GRADIENT ESTIMATES VIA TWO-POINT FUNCTIONS FOR ELLIPTIC EQUATIONS ON MANIFOLDS

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Abstract. We derive estimates relating the values at two points of a solution of a quasilinear isotropic elliptic equation on a compact Riemannian manifold, in terms of the distance between the points and the height at one of them, and a lower bound for the Ricci curvature on the manifold. These estimates imply sharp gradient estimates, amounting to the statement that the largest possible value of the gradient for given height occurs for a particular monotone solution of the corresponding one-dimensional elliptic equation. We also discuss the problem on Finsler manifolds with nonnegative weighted Ricci curvature. Particular cases of our result include gradient estimates of Modica type.

1. Introduction

Let (M^n, g) be a compact Riemannian manifold without boundary. Assume the Ricci curvature of M has lower bound $Ric \geq (n-1)\kappa$. We consider 'isotropic' equations of the following form:

$$\left[\alpha(u,|Du|)\frac{D_iuD_ju}{|Du|^2}+\beta(u,|Du|)(\delta_{ij}-\frac{D_iuD_ju}{|Du|^2})\right]D_iD_ju+q(u,|Du|)=0. \tag{1}$$

We assume that Equation (1) is nonsingular, i.e. the left hand side of (1) is continuous on $\mathbb{R}\times TM\times Sym^2(T^*M)$, and that α and β are nonnegative functions. Our first main result is the following estimate:

Theorem 1. *Let (M^n, g) be a compact Riemannian manifold with nonnegative Ricci curvature, and let u be a viscosity solution of Equation (1). Suppose $\varphi : [a, b] \to [\inf u, \sup u]$ is a C^2 solution of*

$$\alpha(\varphi,\varphi')\varphi''+q(\varphi,\varphi')=0 \quad \text{on } [a,b]; \tag{2}$$

$$\varphi(a)=\inf u;\quad \varphi(b)=\sup u;\quad \varphi'>0 \quad \text{on } [a,b]. \tag{3}$$

Moreover let ψ be the inverse of φ, i.e. $\psi(\varphi(z)) = z$. Then we have

$$\psi(u(y))-\psi(u(x))-d(x,y)\leq 0,\ \forall x,y\in M. \tag{4}$$

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By allowing y to approach x, we deduce the following gradient estimate:

Corollary 2. *Under the assumptions of Theorem 1,* $|Du(x)| \leq \varphi'(\psi(u(x)))$ *for all* $x \in M$.

By applying this result in special situations, we recover several previously known results, known as Modica-type gradient estimates. These originate from the work of Modica [24], who considered bounded solutions on $\mathbb{R}^n$ of the equation

$$\Delta u - Q'(u) = 0, \tag{5}$$

where Q is chosen (by adding a suitable constant) to be non-negative on the range of u. The key gradient estimate of [24] is the following: $|Du(x)|^2 \leq 2Q(u(x))$ for all x. This was proved by differentiating Equation (5) to derive a maximum principle for the function $P = \frac{1}{2}|Du|^2 - Q(u)$, and subsequent works have followed this method (sometimes called the 'P-function method'). We observe that this is a consequence of Corollary 2: In this case $\alpha = 1$, and Equation (2) is equivalent to the statement that $P = \frac{1}{2}(\varphi')^2 - Q(\varphi)$ is constant. We can define φ by solving this with $P = 0$, and then we have $\varphi'(\psi(z)) = \sqrt{2Q(z)}$ for each $z \in [\inf u, \sup u]$, so that the estimate of Corollary 2 is exactly Modica's estimate.

Caffarelli, Garofalo and Segala [5] generalized Modica's result to critical points of energies of the form

$$\mathcal{E}(u) = \int_M \left(\frac{1}{2}\Phi(|\nabla_g u|^2) + Q(u)\right) dv_g, \tag{6}$$

where $\Phi \in C^3(\mathbb{R}^+)$ with $\Phi(0) = 0$ and $Q \in C^2(\mathbb{R})$ (Modica's result corresponds to $\mathit{\Phi}(z) = z$). The Euler-Lagrange equation for $\mathcal{E}$ is given by

$$div_g(\Phi'(|\nabla_g u|^2)\nabla_g u) - Q'(u) = 0. \tag{7}$$

Note that Equation (7) is a special case of (1), with $\alpha = 2\mathit{\Phi}''(z)z + \mathit{\Phi}'(z)$ and $\beta = \mathit{\Phi}'(z)$ where $z = |Du|^2$. In [5] the following estimate was derived: $P \leq 0$, where

$$P = \Phi'(|\nabla u|^2)|\nabla u|^2 - \frac{1}{2}\Phi(|\nabla u|^2) - Q(u) \tag{8}$$

(again Q is chosen to be non-negative on the range of u). As before, this estimate is a direct consequence of our Corollary 2: In this case Equation (2) becomes

$$0 = (2\mathit{\Phi}''((\varphi')^2)(\varphi')^2 + \mathit{\Phi}'((\varphi')^2))\varphi'' - Q'(\varphi).$$

Multiplying by φ' we obtain

$$0 = \left(\mathit{\Phi}'((\varphi')^2)(\varphi')^2 - \frac{1}{2}\mathit{\Phi}((\varphi')^2) - Q(\varphi)\right)' = P'.$$

Defining φ by solving $P = 0$, we obtain a solution of (2) and deduce the claimed inequality from Corollary 2.

Subsequent to [24] and [5], many other authors have considered related problems in $\mathbb{R}^n$, e.g. [6,7,9,12,14,15] and on Riemannian manifolds [11,13,22, 23]. The results of [13] are most relevant to the present paper. The proofs of most of the results mentioned above involve a P-function constructed from the solution u and the first derivatives Du. The gradient estimates amount to pointwise inequalities on the P-function, deduced by application of the maximum principle to an equation resulting from differentiation of the equation satisfied by u. However in this paper we use a different approach, deriving the two-point estimate of Theorem 1 and then deducing the gradient estimate of Corollary 2. The proof is comparatively simple and geometric compared to the calculations involved in the P-function approach. We refer the reader to the papers [2, 3] and the recent survey [1] which gives a discussion of the kinds of methods used in this paper, in a variety of geometric contexts. A further advantage of this method is that our argument does not involve differentiating the equation, and consequently applies (using ideas from [20, 21]) with minimal regularity requirements on the solution u, corresponding to the viscosity solution requirement in Theorem 1. Throughout the paper we use the terminology of viscosity solutions from [8].

The simplicity of our method allows us to extend the proof to a more general situation of manifolds with a negative lower Ricci curvature bound, again by comparison with a suitable one-dimensional 'warped product' solution:

Theorem 3. *Let* (M, g) *be a compact Riemannian manifold (possibly with boundary, in which case we assume the boundary is locally convex and impose the Neumann boundary condition), and* $\kappa < 0$ *such that* $\mathrm{Ric} \geq (n-1)\kappa g$. *Let* $\bar{M} = N \times [a, b]$ *and* $\bar{g} = ds^2 + \rho(s)^2 g^N$ *be such that* $\mathrm{Ric}(\partial_s, \partial_s) = (n-1)\kappa$ *and* ρ'/ρ *is strictly increasing, and let* $\bar{u}(x, s) = \varphi(s)$ *be a solution of* (1) *on* $\bar{M}$, *where* φ *is an increasing* C^2 *diffeomorphism from* $[a, b]$ *to* $[c, d]$. *Let* ψ *be the inverse function of* φ. *Let* u *be a viscosity solution of* (1) *on* M *with range contained in* $[c, d]$. *Then for all* x *and* y *in* M *we have*

$$\psi(u(y)) - \psi(u(x)) - d^M(x, y) \leq 0.$$

As we explain in Section 4, the assumption that $\bar{u}$ is a solution of (1) is equivalent to a certain elliptic equation for φ which involves α and β and also the warping factor ρ.

Corollary 4. *Under the assumptions of Theorem 3,* $|Du(x)| \leq \varphi'(\psi(u(x)))$ *for all* $x \in M$.

It is an interesting question whether such a result holds also in this generality in the case $\kappa > 0$. Our proof does not appear to apply in that situation. However we discuss a different argument which works for general $\kappa \in \mathbb{R}$ under somewhat more stringent assumptions, in Section 5.1.

In addition we notice that the method of two-point functions also applies for equations on Finsler manifolds with nonnegative weighted Ricci curvature, where we only consider the equation corresponding to (7). We address this problem in Section 6. In the final section, we discuss the resulting gradient estimates of Modica type and related rigidity results.

The paper is built up as follows. In Section 2 we recall background material including the definition of viscosity solutions, maximum principles for semicontinuous functions on manifolds, and the first and second variation formulas for the arclength of a curve. In Section 3 we give the proof of Theorem 1 and Corollary 2. The more general result of Theorem 3 is proved in Section 4. Section 5 discusses a different argument which applies for general lower bounds on the Ricci curvature. Section 6 is devoted to the setting of compact Finsler manifolds with nonnegative weighted Ricci curvature. The final section gives the pointwise gradient estimates of Modica type and some rigidity results.

2. PRELIMINARIES

2.1. **Definition of viscosity solutions on manifolds.** Let M be a Riemannian manifold. We use the following notations:

$$USC(M) = \{u : M \to \mathbb{R} | u \text{ is upper semicontinuous}\},$$
$$LSC(M) = \{u : M \to \mathbb{R} | u \text{ is lower semicontinuous}\}.$$

Next we introduce the semijets on manifolds.

Definition 2.1. For a function $u \in USC(M)$, the second order superjet of u at a point $x_0 \in M$ is defined by

$$\mathcal{J}^{2,+}u(x_0) := \big\{(D\varphi(x_0), D^2\varphi(x_0)) : \varphi \in C^2(M), \text{ such that } u - \varphi \text{ attains a local maximum at } x_0\big\}.$$

For $u \in LSC(M)$, the second order subjet of u at $x_0 \in M$ is defined by

$$\mathcal{J}^{2,-}u(x_0) := -\mathcal{J}^{2,+}(-u)(x_0).$$

We also define the closures of $\mathcal{J}^{2,+}u(x_0)$ and $\mathcal{J}^{2,-}u(x_0)$ by

$$\begin{aligned}\bar{\mathcal{J}}^{2,+}u(x_0) = \{&(p, X) \in T_{x_0}M \times Sym^2(T^*_{x_0}M) | \text{there is a sequence } (x_j, p_j, X_j) \\ &\text{such that } (p_j, X_j) \in \mathcal{J}^{2,+}u(x_j) \\ &\text{and } (x_j, u(x_j), p_j, X_j) \to (x_0, u(x_0), p, X) \text{ as } j \to \infty\}.\end{aligned}$$

$$\bar{\mathcal{J}}^{2,-}u(x_0) = -\bar{\mathcal{J}}^{2,+}(-u)(x_0).$$

Now we can define the viscosity solution for the general equation

$$F(x, u, Du, D^2u) = 0 \tag{9}$$

on M: Assume $F \in C(M \times \mathbb{R} \times TM \times Sym^2(T^*M))$ is degenerate elliptic, i.e.

$$F(x, r, p, X) \leq F(x, r, p, Y), \text{ whenever } X \leq Y.$$

Definition 2.2. (1) A function $u \in USC(M)$ is a viscosity subsolution of (9) if for all $x \in M$ and $(p, X) \in \mathcal{J}^{2,+}u(x)$,

$$F(x, u(x), p, X) \geq 0.$$

(2) A function $u \in LSC(M)$ is a viscosity supersolution of (9) if for all $x \in M$ and $(p, X) \in \mathcal{J}^{2,-}u(x)$,

$$F(x, u(x), p, X) \leq 0.$$

(3) A viscosity solution of (9) is a continuous function which is both a viscosity subsolution and a viscosity supersolution of (9).

2.2. **Maximum principle for semicontinuous functions.**

Theorem 5 (Theorem 3.2 in [8])**.** *Let $M_1^{n_1}, \dots, M_k^{n_k}$ be Riemannian manifolds, and $\Omega_i \subset M_i$ open subsets. Let $u_i \in USC(\Omega_i)$ and $\varphi \in C^2(\Omega_1 \times \cdots \times \Omega_k)$. Suppose the function*

$$w(x_1, \dots, x_k) := u_1(x_1) + \cdots + u_k(x_k) - \varphi(x_1, \dots, x_k)$$

attains a maximum at $(\hat{x}_1, \dots, \hat{x}_k)$ on $\Omega_1 \times \cdots \times \Omega_k$. Then for each $\lambda > 0$ there exists $X_i \in Sym^2(T^*_{\hat{x}_i} M_i)$ such that

$$(D_{x_i}\varphi(\hat{x}_1, \dots, \hat{x}_k), X_i) \in \bar{\mathcal{J}}^{2,+}u_i(\hat{x}_i) \text{ for } i = 1, \dots, k,$$

and the block diagonal matrix with entries X_i satisfies

$$-(\frac{1}{\lambda} + ||A||)I \leq \begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{pmatrix} \leq A + \lambda A^2,$$

where $A = D^2\varphi(\hat{x}_1, \dots, \hat{x}_k)$.

2.3. **First and second variation formulae for arclength.** Let $\gamma_0 : [0, l] \to M$ be a geodesic in M parametrised by arc length. Suppose $\gamma(\varepsilon, s)$ is any smooth variation of $\gamma_0(s)$ with $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Then the first variation formula for arclength is

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} L(\gamma(\varepsilon, \cdot)) = \langle \gamma_s, \gamma_\varepsilon \rangle \big|_0^l, \tag{10}$$

where γ_s is the unit tangent vector of γ_0 and $\gamma_\varepsilon = \frac{\partial}{\partial \varepsilon}\gamma$ is the variational vector field.

Furthermore, the second variation formula is given by

$$\frac{\partial^2}{\partial \varepsilon^2}\bigg|_{\varepsilon=0} L(\gamma(\varepsilon, \cdot)) = \int_0^l \Big(|\nabla_{\gamma_s}(\gamma_\varepsilon^\perp)|^2 - R(\gamma_s, \gamma_\varepsilon, \gamma_\varepsilon, \gamma_s) \Big)\, ds + \langle \gamma_s, \nabla_{\gamma_\varepsilon}\gamma_\varepsilon \rangle \big|_0^l, \tag{11}$$

where $\gamma_\varepsilon^\perp$ means the normal part of the variational vector. Here and in the sequel we use the convention on the Riemannian curvature tensor R such that $Ric(X, Y) = tr_g(R(X, \cdot, \cdot, Y))$ for $X, Y \in T_xM$.

3. Riemannian manifolds with nonnegative Ricci curvature

First we prove the following modulus of continuity estimate, which implies Theorem 1 immediately.

Theorem 6. *Let* (M^n, g) *be a compact Riemannian manifold with* $Ric \geq 0$ *and* u *be a viscosity solution of Equation* (1)*. Suppose the barrier* $\varphi : [a,b] \to [\inf u, \sup u]$ *satisfies*

$$\varphi' > 0, \tag{12}$$

$$\frac{d}{dz}\left(\frac{q(\varphi,\varphi') + \varphi''\alpha(\varphi,\varphi')}{\varphi'\beta(\varphi,\varphi')}\right) < 0. \tag{13}$$

Moreover let ψ *be the inverse of* φ*, i.e.* $\psi(\varphi(z)) = z$*. Then we have*

$$\psi(u(y)) - \psi(u(x)) - d(x,y) \leq 0, \ \forall x, y \in M. \tag{14}$$

By letting y approach x, we get the following gradient bound.

Corollary 7. *Under the conditions of Theorem 6, if moreover* $u \in C^1(M)$*, then for every* $x \in M$ *we have*

$$|\nabla u(x)| \leq \varphi'(\psi(u(x))). \tag{15}$$

Now we show how Theorem 6 implies Theorem 1. Let φ satisfy (2) and (3) in Theorem 1. Then for sufficiently small $\delta > 0$, we can solve

$$\begin{gathered} \alpha(\varphi_\delta, \varphi_\delta')\varphi_\delta'' + q(\varphi_\delta, \varphi_\delta') = -\delta z \cdot \varphi_\delta' \cdot \beta(\varphi_\delta, \varphi_\delta'), \\ \varphi_\delta(a) = \varphi(a), \quad \varphi_\delta'(a) = \varphi'(a), \end{gathered}$$

to get φ_δ which satisfies (12) and (13). So by Theorem 6 we have (14) for φ_δ. Letting $\delta \to 0^+$, we finish the proof of Theorem 1.

Next we focus on proving Theorem 6. For that purpose we need a lemma about the behaviour of semijets when composed with an increasing function.

Lemma 8. *Let* u *be a continuous function. Let* $\varphi : \mathbb{R} \to \mathbb{R}$ *be a* C^2 *function with* $\varphi' > 0$*. Let* ψ *be the inverse of* φ*, so that*

$$\varphi(\psi(u(x))) = u(x).$$

1. *Suppose* $(p, X) \in \mathcal{J}^{2,+}(\psi \circ u)(x_0)$*. Then*
$$(\varphi' p, \varphi'' p \otimes p + \varphi' X) \in \mathcal{J}^{2,+}u(x_0),$$
where all derivatives of φ *are evaluated at* $\psi \circ u(x_0)$*.*
2. *Suppose* $(p, X) \in \mathcal{J}^{2,-}(\psi \circ u)(x_0)$*. Then*
$$(\varphi' p, \varphi'' p \otimes p + \varphi' X) \in \mathcal{J}^{2,-}u(x_0),$$
where all derivatives of φ *are evaluated at* $\psi \circ u(x_0)$*.*
3. *The same holds if we replace the semijets by their closures.*

Proof. (1) Recall the definition of the superjet:

$$\mathcal{J}^{2,+}u(x_0) := \{(D\varphi(x_0), D^2\varphi(x_0)) : \varphi \in C^2(M),\ \text{such that } u-\varphi \\ \text{attains a local maximum at } x_0\}.$$

Assume $(p, X) \in \mathcal{J}^{2,+}(\psi \circ u)(x_0)$. Let h be a C^2 function such that $\psi(u(x)) - h(x)$ has a local maximum at x_0 and $(Dh, D^2h)(x_0) = (p, X)$. Since φ is increasing, we know $u(x) - \varphi(h(x)) = \varphi(\psi(u(x))) - \varphi(h(x))$ has a local maximum at x_0. So it follows that

$$(\varphi' p, \varphi'' p \otimes p + \varphi' X) \in \mathcal{J}^{2,+}u(x_0).$$

(2) is similar. (3) follows by approximation. □

Proof of Theorem 6. The proof is by contradiction. Assume there exists some $\varepsilon_0 > 0$ such that

$$\psi(u(y)) - \psi(u(x)) - d(x, y) \le \varepsilon_0,$$

for any $x, y \in M$ and with equality for some $x_0 \neq y_0$.

Next we want to replace $d(x, y)$ by a smooth function $\tilde{d}(x, y)$ on a neighbourhood of (x_0, y_0). To construct this we let γ_0 be the unit speed length-minimizing geodesic joining x_0 and y_0, with length $l = L(\gamma_0)$. Let $\{e_i(s)\}_{i=1}^n$ be parallel orthonormal vector fields along γ_0 with $e_n(s) = \gamma_0'(s)$ for each s. Then in small neighbourhoods U_{x_0} about x_0 and U_{y_0} about y_0, there are mappings $x \mapsto (a_1(x), \dots, a_n(x))$ and $y \mapsto (b_1(y), \dots, b_n(y))$ defined by

$$x = \exp_{x_0}(\sum_1^n a_i(x)e_i(0)), \quad y = \exp_{y_0}(\sum_1^n b_i(y)e_i(l)).$$

Then for some C^2 nonnegative function $f : [0, l] \to \mathbb{R}$ to be determined, we can define a smooth function $\tilde{d}(x, y)$ in $U_{x_0} \times U_{y_0}$ to be the length of the curve

$$\exp_{\gamma_0(s)}\left(\frac{l-s}{l}\sum_1^n a_i(x)\frac{f(s)}{f(0)}e_i(s) + \frac{s}{l}\sum_1^n b_i(y)\frac{f(s)}{f(l)}e_i(s)\right),\ s \in [0, l].$$

It is easy to see that $d(x, y) \le \tilde{d}(x, y)$ in $U_{x_0} \times U_{y_0}$ and with equality at (x_0, y_0). Therefore we have

$$\psi(u(y)) - \psi(u(x)) - \tilde{d}(x, y) \le \varepsilon_0,$$

for any $(x, y) \in U_{x_0} \times U_{y_0}$ and with equality at (x_0, y_0).

Thus we can apply the maximum principle to conclude that for each $\lambda > 0$, there exist $X \in Sym^2(T^*_{x_0}M)$ and $Y \in Sym^2(T^*_{y_0}M)$ such that

$$(D_y\tilde{d}(x_0, y_0), Y) \in \bar{\mathcal{J}}^{2,+}(\psi \circ u)(y_0),$$
$$(D_x\tilde{d}(x_0, y_0), -X) \in \bar{\mathcal{J}}^{2,+}(-\psi \circ u)(x_0),$$
$$\textit{i.e.}\ (-D_x\tilde{d}(x_0, y_0), X) \in \bar{\mathcal{J}}^{2,-}(\psi \circ u)(x_0),$$

and

$$\begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix} \leq M + \lambda M^2,$$

where $M = D^2\tilde{d}(x_0, y_0)$.

Note that $D_y\tilde{d}(x_0, y_0) = e_n(l)$ and $D_x\tilde{d}(x_0, y_0) = -e_n(0)$. By Lemma 8, we have

$$(\varphi'(z_{y_0})e_n(l), \varphi'(z_{y_0})Y + \varphi''(z_{y_0})e_n(l) \otimes e_n(l)) \in \bar{\mathcal{J}}^{2,+}u(y_0),$$
$$(\varphi'(z_{x_0})e_n(0), \varphi'(z_{x_0})X + \varphi''(z_{x_0})e_n(0) \otimes e_n(0)) \in \bar{\mathcal{J}}^{2,-}u(x_0),$$

where $z_{x_0} = \psi(u(x_0))$ and $z_{y_0} = \psi(u(y_0))$.

On the other hand, since u is both a subsolution and a supersolution of (1), we have

$$tr(\varphi'(z_{y_0})A_2Y + \varphi''(z_{y_0})A_2e_n(l) \otimes e_n(l)) + q(\varphi'(z_{y_0}), \varphi(z_{y_0})) \geq 0$$

and

$$tr(\varphi'(z_{x_0})A_1X + \varphi''(z_{x_0})A_1e_n(0) \otimes e_n(0)) + q(\varphi'(z_{x_0}), \varphi(z_{x_0})) \leq 0,$$

where

$$A_1 = \begin{pmatrix} \beta(\varphi(z_{x_0}), \varphi'(z_{x_0})) & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \beta(\varphi(z_{x_0}), \varphi'(z_{x_0})) & 0 \\ 0 & \dots & 0 & \alpha(\varphi(z_{x_0}), \varphi'(z_{x_0})) \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} \beta(\varphi(z_{y_0}), \varphi'(z_{y_0})) & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \beta(\varphi(z_{y_0}), \varphi'(z_{y_0})) & 0 \\ 0 & \dots & 0 & \alpha(\varphi(z_{y_0}), \varphi'(z_{y_0})) \end{pmatrix}.$$

Therefore, first we have

$$0 \leq q(\varphi(z_{y_0}), \varphi'(z_{y_0})) + \varphi''(z_{y_0})\alpha(\varphi(z_{y_0}), \varphi'(z_{y_0})) + \varphi'(z_{y_0})tr\left(\begin{pmatrix} 0 & C \\ C & A_2 \end{pmatrix}\begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix}\right),$$

where C is an $n \times n$ matrix to be determined. Multiplying by $\frac{f^2(l)}{\varphi'(z_{y_0})\beta(\varphi(z_{y_0}), \varphi'(z_{y_0}))}$ gives

$$\begin{aligned} 0 \leq{} & \frac{f^2(l)}{\varphi'(z_{y_0})\beta(\varphi(z_{y_0}), \varphi'(z_{y_0}))}(q(\varphi(z_{y_0}), \varphi'(z_{y_0})) + \varphi''(z_{y_0})\alpha(\varphi(z_{y_0}), \varphi'(z_{y_0}))) \\ & + \frac{f^2(l)}{\beta(\varphi(z_{y_0}), \varphi'(z_{y_0}))}tr\left(\begin{pmatrix} 0 & C \\ C & A_2 \end{pmatrix}\begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix}\right). \end{aligned}$$

Similarly, for the inequality at x_0 we get

$$\begin{aligned}
0 \geq\ & \frac{f^2(0)}{\varphi'(z_{x_0})\beta(\varphi(z_{x_0}),\varphi'(z_{x_0}))}(q(\varphi(z_{x_0}),\varphi'(z_{x_0}))+\varphi''(z_{x_0})\alpha(\varphi(z_{x_0}),\varphi'(z_{x_0})))\\
&-\frac{f^2(0)}{\beta(\varphi(z_{x_0}),\varphi'(z_{x_0}))}tr\left(\begin{pmatrix}A_1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}-X & 0\\ 0 & Y\end{pmatrix}\right).
\end{aligned}$$

Combining them we obtain

$$\begin{aligned}
0 \leq & \frac{f^2(l)}{\varphi'\beta(\varphi,\varphi')}(q(\varphi,\varphi')+\varphi''\alpha(\varphi'))\Big|_{z_{y_0}} - \frac{f^2(0)}{\varphi'\beta(\varphi,\varphi')}(q(\varphi,\varphi')+\varphi''\alpha(\varphi,\varphi'))\Big|_{z_{x_0}}\\
&+\frac{f^2(l)}{\beta(\varphi(z_{y_0}),\varphi'(z_{y_0}))}tr\left(\begin{pmatrix}0 & C\\ C & A_2\end{pmatrix}\begin{pmatrix}-X & 0\\ 0 & Y\end{pmatrix}\right)\\
&+\frac{f^2(0)}{\beta(\varphi(z_{x_0}),\varphi'(z_{x_0}))}tr\left(\begin{pmatrix}A_1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}-X & 0\\ 0 & Y\end{pmatrix}\right).
\end{aligned}$$

Letting

$$C=\begin{pmatrix}\frac{f(0)}{f(l)}\beta(\varphi(z_{y_0}),\varphi'(z_{y_0})) & & & \\ & \ddots & & \\ & & \frac{f(0)}{f(l)}\beta(\varphi(z_{y_0}),\varphi'(z_{y_0})) & \\ & & & 0\end{pmatrix},$$

then the matrix

$$\begin{aligned}
W = \ & \frac{f^2(l)}{\beta(\varphi(z_{y_0}),\varphi'(z_{y_0}))}\begin{pmatrix}0 & C\\ C & A_2\end{pmatrix}+\frac{f^2(0)}{\beta(\varphi(z_{x_0}),\varphi'(z_{x_0}))}\begin{pmatrix}A_1 & 0\\ 0 & 0\end{pmatrix}\\
=&\begin{pmatrix}f^2(0)I_{n-1} & 0 & f(0)f(l)I_{n-1} & 0\\ 0 & f^2(0)\frac{\alpha(\varphi,\varphi')}{\beta(\varphi,\varphi')}\big|_{z_{x_0}} & 0 & 0\\ f(0)f(l)I_{n-1} & 0 & f^2(l)I_{n-1} & 0\\ 0 & 0 & 0 & f^2(l)\frac{\alpha(\varphi,\varphi')}{\beta(\varphi,\varphi')}\big|_{z_{y_0}}\end{pmatrix}
\end{aligned}$$

is positive semidefinite.

So we can use

$$\begin{pmatrix}-X & 0\\ 0 & Y\end{pmatrix}\leq M+\lambda M^2$$

to get

$$\begin{aligned}
0\leq\ & \frac{f^2(l)}{\varphi'\beta(\varphi,\varphi')}(q(\varphi,\varphi')+\varphi''\alpha(\varphi,\varphi'))\Big|_{z_{y_0}}-\frac{f^2(0)}{\varphi'\beta(\varphi,\varphi')}(q(\varphi,\varphi')+\varphi''\alpha(\varphi,\varphi'))\Big|_{z_{x_0}}\\
&+tr(WM)+\lambda\ tr(WM^2).
\end{aligned}$$

Now we compute $tr(WM)$ as follows.

$$\begin{aligned}tr(WM)=&\sum_{i=1}^{n-1}D^2\tilde{d}((f(0)e_i(0),f(l)e_i(l)),(f(0)e_i(0),f(l)e_i(l)))\\&+\frac{\alpha(\varphi,\varphi')}{\beta(\varphi,\varphi')}\Big|_{z_{x_0}}D^2\tilde{d}((f(0)e_n(0),0),(f(0)e_n(0),0))\\&+\frac{\alpha(\varphi,\varphi')}{\beta(\varphi,\varphi')}\Big|_{z_{y_0}}D^2\tilde{d}((0,f(l)e_n(l)),(0,f(l)e_n(l))).\end{aligned}$$

Note that

$$\begin{aligned}&D^2\tilde{d}((f(0)e_i(0),f(l)e_i(l)),(f(0)e_i(0),f(l)e_i(l)))\\=&\frac{d^2}{dt^2}\Big|_{t=0}\tilde{d}(\exp_{x_0}(tf(0)e_i(0)),\exp_{y_0}(tf(l)e_i(l)))\\=&\frac{d^2}{dt^2}\Big|_{t=0}L(\exp_{\gamma_0(s)}(tf(s)e_i(s))_{s\in[0,l]})\\=&\int_0^l[(f'(s))^2-f^2(s)R(e_n,e_i,e_i,e_n)]ds\\=&ff'\big|_0^l-\int_0^l f(f''+fR(e_n,e_i,e_i,e_n))ds,\end{aligned}$$

which implies

$$\begin{aligned}&\sum_{i=1}^{n-1}D^2\tilde{d}((f(0)e_i(0),f(l)e_i(l)),(f(0)e_i(0),f(l)e_i(l)))\\&=(n-1)ff'\big|_0^l-\int_0^l f((n-1)f''+fRic(e_n,e_n))ds\\&\le(n-1)ff'\big|_0^l-\int_0^l(n-1)f(f''+\kappa f)ds.\end{aligned}$$

Similarly we get

$$\begin{aligned}D^2\tilde{d}((f(0)e_n(0),0),(f(0)e_n(0),0))=0,\\D^2\tilde{d}((0,f(l)e_n(l)),(0,f(l)e_n(l)))=0.\end{aligned}$$

In summary, we have

$$\begin{aligned}0\le&\frac{f^2(l)}{\varphi'\beta(\varphi,\varphi')}(q(\varphi,\varphi')+\varphi''\alpha(\varphi,\varphi'))\Big|_{z_{y_0}}-\frac{f^2(0)}{\varphi'\beta(\varphi,\varphi')}(q(\varphi,\varphi')+\varphi''\alpha(\varphi,\varphi'))\Big|_{z_{x_0}}\\&+(n-1)ff'\big|_0^l-\int_0^l(n-1)f(f''+\kappa f)ds+\lambda\ tr(WM^2).\end{aligned}$$

Taking $f(s) \equiv 1$ and $\kappa = 0$, and letting $\lambda \to 0$, we have

$$0 \le \left. \frac{q(\varphi, \varphi') + \varphi'' \alpha(\varphi, \varphi')}{\varphi' \beta(\varphi, \varphi')} \right|_{z_{x_0}}^{z_{y_0}}. \tag{16}$$

Now taking Condition (13) into account, since $z_{y_0} = z_{x_0} + d(x_0, y_0) + \varepsilon_0 > z_{x_0}$, we get a contradiction. Then we must have

$$Z(x, y) = \psi(u(y)) - \psi(u(x)) - d(x, y) \le 0, \tag{17}$$

which is the desired result.

□

4. Manifolds with negative lower Ricci curvature bound

In this section we prove the following modulus of continuity estimate.

Theorem 9. *Let (M^n, g) be a compact Riemannian manifold with Ricci curvature $Ric \ge (n-1)\kappa$, $\kappa < 0$, and u a viscosity solution of Equation (1). Suppose the barrier $\varphi : [a, b] \to [\inf u, \sup u]$ satisfies*

$$\varphi' > 0, \tag{18}$$

$$\left. \frac{q(\varphi, \varphi') + \varphi'' \alpha(\varphi, \varphi')}{\varphi' \beta(\varphi, \varphi')} \right|_{z} + (n-1) \frac{\rho'}{\rho} = 0, \tag{19}$$

where $\rho : [a, b] \to \mathbb{R}^+$ satisfies $\rho'' + \kappa \rho = 0$ and $(\frac{\rho'}{\rho})' > 0$. Moreover let ψ be the inverse of φ, i.e. $\psi(\varphi(z)) = z$. Then we have

$$\psi(u(y)) - \psi(u(x)) - d(x, y) \le 0, \ \forall x, y \in M. \tag{20}$$

By letting y approach x, we get the following gradient bound.

Corollary 10. *Under the conditions of Theorem 9, if moreover $u \in C^1(M)$, then for every $x \in M$ we have*

$$|\nabla u(x)| \le \varphi'(\psi(u(x))). \tag{21}$$

Remark 11. In fact, Theorem 9 is an equivalent statement of Theorem 3. First, in a warped product $\bar{M} = N \times [a, b]$ with $\bar{g} = ds^2 + \rho(s)^2 g^N$, the requirement $Ric(\partial_s, \partial_s) = (n-1)\kappa$ is equivalent to $\rho'' + \kappa\rho = 0$. Second, by the calculation in [1, Sec. 3], $\bar{u}(x, s) = \varphi(s)$ being a solution of (1) means

$$\alpha(\varphi, \varphi')\varphi'' + q(\varphi, \varphi') + (n-1)\varphi' \beta(\varphi, \varphi') \frac{\rho'}{\rho} = 0. \tag{22}$$

Remark 12. Note that $(\frac{\rho'}{\rho})' > 0$ with $\rho'' + \kappa\rho = 0$ implies $\kappa < -(\rho')^2/\rho^2$ and so necessarily $\kappa < 0$. In addition, it is easy to see that $\rho(z) = \cosh(z_0 + z)$ satisfies our conditions; while $\rho(z) = \cos z$ does not, which means for $\kappa > 0$ we need a different argument.

Proof of Theorem 9. The proof is by contradiction. Define

$$Z(x,y)=\psi(u(y))-\psi(u(x))-d(x,y). \tag{23}$$

Assume otherwise

$$\max_{x,y\in M} Z(x,y)=Z(x_0,y_0)=\varepsilon_0>0. \tag{24}$$

As derived in the last section, we have

$$\begin{aligned}0\le{}&\frac{f^2(l)}{\varphi'\beta(\varphi,\varphi')}(q(\varphi,\varphi')+\varphi''\alpha(\varphi,\varphi'))\Big|_{z_{y_0}}-\frac{f^2(0)}{\varphi'\beta(\varphi,\varphi')}(q(\varphi,\varphi')+\varphi''\alpha(\varphi,\varphi'))\Big|_{z_{x_0}}\\&+(n-1)ff'\big|_0^l-\int_0^l(n-1)f(f''+\kappa f)ds.\end{aligned}$$

Take $f(s)=\rho(s)$. And (19) shows that

$$\frac{q(\varphi,\varphi')+\varphi''\alpha(\varphi,\varphi')}{\varphi'\beta(\varphi,\varphi')}\Big|_z=-(n-1)\frac{\rho'(z)}{\rho(z)} \tag{25}$$

is strictly decreasing. So we have

$$\begin{aligned}0<{}&\frac{f^2(l)}{\varphi'\beta(\varphi,\varphi')}(q(\varphi,\varphi')+\varphi''\alpha(\varphi,\varphi'))\Big|_{z_{y_0}}-\frac{f^2(0)}{\varphi'\beta(\varphi,\varphi')}(q(\varphi,\varphi')+\varphi''\alpha(\varphi,\varphi'))\Big|_{z_{x_0}+\varepsilon_0}\\&+(n-1)ff'\Big|_0^l-\int_0^l(n-1)f(f''+\kappa f)ds\\&=\rho^2\left(\frac{q(\varphi,\varphi')+\varphi''\alpha(\varphi,\varphi')}{\varphi'\beta(\varphi,\varphi')}+(n-1)\frac{\rho'}{\rho}\right)\Bigg|_{z_{x_0}+\varepsilon_0}^{z_{y_0}}\\&=0,\end{aligned}$$

which is a contradiction. Then we must have

$$Z(x,y)=\psi(u(y))-\psi(u(x))-d(x,y)\le 0, \tag{26}$$

which is the desired result. ☐

5. RIEMANNIAN MANIFOLDS WITH GENERAL LOWER RICCI BOUND

5.1. **The result and an example.** As observed in Remark 12, the case $\kappa>0$ may not be handled as in the last two sections. In other words, for this case we may not prove that any solution φ to the one-dimensional equation is a barrier. However, we find that for some family of the one-dimensional solutions, the property of being barriers can be extended smoothly in the family, and in fact this phenomenon holds for any $\kappa\in\mathbb{R}$ regardless of its sign. More precisely, we prove the following result.

Theorem 13. *Let (M^n,g) be a compact Riemannian manifold with Ricci curvature $Ric\ge(n-1)\kappa$, $\kappa\in\mathbb{R}$. Assume in Equation* (1) *the coefficients α, β are C^2 functions, and q a continuous function. Let u be a C^3 solution* *of Equation* (1). *Suppose there exists a family of functions* φ_c *from* $[a_c, b_c]$ *to* $[\inf u, \sup u]$ *which satisfies*

$$\frac{\alpha(\varphi,\varphi')\varphi'' + q(\varphi,\varphi')}{\varphi'\beta(\varphi,\varphi')} + (n-1)\frac{\rho'}{\rho} = 0,$$
$$\varphi' > 0,$$

and depends smoothly on $c \in (c_u, +\infty)$. *Here* ρ *is a positive* C^2 *function on* $[a_c, b_c]$ *satisfying* $\rho'' + \kappa\rho = 0$. *Moreover, assume for* $c \gg c_u$, φ_c' *is uniformly large. Then for any* $c > c_u$ *we have*

$$\psi_c(u(y)) - \psi_c(u(x)) - d(x,y) \le 0,\ \forall x, y \in M, \tag{27}$$

where ψ_c *is the inverse of* φ_c.

By letting y approach x, we get the following gradient bound.

Corollary 14. *Under the conditions of Theorem 13, for every* $x \in M$ *we have*

$$|\nabla u(x)| \le \varphi_c'(\psi_c(u(x))),\ c > c_u. \tag{28}$$

Here we give an example in which Theorem 13 applies.

Example 5.1. *Let us consider the following equation:*

$$div_g(\Phi'(|\nabla_g u|^2)\nabla_g u) + q(u) = 0, \tag{29}$$

where $q(u) = Q'(u)$ *for some function* Q, *on a Riemannian manifold with* $Ric \ge n-1$. *Here* Φ *satisfies some structure conditions in Subsection 6.5. We also use the following notations:*

$$K(s) := \Phi'(s)s - \frac{1}{2}\Phi(s),$$
$$\Lambda(s) := 2\Phi''(s)s + \Phi'(s).$$

Note that for this example, $\alpha(u, |\nabla u|) = \Lambda(|\nabla u|^2)$ *and* $\beta(u, |\nabla u|) = \Phi'(|\nabla u|^2)$.

Let $c_u = sup_{r\in[\inf u, \sup u]} Q(r)$ *and suppose* $u_0 \in [\inf u, \sup u]$ *is such that* $Q(u_0) = c_u$. *Then the ODE appearing in Theorem 13 becomes the following: (Here* $c > c_u$ *and* $\rho(z) = \cos z$.*)*

$$\begin{cases}\Lambda((\varphi_c')^2)\varphi_c'' + q(\varphi_c) = (n-1)\tan z\cdot\Phi'((\varphi_c')^2)\varphi_c',\\ \varphi(0) = u_0,\ \varphi'(0) = \sqrt{K^{-1}\circ(c-c_u)},\end{cases} \tag{30}$$

which has a unique solution $\varphi_c : [a,b] \to [\inf u, \sup u]$, *where* $0 \in [a,b] \subset (-\pi/2, \pi/2)$. *Here* K^{-1} *denotes the inverse function of* K. *Equivalently,* φ_c *satisfies:*

$$K((\varphi')^2) + Q(\varphi) = c + \int_0^z (n-1)\tan z\cdot\Phi'((\varphi')^2)(\varphi')^2 dz. \tag{31}$$

Since $\int_0^z (n-1)\tan z\cdot\Phi'((\varphi')^2)(\varphi')^2 dz \ge 0$, *we always have*

$$\varphi_c'(z) \ge \sqrt{K^{-1}\circ(c - Q(\varphi_c))} \ge \sqrt{K^{-1}\circ(c-c_u)} > 0. \tag{32}$$

Moreover, when $c \gg c_u$, φ_c' is uniformly large. So by Theorem 13, for the corresponding inverse ψ_c ($c > c_u$), we have

$$\psi_c(u(y)) - \psi_c(u(x)) - d(x,y) \leq 0,\ \forall x, y \in M. \tag{33}$$

5.2. **The proof.** In this subsection, we first give the outline of the proof and then derive a computation lemma which is needed in the proof.

Proof of Theorem 13. Assume by contradiction that (27) does not hold for some $c_0 > c_u$. Then for any $c > c_0$ we can solve by perturbation

$$\begin{cases} \dfrac{\alpha(\varphi,\varphi')\varphi'' + q(\varphi,\varphi')}{\varphi'\beta(\varphi,\varphi')} = -(n-1)\dfrac{\rho'}{\rho} - \delta(c)\dfrac{z}{\rho^2}, \\ \varphi(a_c) = \varphi_c(a_c),\ \varphi'(a_c) = \varphi_c'(a_c), \end{cases} \tag{34}$$

to get a function $\varphi_{c,\delta(c)}$, where $\delta(c) > 0$ is small, and depends on c with $\lim_{c \to c_0} \delta(c) = 0$.

Denote $D = \{(x,x) : x \in M\}$. Then we consider a manifold $\hat{M}$ with boundary which compactifies $(M \times M) \setminus D$ as follows: As a set, $\hat{M}$ is the disjoint union of $(M \times M) \setminus D$ with the unit sphere bundle $SM = \{(x,v) \in TM : ||v|| = 1\}$. The manifold-with-boundary structure is defined by the atlas generated by all charts for $(M \times M) \setminus D$, together with the charts $\hat{Y}$ from $SM \times (0,r)$ defined by taking a chart Y for SM, and setting $\hat{Y}(z,s) := (\exp(sY(z)), \exp(-sY(z)))$.

In the following we write $\varphi = \varphi_{c,\delta(c)}$ and $\psi = \psi_{c,\delta(c)}$ for short. Let

$$Z(x,y) = \psi(u(y)) - \psi(u(x)) - d(x,y). \tag{35}$$

We define a function $\hat{Z}$ on $\hat{M}$ as follows: For $(x,y) \in (M \times M) \setminus D$, we define

$$\hat{Z}(x,y) = \frac{Z(x,y)}{d(x,y)}. \tag{36}$$

For $(x,v) \in SM$, we define

$$\hat{Z}(x,v) = \frac{1}{\varphi'(z_x)} D_v u(x) - 1. \tag{37}$$

Then one can check that $\hat{Z}$ is a continuous function on $\hat{M}$. And when $c \gg c_0$, we have $\hat{Z} \leq 0$ on $\hat{M}$.

Thus we can define

$$c_1 := \inf\{\bar{c} > c_0 \big| \hat{Z} \leq 0 \text{ on } \hat{M} \text{ for } c \in (\bar{c}, +\infty)\}. \tag{38}$$

By assumption we have $c_1 > c_0$, which we shall prove leads to a contradiction.

In fact for $c = c_1$ there will be two cases.

Case 1: $0 = \hat{Z}(x_0, y_0)$ for some $x_0 \neq y_0$, i.e. $Z(x_0, y_0) = 0$.

Case 2: $Z(x,y) < 0$ for any $x \neq y \in M$ and $\hat{Z}(x_0, v_0) = 0$ for some $(x_0, v_0) \in SM$.

We will rule out these two cases, so $c_1 > c_0$ is impossible.

For that purpose, we need the following computation lemma.

Lemma 15. *Let u be a C^3 solution of Equation* (1)*. For $x \neq y$ with $d(x,y) < inj(M)$, the injectivity radius of M, let $\gamma_0 : [0,l] \to M$ be the length-minimizing geodesic from x to y, choose Fermi coordinates along γ_0 as before, and fix a C^2 function $f : [0,l] \to \mathbb{R}$. Then $Z = Z(x,y)$ satisfies the following equation*

$$\begin{aligned}
\mathcal{L}Z :=& f(l)^2 \frac{\alpha(\varphi,\varphi')}{\beta(\varphi,\varphi')}\bigg|_{z_y} D^2_{(0,e_n),(0,e_n)} Z + f(0)^2 \frac{\alpha(\varphi,\varphi')}{\beta(\varphi,\varphi')}\bigg|_{z_x} D^2_{(e_n,0),(e_n,0)} Z \\
&+ \sum_{i<n} D^2_{(f(0)e_i,f(l)e_i),(f(0)e_i,f(l)e_i)} Z \\
=& -f(l)^2 \frac{\alpha(\varphi,\varphi')\varphi'' + q(\varphi,\varphi')}{\varphi'\beta(\varphi,\varphi')}\bigg|_{z_y} + f(0)^2 \frac{\alpha(\varphi,\varphi')\varphi'' + q(\varphi,\varphi')}{\varphi'\beta(\varphi,\varphi')}\bigg|_{z_x} \\
&- (n-1) f(s) f'(s)\bigg|_0^l + \int_0^l \big((n-1) f f'' + f^2 Ric(e_n)\big)\, ds + DZ * DZ + P \cdot DZ,
\end{aligned}$$

where the coefficients of $DZ * DZ$ and DZ are C^1 functions.

With this lemma at hand, choose

$$f(s) = \rho(z_x + Z + s). \tag{39}$$

Using the condition on Ricci curvature, we can derive

$$\begin{aligned}
&\mathcal{L}Z + DZ * DZ + P \cdot DZ \\
\geq& -f(l)^2 \frac{\alpha(\varphi,\varphi')\varphi'' + q(\varphi,\varphi')}{\varphi'\beta(\varphi,\varphi')}\bigg|_{z_y} + f(0)^2 \frac{\alpha(\varphi,\varphi')\varphi'' + q(\varphi,\varphi')}{\varphi'\beta(\varphi,\varphi')}\bigg|_{z_x} - (n-1) f(s) f'(s)\bigg|_0^l \\
=& -\rho^2 \left(\frac{\alpha(\varphi,\varphi')\varphi'' + q(\varphi,\varphi')}{\varphi'\beta(\varphi,\varphi')} + (n-1)\frac{\rho'}{\rho}\right)\bigg|_{z_x+Z}^{z_y} \\
&+ f(0)^2 \frac{\alpha(\varphi,\varphi')\varphi'' + q(\varphi,\varphi')}{\varphi'\beta(\varphi,\varphi')}\bigg|_{z_x+Z}^{z_x}.
\end{aligned} \tag{40}$$

Let us first consider Case 1. The same computation applies and so by the maximum principle we have at (x_0, y_0)

$$\begin{aligned}
0 \geq& \mathcal{L}Z + DZ * DZ + P \cdot DZ \\
&= -\rho^2 \left(\frac{\alpha(\varphi,\varphi')\varphi'' + q(\varphi,\varphi')}{\varphi'\beta(\varphi,\varphi')} + (n-1)\frac{\rho'}{\rho}\right)\bigg|_{z_x}^{z_y} \\
&= \delta(c_1) z\big|_{z_x}^{z_y} = \delta(c_1) l > 0.
\end{aligned}$$

This contradiction shows that Case 1 cannot occur.

Next we consider Case 2: $Z(x,y)<0$ for any $x\neq y\in M$. In this case by (40) and the choice of $\varphi_{c,\delta(c)}$, we have, for $x\neq y$ close enough to each other,

$$\begin{aligned}&\mathcal{L}Z+DZ*DZ+P\cdot DZ\\&\geq \delta(c_1)z\Big|_{z_x+Z}^{z_y}+f(0)^2\left(-(n-1)\frac{\rho'}{\rho}-\delta(c)\frac{z}{\rho^2}\right)\Bigg|_{z_x+Z}^{z_x}\\&=\delta(c_1)l+C(x,y)Z\geq C(x,y)Z,\end{aligned}$$

where $C(x,y)$ is some bounded function. That is, the inequality above holds near the boundary of $\hat{M}$. Now recall the boundary Hopf maximum principle from [17], applying to any C^2 function Z which has a strict maximum boundary value zero and satisfies $a^{ij}Z_{ij}+b^iZ_i+cZ\geq 0$ with $a^{ij}\in C^2$, $b^i\in C^1$, $c\in L^\infty$ and $[a^{ij}]\geq 0$. Thus we derive for (x_0,v_0)

$$\begin{aligned}0>D_{(0,v)}Z(x,x)&=\lim_{t\to 0}\frac{Z(x,\exp_x(tv))-Z(x,x)}{t}\\&=\lim_{t\to 0}\frac{\psi(u(\exp_x(tv)))-\psi(u(x))-d(x,\exp_x(tv))}{t}\\&=\frac{1}{\varphi'(z_x)}D_vu(x)-1,\end{aligned}$$

which contradicts $\hat{Z}(x_0,v_0)=\frac{1}{\varphi'(z_{x_0})}D_{v_0}u(x_0)-1=0$. So Case 2 is also ruled out. So $c_1>c_0$ is impossible, and we complete the proof of Theorem 13. □

Finally we give the proof of Lemma 15.

The proof of Lemma 15. Recall

$$Z(x,y)=\psi(u(y))-\psi(u(x))-d(x,y). \tag{41}$$

For any $X\in T_xM$ and $Y\in T_yM$, there exists a variation $\gamma(\varepsilon,s)$ of $\gamma_0(s)$ such that $\gamma_\varepsilon(0)=X$ and $\gamma_\varepsilon(l)=Y$. Then the first derivative of Z in the direction (X,Y) is

$$\begin{aligned}D_{(X,Y)}Z&=\psi'(u(y))\langle\nabla u(y),\gamma_\varepsilon(l)\rangle-\psi'(u(x))\langle\nabla u(x),\gamma_\varepsilon(0)\rangle-\langle T(s),\gamma_\varepsilon(s)\rangle|_0^l\\&=\langle\psi'(u)\nabla u-\gamma_s,\gamma_\varepsilon(s)\rangle|_0^l.\end{aligned}$$

Furthermore the second derivative of Z is

$$\begin{aligned}D^2_{(X,Y),(X,Y)}Z&=\psi''(u)\langle\nabla u,\gamma_\varepsilon(l)\rangle^2+\psi'(u)\left(\langle D_{\gamma_\varepsilon}(\nabla u),\gamma_\varepsilon(l)\rangle+\langle\nabla u,D_{\gamma_\varepsilon}\gamma_\varepsilon(l)\rangle\right)\\&\quad-\psi''(u)\langle\nabla u,\gamma_\varepsilon(0)\rangle^2-\psi'(u)\left(\langle D_{\gamma_\varepsilon}(\nabla u),\gamma_\varepsilon(0)\rangle+\langle\nabla u,D_{\gamma_\varepsilon}\gamma_\varepsilon(0)\rangle\right)\\&\quad-\int_0^l\Big(|\nabla_{\gamma_s}(\gamma_\varepsilon^\perp)|^2-R(\gamma_s,\gamma_\varepsilon,\gamma_\varepsilon,\gamma_s)\Big)\,ds-\langle\gamma_s,\nabla_{\gamma_\varepsilon}\gamma_\varepsilon\rangle\big|_0^l.\end{aligned}$$

Note that

$$\psi'\cdot\varphi'=1, \tag{42}$$

$$\psi''\cdot(\varphi')^2+\psi'\cdot\varphi''=0. \tag{43}$$

We get

$$\begin{aligned}D^2_{(X,Y),(X,Y)}Z = &-\frac{\varphi''}{(\varphi')^3}\langle \nabla u, \gamma_\varepsilon(l)\rangle^2 + \frac{\varphi''}{(\varphi')^3}\langle \nabla u, \gamma_\varepsilon(0)\rangle^2\\ &+ \frac{1}{\varphi'}D^2u(\gamma_\varepsilon(l), \gamma_\varepsilon(l)) - \frac{1}{\varphi'}D^2u(\gamma_\varepsilon(0), \gamma_\varepsilon(0))\\ &- \int_0^l \Big(|\nabla_{\gamma_s}(\gamma_\varepsilon^\perp)|^2 - R(\gamma_s, \gamma_\varepsilon, \gamma_\varepsilon, \gamma_s)\Big)\, ds + D_{(D_{\gamma_\varepsilon}\gamma_\varepsilon(0), D_{\gamma_\varepsilon}\gamma_\varepsilon(l))}Z.\end{aligned}$$

Now we choose particular variations to obtain inequalities for particular parts of the Hessian of Z:

(1). Vary y: Choose the variation

$$\gamma(\varepsilon, s) = \gamma_0(s + \varepsilon\frac{s}{l}). \tag{44}$$

So $\gamma_\varepsilon(l) = e_n$, $\gamma_\varepsilon(0) = 0$. Then we get

$$D_{(0,e_n)}Z = \langle \frac{1}{\varphi'}\nabla u - \gamma_s, e_n\rangle(l) = \frac{u_n(y)}{\varphi'} - 1, \tag{45}$$

$$\begin{aligned}D^2_{(0,e_n),(0,e_n)}Z &= -\frac{\varphi'' u_n^2(y)}{(\varphi')^3} + \frac{1}{\varphi'}u_{nn}(y)\\ &= -\frac{\varphi''}{\varphi'}(1 + D_{(0,e_n)}Z)^2 + \frac{1}{\varphi'}u_{nn}(y).\end{aligned} \tag{46}$$

(2). Vary x: Choose the variation

$$\gamma(\varepsilon, s) = \gamma_0(s + \varepsilon\frac{l-s}{l}). \tag{47}$$

So $\gamma_\varepsilon(l) = 0$, $\gamma_\varepsilon(0) = e_n$. Then similarly we get

$$D_{(e_n,0)}Z = -\langle \frac{1}{\varphi'}\nabla u - \gamma_s, e_n\rangle(0) = -\frac{u_n(x)}{\varphi'} + 1, \tag{48}$$

$$D^2_{(e_n,0),(e_n,0)}Z = \frac{\varphi'' u_n^2(x)}{(\varphi')^3} - \frac{1}{\varphi'}u_{nn}(x) = \frac{\varphi''}{\varphi'}(1 - D_{(e_n,0)}Z)^2 - \frac{1}{\varphi'}u_{nn}(x). \tag{49}$$

(3). Vary γ_0 along $e_i(s)$ for fixed $i < n$: Choose

$$\gamma(\varepsilon, s) = \exp_{\gamma_0(s)}(\varepsilon f(s) e_i(s)). \tag{50}$$

So $\gamma_\varepsilon(s) = f(s)e_i(s)$. Therefore

$$\begin{aligned}
D_{(f(0)e_i,f(l)e_i)}Z &= \langle \frac{1}{\varphi'}\nabla u - \gamma_s, f(s)e_i(s)\rangle|_0^l \\
&= \frac{f(l)}{\varphi'}u_i(y) - \frac{f(0)}{\varphi'}u_i(x), \\
D^2_{(f(0)e_i,f(l)e_i),(f(0)e_i,f(l)e_i)}Z &= -\frac{\varphi''}{(\varphi')^3}f^2(l)u_i^2(y) + \frac{\varphi''}{(\varphi')^3}f^2(0)u_i^2(x) \\
&\quad + \frac{1}{\varphi'}f^2(l)u_{ii}(y) - \frac{1}{\varphi'}f^2(0)u_{ii}(x) \\
&\quad - \int_0^l [(f'(s))^2 - f^2(s)R(e_n,e_i,e_i,e_n)]ds.
\end{aligned}$$

Then after summation from $i = 1$ to $i = n-1$ we have

$$\begin{aligned}
\sum_{i<n} D^2_{(f(0)e_i,f(l)e_i),(f(0)e_i,f(l)e_i)}Z &= -\frac{\varphi''}{(\varphi')^3}f^2(l)\sum_{i<n}u_i^2(y) + \frac{\varphi''}{(\varphi')^3}f^2(0)\sum_{i<n}u_i^2(x) \\
&\quad + \frac{1}{\varphi'}f^2(l)\sum_{i<n}u_{ii}(y) - \frac{1}{\varphi'}f^2(0)\sum_{i<n}u_{ii}(x) \\
&\quad - (n-1)f(s)f'(s)|_0^l + \int_0^l [(n-1)ff'' + f^2Ric(e_n)]ds.
\end{aligned}$$

Now recall that at x_0 or y_0 Equation (1) is

$$\alpha(\varphi,\varphi')u_{nn} + \beta(\varphi,\varphi')\sum_{i<n}u_{ii} + q(\varphi) + DZ * DZ + P\cdot DZ = 0. \tag{51}$$

Therefore direct computation yields

$$\begin{aligned}
\mathcal{L}Z := &- f(l)^2\frac{\alpha(\varphi,\varphi')\varphi''}{\varphi'\beta(\varphi,\varphi')}\bigg|_{z_y}(1 + D_{(0,e_n)}Z)^2 + f(0)^2\frac{\alpha(\varphi,\varphi')\varphi''}{\varphi'\beta(\varphi,\varphi')}\bigg|_{z_x}(1 - D_{(e_n,0)}Z)^2 \\
&- \frac{\varphi''}{\varphi'}\bigg|_{z_y}\sum_{i<n}(D_{(0,f(l)e_i)}Z)^2 + \frac{\varphi''}{\varphi'}\bigg|_{z_x}\sum_{i<n}(D_{(f(0)e_i,0)}Z)^2 \\
&- f(l)^2\frac{q(\varphi)}{\varphi'\beta(\varphi,\varphi')}\bigg|_{z_y} + f(0)^2\frac{q(\varphi)}{\varphi'\beta(\varphi,\varphi')}\bigg|_{z_x} - (n-1)f(s)f'(s)\bigg|_0^l \\
&+ \int_0^l \big((n-1)ff'' + f^2Ric(e_n)\big)\,ds + DZ * DZ + P\cdot DZ.
\end{aligned}$$

Putting the terms involving DZ together, we complete the proof of Lemma 15. □

6. CASE FOR FINSLER MANIFOLDS WITH NONNEGATIVE WEIGHTED RICCI CURVATURE

In this section we consider the problem on Finsler manifolds, although only for Equation (76) below of divergence form. We first briefly review the fundamentals of Finsler geometry [4,28] and some developments from [25,27].

Then we give the structure conditions for the equation and regularity of its solutions. Finally we will discuss the modulus of continuity estimates in this Finsler context.

6.1. **Finsler manifolds.** Let M be an n-dimensional connected smooth manifold without boundary. Given a local coordinate $\{x^i\}_{i=1}^n$ on an open set $U \subset M$, let $\{x^i, V^j\}_{i,j=1}^n$ be the coordinate of TU, i.e.

$$V = V^j \frac{\partial}{\partial x^j}, \forall V \in T_xM, x \in U. \tag{52}$$

Definition 6.1 (Finsler structures)**.** A function $F : TM \to [0, \infty)$ is called a Finsler structure if the following three conditions hold:

(1) (Regularity) F is C^∞ on $TM \setminus 0$;
(2) (Positive 1-homogeneity) $F(x, cV) = cF(x, V)$ for all $(x, V) \in TM$ and all $c > 0$;
(3) (Strong convexity) The matrix

$$g_{ij}(x, V) := \frac{\partial^2}{\partial V^i \partial V^j} \left(\frac{1}{2}F^2\right)(x, V) \tag{53}$$

is positive definite for all $(x, V) \in TM \setminus 0$.

We call such a pair (M, F) a smooth Finsler manifold. If moreover a measure m is given on M, we call the triple (M, F, m) a Finsler measure space. Note that for every non-vanishing vector field V, $g_{ij}(x, V)$ induces a Riemannian structure g_V on M by the following formula

$$g_V(X, Y) = g_{ij}(x, V)X^iY^j, \forall X, Y \in T_xM. \tag{54}$$

By the homogeneity of F, $g_V(V, V) = F^2(x, V)$.

For $x, y \in M$, the distance function from x to y is defined by

$$d(x, y) := \inf_\gamma \int_0^1 F(\gamma(t), \dot\gamma(t))dt, \tag{55}$$

where the infimum is taken over all C^1-curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Note that generally $d(x, y) \neq d(y, x)$ since F is only positively homogeneous.

A C^∞-curve $\gamma : [0, 1] \to M$ is called a geodesic if it is locally minimizing and has a constant speed (i.e. $F(\gamma(t), \dot\gamma(t))$ is constant). For $V \in T_xM$, if there exists a geodesic $\gamma : [0, 1] \to M$ with $\dot\gamma(0) = V$, then we define the exponential map by $\exp_x(V) := \gamma(1)$. We say that (M, F) is forward complete if the exponential map is defined on whole TM. Then by Hopf-Rinow theorem (see [4]), any pair of points can be connected by a minimal geodesic.

6.2. **Chern connection and Ricci curvature.** Let $\pi : TM \to M$ be the projection. There exists a unique linear connection on π^*TM, which is called Chern connection. The Chern connection is determined by the following structure equations:

(1) Torsion freeness:

$$D^V_X Y - D^V_Y X = [X, Y]; \tag{56}$$

(2) Almost g-compatibility:

$$Z(g_V(X, Y)) = g_V(D^V_Z X, Y) + g_V(X, D^V_Z Y) + 2C_V(D^V_Z V, X, Y), \tag{57}$$

for $V \in TM \setminus 0, X, Y, Z \in TM$.

Here $D^V_X Y$ is the covariant derivative with respect to the reference vector V and

$$C_V(X, Y, Z) := C_{ijk}(V)X^iY^jZ^k = \frac{1}{4}\frac{\partial^3 F^2(x, V)}{\partial V^i \partial V^j \partial V^k} X^iY^jZ^k \tag{58}$$

is the Cartan tensor of (M, F). Note that $D^{cV}_X Y = D^V_X Y$, $c > 0$ (see e.g. (2.5) in [27]), and $C_V(V, X, Y) = 0$ due to the homogeneity of F.

Given two linear independent vectors $V, W \in T_xM \setminus 0$, the flag curvature is defined by

$$K^V(V, W) = \frac{g_V(R^V(V, W)W, V)}{g_V(V, V)g_V(W, W) - g_V(V, W)^2}, \tag{59}$$

where R^V is the Riemannian curvature given by

$$R^V(X, Y)Z := D^V_X D^V_Y Z - D^V_Y D^V_X Z - D^V_{[X,Y]} Z. \tag{60}$$

Then the Ricci curvature is defined by

$$Ric(V) := \sum_{i=1}^{n-1} K^V(V, e_i), \tag{61}$$

where $\{e_1, \cdots, e_{n-1}, \frac{V}{F(V)}\}$ is the orthonormal basis of T_xM with respect to g_V. (Note that Ric is 0-homogeneous.)

Next we recall the definition of the weighted Ricci curvature on Finsler manifolds introduced by Ohta in [25].

Definition 6.2. Given a unit vector $V \in T_xM$, let $\gamma : (-\varepsilon, \varepsilon) \to M$ be the geodesic with $\gamma(0) = x$ and $\dot\gamma(0) = V$. We decompose the measure dm as $dm = e^{-\Psi} vol_{\dot\gamma}$ along γ, where $vol_{\dot\gamma}$ is the volume form of $g_{\dot\gamma}$. Define the weighted Ricci curvature as

(1) $Ric_n(V) := \begin{cases} Ric(V) + (\Psi \circ \gamma)''(0), \text{ if } (\Psi \circ \gamma)'(0) = 0, \\ -\infty, \text{ otherwise,} \end{cases}$

(2) $Ric_N(V) := Ric(V) + (\Psi \circ \gamma)''(0) - \dfrac{(\Psi \circ \gamma)'(0)^2}{N - n}$ for $N \in (n, \infty)$,

(3) $Ric_\infty(V) := Ric(V) + (\Psi \circ \gamma)''(0)$.

For $c \geq 0$ and $N \in [n, \infty]$, define $Ric_N(cV) := c^2 Ric_N(V)$. (Note that Ric_N is 2-homogeneous.)

We say that $Ric_N \geq K$ for some $K \in \mathbb{R}$ if $Ric_N(V) \geq KF(V)^2$ for all $V \in TM$. It is proved by Ohta [25] that the bound $Ric_N(V) \geq KF(V)^2$ is equivalent to Lott-Villani and Sturm's curvature-dimension condition, which has many interesting applications (see [25, 26]).

6.3. **Gradient and Laplacian.** Given a Finsler structure F on a manifold M, its dual Finsler structure F^* on the cotangent bundle T^*M is defined by

$$F^*(x, \xi) := \sup_{Y \in T_xM \setminus 0} \frac{\xi(Y)}{F(x, Y)}, \forall \xi \in T_x^*M. \tag{62}$$

And the Legendre transformation $\mathcal{L} : TM \to T^*M$ is given by

$$\mathcal{L}(Y) := \begin{cases} g_Y(Y, \cdot), & Y \neq 0, \\ 0, & Y = 0. \end{cases} \tag{63}$$

It is easy to check that $\mathcal{L}$ is a diffeomorphism from $TM \setminus 0$ onto $T^*M \setminus 0$ and $F(Y) = F^*(\mathcal{L}(Y)), \forall Y \in TM$. Moreover, there holds the Cauchy-Schwarz inequality

$$g_Y(Y, Z) \leq F(Y)F(Z), \tag{64}$$

for $\forall Y \neq 0, Z \in TM$.

Now for a smooth function $u : M \to \mathbb{R}$, we define the gradient vector $\nabla u(x)$ as $\nabla u(x) := \mathcal{L}^{-1}(du(x)) \in T_xM$. In a local coordinate system, we can write it as

$$\nabla u(x) = \begin{cases} g^{ij}(x, \nabla u) \dfrac{\partial u}{\partial x^i} \dfrac{\partial}{\partial x^j}, & \text{if } du(x) \neq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{65}$$

where $g^{ij}(x, \nabla u)$ is the inverse of $g_{ij}(x, \nabla u)$. Also note that $g^{ij}(x, \nabla u) = g^{*ij}(x, du)$.

For a differentiable vector field V on M and $x \in M_V := \{x | V(x) \neq 0\}$, we define $\nabla V \in T_x^*M \otimes T_xM$ by

$$\nabla V(Y) := D_Y^V V \in T_xM, \ Y \in T_xM. \tag{66}$$

Then the Hessian of u is given by $\nabla^2 u := \nabla(\nabla u)$ on $M_{\nabla u}$, which can also be seen as in $T_x^*M \otimes T_x^*M$ via

$$\nabla^2 u(X, Y) = g_{\nabla u}(D_X^{\nabla u} \nabla u, Y). \tag{67}$$

One can check that $\nabla^2 u(X, Y)$ is symmetric. See details in [27] or [30].

Now for a given positive C^∞-measure m on M, define the divergence of a differentiable vector field V with respect to m in the weak form by

$$\int_M \phi \, div_m V dm = -\int_M d\phi(V) dm \tag{68}$$

for $\forall \phi \in C_c^\infty(M)$. In a local coordinate system where $dm = \sigma(x)dx$,

$$div_m V = \frac{1}{\sigma(x)} \frac{\partial}{\partial x^i} \left(\sigma(x) V^i\right). \tag{69}$$

Then the Finsler Laplacian of $u \in W_{loc}^{1,2}(M)$ is given by $\Delta_m u = div_m(\nabla u)$. Recall the relationship between $\Delta_m u$ and $\nabla^2 u$ is that (see e.g. [31, Lemma 3.3])

$$\Delta_m u = \sum_{i=1}^n \nabla^2 u(e_i, e_i) - S(\nabla u), \text{ on } M_{\nabla u}. \tag{70}$$

Here $\{e_i\}_{i=1}^n$ is the $g_{\nabla u}$-orthonormal basis and $S : TM \to \mathbb{R}$ is the S-curvature [28] given by

$$S(V) = \frac{d}{dt}\bigg|_{t=0} \Psi \circ \gamma(t), \tag{71}$$

where γ is a geodesic with $\gamma'(0) = V$ and $dm = e^{-\Psi} vol_{\dot\gamma}$. Note that $S(cV) = cS(V)$, for $c > 0$.

On the other hand, for a given smooth non-vanishing vector field V, we can define the weighted gradient vector and the weighted Laplacian on the weighted Riemannian manifold (M, g_V, m) by

$$\nabla^V u(x) = g^{ij}(x, V) \frac{\partial u}{\partial x^j} \frac{\partial}{\partial x^i}, \tag{72}$$

and

$$\Delta_m^V u = div_m(\nabla^V u), \tag{73}$$

respectively. It is worth mentioning that $\nabla^{\nabla u} u = \nabla u$ and $\Delta_m^{\nabla u} u = \Delta_m u$ on $M_{\nabla u}$.

6.4. **The first and second variation formulas for arclength of geodesic.**

Let $\gamma_0 : [0, l] \to M$ be a unit speed geodesic in M. Suppose $\gamma(\varepsilon, s)$ is any variation of $\gamma_0(s)$ with $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Then the first variation formula for arclength is (see e.g. [25])

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} L(\gamma(\varepsilon, \cdot)) = g_{\gamma_s}(\gamma_s, \gamma_\varepsilon)\big|_0^l, \tag{74}$$

where γ_s is the unit tangent vector of γ_0 and $\gamma_\varepsilon = \frac{\partial}{\partial\varepsilon}\gamma$ is the variational vector field.

Furthermore, the second variation formula is given by (see e.g. [25])

$$\begin{aligned}\frac{\partial^2}{\partial \varepsilon^2}\bigg|_{\varepsilon=0} L(\gamma(\varepsilon, \cdot)) = &\int_0^l \{g_{\gamma_s}(D_{\gamma_s}^{\gamma_s}(\gamma_\varepsilon{}^\perp), D_{\gamma_s}^{\gamma_s}(\gamma_\varepsilon{}^\perp)) - g_{\gamma_s}(R^{\gamma_s}(\gamma_s, \gamma_\varepsilon)\gamma_\varepsilon, \gamma_s)\} ds \\ &+ g_{\gamma_s}(D_{\gamma_\varepsilon}^{\gamma_s}\gamma_\varepsilon, \gamma_s)\big|_0^l,\end{aligned}$$

where $\gamma_\varepsilon^\perp$ means the normal part of the variational vector.

6.5. **Euler-Lagrange equation of energy functional, structure conditions and notations.** Let (M^n, F, m) be a compact Finsler measure space without boundary. We consider the following energy functional

$$\mathcal{E}(u) = \int_M \left(\frac{1}{2}\Phi(F^2(\nabla u)) - Q(u) \right) dm, \tag{75}$$

where $\Phi \in C^3(\mathbb{R}^+)$ with $\Phi(0) = 0$ and $Q \in C^2(\mathbb{R})$. The Euler-Lagrange equation for $\mathcal{E}$ is given by

$$div_m(\Phi'(F^2(\nabla u))\nabla u) + q(u) = 0, \tag{76}$$

where $q(u) = Q'(u)$. We always assume Φ satisfies the following structure conditions: $\Phi(0) = 0$, and there exist $p > 1$, $\tau \geq 0$ and $c_1, c_2 > 0$ such that for any $V, W \in TM \setminus 0$,

$$c_1(\tau + F(V))^{p-2} \leq \Phi'(F^2(V)) \leq c_2(\tau + F(V))^{p-2} \tag{77}$$

and

$$c_1(\tau + F(V))^{p-2}F^2(W) \leq a(V)(W, W) \leq c_2(\tau + F(V))^{p-2}F^2(W), \tag{78}$$

where $a(V) = 2\Phi''(F^2(V))\mathcal{L}(V) \otimes \mathcal{L}(V) + \Phi'(F^2(V))g_V$.

Remark 16. If we choose

$$\Phi(s) = \frac{2}{p}s^{\frac{p}{2}}, \tag{79}$$

we get the case of p-Finsler-Laplace operator, which is the model for our structure conditions.

We also use the following notations:

$$K(s) := \Phi'(s)s - \frac{1}{2}\Phi(s),$$
$$\Lambda(s) := 2\Phi''(s)s + \Phi'(s).$$

Note that $K'(s) = \frac{1}{2}\Lambda(s)$. And taking in (78) $W = V$, we have $\Lambda(s) \geq c_1(\tau + s^{\frac{1}{2}})^{p-2} > 0$, for $s > 0$.

6.6. **The regularity of the solutions to the equation.** By Theorem 1 in [29] (see also [10] and [19]), any solution u of (76) satisfies $u \in C^{1,\alpha}(M)$ and

$$||u||_{C^{1,\alpha}(M)} \leq C(|u|_\infty, M). \tag{80}$$

Furthermore, on any domain Ω such that $\inf_{\bar{\Omega}} F(\nabla u) > 0$, Equation (76) is uniformly elliptic in Ω. Then Theorem 6.3 in [18, Chap.4] implies $u \in C^{2,\alpha}(\Omega)$.

6.7. **Modulus of continuity estimate.** We shall prove the following modulus of continuity estimate.

Theorem 17. *Let* (M^n, F, m) *be a compact Finsler measure space with nonnegative weighted Ricci curvature* Ric_∞ *and* $u \in W^{1,p}(M)$ *a solution of Equation* (76)*. Suppose the barrier function* $\varphi : [a,b] \to [\inf u, \sup u]$ *satisfies*

$$\varphi' > 0, \tag{81}$$

$$\frac{d}{ds}\left(\frac{\Lambda((\varphi')^2)\varphi'' + q(\varphi)}{\varphi'\Phi'((\varphi')^2)}\right) < 0. \tag{82}$$

Moreover let ψ *be the inverse of* φ*, i.e.* $\psi(\varphi(s)) = s$*. Then we have*

$$\psi(u(y)) - \psi(u(x)) - d(x,y) \le 0,\ \forall x, y \in M. \tag{83}$$

Remark 18. In the result above, by simple perturbation, we can replace Condition (82) by the following equality

$$\Lambda((\varphi')^2)\varphi'' + q(\varphi) = 0. \tag{84}$$

By letting y approach x, we get the following gradient bound.

Corollary 19. *Under the conditions of Theorem 17, for every* $x \in M$ *we have*

$$F(\nabla u(x)) \le \varphi'(\psi(u(x))). \tag{85}$$

Proof of Theorem 17. The proof is by contradiction. Define

$$Z(x,y) = \psi(u(y)) - \psi(u(x)) - d(x,y). \tag{86}$$

Assume otherwise

$$\max_{x,y\in M} Z(x,y) = Z(x_0,y_0) = \varepsilon_0 > 0. \tag{87}$$

Obviously $x_0 \ne y_0$ and for any smooth unit speed curve $\gamma : [0,l] \to M$

$$\mathcal{Z}(\gamma) := \psi(u(\gamma(l))) - \psi(u(\gamma(0))) - L(\gamma) \le Z(\gamma(0),\gamma(l)) \le \varepsilon_0, \tag{88}$$

with equality when $\gamma = \gamma_0$, a length-minimising geodesic from x_0 to y_0.

Let $\gamma(\varepsilon, s)$ be any variation of $\gamma_0(s)$. Then the first derivative condition yields

$$0 = \psi'(u(y_0))g_{\nabla u}(\nabla u(y_0), \gamma_\varepsilon(l)) - \psi'(u(x_0))g_{\nabla u}(\nabla u(x_0), \gamma_\varepsilon(0)) - g_{\gamma_s}(\gamma_s, \gamma_\varepsilon)\big|_0^l.$$

Since the variation is arbitrary, we have

$$\psi'(u(y_0))\nabla u(y_0) = T(l),$$
$$\psi'(u(x_0))\nabla u(x_0) = T(0),$$

or equivalently

$$\nabla u(y_0) = \varphi'(z_y)T(l),$$
$$\nabla u(x_0) = \varphi'(z_x)T(0),$$

where T is the unit tangent vector of the curve, $z_y = \psi(u(y_0))$ and $z_x = \psi(u(x_0))$.

Next we will obtain some information from the following second derivative condition:

$$\begin{aligned}0 &\geq \psi''(u)g_{\nabla u}(\nabla u, \gamma_\varepsilon(l))^2 + \psi'(u)\left(g_{\nabla u}(D^{\nabla u}_{\gamma_\varepsilon}\nabla u, \gamma_\varepsilon(l)) + g_{\nabla u}(\nabla u, D^{\nabla u}_{\gamma_\varepsilon}\gamma_\varepsilon(l))\right)\\ &- \psi''(u)g_{\nabla u}(\nabla u, \gamma_\varepsilon(0))^2 - \psi'(u)\left(g_{\nabla u}(D^{\nabla u}_{\gamma_\varepsilon}\nabla u, \gamma_\varepsilon(0)) + g_{\nabla u}(\nabla u, D^{\nabla u}_{\gamma_\varepsilon}\gamma_\varepsilon(0))\right)\\ &- \int_0^l \{g_{\gamma_s}(D^{\gamma_s}_{\gamma_s}({\gamma_\varepsilon}^\perp), D^{\gamma_s}_{\gamma_s}({\gamma_\varepsilon}^\perp)) - g_{\gamma_s}(R^{\gamma_s}(\gamma_s, \gamma_\varepsilon)\gamma_\varepsilon, \gamma_s)\}ds - g_{\gamma_s}(D^{\gamma_s}_{\gamma_\varepsilon}\gamma_\varepsilon, \gamma_s)\Big|_0^l,\end{aligned}$$

where we have suppressed some notations.

Choose at x_0 an orthonormal basis $\{e_1, \cdots, e_{n-1}, e_n = T(0)\}$ with respect to $g_{\gamma_s(0)}$. Then parallel transport along γ_0 to produce an orthonormal basis $\{e_i(s)\}$ for each tangent space $T_{\gamma_0(s)}M$ (So $D^{\gamma_s}_{\gamma_s}e(s) = 0$). Note that $e_n(s) = T(s)$ for each s. Then we consider the following three variations.

(1) Vary y_0. Choose the variation

$$\gamma(\varepsilon, s) = \gamma_0(s + \varepsilon\frac{s}{l}). \tag{89}$$

So $\gamma_\varepsilon(l) = e_n$ and $\gamma_\varepsilon(0) = 0$. Then we get

$$\begin{aligned}0 &\geq \psi''(u)g_{\nabla u}(\nabla u, \gamma_\varepsilon(l))^2 + \psi'(u)g_{\nabla u}(D^{\nabla u}_{\gamma_\varepsilon}\nabla u, \gamma_\varepsilon(l))\\ &= \psi''u_n(y_0)^2 + \psi'u_{nn}(y_0)\\ &= \frac{u_{nn}(y_0) - \varphi''(z_y)}{\varphi'(z_y)}.\end{aligned} \tag{90}$$

(2) Vary x_0. Choose the variation

$$\gamma(\varepsilon, s) = \gamma_0(s + \varepsilon\frac{l-s}{l}). \tag{91}$$

So $\gamma_\varepsilon(l) = 0$ and $\gamma_\varepsilon(0) = e_n$. Then similarly we get

$$0 \geq -\frac{u_{nn}(x_0) - \varphi''(z_x)}{\varphi'(z_x)}. \tag{92}$$

(3) Vary γ_0 along $e_i(s)$ for fixed $i < n$. Choose

$$\gamma(\varepsilon, s) = \exp_{\gamma_0(s)}(\varepsilon e_i(s)). \tag{93}$$

So $\gamma_\varepsilon(s) = e_i(s)$. Therefore

$$\begin{aligned}0 &\geq \psi'(u(y_0))u_{ii}(y_0) - \psi'(u(x_0))u_{ii}(x_0) + \int_0^l g_{\gamma_s}(R^{\gamma_s}(\gamma_s, e_i)e_i, \gamma_s)ds\\ &= \frac{u_{ii}(y_0)}{\varphi'(z_y)} - \frac{u_{ii}(x_0)}{\varphi'(z_x)} + \int_0^l g_{\gamma_s}(R^{\gamma_s}(\gamma_s, e_i)e_i, \gamma_s)ds.\end{aligned}$$

Then after summation from $i = 1$ to $i = n - 1$ and noting

$$Ric_\infty(\gamma_s) = Ric(\gamma_s) + (\Psi \circ \gamma)'' \geq 0, \tag{94}$$

we have

$$0 \geq \frac{\sum_{i<n} u_{ii}(y_0)}{\varphi'(z_y)} - \frac{\sum_{i<n} u_{ii}(x_0)}{\varphi'(z_x)} - \int_0^l (\Psi \circ \gamma)'' ds$$
$$= \frac{\sum_{i<n} u_{ii}(y_0)}{\varphi'(z_y)} - \frac{\sum_{i<n} u_{ii}(x_0)}{\varphi'(z_x)} - (\Psi \circ \gamma)'\big|_0^l. \tag{95}$$

Now recall Equation (76) is

$$0 = \Phi''(F^2(\nabla u))2u_{ij}u_iu_j + \Phi'(F^2(\nabla u))\Delta_m u + q(u) \tag{96}$$
$$= \Phi''(F^2(\nabla u))2u_{ij}u_iu_j + \Phi'(F^2(\nabla u))\left(\sum_{i=1}^n u_{ii} - S(\nabla u)\right) + q(u) \tag{97}$$

on $M_{\nabla u}$. In particular, at x_0 or y_0

$$\Phi''((\varphi')^2)2u_{nn}(\varphi')^2 + \Phi'((\varphi')^2)\left(\sum_{i=1}^n u_{ii} - \varphi'(\Psi\circ\gamma)'\right) + q(\varphi) = 0. \tag{98}$$

As a result we can solve

$$\sum_{i<n} u_{ii} = -\frac{\Lambda((\varphi')^2)u_{nn} + q(\varphi)}{\Phi'((\varphi')^2)} + \varphi'(\Psi\circ\gamma)' \tag{99}$$

at x_0 or y_0.

Plugging (99) into (95) and using (90) and (92), finally we have

$$0 \geq -\frac{\Lambda((\varphi')^2)\varphi'' + q(\varphi)}{\varphi'\Phi'((\varphi')^2)}\bigg|_{z_y} + \frac{\Lambda((\varphi')^2)\varphi'' + q(\varphi)}{\varphi'\Phi'((\varphi')^2)}\bigg|_{z_x}. \tag{100}$$

Now taking (82) into account, since $z_y = z_x + d(x_0, y_0) + \varepsilon_0 > z_x$, we get a contradiction. Then we must have

$$Z(x,y) = \psi(u(y)) - \psi(u(x)) - d(x,y) \leq 0, \tag{101}$$

which is the desired result. □

7. POINTWISE GRADIENT ESTIMATES AND RIGIDITY RESULTS FOR FINSLER MANIFOLDS WITH NONNEGATIVE WEIGHTED RICCI CURVATURE

In this section we continue the study on Finsler manifolds in Section 6. More precisely, we derive the gradient estimates of Modica type and some standard rigidity results.

Define

$$c_u := \sup_{r\in[\inf u, \sup u]} Q(r). \tag{102}$$

Then we shall prove the following gradient estimates of Modica type.

Theorem 20. *Let (M^n, F, m) be a compact Finsler measure space with nonnegative weighted Ricci curvature Ric_∞ and $u \in W^{1,p}(M)$ a solution of Equation (76). Then for all $x \in M$, there holds*

$$\Phi'(F^2(\nabla u))F^2(\nabla u) - \frac{1}{2}\Phi(F^2(\nabla u)) \leq c_u - Q(u). \tag{103}$$

Proof of Theorem 20. Fix any

$$c > c_u = \sup_{r\in[\inf u, \sup u]} Q(r). \tag{104}$$

Then we can solve

$$K((\varphi')^2) = c - Q(\varphi) \tag{105}$$

to get a solution φ_c with $\varphi_c' > 0$ and its image being $[\inf u, \sup u]$. In fact,

$$s = s_0 + \int_{\inf u}^{\varphi_c} \frac{d\varphi}{\sqrt{K^{-1} \circ (c - Q(\varphi))}} \tag{106}$$

for $\varphi_c \in [\inf u, \sup u]$. Differentiating (105) we know φ_c solves

$$\Lambda((\varphi')^2)\varphi'' + q(\varphi) = 0. \tag{107}$$

Now noting Remark 18, we can apply Corollary 19 to get

$$K(F^2(\nabla u)) + Q(u) \leq K((\varphi_c')^2) + Q(\varphi_c) = c. \tag{108}$$

Finally since $c > c_u$ is arbitrary we have

$$K(F^2(\nabla u)) + Q(u) \leq c_u. \tag{109}$$

So we complete the proof of Theorem 20.

□

Another application of our modulus of continuity estimate, Theorem 17, is a rigidity result for u concerning c_u as follows.

Theorem 21. *Let u be as in Theorem 20. Suppose $\tau = 0$ in the structure conditions (77) and (78). Moreover, when $p > 2$, we assume at any r_0 with $Q(r_0) = c_u$ and $Q'(r_0) = 0$ there holds $Q(r) = Q(r_0)+O(|r-r_0|^p)$ as $r \to r_0$. If there exists a point $x_0 \in M$ satisfying $Q(u(x_0)) = c_u$ and $Q'(u(x_0)) = 0$, then u is constant.*

Remark 22. Here we provide an example to illustrate some aspects of Theorem 21: Let $p = 2$ and $Q(u) = \sin u$. Then Theorem 21 indicates that the image $[\inf u, \sup u]$ of any non-constant solution u can not contain points in $\{2k\pi + \frac{\pi}{2} : k \in \mathbb{N}\}$, which gives a restriction on the solutions.

Proof of Theorem 21. Assume u is not a constant function. Then $[\inf u, \sup u]$ has nonempty interior. Without loss of generality we may assume $u(x_0) > \inf u$. Then using (106) we get

$$\psi_c(u(x_0)) - \psi_c(\inf u) = \int_{\inf u}^{u(x_0)} \frac{d\varphi}{\sqrt{K^{-1} \circ (c - Q(\varphi))}}, \quad c > c_u. \tag{110}$$

Next we claim

$$\lim_{c\to c_u^+}\int_{\inf u}^{u(x_0)}\frac{d\varphi}{\sqrt{K^{-1}\circ(c-Q(\varphi))}}=+\infty. \tag{111}$$

So when c is close enough to c_u^+, we get a contradiction to the modulus of continuity estimate

$$\psi_c(u(y))-\psi_c(u(x))\le d(x,y)<+\infty,\quad x,y\in M. \tag{112}$$

To prove the claim, first we observe that

$$s^{\frac{p}{2}}\le\frac{2}{\varepsilon_0}K(s),0<\varepsilon_0<\frac{2}{p}c_1. \tag{113}$$

In fact, define $G(s)=2K(s)-\varepsilon_0 s^{\frac{p}{2}}$ with $\varepsilon_0<\frac{2}{p}c_1$. Then $G(0)=0$ and

$$G'(s)=\Lambda(s)-\varepsilon_0\frac{p}{2}s^{\frac{p}{2}-1}>0 \tag{114}$$

by the assumption. Thus $G(s)\ge 0$, which is (113).

Therefore, we obtain

$$\sqrt{K^{-1}\circ(c_u-Q(\varphi))}\le c(c_u-Q(\varphi))^{1/p}=c(Q(u(x_0))-Q(\varphi))^{1/p}. \tag{115}$$

Note that for $1<p\le 2$, by Taylor expansion $Q(r)-Q(r_0)=O(|r-r_0|^2)=O(|r-r_0|^p)$. For $p>2$, by the assumption $Q(r)-Q(r_0)=O(|r-r_0|^p)$. In either case, we can conclude

$$\sqrt{K^{-1}\circ(c_u-Q(\varphi))}\le\tilde{c}(u(x_0)-\varphi),\quad \varphi\le u(x_0),$$

which implies that

$$\int_{\inf u}^{u(x_0)}\frac{d\varphi}{\sqrt{K^{-1}\circ(c_u-Q(\varphi))}}=+\infty. \tag{116}$$

So we obtain the desired claim and finish the proof of Theorem 21. □

Also we can give a characterization of c_u.

Theorem 23. *Under the same assumption of Theorem 21, we have*

$$c_u=\max\{Q(\inf u),Q(\sup u)\}. \tag{117}$$

Moreover, if there exists a point $x_0\in M$ satisfying $Q(u(x_0))=c_u$, then either $u(x_0)=\inf u$ or $u(x_0)=\sup u$.

Proof of Theorem 23. Without loss of generality, we assume that u is not a constant. Assume $c_u>\max\{Q(\inf u),Q(\sup u)\}$. Then there exists $r_0\in(\inf u,\sup u)$ such that

$$\sup_{r\in[\inf u,\sup u]}Q(r)=c_u=Q(r_0). \tag{118}$$

So r_0 is a local maximum point for Q and $Q'(r_0)=0$.

Meanwhile by the continuity of u, there exists a point y_0 such that $u(y_0) = r_0$. Thus $Q(u(y_0)) = c_u$ and $Q'(u(y_0)) = 0$. So u is constant by Theorem 21, which is a contradiction. Therefore we complete the proof.

$\square$

References

[1] Ben Andrews, *Moduli of continuity, isoperimetric profiles, and multi-point estimates in geometric heat equations*, Surveys in Differential Geometry **19** (2014), 1–47.

[2] Ben Andrews and Julie Clutterbuck, *Time-interior gradient estimates for quasilinear parabolic equations*, Indiana Univ. Math. J. **58** (2009), no. 1, 351–380.

[3] Ben Andrews and Julie Clutterbuck, *Sharp modulus of continuity for parabolic equations on manifolds and lower bounds for the first eigenvalue*, Anal. PDE **6** (2013), no. 5, 1013–1024.

[4] D. Bao, S.-S. Chern and Z. Shen, *An introduction to Riemann-Finsler geometry*, Springer-Verlag, New York (2000).

[5] L. Caffarelli, N. Garofalo and F. Segala, *A gradient bound for entire solutions of quasilinear equations and its consequences*, Comm. Pure Appl. Math. **47** (1994), 1457–1473.

[6] D. Castellaneta, A. Farina and E. Valdinoci, *A pointwise gradient estimate for solutions of singular and degenerate PDEs in possibly unbounded domains with nonnegative mean curvature*, Commun. Pure Appl. Anal. **11** (2012), 1983–2003.

[7] M. Cozzi, A. Farina and E. Valdinoci, *Gradient bounds and rigidity results for singular, degenerate, anisotropic partial differential equations*, Comm. Math. Phys. **331** (2014), no. 1, 189–214.

[8] M. G. Crandall, H. Ishii and P. L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. **27** (1992), 1–67.

[9] D. Danielli and N. Garofalo, *Properties of entire solutions of non-uniformly elliptic equations arising in geometry and in phase transitions*, Calc. Var. Partial Differential Equations **15** (2002), no. 4, 451–491.

[10] E. Di Benedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. **7** (1983), 827–850.

[11] A. Farina, Y. Sire and E. Valdinoci, *Stable solutions of elliptic equations on Riemannian manifolds*, J. Geom. Anal. **23** (2013), 1158–1172.

[12] A. Farina and E. Valdinoci, *A pointwise gradient estimate in possibly unbounded domains with nonnegative mean curvature*, Adv. Math. **225** (2010), no. 5, 2808–2827.

[13] A. Farina and E. Valdinoci, *A pointwise gradient bound for elliptic equations on compact manifolds with nonnegative Ricci curvature*, Discrete Contin. Dyn. Syst. **30** (2011), no. 4, 1139–1144.

[14] A. Farina and E. Valdinoci, *Pointwise estimates and rigidity results for entire solutions of nonlinear elliptic PDE's*, ESAIM Control Optim. Calc. Var. **19** (2013), no. 2, 616–627.

[15] A. Farina and E. Valdinoci, *Gradient bounds for anisotropic partial differential equations*, Calc. Var. Partial Differ. Equ. **49** (2014), no. 3-4, 923–936.

[16] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer-Verlag, Berlin, (1983).

[17] C. D. Hill, *A sharp maximum principle for degenerate elliptic-parabolic equations*, Indiana Univ. Math. J. **20** (1970), 213–229.

[18] O. A. Ladyzhenskaya and N. N. Uraltseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, (1968).

[19] J. L. Lewis, *Regularity of the derivatives of solutions to certain degenerate elliptic equations*, Indiana Univ. Math. J. **32** (1983), 849–858.

[20] X. Li, *Moduli of continuity for viscosity solutions*, Proc. Amer. Math. Soc. **144** (2016), 1717–1724.

[21] X. Li and K. Wang, *Moduli of continuity for viscosity solutions on manifolds*, J. Geom. Anal. **27** (2017), no. 1, 557–576.

[22] L. Ma and I. Witt, *Liouville theorem for the nonlinear Poisson equation on manifolds*, J. Math. Anal. Appl. **416** (2014), 800–804.

[23] P. Mastrolia and M. Rigoli, *Diffusion-type operators, Liouville theorems and gradient estimates on complete manifolds*, Nonlinear Anal. **72** (2010), 3767–3785.

[24] L. Modica, *A gradient bound and a Liouville theorem for nonlinear Poisson equations*, Comm. Pure Appl. Math. **38** (1985), 679–684.

[25] S. Ohta, *Finsler interpolation inequalities*, Calc. Var. Partial Differential Equations **36** (2009), 211–249.

[26] S. Ohta and K. Sturm, *Heat flow on Finsler manifolds*, Comm. Pure Appl. Math. **62** (2009), 1386–1433.

[27] S. Ohta and K. Sturm, *Bochner-Weitzenböck formula and Li-Yau estimates on Finsler manifolds*, Adv. Math. **252** (2014), 429–448.

[28] Z. Shen, *Lectures on Finsler geometry*, World Scientific Publishing Co., Singapore (2001).

[29] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*. J. Diff. Equ. **51** (1984), 126–150.

[30] G. Wang and C. Xia, *A sharp lower bound for the first eigenvalue on Finsler manifolds*, Ann. Inst. H. Poincaré Anal. Non Linéaire **30** (2013), 983–996.

[31] B. Y. Wu and Y. L. Xin, *Comparison theorems in Finsler geometry and their applications*, Math. Ann. **337** (2007), 177–196.

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