1}{2}\left[\phi_{1,n+1} + 2\phi_{0,n+1}\right]\chi_{\left[\frac{0}{2^{n+1}},\frac{1}{2^{n+1}}\right]} + \frac{1}{2}\phi_{1,n+1}\chi_{\left[\frac{1}{2^{n+1}},\frac{2}{2^{n+1}}\right]}.\end{aligned} \tag{4.21}$$

  For $i = 1, 2, \ldots 2^{n+1} - 1$

$$\phi_{i,n}(x) = \begin{cases} \lambda_{i-1,n} = \frac{1}{2}\lambda_{2(i-1),n+1} + \frac{1}{2}\left[2\lambda_{2i-1,n+1} + \Lambda_{2i-1,n+1}\right] \\ \Lambda_{i,n} = \frac{1}{2}\left[\lambda_{2i,n+1} + 2\Lambda_{2i,n+1}\right] + \frac{1}{2}\Lambda_{2i+1,n+1}. \end{cases}$$

$$\phi_{i,n}(x) = \begin{cases} \lambda_{i-1,n} = \frac{1}{2}\phi_{2i-1,n+1}\chi_{[\frac{i}{2^{n+1}},\frac{i+1}{2^{n+1}}]} + \frac{1}{2}\left[2\phi_{2i,n+1} + \phi_{2i-1,n+1}\right]\chi_{[\frac{i+1}{2^{n+1}},\frac{i+2}{2^{n+1}}]} \\ \Lambda_{i,n} = \frac{1}{2}\left[\phi_{2i+1,n+1} + 2\phi_{2i,n+1}\right]\chi_{[\frac{i}{2^{n+1}},\frac{i+1}{2^{n+1}}]} + \frac{1}{2}\phi_{2i+1,n+1}\chi_{[\frac{i+1}{2^{n+1}},\frac{i+2}{2^{n+1}}]}. \end{cases} \tag{4.22}$$

$$\begin{aligned} \phi_{2^n,n}(x) = \lambda_{2^n-1,n} &= \frac{1}{2}\lambda_{2^{n+1}-2,n+1} + \frac{1}{2}\left[2\lambda_{2^{n+1}-1,n+1} + \Lambda_{2^{n+1}-1,n+1}\right] \\ &= \frac{1}{2}\phi_{2^{n+1}-1,n+1}\chi_{[\frac{2^{n+1}-2}{2^{n+1}},\frac{2^{n+1}-1}{2^{n+1}}]} + \frac{1}{2}\left[2\phi_{2^{n+1},n+1} + \phi_{2^{n+1}-1,n+1}\right]\chi_{[\frac{2^{n+1}-1}{2^{n+1}},\frac{2^{n+1}}{2^{n+1}}]}. \end{aligned} \tag{4.23}$$

- Hierarchical element:

$$\begin{aligned} \phi_{i+\frac{1}{2},n}(x) = \Psi_{i,n}(x) = (\sigma\phi_{i+\frac{1}{2},n+1} + \phi_{i+1,n+1})\chi_{[\frac{i}{2^{n+1}},\frac{i+\frac{1}{2}}{2^{n+1}}]} \\ + (\sigma\phi_{i+\frac{3}{2},n+1} + \phi_{i+1,n+1})\chi_{[\frac{i+\frac{1}{2}}{2^{n+1}},\frac{i+1}{2^{n+1}}]}. \end{aligned} \tag{4.24}$$

The above identities imply that the functions $\phi_{i,n}$ and $\phi_{i+\frac{1}{2},n}$ for $i = \{0, 1, \dots, 2^n\}$ can be expressed as a linear combination of $\phi_{i,n+1}$ and $\phi_{i+\frac{1}{2},n+1}$ respectively for $i = \{0, 1, \dots, 2^{n+1}\}$.

Consequently,

$$V^h_{2^n} \subset V^h_{2^{n+1}}.$$

Thus, for all $n \in \mathbb{N} \cup \{0\}$ it is apparent that

$V^h_1 \subset V^h_2 \dots \subset V^h_{2^n} \subset V^h_{2^{n+1}} \subset \dots$. □

Hierarchical decomposition of the finite element space

The finite element space $V^h_{2^n}$ has a direct sum hierarchical decomposition

$$V^h_{2^n} = U \bigoplus W$$

where U is a space of piecewise linear nodal basis functions and W is a space of piecewise fractal functions which are zero at nodes. Thus for $v_h \in V^h_{2^n}$ there is a unique decomposition,

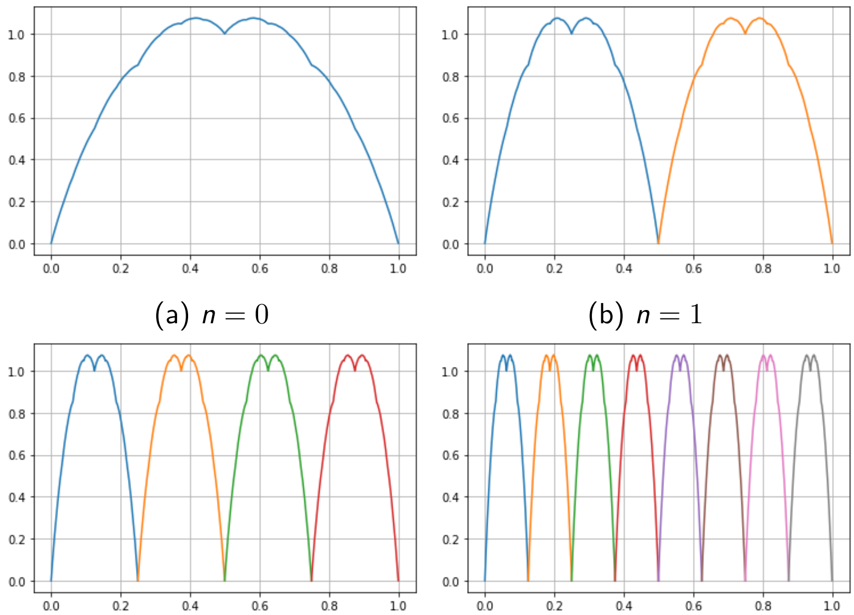
$v_h = u \bigoplus w$ where $u \in U$ and $w \in V$. A basis for U is a usual nodal basis of piecewise linear polynomials. A basis of W consists of all fractal functions which are associated with the midpoints of the elements in the mesh. This is the hierarchical basis for the space of fractal functions. Thus, the space

$$U = \{\phi_{i,n} \mid \text{ for } i = 0, 1, \ldots 2^n\}.$$

and

$$W = \{\phi_{i+\frac{1}{2},n} \mid \text{ for } i = 0, 1, \ldots 2^n - 1\}$$

where the functions $\phi_{i,n}$ and $\phi_{i+\frac{1}{2},n}$ are represented by equations (4.16) - (4.19). Figure (4.2) represents the hierarchical fractal element over the finite element mesh for $\sigma = 0.35$.



(a) $n = 0$ (b) $n = 1$ (c) $n = 2$ (d) $n = 3$

Figure 4.2: Hierarchical fractal element for $\sigma = 0.35$ over the finite element mesh.

Remark. 1. *For $\sigma = \frac{1}{4}$, the fractal functions $\phi_{i+\frac{1}{2},n}$ for $i = 0, 1, \ldots 2^n - 1$ are quadratic functions. Therefore, W consists of span of quadratic functions associated with the midpoints of each element. This is the hierarchical basis for a piecewise quadratic polynomial space.*

2. *For $\sigma = 0$, the fractal functions $\phi_{i+\frac{1}{2},n}$ for $i = 0, 1, \ldots 2^n - 1$ are linear functions.*

In this case, the space U is the space of continuous piecewise linear functions associated with a coarser mesh, while W consists of a span of fine grid nodal basis functions associated with the nodes which are not in the coarser mesh.

Let $u_h \in V_h$ be an approximate solution of equation (4.2). Then the approximate solution over the entire domain in terms of fractal basis functions is expressed as follows.

$$u_h(x) = \sum_{i=0}^{2^n} u_i \phi_{i,n}(x) + \sum_{i=0}^{2^n-1} \Delta u_{i+\frac{1}{2}} \phi_{i+\frac{1}{2},n}(x) \tag{4.25}$$

where u_i denotes the value of u_h at the global node i and $\phi_{i,n}(x)$ are the interpolation function associated with the node i. That is, $u_i = u_h(x_i)$ and $u_{i+1} = u_h(x_{i+1})$ and $\Delta u_{i+\frac{1}{2}}$ represent the additive refinements. The value of $u_h(x)$ at $x = x_{i+\frac{1}{2}}$ is calculated from $\Delta u_{i+\frac{1}{2}}$ using formula

$$u_h(x_{i+\frac{1}{2}}) = \frac{u_i + u_{i+1}}{2} + \Delta u_{i+\frac{1}{2}}.$$

4.4 Interpolation error estimates

The interpolation error estimates describe behavior of errors depending on the mesh size h and an order of approximation p. For a function $u \in C^0(0,1)$, let πu be its fractal interpolant. This fractal interpolant is a unique continuous fractal function that takes the same value as u at all the mesh vertices. Consequently, $u(x_i) = \pi u(x_i)$ for all mesh vertices x_i.

The interpolation error is a difference $u - \pi u$ such that $u(x_i) = \pi u(x_i)$ for all mesh vertices x_i. The interpolation error is controlled elementwise before being controlled globally on the interval $[0,1]$. This motivates an introduction of local interpolant over each element. Thus, the fractal interpolant over each element $[x_i, x_{i+1}]$ is as stated below:

$$\pi u = u_i \Lambda_{i,n}(x) + u_{i+1} \lambda_{i,n}(x) + \Delta u_{i+\frac{1}{2}} \Psi_{i,n}(x) \tag{4.26}$$

That is

$$\pi u = u_i((1+i) - 2^n x) + u_{i+1}(2^n x - i) + \Delta u_{i+\frac{1}{2}} \Psi(2^n x - i).$$

The derivative of fractal interpolant is expressed as follows.

$$(\pi u)' = -u_i 2^n + u_{i+1} 2^n + \Delta u_{i+\frac{1}{2}} 2^n \psi'(2^n x - i).$$

The function $\psi'(2^n x - i)$ is also a fractal function. This fractal function is discontinuous at the end points of the interval $[x_i, x_{i+1}]$ and at every dyadic point in the interval $[x_i, x_{i+1}]$. Therefore, the traditional techniques used to find the bounds for the interpolation error theoretically are not applicable. Hence, to find the bounds for interpolation error theoretically, we use the triangle inequality property of the norm. To this end, consider a quadratic interpolant u_q of u. Then $u(x_i) = u_q(x_i)$ and $u(x_{i+1}) = u_q(x_{i+1})$.

Using the triangle inequality property of the norm, one has

$$\|u - \pi u\| = \left\|u - u_q + u_q - \pi u\right\| \le \left\|u - u_q\right\| + \left\|u_q - \pi u\right\|.$$

Thus, to find bounds for interpolation error in L^2 norm and in H^1 seminorm, bounds on $\left\|u - u_q\right\|_{L^2}, \left\|u_q - \pi u\right\|_{L^2}, |u - u_q|_{H^1}, |u_q - \pi u|_{H^1}$ are calculated as follows.

4.4.1 Bound on $\left\|u - u_q\right\|_{L^2}$ and $|u - u_q|_{H^1}$:

Let an infinite dimensional space $V = H^1$ be approximated by quadratic elements of order less than or equal to 2. Then, the following theorem provides the interpolation bound for the quadratic element.

Theorem 25 (Interpolation error with Langrange quadratic element)**.** *Let the function space $V_{2^n}^{h^*}$ be constructed with Lagrange finite elements of order at most 2 on a uniform mesh with mesh size h. Let $\pi' : H^3([0,1]) \to V_{2^n}^{h^*}$ be an interpolation operator associated with $V_{2^n}^{h^*}$ and let $u \in H^3([0,1])$. Then there exists a constant $C < \infty$ independent of h such that*

$$|u - \pi' u|_{H^1([0,1])} \le (Ch)^2 |u|_{H^3([0,1])}. \tag{4.27}$$

Moreover,

$$\left\|u - \pi' u\right\|_{L^2[0,1]} + C\,h\,|u - \pi' u|_{H^1([0,1])} \le 2(Ch)^3 |u|_{H^3([0,1])}. \tag{4.28}$$

Proof. Recalling the definitions of the two norms, it is sufficient to prove the estimate piecewise. That is,

$$|u - \pi' u|_{H^1[x_i,x_{i+1}]} \leq (Ch)^2 |u|_{H^3[x_i,x_{i+1}]}$$

as the stated result represented by equation (4.27) follows by summing over i.

Let $e = u - \pi' u$ be an error. Let, $\pi' u$ be a quadratic interpolant.

Then,

$$u(x_i) = \pi' u(x_i) \text{ and } u(x_{i+1}) = \pi' u(x_{i+1}).$$

It is apparent that

$$e(x_i) = e(x_{i+1}) = 0.$$

Thus, for any $x \in [x_i, x_{i+1}]$

$$e(x) = e(x) - e(x_i) = \int_{x_i}^{x} e'(t) dt.$$

Therefore,

$$\begin{aligned}
|e(x)|^2 &\leq \int_{x_i}^{x} |e'(t)|^2 dt = \int_{x_i}^{x} |e'(t)\ 1|^2 dt \\
&= \int_{x_i}^{x} |1|^2 dx \int_{x_i}^{x} |e'(t)|^2 dt \\
&= (x - x_i) \int_{x_i}^{x} |e'(t)|^2 dt \\
&\leq (x - x_i) \int_{x_i}^{x_{i+1}} |e'(t)|^2 dt \\
&= (x - x_i) \left\| e' \right\|^2_{L^2[x_i,x_{i+1}]} .
\end{aligned}$$

Integrating with respect to x on the interval $[x_i, x_{i+1}]$,

$$\int_{x_i}^{x_{i+1}} |e|^2 dx \leq \left\| e' \right\|^2_{L^2[x_i,x_{i+1}]} \int_{x_i}^{x_{i+1}} (x - x_i) dx.$$

$$\|e\|^2_{L^2[x_i,x_{i+1}]} \leq \frac{h^2}{2} \left\| e' \right\|^2_{L^2[x_i,x_{i+1}]} .$$

Thus,

$$\|e\|_{L^2[x_i,x_{i+1}]} \leq \frac{h}{\sqrt{2}}\left\|e'\right\|_{L^2[x_i,x_{i+1}]}. \tag{4.29}$$

Since $e(x_i) = e(x_{i+1}) = 0$ by Rolle's mean value theorem, there exists at least one $\xi \in [x_i, x_{i+1}]$ such that $e'(\xi) = 0$. Thus,

$$e'(x) = e'(x) - e'(\xi) = \int_{\xi}^{x} e''(t)dt$$

$$\begin{aligned}
|e'(x)|^2 &\leq \int_{\xi}^{x} |e''(t)|^2(t)dt \\
&= \int_{\xi}^{x} |e''(t)1|^2(t)dt \\
&= \int_{\xi}^{x} |1|^2 dt \int_{\xi}^{x} |e''(t)|^2 dt \\
&= (x-\xi)\int_{x_i}^{x_{i+1}} |e''(t)|^2 dt \\
&= (x-\xi)\left\|e''\right\|^2_{L^2[x_i,x_{i+1}]}.
\end{aligned}$$

Integrating with respect to x on the interval $[x_i, x_{i+1}]$,

$$\int_{x_i}^{x_{i+1}} |e'(x)|^2 dx \leq \left\|e''\right\|^2_{L^2[x_i,x_{i+1}]} \int_{x_i}^{x_{i+1}} (x-\xi)dx.$$

$$\left\|e'\right\|^2_{L^2[x_i,x_{i+1}]} \leq \frac{h^2}{2}\left\|e''\right\|^2_{L^2[x_i,x_{i+1}]}.$$

Thus,

$$\left\|e'\right\|_{L^2[x_i,x_{i+1}]} \leq \frac{h}{\sqrt{2}}\left\|e''\right\|_{L^2[x_i,x_{i+1}]}. \tag{4.30}$$

The construction of quadratic interpolant on the interval $[x_i, x_{i+1}]$ requires three nodes namely, $x_i, x_{i+1}, x_{i+\frac{1}{2}}$ where $x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}$.

Thus,

$$u(x_i) = \pi' u(x_i) \quad u(x_{i+\frac{1}{2}}) = \pi' u(x_{i+\frac{1}{2}}) \quad u(x_{i+1}) = \pi' u(x_{i+1}).$$

Consequently,

$$e(x_i) = e(x_{i+1}) = e(x_{i+\frac{1}{2}}).$$

By Rolle's mean value theorem there exists $c_1 \in (x_i, x_{i+\frac{1}{2}})$ such that

$$e'(c_1) = \frac{e(x_{i+\frac{1}{2}}) - e(x_i)}{x_{i+\frac{1}{2}} - x_i} = 0 \tag{4.31}$$

Similarly, by Rolle's mean value theorem, there exists another point $c_2 \in (x_{i+\frac{1}{2}}, x_{i+1})$ such that

$$e'(c_2) = \frac{e(x_{i+\frac{1}{2}}) - e(x_i)}{x_{i+1} - x_{i+\frac{1}{2}}} = 0. \tag{4.32}$$

Thus, $e'(c_1) = e'(c_2) = 0$.

Therefore, by Rolle's mean value theorem, there exists $c \in (c_1, c_2) \subset (x_i, x_{i+1})$ such that

$$e''(c) = \frac{e'(c_2) - e'(c_1)}{c_2 - c_1} = 0. \tag{4.33}$$

Thus,

$$e''(x) = e''(x) - e''(c) = \int_c^x e'''(t)dt.$$

$$\begin{aligned}
|e''(x)|^2 &\leq \int_c^x |e'''(t)|^2(t)dt \\
&= \int_c^x |e'''(t)1|^2(t)dt \\
&= \int_c^x |1|^2 dt \int_c^x |e'''(t)|^2 dt \\
&\leq (x-c)\int_{x_i}^{x_{i+1}} |e'''(t)|^2 dt \\
&= (x-c)\left\|e'''\right\|^2_{L^2[x_i, x_{i+1}]}.
\end{aligned}$$

Putting $e''' = u'''$ and integrating wrt x on the interval $[x_i, x_{i+1}]$ one has

$$\int_{x_i}^{x_{i+1}} |e''(x)|^2 dx \leq \left\|u'''\right\|^2_{L^2[x_i, x_{i+1}]} \int_{x_i}^{x_{i+1}} (x-\xi)dx.$$

$$\left\|e''\right\|^2_{L^2[x_i, x_{i+1}]} \leq \frac{h^2}{2}\left\|u'''\right\|^2_{L^2[x_i, x_{i+1}]}.$$

Thus,

$$\left\|e''\right\|_{L^2[x_i, x_{i+1}]} \leq \frac{h}{\sqrt{2}}\left\|u'''\right\|_{L^2[x_i, x_{i+1}]}. \tag{4.34}$$

From equations (4.29), (4.30) and (4.34) one has

$$|u - \pi' u|_{H^1[x_i, x_{i+1}]} \leq (Ch)^2 |u|_{H^3[x_i, x_{i+1}]}$$

and

$$\left\| u - \pi' u \right\|_{L^2[x_i, x_{i+1}]} \leq (Ch)^3 |u|_{H^3[x_i, x_{i+1}]}.$$

Thus,

$$\left\| u - \pi' u \right\|_{L^2[x_i, x_{i+1}]} + C\, h\, |u - \pi' u|_{H^1[x_i, x_{i+1}]} \leq 2(Ch)^3 |u|_{H^3[x_i, x_{i+1}]}.$$

□

Corollary 9. *Let $u_q = 4x(1-x)$ be a quadratic interpolant on a uniform mesh of size h and $u \in H^3[0,1]$. Then, there exists a constant $C < \infty$ independent of h such that,*

$$|u - u_q|_{H^1[0,1]} \leq Ch^2 |u|_{H^3[0,1]}. \tag{4.35}$$

Moreover,

$$\left\| u - u_q \right\|_{L^2[0,1]} + C\, h\, |u - u_q|_{H^1[0,1]} \leq 2Ch^3 |u|_{H^3[0,1]}. \tag{4.36}$$

Proof. Let, $\pi' u = u_q = 4x(1-x)$ ba a quadratic interpolant on a uniform size h.
Then, the results in equations (4.35) and (4.36) are obtained by substituting $\pi' u = u_q$ in the identities (4.27) and (4.28). □

4.4.2 Bound on $\left\| u_q - \pi u \right\|_{L^2}$ and $|u_q - \pi u|_{H^1}$

To find bound on $\left\| u_q - \pi u \right\|_{L^2}$, let $u_q(x) = 4x(1-x)$ be the quadratic interpolant on the interval $[0,1]$. The exact representation of u_q on the interval $[0,1]$ is

$$u_q = \Psi_q(x) = \begin{cases} \frac{1}{4}\Psi_q(2x) + 2x, \text{ if } x \in [0, \frac{1}{2}) \\ = \frac{1}{4}\Psi_q(2x-1) + 2 - 2x, \text{ if } x \in [\frac{1}{2}, 1]. \end{cases} \tag{4.37}$$

The exact representation of u_q on the interval $[x_i, x_{i+1}]$ is

$$u_q = u_i\Lambda_{i,n}(x) + u_{i+1}\lambda_{i,n}(x) + \Delta u_{i+\frac{1}{2}}\Psi_{q_{i,n}}(x) \tag{4.38}$$

where

$$\Psi_{q_{i,n}}(x) = \Psi_q(2^n x - i).$$

That is

$$\pi u = u_i\Lambda(2^n x - i) + u_{i+1}\lambda(2^n x - i) + \Delta u_{i+\frac{1}{2}}\Psi_q(2^n x - i).$$

Since u_q and πu are interpolants of the exact solution u, one has

$$u(x_i) = u_q(x_i) = \pi u(x_i) \text{ and } u(x_{i+1}) = u_q(x_{i+1}) = \pi u(x_{i+1}).$$

Lemma 7.

$$\Delta u_{i+\frac{1}{2}} = h^2.$$

Proof. Using the identity, $u_q(x) = 4x - 4x^2$,

one has

$$u_i = u_q(x_i) = 4x_i - 4x_i^2.$$

$$u_{i+1} = u_q(x_{i+1}) = 4x_{i+1} - 4x_{i+1}^2.$$

Therefore,

$$\frac{u_i + u_{i+1}}{2} = \frac{4x_i - 4x_i^2 + 4x_{i+1} - 4x_{i+1}^2}{2}$$

$$= 4\frac{x_i + x_{i+1}}{2} - 4\frac{x_i^2 + x_{i+1}^2 + 2x_i x_{i+1}}{2} + 4x_i x_{i+1}$$

$$= 4x_{i+\frac{1}{2}} - 8\frac{x_i^2 + x_{i+1}^2 + 2x_i x_{i+1}}{4} + 4x_i x_{i+1}$$

$$= 4x_{i+\frac{1}{2}} - 8x_{i+\frac{1}{2}}^2 + 4x_i x_{i+1}$$

$$= (4x_{i+\frac{1}{2}} - 4x_{i+\frac{1}{2}}^2) - 4x_{i+\frac{1}{2}}^2 + 4(x_{i+\frac{1}{2}} - \frac{h}{2})(x_{i+\frac{1}{2}} + \frac{h}{2})$$

$$= u_{i+\frac{1}{2}} - 4x_{i+\frac{1}{2}}^2 + 4x_{i+\frac{1}{2}}^2 - 4\frac{h^2}{4}$$

$$= \Delta u_{i+\frac{1}{2}} - h^2.$$

Thus,

$$\frac{u_i + u_{i+1}}{2} = \Delta u_{i+\frac{1}{2}} - h^2.$$

Since

$$\frac{u_i + u_{i+1}}{2} = \Delta u_{i+\frac{1}{2}} - \Delta u_{i+\frac{1}{2}}$$

it follows that

$$\Delta u_{i+\frac{1}{2}} = h^2.$$

□

Lemma 8. *1.*

$$\int_0^1 \Psi^2(x)dx = \left[\frac{2+\sigma}{6(1-\sigma)^2(1+\sigma)}\right]. \tag{4.39}$$

2.

$$\int_0^1 \Psi_q^2(x)dx = \frac{8}{15}. \tag{4.40}$$

3.

$$\int_0^1 \Psi_q(x)\,\Psi(x)dx = \left[\frac{5-2\sigma}{12(1-\sigma)(1-\frac{\sigma}{4})}\right]. \tag{4.41}$$

Proof. 1. From equation (3.30), it follows that

$$\int_0^1 \Psi^2(x)dx = \left[\frac{2+\sigma}{6(1-\sigma)^2(1+\sigma)}\right].$$

2. Substituting $\sigma = \frac{1}{4}$ in equation (3.30)

$$\int_0^1 \Psi_q^2(x)dx = \frac{8}{15}.$$

3.

$$\int_0^1 \Psi_q(x)\,\Psi(x)\,dx = \int_0^{\frac{1}{2}} (\frac{1}{4}\Psi_q(2x)+2x)(\sigma\Psi(2x)+2x)$$

$$+\int_{\frac{1}{2}}^1 (\frac{1}{4}\Psi_q(2x-1)+2-2x)(\sigma\Psi(2x-1)+2-2x)\,dx$$

$$= \int_0^{\frac{1}{2}} (\frac{\sigma}{4}\Psi_q(2x)\Psi(2x)+\frac{1}{4}(2x)\Psi_q(2x)dx+\sigma(2x)\,\Psi(2x)+4x^2)dx$$

$$+\int_{\frac{1}{2}}^1 (\frac{\sigma}{4}\Psi_q(2x-1)\Psi(2x-1)+\frac{1}{4}(2-2x)\Psi_q(2x-1)dx+\sigma(2-2x)\,\Psi(2x-1)+4(1-x)^2)dx.$$

Substituting $2x = t$ in the first integral and $2x-1 = t$ in the second integral

$$\int_0^1 \Psi_q(x)\,\Psi(x)\,dx = \int_0^1 \frac{\sigma}{8}\Psi_q(t)\Psi(t)dt+\int_0^1 \frac{1}{8}(t)\Psi_q(t)dt+\int_0^1 \frac{\sigma}{2}(t)\,\Psi(t)dt+\left(\frac{4}{3}\right)\left(\frac{1}{2^3}\right)$$

$$+\int_0^1 \frac{\sigma}{8}\Psi_q(t)\Psi(t)dt+\int_0^1 \frac{1}{8}(1-t)\Psi_q(t)dt+\int_0^1 \frac{\sigma}{2}(1-t)\,\Psi(t)dt+\left(\frac{4}{3}\right)\left(\frac{1}{2^3}\right).$$

Therefore,

$$\left(1-\frac{\sigma}{4}\right)\int_0^1 \Psi_q(x)\,\Psi(x)\,dx = \frac{1}{3}+\frac{1}{8}\int_0^1 \Psi_q(t)dt+\frac{\sigma}{2}\int_0^1 \Psi(t)dt$$

$$= \frac{1}{3}+\frac{1}{12}+\frac{\sigma}{2}\frac{1}{2(1-\sigma)}$$

$$= \frac{5}{12}+\frac{\sigma}{4(1-\sigma)}$$

$$= \frac{5-2\sigma}{12(1-\sigma)}.$$

Thus,

$$\int_0^1 \Psi_q(x)\,\Psi(x)dx = \left[\frac{5-2\sigma}{12(1-\sigma)(1-\frac{\sigma}{4})}\right].$$

□

The following proposition provides the bound on $\left\|u_q-\pi u\right\|_{L^2}$.

Proposition 13.

$$\left\|u_q-\pi u\right\|_{L^2[0,1]} = C^{**}\,h^{\frac{5}{2}}. \tag{4.42}$$

where,

$$C^{**} = \sqrt{\frac{8}{15} - \left[\frac{5-2\sigma}{6(1-\sigma)(1-\frac{\sigma}{4})}\right] + \left[\frac{2+\sigma}{6(1-\sigma)^2(1+\sigma)}\right]}. \tag{4.43}$$

Proof. Using exact representations of u_q and πu in the interval $[x_i, x_{i+1}]$ one has

$$u_q - \pi u = \Delta u_{i+\frac{1}{2}}(\Psi_q(2^n x - i) - \Psi(2^n x - i))$$

Therefore,

$$\begin{aligned}\left\|u_q - \pi u\right\|^2_{L^2[x_i,x_{i+1}]} &= \int_{x_i}^{x_{i+1}} (\Delta u_{i+\frac{1}{2}}(\Psi_q(2^n x - i) - \Psi(2^n x - i)))^2 dx \\ &= \int_{x_i}^{x_{i+1}} (\Delta u_{i+\frac{1}{2}})^2(\Psi_q(2^n x - i) - \Psi(2^n x - i))^2 dx \\ &= (h^2)^2 \int_0^1 (\Psi_q(t) - \Psi(t))^2\, h\, dt \\ &= h^5 \int_0^1 \Psi_q^2(t)dt - 2\int_0^1 \Psi(t)\,\Psi_q(t)dt + \int_0^1 \Psi^2(t))\, dt \\ &= h^5\left\{\frac{8}{15} - \left[\frac{5-2\sigma}{6(1-\sigma)(1-\frac{\sigma}{4})}\right] + \left[\frac{2+\sigma}{6(1-\sigma)^2(1+\sigma)}\right]\right\}. \end{aligned} \tag{4.44}$$

Therefore,

$$\left\|u_q - \pi u\right\|^2_{L^2[x_i,x_{i+1}]} = h^5\left\{\frac{8}{15} - \left[\frac{5-2\sigma}{6(1-\sigma)(1-\frac{\sigma}{4})}\right] + \left[\frac{2+\sigma}{6(1-\sigma)^2(1+\sigma)}\right]\right\}.$$

Thus,

$$\left\|u_q - \pi u\right\|_{L^2[x_i,x_{i+1}]} = C^{**}\, h^{\frac{5}{2}}.$$

where

$$C^{**} = \sqrt{\frac{8}{15} - \left[\frac{5-2\sigma}{6(1-\sigma)(1-\frac{\sigma}{4})}\right] + \left[\frac{2+\sigma}{6(1-\sigma)^2(1+\sigma)}\right]}.$$

Summing over 2^n intervals,

$$\left\|u_q - \pi u\right\|_{L^2[0,1]} = C^{**}\, h^{\frac{5}{2}}.$$

□

Remark.

- *For $\sigma = \frac{1}{4}$, $C^{**} = 0$. From equation (4.42) it follows that*

$$\left\| u_q - \pi u \right\|_{L^2[0,1]} = 0.$$

- *For $\sigma = 0$, $C^{**} = \sqrt{\frac{1}{30}}$. From equation (4.42) it follows that*

$$\left\| u_q - \pi u \right\|_{L^2[0,1]} = \sqrt{\frac{1}{30}}\, h^{\frac{5}{2}}.$$

Bound on $|u_q - \pi u|_{H^1}$:

To find the bound on $|u_q - \pi u|_{H^1}$ we prove the following results:

Lemma 9. *1.*

$$\int_0^1 \Psi_q'(x)\Psi'(x)dx = \frac{4}{1-\sigma}. \tag{4.45}$$

2.

$$\int_0^1 \Psi'^2(x)dx = \frac{4}{1-4\sigma^2}. \tag{4.46}$$

3.

$$\int_0^1 \Psi_q'^2(x)dx = \frac{16}{3}. \tag{4.47}$$

Proof. 1.

$$\begin{aligned}
\int_0^1 \Psi_q'(x)\Psi'(x)dx &= \int_0^{\frac{1}{2}} \left(\frac{1}{2}\Psi_q'(2x) + 2\right)\left(2\sigma\Psi'(2x) + 2\right) dx \\
&+ \int_{\frac{1}{2}}^1 \left(\frac{1}{2}\Psi_q'(2x-1) - 2\right)\left(2\sigma\Psi'(2x-1) - 2\right) dx \\
&= \int_0^{\frac{1}{2}} \left(\sigma\Psi_q'(2x)\Psi'(2x) + \Psi_q'(2x) + 4\sigma\Psi'(2x) + 4\right) dx \\
&+ \int_{\frac{1}{2}}^1 \left(\sigma\Psi_q'(2x-1)\Psi'(2x-1) - \Psi_q'(2x-1) - 4\sigma\Psi'(2x-1) + 4\right) dx
\end{aligned}$$

$$= \sigma \int_0^1 \Psi_q'(x)\Psi'(x)dx + 4.$$

Thus,

$$(1-\sigma)\int_0^1 \Psi_q'(x)\Psi'(x)dx = 4.$$

$$\int_0^1 \Psi_q'(x)\Psi'(x)dx = \frac{4}{1-\sigma}.$$

2. From equation (3.28) it follows that

$$\int_0^1 \Psi'^2(x)dx = \frac{4}{1-4\sigma^2}.$$

3. Substituting $\sigma = \frac{1}{4}$ in $\int_0^1 \Psi'^2(x)dx = \frac{4}{1-4\sigma^2}$, we get,

$$\int_0^1 \Psi_q'^2(x)dx = \frac{16}{3}.$$

□

The following proposition provides the bound on $|u_q - \pi u|_{H^1}$.

Proposition 14.

$$|u_q - \pi u|_{H^1} = c^* \, h^{\frac{3}{2}}. \tag{4.48}$$

where

$$C^* = \sqrt{\left\{\frac{16}{3} - 2\frac{4}{1-\sigma} + \frac{4}{1-4\sigma^2}\right\}}. \tag{4.49}$$

Proof. Using exact representations of u_q and πu in the interval $[x_i, x_{i+1}]$, one has

$$u_q - \pi u = \Delta u_{i+\frac{1}{2}}(\Psi_q(2^n x - i) - \Psi(2^n x - i)).$$

Therefore,

$$\begin{aligned}
|u_q - \pi u|^2_{H^1[x_i,x_{i+1}]} &= \int_{x_i}^{x_{i+1}} (\Delta u_{i+\frac{1}{2}}(\Psi_q(2^n x - i) - \Psi(2^n x - i)))'^2 \, dx \\
&= \int_{x_i}^{x_{i+1}} (\Delta u_{i+\frac{1}{2}})^2 (2^n \Psi_q'(2^n x - i) - 2^n \Psi'(2^n x - i))^2 dx \\
&= \frac{(h^2)^2}{h^2} \int_0^1 (\Psi_q'(t) - \Psi'(t))^2 \, h \, dt \\
&= h^3 \int_0^1 \Psi_q'^2(t)dt - 2\int_0^1 \Psi'(t)\,\Psi_q'(t)dt + \int_0^1 \Psi'^2(t))\,dt \qquad (4.50) \\
&= h^3 \left\{\frac{16}{3} - 2\frac{4}{1-\sigma} + \frac{4}{1-4\sigma^2}\right\}.
\end{aligned}$$

Therefore,

$$|u_q - \pi u|_{H^1[x_i,x_{i+1}]} = \sqrt{\left\{\frac{16}{3} - 2\frac{4}{1-\sigma} + \frac{4}{1-4\sigma^2}\right\}}\, h^{\frac{3}{2}}.$$

Thus,

$$|u_q - \pi u|_{H^1[x_i,x_{i+1}]} = C^*\, h^{\frac{3}{2}}.$$

Summing over 2^n intervals

$$\left\|u_q - \pi u\right\|_{H^1[0,1]} = C^*\, h^{\frac{3}{2}}.$$

where

$$C^* = \sqrt{\left\{\frac{16}{3} - 2\frac{4}{1-\sigma} + \frac{4}{1-4\sigma^2}\right\}}.$$

□

Remark.

- *For* $\sigma = \frac{1}{4},\ C^* = 0.$
- *For* $\sigma = 0,\ C^* = \frac{2}{\sqrt{3}}.$

4.4.3 Bound for $\left\|u - \pi u\right\|_{L^2}$ and $|u - \pi u|_{H^1}$:

The following theorem provides bounds for an interpolation error in the H^1 norm.

Theorem 26. *Let the function space $V^h_{2^n}$ be constructed with the piecewise continuous fractal functions with the uniform mesh size h. Let $u \in H^3(I)$ and let $\pi : H^1(I) \to V^h_{2^n}$ be interpolation operator associated with $V^h_{2^n}$. Then there exists a constant $C < \infty$ independent of u and h such that*

$$\left\|u - \pi u\right\|_{L^2(I)} \leq (Ch)^3 |u|_{H^3(I)} + C^{**}\, h^{\frac{5}{2}} \tag{4.51}$$

where

$$C^{**} = \sqrt{\frac{8}{15} - \left[\frac{5-2\sigma}{6(1-\sigma)(1-\frac{\sigma}{4})}\right] + \left[\frac{2+\sigma}{6(1-\sigma)^2(1+\sigma)}\right]}.$$

Moreover,

$$|u - \pi u|_{H^1(I)} \leq (Ch)^2 |u|_{H^3(I)} + C^*\, h^{\frac{3}{2}} \tag{4.52}$$

where,

$$C^* = \sqrt{\left\{\frac{16}{3} - 2\frac{4}{1-\sigma} + \frac{4}{1-4\sigma^2}\right\}}.$$

Proof. Let $u(x)$ be an exact solution, $u_q(x) = 4x(1-x)$ be a quadratic interpolant and πu be a fractal interpolant on the interval $[0,1]$. Using triangle inequality property of norms, one has

$$\|u - \pi u\|_{L^2} = \left\|u - u_q + u_q - \pi u\right\|_{L^2}$$

$$\leq \left\|u - u_q\right\|_{L^2} + \left\|u_q - \pi u\right\|_{L^2}.$$

From equations (4.36) and (4.42), it follows that

$$\leq (Ch)^3 |u|_{H^3(I)} + C^{**}\, h^{\frac{5}{2}}.$$

Similarly,

$$|u - \pi u|_{H^1(I)} = |u - u_q + u_q - \pi u|_{H^1(I)}$$

$$\leq |u - u_q|_{H^1(I)} + |u_q - \pi u|_{H^1(I)}.$$

From equations (4.35) and (4.48), it follows that

$$\leq (Ch)^2 |u|_{H^3(I)} + C^{*}\, h^{\frac{3}{2}}.$$

□

Remark. *1. For $\sigma = \frac{1}{4}$ values of constants C^* and C^{**} calculated from equations (4.49) and (4.43) respectively are zero.*

$$C^* = 0 \ \text{ and } \ C^{**} = 0.$$

Therefore, from equations (4.51) and (4.52), It follows that

$$\|u - \pi u\|_{L^2(I)} \leq (Ch)^3 |u|_{H^3(I)}$$

and

$$|u - \pi u|_{H^1(I)} \leq (Ch)^2 |u|_{H^3(I)}.$$

2. *For $\sigma = 0$ values of constants C^* and C^{**} calculated from equations (4.49) and (4.43) respectively are as follows.*

$$C^* = \frac{2}{\sqrt{3}} \text{ and } C^{**} = \frac{1}{\sqrt{30}}.$$

Therefore, from equations (4.51) and (4.52), It follows that

$$\|u - \pi u\|_{L^2(I)} \leq (Ch)^3 |u|_{H^3(I)} + \frac{1}{\sqrt{30}} h^{\frac{5}{2}}$$

and

$$|u - \pi u|_{H^1(I)} \leq (Ch)^2 |u|_{H^3(I)} + \frac{2}{\sqrt{3}} h^{\frac{3}{2}}.$$

3. *The fractal function $(\pi u)'$ is a weak derivative of the fractal function πu. This fractal function $(\pi u)'$ is discontinuous at every dyadic point in the interval $[x_i, x_{i+1}]$ and at its end points. Therefore, the usual methods of numerical integration can not be used to calculate the error in H^1 seminorm. Hence, it is necessary to note that the bounds for an interpolation error in H^1 seminorm can not be calculated practically in the numerical experiments. Further investigations are required to calculate the error in H^1 seminorm. However, the bounds of interpolation error in H^1 seminorm are established theoretically. Please refer theorem 26.*

4.5 Finite Element Model

Let u_h represented by equation (4.25) be an approximate solution of a given boundary value problem where $u_i = u_h(x_i)$ are to be determined.
A weak form of partial differential equation is stated below:

$$a(u_h, \phi_{i,n}) = f(\phi_{i,n}) \text{ for } i = 0, \frac{1}{2}, 1, \frac{3}{3}, \dots 2^n.$$

Thus,

$$\sum_{j=0}^{2^n} a(\phi_{j,n}, \phi_{i,n}) u_j = f(\phi_{i,n}) \text{ for } i = 0, \frac{1}{2}, 1, \frac{3}{3}, \dots 2^n.$$

The above system can be expressed in the matrix form as, $AU = F$ where the stiffness matrix A is,

$A = a(i,j)_{(2^{n+1}+1)\times(2^{n+1}+1)}$; where $a(i,j) = a(\phi_{j,n}, \phi_{i,n}) = \int \phi'_{j,n}\phi'_{i,n}$

and the load vector $F = f(\phi_{i,n}) = (f, \phi_{i,n}) = \int f\, \phi_{i,n}$, for $i = 0, 1, \dots, 2^{n+1}$. To calculate the stiffness matrix and the source vector over the entire domain, the coefficient matrix and the source vector for an element are calculated. Let K^i and F^i be the coefficient matrix and the source vector respectively for the typical element I_i. The element I_i is located between the nodes x_i and x_{i+1}. Since interpolation functions are compactly supported, there are three non-zero interpolation functions over each element. The equations (4.10), (4.11), (4.12) represent non-zero interpolation functions on the interval I_i. The entries of the coefficient matrix on the interval I_i are calculated as follows.

Let, $K^i = \begin{bmatrix} K^i_{11} & K^i_{12} & K^i_{13} \\ K^i_{21} & K^i_{22} & K^i_{23} \\ K^i_{31} & K^i_{32} & K^i_{33} \end{bmatrix} = \begin{bmatrix} a(\varphi^i_1, \varphi^i_1) & a(\varphi^i_1, \varphi^i_2) & a(\varphi^i_1, \varphi^i_3) \\ a(\varphi^i_2, \varphi^i_1) & a(\varphi^i_2, \varphi^i_2) & a(\varphi^i_2, \varphi^i_3) \\ a(\varphi^i_3, \varphi^i_1) & a(\varphi^i_3, \varphi^i_2) & a(\varphi^i_3, \varphi^i_3) \end{bmatrix}$.

Thus, $K^i = \begin{bmatrix} \int_{I_i} {\varphi^i_1}'{\varphi^i_1}'dx & \int_{I_i} {\varphi^i_1}'{\varphi^i_2}'dx & \int_{I_i} {\varphi^i_1}'{\varphi^i_3}'dx \\ \int_{I_i} {\varphi^i_2}'\varphi'_1 dx & \int_{I_i} {\varphi^i_2}'{\varphi^i_2}'dx & \int_{I_i} {\varphi^i_2}'{\varphi^i_3}'dx \\ \int_{I_i} {\varphi^i_3}'\varphi'_1 dx & \int_{I_i} {\varphi^i_3}'{\varphi^i_2}'dx & \int_{I_i} {\varphi^i_3}'{\varphi^i_3}'dx \end{bmatrix}$.

The entries in the coefficient matrix are calculated as follows.

1.

$$K^i_{11} = a(\varphi^i_1, \varphi^i_1) = \int_{I_i} {\varphi^i_1}'{\varphi^i_1}'dx = \int_{x_i}^{x_{i+1}} \left(\frac{1}{h}\right)\left(\frac{1}{h}\right) dx$$
$$= 2^{2n}\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} dx = 2^{2n}\frac{1}{2^n} = 2^n.$$

2.

$$K^i_{12} = a(\varphi^i_1, \varphi^i_2) = K^i_{21} = a(\varphi_2, \varphi_1) = \int_{I_i} {\varphi^i_1}'{\varphi^i_2}'dx = \int_{x_i}^{x_{i+1}} \frac{1}{h}\Psi'_{i,n}(x)dx$$
$$= \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} 2^{2n}\Psi'(2^n x - i)dx = 2^n\int_0^1 \psi'(x)dx = 0.$$

3.

$$K_{13}^i = a(\varphi_1^i, \varphi_3^i) = K_{31}^i = a(\varphi_3, \varphi_1) = \int_{I_i} {\varphi_1^i}' {\varphi_3^i}' dx = \int_{x_i}^{x_{i+1}} \left(\frac{1}{h}\right)\left(\frac{-1}{h}\right) dx$$
$$= -2^{2n} \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} dx = -2^n.$$

4.

$$K_{22}^i = a(\varphi_2^i, \varphi_2^i) = \int_{I_i} {\varphi_2^i}' {\varphi_2^i}' dx = \int_{x_i}^{x_{i+1}} {\Psi'}^2_{i,n}(x) dx$$
$$= \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} 2^{2n} {\Psi'}^2 (2^n x - i) dx = 2^n \int_0^1 {\Psi'}^2(x) dx = 2^n \left(\frac{4}{1 - 4\sigma^2}\right).$$

5.

$$K_{23}^i = a(\varphi_2^i, \varphi_3^i) = K_{32}^i = a(\varphi_3^i, \varphi_2^i) = \int_{I_i} {\varphi_2^i}' {\varphi_3^i}' dx = \int_{x_i}^{x_{i+1}} \Psi'_{i,n}(x)(-1) dx$$
$$= \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} -2^n \Psi'(2^n x - i) dx = -2^n \int_0^1 \Psi'(x) dx = 0.$$

6.

$$K_{33}^i = a(\varphi_3^i, \varphi_3^i) = \int_{I_i} {\varphi_3^i}' {\varphi_3^i}' dx = \int_{x_i}^{x_{i+1}} \left(\frac{-1}{h}\right)\left(\frac{-1}{h}\right) dx$$
$$= 2^{2n} \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} dx = 2^{2n} \frac{1}{2^n} = 2^n.$$

Thus, the element matrix is:

$$K_{ij} = \begin{bmatrix} 2^n & 0 & -2^n \\ 0 & 2^n \left(\frac{4}{1-4\sigma^2}\right) & 0 \\ -2^n & 0 & 2^n \end{bmatrix} = 2^n \begin{bmatrix} 1 & 0 & -1 \\ 0 & \left(\frac{4}{1-4\sigma^2}\right) & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

The entries of the source vector on the interval I_i are calculated as follows.

$$F_i = \begin{bmatrix} \int_{I_i} f\,\varphi_1^i\,dx \\ \int_{I_i} f\,\varphi_2^i\,dx \\ \int_{I_i} f\,\varphi_3^i\,dx \end{bmatrix} = \begin{bmatrix} \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\Lambda_{i,n}(x)dx \\ \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\Psi_{i,n}(x)dx \\ \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\lambda_{i,n}(x)dx \end{bmatrix} = \begin{bmatrix} \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)((i+1)-2^n x)dx \\ \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\Psi(2^n x - i)dx \\ \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)(2^n x - i)dx \end{bmatrix}.$$

The fractal functions are integrated explicitly using the self-similarity property. We use the trapezoidal rule for the numerical integration of the fractal function [48] where the explicit integration of fractal functions is unattainable.

4.5.1 Assembly of element matrices

The assembly of the finite element matrices is carried out by imposing inter-element continuity of the variables and boundary conditions.

Thus, the assembled coefficient matrix is as follows:

$$A = \begin{bmatrix} K_{11}^0 & K_{12}^0 & K_{13}^0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ K_{21}^0 & K_{22}^0 & K_{23}^0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ K_{31}^0 & K_{32}^0 & K_{33}^0 + K_{11}^1 & K_{12}^1 & K_{13}^1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & K_{21}^1 & K_{22}^1 & K_{23}^1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & K_{31}^1 & K_{32}^1 & K_{33}^1 + K_{11}^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & K_{33}^{2^{n+1}-1} + K_{11}^{2^{n+1}} & K_{12}^{2^{n+1}} & K_{13}^{2^{n+1}} \\ 0 & 0 & 0 & 0 & 0 & \cdots & K_{21}^{2^{n+1}} & K_{22}^{2^{n+1}} & K_{13}^{2^{n+1}} \\ 0 & 0 & 0 & 0 & 0 & \cdots & K_{31}^{2^{n+1}} & K_{32}^{2^{n+1}} & K_{33}^{2^{n+1}} \end{bmatrix}.$$

The assembled source matrix is as follows:

$$F = \begin{bmatrix} f_1^0 \\ f_2^0 \\ f_3^0 + f_1^1 \\ f_2^1 \\ f_3^1 + f_1^2 \\ \vdots \\ f_3^{2^{n+1}} + f_1^{2^{n+1}+1} \\ f_2^{2^{n+1}+1} \\ f_3^{2^{n+1}+1} \end{bmatrix}.$$

Dirichlet boundary conditions are then imposed on these two matrices to reduce the assembled set of equations equal to the number of unknowns. After imposing Dirichlet boundary conditions, the dimensions of system matrices A, U, F are $(2^{n+1}-1)\times(2^{n+1}-1)$, $(2^{n+1}-1)\times 1$ and $(2^{n+1}-1)\times 1$ respectively. We use the Gaussian elimination method to solve the linear system of equations $AU = F$ for the unknown vector U. Thus,

$$U = \begin{bmatrix} \Delta u_{\frac{1}{2}} \\ u_1 \\ \Delta u_{\frac{3}{2}} \\ u_2 \\ \vdots \\ u_{2^n-1} \\ \Delta_{2^n-\frac{1}{2}} \end{bmatrix}.$$

Since Dirichlet boundary conditions are imposed, it follows that

$$u_0 = u_h(0) = 0 \quad \text{and} \quad u_{2^n} = u_h(1) = 0.$$

The approximate solution u_h is obtained using the formula

$$u(x) \approx u_h(x) = \sum_{i=0}^{2^n-1} u_i \phi_{i,n}(x) + \sum_{i=0}^{2^n-1} \Delta u_{i+\frac{1}{2}} \phi_{i+\frac{1}{2},n} \tag{4.53}$$

where u_i denotes the value of u_h at the global node i . That is, $u_i = u_h(x_i)$ and $u_{i+1} = u_h(x_{i+1})$. Further, the functions $\phi_{i,n}(x)$ are interpolation functions associated with the node i.

The value of $u_h(x)$ at $x = x_{i+\frac{1}{2}}$ is calculated from $\Delta u_{i+\frac{1}{2}}$ using formula

$$u_h(x_{i+\frac{1}{2}}) = \frac{u_i + u_{i+1}}{2} + \Delta u_{i+\frac{1}{2}}.$$

Chapter 5

Solution of Poisson's equation with Dirichlet boundary condition

This chapter demonstrates illustrations of the finite element method with the hierarchical fractal basis. This chapter contributes to original work. We restrict our discussion to a second-order elliptic partial differential equation with Dirichlet boundary conditions in 1-dimension.

5.1 Model problem

The Poisson equation with Dirichlet boundary conditions in 1-dimension represented by the following equation:

$$-\frac{d^2u}{dx^2} = f, \quad \text{for } x \in (0,1) \tag{5.1}$$

subject to Dirichlet boundary conditions,

$$u(0) = 0 \quad u(1) = 0. \tag{5.2}$$

The Poisson equation is reformulated in the variational formulation as follows.

$$-\frac{d^2u}{dx^2} = f$$

Multiplying the equation 5.1 by a test function $v(x) \in H_0^1(0,1)$,

$$\int_0^1 -\frac{d^2u}{dx^2}(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx$$

Integrating by parts,

$$-[\frac{du}{dx}v(x)]_0^1 + \int_0^1 \frac{du}{dx}\frac{dv}{dx}\,dx = \int_0^1 f(x)v(x)\,dx$$

Since, $v(0) = 0$ and $v(1) = 0$, one has

$$\int_0^1 \frac{du}{dx}\frac{dv}{dx}\,dx = \int_0^1 f(x)v(x)\,dx$$

$$a(u,v) = f(v) \tag{5.3}$$

where,

$$a(u,v) = \int_0^1 \frac{du}{dx}\frac{dv}{dx}\,dx$$

and

$$f(v) = \int_0^1 f(x)v(x)\,dx.$$

We use the finite dimensional subspace of continuous fractal functions $V_{2^n}^h$ as the approximation space in the Ritz Galerkin approximation. To illustrate the use of fractal functions for approximation in the finite element method, we choose functions $f(x)$ in (5.1) and solve the Poisson equation with Dirichlet boundary conditions represented by equations (5.2).

5.2 Illustrations

5.2.1 Solution to the Poisson equation $-\frac{d^2u}{dx^2} = 1$ with Dirichlet boundary conditions

Assume that

$$f(x) = 1$$

in the Poisson equation (5.1).Then the Poisson equation is stated as below:

$$-\frac{d^2u}{dx^2} = 1, \qquad \text{for } x \in (0,1)$$

subject to homogeneous Dirichlet boundary conditions,

$$u(0) = 0 \quad u(1) = 0.$$

Let $n = 2$ and an interval $(0,1)$ is discretized into $2^n = 2^2 = 4$ equal subintervals.

To determine matrix of the system, we find element matrices and assemble them. The coefficient matrix over each element is represented by matrix equation (4.5).

Substituting $\sigma = 0.25$,

$$K_{ij} = 4 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0.53333 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

The global stiffness matrix A after imposing boundary conditions is as follows:

$$A = \begin{bmatrix} 21.333 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 21.333 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 8 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 21.333 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 21.333 \end{bmatrix}.$$

To find the entries of the source vector, the self-similarity property of the fractal functions is used to evaluate the integrals of the fractal function.

The entries of the source vector over the subinterval I_i are calculated as follows.

1.

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\frac{x_{i+1}-x}{x_{i+1}-x_i}dx \quad = \quad \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} \frac{x_{i+1}-x}{h}dx \quad = \quad h\int_0^1 xdx \quad = \quad \frac{h}{2}.$$

2.

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\Psi_{i,n}(x)dx = \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} \Psi_{i,n}(x)dx = \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} \Psi(2^n x - i)dx.$$

$$= h\int_0^1 \Psi(x)dx = h\left[\frac{1}{2(1-\sigma)}\right] = \left[\frac{h}{2(1-\sigma)}\right].$$

3.

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\frac{x-x_i}{x_{i+1}-x_i}dx \quad = \quad \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} \frac{x-x_i}{h}dx \quad = \quad h\int_0^1 xdx \quad = \quad \frac{h}{2}.$$

Therefore, the source vector (F) over the subinterval I_i is as follows:

$$F_i = \begin{bmatrix} \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\frac{x_{i+1}-x}{x_{i+1}-x_i}dx \\ \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\Psi_{i,n}(x)dx \\ \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\frac{x-x_i}{x_{i+1}-x_i}dx \end{bmatrix} = \begin{bmatrix} \frac{h}{2} \\ \left[\frac{h}{2(1-\sigma)}\right] \\ \frac{h}{2} \end{bmatrix}.$$

The entries in the assembled source matrix after imposing boundary conditions are as follows:

$$F = \begin{bmatrix} \frac{h}{2(1-\sigma)} \\ \frac{h}{2}+\frac{h}{2} \\ \frac{h}{2(1-\sigma)} \\ \frac{h}{2}+\frac{h}{2} \\ \frac{h}{2(1-\sigma)} \\ \frac{h}{2}+\frac{h}{2} \\ \frac{h}{2(1-\sigma)} \end{bmatrix} = \begin{bmatrix} \frac{h}{2(1-\sigma)} \\ h \\ \frac{h}{2(1-\sigma)} \\ h \\ \frac{h}{2(1-\sigma)} \\ h \\ \frac{h}{2(1-\sigma)} \end{bmatrix} = \begin{bmatrix} 0.09375 \\ 0.25 \\ 0.09375 \\ 0.25 \\ 0.09375 \\ 0.25 \\ 0.09375 \end{bmatrix}.$$

The system of linear equations after imposing boundary conditions is as follows:

$$\begin{bmatrix} 21.333 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 21.333 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 8 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 21.333 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 21.333 \end{bmatrix} \begin{bmatrix} \Delta u_{\frac{1}{2}} \\ u_1 \\ \Delta u_{\frac{3}{2}} \\ u_2 \\ \Delta u_{\frac{5}{2}} \\ u_3 \\ \Delta u_{\frac{7}{2}} \end{bmatrix} = \begin{bmatrix} 0.09375 \\ 0.25 \\ 0.09375 \\ 0.25 \\ 0.09375 \\ 0.25 \\ 0.09375 \end{bmatrix}.$$

The solution (U) of above system of equations is obtained by the Gaussian elimination method.

$$U = \begin{bmatrix} \Delta u_{\frac{1}{2}} \\ u_1 \\ \Delta u_{\frac{3}{2}} \\ u_2 \\ \Delta u_{\frac{5}{2}} \\ u_3 \\ \Delta u_{\frac{7}{2}} \end{bmatrix} = \begin{bmatrix} 0.00439453 \\ 0.09375 \\ 0.00439453 \\ 0.125 \\ 0.00439453 \\ 0.09375 \\ 0.00439453 \end{bmatrix}.$$

where u_i denotes the value of u_h at a global node i. That is, $u_i = u_h(x_i)$ for $i = 0, 1, 2, 3$ and $\Delta u_{i+\frac{1}{2}}$ for $i = 0, 1, 2, 3$ denote the additive increments.

Therefore, the value of u_h at $x = x_{i+\frac{1}{2}}$ for $i = 1, 2, 3$ is calculated from $\Delta u_{i+\frac{1}{2}}$ using formula

$$u_h(x_{i+\frac{1}{2}}) = \frac{u_i + u_{i+1}}{2} + \Delta u_{i+\frac{1}{2}}.$$

Thus,

$$u_h(x_{\frac{1}{2}}) = 0.05126953 \quad u_h(x_{\frac{3}{2}}) = 0.11376953$$

$$u_h(x_{\frac{5}{2}}) = 0.11376953 \quad u_h(x_{\frac{7}{2}}) = 0.05126953$$

and

$$u_h = \begin{bmatrix} 0 \\ 0.05126953 \\ 0.09375 \\ 0.11376953 \\ 0.125 \\ 0.11376953 \\ 0.09375 \\ 0.05126953 \\ 0 \end{bmatrix}.$$

Further, the exact solution (u) is obtained using formula $u = \frac{x}{2} - \frac{x^2}{2}$

Table (5.1) compares the values of the approximate solution (u_h) and the exact solution (u) for $\sigma = 0.25$.

Exact solution(u)	Approximate Solution(u_h)
0	0
0.0546875	0.05126953
0.09375	0.09375
0.1171875	0.11376953
0.125	0.125
0.1171875	0.11376953
0.09375	0.09375
0.0546875	0.05126953
0	0

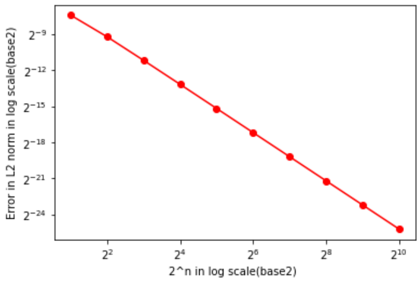
Table 5.1: Values of the approximate solution (u_h) and the exact solution (u) for $\sigma = 0.25$.

Error Estimates:

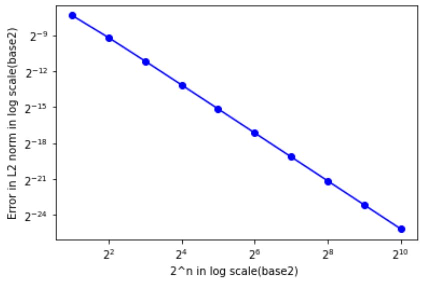
Error in the L^2 norm: The error in the L^2 norm for different values of σ over the finite element mesh having 2^n elements for $n = 1, 2, ... 10$ is listed in table (5.2) and represented graphically in figure (5.1).

2^n	$\sigma = 0$	$\sigma = 0.25$	$\sigma = 0.35$	$\sigma = 0.45$
2	7.81250000000000E-03	5.69428895980986E-03	6.13624761873523E-03	1.00132211538461E-02
4	2.07160189800746E-03	1.64001816925590E-03	1.70482629883934E-03	3.54851959875516E-03
8	5.25044266787268E-04	4.23173568173015E-04	4.38526766830438E-04	8.89829912091995E-04
16	1.31703769696291E-04	1.06611767525420E-04	1.10509845744672E-04	2.22716041209761E-04
32	3.29535532821468E-05	2.67040200421178E-05	2.76903114100869E-05	5.57042624711434E-05
64	8.24011309577248E-06	6.67919630795449E-06	6.92714805730644E-06	1.39286152115573E-05
128	2.06013605850696E-06	1.66999851612620E-06	1.73212696912371E-06	3.48241980966117E-06
256	5.15040750499890E-07	4.17512094078282E-07	4.33057680814105E-07	8.70633442045876E-07
512	1.28760607702904E-07	1.04378803367222E-07	1.08266448678938E-07	2.17661465682749E-07
1024	3.21901792463067E-08	2.60947438852053E-08	2.70667786150476E-08	5.44157244496846E-08

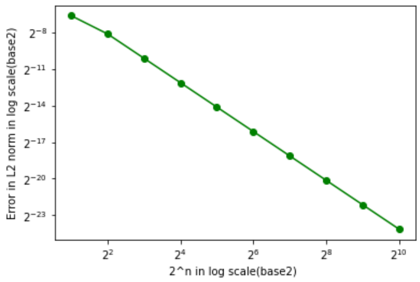
Table 5.2: Error in the L^2 norm for different values of $\sigma \in [0, \frac{1}{2})$.



(a) $\sigma = 0.25$



(b) $\sigma = 0.35$



(c) $\sigma = 0.45$

Figure 5.1: Rate of convergence in L^2 norm for different values of $\sigma \in [0, \frac{1}{2})$.

It is observed that:

- when the number of elements is doubled, the error in the L^2 norm decreased approximately by a factor of four.
- the error in L^2 norm is minimum for $\sigma = \frac{1}{4} = 0.25$ since for $\sigma = \frac{1}{4}$ the fractal function $\Psi(x)$ is a quadratic function $4x(1-x)$ and is C^2-smooth.

5.2.2 Solution to the Poisson equation $-\frac{d^2u}{dx^2} = x$ with Dirichlet boundary conditions

Assume that

$$f(x) = x$$

in the Poisson equation represented by equation (5.1). Then the Poisson equation is stated as below:

$$-\frac{d^2u}{dx^2} = x, \qquad \text{for } x \in (0,1),$$

subject to boundary conditions,

$$u(0) = 0 \quad u(1) = 0.$$

Let $n = 2$. We discretize the interval $(0,1)$ in $2^n = 2^2 = 4$ subintervals.

To determine the matrix of the system, we find element matrices and assemble them.

For $\sigma = 0.35$, the coefficient matrix over each element is as follows:

$$K_{ij} = 4\begin{bmatrix} 1 & 0 & -1 \\ 0 & 7.8431372549019605 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

The global stiffness matrix after imposing boundary conditions is as follows:

$$A = \begin{bmatrix} 31.37254902 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 31.37254902 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 8 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 31.37254902 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 31.37254902 \end{bmatrix}.$$

To find the entries of the source vector, the self-similarity property of the fractal functions is used to evaluate the integrals of the fractal function. The entries of the source vector over the subinterval I_i are calculated as follows.

1.
$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\frac{x_{i+1}-x}{x_{i+1}-x_i}dx = \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} x\left(\frac{x_{i+1}-x}{h}\right)dx.$$

To find value of the integral, let

$$\frac{x_{i+1}-x}{h} = t.$$

Then, $dx = -h\,dt$ and $x = x_{i+1} - ht$.

Therefore,

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\frac{x_{i+1}-x}{x_{i+1}-x_i}dx = \int_0^1 t\,(x_{i+1}-ht)\,h\,dt$$
$$= \int_0^1 x_{i+1}\,h\,t\,dt - \int_0^1 h^2\,t^2\,dt = \frac{(i+1)h^2}{2} - \frac{h^3}{3}$$
$$= \frac{h^2(3i+1)}{6}.$$

2.
$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\Psi_{i,n}(x)dx = \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} x\,\Psi_{i,n}(x)dx$$
$$= \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} x\,\Psi(2^n x - i)dx.$$

To find value of the integral, let $2^n x - i = t$.

Then, $x = \frac{t+i}{2^n} = h\,(t+i)$ and $dx = h\,dt$.

Therefore,

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\Psi_{i,n}(x)dx = \int_0^1 h\,(t+i)\Psi(t)\,h\,dt = h^2\int_0^1 (t+i)\Psi(t)dt$$

$$= h^2\int_0^1 t\,\Psi(t)\,dt + h^2\int_0^1 i\,\Psi(t)\,dt = h^2\left[\frac{2-\sigma}{8(1-\sigma)(1-\frac{\sigma}{2})} + \frac{i}{2(1-\sigma)}\right]$$

$$= h^2\left[\frac{(2+4\,i)-\sigma(1+2\,i)}{8(1-\sigma)(1-\frac{\sigma}{2})}\right].$$

3.

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\frac{x-x_i}{x_{i+1}-x_i}dx = \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} \frac{x-x_i}{h}dx.$$

To find value of the integral, let

$$\frac{x-x_i}{h} = t.$$

Then, $dx = h\,dt$ and $x = x_i + ht$.

Therefore,

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\frac{x-x_i}{x_{i+1}-x_i}dx = \int_0^1 (h\,t + x_i)t\,h\,dt$$

$$= \int_0^1 x_i\,h\,t\,dt + \int_0^1 h^2\,t^2\,dt = \frac{i\,h^2}{2} + \frac{h^2}{3}$$

$$= \frac{h^2(3i+2)}{6}.$$

Therefore, the source vector (F) over the subinterval I_i is as follows:

$$F_i = \begin{bmatrix} \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\frac{x_{i+1}-x}{x_{i+1}-x_i}dx \\ \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\Psi_{i,n}(x)dx \\ \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\frac{x-x_i}{x_{i+1}-x_i}dx \end{bmatrix} = \begin{bmatrix} \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} x\,\frac{x_{i+1}-x}{x_{i+1}-x_i}dx \\ \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} x\,\Psi_{i,n}(x)dx \\ \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} x\,\frac{x-x_i}{x_{i+1}-x_i}dx \end{bmatrix} = \begin{bmatrix} \frac{h^2(3i+1)}{6} \\ h^2\left[\frac{(2+4\,i)-\sigma(1+2\,i)}{8(1-\sigma)(1-\frac{\sigma}{2})}\right] \\ \frac{h^2(3i+2)}{6} \end{bmatrix}.$$

The entries in the assembled source matrix after imposing boundary conditions are as follows:

$$F = \begin{bmatrix} h^2\left[\frac{(2+4\,i)-\sigma(1+2\,i)}{8(1-\sigma)(1-\frac{\sigma}{2})}\right] \\ \frac{h^2(3i+2)}{6}+\frac{h^2(3i+1)}{6} \\ h^2\left[\frac{(2+4\,i)-\sigma(1+2\,i)}{8(1-\sigma)(1-\frac{\sigma}{2})}\right] \\ \frac{h^2(3i+2)}{6}+\frac{h^2(3i+1)}{6} \\ h^2\left[\frac{(2+4\,i)-\sigma(1+2\,i)}{8(1-\sigma)(1-\frac{\sigma}{2})}\right] \\ \frac{h^2(3i+2)}{6}+\frac{h^2(3i+1)}{6} \\ h^2\left[\frac{(2+4\,i)-\sigma(1+2\,i)}{8(1-\sigma)(1-\frac{\sigma}{2})}\right] \end{bmatrix} = \begin{bmatrix} h^2\left[\frac{(2+4\,i)-\sigma(1+2\,i)}{8(1-\sigma)(1-\frac{\sigma}{2})}\right] \\ \frac{h^2(2i+1)}{2} \\ h^2\left[\frac{(2+4\,i)-\sigma(1+2\,i)}{8(1-\sigma)(1-\frac{\sigma}{2})}\right] \\ \frac{h^2(2i+1)}{2} \\ h^2\left[\frac{(2+4\,i)-\sigma(1+2\,i)}{8(1-\sigma)(1-\frac{\sigma}{2})}\right] \\ \frac{h^2(2i+1)}{2} \\ h^2\left[\frac{(2+4\,i)-\sigma(1+2\,i)}{8(1-\sigma)(1-\frac{\sigma}{2})}\right] \end{bmatrix}.$$

The system of linear equations after imposing boundary conditions is as follows:

$$\begin{bmatrix} 31.3725 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 31.3725 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 8 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 31.3725 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 31.3725 \end{bmatrix} \begin{bmatrix} \Delta u_{\frac{1}{2}} \\ u_1 \\ \Delta u_{\frac{3}{2}} \\ u_2 \\ \Delta u_{\frac{5}{2}} \\ u_3 \\ \Delta u_{\frac{7}{2}} \end{bmatrix} = \begin{bmatrix} 0.26442308 \\ 0.125 \\ 0.3125 \\ 0.1875 \\ 0.36057692 \\ 0.25 \\ 0.40865385 \end{bmatrix}.$$

The solution (U) of the above system of equations is obtained by the Gaussian elimination method.

$$U = \begin{bmatrix} \Delta u_{\frac{1}{2}} \\ u_1 \\ \Delta u_{\frac{3}{2}} \\ u_2 \\ \Delta u_{\frac{5}{2}} \\ u_3 \\ \Delta u_{\frac{7}{2}} \end{bmatrix} = \begin{bmatrix} 0.00842849 \\ 0.0625 \\ 0.00996094 \\ 0.09375 \\ 0.01149339 \\ 0.078125 \\ 0.01302584 \end{bmatrix}$$

where u_i denotes the value of u_h at a global node i. That is, $u_i = u_h(x_i)$ for $i = 0, 1, 2, 3$ and $\Delta u_{i+\frac{1}{2}}$ for $i = 0, 1, 2, 3$ denote the additive increments.

Therefore, the value of u_h at $x = x_{i+\frac{1}{2}}$ for $i = 1, 2, 3$ is calculated from $\Delta u_{i+\frac{1}{2}}$ using formula

$$u_h(x_{i+\frac{1}{2}}) = \frac{u_i + u_{i+1}}{2} + \Delta u_{i+\frac{1}{2}}.$$

Thus,

$$u_h(x_{\frac{1}{2}}) = 0.03967849 \quad u_h(x_{\frac{3}{2}}) = 0.08808594$$

$$u_h(x_{\frac{5}{2}}) = 0.09743089 \quad u_h(x_{\frac{7}{2}}) = 0.05208834$$

and

$$u_h = \begin{bmatrix} 0 \\ 0.03967849 \\ 0.0625 \\ 0.08808594 \\ 0.09375 \\ 0.09743089 \\ 0.078125 \\ 0.05208834 \\ 0 \end{bmatrix}.$$

Further, the exact solution is obtained using formula

$$u = \frac{x}{2} - \frac{x^2}{2}.$$

Table (5.3) compares the values of the approximate solution (u_h) and the exact solution (u) for $\sigma = 0.35$.

Exact solution (u)	Approximate Solution (u_h)
0	0
0.0205078125	0.03967849
0.0390625	0.0625
0.0537109375	0.08808594
0.0625	0.09375
0.0634765625	0.09743089
0.0546875	0.078125
0.0341796875	0.05208834
0	0

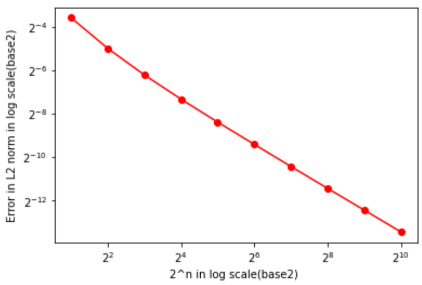
Table 5.3: Values of the approximate solution u_h and the exact solution u for $\sigma = 0.35$.

Error Estimates:

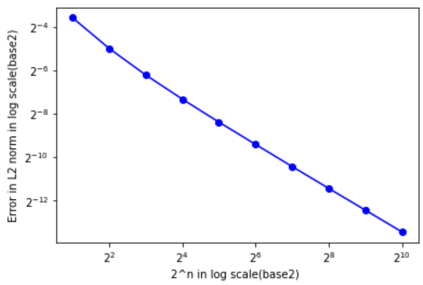
Error in the L^2 norm: The error in the L^2 norm for different values of σ over the finite element mesh having 2^n elements for $n = 1, 2, \ldots 10$ is listed in table (5.4) and represented graphically in figure (5.2).

2^n	$\sigma = 0.0$	$\sigma = 0.25$	$\sigma = 0.35$	$\sigma = 0.45$
2	1.91366386154935E-02	1.91366386154935E-02	1.79082505117203E-02	1.64858422073342E-02
4	1.09618868753142E-02	1.09618868753142E-02	1.06024161471505E-02	9.97417425379824E-03
8	5.64962718037409E-03	5.64962718037409E-03	5.55601303077984E-03	5.37702890258271E-03
16	2.84575393306746E-03	2.84575393306746E-03	2.82201242102854E-03	2.77503150967745E-03
32	1.42549014957353E-03	1.42549014957353E-03	1.41951993126662E-03	1.40752197989843E-03
64	7.13071587733554E-04	7.13071587733554E-04	7.11575124644722E-04	7.08545586974459E-04
128	3.56576603778381E-04	3.56576603778381E-04	3.56202026927898E-04	3.55440979084511E-04
256	1.78293402998426E-04	1.78293402998426E-04	1.78199702857006E-04	1.78008988664750E-04
512	8.91473391356376E-05	8.91473391356376E-05	8.91239072154556E-05	8.90761724913701E-05
1024	4.45737492604231E-05	4.45737492604231E-05	4.45678904264053E-05	4.45559497453401E-05

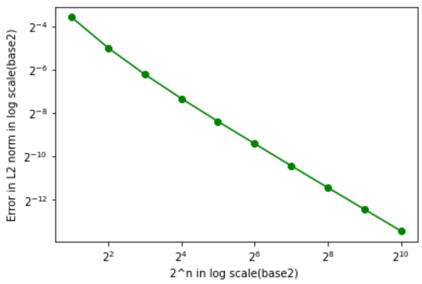
Table 5.4: Error in the L^2 norm for different values of $\sigma \in [0, \frac{1}{2})$.



(a) $\sigma = 0.25$



(b) $\sigma = 0.35$



(c) $\sigma = 0.45$

Figure 5.2: Rate of convergence in L^2 norm for different values of $\sigma \in [0, \frac{1}{2})$.

It is observed that

- when the number of elements is doubled, the Error in the L^2 norm decreased approximately by a factor of two.

- the error in L^2 norm depends on the scaling factor σ. It decreases as σ is increased from $\sigma = 0$ to $\sigma = 0.45$.

5.2.3 Solution to the Poisson equation $-\frac{d^2u}{dx^2} = e^x$ with Dirichlet boundary conditions

Assume that

$$f(x) = e^x$$

in the Poisson equation represented by equation (5.1). Then the Poisson equation is stated as below:

$$-\frac{d^2u}{dx^2} = e^x, \quad \text{for } x \in (0,1)$$

subject to homogeneous Dirichlet boundary conditions,

$$u(0) = 0 \quad u(1) = 0.$$

Let $n = 2$. We discretize the interval $(0,1)$ in $2^2 = 4$ subintervals.

To determine matrix of the system, we find element matrices and assemble them. For $\sigma = 0.45$, the coefficient matrix over each element is as follows:

$$K_{ij} = 4 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0.53333 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

The global stiffness matrix after imposing boundary conditions is as follows:

$$A = \begin{bmatrix} 84.2105 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 84.2105 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 8 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 84.2105 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 84.2105 \end{bmatrix}.$$

The explicit integral of the fractal function is unattainable hence to find the entries of the source vector, the self-similarity property of the fractal function and the trapezoidal rule for the numerical integration are used to evaluate the integrals of fractal functions. The entries of the source vector over the subinterval I_i are calculated as follows.

For $f(x) = e^x$,

1.
$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\frac{x_{i+1}-x}{x_{i+1}-x_i}dx = \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} e^x\left(\frac{x_{i+1}-x}{h}\right)dx.$$

To find value of the integral, let $\frac{x_{i+1}-x}{h} = t$.

Then, $dx = -h\,dt$ and $x = x_{i+1} - ht$.

Therefore,
$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\frac{x_{i+1}-x}{x_{i+1}-x_i}dx = \int_0^1 t\,e^{(x_{i+1}-ht)}\,h\,dt$$

$$= e^{x_{i+1}}\,h\int_0^1 e^{-ht}\,t\,dt = e^{(i+1)h}\,h\left[t\,\frac{e^{-ht}}{-h} - \frac{e^{-ht}}{h^2}\right]_0^1 = e^{(i+1)h}\left[\frac{1}{h} - \frac{e^{-h}}{h} - e^{-h}\right].$$

2.
$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\frac{x-x_i}{x_{i+1}-x_i}dx = \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} e^x\,\frac{x-x_i}{h}dx.$$

To find the value of the integral, let

$$\frac{x-x_i}{h} = t.$$

Then, $dx = h\,dt$ and $x = x_i + ht$.

Therefore,

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\frac{x - x_i}{x_{i+1} - x_i} dx = \int_0^1 e^{(h\,t + x_i)}\; t\,h\,dt$$

$$= e^{ih}\,h \int_0^1 e^t\; t\,dt = e^{ih}\,h \left[\frac{t\,e^{ht}}{h} - \frac{e^{ht}}{h^2}\right]_0^1 = e^{ih}\left[e^h - \frac{e^h}{h} + \frac{1}{h}\right].$$

3.

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\Psi_{i,n}(x)dx = \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} e^x\,\Psi_{i,n}(x)dx = \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} e^x\,\Psi(2^n x - i)dx.$$

To find value of the integral, let $2^n x - i = t$. Then, $x = \frac{t+i}{2^n} = h\,(t+i)$ and $dx = h\,dt$.

Therefore,

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\Psi_{i,n}(x)dx = \int_0^1 e^{h\,(t+i)}\Psi(t)\,h\,dt = h\,e^{ih}\int_0^1 e^{ht}\,\Psi(t)\,dt.$$

Using definition of $\Psi(t)$, one has

$$= e^{ih}\int_0^{\frac{1}{2}} e^{ht}\left[\sigma\,\Psi(2t) + 2t\right]\,h\,dt + e^{ih}\int_{\frac{1}{2}}^1 e^{ht}\left[\sigma\Psi(2t-1) + (2-2t)\right]\,h\,dt$$

$$= h\,e^{ih}\left[\int_0^{\frac{1}{2}} e^{ht}\sigma\,\Psi(2t)dt + \int_0^{\frac{1}{2}} e^{ht}\,2t\,dt + \int_{\frac{1}{2}}^1 e^{ht}\,\sigma\Psi(2t-1)dt + \int_{\frac{1}{2}}^1 e^{ht}\,(2-2t)\,dt\right].$$

Using the trapezoidal rule of numerical integration for $\int_0^{\frac{1}{2}} e^{ht}\sigma\,\Psi(2t)dt$ and $\int_{\frac{1}{2}}^1 e^{ht}\,\sigma\Psi(2t-1)dt$, one has,

$$= \frac{\sigma}{8}\left[e^{\frac{h}{8}}\,\Psi(\frac{1}{4}) + e^{\frac{h}{4}}\,\Psi(\frac{1}{2}) + [e^{\frac{3h}{8}}\,\Psi(\frac{3}{4})\right] + \frac{\sigma}{8}\left[e^{\frac{5h}{8}}\,\Psi(\frac{1}{4}) + e^{\frac{6h}{8}}\,\Psi(\frac{1}{2}) + [e^{\frac{7h}{8}}\,\Psi(\frac{3}{4})\right]$$
$$+ \frac{1}{h}\left[e^{\frac{h}{2}} - \frac{2\,e^{\frac{h}{2}}}{h} + \frac{2}{h}\right] + \frac{1}{h}\left[\frac{e^h}{h} - e^{\frac{h}{2}} - \frac{2\,e^{\frac{h}{2}}}{h}\right].$$

Since $\Psi(\frac{1}{2}) = 1$ and $\Psi(\frac{1}{4}) = \Psi(\frac{3}{4}) = \sigma + \frac{1}{2}$, it follows that,

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} e^x \Psi_{i,n}(x)dx = e^{ih}\, h\, \frac{\sigma}{8} \left[\left(e^{\frac{h}{8}} + e^{\frac{3h}{8}} + e^{\frac{5h}{8}} + e^{\frac{7h}{8}} \right) \left(\sigma + \frac{1}{2} \right) + (e^{\frac{h}{4}} + e^{\frac{3h}{4}}) \right] + e^{ih} \left[\frac{-4\, e^{\frac{h}{2}}}{h} + \frac{2}{h} + \frac{2e^h}{h} \right].$$

Therefore, the source vector (F) over the subinterval I_i is as follows:

$$F_i = \begin{bmatrix} \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x) \frac{x_{i+1}-x}{x_{i+1}-x_i} dx \\ \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\Psi_{i,n}(x)dx \\ \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x) \frac{x-x_i}{x_{i+1}-x_i} dx \end{bmatrix} = \begin{bmatrix} \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} e^x \frac{x_{i+1}-x}{x_{i+1}-x_i} dx \\ \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} e^x\, \Psi_{i,n}(x)dx \\ \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} e^x \frac{x-x_i}{x_{i+1}-x_i} dx \end{bmatrix}.$$

$$F_i = \begin{bmatrix} e^{(i+1)h} \left[\frac{1}{h} - \frac{e^{-h}}{h} - e^{-h} \right] \\ e^{ih}\, h\, \frac{\sigma}{8} \left[\left(e^{\frac{h}{8}} + e^{\frac{3h}{8}} + e^{\frac{5h}{8}} + e^{\frac{7h}{8}} \right) \left(\sigma + \frac{1}{2} \right) + (e^{\frac{h}{4}} + e^{\frac{3h}{4}}) \right] + e^{ih} \left[\frac{-4\, e^{\frac{h}{2}}}{h} + \frac{2}{h} + \frac{2e^h}{h} \right] \\ e^{ih} \left[e^h - \frac{e^h}{h} + \frac{1}{h} \right] \end{bmatrix}.$$

The entries in the assembled source matrix after imposing boundary conditions are as follows:

$$F = \begin{bmatrix} e^{ih}\, h\, \frac{\sigma}{8} \left[\left(e^{\frac{h}{8}} + e^{\frac{3h}{8}} + e^{\frac{5h}{8}} + e^{\frac{7h}{8}} \right) \left(\sigma + \frac{1}{2} \right) + (e^{\frac{h}{4}} + e^{\frac{3h}{4}}) \right] + e^{ih} \left[\frac{-4\, e^{\frac{h}{2}}}{h} + \frac{2}{h} + \frac{2e^h}{h} \right] \\ e^{ih} \left[e^h - \frac{e^h}{h} + \frac{1}{h} \right] + e^{(i+1)h} \left[\frac{1}{h} - \frac{e^{-h}}{h} - e^{-h} \right] \\ e^{ih}\, h\, \frac{\sigma}{8} \left[\left(e^{\frac{h}{8}} + e^{\frac{3h}{8}} + e^{\frac{5h}{8}} + e^{\frac{7h}{8}} \right) \left(\sigma + \frac{1}{2} \right) + (e^{\frac{h}{4}} + e^{\frac{3h}{4}}) \right] + e^{ih} \left[\frac{-4\, e^{\frac{h}{2}}}{h} + \frac{2}{h} + \frac{2e^h}{h} \right] \\ e^{ih} \left[e^h - \frac{e^h}{h} + \frac{1}{h} \right] + e^{(i+1)h} \left[\frac{1}{h} - \frac{e^{-h}}{h} - e^{-h} \right] \\ e^{ih}\, h\, \frac{\sigma}{8} \left[\left(e^{\frac{h}{8}} + e^{\frac{3h}{8}} + e^{\frac{5h}{8}} + e^{\frac{7h}{8}} \right) \left(\sigma + \frac{1}{2} \right) + (e^{\frac{h}{4}} + e^{\frac{3h}{4}}) \right] + e^{ih} \left[\frac{-4\, e^{\frac{h}{2}}}{h} + \frac{2}{h} + \frac{2e^h}{h} \right] \\ e^{ih} \left[e^h - \frac{e^h}{h} + \frac{1}{h} \right] + e^{(i+1)h} \left[\frac{1}{h} - \frac{e^{-h}}{h} - e^{-h} \right] \\ e^{ih}\, h\, \frac{\sigma}{8} \left[\left(e^{\frac{h}{8}} + e^{\frac{3h}{8}} + e^{\frac{5h}{8}} + e^{\frac{7h}{8}} \right) \left(\sigma + \frac{1}{2} \right) + (e^{\frac{h}{4}} + e^{\frac{3h}{4}}) \right] + e^{ih} \left[\frac{-4\, e^{\frac{h}{2}}}{h} + \frac{2}{h} + \frac{2e^h}{h} \right] \end{bmatrix}.$$

The system of linear equations after imposing boundary conditions is as follows:

$$\begin{bmatrix} 84.2105 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 84.2105 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 8 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 84.2105 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 84.2105 \end{bmatrix} \begin{bmatrix} \Delta u_{\frac{1}{2}} \\ u_1 \\ \Delta u_{\frac{3}{2}} \\ u_2 \\ \Delta u_{\frac{5}{2}} \\ u_3 \\ \Delta u_{\frac{7}{2}} \end{bmatrix} = \begin{bmatrix} 0.81834826 \\ 0.36469585 \\ 1.05077997 \\ 0.46827875 \\ 1.34922819 \\ 0.60128181 \\ 1.73244329 \end{bmatrix}.$$

The solution of above system of equations is obtained by the Gaussian elimination. Therefore, the matrix

$$U = \begin{bmatrix} \Delta u_{\frac{1}{2}} \\ u_1 \\ \Delta u_{\frac{3}{2}} \\ u_2 \\ \Delta u_{\frac{5}{2}} \\ u_3 \\ \Delta u_{\frac{7}{2}} \end{bmatrix} = \begin{bmatrix} 0.00971789 \\ 0.16449543 \\ 0.01247801 \\ 0.23781689 \\ 0.01602208 \\ 0.19406867 \\ 0.02057276 \end{bmatrix}.$$

where u_i denotes the value of u_h at a global node i. That is, $u_i = u_h(x_i)$ for $i = 0, 1, 2, 3$ and $\Delta u_{i+\frac{1}{2}}$ for $i = 0, 1, 2, 3$ denote the additive increments.

Therefore, the value of u_h at $x = x_{i+\frac{1}{2}}$ for $i = 1, 2, 3$ is calculated from $\Delta u_{i+\frac{1}{2}}$ using formula

$$u_h(x_{i+\frac{1}{2}}) = \frac{u_i + u_{i+1}}{2} + \Delta u_{i+\frac{1}{2}}.$$

Thus,

$$u_h(x_{\frac{1}{2}}) = 0.0919656 \quad u_h(x_{\frac{3}{2}}) = 0.21363417$$

$$u_h(x_{\frac{5}{2}}) = 0.23196487 \quad u_h(x_{\frac{7}{2}}) = 0.1176071$$

and

$$u_h = \begin{bmatrix} 0 \\ 0.0919656 \\ 0.16449543 \\ 0.21363417 \\ 0.23781689 \\ 0.23196487 \\ 0.19406867 \\ 0.1176071 \\ 0 \end{bmatrix}.$$

Further the exact solution (u) is obtained using formula $u = (1 - e^x) + (e - 1) * x)$.

The following table (5.5) compares the values of the approximate solution (u_h) and the exact solution (u) for $\sigma = 0.45$.

Exact solution (u)	Approximate Solution (u_h)
0	0
0.08163677549055431	0.0919656
0.14554504042701988	0.16449543
0.1893642710539407	0.21363417
0.21041964352939435	0.23781689
0.2056801853546808	0.23196487
0.17171135473160914	0.19406867
0.10462130593456642	0.1176071
0	0

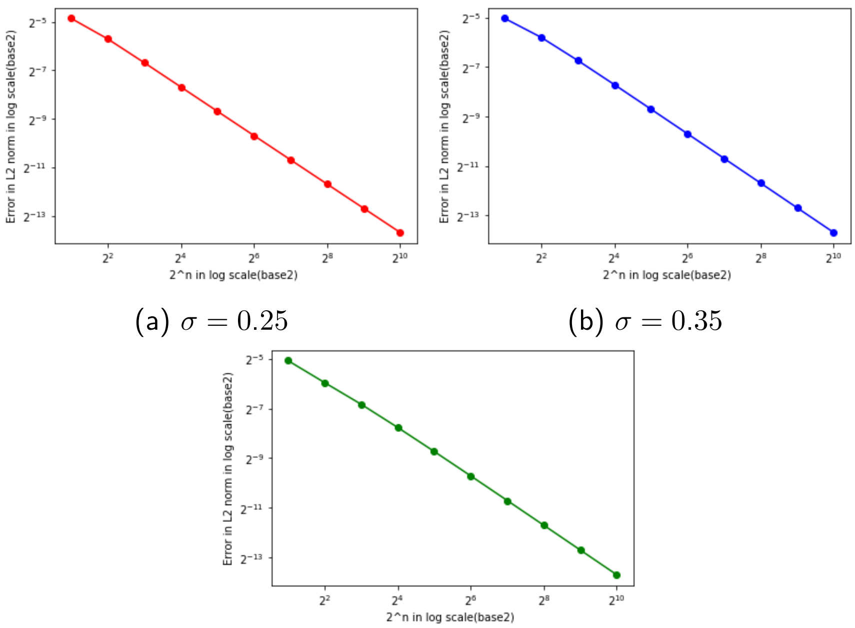
Table 5.5: Values of the approximate solution u_h and the exact solution u for $\sigma = 0.45$.

Error Estimates:

Error in the L^2 norm: The error in the L^2 norm for different values of σ over the finite element mesh having 2^n elements for $n = 1, 2, \dots 10$ is listed in table (5.6) and represented graphically in figure (5.3).

2^n	$\sigma = 0.0$	$\sigma = 0.25$	$\sigma = 0.35$	$\sigma = 0.45$
2	3.48871613434091E-02	3.45689164035987E-02	3.05398711084345E-02	2.97443162780337E-02
4	1.93009821781688E-02	1.92094696752423E-02	1.79227325363631E-02	1.61273148178102E-02
8	8.81243781061457E-03	9.73096798428947E-03	9.38838646262191E-03	8.81243781061457E-03
16	4.86391858787689E-03	4.85796022791724E-03	4.77030953958756E-03	4.61380969902463E-03
32	2.42393602008081E-03	2.42244293914433E-03	2.40031250451123E-03	2.35979637588661E-03
64	1.20939038460854E-03	1.20901684996735E-03	1.20345894977332E-03	1.19316576812038E-03
128	6.03980039935934E-04	6.03886633962749E-04	6.02494114387850E-04	5.99900878516157E-04
256	3.01802432630233E-04	3.01779079018707E-04	3.01430575757695E-04	3.00779809439733E-04
512	1.50853221609713E-04	1.50847382983955E-04	1.50760210944353E-04	1.50597214367949E-04
1024	7.54144931551141E-05	7.54130334734484E-05	7.53912347133224E-05	7.53504475794685E-05

Table 5.6: Error in the L^2 norm for different values of $\sigma \in [0, \frac{1}{2})$.



(a) $\sigma = 0.25$

(b) $\sigma = 0.35$

(c) $\sigma = 0.45$

Figure 5.3: Rate of convergence in L^2 norm for different values of $\sigma \in [0, \frac{1}{2})$.

It is observed that

- when the number of elements are doubled, the error in the L^2 norm decreased by a factor of two.
- the error in L^2 norm depends on the scaling factor σ. It decreases as σ is increased from $\sigma = 0$ to $\sigma = 0.45$.

5.2.4 Solution to the differential equation $-\frac{d^2u}{dx^2} + u = cos(\pi x)$ with Dirichlet boundary conditions

The differential equation is stated as below:

$$-\frac{d^2u}{dx^2} + u = cos(\pi x), \quad \text{for } x \in (0, 1) \tag{5.4}$$

Subject to boundary conditions,

$$u(0) = 0 \quad u(1) = 0.$$

The differential equation represented by equation (5.4) is reformulated in the variational formulation as follows.

Multiplying the equation (5.4) by a test function $v(x) \in H_0^1(0, 1)$,

$$\int_0^1 \left[-\frac{d^2u}{dx^2}(x) + u(x)\right] v(x)\, dx = \int_0^1 cos(\pi x) v(x)\, dx.$$

Integrating by parts,

$$-\left[\frac{du}{dx} v(x)\right]_0^1 + \int_0^1 \frac{du}{dx}\frac{dv}{dx}\, dx + \int_0^1 u(x)\, v(x)\, dx = \int_0^1 cos(\pi x) v(x)\, dx.$$

Since, $v(0) = 0$ and $v(1) = 0$, one has

$$\int_0^1 \frac{du}{dx}\frac{dv}{dx}\, dx + \int_0^1 u(x)\, v(x)\, dx = \int_0^1 cos(\pi x) v(x)\, dx.$$

$$a(u, v) + (u.v) = f(v) \tag{5.5}$$

where,

$$a(u, v) = \int_0^1 \frac{du}{dx}\frac{dv}{dx}\, dx,$$

$$(u, v) = \int_0^1 u(x)\, v(x) dx$$

and

$$f(v) = \int_0^1 f(x)\, v(x)\, dx.$$

We use the finite dimensional subspace of continuous fractal functions $V^h_{2^n}$ as approximation space in the Ritz Galerkin approximation. Let $n = 2$ and the interval $(0, 1)$ is discretized into $2^n = 2^2 = 4$ equal subintervals. To determine matrix of the system, we find element matrices and assemble them. The coefficient matrix over each element is calculated below.

For $\sigma = 0.25$, the matrix A is as follows:

$$A = a(u, v) = 4 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0.53333 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

The matrix $A^{'}$ is as follows:

$$A^{'} = (u, v) = \begin{bmatrix} \frac{h}{6} & \frac{h(2-\sigma)}{8(1-\sigma)(1-\frac{\sigma}{2})} & \frac{h}{3} \\ \frac{h(2-\sigma)}{8(1-\sigma)(1-\frac{\sigma}{2})} & \frac{h(2+\sigma)}{6(1-\sigma)^2(1+\sigma)} & \frac{h(2-\sigma)}{8(1-\sigma)(1-\frac{\sigma}{2})} \\ \frac{h}{3} & \frac{h(2-\sigma)}{8(1-\sigma)(1-\frac{\sigma}{2})} & \frac{h}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 0.08333333333333333 & 0.06380208333333334 & 0.041666666666666664 \\ 0.06380208333333334 & 0.13333333333333333 & 0.06380208333333334 \\ 0.041666666666666664 & 0.06380208333333334 & 0.08333333333333333 \end{bmatrix}.$$

The element matrix K_{ij} is obtained by adding the two element metrices A and $A^{'}$. Thus, the element matrix is as follows:

$$K_{ij} = A + A^{'} = \begin{bmatrix} 1.08333333 & 0.06380208 & -0.95833333 \\ 0.06380208 & 5.46666667 & 0.06380208 \\ -0.95833333 & 0.06380208 & 1.08333333 \end{bmatrix}.$$

The global stiffness matrix A after imposing boundary conditions is as follows:

$$A = \begin{bmatrix} 21.8666 & 0.2552 & 0. & 0. & 0. & 0. & 0. \\ 0.2552 & 8.6666 & 0.2552 & -3.8333 & 0. & 0. & 0. \\ 0. & 0.2552 & 21.8666 & 0.2552 & 0. & 0. & 0. \\ 0. & -3.8333 & 0.2552 & 8.6666 & 0.2552 & -3.8333 & 0. \\ 0. & 0. & 0. & 0.2552 & 21.8666 & 0.2552 & 0. \\ 0. & 0. & 0. & -3.8333 & 0.2552 & 8.6666 & 0.2552 \\ 0. & 0. & 0. & 0. & 0. & 0.2552 & 21.8666 \end{bmatrix}.$$

To find the entries of the source vector, the self-similarity property of the fractal functions is used to evaluate the integrals of the fractal function.

The entries of the source vector over the subinterval I_i are calculated as follows.

1.

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\frac{x_{i+1}-x}{x_{i+1}-x_i}dx = \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} ((1+i)-2^n x)\, cos(\pi\, x)\, dx$$

$$= h\int_0^1 (1-x)\, cos[\pi\, h\,(x+i)]\, dx = h\left\{\left[(1-x)\frac{sin[\pi h(x+i)]}{\pi h}\right] - \left[\frac{cos[\pi h(x+i)]}{\pi^2 h^2}\right]\right\}_0^1$$

$$= -\frac{cos[\pi h(1+i)]}{\pi^2 h} - \frac{sin(\pi hi)}{\pi} + \frac{cos(\pi hi)}{\pi^2 h}.$$

2.

$$Let, I = \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\Psi_{i,n}(x)dx = \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} cos(\pi\, x)\Psi(2^n x - i)dx = h\int_0^1 cos[\pi\, h(x+i)]\Psi(x)dx.$$

Using the self similarity property of fractals, one has,

$$= h\left\{\int_0^{\frac{1}{2}} cos[\pi\, h(x+i)][\sigma\Psi(2x)+2x]dx + \int_{\frac{1}{2}}^1 cos[\pi\, h(x+i)][\sigma\Psi(2x-1)+(2-2x)]dx\right\}$$

$$= h\,\sigma\int_0^{\frac{1}{2}} cos[\pi\, h(x+i)]\,\Psi(2x)dx + h\int_0^{\frac{1}{2}}(2x)\, cos[\pi\, h(x+i)]dx$$
$$+ h\,\sigma\int_{\frac{1}{2}}^1 cos[\pi\, h(x+i)]\,\Psi(2x-1)dx + h\int_{\frac{1}{2}}^1 (2x-2)\, cos[\pi\, h(x+i)]dx.$$

The integrals $\int_0^{\frac{1}{2}} cos[\pi\, h(x{+}i)]\, \Psi(2x) dx$ and $\int_{\frac{1}{2}}^{1} cos[\pi\, h(x{+}i)]\, \Psi(2x{-}1) dx$ are evaluated using the trapezoidal rule of numerical integration. Thus,

$$h\,\sigma \int_0^{\frac{1}{2}} cos[\pi\, h(x{+}i)]\, \Psi(2x) dx = \frac{h\,\sigma}{8}\left(\sigma + \frac{1}{2}\right)\left\{cos\left[\pi\, h(\frac{1}{8}+i)\right] + cos\left[\pi\, h(\frac{3}{8}+i)\right]\right\} + cos\left[\pi\, h(\frac{1}{4}+i)\right].$$

Similarly,

$$h\,\sigma \int_{\frac{1}{2}}^{1} cos[\pi\, h(x{+}i)]\, \Psi(2x{-}1) dx = \frac{h\,\sigma}{8}\left(\sigma + \frac{1}{2}\right)\left\{cos\left[\pi\, h(\frac{5}{8}+i)\right] + cos\left[\pi\, h(\frac{7}{8}+i)\right]\right\} + cos\left[\pi\, h(\frac{3}{4}+i)\right].$$

The integrals $\int_0^{\frac{1}{2}} (2x)\, cos[\pi\, h(x+i)] dx$ and $\int_{\frac{1}{2}}^{1} (2x-2)\, cos[\pi\, h(x+i)] dx$ are evaluated using integration by parts. Thus,

$$h \int_0^{\frac{1}{2}} (2x)\, cos[\pi\, h(x{+}i)] dx = \frac{1}{\pi} sin\left[\pi\, h(\frac{1}{2}+i)\right] + \frac{2}{\pi^2\, h} cos\left[\pi\, h(\frac{1}{2}+i)\right] - \frac{2}{\pi^2\, h} cos(\pi\, hi).$$

Similarly,

$$h \int_{\frac{1}{2}}^{1} (2x-2)\, cos[\pi\, h(x+i)] dx = -\frac{1}{\pi} sin\left[\pi\, h(\frac{1}{2}+i)\right] + \frac{2}{\pi^2\, h} cos\left[\pi\, h(\frac{1}{2}+i)\right] - \frac{2}{\pi^2\, h} cos\left[\pi\, h\,(1+i)\right].$$

Adding four integrals,

$$I = \frac{h\,\sigma}{8}\left(\sigma + \frac{1}{2}\right)\left\{cos\left[\pi\, h(\frac{1}{8}+i)\right] + cos\left[\pi\, h(\frac{3}{8}+i)\right] + cos\left[\pi\, h(\frac{5}{8}+i)\right] + cos\left[\pi\, h(\frac{7}{8}+i)\right]\right\}$$
$$+ \frac{h\,\sigma}{8}\left\{cos\left[\pi\, h(\frac{1}{4}+i)\right] + cos\left[\pi\, h(\frac{3}{4}+i)\right]\right\} + \frac{4}{\pi^2\, h} cos\left[\pi\, h(\frac{1}{2}+i)\right] - \frac{2}{\pi^2\, h} cos(\pi\, h\, i)$$
$$- \frac{2}{\pi^2\, h} cos[\pi\, h\,(1+i)]. \quad (5.6)$$

3.

$$\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\frac{x-x_i}{x_{i+1}-x_i}dx = \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} (2^n x - i)\, cos(\pi\, x)\, dx$$
$$= h\int_0^1 x\, cos[\pi\, h\,(x+i)]\, dx = h\left\{x\,\frac{sin[\pi\, h(x+i)]}{\pi\, h} + \frac{cos[\pi\, h(x+i)]}{\pi^2\, h^2}\right\}_0^1$$
$$= \frac{sin[\pi\, h(1+i)]}{\pi} + \frac{cos[\pi\, h(1+i)]}{\pi^2\, h} - \frac{cos(\pi\, h\, i)}{\pi^2\, h}.$$

Therefore, the source vector (F) over the subinterval I_i is as follows:

$$F_i = \begin{bmatrix} \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\frac{x_{i+1}-x}{x_{i+1}-x_i}dx \\ \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\Psi_{i,n}(x)dx \\ \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(x)\frac{x-x_i}{x_{i+1}-x_i}dx \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{cos[\pi h(1+i)]}{\pi^2 h} - \frac{sin(\pi hi)}{\pi} + \frac{cos(\pi hi)}{\pi^2 h} \\ I \\ \frac{sin[\pi\, h(1+i)]}{\pi} + \frac{cos[\pi\, h(1+i)]}{\pi^2\, h} - \frac{cos(\pi\, h\, i)}{\pi^2\, h} \end{bmatrix}.$$

where,

$$I = \frac{h\,\sigma}{8}\left(\sigma+\frac{1}{2}\right)\left\{cos\left[\pi\, h(\frac{1}{8}+i)\right] + cos\left[\pi\, h(\frac{3}{8}+i)\right] + cos\left[\pi\, h(\frac{5}{8}+i)\right] + cos\left[\pi\, h(\frac{7}{8}+i)\right]\right\}$$
$$+ \frac{h\,\sigma}{8}\left\{cos\left[\pi\, h(\frac{1}{4}+i)\right] + cos\left[\pi\, h(\frac{3}{4}+i)\right]\right\} + \frac{4}{\pi^2\, h}cos\left[\pi\, h(\frac{1}{2}+i)\right] - \frac{2}{\pi^2\, h}cos(\pi\, h\, i)$$
$$- \frac{2}{\pi^2\, h}cos[\pi\, h\,(1+i)]. \quad (5.7)$$

The entries in the assembled source matrix after imposing boundary conditions are as follows:

$$F = \begin{bmatrix} I \\ -\frac{cos[\pi h(1+i)]}{\pi^2 h} - \frac{sin(\pi h i)}{\pi} + \frac{cos(\pi h i)}{\pi^2 h} + \frac{sin[\pi\, h(1+i)]}{\pi} + \frac{cos[\pi\, h(1+i)]}{\pi^2\, h} - \frac{cos(\pi\, h\, i)}{\pi^2\, h} \\ I \\ -\frac{cos[\pi h(1+i)]}{\pi^2 h} - \frac{sin(\pi h i)}{\pi} + \frac{cos(\pi h i)}{\pi^2 h} + \frac{sin[\pi\, h(1+i)]}{\pi} + \frac{cos[\pi\, h(1+i)]}{\pi^2\, h} - \frac{cos(\pi\, h\, i)}{\pi^2\, h} \\ I \\ -\frac{cos[\pi h(1+i)]}{\pi^2 h} - \frac{sin(\pi h i)}{\pi} + \frac{cos(\pi h i)}{\pi^2 h} + \frac{sin[\pi\, h(1+i)]}{\pi} + \frac{cos[\pi\, h(1+i)]}{\pi^2\, h} - \frac{cos(\pi\, h\, i)}{\pi^2\, h} \\ I \end{bmatrix}$$

$$= \begin{bmatrix} I \\ \frac{1}{\pi}\{sin[\pi\, h\,(1+i)] - sin(\pi\, h\, i)\} \\ I \\ \frac{1}{\pi}\{sin[\pi\, h\,(1+i)] - sin(\pi\, h\, i)\} \\ I \\ \frac{1}{\pi}\{sin[\pi\, h\,(1+i)] - sin(\pi\, h\, i)\} \\ I \end{bmatrix} = \begin{bmatrix} 0.06184282 \\ -0.09323081 \\ -0.14930179 \\ -0.22507908 \\ -0.06184282 \\ 0.09323081 \\ 0.14930179 \end{bmatrix}.$$

where,

$$I = \frac{h\,\sigma}{8}\left(\sigma + \frac{1}{2}\right)\left\{cos\left[\pi\, h(\frac{1}{8}+i)\right] + cos\left[\pi\, h(\frac{3}{8}+i)\right] + cos\left[\pi\, h(\frac{5}{8}+i)\right] + cos\left[\pi\, h(\frac{7}{8}+i)\right]\right\}$$
$$+ \frac{h\,\sigma}{8}\left\{cos\left[\pi\, h(\frac{1}{4}+i)\right] + cos\left[\pi\, h(\frac{3}{4}+i)\right]\right\} + \frac{4}{\pi^2\, h} cos\left[\pi\, h(\frac{1}{2}+i)\right] - \frac{2}{\pi^2\, h} cos(\pi\, h\, i)$$
$$- \frac{2}{\pi^2\, h} cos[\pi\, h\,(1+i)]. \quad (5.8)$$

The system of linear equations after imposing boundary conditions is as follows:

$$\begin{bmatrix} 21.8666 & 0.2552 & 0. & 0. & 0. & 0. & 0. \\ 0.2552 & 8.6666 & 0.2552 & -3.8333 & 0. & 0. & 0. \\ 0. & 0.2552 & 21.8666 & 0.2552 & 0. & 0. & 0. \\ 0. & -3.8333 & 0.2552 & 8.6666 & 0.2552 & -3.8333 & 0. \\ 0. & 0. & 0. & 0.2552 & 21.8666 & 0.2552 & 0. \\ 0. & 0. & 0. & -3.8333 & 0.2552 & 8.6666 & 0.2552 \\ 0. & 0. & 0. & 0. & 0. & 0.2552 & 21.8666 \end{bmatrix} \begin{bmatrix} \Delta u_{\frac{1}{2}} \\ u_1 \\ \Delta u_{\frac{3}{2}} \\ u_2 \\ \Delta u_{\frac{5}{2}} \\ u_3 \\ \Delta u_{\frac{7}{2}} \end{bmatrix} = \begin{bmatrix} 0.06184282 \\ -0.09323081 \\ -0.14930179 \\ -0.22507908 \\ -0.06184282 \\ 0.09323081 \\ 0.14930179 \end{bmatrix}.$$

The solution (U) of above system of equations is obtained by the Gaussian elimination method.

$$U = \begin{bmatrix} \Delta u_{\frac{1}{2}} \\ u_1 \\ \Delta u_{\frac{3}{2}} \\ u_2 \\ \Delta u_{\frac{5}{2}} \\ u_3 \\ \Delta u_{\frac{7}{2}} \end{bmatrix} = \begin{bmatrix} 0.00317115 \\ -0.02938641 \\ -0.0059911 \\ -0.04230551 \\ -0.00223998 \\ -0.00809253 \\ 0.0069222 \end{bmatrix}$$

where u_i denotes the value of u_h at a global node i. That is, $u_i = u_h(x_i)$ for $i = 0, 1, 2, 3$ and $\Delta u_{i+\frac{1}{2}}$ for $i = 0, 1, 2, 3$ denote the additive increments.

Therefore, the value of u_h at $x = x_{i+\frac{1}{2}}$ for $i = 1, 2, 3$ is calculated from $\Delta u_{i+\frac{1}{2}}$ using formula

$$u_h(x_{i+\frac{1}{2}}) = \frac{u_i + u_{i+1}}{2} + \Delta u_{i+\frac{1}{2}}.$$

Thus,

$$u_h(x_{\frac{1}{2}}) = -0.01152206 \quad u_h(x_{\frac{3}{2}}) = -0.04183706$$

$$u_h(x_{\frac{5}{2}}) = -0.02743899 \quad u_h(x_{\frac{7}{2}}) = 0.00287601$$

and

$$u_h = \begin{bmatrix} 0 \\ -0.01152206 \\ -0.02938641 \\ -0.04183706 \\ -0.04230551 \\ -0.02743899 \\ -0.00809253 \\ 0.00287601 \\ 0 \end{bmatrix}.$$

Further, the exact solution (u) is obtained using formula

$$u = \frac{1}{1+\pi^2}\left[-\frac{e^x}{1-e} + \frac{e\,(e^{-x})}{1-e} + cos(\pi\, x)\right].$$

Table (5.7) compares the values of the approximate solution (u_h) and the exact solution (u) for $\sigma = 0.25$.

Exact solution (u)	Approximate Solution (u_h)
0	0
0.01722748933063872	-0.01152206
0.020454743191404234	-0.02938641
0.013080412974840881	-0.041837 06
1.5847369974441745e-17	-0.04230551
-0.01308041297484087	-0.02743899
-0.020454743191404223	-0.00809253
-0.01722748933063872	0.00287601
0	0

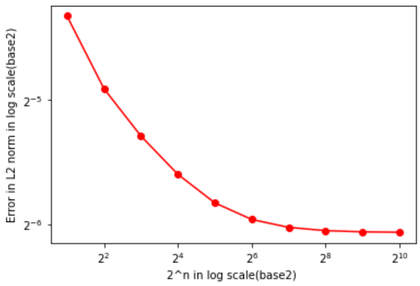
Table 5.7: Values of the approximate solution (u_h) and the exact solution (u) for $\sigma = 0.25$.

Error Estimates:

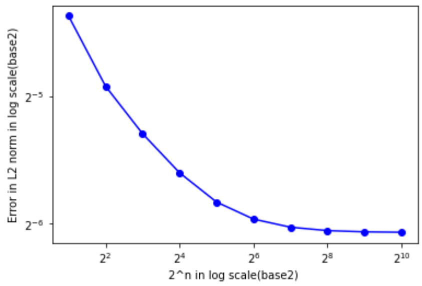
1. **Error in the L^2 norm:** The error in the L^2 norm for different values of σ over the finite element mesh having 2^n elements for $n = 1, 2, ... 10$ is listed in table (5.8) and represented graphically in figure (5.4).

2^n	$\sigma = 0$	$\sigma = 0.25$	$\sigma = 0.35$	$\sigma = 0.45$
2	5.01198017052233E-02	4.99927781406081E-02	4.83688236215456E-02	4.57354457814678E-02
4	3.32978855370582E-02	3.33157350752760E-02	3.30139658766016E-02	3.24769855363458E-02
8	2.55541794153381E-02	2.55636495456151E-02	2.55087452689421E-02	2.54047271193684E-02
16	2.06313881126920E-02	2.06339740885714E-02	2.06244758773473E-02	2.06058966009064E-02
32	1.75892082406128E-02	1.75897759734864E-02	1.75882451611767E-02	1.75851971232521E-02
64	1.60306574659007E-02	1.60307699863851E-02	1.60305625796286E-02	1.60301409741419E-02
128	1.53385451659065E-02	1.53385636577680E-02	1.53385411413071E-02	1.53384943655201E-02
256	1.50592654600481E-02	1.50592677870355E-02	1.50592656791484E-02	1.50592613482241E-02
512	1.49582745250002E-02	1.49582747545823E-02	1.49582745603341E-02	1.49582741961595E-02
1024	1.49257199302213E-02	1.49257199489166E-02	1.49257199297617E-02	1.49257199000758E-02

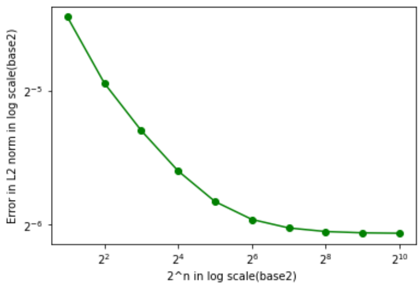
Table 5.8: Error in the L^2 norm for different values of $\sigma \in [0, \frac{1}{2})$.



(a) $\sigma = 0.25$



(b) $\sigma = 0.35$



(c) $\sigma = 0.45$

Figure 5.4: Rate of convergence in L^2 norm for different values of $\sigma \in [0, \frac{1}{2})$.

It is observed that:

- when the number of elements doubled, the error in the L^2 norm decreased.

- the error in L^2 norm for $\sigma = 0$ is less than the error for $\sigma = 0.25$. Moreover, the error in L^2 norm for $\sigma = 0.35, 0.45$ is lower than the error in L^2 norm for $\sigma = 0.25$ and $\sigma = 0$. The error in L^2 norm decreases further as $\sigma \to 0.49$.

2. **Ratio of error in L^1 norm to error in L^∞ norm:** For $2^n = 2^2 = 4$ subintervals and for different values of $\sigma \in [0, \frac{1}{2})$, error in L^1 norm, error in L^∞ norm and the ratio of error in L^1 norm to error in L^∞ norm is listed in the tables (5.9), (5.10), (5.11) respectively.

Subinterval	$\sigma = 0$	$\sigma = 0.25$	$\sigma = 0.35$	$\sigma = 0.45$
1	3.57386202638598E-03	3.59369315733388E-03	3.69266351318484E-03	3.87373623380698E-03
2	1.30874388070286E-02	1.30948285395444E-02	1.29383435070841E-02	1.26357865976112E-02
3	7.07481502051572E-03	7.08301094676120E-03	7.03495586395339E-03	6.94044428010211E-03
4	4.07733040367598E-03	4.05821399741908E-03	3.85375415271279E-03	3.46983089075259E-03

Table 5.9: Error in L^1 norm for different values of $\sigma \in [0, \frac{1}{2})$.

Subinterval	$\sigma = 0$	$\sigma = 0.25$	$\sigma = 0.35$	$\sigma = 0.45$
1	2.85908962110878E-02	2.87495452586710E-02	2.95413081054787E-02	3.09898898704558E-02
2	5.48351473864320E-02	5.49174730689235E-02	5.36189159238213E-02	5.10911617932100E-02
3	5.48351473864320E-02	5.49174730689235E-02	5.36189159238213E-02	5.10911617932100E-02
4	5.48351473864320E-02	5.49174730689235E-02	5.36189159238213E-02	5.10911617932100E-02

Table 5.10: Error in L^∞ norm for different values of $\sigma \in [0, \frac{1}{2})$.

Subinterval	$\sigma = 0$	$\sigma = 0.25$	$\sigma = 0.35$	$\sigma = 0.45$
1	1.25000000000000E-01	1.25000000000000E-01	1.25000000000000E-01	1.25000000000000E-01
2	2.38668799680602E-01	2.38445576749496E-01	2.41301848128861E-01	2.47318443232005E-01
3	1.29019713773328E-01	1.28975543683005E-01	1.31202873887795E-01	1.35844322902528E-01
4	7.43561492584744E-02	7.38965901130569E-02	7.18730337291411E-02	6.79145035847223E-02

Table 5.11: Ratio of error in L^1 norm and error in L^∞ norm for different values of $\sigma \in [0, \frac{1}{2})$.

It is observed that the ratio of error in L^1 norm to error in L^∞ norm is maximum in the second subinterval for $\sigma = 0, 0.25, 0.35, 0.45$. Further investigations are required to determine viability of this ratio to perform adaptive mesh refinement.

5.3 Python code

```
#Program to find the approximate solution of the Poisson equation with f
    (x)=1 and b.c u(0)=u(1)=0

import numpy as np
from scipy import *
import matplotlib.pyplot as plt
import math

n=10
#Solution by finite element method.
k=(2**(n+1))+1
x=np.linspace(0.0,1,k,endpoint=True) #Division of interval [0,1] into
    equidistant points

h=(1.0/(2**n)) #Length of each subinterval
sigma = 0.35

print("h=",h)

A=np.zeros((k,k)) #Initially, Stiffness Matrix is set to zero matrix.
B=np.zeros((k,1)) # Applied load vector (matrix on RHS) is set to zero
    vector.

#Finding element matrices

ael1= (4/(1-(4*(sigma**2))))
Ael=[[1,-1,0],[-1,1,0],[0,0,ael1]] #element matrix

print("\nElement Matrix=\n",Ael)

#Finding global stiffness matrix A

D=np.zeros((k))
U=np.zeros((k-1))
```

```

#Finding tridiagonal entries of the stiffness matrix.
D[:2**n]=D[:2**n]+((1/h)*Ael[0][0])
D[1:]=D[1:]+((1/h)*Ael[1][1])
D[2**n+1:]=((1/h)*Ael[2][2])
U[:2**n]=((1/h)*Ael[0][1])

A=np.diag(U,-1)+np.diag(D,0)+np.diag(U,1) #Forming tridiagonal matrix
    without imposing boundary conditions.

print("\nThe stiffness matrix A without imposing boundary conditions=\n"
    ,A)


#imposing boundary condition

A[0][0]=1
A[0][1]=0
A[1][0]=0
A[2**n][2**n]=1
A[2**n-1][2**n]=0
A[2**n][2**n-1]=0

print("\nThe global stiffness matrix with boundary conditions is.....A=\
    n",A)

#Finding element matrices F.


f=1 # RHS of pde f(x)=1
#print("f=",f)

#calculating values of RHS of the system AU=B by integrating fractal
    functions explicitly.
B[0]=h/2 #first element of the matrix B
```

```
B[2**n]=h/2 #last element of the matrix B


for i in range(0,2**n+1):
B[i]=h #entries of matrix B

B[2**n+1:]=(h/(2*(1-sigma)))

B[0]=0 #imposing boundary condition
B[2**n]=0 #imposing boundary condition
print("\nThe global force matrix with boundary conditions is.....B=\n",B
    )

F=B
Y = np.linalg.solve(A,F)

#Gauss elimination method to solve the system of equations
#Elimination

for j in range(0,k):
for i in range(j+1,k):
if A[i,j]!=0:
r=A[i,j]/A[j,j]
A[i,j+1:k]=A[i,j+1:k]-r*A[j,j+1:k]
B[i]=B[i]-r*B[j]

#back Substitution

for j in range(k-1,-1,-1):
B[j]=(B[j]-np.dot(A[j,j+1:k-1],B[j+1:k-1]))/A[j,j]

print("\nThe approximate value of the function u at each node=B \n ",B)

Yapp=np.zeros(k)
Ym=np.zeros(2**n)
```

```
Ye = Y[2**n+1:]
Ya=Y[:2**n+1]
print('Ye=',Ye)
print('Ya=',Ya)
for i in range(0,2**n):
Ym[i]=((Ya[i]+Ya[i+1])/2)+Ye[i]
print("Ym=",Ym)

j=0
for i in range(0,k-1,2):
Yapp[i]=Y[j]
Yapp[i+1]=Ym[j]
j=j+1
print("Yapp=",Yapp)
plt.plot(x,Yapp,color='red', marker='o')
Xexact=np.linspace(0.0,1,k,endpoint=True)
Yexact=[ (x/2)-(x**2/2) for x in Xexact] #Exact solution is, y=x(1-x)/2
print("\n Yexact=",Yexact)
print("\n The graph of exact solution:\n")
plt.plot(Xexact,Yexact,color='blue',marker='')
plt.show()


#.1.finding error in L2 norm
e=np.zeros(k,dtype='g')
esq=np.zeros(k,dtype='g')
e1= np.zeros(k,dtype='g')
L2error=0
sum1=0
for i in range(0,k):
e[i]=np.abs(Yexact[i]-Yapp[i])
esq[i]=e[i]*e[i] #error square
sum1=sum1+esq[i]
L2error=sum1/(k-1)
L=math.sqrt(L2error) #L2 error
```

```
print("Error in L2 norm =",L)

#3. error in max norm.
print("Error in max norm=", max(e))

# values of psi fractal bubble at dyadic points.
r=k+(2**(n+1))
x1=np.linspace(0.0,1,r,endpoint=True)

psi = np.zeros(r)
psi[-1] = psi[0]= 0
psi[2**(n+1)]=1
n1=n+1
for i in range(0,n1):
psi2 = psi[::2]
psi[:2**(n+1)+1] = sigma*psi2 + 2*x1[:2**(n+1)+1]
psi[2**(n+1):] = sigma*psi2 - 2*x1[2**(n+1):] + 2
print("psi=",psi)
print("psi2=",psi2)

#Calculations for L2-norm of the error

#.1. Finding values of y at different dyadic points of the interval [xi,
    xi+1]
Yrem=np.zeros(2**(n+1))
Yfem=np.zeros(r)
j=0
p=0
for i in range(0,2**(n+1),2):
Yrem[i]=(((3*Ya[j])+Ya[j+1])/4)+(Ye[p]*psi[(2*i)+1])
Yrem[i+1]=(((3*Ya[j+1])+Ya[j])/4)+(Ye[p]*psi[(2*i)+3])
j=j+1
p=p+1
print("Yrem=",Yrem)

```

```
l=0
for i in range(0,r-1,2):
Yfem[i]=Yapp[l]
Yfem[i+1]=Yrem[l]
l=l+1
print("Yfem=",Yfem)

#2.finding error in L2 norm
e=np.zeros(r,dtype='g')
esq=np.zeros(r,dtype='g')
e1= np.zeros(r,dtype='g')
L2error=0
sum1=0
Yex=[(x/2)-(x**2/2) for x in x1]
for i in range(0,r):
e[i]=np.abs(Yfem[i]-Yex[i])
print("Yex[i],Yfem[i]",Yex[i],Yfem[i])
print("e[i]=",e[i])
esq[i]=e[i]*e[i] #error square
sum1=sum1+esq[i]
L2error=sum1/(r-1)
L=np.sqrt(L2error) #L2 error
print("Error in L2 norm, max norm =",L,max(e))
```

Listing 5.1: Solution to the Poisson equation with Dirichlet boundary conditions using fractal functions.

Chapter 6

Conclusion and future work

6.1 Conclusion

In this thesis, the space of continuous fractal functions was utilized successfully as the finite element space in the finite element method. To this end, univariate, $\mathbb{R}$ - valued, continuous fractal functions were constructed and used as hierarchical basis for interpolation in the finite element method. This hierarchical fractal basis consisted of three interpolation functions: two linear nodal basis functions and a fractal bubble.

The numerical illustrations demonstrated that the approximate solution and hence the interpolation error in L^2 norm were controlled by the smoothness of the fractal bubble. The smoothness of the fractal bubble was determined by the vertical scaling factor σ of the fractal bubble.

For $\sigma = 0$, the fractal bubble was a linear function and was C^1 smooth. Whereas for $\sigma = \frac{1}{4} = 0.25$, the fractal bubble was a quadratic function $4x(1-x)$ and was C^2 smooth. Additionally, for $0 \leq \sigma \leq 0.49$ and $\sigma \neq 0, 0.25$, the fractal bubble was C^0 smooth.

For the Poisson equation with $f(x) = 1$ and with Dirichlet boundary conditions $u(0) = u(1) = 0$, the error in L_2 norm was minimum when $\sigma = \frac{1}{4} = 0.25$. This was as expected since the fractal bubble for $\sigma = \frac{1}{4} = 0.25$ was C^2 smooth.

Further, for the Poisson equation with $f(x) = x$ and $f(x) = e^x$ and with Dirichlet boundary conditions $u(0) = u(1) = 0$, the error in L^2 norm error was expected to be minimum for $\sigma = \frac{1}{4} = 0.25$. However, the error in L^2 norm for $\sigma = 0.35, 0.45$ was lower than the error in L^2

norm for $\sigma = 0.25$. Thus, for $0 \leq \sigma \leq 0.45$, the error in L^2 norm was minimum for $\sigma = 0.45$ and decreased further as $\sigma \to 0.49$.

Likewise, for a differential equation,

$$-\frac{d^2u}{dx^2} + \frac{du}{dx} = cos(\pi x),\ 0 \leq x \leq 1$$

with Dirichlet boundary conditions $u(0) = u(1) = 0$, the interpolation error in L^2 norm was found to be decreasing as $\sigma \to 0.49$.

Thus, for the Poisson equation with $f(x) = x$ and $f(x) = e^x$ and for a differential equation,

$$-\frac{d^2u}{dx^2} + \frac{du}{dx} = cos(\pi x),\ 0 \leq x \leq 1$$

with Dirichlet boundary conditions, C^0 smooth fractal bubble can yield better approximation than C^2 smooth fractal bubble in the hierarchical finite element method.

The viability of utilization of C^0 smooth fractal bubble in the hierarchical finite element method follows from the fact that for $\sigma \in [0, \frac{1}{2})$, the interpolation error decreased with an increase in the number of finite elements.

The numerical demonstrations have also substantiated the bounds for an interpolation error in L^2 norm theoretically. Additionally, the bounds for an interpolation error in H^1 seminorm were established theoretically. However, the experimental orders of convergence can not be verified with the theoretical bounds of error in H^1 seminorm. Hence, further investigations are required to compute an exact error and the bounds for an interpolation error in H^1 seminorm.

Finally, we conclude that the space of continuous fractal functions, being a larger space can provide better approximation in the hierarchical finite element method.

6.2 Future Work

As a next step, it would be of interest to investigate the utilization of fractal functions to find the numerical solution of elliptic partial differential equations in two and three dimensions. For example, a tensor product of two continuous fractal functions defined on an interval $[0, 1]$

may be used as a fractal bubble in two dimensions. The hierarchical fractal basis in two dimensions may then be constructed using standard shape functions in two dimensions and the fractal bubble. The discontinuous Galerkin method may be used if the fractal functions in two dimensions do not follow interelement continuity condition. Likewise, the hierarchical fractal basis can be constructed in three dimensions in the finite element approximations.

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