Coalgebraic Predicate Logic: Equipollence Results and Proof Theory

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Abstract. The recently introduced Coalgebraic Predicate Logic (CPL) provides a general first-order syntax together with extra modal-like operators that are interpreted in a coalgebraic setting. The universality of the coalgebraic approach allows us to instantiate the framework to a wide variety of situations, including probabilistic logic, coalition logic or the logic of neighbourhood frames. The last case generalises a logical setup proposed by C.C. Chang in early 1970's. We provide further evidence of the naturality of this framework. We identify syntactically the fragments of CPL corresponding to extended modal formalisms and show that the full CPL is equipollent with coalgebraic hybrid logic with the downarrow binder and the universal modality. Furthermore, we initiate the study of structural proof theory for CPL by providing a sequent calculus and a cut-elimination result.

1 Introduction

Coalgebras over sets provide an universal framework for state-based systems, such as (labelled or unlabelled) transition systems, multigraphs, conditional frames, game frames or (monotone and general) neighbourhood frames. They provide a natural semantics for a wide range of modal logics, ranging from conditional and probabilistic to coalition logic. The development of a full-blown coalgebraic model and correspondence theory is hindered by the lack of a formalism that allows both direct reference to individual states and supports universal quantification and binding: a coalgebraic counterpart of first-order (and higher-order) predicate logic. The framework of coalgebraic predicate logic (CPL) was introduced recently in [9] in a joint paper with Lutz Schröder, where we have provided a complete Hilbert axiomatisation, a modal correspondence theorem and some basic model-theoretic constructions. The present paper is intended as a companion to *op.cit.* presenting more evidence that coalgebraic predicate logic is a natural extension of both (coalgebraic) modal logic and first-order logic.

As explained in *op.cit.*, our approach can be traced back to an unjustly forgotten paper [7] by C. C. Chang. The original motivation was to simplify model theory for what

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Montague called *pragmatics* and replace Montague's many-sorted setting by one without sorts. In contemporary terminology, Chang's paper deals with model and correspondence theory for *neighbourhood frames*: coalgebras for double contravariant powerset functor (see [8] for a coalgebraic treatment in a two-sorted setting). His constructions and results include both suitable notions of (elementary) submodel/extension, elementary chain of models and ultraproduct and suitable variants of Tarski-Vaught, downward and upward Löwenheim-Skolem and compactness theorems. Curiously, Chang did not prove any completeness results. Apart from these technical developments, advantages of [7] include its lucid, intuitive motivation and examples. But the biggest interest lies in the syntax itself, with only one sort of variables for elements of the state space and no need for explicit quantification over neighbourhood or successors. Apart from a number of papers (e.g., Sgro [16]) on interior operator logic in topology, we are not aware of any work in a similar setup.

Our CPL (Coalgebraic Predicate Logic) is based on a notational variant of Chang's syntax. The interpretation of CPL in coalgebras for arbitrary Set-functors is parametric in the notion of *predicate lifting*; if Λ is a supply of modal operators, then the supply of functors and predicate liftings interpreting modalities in Λ is called Λ-*structure*. In [9], the authors together with Lutz Schröder have proved completeness results for two classes of Λ-structures—the first one (S1SC) generalizing neighbourhood frames, another one (k-bounded) generalizing Kripke structures. Furthermore, it was shown that there are limitations to the possible scope of more general completeness results. We also proved an analogue of the Van Benthem-Rosen theorem, characterizing coalgebraic modal logic (CML) as the behavioural-invariant fragment of CPL. Finally, we provided foundations of model theory for CPL, significantly generalizing the scope of Chang's model-theoretic results discussed above.

This paper is intended as a companion paper to [9]. In the first part, we generalize the results of [4] characterizing the correspondence between predicate logic and extended hybrid formalisms $\mathcal{CHL}_{\Lambda}^{\downarrow,\mathsf{A}}$ and $\mathcal{CHL}_{\Lambda}^{\forall,@}$. We take it as yet another indication that CPL is natural and well-designed both as a generalization of FOL and "the" predicate relative of existing coalgebraic formalisms. Furthermore, due to a somewhat modal character of the CPL syntax, the correspondence is even closer and more natural than for ordinary FOL and additional results on the correspondence between sublanguages of CPL and various extensions of coalgebraic modal/hybrid logic become available. In the second part, we initiate the study of proof theory of CPL. We provide cut-free sequent systems for strongly one-step complete (S1SC) Λ-structures, which generalize the neighbourhood logic (and hence Chang's original setup). Our proof of cut-elimination is entirely syntactic and constructive.

2 Syntax and Semantics

Throughout the paper, we fix a modal similarity type Λ consisting of modal operators with associated arities. We also fix a set Σ of *predicate symbols* (with associated arities) and a set V^i of *individual variables*. The *language* $\mathcal{CPL}_\Lambda$ of coalgebraic predicate logic is given by the grammar

$$\mathcal{CPL}_\Lambda \ni \phi, \psi ::= x = y \mid P(x_1, \ldots, x_n) \mid \bot \mid \phi \to \psi \mid \forall x \phi \mid x \heartsuit [x_1 : \phi_1] \ldots [x_n : \phi_n]$$

where $\heartsuit \in \Lambda$ is an n-ary modal operator, $P \in \Sigma$ is an n-ary predicate symbol and $x, y, x_1, \ldots, x_n \in \mathrm{V}^i$ are individual variables. For simplicity, we ignore function symbols which can be added in the same way as in [7].

In a formula $x\heartsuit[x_1 : \phi_1] \ldots [x_n : \phi_n]$, $\heartsuit$ is an n-ary modal operator, applied to n arguments $[x_i : \phi_i]$, for $i = 1, \ldots, n$. Here, x_i is a *comprehension variable*. Given a first-order structure with carrier set W and variable assignment ϑ, $[x : \phi]$ denotes the set of all those states $w \in W$ such that ϕ holds under the modified assignment $\vartheta[x \mapsto w]$. Our informal reading of $[x : \phi]$ is 'the set of all x such that ϕ'. As a consequence, the n-tuple $[x_1 : \phi_1] \ldots [x_n : \phi_n]$ denotes an n-tuple of predicates on the carrier set, to which we can apply an n-ary modal operator $\heartsuit$ in the usual way. The formula $x\heartsuit[x_1 : \phi_1] \ldots [x_n : \phi_n]$ is then best understood as expressing that the property $\heartsuit[x_1 : \phi_1] \ldots [x_n : \phi_n]$ is true relative to the (interpretation of) x in a first-order structure.

Example 1. We have provided a number of examples of the use of CPL in a variety of situations already in [9] where consider CPL over probabilistic modal logic, over coalition logic and Pressburger modal logic. Here, we content ourselves with the following:

1. As originally noted by by Chang himself, coalgebraic predicate logic is particularly well-suited for reasoning about social situations and relationships between an individual and sets of individuals. Indeed, Chang's examples suggest an *intensional* reading of $\heartsuit$ as 'useful' or 'enjoyable'. Given a unary modality $\Box$ and a binary relation $S(x, y)$ that we read as 'x speaks to y', the formula $x\Box[z : S(z, y)]$ reads as 'x finds it enjoyable to speak to y' where x determines the truth of this sentence by inspecting the the set '$\{z : S(z, y)\}$' of people speaking to y. The fact that whether or not x finds it enjoyable to speak to y may depend *non-monotonically* on the set of people y converses with suggests to interpret $\Box$ as a neighbourhood modality (as we will in fact do in Example 4).

2. Coalgebraic predicate logic can also be used to speak about 'losers', 'jerks' and 'politicians'. In [2], these terms are defined using hybrid logic over Kripke semantics where the underlying binary relation is understood as '*respects*' or '*admires*'. For example, a loser is understood as a person who lacks self-respect. In coalgebraic predicate logic, the fact that x is a loser is expressed by the formula $x\Box[y : \neg(y = x)]$. We read this formula as 'everybody whom x respects has the property of being distinct from x', i.e. x lacks self-respect. Accordingly, our interpretation of $\Box$ (given in Detail in Example 4) in this example will be relational, and coincides with the Kripke-interpretation over relational models. We leave it to the reader to express their own (or [2]'s) notions of 'jerks' and 'politicians' in coalgebraic predicate logic. In Section 3, we will show that coalgebraic hybrid logic is in fact equi-expressive to coalgebraic predicate logic.

3. Coalgebraic predicate logic also extends, for instance, majority logic [11] to a first-order setting. If we take x to be a politician if the majority of people known to them distrusts x, then the fact that x is a politician is expressed by the formula $x\mathrm{M}[y : D(x, y)]$ where M is the majority operator that we read 'the majority of' (and assume that majorities are taken among people that are known to an individual) and $D(x, y)$ is a

binary relation that expresses that x distrusts y. We will make this semantically precise, in the next example, by interpreting M as the majority operator of [11].

The semantics of coalgebraic predicate logic is given, as usual, in terms of a *first order structure* and a *variable assignment*. Crucially, the first-order structure must accommodate for the interpretation of the modalities present in the similarity type Λ and must provide a *uniform* interpretation of modalities. We therefore understand first-order structures for Λ as an enrichment of the standard notion of first-order structure with a device to interpret modalities. In our (coalgebraic) context, the interpretation of modal operators is given by Λ-structures.

Definition 2. A Λ-structure is an endofunctor T : Set $\to$ Set on the category of sets, together with an assignment of *predicate liftings*, that is, a set-indexed family of maps

$$[\![\heartsuit]\!]_X : (\mathcal{Q}X)^n \to \mathcal{Q}(TX) \qquad (X \in \mathsf{Set})$$

for every n-ary modal operator $\heartsuit \in \Lambda$. Here $\mathcal{Q}$ is the *contravariant* powerset functor, and we require naturality of $[\![\heartsuit]\!]$, that is, $(Tf)^{-1} \circ [\![\heartsuit]\!]_Y = \heartsuit_X \circ (f^{-1})^n$ for every function $f : X \to Y$. We usually denote Λ-structures by the underlying endofunctor T, when the underlying assigned predicate liftings are clear.

In the remainder of this paper, we assume that our chosen set Λ of modal operators comes equipped with a Λ-structure. We now take first-order structures to be T-*coalgebras* that are additionally equipped with an interpretation of the given relation symbols.

Definition 3. A *first-order structure* (for Λ relative to a Λ-structure T) is a triple $\mathfrak{M} = (C, \gamma, \pi)$ where (C, γ) is a T-*coalgebra*, i.e. C is a set and $\gamma : C \to TC$ a (transition) function, and π is an interpretation of the predicate symbols, that is, $\pi(P) \subseteq C^n$ for all n-ary predicate symbols $P \in \Sigma$. We silenty identify T-coalgebras with first-order structures and leave the interpretation of predicate symbols implicit whenever this does not cause confusion.

The semantics of coalgebraic predicate logic is best explained as the algamation of coalgebraic modal logic and (standard) first-order logic. Given a T-coalgebra (C, γ), formula ϕ of coalgebraic modal logic are interpreted as subsets $[\![\phi]\!] \subseteq C$. The crucial clause for modal operators is $[\![\heartsuit(\phi_1, \ldots, \phi_n)]\!] = \{c \in C \mid \gamma(c) \in [\![\heartsuit]\!]_C([\![\phi_1]\!], \ldots, [\![\phi_n]\!])\}$, discussed in detail e.g. in [13]. Informally speaking, the (coalgebraic) interpretation of $\heartsuit(\phi_1, \ldots, \phi_n)$ is the set of individuals $c \in C$ that enjoy property $\heartsuit$ (which depends on $\phi_1, \ldots, \phi_n$). In coalgebraic *predicate* logic, this interpretation is relativised to *individuals*: in a first-order structure $\mathfrak{M} = (C, \gamma, \pi)$, the formula $x\heartsuit[x_1 : \phi_1] \ldots [x_n : \phi_n]$ is true under a variable assigment ϑ if the individual $\vartheta(x)$ has property $\heartsuit$ which now depends on the sets of individuals x_i that have property $\phi_i(x_i)$.

Formally, we define *truth* $\mathfrak{M}, \vartheta \models \phi$ of a formula $\phi \in \mathcal{CPL}_\Lambda$ in a first-order structure $\mathfrak{M} = (C, \gamma, \pi)$ relative to a variable assignment $\vartheta : \mathsf{V}^i \to C$ by the standard clauses for propositional and first-order connectives, augmented with

$$\mathfrak{M}, \vartheta \models x\heartsuit[x_1 : \phi_1] \ldots [x_n : \phi_n] \iff \gamma(\vartheta(x)) \in [\![\heartsuit]\!]_X([\![\phi_1]\!]^{x_1}_{\mathfrak{M},\vartheta}, \ldots, [\![\phi_n]\!]^{x_n}_{\mathfrak{M},\vartheta})$$

where $[\![\phi]\!]^y_{\mathfrak{M},\vartheta} = \{c \in C \mid \mathfrak{M}, \vartheta[y \mapsto c] \models \phi\}$ (we usually drop the subscript $\mathfrak{M}, \vartheta$) and $\vartheta[y \mapsto c]$ is the same variable assignment as ϑ except it maps y to c.

Example 4. We continue Example 1 and describe the structures that give rise to the interpretation put forward above.

1. Chang's original attempt generalises the neighbourhood interpretation of modal logics to the setting of full first-order logic. For a similarity type $\Lambda = \{\Box\}$ containing a single unary operator, neighbourhood semantics is captured coalgebraically by the Λ-structure $\mathcal{N} = \mathcal{Q} \circ \mathcal{Q}$ together with the predicate lifting defined by

$$[\![\Box]\!]_X(A) = \{N \in \mathcal{N}X \mid A \in N\}$$

which ensures that the standard modal neighbourhood semantics conincides with the coalgebraic semantics of modal formulae. In a first-order setting, this exhibits Chang's original language (and its interpretation) as a special case of coalgebraic predicate logic. In a first-order structure (C, γ), every individual $c \in C$ induces a set $\gamma(c) \subseteq \mathcal{P}(C)$ of neighbourhoods such that – in the spirit of the example – x finds it enjoyable to speak to y if the set $[z : S(z, y)]$ is a (social) neighbourhood of x.
2. In [2], hybrid logic is used to define *losers, jerks* and *politicians*, where notions like *respect* or *admiration* are modelled by binary relations between individuals. We replace relations by T-coalgebras for $TC := \mathcal{P}C$ ($\mathcal{P}$ is the *covariant* powerset functor) and interpret the (unary) model operator $\Box$ by $[\![\Box]\!]_C(A) := \{B \in \mathcal{P}C \mid B \subseteq A\}$ which gives the standard semantics. The formula $x\Box[y : \neg(y = x)]$ then expresses that x is a loser, i.e. lacks an arc along the relation expressing self-respect. We leave it to the reader to express the definitions of jerks and politicians given in [2]). Indeed, hybrid logic is translatable to the language discussed here, see below.
3. As a slight variation, we may consider a predicate version of majority logic [11] where we again co-algebraise the relational semantics. We interpret formulae involving an operator M (read 'the majority ...') over coalgebras of type $(C, \gamma : C \to \mathcal{B}C)$ where $\mathcal{B}C := \{f : C \to \mathbb{N} \mid f(c) \neq 0 \text{ only finitely often}\}$ using

$$[\![\mathrm{M}]\!]_C(A) := \{f \in \mathcal{B}C \mid \sum_{x \in A} f(x) > \sum_{x \notin A} f(x)\}.$$

This differs from the original semantics of *op.cit.* but induces the same set of true sentences. If we read M as *the majority of people someone knows ...* and $R(x, y)$ as *likes*, an unpopular person could be characterised by the sentence $x\mathrm{M}\{y : \neg R(y, x)\}$ stipulating that the majority of people x knows don't like x.
4. Frame classes can be combined: instead of using the relation symbol R in the previous example, we could consider coalgebras $(C, \gamma : C \to TC)$ where $TC := \mathcal{B}C \times \mathcal{P}C$ gives a majority structure and a relational structure, and interpret the operators M and $\Box$ by projecting out the components. Unpopular individuals are then characterised as satisfying $x\mathrm{M}[y : y\Box[z : \neg(z = x)]]$.

3 Equipollence Results

3.1 Coalgebraic Standard Translation for $\mathcal{CML}_\Lambda$

The formulas $\mathcal{CML}_\Lambda(\Sigma)$ of pure (coalgebraic) modal logic in the modal signature Λ over Σ (now all elements of Σ are assumed to be of arity 1) are given by the grammar:

$$\mathcal{CML}_\Lambda \quad \phi, \psi ::= P \mid \bot \mid \phi \to \psi \mid \heartsuit(\phi_1, \ldots, \phi_n),$$

where $P \in \Sigma$.

Satisfaction is defined with respect to $\mathfrak{M} = (C, \gamma, \pi)$ and a specific point $c \in C$ in a standard way, see e.g. [14,15].

Proposition and Definition 5. *Define the* coalgebraic standard translation *as*

$$\begin{aligned} ST_x(P) &:= P(x), \\ ST_x(\heartsuit(\phi_1, \ldots, \phi_n)) &:= x\heartsuit[x : ST_x(\phi_1)] \ldots [x : ST_x(\phi_n)], \\ ST_x(\bot) &:= \bot, \\ ST_x(\phi \to \psi) &:= ST_x(\phi) \to ST_x(\psi). \end{aligned}$$

Then for any $\phi \in \mathcal{CML}_\Lambda(\Sigma)$ and any $\mathfrak{M} = (C, \gamma, \pi), \vartheta, c$, we have $\mathfrak{M}, c \models \phi$ iff $\mathfrak{M}, \vartheta[x \mapsto c] \models ST_x(\phi)$.

This definition is more straightforward than the standard translation into FOL of modal logic over ordinary Kripke frames. Moreover, ST_x uses only one variable from V^i, namely x itself. In fact, we can immediately observe that

Proposition 6. *Whenever Σ consists entirely of* unary *predicate symbols, the subset of $\phi \in \mathcal{CPL}_\Lambda(\Sigma)$ obtained as the image of ST_x for a fixed $x \in \mathsf{V}^i$ consists precisely of equality-free and quantifier-free formulae in the variable x.*

Whereas the Van Benthem-Rosen theorem provided in [9] is a semantic characterization of $\mathcal{CML}_\Lambda$ wrt $\mathcal{CPL}_\Lambda$, Proposition 6 above is its syntactic counterpart. In fact, we can combine the two results to obtain.

Corollary 7. *Whenever Σ consists entirely of* unary *predicate symbols (and there are no function symbols), the behaviourally-invariant (over finite structures) formulas of $\mathcal{CPL}_\Lambda$ in one-free variable are up to equivalence (over finite structures) precisely the equality-free and quantifier-free formulas in the single-variable fragment of $\mathcal{CPL}_\Lambda$.*

No such syntactic characterization exists for formulas of ordinary first-order logic invariant under bisimulation. Of course, we can do better thanks to the somewhat more modal character of CPL syntax as compared to ordinary FOL.

3.2 Hybrid Languages

In this section, we generalize the results of [4]. Our ultimate goal is Theorem 13 below which establishes the equivalence of CPL with the hybrid languages $\mathcal{CHL}_\Lambda^{\downarrow\mathsf{A}}$ and

$\mathcal{CHL}_\Lambda^{\forall,@}$. Both correspondences also hold for ordinary predicate logic over relational structures (FOL) and extend to CPL. We take this as yet another indication that CPL is natural and well-designed both as a generalization of FOL and "the" predicate logic cousin of existing coalgebraic formalisms.

This is our main, but not the only motivation. We progress towards this result step-by-step, extending the modal language gradually with new hybrid constructs. In this way, we reveal that a similar correspondence exists between natural fragments of CPL and weaker hybrid languages, most importantly between quantifier-free CPL and $\mathcal{CHL}_\Lambda^{\downarrow,@}$—something which has no analogue in the FOL case.

Again, obviously the correspondence between fragments of CPL and extensions of CML is tighter than in the case of FOL and ML only due to the modal flavour of CPL. However, results such as Corollary 10 are useful spadework: any model-theoretic tool to be developed—say, a variant of E-F games—would be adequate for an extended coalgebraic modal formalism (e.g., $\mathcal{CHL}_\Lambda^{\downarrow,@}$) iff it is adequate for the corresponding fragment of CPL (e.g., the variable-free fragment), so we are free to work with whichever formalism we find more convenient at a given moment. This is closely related to our present research efforts. The straightforward correspondence also provides a good starting point for an extension of research programme sketched in [5]—see Remark 14 at the end of this section.

Given a supply of *world variables* V^w that we are going to keep fixed and implicit, we define the following *coalgebraic hybrid languages*

$\mathcal{CHL}_\Lambda^{\downarrow,@}$	$\phi,\psi ::= z \mid P \mid \bot \mid \phi\to\psi \mid \heartsuit(\phi_1,\dots,\phi_n) \mid @_z\phi \mid {\downarrow} z.\phi$
$\mathcal{CHL}_\Lambda^{\downarrow,\mathsf{A}}$	$\phi,\psi ::= z \mid P \mid \bot \mid \phi\to\psi \mid \heartsuit(\phi_1,\dots,\phi_n) \mid \mathsf{A}\phi \mid {\downarrow} z.\phi$
$\mathcal{CHL}_\Lambda^{\forall,@}$	$\phi,\psi ::= z \mid P \mid \bot \mid \phi\to\psi \mid \heartsuit(\phi_1,\dots,\phi_n) \mid @_z\phi \mid \forall z.\phi$

where $z \in \mathsf{V}^w$. We refer the reader to, e.g, [15,4,5] for the semantics. The extension of the standard translation to these formalism is unproblematic in some cases, just like in the case of ordinary hybrid logic over Kripke frames:

$$ST_x(z) := x = z, \quad ST_x(\mathsf{A}\phi) := \forall x.ST_x(\phi), \quad ST_x(\forall z.\phi) := \forall z.ST_x(\phi).$$

One is tempted to put forward also

$$ST_x(@_z\phi) := ST_x(\phi)[z/x], \quad ST_x({\downarrow} z.\phi) := ST_x(\phi)[x/z].$$

However, with other clauses remaining the same, this could work only if $[z/x]$ denotes *capture-avoiding* substitution. Sadly, this in turn would entail forsaking the luxury of using just one designated variable for comprehension. Guillame Malod (see [6]) observed that if we restrict the supply of variables, a translation along the above lines—indeed first proposed in the literature (which shows that the present discussion is less trivial than it might seem)—would fail even when embedding the hybrid logic over Kripke frames in the two-variable fragment of FOL. Malod's counterexample used nesting of modalities of level two, but as our translation uses just one designated variable, ST would go wrong already on formulas of depth one. Just consider $ST_x({\downarrow} z.\Diamond z)$: we would obtain $x\Diamond[x : x = x]$, which is a formula with a completely different meaning.

There are two ways out. First is to redefine

$$STmod_x(@_z\phi) := \forall x.(x = z \to ST_x(\phi)), \tag{1}$$
$$STmod_x(\downarrow z.\phi) := \forall z.(x = z \to ST_x(\phi)). \tag{2}$$

The second is to keep ST for hybrid formulas as defined above and change the modal clause instead:

$$ST_x(\heartsuit(\phi_1, \ldots, \phi_n)) := x\heartsuit[y : ST_y(\phi_1)] \ldots [y : ST_y(\phi_n)], \tag{3}$$

where y is the first (in some fixed enumeration) variable *not used* in $ST_x(\phi_1), \ldots, ST_x(\phi_n)$; by *not used* here we mean both free and bound usage. Furthermore, to ensure that the translation works correctly, we have to assume that *neither* x *nor* y appears in V^w. While the requirement to use more bound variables can be cumbersome—particularly for infinite sets of formulas—we prefer this option, as it makes it easier to characterize weaker hybrid languages as suitable syntactic fragments of CPL.

We can now state a generalization of both Proposition 5 and corresponding results from the hybrid logic literature—see, e.g., [4] for references:

Proposition 8. *For any* $\phi \in \mathcal{CHL}_\Lambda$ *and any* $\mathfrak{M} = (C, \gamma, \pi), \vartheta, c$, *we have* $\mathfrak{M}, \vartheta, c \models \phi$ *iff* $\mathfrak{M}, \vartheta[x \mapsto c] \models ST_x(\phi)$.

As is well-known in the hybrid logic community—see again [4] for references—there is also a translation in the reverse direction for sufficiently expressive hybrid languages. This also generalizes to our setting, see Table 1.

Table 1. Coalgebraic Hybrid Translation from quantifier-free CPL to $\mathcal{CHL}_\Lambda^{\downarrow,@}$

$HT(P(x)) := @_x P$	$HT(x = y) := @_x y$
$HT(\bot) := \bot$	$HT(\phi \to \psi) := HT(\phi) \to HT(\psi)$
$HT(x\heartsuit[y_1 : \phi_1] \ldots [y_n : \phi_n])$	$:= @_x\heartsuit(\downarrow y_1.HT(\phi_1), \ldots, \downarrow y_n.HT(\phi_n))$

Proposition 9. *For any* $\phi \in \mathcal{CPL}_\Lambda$ *and any* $\mathfrak{M} = (C, \gamma, \pi), \vartheta, c$, *we have*

$$\mathfrak{M}, \vartheta, c \models HT(\phi) \textit{ iff } \mathfrak{M}, \vartheta[x \mapsto c] \models \phi.$$

Combining Propostions 9 and 8, we get:

Corollary 10. *Whenever* Σ *consists purely of unary predicates (and no function symbols),* $\mathcal{CHL}_\Lambda^{\downarrow,@}$ *is expressively equivalent to the quantifier-free fragment of* $\mathcal{CPL}_\Lambda$, *assuming* V^i *contains* V^w *plus a disjoint infinite supply of additional individual variables (used for comprehension).*

Remark 11 (Quantifier-free CPL as the bounded fragment of FOL). In the case of ordinary FOL, the fragment equivalent to $\mathcal{CHL}_\Lambda^{\downarrow,@}$ is characterized as the *bounded*