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The local index formula in semifinite von Neumann algebras II: The even case

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Abstract

We generalise the even local index formula of Connes and Moscovici to the case of spectral triples for a $*$-subalgebra $\mathscr{A}$ of a general semifinite von Neumann algebra. The proof is a variant of that for the odd case which appears in Part I. To allow for algebras with a non-trivial centre we have to establish a theory of unbounded Fredholm operators in a general semifinite von Neumann algebra and in particular prove a generalised McKean–Singer formula.

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1. Introduction

There have been two new proofs of the local index theorem in the non-commutative geometry of Connes and Moscovici [CoM], by Higson [H] and, for the odd case, by the present authors in part I of this two part series of papers [CPRS2]. The novelty in [CPRS2] is consideration of spectral triples "inside" a general semifinite von Neumann algebra and in the introduction of a new odd cocycle (in the (b, B) bicomplex of cyclic cohomology) which provides a substitute in our approach for the JLO cocycle [Co4] used in [CoM]. Our new cocycle is reminiscent of, but distinct from, Higson's 'improper cocycle' [H]. In subsequent work [CPRS4], we will relate these two cocycles showing how to obtain a renormalised version of Higson's cocycle from our resolvent cocycle.

The present paper is concerned with two primary results, the even semifinite local index formula proved via the even resolvent cocycle and a prerequisite, a general theory of Fredholm operators in von Neumann algebras which may have non-trivial centre. This extension is essential to encompass examples such as arise in the L^2 index theorem of Atiyah. (Other applications are referenced in Part I.)

For a finitely summable even spectral triple with spectral dimension q (see [CPRS2] for the latter terminology) we use the even resolvent cocycle to obtain an expression for the index. The even resolvent cocycle is a (b, B) cocycle with values in functions defined and holomorphic in a certain half-plane modulo those functions holomorphic in a larger half-plane containing the critical point $r = (1 - q)/2$. By taking residues at the critical point as in [CPRS2] we prove the even case of a local index formula for smooth finitely summable semifinite spectral triples. Thus as in [CPRS2] we need the property of 'isolated spectral dimension' to analytically continue our resolvent cocycle term-by-term to a deleted neighbourhood of $r = (1 - q)/2$. This then defines a generalisation of the Connes–Moscovici even residue cocycle in the finite (b, B) bicomplex.

There remains one gap in our treatment in that we do not prove that the residue cocycle represents the Chern character of our semifinite spectral triple. This gap will be filled in a subsequent paper, [CPRS4], as the proof is not short.

Our exposition is organised as follows. We assume all of the notation of the first part [CPRS2] but include additional preliminary material, notation and definitions needed for this paper in Section 2. Our main theorem starts from a version of the McKean–Singer formula for the index. However, we found that Fredholm theory in semifinite von Neumann algebras with a non-trivial centre did not exist in a form that was suitable for this purpose. In particular, the case of an operator which is Fredholm from the range of one projection to the range of another projection (which is the case of the McKean–Singer formula) had not been touched in this setting, and is rather subtle. Thus Section 3 establishes such a theory. We note that in this paper we fix a faithful normal semifinite trace τ on our algebra once and for all. Thus strictly speaking we deal always with τ-Fredholm theory, and do not give a full treatment involving centre-valued traces and related machinery. Those expert in all these matters can move straight to Section 4 where we state our main theorem, the local index theorem for even semi-finite spectral triples.

The main theorem has three parts. The first expresses an index pairing as the *residue* of the pairing between the resolvent cocycle and the Chern character of a projection. This residue exists with no assumptions concerning analytic continuations. The second statement is similar to the first, but the index is expressed as the residue of a sum of zeta functions. The third part finally assumes that we can analytically continue the individual zeta functions, so that we express the index pairing as a sum of residues of zeta functions. These residues assemble to form a (b, B) cocycle, called the residue cocycle.

The proof has a number of important differences with that of the odd case and these are highlighted in Section 5.1 where we establish an analytic formula for the even index which is the starting point for our proof. The rest of Section 5 contains the computations needed to prove the main theorem. By Section 5.6 we have enough to prove part (2) of the main theorem and the index formula of part (3). To prove part (1) and the cohomological part of (3), we introduce the resolvent cocycle for the even case in Section 6.

We conclude this introduction with some general comments on the existing proofs of the Local Index theorem which may help put our results in context. Connes and Moscovici begin with a representative of the Chern character (the JLO cocycle) and deform it to obtain the unrenormalised residue cocycle. It is automatically a representative of the Chern character, and so an index cocycle. While this cocycle can be renormalised, it is unclear to us whether a procedure exists to modify the JLO cocycle so that it yields the renormalised version automatically.

Higson *writes down* a function valued cocycle, proves that it is an index cocycle and then proves it is in the class of the Chern character [H]. The unrenormalised local index theorem follows from Higson's cocycle and the pseudodifferential calculus. We show in [CPRS4] that there is a simple modification of Higson's cocycle which leads directly to the renormalised residue cocycle. In this paper, as in [CPRS2], we begin with an analytic formula for the index pairing and apply perturbation theory and the pseudodifferential calculus to obtain the renormalised residue cocycle directly. As part of this process we also obtain a function valued (almost) cocycle similar to Higson's, but with superior holomorphy properties. Our cocycles are automatically index cocycles, and so we need only show that they are in the class of the Chern character. This will be shown in [CPRS4], closely following Higson's methods.

2. Definitions and background

We adopt the notational conventions of [CPRS2]. Thus $\mathcal{N}$ is semifinite von Neuman algebra acting on a Hilbert space $\mathcal{H}$ and τ is a faithful normal semifinite trace on $\mathcal{N}$. An even semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by a $*$-algebra $\mathcal{A} \subset \mathcal{N}$, a densely defined unbounded operator $\mathcal{D}$ affiliated with $\mathcal{N}$ on $\mathcal{H}$ and in addition to the properties of Definition 2.1 of [CPRS2], has a grading $\gamma \in \mathcal{N}$ such that $\gamma^* = \gamma$, $\gamma^2 = 1$, $a\gamma = \gamma a$ for all $a \in \mathcal{A}$ and $\mathcal{D}\gamma + \gamma\mathcal{D} = 0$. As in [CPRS2] we deal only with unital algebras $\mathcal{A}$ where the identity of $\mathcal{A}$ is that of $\mathcal{N}$. We write $P = (1 + \gamma)/2$ and

$\mathcal{D}^+ = (1-P)\mathcal{D}P = P^\perp\mathcal{D}P$. The operator $\mathcal{D}^+ : \mathcal{H}^+ = P(\mathcal{H}) \to \mathcal{H}^- = P^\perp(\mathcal{H})$ is, as we shall see, an unbounded Breuer–Fredholm operator.

The numerical index discussed here is the result of a pairing between an even K-theory class represented by a projection p, and an even K-homology class represented by $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, [Co4, Chapter III,IV]. This point of view also makes sense in the general semifinite setting after suitably interpreting K-homology classes, [CPRS1,CP2]. The pairing of (b, B) cocycles with K-homology classes is written in the even case as

$$\langle [p], [(\mathcal{A}, \mathcal{H}, \mathcal{D})]\rangle = \langle [Ch_*(p)], [Ch^*(\mathcal{A}, \mathcal{H}, \mathcal{D})]\rangle, \tag{1}$$

where $[p] \in K_0(\mathcal{A})$ is a K-theory class with representative p and $[(\mathcal{A}, \mathcal{H}, \mathcal{D})]$ is the K-homology class of the even spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. On the right-hand side, $Ch_*(p)$ is the Chern character of p, and $[Ch_*(p)]$ its periodic cyclic homology class. Similarly $[Ch^*(\mathcal{A}, \mathcal{H}, \mathcal{D})]$ is the periodic cyclic cohomology class of the Chern character of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. *The analogue of Eq.* (1), *for a suitable cocycle associated to* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, *in the general semifinite case is part of our main result.*

We refer to [Co4,Lo,CPRS2] for the definition of the (b, B) bicomplex. The (b, B) Chern character of a projection in an algebra $\mathcal{A}$ is an even (b, B) cycle with $2m$th term, $m \geqslant 1$, given by

$$Ch_{2m}(p) = (-1)^m \frac{(2m)!}{2(m!)}(2p-1) \otimes p^{\otimes 2m}.$$

For $m = 0$ the definition is $Ch_0(p) = p$.

3. Fredholm theory in semifinite von Neumann algebras

We need to generalise the real-valued Fredholm index theory outlined in [PR, Appendix B].

In particular, we must study Fredholm operators in a "skew-corner" of our semifinite von Neumann algebra $\mathcal{N}$. That is, if P and Q are projections in $\mathcal{N}$ (not necessarily infinite and not necessarily equivalent) we will extend the notion of τ-index and τ-Fredholm to operators $T \in P\mathcal{N}Q$. If $\mathcal{N}$ is a factor, this is *much easier* and is done in Appendix A of [Ph1]. We simply refer to them as $(P \cdot Q)$-Fredholm operators. Most results work in this setting; however the ploy used in [Ph1] of invoking the existence of a partial isometry from P to Q to reduce to the case $P\mathcal{N}P$ (solved in [PR]) is not available. In fact, because of examples to which our version of the McKean–Singer Theorem applies, P and Q are not generally equivalent. One notable result that is different in the non-factor setting (even if $P = Q$) is that the set of $(P \cdot Q)$-Fredholm operators with a given index *is* open but is *not* generally connected: information is lost when one fixes a trace to obtain a real-valued index. That the set of $(P \cdot Q)$-Fredholm operators with a given index is open (and other facts) is very sensitive to the order in which the expected results are proved. As the Fredholm alternative is *not* available in the $(P \cdot Q)$ setting, we take a novel approach and deduce many facts from the formula for the index of a product. We also study unbounded operators affiliated to a "skew-corner".

Notation: If T is an operator in the von Neumann algebra $\mathcal{N}$ (or T is closed and affiliated to $\mathcal{N}$) then we let R_T and N_T be the projections on the closure of the range of T and the kernel of T, respectively. If $T \in P\mathcal{N}Q$, (or T is closed and affiliated to $P\mathcal{N}Q$) then we will denote the projection on $ker_Q(T) = ker(T_{|_{Q(\mathcal{H})}}) = ker(T) \cap Q(\mathcal{H})$ by N_T^Q and observe that $N_T^Q = QN_T = N_TQ \leqslant Q$ while $R_T \leqslant P$.

Definition 3.1. With the usual assumptions on $\mathcal{N}$ let P and Q be projections (*not* necessarily infinite, or equivalent) in $\mathcal{N}$, and let $T \in P\mathcal{N}Q$. Then T is called $(P \cdot Q)$-*Fredholm* if and only if

(1) $\tau(N_T^Q) < \infty$, and $\tau(N_{T^*}^P) < \infty$, and
(2) There exists a τ-finite projection $E \leqslant P$ with $range(P - E) \subseteq range(T)$.
If T is $(P \cdot Q)$-Fredholm then the $(P \cdot Q)$-*index* of T is

$$Ind(T) = \tau(N_T^Q) - \tau(N_{T^*}^P).$$

Lemma 3.2. *With the usual assumptions on* $\mathcal{N}$, *let* $T \in P\mathcal{N}Q$. *Then*,

(1) *If* $P_1 = R_T$ *is* τ-*cofinite in* P *and* $Q_1 = Q - N_T^Q = supp(T) = R_{T^*}$ *is* τ-*cofinite in* Q, *then* T is $(P \cdot Q)$-*Fredholm if and only if* T *is* $(P_1 \cdot Q_1)$-*Fredholm. Then the* $(P_1 \cdot Q_1)$-*Index of* T *is* 0, *while the* $(P \cdot Q)$-*Index of* T is $\tau(Q - Q_1) - \tau(P - P_1)$.
(2) *If* T *is* $(P \cdot Q)$-*Fredholm*, *then* T^* *is* $(Q \cdot P)$-*Fredholm and* $Ind(T^*) = -Ind(T)$. *If* $T = V|T|$ *is the polar decomposition*, *then* V *is* $(P \cdot Q)$-*Fredholm with* $Ind(V) = Ind(T)$ *and* $|T|$ *is* $(Q \cdot Q)$-*Fredholm of index* 0.
(3) *If* $T = V|T|$ *is* $(P \cdot Q)$-*Fredholm*, *then there exists a spectral projection* $Q_0 \leqslant Q$ *for* $|T|$ *so that* $\tau(Q - Q_0) < \infty$, *and* $P_0 = VQ_0V^*$ *satisfies*: $\tau(P - P_0) < \infty$, $P_0(\mathcal{H}) = range(TQ_0) \subset range(T)$, $Q_0(\mathcal{H}) \subset range(T^*)$, $TQ_0 = P_0TQ_0 : Q_0(\mathcal{H}) \to P_0(\mathcal{H})$ *and* $T^*P_0 = Q_0T^*P_0 : P_0(\mathcal{H}) \to Q_0(\mathcal{H})$ *are invertible as bounded linear operators*.
(4) *The set of all* $(P \cdot Q)$-*Fredholm operators in* $P\mathcal{N}Q$ *is open in the norm topology*.

Proof. (1) is straightforward, noting that $Q_1 = 1 - N_T = R_{T^*} = supp(T)$.

(2) In the notation of part (1), $VV^* = P_1$ and $V^*V = Q_1$ so that V is $(P \cdot Q)$-Fredholm with $Ind(V) = Ind(T)$. Since both T^* and $|T|$ have τ-finite kernel and cokernel, it suffices to observe that if $\tilde{P} \leqslant P$ is τ-cofinite in P and $\tilde{P}(\mathcal{H}) \subseteq T(\mathcal{H})$ then $\tilde{Q} := V^*\tilde{P}V$ is τ-cofinite in Q and satisfies $\tilde{Q}(\mathcal{H}) \subseteq T^*(\mathcal{H}) = |T|(\mathcal{H})$. The index statements are clear.

(3) By part (1), we can assume that $P = R_T$ and $Q = R_{T^*} = supp(T)$. Now $|T| \geqslant 0$ is 1:1 and τ-Fredholm in $Q\mathcal{N}Q$. As $|T|$ is invertible modulo $\mathcal{K}_{Q\mathcal{N}Q}$ by Theorem B1 of [PR], the argument of Lemma 3.7 of [CP0] shows that there exists a spectral projection $Q_0 \leqslant Q$ for $|T|$ with $\tau(Q - Q_0) < \infty$ and $|T|Q_0$ is bounded below on $Q_0(\mathcal{H})$. Let

$P_0 = VQ_0V^*$ this satisfies: $\tau(P - P_0) < \infty$. Now, $TQ_0 = V|T|Q_0 = \cdots = P_0TQ_0$, and similarly, $T^*P_0 = \cdots = Q_0T^*P_0$. Since, $TQ_0(\mathcal{H}) = V|T|Q_0(\mathcal{H}) = V(Q_0(\mathcal{H})) = P_0(\mathcal{H})$, we see $P_0(\mathcal{H}) = range(TQ_0) \subset range(T)$ and $TQ_0 = P_0TQ_0 : Q_0(\mathcal{H}) \to P_0(\mathcal{H})$ is invertible as a bounded operator. The remaining bits are similar.

(4) Using (1), we have that T is $(P_1 \cdot Q_1)$-Fredholm of index 0 and V is a partial isometry in $\mathcal{N}$ with $VV^* = P_1$ and $V^*V = Q_1$. By part (3) choose $Q_0 \leqslant Q_1$ such that $\tau(Q_1 - Q_0) < \infty$ and $P_0 = VQ_0V^*$ so that $\tau(P_1 - P_0) < \infty$ satisfies $P_0(\mathcal{H}) \subset range(T)$, and $TQ_0 = P_0TQ_0$ is invertible as a bounded operator from $Q_0(\mathcal{H})$ to $P_0(\mathcal{H})$. In particular, there exists $c > 0$ so that for all $x \in \mathcal{H}$:

$$\|TQ_0x\| = \|P_0TQ_0x\| \geqslant c\|Q_0x\| \ \ \& \ \ \|T^*P_0x\| = \|Q_0T^*P_0x\| \geqslant c\|P_0x\|.$$

So if $A \in P\mathcal{N}Q$ and $\|T - A\| < c/3$, then for all $x \in \mathcal{H}$:

$$\|AQ_0x\| \geqslant 2c/3\|Q_0x\| \ \ \& \ \ \|A^*P_0x\| \geqslant 2c/3\|P_0x\|.$$

Now clearly, TQ_0 and T^*P_0 have closed ranges P_0 and Q_0, respectively. Let $\tilde{P}_0$ and $\tilde{Q}_0$ be the closed ranges of AQ_0 and A^*P_0, respectively. Now, if $y \in P_0(\mathcal{H})$ is a unit vector, then $y = TQ_0x$ and $\|Q_0x\| \leqslant 1/c$. Letting $y_1 = AQ_0x \in \tilde{P}_0(\mathcal{H})$, we have $\|y - y_1\| \leqslant (c/3)(1/c) = 1/3$. Similarly, if $z \in \tilde{P}_0(\mathcal{H})$ is a unit vector, we find $z_1 \in P_0(\mathcal{H})$ with $\|z_1 - z\| \leqslant (c/3)(3/2c) = 1/2$. One concludes that $\|P_0 - \tilde{P}_0\| \leqslant 1/3 + 1/2 < 1$, and so P_0 and $\tilde{P}_0$ are unitarily equivalent by a unitary in $P\mathcal{N}P$ that fixes P. Hence, $P - \tilde{P}_0$ is τ-cofinite and not only is $\tilde{P}_0(\mathcal{H}) \subset range(A)$, but also $N^P_{A^*} = P - R_A \leqslant (P - \tilde{P}_0)$ is τ-finite. Similarly, $N^Q_A \leqslant (Q - \tilde{Q}_0)$ is τ-finite and A is $(P \cdot Q)$-Fredholm. □

Definition 3.3. If $T \in P\mathcal{N}Q$, then a *parametrix* for T is an operator $S \in Q\mathcal{N}P$ satisfying $ST = Q + k_1$ and $TS = P + k_2$ where $k_1 \in \mathcal{K}_{Q\mathcal{N}Q}$ and $k_2 \in \mathcal{K}_{P\mathcal{N}P}$.

Lemma 3.4. *With the usual assumptions on $\mathcal{N}$, then $T \in P\mathcal{N}Q$ is $(P \cdot Q)$-Fredholm if and only if T has a parametrix $S \in Q\mathcal{N}P$. Moreover, any such parametrix is $(Q \cdot P)$-Fredholm.*

Proof. Let S be a parametrix for T. Then $TS = P + k_2$ is Fredholm in $P\mathcal{N}P$ by Appendix B of [PR]. Hence there exists a projection $P_1 \leqslant P$ with $\tau(P - P_1) < \infty$ and $P_1(\mathcal{H}) \subset range(TS) \subset range(T)$. So, $N^P_{T^*} = P - R_T \leqslant P - P_1$ is τ-finite. On the other hand, $T^*S^* = (ST)^* = Q + k_1^*$ is Fredholm in $Q\mathcal{N}Q$ again by Appendix B of [PR] and so by the same argument N^Q_T is also τ-finite. That is, T is $(P \cdot Q)$-Fredholm and similarly S is $(Q \cdot P)$-Fredholm.

Now suppose that T is $(P \cdot Q)$-Fredholm. By part (3) of Lemma 3.2, there exist projections Q_0 and P_0 which are τ-cofinite in Q and P, respectively, so that $TQ_0 = P_0TQ_0 : Q_0(\mathcal{H}) \to P_0(\mathcal{H})$ is invertible as a bounded linear operator. Let S be its inverse. Then $S \in \mathcal{N}$ so that $S = Q_0SP_0 \in Q\mathcal{N}P$, and $STQ_0 = Q_0$ and $TQ_0S = P_0$.

Finally,

$$ST = STQ_0 + ST(Q - Q_0) = Q_0 + k = Q + k_1 \ \ and \ \ TS = TQ_0S = P_0 = P + k_2,$$

where $k_1 \in \mathcal{K}_{Q\mathcal{N}Q}$ and $k_2 \in \mathcal{K}_{P\mathcal{N}P}$. That is, S is a parametrix for T. □

Lemma 3.5. *We retain the usual assumptions on $\mathcal{N}$.*

(1) *Let $T \in P\mathcal{N}Q$ be $(P \cdot Q)$-Fredholm. If $k \in P\mathcal{K}_{\mathcal{N}}Q$ then $T + k$ is also $(P \cdot Q)$-Fredholm.*
(2) *If $T \in P\mathcal{N}Q$ is $(P \cdot Q)$-Fredholm and $S \in G\mathcal{N}P$ is $(G \cdot P)$-Fredholm, then ST is $(G \cdot Q)$-Fredholm.*

Proof. One checks that if S is a parametrix for T then S is also a parametrix for $T+k$ and that if T_1 is a parametrix for T and S_1 is a parametrix for S, then T_1S_1 is a parametrix for ST. □

Proposition 3.6. *Let G, P, Q be projections in $\mathcal{N}$ (with trace τ) and let $T \in P\mathcal{N}Q$ be $(P \cdot Q)$-Fredholm and $S \in G\mathcal{N}P$ be $(G \cdot P)$-Fredholm, respectively. Then, ST is $(G \cdot Q)$-Fredholm and*

$$Ind(ST) = Ind(S) + Ind(T).$$

We follow Breuer in [B2] indicating the changes needed in this generality. Before proving the proposition we require a Lemma.

Lemma 3.7 (*Cf. Lemma 1 of Breuer [B2]*). *With the hypotheses of the Proposition*:

$$N_{ST}^{Q} - N_{T}^{Q} \sim inf(R_T, N_S^P).$$

Proof. We follow Breuer's arguments replacing $ker(T)$ with $ker_Q(T) = ker(T) \cap Q(\mathcal{H})$; $ker(ST)$ with $ker_Q(ST)$; and $ker(S)$ with $ker_P(S)$. Noting $N_T^Q = QN_T = N_TQ$ and similar identities, we read Breuer until we choose projections $E_1 \leqslant E_2 \leqslant E_3 \leqslant \cdots$ as in Lemma 13 of [B1] satisfying each $(P - E_n)$ is τ-finite, $E_n(\mathcal{H}) \subset range(T)$, and $sup\{E_n|\ n = 1, 2, \ldots\} = R_T$. We continue reading carefully, replacing 1 with P at crucial points. Finally, we get the conclusion from:

$$N_{ST}^{Q} - N_{T}^{Q} = R_{(N_{ST}^{Q} - N_{T}^{Q})T^*} \sim R_{T(N_{ST}^{Q} - N_{T}^{Q})} = inf(R_T, N_S^P). \qquad \square$$

Proof of Proposition 3.6. Now, S^*, T^*, ST, and $(ST)^* = T^*S^*$ are all Fredholm, and the above lemma implies:

$$N_{ST}^{Q} - N_{T}^{Q} \sim inf(R_T, N_S^P) \ \ and \ \ N_{(ST)^*}^{G} - N_{S^*}^{G} \sim inf(R_{S^*}, N_{T^*}^P).$$

The projections on the RHS of the two similarities are in $P\mathcal{N}P$, and so by [Dix, Cor. 1, p. 216]:

$$N_S^P - inf(P - N_{T^*}^P, N_S^P) \sim N_{T^*}^P - inf(P - N_S^P, N_{T^*}^P).$$

Since $P - N_{T^*}^P = R_T$ and $P - N_S^P = R_{S^*}$, we get:

$$N_S^P - inf(R_T, N_S^P) \sim N_{T^*}^P - inf(R_{S^*}, N_{T^*}^P).$$

Using these similarities we calculate:

$$\begin{aligned} Ind(ST) &= \tau(N_{ST}^Q) - \tau(N_{(ST)^*}^G) \\ &= \tau(N_{ST}^Q - N_T^Q) - \tau(N_{(ST)^*}^G - N_{S^*}^G) + \tau(N_T^Q) - \tau(N_{S^*}^G) \\ &= \cdots = \tau(N_S^P) - \tau(N_{S^*}^G) + \tau(N_T^Q) - \tau(N_{T^*}^P) = Ind(S) + Ind(T). \qquad \square \end{aligned}$$

Corollary 3.8 (*Invariance properties of the $(P \cdot Q)$-Index*). *Let $T \in P\mathcal{N}Q$.*

(1) *If T is $(P\cdot Q)$-Fredholm then there exists $\delta > 0$ so that if $S \in P\mathcal{N}Q$ and $\|T-S\| < \delta$ then S is $(P \cdot Q)$-Fredholm and $Ind(S) = Ind(T)$.*
(2) *If T is $(P \cdot Q)$-Fredholm and $k \in P\mathcal{K}_{\mathcal{N}}Q$ then $T + k$ is $(P \cdot Q)$-Fredholm and $Ind(T + k) = Ind(T)$.*

Proof. (1) By the Proposition and part (2) of Lemma 3.2, TT^* is Fredholm of index 0 in $P\mathcal{N}P$. So by Corollary B2 of [PR] there exists $\varepsilon_1 > 0$ so that if $A \in P\mathcal{N}P$ satisfies $\|A - TT^*\| < \varepsilon_1$ then A is Fredholm of index 0. Moreover, by part (4) of Lemma 3.2 there exists $\varepsilon_2 > 0$ so that the ball of radius ε_2 about T in $P\mathcal{N}Q$ is contained in the $(P \cdot Q)$-Fredholms. Let $\delta = \min\{\varepsilon_2, \varepsilon_1/\|T\|\}$. Then if $S \in P\mathcal{N}Q$ and $\|T - S\| < \delta$ then S is $(P \cdot Q)$-Fredholm and $\|ST^* - TT^*\| < \varepsilon_1$ so that ST^* is $(P \cdot P)$-Fredholm of index 0. By the Proposition and part (2) of Lemma 3.2:

$$0 = Ind(ST^*) = Ind(S) - Ind(T).$$

(2) This is similar to part (1) but uses Lemma 3.5 part (1) in place of Lemma 3.2 part (4). $\square$

In [Ph1] *spectral flow* is defined in a semifinite *factor* using the index of Breuer–Fredholm operators in a skew-corner $P\mathcal{N}Q$ (in particular the operator PQ) and uses the product theorem for the index and other standard properties. The non-factor case for Toeplitz operators ($P = Q$) is covered in [PR] but the more subtle "skew-corner" case has not appeared in the literature. This section enables one to extend [Ph1] to the non-factor setting where it was needed for [CP2,CPS2,CPRS2]. For use in the present paper we generalise some of these results to closed, densely defined operators affiliated to $P\mathcal{N}Q$ by studying the map $T \mapsto T(1+|T|^2)^{-1/2}$.

Definition 3.9. A closed, densely defined operator T affiliated to $P\mathcal{N}Q$ is $(P \cdot Q)$-*Fredholm* if

(1) $\tau(N_T^Q) < \infty$, and $\tau(N_{T^*}^P) < \infty$, and
(2) There exists a τ-finite projection $E \leqslant P$ with $range(P-E) \subset range(T)$.
If T is $(P \cdot Q)$-Fredholm then the $(P \cdot Q)$-*index* of T is: $Ind(T) = \tau(N_T^Q) - \tau(N_{T^*}^P)$.

Remark. Using the equalities: $range(1+|T|^2)^{-1/2} = dom(1+|T|^2)^{1/2} = dom(|T|) = dom(T)$ one can show that: $range(T) = range(T(1+|T|^2)^{-1/2})$; $ker(T) = ker(T(1+|T|^2)^{-1/2})$ and $ker(T^*) = ker([T(1+|T|^2)^{-1/2}]^*)$. A little more thought completes the following:

Proposition 3.10 (*Index*). *If T is a closed, densely defined operator affiliated to $P\mathcal{N}Q$, then T is $(P \cdot Q)$-Fredholm if and only if the operator $T(1+|T|^2)^{-1/2}$ is $(P \cdot Q)$-Fredholm in $P\mathcal{N}Q$. In this case,*

$$Ind(T) = Ind(T(1+|T|^2)^{-1/2}).$$

Proposition 3.11 (*Continuity*). *If T is a closed, densely defined operator affiliated to $P\mathcal{N}Q$, and $A \in P\mathcal{N}Q$ then $T+A$ is also closed, densely defined, and affiliated to $P\mathcal{N}Q$ and*

$$\|T(1+|T|^2)^{-1/2} - (T+A)(1+|T+A|^2)^{-1/2}\| \leqslant \|A\|.$$

Proof. We define the following self-adjoint operators:

$$D = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}.$$

Then, D is affiliated to $M_2(\mathcal{N})$ and $B \in M_2(\mathcal{N})$. By [CP1, Theorem 8, Appendix A], we have:

$$\|D(1+D^2)^{-1/2} - (D+B)(1+(D+B)^2)^{-1/2}\| \leqslant \|B\|.$$

A little calculation yields:

$$\|T(1+|T|^2)^{-1/2} - (T+A)(1+|T+A|^2)^{-1/2}\|$$
$$\leqslant \|D(1+D^2)^{-1/2} - (D+B)(1+(D+B)^2)^{-1/2}\| \leqslant \|B\| = \|A\|. \qquad \square$$

Corollary 3.12 (*Index continuity*). *If T is affiliated to $P\mathcal{N}Q$ and T is $(P\cdot Q)$-Fredholm then there exists $\varepsilon > 0$ so that if $A \in P\mathcal{N}Q$ and $\|A\| < \varepsilon$, then $T + A$ is $(P \cdot Q)$-Fredholm and*

$$Ind(T + A) = Ind(T).$$

Proposition 3.13 (*Compact perturbation*). *Let T be any closed, densely defined operator affiliated to $P\mathcal{N}Q$.*

(1) *If $k \in P\mathcal{K}_{\mathcal{N}}Q$, then the difference $T(1+|T|^2)^{-1/2} - (T+k)(1+|T+k|^2)^{-1/2}$ is in $P\mathcal{K}_{\mathcal{N}}Q$!*
(2) *If T is $(P \cdot Q)$-Fredholm then for all $k \in P\mathcal{K}_{\mathcal{N}}Q$, $T + k$ is $(P \cdot Q)$-Fredholm and*

$$Ind(T + k) = Ind(T).$$

Proof. We prove the surprisingly subtle (and rather surprising!) first statement, since part (2) is an immediate corollary by Proposition 3.10 and Corollary 3.8. By the 2×2 matrix trick, we can assume that T and k are self-adjoint and that $P = Q = 1$. By the resolvent equation:

$$(T + i1)^{-1} - (T + k + i1)^{-1} = (T + i1)^{-1}k(T + k + i1)^{-1} \in \mathcal{K}_{\mathcal{N}}.$$

However the identity, $(T+i1)^{-1} = T(1+T^2)^{-1} - i(1+T^2)^{-1}$ and the corresponding identity for $(T+k+i1)^{-1}$ imply that:

$$T(1 + T^2)^{-1} - (T + k)(1 + (T + k)^2)^{-1} \in \mathcal{K}_{\mathcal{N}}$$

since this difference is the self-adjoint part of an element in the C^*-algebra $\mathcal{K}_{\mathcal{N}}$. Now for $\mu > 0$ real we can replace T with μT and $(T+k)$ with $\mu(T+k)$ and get:

$$\mu\left\{\mu T(1 + (\mu T)^2)^{-1} - \mu(T + k)(1 + (\mu(T + k))^2)^{-1}\right\} \in \mathcal{K}_{\mathcal{N}}.$$

For any real $\lambda \geqslant 0$, let $\mu = (1+\lambda)^{-1/2}$ and a little calculation yields:

$$T(1 + T^2 + \lambda)^{-1} - (T + k)(1 + (T + k)^2 + \lambda)^{-1} \in \mathcal{K}_{\mathcal{N}}.$$

By [CP1, Lemma 6, Appendix A] we have the estimate:

$$\|T(1 + T^2 + \lambda)^{-1} - (T + k)(1 + (T + k)^2 + \lambda)^{-1}\| \leqslant \frac{\|k\|}{1 + \lambda},$$

and hence the following integral converges absolutely in operator norm to an element in the C^*-algebra $\mathcal{K}_{\mathcal{N}}$:

$$\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\left(T(1+T^2+\lambda)^{-1}-(T+k)(1+(T+k)^2+\lambda)^{-1}\right)d\lambda.$$

If we call this element k_0, then by [CP1, Lemma 4, Appendix A], we have for all $\xi \in dom(T) = dom(T+k)$ that the following integrals *converge in* $\mathcal{H}$ and:

$$\begin{aligned}
&T(1+T^2)^{-1/2}(\xi)-(T+k)(1+(T+k)^2)^{-1/2}(\xi)\\
&\quad=\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}T(1+T^2+\lambda)^{-1}(\xi)\,d\lambda\\
&\qquad-\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(T+k)(1+(T+k)^2+\lambda)^{-1}(\xi)\,d\lambda\\
&\quad=\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\left(T(1+T^2+\lambda)^{-1}-(T+k)(1+(T+k)^2+\lambda)^{-1}\right)(\xi)\,d\lambda=k_0(\xi).
\end{aligned}$$

As both side of this equation are bounded operators, we have:

$$T(1+T^2)^{-1/2}-(T+k)(1+(T+k)^2)^{-1/2}=k_0\in\mathcal{K}_{\mathcal{N}}. \qquad \square$$

Definition 3.14. For many geometric examples, the following is a useful notion. If T is a closed, densely defined, unbounded operator affiliated to $P\mathcal{N}Q$ then a *parametrix* for T is a bounded everywhere defined operator $S \in Q\mathcal{N}P$ so that:

(1) $\overline{TS} = P + k_1$ for $k_1 \in P\mathcal{K}_{\mathcal{N}}P$,
(2) $\overline{ST} = Q + k_2$ for $k_2 \in Q\mathcal{K}_{\mathcal{N}}Q$.

Note, since T is closed and S is bounded, $TS = \overline{TS}$ is everywhere defined and bounded by (1). For example, if D is an unbounded self-adjoint operator and $(1+D^2)^{-1} \in \mathcal{K}_{\mathcal{N}}$ then $D(1+D^2)^{-1}$ is a parametrix for D since $\overline{D(1+D^2)^{-1}D} = D^2(1+D^2)^{-1} = 1-(1+D^2)^{-1}$.

Lemma 3.15. *If T is a closed, densely defined, unbounded operator affiliated to $P\mathcal{N}Q$ then T has a parametrix if and only if T is $(P\cdot Q)$-Fredholm.*

Proof. If S is a parametrix for T then by (1) TS is everywhere defined and Fredholm in $P\mathcal{N}P$. So there exists a projection $E \leqslant P$ with $\tau(E) < \infty$ and: $range(P-E) \subset range(TS) \subset range(T)$. In particular, this implies (since TS is bounded) that $N^P_{(TS)^*}$ is τ-finite. But $S^*T^* \subseteq (TS)^*$ and so $N^P_{T^*} \leqslant N^P_{(TS)^*}$. That is, $\tau(N^P_{T^*}) < \infty$. Now, $\overline{ST} = Q + k_2$ is $(Q\cdot Q)$-Fredholm and so has a τ-finite Q-kernel. But $N^Q_T \leqslant N^Q_{\overline{ST}}$. That is, $\tau(N^Q_T) < \infty$ and T is $(P\cdot Q)$-Fredholm.

If $T = V|T|$ is $(P \cdot Q)$-Fredholm then $|T|(1+|T|^2)^{-1/2}$ is bounded and $(Q \cdot Q)$-Fredholm and so has a parametrix S which we can take to be a function of $|T|(1+|T|^2)^{-1/2}$. Thus S commutes with $(1+|T|^2)^{-1/2}$. One then checks that $(1+|T|^2)^{-1/2}SV^*$ is a parametrix for T. $\square$

Remark. In general a parametrix for a genuinely unbounded Fredholm operator is *not* Fredholm as its range cannot contain the range of a cofinite projection.

Theorem 3.16 (*McKean–Singer*). *Let D be an unbounded self-adjoint operator affiliated to the semifinite von Neumann algebra $\mathcal{N}$ (with faithful normal semifinite trace τ). Let γ be a self-adjoint unitary in $\mathcal{N}$ which anticommutes with D. Finally, let f be a continuous even function on $\mathbf{R}$ with $f(0) \neq 0$ and $f(D)$ trace-class. Let $D^+ = P^\perp D P$ where $P = (\gamma+1)/2$ and $P^\perp = 1 - P$. Then as an operator affiliated to $P^\perp \mathcal{N} P$, D^+ is $(P^\perp \cdot P)$-Fredholm and*

$$Ind(D^+) = \frac{1}{f(0)} \tau(\gamma f(D)).$$

Proof. Let $D^- = PDP^\perp$. Since $\{D, \gamma\} = 0$, we see that relative to the decomposition $1 = P \oplus P^\perp$:

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \quad D^2 = \begin{pmatrix} D^-D^+ & 0 \\ 0 & D^+D^- \end{pmatrix},$$

$$|D| = \begin{pmatrix} |D^+| & 0 \\ 0 & |D^-| \end{pmatrix}.$$

We have already observed that $D(1+D^2)^{-1}$ is a parametrix for D. But, then:

$$D(1+D^2)^{-1} = \begin{pmatrix} 0 & D^-(P^\perp + |D^-|^2)^{-1} \\ D^+(P + |D^+|^2)^{-1} & 0 \end{pmatrix}.$$

Hence $D^-(P^\perp + |D^-|^2)^{-1}$ is a parametrix for D^+ and so D^+ is $(P^\perp \cdot P)$-Fredholm. Let $D^+ = V|D^+|$ be the polar decomposition of D^+ so that $D^- = D^{+*} = |D^+|V^*$. Then $V \in \mathcal{N}$ is a partial isometry with initial space $P_1 = V^*V = supp(D^+) \leqslant P$ and final space $Q_1 = VV^* = range(D^+)^- = supp(D^-) \leqslant P^\perp$. Then, $ker(D^+) = P_0(\mathcal{H})$ as an operator on $P(\mathcal{H})$ where $P_0 = P - P_1$. Similarly, $coker(D^+) = ker(D^-) = Q_0(\mathcal{H})$ where $Q_0 = P^\perp - Q_1$.

Now, $|D^+|^2 = D^-D^+ = D^-D^{-*} = V^*|D^-|^2V$ so that $|D^+| = V^*|D^-|V$ and if g is any bounded continuous function then, $g(|D^+||_{P_1(\mathcal{H})}) = V^*g(|D^-||_{Q_1(\mathcal{H})})V$. But, as operators on $P(\mathcal{H})$, and respectively, $P^\perp(\mathcal{H})$ we have:

$$g(|D^+|) = P_1 g(|D^+|)P_1 \oplus g(0)P_0 = g(|D^+||_{P_1(\mathcal{H})}) \oplus g(0)P_0 \text{ and}$$

$$g(|D^-|) = Q_1 g(|D^-|)Q_1 \oplus g(0)Q_0 = g(|D^-||_{Q_1(\mathcal{H})}) \oplus g(0)Q_0.$$

Finally, since f is even, we have $f(D) = f(|D|)$ and so:

$$\gamma f(D) = \begin{pmatrix} f(|D^+|) & 0 \\ 0 & -f(|D^-|) \end{pmatrix}$$
$$= \begin{pmatrix} f(|D^+|_{|P_1(\mathcal{H})}) \oplus f(0)P_0 & 0 \\ 0 & -f(|D^-|_{|Q_1(\mathcal{H})}) \oplus -f(0)Q_0 \end{pmatrix}$$
$$= \begin{pmatrix} V^* f(|D^-|_{|Q_1(\mathcal{H})})V \oplus f(0)P_0 & 0 \\ 0 & -f(|D^-|_{|Q_1(\mathcal{H})}) \oplus -f(0)Q_0 \end{pmatrix}.$$

Hence,

$$\tau(\gamma f(D)) = f(0)\tau(P_0) - f(0)\tau(Q_0) = f(0)Ind(D^+). \qquad \square$$

Corollary 3.17. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an even spectral triple with grading γ, $(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^n(\mathcal{N})$ and $p \in \mathcal{A}$, a projection. Then, relative to the decomposition afforded by γ as above, we have*:

$$p = \begin{pmatrix} p^+ & 0 \\ 0 & p^- \end{pmatrix}, \quad where \ \ p^+ = PpP = Pp \ \ and \ \ p^- = P^\perp p P^\perp = pP^\perp.$$

So, $p\mathcal{D}^+p = pP^\perp \mathcal{D} P p = p^- \mathcal{D} p^+$ is an operator affiliated to $p^- \mathcal{N} p^+$ and we have that $p^-\mathcal{D}^+p^+$ is $(p^- \cdot p^+)$-Fredholm and for any fixed $a \geqslant 0$ its $(p^- \cdot p^+)$-index is given by

$$Ind(p\mathcal{D}^+p) = Ind(p^-\mathcal{D}^+p^+) = (1+a)^{n/2}\tau\left(\gamma p\left(p + a + (p\mathcal{D}p)^2\right)^{-n/2}\right).$$

Proof. In the above version of the McKean–Singer theorem, we replace $\mathcal{A}$ with $p\mathcal{A}p$ which is a unital subalgebra of the semifinite von Neumann algebra $p\mathcal{N}p$. Moreover, the operator $p\mathcal{D}p$ is self-adjoint and affiliated to $p\mathcal{N}p$, and $p\gamma$ is a grading in $p\mathcal{N}p$. One easily checks that

$$(p\mathcal{D}p)^+ = p^-\mathcal{D}^+p^+.$$

Letting $f(x) = (1 + a + x^2)^{-n/2}$, we can apply the McKean–Singer theorem once we show that $(p + a + (p\mathcal{D}p)^2)^{-1/2} \in \mathcal{L}^n(p\mathcal{N}p)$. It suffices to do this for $a = 0$ since

$$(p + a + (p\mathcal{D}p)^2)^{-1/2} \leqslant (p + (p\mathcal{D}p)^2)^{-1/2}.$$

This is a careful calculation:

$$
\begin{aligned}
& p(1+\mathcal{D}^2)^{-1}p - (p+(p\mathcal{D}p)^2)^{-1} \\
&\quad = p[(1+\mathcal{D}^2)^{-1} - \{(p+(p\mathcal{D}p)^2)+(1-p)\}^{-1}]p \\
&\quad = p(1+\mathcal{D}^2)^{-1}\Big((p+(p\mathcal{D}p)^2)+(1-p)-(1+\mathcal{D}^2)\Big) \\
&\qquad \times\{(p+(p\mathcal{D}p)^2)+(1-p)\}^{-1}p \\
&\quad = p(1+\mathcal{D}^2)^{-1}\Big((p\mathcal{D}p)^2-\mathcal{D}^2\Big)p(p+(p\mathcal{D}p)^2)^{-1}p \\
&\quad = p(1+\mathcal{D}^2)^{-1}\left([p,\mathcal{D}]p\mathcal{D}p+\mathcal{D}p[\mathcal{D},p]+\mathcal{D}[p,\mathcal{D}]\right)p(p+(p\mathcal{D}p)^2)^{-1}p \\
&\quad = p(1+\mathcal{D}^2)^{-1}[p,\mathcal{D}]p\mathcal{D}p(p+(p\mathcal{D}p)^2)^{-1}p \\
&\qquad +p(1+\mathcal{D}^2)^{-1}\mathcal{D}p[\mathcal{D},p]p(p+(p\mathcal{D}p)^2)^{-1}p \\
&\qquad +p(1+\mathcal{D}^2)^{-1}\mathcal{D}[p,\mathcal{D}]p(p+(p\mathcal{D}p)^2)^{-1}p.
\end{aligned}
$$

Now, since $|\mathcal{D}(1+\mathcal{D}^2)^{-1}| \leqslant (1+\mathcal{D}^2)^{-1/2}$, we have that three terms in the last lines are in $\mathcal{L}^{n/2}$, $\mathcal{L}^n$, and $\mathcal{L}^n$, respectively, and so their sum is in $\mathcal{L}^n$. Since, $p(1+\mathcal{D}^2)^{-1}p \in \mathcal{L}^{n/2}$, we see from the first line in the displayed equations that $(p+(p\mathcal{D}p)^2)^{-1}$ is in $\mathcal{L}^n$.

Now, armed with this new information, we look at the three terms in the last line again, and see that they are in $\mathcal{L}^{n/2}$, $\mathcal{L}^n \cdot \mathcal{L}^n$, and $\mathcal{L}^n \cdot \mathcal{L}^n$, respectively, and so their sum is in $\mathcal{L}^{n/2}$. Thus, $(p+(p\mathcal{D}p)^2)^{-1}$ is, in fact, in $\mathcal{L}^{n/2}$: in other words, $(p+(p\mathcal{D}p)^2)^{-1/2}$ is in $\mathcal{L}^n$ as claimed. □

From now on, we follow convention and denote the above index by $Ind(p\mathcal{D}^+p)$; effectively disguising the fact that $p\mathcal{D}^+p$ is, in fact, Fredholm relative to the "skew-corner," $p^-\mathcal{N}p^+$.

Remark. The ideal $\mathcal{L}^n(\mathcal{N})$ can be replaced by any symmetric ideal $\mathcal{I} \subset \mathcal{K}_N$ provided we use an even function f satisfying $f(|T|) \in \mathcal{L}^1$ for all $T \in \mathcal{I}$. The formula then becomes:

$$Ind(p\mathcal{D}^+p) = (1/f(0))\tau\left(\gamma p f\left(\left(p+(p\mathcal{D}p)^2\right)^{-1/2}\right)\right).$$

In particular, if $(\mathcal{A},\mathcal{H},\mathcal{D})$ is θ-summable, and $f(x)=e^{-tx^2}$, $t>0$, the formula becomes:

$$Ind(p\mathcal{D}^+p) = \tau\left(\gamma p e^{-t(p\mathcal{D}p)^2}\right).$$

4. Statement of the main result

We use the notation of [CPRS2]. Denote multi-indices by $(k_1,\ldots,k_m)$, $k_i=0,1,2,\ldots$, whose length m will always be clear from the context and let $|k|=k_1+\cdots+k_m$. Define

$$\alpha(k)=(k_1!k_2!\cdots k_m!(k_1+1)(k_1+k_2+2)\cdots(|k|+m))^{-1}$$

and $\sigma_{n,j}$ (the elementary symmetric functions of $\{1,\ldots,n\}$) by $\prod_{j=0}^{n-1}(z+j)=\sum_{j=1}^{n}z^j\sigma_{n,j}$. If $(\mathcal{A},\mathcal{H},\mathcal{D})$ is a QC^∞ spectral triple and $T\in\mathcal{N}$ then $T^{(n)}$ is the nth iterated commutator with $\mathcal{D}^2$, that is, $[\mathcal{D}^2,[\mathcal{D}^2,[\cdots,[\mathcal{D}^2,T]\cdots]]]$.

We let $q=\inf\{k\in\mathbf{R}:\tau((1+\mathcal{D}^2)^{-k/2})<\infty\}$ be the *spectral dimension* of $(\mathcal{A},\mathcal{H},\mathcal{D})$ and we assume it is *isolated*, *i.e.*, for

$$b=a_0[\mathcal{D},a_1]^{(k_1)}\cdots[\mathcal{D},a_m]^{(k_m)}(1+\mathcal{D}^2)^{-m/2-|k|}$$

the zeta functions

$$\zeta_b(z-(1-q)/2)=\tau(b(1+\mathcal{D}^2)^{-z+(1-q)/2})$$

have analytic continuations to a deleted neighbourhood of $z=(1-q)/2$. As in [CPRS2] we let $\tau_j(b)=res_{z=(1-q)/2}(z-(1-q)/2)^j\zeta_b(z-(1-q)/2)$. Our main result is:

Theorem 4.1 (*Semifinite even local index theorem*). *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be an even QC^∞ spectral triple with spectral dimension $q\geqslant 1$. Let $N=[\frac{q+1}{2}]$, where $[\cdot]$ denotes the integer part, and let $p\in\mathcal{A}$ be a self-adjoint projection. Then*

(1) *$Ind(p\mathcal{D}^+p)=res_{r=(1-q)/2}\left(\sum_{m=0,even}^{2N}\phi_m^r(Ch_m(p))\right)$, where for $a_0,\ldots,a_m\in\mathcal{A}$, $l=\{a+iv:v\in\mathbf{R}\}$, $0<a<1/2$, $R_s(\lambda)=(\lambda-(1+s^2+\mathcal{D}^2))^{-1}$ and $r>1/2$ we define $\phi_m^r(a_0,a_1,\ldots,a_m)$ to be*

$$\frac{(m/2)!}{m!}\int_0^\infty 2^{m+1}s^m\tau\left(\gamma\frac{1}{2\pi i}\int_l\lambda^{-q/2-r}a_0R_s(\lambda)[\mathcal{D},a_1]R_s(\lambda)\cdots[\mathcal{D},a_m]R_s(\lambda)\,d\lambda\right)ds.$$

In particular the sum on the right-hand side of (1) *analytically continues to a deleted neighbourhood of $r=(1-q)/2$ with* at worst *a simple pole at $r=(1-q)/2$. Moreover, the complex function-valued cochain $(\phi_m^r)_{m=0,even}^{2N}$ is a (b,B) cocycle for $\mathcal{A}$ modulo functions holomorphic in a half-plane containing $r=(1-q)/2$.*

(2) *The index*, $Ind(p\mathcal{D}^+p)$ *is also the residue of a sum of zeta functions*:

$$res_{r=(1-q)/2}\left(\sum_{m=0,even}^{2N}\sum_{|k|=0}^{2N-m}\sum_{j=1}^{|k|+m/2}(-1)^{|k|+m/2}\alpha(k)\frac{(m/2)!}{2m!}\sigma_{|k|+m/2,j}\right.$$

$$\times(r-(1-q)/2)^j\tau\Big(\gamma(2p-1)[\mathcal{D},p]^{(k_1)}[\mathcal{D},p]^{(k_2)}$$

$$\left.\times\cdots[\mathcal{D},p]^{(k_m)}(1+\mathcal{D}^2)^{-m/2-|k|-r+(1-q)/2}\Big)\right),$$

(*for* $m=0$ *we replace* $(2p-1)$ by $2p$). *In particular the sum of zeta functions on the right-hand side analytically continues to a deleted neighbourhood of* $r=(1-q)/2$ *and has* at worst *a simple pole at* $r=(1-q)/2$.

(3) *If* $(\mathcal{A},\mathcal{H},\mathcal{D})$ *also has isolated spectral dimension then*

$$Ind(p\mathcal{D}^+p)=\sum_{m=0,even}^{2N}\phi_m(Ch_m(p)),$$

where for $a_0,\ldots,a_m\in\mathcal{A}$ *we have* $\phi_0(a_0)=res_{r=(1-q)/2}\phi_0^r(a_0)=\tau_{-1}(\gamma a_0)$ *and for* $m\geqslant 2$

$$\phi_m(a_0,\ldots,a_m)=res_{r=(1-q)/2}\phi_m^r(a_0,\ldots,a_m)=\sum_{|k|=0}^{2N-m}(-1)^{|k|}\alpha(k)$$

$$\times\sum_{j=1}^{|k|+m/2}\sigma_{(|k|+m/2),j}\tau_{j-1}\left(\gamma a_0[\mathcal{D},a_1]^{(k_1)}\cdots[\mathcal{D},a_m]^{(k_m)}(1+\mathcal{D}^2)^{-|k|-m/2}\right),$$

and $(\phi_m)_{m=0,even}^{2N}$ *is a* (b,B) *cocycle for* $\mathcal{A}$. *When* $[q]=2n+1$ *is odd*, *the term with* $m=2N$ *is zero*, *and for* $m=0,2,\ldots,2N-2$, *all the top terms with* $|k|=2N-m$ *are zero*.

Corollary 4.2. *For* $1\leqslant q<2$, *the statements in* (3) *of Theorem* 4.1 *are true without the assumption of isolated dimension spectrum*.

5. The local index theorem in the even case

The main technical device that improves the proof of the local index theorem of [CoM] for odd spectral triples stems from our use in [CPRS2] of the resolvent cocycle to reduce the hypotheses needed for the theorem and most importantly to provide a simple proof that the (renormalised) residue cocycle of Connes–Moscovici is an index cocycle. We will see that these improvements also apply in the even case.

In this Section we will derive the formulae for the index appearing in parts (2) and (3) of Theorem 4.1. The exposition is broken down into six subsections. Each subsection ends with a new formula for the index which the next subsection builds on until we eventually obtain, in Section 5.5, part (2) of the main theorem. In Section 5.6 we will prove the index formula in part (3) of Theorem 4.1. Our starting point is the McKean–Singer formula (Corollary 3.17) for the index while in [CPRS2] the starting point was the spectral flow formula of Carey–Phillips [CP2].

5.1. Exploiting Clifford–Bott periodicity

We utilise an idea of Getzler from [G] adapted to a more functional analytic setting based on [CP0]. We begin with an even semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ with $\mathbf{Z}_2$-grading γ. We will assume that this spectral triple is n-summable for any $n > q$ with $q \geqslant 1$ fixed once and for all. If $p \in \mathcal{A}$ then our aim is to derive from McKean–Singer a new formula for the index of $p\mathcal{D}^+ p = p_-\mathcal{D}^+ p_+$ where $\mathcal{D}^+ = (1-\gamma)\mathcal{D}(1+\gamma)/4 = P^\perp \mathcal{D} P$ and $p_+ = PpP$ and $p_- = P^\perp p P^\perp$. (Note that what follows differs significantly from what is done in [CPRS2].)

Definition 5.1. Form the Hilbert space $\tilde{\mathcal{H}} = \mathbf{C}^2 \otimes \mathcal{H}$ on which acts the semifinite von Neumann algebra $\tilde{\mathcal{N}} = M_2(\mathbf{C}) \otimes \mathcal{N}$. Introduce the two-dimensional Clifford algebra in the form

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let 1_2 denote the 2×2 identity matrix and define the grading in $\tilde{\mathcal{N}}$ by $\tilde{\gamma} = \sigma_3 \otimes \gamma$ and a Clifford element $\tilde{\sigma}_2 = \sigma_2 \otimes 1 \in \tilde{\mathcal{N}}$ which anticommutes with $\tilde{\gamma}$ where 1 is the identity operator in $\mathcal{N}$.

Let $p \in \mathcal{A}$ be a projection. Introduce the following operators affiliated to $\tilde{\mathcal{N}}$ on $\tilde{\mathcal{H}}$:

$$\tilde{\mathcal{D}} = \sigma_3 \otimes \mathcal{D}, \qquad \mathcal{D}_p = p\mathcal{D}p + (1-p)\mathcal{D}(1-p) = \mathcal{D} + [\mathcal{D}, p](1-2p),$$

$$\begin{aligned} \mathcal{D}_w &= (1-w)\mathcal{D} + w(p\mathcal{D}p + (1-p)\mathcal{D}(1-p)) = (1-w)\mathcal{D} + w\mathcal{D}_p \\ &= \mathcal{D} + w[\mathcal{D}, p](1-2p), \end{aligned}$$

and noting that $\tilde{\sigma}_2\,(1_2 \otimes (2p-1)) = \sigma_2 \otimes (2p-1)$, we define:

$$\begin{aligned} &\tilde{\mathcal{D}}_{w,s} = \sigma_3 \otimes \mathcal{D}_w + s(\sigma_2 \otimes (2p-1)) =: \tilde{\mathcal{D}}_w + s(\sigma_2 \otimes (2p-1)), \\ &w \in [0,1], s \in (-\infty, \infty). \end{aligned}$$

Note that $\sigma_3 \otimes \mathcal{D}_w$ is odd (i.e., anticommutes with the grading $\tilde{\gamma}$) and that $\tilde{\sigma}_2$ and $\sigma_3 \otimes \mathcal{D}_w$ anticommute. Notice that $\frac{d}{ds}\tilde{\mathcal{D}}_{w,s} = \sigma_2 \otimes (2p-1)$.

We extend the trace τ on $\mathcal{N}$ to $\tau_2 := Tr_2 \otimes \tau$ on $\tilde{\mathcal{N}}$ by taking the matrix trace Tr_2 in the first tensor factor. There is a graded Clifford trace (super trace) on $\tilde{\mathcal{N}}$ which we write as $S\tau(T) = \frac{1}{2}\tau_2((\sigma_3 \otimes 1)\tilde{\gamma}T)$, $T \in \tilde{\mathcal{N}}$, and note that this reduces to $\tau(\gamma S)$ for $T = 1_2 \otimes S \in \tilde{\mathcal{N}}$.

Now

$$\begin{aligned}\tilde{\mathcal{D}}^2_{w,s} &= 1_2 \otimes \mathcal{D}_w^2 - s\sigma_3\sigma_2 \otimes (2p-1)\mathcal{D}_w + s\sigma_3\sigma_2 \otimes \mathcal{D}_w(2p-1) + s^2 \\ &= 1_2 \otimes \mathcal{D}_w^2 + 2s(1-w)\sigma_3\sigma_2 \otimes [\mathcal{D}, p] + s^2.\end{aligned}$$

Here we used $\mathcal{D}_w(2p-1) - (2p-1)\mathcal{D}_w = 2(1-w)[\mathcal{D}, p]$. At $w = 0$ we have

$$\tilde{\mathcal{D}}^2_{0,s} = \begin{pmatrix} \mathcal{D}^2 + s^2 & -i2s[\mathcal{D}, p] \\ -i2s[\mathcal{D}, p] & \mathcal{D}^2 + s^2 \end{pmatrix} = \tilde{\mathcal{D}}^2 + s^2 + 2s\sigma_3\sigma_2 \otimes [\mathcal{D}, p]$$

and at $w = 1$:

$$\begin{aligned}\tilde{\mathcal{D}}^2_{1,s} &= \begin{pmatrix} (p\mathcal{D}p + (1-p)\mathcal{D}(1-p))^2 + s^2 & 0 \\ 0 & (p\mathcal{D}p + (1-p)\mathcal{D}(1-p))^2 + s^2 \end{pmatrix} \\ &= \tilde{\mathcal{D}}_p^2 + s^2,\end{aligned}$$

where, $\tilde{\mathcal{D}}_p := \sigma_3 \otimes \mathcal{D}_p$. Note that

$$\frac{d}{dw}\mathcal{D}_w = p\mathcal{D}p + (1-p)\mathcal{D}(1-p) - \mathcal{D} = [\mathcal{D}, p](1-2p).$$

Lemma 5.2. *Consider the affine space Φ of perturbations, $\hat{\mathcal{D}}$, of $\tilde{\mathcal{D}} = \sigma_3 \otimes \mathcal{D}$ given by*

$$\Phi = \{\hat{\mathcal{D}} = \tilde{\mathcal{D}} + X \ \mid X \in \tilde{\mathcal{N}}_{sa} \text{ and } [X, \sigma_2 \otimes \gamma] = 0\}.$$

Notice that each $\hat{\mathcal{D}}$ commutes with $\sigma_2 \otimes \gamma$. Let, for any $n > q$ and $\hat{\mathcal{D}} \in \Phi$

$$\alpha_{\hat{\mathcal{D}}}(Y) = \tau_2\left((\sigma_2 \otimes \gamma)Y(1 + \hat{\mathcal{D}}^2)^{-n/2}\right).$$

Then $\hat{\mathcal{D}} \mapsto \alpha_{\hat{\mathcal{D}}}$ is an exact one-form (i.e., an exact section of the cotangent bundle to Φ).

The proof of this Lemma is a trivial variation of the proof of Lemma 5.6 of [CPRS2]: in the notation of that lemma, let $\Gamma = \sigma_2 \otimes \gamma$ and multiply $S\tau$ by 2.

Now, for $n > q+1$ we introduce the function

$$a(w) = \frac{1}{4}\int_{-\infty}^{\infty} \tau_2\left((1_2 \otimes \{\gamma(2p-1)\})(1+\tilde{\mathcal{D}}_{w,s}^2)^{-n/2}\right) ds.$$

This integral converges absolutely due to the following two estimates. The first is from Lemma 5.2 of [CPRS2] (together with the remark immediately preceding that lemma) with $n = q + 2r$ and all $s \geqslant 2$:

$$\begin{aligned}\|(1+\tilde{\mathcal{D}}_{w,s}^2)^{-n/2}\|_1 &= \|(1+\tilde{\mathcal{D}}_{w,s}^2)^{-q/2-r}\|_1 \\ &\leqslant \|(1/2+\tilde{\mathcal{D}}_{w}^2)^{-(q/2+\varepsilon)}\|_1 \left(1/2+(1/2)(s^2)\right)^{-r+\varepsilon},\end{aligned}$$

where $r = \frac{n-q}{2} > \frac{1}{2} + \varepsilon$.

The second is from Corollary 8 of Appendix B of [CP1], letting $\tilde{\mathcal{D}}_w = \tilde{\mathcal{D}} + wA \in \Phi$ where A is in $\tilde{\mathcal{N}}_{sa}$. The cited result gives us a constant $C = C(A, q, \varepsilon) > 0$ such that

$$\|(1/2+\tilde{\mathcal{D}}_{w}^2)^{-(q/2+\varepsilon)}\|_1 \leqslant \tau_2\left((2(1+\tilde{\mathcal{D}}_{w}^2)^{-1})^{(q/2+\varepsilon)}\right) \leqslant C\tau_2\left((1+\tilde{\mathcal{D}}^2)^{-(q/2+\varepsilon)}\right).$$

So, with $n = q + 2r$ and $r > 1/2$, the function $a(w)$ is well-defined.

With that settled, we now observe that

$$\begin{aligned}a(w) &= \frac{1}{4}\int_{-\infty}^{\infty} \tau_2\left((\sigma_2\otimes\gamma)(\sigma_2\otimes(2p-1))(1+\tilde{\mathcal{D}}_{w,s}^2)^{-n/2}\right) ds \\ &= \frac{1}{4}\int_{-\infty}^{\infty} \tau_2\left((\sigma_2\otimes\gamma)\left(\frac{d}{ds}\tilde{\mathcal{D}}_{w,s}\right)(1+\tilde{\mathcal{D}}_{w,s}^2)^{-n/2}\right) ds\end{aligned}$$

with the last expression designed to link with the result of the previous lemma. In fact, $a(w)$ does not really depend on w as we now prove.

Lemma 5.3. *We have that $a(w)$ is constant, in particular, $a(0) = a(1)$.*

Proof. Exactness of the one-form α means that integral of α along any continuous piecewise smooth closed path in Φ must be 0. Consider the closed (rectangular) path β given by the four linear paths beginning with:

$$\beta_{0,N}(s) = \tilde{\mathcal{D}}_{0,s} \ \textit{for}\ \ s \in [-N, N];\ \text{then}\ \beta_N(w) = \tilde{\mathcal{D}}_{w,N}\ \textit{for}\ \ w \in [0,1]; \quad \text{then}$$

$$\beta_{1,N}(s) = \tilde{\mathcal{D}}_{1,-s}\ \textit{for}\ \ s \in [-N, N];\ \text{then}\ \beta_{-N}(w) = \tilde{\mathcal{D}}_{1-w,-N}\ \textit{for}\ \ w \in [0,1].$$

Then, the integral of α around β is 0. For example, the integral of α along β_N is

$$\int_0^1 \tau_2\left((\sigma_2\otimes\gamma)\frac{d}{dw}(\tilde{\mathcal{D}}_{w,N})(1+\tilde{\mathcal{D}}_{w,N}^2)^{-n/2}\right)dw = \int_0^1 \tau_2\left(B(1+\tilde{\mathcal{D}}_{w,N}^2)^{-n/2}\right)dw,$$

where $B=\sigma_2\sigma_3\otimes\gamma[\mathcal{D},p](1-2p)\in\tilde{\mathcal{N}}$. Now by the above estimates we have:

$$\|B(1+\tilde{\mathcal{D}}_{w,N}^2)^{-n/2}\|_1 \leqslant C\|B\|\tau_2\left((1+\tilde{\mathcal{D}}^2)^{-(q/2+\varepsilon)}\right)\left(1/2+(1/2)(N^2)\right)^{-r+\varepsilon},$$

which for $r>\varepsilon$ goes to 0 as $N\to\infty$. Similarly, the integral along β_{-N} goes to 0 as $N\to\infty$.
Now the integral of α along $\beta_{0,N}$ is:

$$\int_{-N}^N \tau_2\left((1_2\otimes\gamma(2p-1))(1+\tilde{\mathcal{D}}_{0,s}^2)^{-n/2}\right)ds \to 4a(0) \ \ as \ \ N\to\infty.$$

Similarly, the integral of α along $\beta_{1,N}$ converges to $-4a(1)$ as $N\to\infty$. That is, $4a(0)-4a(1)=0$ or $a(0)=a(1)$. Similarly, $a(0)=a(w)$ for any $w\in[0,1]$. $\square$

Using the preceding lemma we obtain

$$a(1)=a(0)=\frac{1}{4}\int_{-\infty}^{\infty}\tau_2\left((1_2\otimes\gamma(2p-1))(1+\tilde{\mathcal{D}}_{0,s}^2)^{-n/2}\right)ds$$

and thus we can calculate $a(0)$ and $a(1)$ to obtain two different expressions for the same quantity.

For the next calculation, observe that the definition of $a(w)$ gives

$$a(1)=\frac{1}{4}\int_{-\infty}^{\infty}\tau_2\left((1_2\otimes\gamma(2p-1))(1+\tilde{\mathcal{D}}_{1,s}^2)^{-n/2}\right)ds$$

and inserting $\tilde{\mathcal{D}}_{1,s}^2=\tilde{\mathcal{D}}_p^2+s^2$, we get by an application of McKean–Singer (Corollary 3.17):

$$\begin{aligned} a(1) &= \frac{1}{4}\int_{-\infty}^{\infty}\tau_2\left((1_2\otimes\gamma(2p-1))(1+(p\tilde{\mathcal{D}}p+(1-p)\tilde{\mathcal{D}}(1-p))^2+s^2)^{-n/2}\right)ds \\ &= \int_{-\infty}^{\infty}\tau\left(\gamma p(1+(p\mathcal{D}p)^2+s^2)^{-n/2}\right)ds \end{aligned}$$

$$-\frac{1}{2}\int_{-\infty}^{\infty}\tau\left(\gamma(1+(p\mathcal{D}p+(1-p)\mathcal{D}(1-p))^2+s^2)^{-n/2}\right)ds$$

$$= Ind(p\mathcal{D}^+p)\int_{-\infty}^{\infty}(1+s^2)^{-n/2}\,ds-\frac{1}{2}\int_{-\infty}^{\infty}\tau\left(\gamma(1+\mathcal{D}_p^2+s^2)^{-n/2}\right)ds.$$

We put Lemma 5.2 to work again to get rid of the subscript p in the last integral above.

Lemma 5.4. *With the hypotheses as above and* $n=q+2r>q+1$, *we have*:

$$\int_{-\infty}^{\infty}\tau\left(\gamma(1+\mathcal{D}_p^2+s^2)^{-n/2}\right)ds=\int_{-\infty}^{\infty}\tau\left(\gamma(1+\mathcal{D}^2+s^2)^{-n/2}\right)ds.$$

Proof. For $w\in[0,1]$ and $s\in\mathbf{R}$ we let:

$$\hat{\mathcal{D}}_{w,s}=\tilde{\mathcal{D}}_w+s(\sigma_2\otimes 1)=\tilde{\mathcal{D}}+wA+s(\sigma_2\otimes 1),$$

where $A=\sigma_3\otimes([\mathcal{D},p](1-2p))$, so that $\tilde{\mathcal{D}}+A=\tilde{\mathcal{D}}_p$. Then both perturbations of $\tilde{\mathcal{D}}$ commute with $\sigma_2\otimes\gamma$ and therefore $\hat{\mathcal{D}}_{w,s}\in\Phi$. Moreover $\sigma_2\otimes 1$ anticommutes with $\tilde{\mathcal{D}}$ and with A, and so $\hat{\mathcal{D}}_{w,s}^2=\tilde{\mathcal{D}}_w^2+s^2$. In particular, $\hat{\mathcal{D}}_{0,s}^2=\tilde{\mathcal{D}}^2+s^2$ and $\hat{\mathcal{D}}_{1,s}^2=\tilde{\mathcal{D}}_p^2+s^2$. One now applies Lemma 5.2 to the closed rectangular path in Φ described as follows:

$$\beta_{0,N}(s)=\hat{\mathcal{D}}_{0,s}\ \textit{for}\ s\in[-N,N];\ \text{then}\ \beta_N(w)=\hat{\mathcal{D}}_{w,N}\ \textit{for}\ w\in[0,1];\quad\text{then}$$

$$\beta_{1,N}(s)=\hat{\mathcal{D}}_{1,-s}\ \textit{for}\ s\in[-N,N];\ \text{then}\ \beta_{-N}(w)=\hat{\mathcal{D}}_{1-w,-N}\ \textit{for}\ w\in[0,1].$$

As in the previous lemma, the integral of the one-form along β_N equals:

$$\int_0^1\tau_2\left((\sigma_2\otimes\gamma)A(1+\hat{\mathcal{D}}_{w,N}^2)^{-n/2}\right)dw$$

and converges to 0 as $N\to\infty$. Similarly, the integral along β_{-N} goes to 0 as $N\to\infty$.

Moreover, the integral along $\beta_{0,N}$ equals:

$$\int_{-N}^{N}\tau_2\left((\sigma_2\otimes\gamma)(\sigma_2\otimes 1)(1+\tilde{\mathcal{D}}^2+s^2)^{-n/2}\right)ds=2\int_{-N}^{N}\tau\left(\gamma(1+\mathcal{D}^2+s^2)^{-n/2}\right)ds,$$

which as $N\to\infty$ converges to $2\int_{-\infty}^{\infty}\tau\left(\gamma(1+\mathcal{D}^2+s^2)^{-n/2}\right)ds$. Similarly, the integral along $\beta_{1,N}$ converges to: $-2\int_{-\infty}^{\infty}\tau\left(\gamma(1+\mathcal{D}_p^2+s^2)^{-n/2}\right)ds$. The proof is completed by observing that the integral around the closed path β is 0. □

This establishes the main formula of this section:

Lemma 5.5. *For $n = q + 2r > q + 1$ we have*:

$$
\begin{aligned}
Ind(p\mathcal{D}^+ p)C_{n/2} &= a(1) + \frac{1}{2}\int_{-\infty}^{\infty} \tau\left(\gamma(1+\mathcal{D}^2+s^2)^{-n/2}\right) ds \\
&= \frac{1}{4}\int_{-\infty}^{\infty} \tau_2\left((1_2 \otimes \gamma(2p-1))(1+\tilde{\mathcal{D}}_{0,s}^2)^{-n/2}\right) ds \\
&\quad + \int_0^{\infty} \tau\left(\gamma(1+\mathcal{D}^2+s^2)^{-n/2}\right) ds
\end{aligned}
$$

where

$$
C_{n/2} = \int_{-\infty}^{\infty} (1+s^2)^{-n/2}\, ds = \frac{\Gamma(1/2)\Gamma(n/2-1/2)}{\Gamma(n/2)}.
$$

Note that $C_{n/2}$ is the normalisation 'constant' that appeared in [CPRS2]. Given the expression in terms of Γ functions we may take n as a complex variable and see that the first pole is at $n = 1$. If we write $n = q + 2r$ then the pole is at $r = (1-q)/2$ which is the origin of the critical point in the zeta functions in our main theorem. We reiterate that the above formula is only valid for $r > 1/2$ but that the LHS gives an analytic continuation of the RHS to a deleted neighbourhood of this critical point $r = (1-q)/2$.

5.2. Resolvent expansion of the index

In this subsection we will take the index formula of the preceding lemma and apply a resolvent expansion to the integrand. We begin with some notation. Let $N = [(q+1)/2]$, where $[\cdot]$ denotes the integer part. If q is an even integer, then $N = q/2$. If q is an odd integer, then $N = (q+1)/2$. In general, since $N \leqslant (q+1)/2 < N+1$ we have $2N - 1 \leqslant q < 2N + 1$, so that $2N - 1$ is the greatest odd integer in q. Also, $2N \leqslant q$ whenever $2n \leqslant q < 2n + 1$ for some positive integer n. In all cases $2N + 2 > q$.

We allow $q \geqslant 1$, so $(1+\mathcal{D}^2)^{-n/2} \in \mathcal{L}^1(\mathcal{N})$ for all $n > q$. By scale invariance of the index, we may replace $\mathcal{D}$ by $\varepsilon\mathcal{D}$ without changing the index. Since we need $\|[\mathcal{D}, p]\| < \sqrt{2}$ below, we assume this without further comment. We now make use of the Clifford structure. It allows us to employ the resolvent expansion to study $a(0)$, and we need only retain the even terms.

Lemma 5.6. *Let l be the line $\{\lambda = a + iv : -\infty < v < \infty\}$ where $0 < a < 1/2$ is fixed. There exists $1 > \delta > 0$ such that for $r > 1/2$*

$$
\begin{aligned}
a(1) = a(0) &= \frac{1}{2}\int_{-\infty}^{\infty} S\tau(\{1_2 \otimes (2p-1)\}(1+\tilde{\mathcal{D}}_{0,s}^2)^{-q/2-r})\, ds \\
&= \frac{1}{2}\int_{-\infty}^{\infty} S\tau\left(\frac{1}{2\pi i}\int_l \lambda^{-q/2-r}(1_2 \otimes (2p-1))(\lambda - (1+\tilde{\mathcal{D}}_{0,s}^2))^{-1}\, d\lambda\right) ds
\end{aligned}
$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} S\tau\left(\frac{1}{2\pi i}\int_l \lambda^{-q/2-r}(1_2 \otimes (2p-1))\right.$$
$$\times \sum_{m=0}^{2N}\Big((\lambda-(1+s^2+\tilde{\mathcal{D}}^2))^{-1}2s\sigma_3\sigma_2 \otimes [\mathcal{D}, p]\Big)^m$$
$$\left.\times (\lambda-(1+s^2+\tilde{\mathcal{D}}^2))^{-1}\, d\lambda\right) ds + holo, \tag{2}$$

where holo is a function of r holomorphic for $Re(r) > (1-q)/2 - \delta/2$.

Proof. The first equality is just Cauchy's formula $f(z) = \frac{1}{2\pi i}\int_l f(\lambda)(\lambda - z)^{-1}\, d\lambda$ (see the introductory remarks of Section 6.2 of [CPRS2] addressing the issue of convergence). The expansion in the statement of the Lemma is just the resolvent expansion:

$$\tilde{R}_s(\lambda) = \sum_{m=0}^{2N}(R_s(\lambda)2s\sigma_3\sigma_2 \otimes [\mathcal{D}, p])^m\, R_s(\lambda) + (R_s(\lambda)2s\sigma_3\sigma_2 \otimes [\mathcal{D}, p])^{2N+1}\, \tilde{R}_s(\lambda),$$

where $\tilde{R}_s(\lambda) = (\lambda-(1+s^2+\tilde{\mathcal{D}}^2+2s\sigma_3\sigma_2\otimes[\mathcal{D}, p]))^{-1}$ and $R_s(\lambda) = (\lambda-(1+s^2+\tilde{\mathcal{D}}^2))^{-1}$. The remainder term in the resolvent expansion is

$$\frac{1}{2}\int_{-\infty}^{\infty}(2s)^{2N+1}S\tau\left(\frac{1}{2\pi i}\int_l \lambda^{-q/2-r}(1_2 \otimes (2p-1))\right.$$
$$\left.\times \Big(\sigma_3\sigma_2 \otimes R_s(\lambda)[\mathcal{D}, p]\Big)^{2N+1}\tilde{R}_s(\lambda)\, d\lambda\right) ds. \tag{3}$$

By Hölder's inequality

$$\|(R_s(\lambda)[\mathcal{D}, p])^{2N+1}\|_1 \leqslant \|[\mathcal{D}, p]\|^{2N+1}\|R_s(\lambda)\|_{2N+1}^{2N+1} = C\|R_s(\lambda)^{2N+1}\|_1,$$

and by [CPRS2, Lemma 5.3] for all sufficiently small $\varepsilon > 0$ and $q \geqslant 1$.

$$\|R_s(\lambda)^{2N+1}\|_1 \leqslant C_\varepsilon((1/2+s^2-a)^2+v^2)^{-(2N+1)/2+(q+\varepsilon)/4},$$

where $\lambda = a + iv$. Moreover for $\|[\mathcal{D}, p]\| < \sqrt{2}$ we have by [CPRS2, Lemma 5.1]

$$\|\tilde{R}_s(\lambda)\| \leqslant C'((1+s^2-a-s\|[\mathcal{D}, p]\|)^2+v^2)^{-1/2}.$$

We put these estimates together to obtain an estimate for the trace norm of the remainder term (3). We find

$$\|(3)\|_1 \leqslant C''_\varepsilon \int_{-\infty}^{\infty} s^{2N+1} \int_{-\infty}^{\infty} \sqrt{a^2+v^2}^{\,-q/2-r}((1/2+s^2-a)^2+v^2)^{-(2N+1)/2+(q+\varepsilon)/4} \times((1+s^2-a-s\|[\mathcal{D},p]\|)^2+v^2)^{-1/2}\,dv\,ds.$$

Applying [CPRS2, Lemma 5.4] (one easily checks that one can integrate from $-\infty$ instead of 0 there) we find that this integral is finite provided $q+\varepsilon < 2N+2$ and $-2N-2r+\varepsilon<0$. The first condition is always satisfied by virtue of our choice of $2N$ and $\varepsilon\leqslant 1$. For the second condition to be true at $q+2r=1-\delta$ requires that $\varepsilon+\delta+q < 2N+1$ and a δ satisfying this condition can always be found since $2N-1\leqslant q < 2N+1$. That (3) defines a holomorphic function of r for $Re(r) > (1-q)/2-\delta/2$ can be seen by an argument essentially identical to the one in the proof of [CPRS2, Lemma 7.4]. $\square$

Observation. Since $1_2\otimes\gamma$ commutes with $\tilde{\mathcal{D}}^2$ and anticommutes with $1_2\otimes[\mathcal{D},p]$, all the terms in the expansion with m odd vanish. On the other hand, each of the integrands with m even is an even function of s and so we may replace $\frac{1}{2}\int_{-\infty}^{\infty}$ by $\int_0^{\infty}$ in the above expansion.

Observation. Using [CPRS2, Lemma 7.2], we find that for $Re(r)>0$ each term in the above sum is in fact trace class, so we may interchange the trace and the sum. Having done this, we examine the $m=0$ term. The $m=0$ term in the above expansion is given by

$$2\int_0^{\infty}\tau(\gamma p(1+s^2+\mathcal{D}^2)^{-q/2-r})\,ds - \int_0^{\infty}\tau\left(\gamma(1+\mathcal{D}^2+s^2)^{-q/2-r}\right)ds,$$

where the second term is the same (except for sign) as the second term in Lemma 6.5. Hence if we write $R_s(\lambda)=(\lambda-(1+s^2+\tilde{\mathcal{D}}^2))^{-1}$ we have for $r>1/2$

$$\begin{aligned} Ind(p\mathcal{D}^+p)C_{q/2+r} &= 2\int_0^{\infty}\tau(\gamma p(1+s^2+\mathcal{D}^2)^{-q/2-r})\,ds \\ &\quad + \sum_{m=2,even}^{2N}\int_0^{\infty} S\tau\left(\frac{1}{2\pi i}\int_l \lambda^{-q/2-r}(1_2\otimes(2p-1))\right. \\ &\quad \left.\times\,(R_s(\lambda)2s\sigma_3\sigma_2\otimes[\mathcal{D},p])^m R_s(\lambda)\,d\lambda\right)ds + holo, \end{aligned} \tag{4}$$

where $holo$ is a function of r holomorphic for $Re(r) > (1-q)/2-\delta/2$.

The left-hand side of Eq. (4) provides an analytic continuation of the right-hand side which is otherwise only defined for $Re(r) > 1/2$. The simple pole at $r = (1-q)/2$ has residue equal to $Ind(p\mathcal{D}^+p)$. We intend to compute this residue in terms of the analytic continuations of the integrals appearing on the right-hand side.

5.3. Pseudodifferential expansion of the index

In this section, we use ideas of [CPRS2] and Connes–Moscovici's pseudodifferential calculus to rewrite Eq. (4) in a form in which all the resolvents in the integrand are commuted to the right. In this new form we will be in a position to calculate residues explicitly term by term. Our aim is to prove the following:

Lemma 5.7. *There exists a* $1 > \delta > 0$ *such that for* $r > 1/2$

$$
\begin{aligned}
Ind(p\mathcal{D}^+p)C_{q/2+r} &= 2\int_0^\infty \tau(\gamma p(1+s^2+\mathcal{D}^2)^{-q/2-r})\,ds \\
&\quad + \sum_{m=2,even}^{2N}\sum_{|k|=0}^{2N-m}(-1)^{m/2}C(k) \\
&\quad \times \int_0^\infty (2s)^m S\tau\left(\frac{1}{2\pi i}\int_l \lambda^{-q/2-r}\left(1_2\otimes\{(2p-1)[\mathcal{D},p]^{(k_1)}\right.\right. \\
&\quad \left.\left.\times\cdots[\mathcal{D},p]^{(k_m)}\}\right)R_s(\lambda)^{|k|+m+1}\,d\lambda\right)ds + holo,
\end{aligned}
$$

where $R_s(\lambda) = (\lambda - (1+s^2+\tilde{\mathcal{D}}^2))^{-1}$, *holo is a function of r holomorphic for* $Re(r) > (1-q)/2 - \delta/2$ *and* $C(k) = (|k|+m)!\alpha(k)$.

Proof. This is an application of our adaptation of Higson's version of the pseudodifferential expansion, and the observation that $(\sigma_3\sigma_2)^2 = -1$. By [CPRS2, Lemma 6.11], the remainder from the pseudodifferential expansion (applied to the mth term in the resolvent expansion) is of order at most $-2m-(2N-m)-3 = -m-2N-3$. By [CPRS2, Lemma 6.12], the remainder $P_{m,N}$ satisfies

$$\|P_{m,N}(s,\lambda)(\lambda-(1+s^2+\tilde{\mathcal{D}}^2))^{(m+2N+3)/2}\| \leqslant C,$$

where the bound is uniform in s, λ and square roots use the principal branch of log. We use this to replace $P_{m,N}$ by powers of the resolvent to estimate the trace norm of the remainder. We obtain

$$\begin{aligned}
&\left\| \int_0^\infty s^m \int_l \lambda^{-q/2-r} P_{m,N}\, d\lambda\, ds \right\|_1 \\
&\leqslant C \int_0^\infty s^m \int_{-\infty}^\infty \sqrt{a^2+v^2}^{-q/2-r} \| R_s(\lambda)^{(m+2N+3)/2}\|_1\, dv\, ds \\
&\leqslant C' \int_0^\infty s^m \sqrt{a^2+v^2}^{-q/2-r} \sqrt{(1/2+s^2-a)^2+v^2}^{-(m+2N+3)/2+(q+\varepsilon)/2} dv\, ds,
\end{aligned}$$

where the final estimate comes from [CPRS2, Lemma 5.3]. Applying [CPRS2, Lemma 5.4] we find that this integral is finite provided $m-2((m+2N+3)/2-(q+\varepsilon)/2) < -1$ and $m-2((m+2N+3)/2-(q+\varepsilon)/2)+1-q-2r < -2$. The former condition requires $q+\varepsilon < 2+2N$, which is true by our choice of N. For the second condition to be true at $q+2r = 1-\delta$ requires $\varepsilon+\delta+q < 2N+1$, and since $2N-1 \leqslant q < 2N+1$, for sufficiently small $\varepsilon > 0$ there exists a $1 > \delta > 0$ satisfying this condition. □

5.4. Integrating out the parameter dependence

The formula of the last lemma has two integrals: one over the resolvent parameter λ and the other over $s \in [0, \infty)$. The λ integral can be performed by a simple application of Cauchy's formula for derivatives.

Lemma 5.8. *There exists $1 > \delta > 0$ such that for $r > 1/2$*

$$\begin{aligned}
&Ind(p\mathcal{D}^+p)C_{q/2+r} - 2\int_0^\infty \tau(\gamma p(1+s^2+\mathcal{D}^2)^{-q/2-r})\, ds \\
&= \sum_{m=2,even}^{2N} \sum_{|k|=0}^{2N-m} (-1)^{m/2+|k|} C(k) \frac{\Gamma(q/2+r+|k|+m)}{\Gamma(q/2+r)(|k|+m)!} \\
&\times \int_0^\infty (2s)^m S\tau\Big(1_2 \otimes \{(2p-1)[\mathcal{D},p]^{(k_1)} \cdots [\mathcal{D},p]^{(k_m)}\} \\
&\times (1+s^2+\tilde{\mathcal{D}}^2)^{-q/2-r-|k|-m}\Big)\, ds + holo,
\end{aligned}$$

where holo is a function of r holomorphic for $Re(r) > (1-q)/2 - \delta/2$.

Proof. After "pulling" the unbounded operator $1_2 \otimes \{(2p-1)[\mathcal{D},p]^{(k_1)} \cdots [\mathcal{D},p]^{(k_m)}\}$ out of the integral (how to do this is explained in the proof of [CPRS2, Lemma 7.2]) we just apply Cauchy's Formula in the operator setting (also discussed in

[CPRS2, Lemma 7.2]):

$$\frac{1}{2\pi i}\int_l \lambda^{-q/2-r} R_s(\lambda)^{|k|+m+1}\,d\lambda$$
$$= \frac{1}{(|k|+m)!}\left(\frac{d^{|k|+m}}{d\lambda^{|k|+m}}\lambda^{-q/2-r}\right)\Bigg|_{\lambda=(1+s^2+\tilde{\mathcal{D}}^2)}$$
$$= (-1)^{|k|+m}\frac{\Gamma(q/2+r+|k|+m)}{\Gamma(q/2+r)(|k|+m)!}(1+s^2+\tilde{\mathcal{D}}^2)^{-q/2-r-|k|-m}. \qquad \square$$

The remaining s-integral is not difficult either.

Lemma 5.9. *There exists $1>\delta>0$ such that for $r>1/2$*

$$Ind(p\mathcal{D}^+p)C_{q/2+r} - C_{q/2+r}\tau(\gamma p(1+\mathcal{D}^2)^{(1-q)/2-r})$$
$$= \sum_{m=2,even}^{2N}\sum_{|k|=0}^{2N-m}(-1)^{m/2+|k|}C(k)\frac{2^{m-1}\Gamma((m+1)/2)\Gamma(q/2+r+|k|+(m-1)/2)}{\Gamma(q/2+r)(|k|+m)!}$$
$$\times S\tau\left(1_2\otimes\{(2p-1)[\mathcal{D},p]^{(k_1)}\cdots[\mathcal{D},p]^{(k_m)}\}(1+\tilde{\mathcal{D}}^2)^{-q/2-r-|k|-(m-1)/2}\right)$$
$$= \sum_{m=2,even}^{2N}\sum_{|k|=0}^{2N-m}(-1)^{m/2+|k|}C(k)\frac{2^{m-1}\Gamma((m+1)/2)\Gamma(q/2+r+|k|+(m-1)/2)}{\Gamma(q/2+r)(|k|+m)!}$$
$$\times\tau\left(\gamma(2p-1)[\mathcal{D},p]^{(k_1)}\cdots[\mathcal{D},p]^{(k_m)}(1+\mathcal{D}^2)^{-q/2-r-|k|-(m-1)/2}\right) + holo,$$

where holo is a function of r holomorphic for $Re(r)>(1-q)/2-\delta/2$.

Proof. The integral is a Bochner integral (for a discussion of the subtleties see the proof of [CPRS2, Proposition 8.2]), and so we can move the s-integral past the supertrace. Then using the Laplace Transform argument of [CPRS2, Proposition 8.2], we have:

$$\int_0^\infty (2s)^m(1+s^2+\tilde{\mathcal{D}}^2)^{-|k|-m-q/2-r}\,ds$$
$$= \frac{1}{\Gamma(q/2+r+m+|k|)}\int_0^\infty\int_0^\infty u^{|k|+m+q/2+r-1}(2s)^m e^{-(1+\tilde{\mathcal{D}}^2)u}e^{-s^2u}\,ds\,du$$
$$= \frac{\Gamma((m+1)/2)2^m}{2\Gamma(|k|+m+q/2+r)}\int_0^\infty u^{|k|+(m-1)/2+q/2+r-1}e^{-(1+\tilde{\mathcal{D}}^2)u}\,du$$
$$= \frac{2^{m-1}\Gamma((m+1)/2)\Gamma(|k|+(m-1)/2+q/2+r)}{\Gamma(|k|+m+q/2+r)}(1+\tilde{\mathcal{D}}^2)^{-|k|-(m-1)/2-q/2-r}.$$

Substituting this into the result of the last lemma almost gives the result. The only extra things to do are to note the value of the constant arising from the integration for $m = 0$ and to trace out the Clifford variables (which could have been done earlier). Removing the Clifford variables is easy, because it is just a trace over the 2×2 identity matrix, and the factor of $1/2$ in the definition of the super trace cancels it out. Hence the result. □

5.5. Simplifying the constants

To obtain the constants that appear before the residues of the zeta functionsc in the statement of our main theorem requires us to manipulate the constants in front of the zeta functions in the statement of the last lemma of the preceding subsection. Legendre's duplication formula for the Gamma function [A, p. 200] says

$$2^{m-1}\Gamma((m+1)/2) = \sqrt{\pi}\Gamma(m)/\Gamma(m/2).$$

For $m = 0$ replace the right-hand side with $\sqrt{\pi}/2$. Since for $m > 0$ and even, $\frac{\Gamma(m)}{\Gamma(m/2)} = \frac{1}{2}\frac{m!}{(m/2)!}$, we have $2^m\Gamma((m+1)/2) = \sqrt{\pi}m!/(m/2)!$. The functional equation for the Gamma function says

$$\begin{aligned}
&\frac{\sqrt{\pi}\Gamma(r+(q-1)/2+|k|+m/2)}{\Gamma(q/2+r)}\\
&\quad= \frac{\sqrt{\pi}\Gamma(r+(q-1)/2)}{\Gamma(q/2+r)}\prod_{j=0}^{|k|+m/2-1}(r+(q-1)/2+j)\\
&\quad= C_{q/2+r}\sum_{j=1}^{|k|+m/2}(r+(q-1)/2)^j\sigma_{(|k|+m/2),j},
\end{aligned}$$

where the $\sigma_{(|k|+m/2),j}$ are the elementary symmetric functions of the integers $1, 2, \ldots, |k| + m/2$. Substituting these oddments into the formula from Lemma 5.9 for $r > 1/2$ and with $h = m/2 + |k|$ gives, *modulo functions of r holomorphic for $Re(r) > (1-q)/2-\delta$*:

$$\begin{aligned}
&Ind(p\mathcal{D}^+p)C_{q/2+r} - C_{q/2+r}\tau(\gamma p(1+\mathcal{D}^2)^{(1-q)/2-r})\\
&\quad= \sum_{m=2,even}^{2N}\sum_{|k|=0}^{2N-m}(-1)^{m/2+|k|}C(k)\frac{\sqrt{\pi}m!\Gamma(q/2+r+|k|+(m-1)/2)}{2(m/2)!\Gamma(q/2+r)(|k|+m)!}\\
&\qquad\times\tau\left(\gamma(2p-1)[\mathcal{D},p]^{(k_1)}\cdots[\mathcal{D},p]^{(k_m)}(1+\mathcal{D}^2)^{-q/2-r-|k|-(m-1)/2}\right)\\
&\quad= \sum_{m=2,even}^{2N}\sum_{|k|=0}^{2N-m}(-1)^{m/2+|k|}\frac{m!}{(m/2)!}\frac{\alpha(k)}{2}\frac{\sqrt{\pi}\Gamma(q/2+r+|k|+(m-1)/2)}{\Gamma(q/2+r)}
\end{aligned}$$

$$\times\tau\left(\gamma(2p-1)[\mathcal{D},p]^{(k_1)}\cdots[\mathcal{D},p]^{(k_m)}(1+\mathcal{D}^2)^{-q/2-r-|k|-(m-1)/2}\right)$$

$$= \sum_{m=2,even}^{2N}\sum_{|k|=0}^{2N-m}(-1)^{m/2+|k|}\frac{m!}{(m/2)!}\frac{\alpha(k)}{2}C_{q/2+r}$$

$$\times\sum_{j=1}^{h}\sigma_{h,j}(r+(q-1)/2)^j\tau\left(\gamma(2p-1)[\mathcal{D},p]^{(k_1)}\right.$$

$$\left.\times\cdots[\mathcal{D},p]^{(k_m)}(1+\mathcal{D}^2)^{(1-q)/2-r-|k|-m/2}\right). \tag{5}$$

Observe that we have not used the isolated spectral dimension assumption at any point in this calculation. Despite this, the above sum of zeta functions (which includes the $m=0$ term which we have written once on the LHS of the first equality to save space) has a simple pole at $r=(1-q)/2$ with residue equal to $Ind(p\mathcal{D}^+p)$. This proves part (2) of Theorem 4.1.

To proceed further, we need to assume that the individual zeta functions have analytic continuations.

5.6. Taking the residues

This step will prove the index formula in part (3) of Theorem 4.1. We now have to *assume isolated spectral dimension*. Then, denoting:

$$\zeta_{m,k}(z)=\tau\left(\gamma(2p-1)[\mathcal{D},p]^{(k_1)}\cdots[\mathcal{D},p]^{(k_m)}(1+\mathcal{D}^2)^{-z-|k|-m/2}\right),$$

we have for $r>1/2$

$$\begin{aligned} Ind(p\mathcal{D}^+p)C_{q/2+r} &= C_{q/2+r}\tau(\gamma p(1+\mathcal{D}^2)^{-(q-1)/2-r}) \\ &\quad+\sum_{m=2,even}^{2N}\sum_{|k|=0}^{2N-m}(-1)^{m/2+|k|}\frac{m!}{(m/2)!}\frac{\alpha(k)}{2}C_{q/2+r} \\ &\quad\times\sum_{j=1}^{h}\sigma_{h,j}(r+(q-1)/2)^j\zeta_{m,k}((q-1)/2+r)+holo, \end{aligned}$$

where $h=|k|+m/2$. Now, divide through by $C_{q/2+r}$, and multiply by $1/(r+(q-1)/2)$. The remainder term is now

$$\frac{holo}{C_{q/2+r}(r+(q-1)/2)},$$

which is still holomorphic at the critical point (since it has a removable singularity). Denote the analytic continuation of $\zeta_{m,k}((q-1)/2+r)$ by $\mathbf{Z}_{m,k}((q-1)/2+r)$. Define for $j=-1,0,1,\dots$

$$\tau_j\left(\gamma(2p-1)[\mathcal{D},p]^{(k_1)}\cdots[\mathcal{D},p]^{(k_m)}(1+\mathcal{D}^2)^{-m/2-|k|}\right)$$
$$= res_{(1-q)/2}(r-(1-q)/2)^j\mathbf{Z}_{m,k}((q-1)/2+r),$$

(we replace $2p-1$ by $2p$ when $m=0$). Thus taking the residues of the left- and right-hand sides of Eq. (5) we obtain (setting $h=|k|+m/2$)

$$\begin{aligned}
Ind(p\mathcal{D}^+p) &= res_{r=(1-q)/2}\frac{1}{(r+(q-1)/2)}Ind(p\mathcal{D}^+p)\\
&= res_{r=(1-q)/2}\frac{1}{(r+(q-1)/2)}\tau(\gamma p(1+\mathcal{D}^2)^{-(r-(1-q)/2)})\\
&\quad+\sum_{m=2,even}^{2N}\sum_{|k|=0}^{2N-m}(-1)^h\frac{m!}{(m/2)!}\frac{\alpha(k)}{2}\sum_{j=1}^{h}\sigma_{h,j}res_{r=(1-q)/2}\\
&\quad\times(r+(q-1)/2)^{j-1}\mathbf{Z}_{m,k}((q-1)/2+r)\\
&\quad=\tau_{-1}(\gamma p)+\sum_{m=2,even}^{2N}\sum_{|k|=0}^{2N-m}(-1)^h\frac{m!}{(m/2)!}\frac{\alpha(k)}{2}\sum_{j=1}^{h}\sigma_{h,j}\tau_{j-1}\\
&\quad\times\left(\gamma(2p-1)[\mathcal{D},p]^{(k_1)}\cdots[\mathcal{D},p]^{(k_m)}(1+\mathcal{D}^2)^{-m/2-|k|}\right).
\end{aligned}\qquad(6)$$

Observe that since $j-1$ runs from 0 to $|k|+m/2-1$, at worst, we only need to consider the first $|k|+m/2-1$ terms in the principal part of the Laurent series for $\mathbf{Z}_{m,k}$ at $r=(1-q)/2$, as well as the constant term. Moreover, this number is bounded by

$$|k|+m/2-1\leqslant 2N-m+m/2-1=2N-m/2-1\leqslant 2N-1\leqslant q$$

since $2N-1\leqslant q<2N+1$. Hence $|k|+m/2-1\leqslant q$. Furthermore, since $\gamma(2p-1)[\mathcal{D},p]^{(k_1)}\cdots[\mathcal{D},p]^{(k_m)}\in OP^{|k|}$, it equals $B(1+\mathcal{D}^2)^{|k|/2}$ for some B bounded, and so:

$$\gamma(2p-1)[\mathcal{D},p]^{(k_1)}\cdots[\mathcal{D},p]^{(k_m)}(1+\mathcal{D}^2)^{-m/2-|k|-r-(q-1)/2}$$
$$=B(1+\mathcal{D}^2)^{-m/2-|k|/2-r-(q-1)/2}.$$

The right-hand side has finite trace for

$$Re(r)>(1-m-|k|)/2=(1-q)/2+(q-m-|k|)/2.$$

Thus whenever $m+|k|>q$ we obtain a term which is holomorphic at $r=(1-q)/2$. If $[q]$ is *odd* then there exists $n\in\mathbf{N}$ with $2n+1\leqslant q<2n+2$ and so $N=n+1$ and $2N=2n+2>q$. Hence the residues of the terms with $m=2N$ all vanish, and similarly for any $m=2,\dots,2N-2$ the residues of the top terms with $|k|=2N-m$ vanish.

This computation, which has produced Eq. (6) has actually proved the index formula in part (3) of Theorem 4.1. To prove (1) and the remainder of (3), we need to study the resolvent cocycle.

6. The resolvent cocycle in the even case

Part (3) of Theorem 4.1 claims that the index is actually a pairing of a (b,B) cocycle with the Chern character of the idempotent p. Similarly, in (1) we have an 'almost' cocycle, and the residue of the pairing computes the index. In order to show this we introduce an auxiliary function-valued (b,B)-cochain called the *resolvent cocycle* (cf [CPRS2, Section 7]). The definition is inspired by the resolvent expansion, and we show that it is a (b,B)-cocycle *modulo functions of r holomorphic in an open half-plane containing* $r=(1-q)/2$. We use the resolvent cocycle to complete the proof of Theorem 4.1 in Section 6.1.

Our starting point for this section is the expansion of $a(0)$ obtained in Eq. (4) at the end of Section 5.2. We have

$$\begin{aligned}Ind(p\mathcal{D}^+p)C_{q/2+r} &= 2\int_0^\infty \tau(\gamma p(1+s^2+\mathcal{D}^2)^{-q/2-r})\,ds\\&\quad+\sum_{m=2,even}^{2N}\int_0^\infty S\tau\left(\frac{1}{2\pi i}\int_l \lambda^{-q/2-r}(2p-1)\right.\\&\quad\left.\times(R_s(\lambda)2s\sigma_3\sigma_2\otimes[\mathcal{D},p])^m\,R_s(\lambda)\,d\lambda\right)ds+holo,\end{aligned}$$

where *holo* is a function of r holomorphic for $r>(1-q)/2-\delta/2$ where $1>\delta>0$. If we now perform the 'super bit' of the trace we obtain

$$\begin{aligned}Ind(p\mathcal{D}^+p)C_{q/2+r} &= 2\int_0^\infty \tau(\gamma p(1+s^2+\mathcal{D}^2)^{-q/2-r})\,ds\\&\quad+\sum_{m=2,even}^{2N}(-1)^{m/2}\int_0^\infty \tau\left(\frac{1}{2\pi i}\int_l \lambda^{-q/2-r}\gamma(2p-1)\right.\\&\quad\left.\times(R_s(\lambda)2s[\mathcal{D},p])^m\,R_s(\lambda)\,d\lambda\right)ds+holo,\end{aligned}$$

where by abuse of notation we have written $R_s(\lambda) = (\lambda - (1 + s^2 + \mathcal{D}^2))^{-1}$ (as opposed to $\tilde{\mathcal{D}}^2$).

Assuming that the right-hand side is (almost) the pairing of a cocycle with the Chern character of the projection p, to obtain a formula for the cocycle we expect to remove the normalisations coming from the Chern character of p, and that is all. Including the powers of two in the normalisation gives the next definition.

Definition 6.1. For m even, $0 \leqslant m \leqslant 2N$, $a_0, \ldots, a_m \in \mathcal{A}$, and $\eta_m = 2^{m+1}\frac{(m/2)!}{m!}$ define the following function of r for $r > (1-m)/2$:

$$\phi_m^r(a_0, \ldots, a_m) = \frac{\eta_m}{2\pi i} \int_0^\infty s^m \tau\left(\gamma \int_l \lambda^{-q/2-r} a_0 R_s(\lambda)[\mathcal{D}, a_1] R_s(\lambda) \times \cdots R_s(\lambda)[\mathcal{D}, a_m] R_s(\lambda)\, d\lambda\right) ds.$$

Observe that the definition for $m = 0$ and the Cauchy formula gives

$$\phi_0^r(a_0) = 2 \int_0^\infty \tau(\gamma a_0 (1 + s^2 + \mathcal{D}^2)^{-q/2-r})\, ds.$$

Proposition 6.2. *For $0 \leqslant m \leqslant 2N-2$, $B\phi_{m+2}^r + b\phi_m^r = 0$ and there exists a $1 > \delta > 0$ such that $(b\phi_{2N}^r)(a_0, \ldots, a_{2N+1})$ is holomorphic for $Re(r) > (1-q)/2 - \delta/2$.*

Proof. We use the discussion of [CPRS2, Subsection 7.2], noting some minor differences which arise due to the grading, γ. We begin by computing $B\phi_{m+2}^r$. Applying the definitions we have,

$$\begin{aligned}
(B\phi_{m+2}^r)(a_0, \ldots, a_{m+1}) &= \sum_{j=0}^{m+1} (-1)^j \phi_{m+2}^r(1, a_j, \ldots, a_{m+1}, a_0, \ldots, a_{j-1}) \\
&= \sum_{j=0}^{m+1} \frac{(-1)^j \eta_{m+2}}{2\pi i} \int_0^\infty s^{m+2} \tau\left(\gamma \int_l \lambda^{-q/2-r} R_s(\lambda)[\mathcal{D}, a_j] \right. \\
&\quad \left. \times \cdots [\mathcal{D}, a_{m+1}] R_s(\lambda) \cdots [\mathcal{D}, a_{j-1}] R_s(\lambda)\, d\lambda\right) ds \\
&= \sum_{j=0}^{m+1} \frac{\eta_{m+2}}{2\pi i} \int_0^\infty s^{m+2} \tau\left(\gamma \int_l \lambda^{-q/2-r} [\mathcal{D}, a_0] R_s(\lambda) \right. \\
&\quad \left. \times \cdots R_s(\lambda) 1 R_s(\lambda) \cdots [\mathcal{D}, a_{m+1}] R_s(\lambda)\, d\lambda\right) ds.
\end{aligned}$$

The last line follows from [CPRS2, Lemma 7.7] modified by the fact that while $R_s(\lambda)$ commutes with γ, $[\mathcal{D}, a_i]$ anticommutes with γ. We now employ [CPRS2, Lemma 7.6]:

$$-k\int_0^\infty s^{k-1}\frac{1}{2\pi i}\tau\left(\gamma\int_l \lambda^{-q/2-r}A_0R_s(\lambda)A_1R_s(\lambda)\cdots A_mR_s(\lambda)\,d\lambda\right)ds$$
$$=2\sum_{j=0}^m\int_0^\infty s^{k+1}\frac{1}{2\pi i}\tau\left(\gamma\int_l \lambda^{-q/2-r}A_0R_s(\lambda)A_1R_s(\lambda)\cdots A_jR_s(\lambda)1R_s(\lambda)A_{j+1}\right.$$
$$\left.\times\cdots A_mR_s(\lambda)\,d\lambda\right)ds.$$

Applying this formula to our computation for $B\phi_{m+2}^r$ yields

$$(B\phi_{m+2}^r)(a_0,\dots,a_{m+1})$$
$$=-\tfrac{1}{2}(m+1)\frac{\eta_{m+2}}{2\pi i}\int_0^\infty s^m\tau\left(\gamma\int_l \lambda^{-q/2-r}R_s(\lambda)[\mathcal{D},a_0]\right.$$
$$\left.\times\cdots R_s(\lambda)[\mathcal{D},a_{m+1}]R_s(\lambda)\,d\lambda\right)ds$$
$$=-\frac{\eta_m}{2\pi i}\int_0^\infty s^m\tau\left(\gamma\int_l \lambda^{-q/2-r}R_s(\lambda)[\mathcal{D},a_0]\cdots[\mathcal{D},a_{m+1}]R_s(\lambda)\,d\lambda\right)ds.$$

Here we used $(m+1)/2\times\eta_{m+2}=\eta_m$. Next one expands the first commutator on the right-hand side, $[\mathcal{D},a_0]=\mathcal{D}a_0-a_0\mathcal{D}$, and anticommutes the second $\mathcal{D}$ through the remaining $[\mathcal{D},a_j]$ using $\mathcal{D}[\mathcal{D},a_j]+[\mathcal{D},a_j]\mathcal{D}=[\mathcal{D}^2,a_j]$. Recalling that $\mathcal{D}$ anticommutes with γ, we find from the proof of [CPRS2, Proposition 7.10] that

$$(B\phi_{m+2}^r)(a_0,\dots,a_{m+1})$$
$$=\frac{\eta_m}{2\pi i}\int_0^\infty s^m\sum_{j=1}^{m+1}(-1)^{j+1}\tau\left(\gamma\int_l \lambda^{-q/2-r}R_s(\lambda)a_0R_s(\lambda)[\mathcal{D},a_1]\cdots[\mathcal{D}^2,a_j]\right.$$
$$\left.\times\cdots[\mathcal{D},a_{m+1}]R_s(\lambda)\,d\lambda\right)ds.$$

We recall that $B\phi_0^r=0$, by definition. The computation of $b\phi_m^r$ is precisely the same as [CPRS2, Proposition 7.10], and gives

$$(b\phi_m^r)(a_0,\dots,a_{m+1})$$
$$=\frac{\eta_m}{2\pi i}\int_0^\infty s^m\sum_{j=1}^{m+1}(-1)^{j}\tau\left(\gamma\int_l \lambda^{-q/2-r}R_s(\lambda)a_0R_s(\lambda)[\mathcal{D},a_1]\cdots[\mathcal{D}^2,a_j]\right.$$
$$\left.\times\cdots[\mathcal{D},a_{m+1}]R_s(\lambda)\,d\lambda\right)ds.$$

Hence $B\phi^r_{m+2} + b\phi^r_m = 0$ for $0 \leqslant m \leqslant 2N-2$ (indeed for all $m \geqslant 0$).

For $m = 2N$, we use Hölder's inequality (together with [CPRS2, Lemma 6.10] to see that $|[\mathcal{D}^2, a_j]R_s(\lambda)| \leqslant C'|R_s(\lambda)|^{1/2}$) which yields a constant C independent of s and λ so that:

$$\begin{aligned}&\|R_s(\lambda)a_0R_s(\lambda)[\mathcal{D}, a_1]\cdots R_s(\lambda)[\mathcal{D}^2, a_j]R_s(\lambda)\cdots R_s(\lambda)[\mathcal{D}, a_{2N+1}]R_s(\lambda)\|_1\\ &\quad\leqslant C\|R_s(\lambda)^{2N+5/2}\|_1.\end{aligned}$$

Consequently, we have the estimate (using [CPRS2, Lemma 5.3])

$$\begin{aligned}&|(b\phi^r_{2N})(a_0, \ldots, a_{2N+1})|\\ &\quad\leqslant C\int_0^\infty s^{2N}\int_{-\infty}^{\infty}\sqrt{a^2+v^2}^{\,-q/2-r}\|R_s(\lambda)^{2N+5/2}\|_1\,dv\,ds\\ &\quad\leqslant C_\varepsilon\int_0^\infty s^{2N}\int_{-\infty}^{\infty}\sqrt{a^2+v^2}^{\,-q/2-r}\sqrt{(1/2+s^2-a)^2+v^2}^{\,-2N-5/2+(q+\varepsilon)/2}\,dv\,ds.\end{aligned}$$

Consulting [CPRS2, Lemma 5.4] we find that this integral is convergent when $2r = 1-q-\delta$ provided $q+\varepsilon-4 < 2N$, which is true, and $q+\varepsilon+\delta < 2N+3$, which again is true. As for the case of the remainder term in the proof of [CPRS2, Lemma 7.4] this shows that the above formula for $(b\phi^r_{2N})(a_0, \ldots, a_{2N+1})$ gives a holomorphic function of r in a neighbourhood of $(1-q)/2$ as claimed. □

Observe that together with Eq. (4) the above result proves part (1) of Theorem 4.1.

6.1. The residue cocycle

In this subsection we complete the proof of Theorem 4.1. First we need to define the residue cocycle.

Definition 6.3. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞ finitely summable spectral triple with isolated spectral dimension $q \geqslant 1$. For $m = 2, \ldots, 2N$ and $a_0, \ldots, a_m \in \mathcal{A}$ define functionals

$$\begin{aligned}&\phi_m(a_0, \ldots, a_m)\\ &\quad= \sum_{|k|=0}^{2N-m}(-1)^{|k|}\alpha(k)\sum_{j=1}^{|k|+m/2}\sigma_{(|k|+m/2),j}\\ &\qquad\times\tau_{j-1}\Big(\gamma a_0[\mathcal{D}, a_1]^{(k_1)}\cdots[\mathcal{D}, a_m]^{(k_m)}(1+\mathcal{D}^2)^{-m/2-|k|}\Big),\end{aligned}$$

and for $m = 0$ define $\phi_0(a_0) = \tau_{-1}(\gamma a_0)$.

Theorem 6.4. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞ finitely summable spectral triple with isolated spectral dimension $q \geqslant 1$. When evaluated on any $a_0, \ldots, a_m \in \mathcal{A}$, the components ϕ_m^r of the resolvent cocycle (ϕ^r) analytically continue to a deleted neighbourhood of $r = (1-q)/2$. Moreover, if we denote this continuation by $\mathbf{\Phi}_m^r(a_0, \ldots, a_m)$ then*

$$res_{r=(1-q)/2} \frac{1}{C_{q/2+r}(r+(q-1)/2)} \mathbf{\Phi}_m^r(a_0, \ldots, a_m) = \phi_m(a_0, \ldots, a_m).$$

Remark. Observe that, as a function of r, $[C_{q/2+r}(r+(q-1)/2)]^{-1}$ has a removable singularity at $r = (1-q)/2$. Thus all the statements concerning the resolvent cocycle also apply to the resolvent cocycle multiplied by this function.

Proof. For m even, evaluate ϕ_m^r on $a_0, \ldots, a_m \in \mathcal{A}$ and apply the pseudodifferential expansion. This yields (modulo functions holomorphic for $Re(r) > (1-q)/2 - \delta$)

$$\begin{aligned}
&\phi_m^r(a_0, \ldots, a_m) \\
&\quad = \sum_{|k|=0}^{2N-m} C(k) \frac{\eta_m}{2\pi i} \int_0^\infty s^m \tau\left(\gamma \int_l \lambda^{-q/2-r} a_0 [\mathcal{D}, a_1]^{(k_1)}\right. \\
&\qquad \left. \times \cdots [\mathcal{D}, a_m]^{(k_m)} R_s(\lambda)^{m+|k|+1} \, d\lambda\right) ds.
\end{aligned}$$

Proceeding according to our previous computations we have

$$\begin{aligned}
&\phi_m^r(a_0, \ldots, a_m) \\
&= \sum_{|k|=0}^{2N-m} \frac{(-1)^{m+|k|} C(k) \Gamma(q/2+r+m+|k|)}{\Gamma(q/2+r)(m+|k|)!} 2^{m+1} \frac{(m/2)!}{m!} \\
&\times \int_0^\infty s^m \tau\left(\gamma a_0 [\mathcal{D}, a_1]^{(k_1)} \cdots [\mathcal{D}, a_m]^{(k_m)} (1+s^2+\mathcal{D}^2)^{-(m+|k|+q/2+r)}\right) ds \\
&= \sum_{|k|=0}^{2N-m} \frac{(-1)^{|k|} C(k) 2^m \Gamma((m+1)/2) \Gamma(q/2+r+m/2-1/2+|k|)}{\Gamma(q/2+r)(m+|k|)!} \frac{(m/2)!}{m!} \\
&\times \tau\left(\gamma a_0 [\mathcal{D}, a_1]^{(k_1)} \cdots [\mathcal{D}, a_m]^{(k_m)} (1+\mathcal{D}^2)^{-(m/2-1/2+|k|+q/2+r)}\right)
\end{aligned}$$

$$= \sum_{|k|=0}^{2N-m} \frac{(-1)^{|k|}C(k)\sqrt{\pi}\Gamma(q/2+r+m/2-1/2+|k|)}{\Gamma(q/2+r)(m+|k|)!}$$

$$\times\tau\left(\gamma a_0[\mathcal{D},a_1]^{(k_1)}\cdots[\mathcal{D},a_m]^{(k_m)}(1+\mathcal{D}^2)^{-(m/2-1/2+|k|+q/2+r)}\right)$$

$$= \sum_{|k|=0}^{2N-m}(-1)^{|k|}C_{q/2+r}\alpha(k)\sum_{j=0}^{h}\sigma_{h,j}(r+(q-1)/2)^j$$

$$\times\tau\left(\gamma a_0[\mathcal{D},a_1]^{(k_1)}\cdots[\mathcal{D},a_m]^{(k_m)}(1+\mathcal{D}^2)^{-(m/2-1/2+|k|+q/2+r)}\right),$$

where $h=|k|+m/2$. The result is now clear. □

Corollary 6.5. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a QC^∞ finitely summable spectral triple with isolated spectral dimension $q\geqslant 1$. The cochain (ϕ) with components ϕ_m, $m=0,2,\ldots,2N$, is a (b,B)-cocycle. For any projection $p\in\mathcal{A}$ we have*

$$Ind(p\mathcal{D}^+p)=\sum_{m=0,even}^{2N}\phi_m(Ch_m(p)).$$

Proof. The first statement follows because $\left((C_{q/2+r}(r+(q-1)/2))^{-1}\phi^r\right)$ is a (b,B)-cocycle modulo functions holomorphic at $r=(1-q)/2$ and hence so is its analytic continuation, $\left((C_{q/2+r}(r+(q-1)/2))^{-1}\mathbf{\Phi}^r\right)$. For the second statement we recall that

$$Ch_0(p)=p,\qquad Ch_m(p)=(-1)^{m/2}\frac{m!}{2(m/2)!}(2p-1)\otimes p^{\otimes m}.$$

Thus $\sum_{m=0}^{2N}\phi_m(Ch_m(p))$ is given precisely by the formula on the right-hand side of Eq. (6), the left-hand side of which is $Ind(p\mathcal{D}^+p)$. This completes the proof. □

We have now completed the proof of Theorem 4.1. We present the easy proof of Corollary 4.2.

Corollary 6.6. *For $1\leqslant q<2$, we do not need to assume isolated spectral dimension to compute the index pairing.*

Proof. For $1\leqslant q<2$ we have $N=1$, but as we observed after Eq. (6), the term with $m=2N$ is holomorphic at $r=(1-q)/2$ when $[q]$ is an odd integer. Hence we have only the $m = 0$ term. So,

$$Ind(p\mathcal{D}^+p) = \tau(\gamma p(1+\mathcal{D}^2)^{-(q-1)/2-r}) + \frac{holo}{C_{q/2+r}}.$$

By the remark in Theorem 6.4:

$$\frac{1}{r+(q-1)/2} Ind(p\mathcal{D}^+p) = \frac{1}{r+(q-1)/2}\tau(\gamma p(1+\mathcal{D}^2)^{-(q-1)/2-r}) + holo.$$

Taking residues we have

$$Ind(p\mathcal{D}^+p) = res_{r=(1-q)/2}\tau(\gamma p(1+\mathcal{D}^2)^{-(q-1)/2-r}),$$

and the residue on the right necessarily exists, and is equal to $\tau_{-1}(\gamma p)$. Hence the individual terms in the expansion of the index analytically continue to a punctured neighbourhood of $r = (1-q)/2$ with no need to invoke the isolated spectral dimension hypothesis. The single term ϕ_0 forms a (b, B) cocycle for $\mathcal{A}$ since $B\phi_0 = 0$ and $b\phi_0^r$ is holomorphic at $r = (1-q)/2$. $\square$

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