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## ABSTRACT

We explore the existence of quasisymmetric magnetic fields in asymmetric toroidal domains. These vector fields can be identified with a class of magnetohydrodynamic equilibria in the presence of pressure anisotropy. First, using Clebsch potentials, we derive a system of two coupled nonlinear first order partial differential equations expressing a family of quasisymmetric magnetic fields in bounded domains. In regions where flux surfaces and surfaces of constant field strength are not tangential, this system can be further reduced to a single degenerate nonlinear second order partial differential equation with externally assigned initial data. Subclasses of solutions are then constructed by specifying as input the form the flux function, which enforces boundary shape and nested flux surfaces. In particular, we exhibit smooth quasisymmetric vector fields, which correspond to local solutions of anisotropic magnetohydrodynamics in asymmetric toroidal domains such that tangential boundary conditions are fulfilled on a portion of the bounding surface. These solutions are local because they lack periodicity in the toroidal angle. The problems of boundary shape and locality are also discussed. We find that magnetic fields with Euclidean isometries can be fitted into asymmetric domains and that the mathematical difficulty encountered in the derivation of global quasisymmetric magnetic fields lies in the topological obstruction toward global extension affecting local solutions of the governing nonlinear first order partial differential equations.





## I. INTRODUCTION

In the context of ideal magnetohydrodynamics with isotropic pressure, plasma equilibria are described by balance between Lorentz force and pressure gradient,

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = \nabla P, \quad \nabla \cdot \boldsymbol{B} = 0 \quad \text{in } \Omega, \tag{1a}$$

$$\boldsymbol{B} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega. \tag{1b}$$

In the notation above, $\boldsymbol{B}$ is the magnetic field, $P$ is the pressure, $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, and $\boldsymbol{n}$ is the unit outward normal to the bounding surface $\partial\Omega$. As a system of nonlinear first order partial differential equations for the Cartesian components $B_x$, $B_y$, $B_z$ and the pressure $P$, Eq. (1) is twice elliptic and twice hyperbolic, in the sense that it can be regarded as the combination of two elliptic equations and two hyperbolic equations (see Refs. 1 and 2 or Appendix on this point). Such mixed behavior makes (1) one of the hardest equations in mathematical physics. Indeed, while in the presence of a continuous Euclidean symmetry, the hyperbolic part can be removed to obtain an elliptic nonlinear second order partial differential equation (the Grad–Shafranov equation[3–5]) for the flux function (level sets of the flux function define flux surfaces, i.e., surfaces through which the magnetic flux is zero), the existence of regular solutions of (1) without such isometries remains an unsolved problem in the theory of partial differential equations.[2]

In the following, we shall always use the word symmetry to refer to continuous transformations of three dimensional Euclidean space that preserve the Euclidean distance between points. These continuous transformations are *Euclidean isometries* and consist of a superposition of translations and rotations. Asymmetry will then imply the absence of such Euclidean isometries. Furthermore, a certain equilibrium will be deemed symmetric as long as the magnetic field is, without any requirements on other fields or boundary shape. It should be noted that a symmetric solution $(\boldsymbol{B}, P)$ of (1) does not necessarily imply a symmetric boundary $\partial\Omega$. For an example of symmetric

magnetic field satisfying (1) within an asymmetric bounded domain, see Ref. 6. A symmetric solution always implies a symmetric boundary when the boundary corresponds to an isobaric (flux) surface. According to a conjecture due to H. Grad, well-behaved solutions of (1) with non-constant pressure in $\Omega$ and constant pressure on $\partial\Omega$ should possess a high degree of symmetry.[7] Here, the reasoning is the following. First, the boundary condition for the pressure $P$ effectively forces the magnetic field $\boldsymbol{B}$ and the current $\nabla \times \boldsymbol{B}$ to lie on toroidal surfaces (in this regard see also Arnold's structure theorem[8]). In particular, note that the level sets of the pressure (which also assign the bounding surface in a standard isotropic equilibrium) cannot be spherical due to the Poincaré–Hopf theorem (of which the hairy ball theorem[9] is a special case). Indeed, a consequence of this theorem is that there is no continuous nonvanishing tangent vector field to a sphere, a fact that is incompatible with $\nabla P = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \neq \boldsymbol{0}$ (which implies $\boldsymbol{B} \neq \boldsymbol{0}$, $\nabla \times \boldsymbol{B} \neq \boldsymbol{0}$, $\boldsymbol{B} \cdot \nabla P = 0$, and $\nabla \times \boldsymbol{B} \cdot \nabla P = 0$). Then, for both $\boldsymbol{B}$ and $\nabla \times \boldsymbol{B}$ to lie consistently on toroidal surfaces, they must be periodic in the toroidal and poloidal angles. This requirement cannot be easily satisfied in the absence of axial symmetry, which would guarantee periodicity of solutions in the toroidal direction. Indeed, the shape of an asymmetric torus is, in general, inconsistent with the integral curves generated by the characteristic system of ordinary differential equations associated with isotropic force balance, and solutions fail to be periodic. A workaround for this problem consists in expanding the class of admissible solutions so that controlled discontinuities are allowed. In this construction, the domain $\Omega$ is partitioned into a given number of subdomains. Within each subdomain, a constant pressure is assumed, and the corresponding magnetic field is a Beltrami field (an eigenstate of the curl operator, see Refs. 10–12 for the existence of such solutions), while pressure jumps occur at the boundaries among subdomains. This approach has proven effective for the modeling of three-dimensional magnetohydrodynamic equilibria aimed at stellarator design.[13–15] (Stellarators are candidate magnetic confinement devices for nuclear fusion applications that do not exhibit boundary symmetry.)

Compared to an axially symmetric tokamak, a stellarator offers the advantage that the field line twist required to trap charged particles is mainly sustained through asymmetric coils instead of plasma currents, thus enabling steady state operation of the machine (although one could still have currents in stellarators that contribute to the twist). The lack of symmetry comes at the price of deteriorated confinement and complicated coil designs to provide a quasisymmetric confining magnetic field,[16] i.e., a magnetic field $\boldsymbol{B}$ whose strength $B = \sqrt{\boldsymbol{B} \cdot \boldsymbol{B}}$ is independent of a direction $\boldsymbol{u}$ in space, $\boldsymbol{u} \cdot \nabla B = 0$ (see below for a rigorous definition of quasisymmetric vector field). Quasisymmetry is desirable because it gives rise to an approximate invariance (we use the word invariance instead of symmetry to avoid confusion with Euclidean symmetries) of the guiding center Hamiltonian, which depends on the magnetic field only through the field strength $B$,[17] and, thus, to a conserved momentum at first order in the guiding center ordering. The conserved quantity is expected to enhance particle confinement along the magnetic field. The type of quasisymmetry described above is called weak quasisymmetry,[18,19] and it will be the one discussed in the paper. One speaks of strong quasisymmetry if it corresponds to an exact invariance of the guiding center Hamiltonian. We refer the reader to Refs. 20–22 for further details on the notion of quasisymmetry. For completeness, it should also be noted that beyond quasisymmetric configurations more general classes of confinement devices called omnigeneous have been proposed with the property that radial particle drifts vanish on average.[17,23]

A quasisymmetric magnetic field is mathematically characterized by the property that there exists a solenoidal vector field $\boldsymbol{u}$ such that both $\boldsymbol{B}$ and the field strength $B$ are Lie-invariant along $\boldsymbol{u}$, i.e.,

$$\mathfrak{L}_{\boldsymbol{u}}\boldsymbol{B} = \boldsymbol{0}, \quad \mathfrak{L}_{\boldsymbol{u}}B = 0, \quad \mathfrak{L}_{\boldsymbol{u}}dV = 0, \tag{2}$$

where $dV = dxdydz$ denotes the volume element in $\mathbb{R}^3$ and $\mathfrak{L}$ is the Lie derivative. The Lie-derivative of a tensor field along a vector field $\boldsymbol{u}$ at a point $\boldsymbol{x}$ quantifies the difference (in an infinitesimal sense) between the tensor field at $\boldsymbol{x}$ and the value that the tensor field would have if it were advected along the flow of $\boldsymbol{u}$.[24] When the Lie derivative of a tensor vanishes, the tensor is constant along $\boldsymbol{u}$. The existence of invariant tensors is a special property of the flow of $\boldsymbol{u}$ and, thus, a form of symmetry.

For a vector field $\boldsymbol{B}$, a scalar field $B$, and a volume element $dV$, the action of the Lie derivative is given in local Cartesian coordinates, respectively, by

$$\mathfrak{L}_{\boldsymbol{u}}\boldsymbol{B} = \boldsymbol{u} \cdot \nabla \boldsymbol{B} - \boldsymbol{B} \cdot \nabla \boldsymbol{u}, \quad \mathfrak{L}_{\boldsymbol{u}}B = \boldsymbol{u} \cdot \nabla B, \quad \mathfrak{L}_{\boldsymbol{u}}dV = (\nabla \cdot \boldsymbol{u})dV. \tag{3}$$

Notice that $\nabla \cdot \boldsymbol{u}$ measures the relative change of the volume element along $\boldsymbol{u}$; it is in this sense that a solenoidal vector field is volume preserving. A symmetric magnetic field is also quasisymmetric. In this case, $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$ is the generator of continuous Euclidean isometries, with $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ constant vectors and $\boldsymbol{x}$ the position vector in $\mathbb{R}^3$.[5] Note that (1a) can also be expressed through the Lie derivative as $\mathfrak{L}_{\nabla \times \boldsymbol{B}}\boldsymbol{B} = \boldsymbol{0}$ and $\mathfrak{L}_{\boldsymbol{B}}dV = 0$. The quasisymmetry condition (2) is written in vector notation as[18]

$$\boldsymbol{B} \times \boldsymbol{u} = \nabla\zeta, \quad \boldsymbol{u} \cdot \nabla B = 0, \quad \nabla \cdot \boldsymbol{u} = 0. \tag{4}$$

Here, $\zeta$ is some scalar function. Furthermore, in going from (2) to (4), we used the vector identity $\nabla \times (\boldsymbol{B} \times \boldsymbol{u}) = \boldsymbol{B}\nabla \cdot \boldsymbol{u} - \boldsymbol{u}\nabla \cdot \boldsymbol{B} + \boldsymbol{u} \cdot \nabla \boldsymbol{B} - \boldsymbol{B} \cdot \nabla \boldsymbol{u}$, the vanishing of $\nabla \cdot \boldsymbol{B}$ and $\nabla \cdot \boldsymbol{u}$, and the fact that if a field is curl free, $\nabla \times (\boldsymbol{B} \times \boldsymbol{u}) = \boldsymbol{0}$, and the first homology is trivial, then it can be identified with the gradient of a function, $\boldsymbol{B} \times \boldsymbol{u} = \nabla\zeta$ (locally this is always true due to the Poincaré lemma). Nonetheless, in stellarator applications, it is customary to identify $\zeta$ with the flux function $\Psi$ taking constant values on the bounding surface, so that both the magnetic field $\boldsymbol{B}$ and the quasisymmetry $\boldsymbol{u}$ lie on flux surfaces, and the conserved momentum is well approximated by the flux function, at least in the limit of small parallel velocity (see Ref. 18 for the expression of the conserved momentum).

Unfortunately, the analysis of Ref. 25 suggests that fulfilling (1) and (4) simultaneously with a constant pressure at the bounding surface in an asymmetric configuration should not be possible (see also Refs. 26–29). In fact, at present no quasisymmetric solutions of system (1) are known to exist. Nevertheless, design and optimization of stellarators with approximately quasisymmetric or omnigeneous magnetic fields represent active areas of research.[30] Furthermore, experimental devices have been built, including the Helically Symmetric eXperiment (HSX),[31] which aims at a quasihelical magnetic field, and Wendelstein 7-X,[32] which targets omnigenity. A possible strategy to overcome the difficulty in achieving exact quasisymmetric configurations within the framework of ideal isotropic magnetohydrodynamics is to modify

the force balance equation [the first equation in (1a)] by introducing pressure anisotropy. In this regard, it has been suggested that pressure anisotropy could remove the problem of overdetermination encountered in the isotropic setting.[33,34] In the anisotropic case, anisotropic magnetohydrodynamic equilibria are described by the boundary value problem

$$\begin{aligned}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}=\nabla\cdot\Pi,\quad \nabla\cdot\boldsymbol{B}=0\ \text{ in }\Omega,\\ \boldsymbol{B}\cdot\boldsymbol{n}=0\ \text{ on }\partial\Omega,\end{aligned}\tag{5}$$

where in Cartesian coordinates $(x^1,x^2,x^3)=(x,y,z)$, the twice contravariant symmetric (here, symmetric does not refer to continuous Euclidean isometries but to the property that the transpose tensor equals the tensor itself) pressure tensor $\Pi$ has components

$$\Pi^{ij}=P_\perp\delta^{ij}+\frac{(P_\parallel-P_\perp)}{B^2}B^iB^j,\quad i,j=1,2,3.\tag{6}$$

The functions $P_\perp$ and $P_\parallel$ are pressure fields associated with the perpendicular and parallel directions to the magnetic field $\boldsymbol{B}$. Indeed, setting $b^i=B^i/B$, $i=1,2,3$, one has

$$\nabla\cdot\Pi=\boldsymbol{b}\times(\nabla P_\perp\times\boldsymbol{b})+(\boldsymbol{b}\cdot\nabla P_\parallel)\boldsymbol{b}+(P_\parallel-P_\perp)\nabla\cdot(\boldsymbol{bb}).\tag{7}$$

Hence, whenever $\nabla\cdot(\boldsymbol{bb})=\boldsymbol{0}$ (e.g., a straight magnetic field) the gradient of $P_\perp$ generates the pressure force in the direction perpendicular to $\boldsymbol{B}$, while the gradient of $P_\parallel$ generates the pressure force in the direction parallel to $\boldsymbol{B}$. Isotropic magnetohydrodynamic equilibria can be recovered when $P_\perp=P_\parallel=P$, with $P$ the scalar pressure. We also remark that the form of the pressure tensor (6) is rooted in kinetic theory: The pressure fields $P_\perp$ and $P_\parallel$ represent the averaged squared deviations of the charged particle velocities in the guiding center approximation across and along the magnetic field with respect to the mean perpendicular and parallel flows.[7,35–37]

Our aim in this paper is to study the boundary value problem (5) with the quasisymmetry condition (4) for the magnetic field. To the best of our knowledge, exception made for symmetric solutions, the existence of quasisymmetric solutions of this system is not known at present. By using a simplifying assumption on the values of $P_\perp$ and $P_\parallel$, we will construct examples of regular quasisymmetric magnetic fields, i.e., asymmetric solutions $\boldsymbol{B}$ of system (4), (5) in a neighborhood $U\subset\Omega$ within an asymmetric toroidal volume $\Omega$ such that $\partial U\cap\partial\Omega\neq\varnothing$. We remark that for the time being we shall not be concerned with the feasibility of the obtained equilibria in the laboratory or their stability, but simply focus on their existence.

The present paper is organized as follows. In Sec. II, general considerations on the geometrical implications of quasisymmetry are given. In particular, Eq. (4) is written in terms of a set of Clebsch potentials.[38] This form will be useful to build explicit solutions. In Sec. III, we discuss the representation of asymmetric tori in terms of special sets of coordinates. In Sec. IV, we derive regular symmetric equilibria that solve system (5) within asymmetric toroidal domains. In Sec. V, we construct regular quasisymmetric solutions of system (5) in a neighborhood $U\subset\Omega$ of an asymmetric toroidal volume $\Omega$ that satisfy tangential boundary conditions on a portion of the boundary $\partial U\cap\partial\Omega\neq\varnothing$. Concluding remarks are given in Sec. VI.

## II. GENERAL REMARKS ON QUASISYMMETRY

Consider Cartesian coordinates $(x^1,x^2,x^3)=(x,y,z)$ in $\Omega$. It is convenient to write the pressure fields $P_\perp$ and $P_\parallel$ as follows:

$$P_\perp=\mathcal{P}-\frac{1}{2}\sigma B^2,\quad P_\parallel=\mathcal{P}+\frac{1}{2}\sigma B^2.\tag{8}$$

Here, the functions $\mathcal{P}$ and $\sigma$ can be interpreted as a reference pressure field and a deviation (anisotropy) term, respectively. The Cartesian components of the pressure tensor become

$$\Pi^{ij}=\left(\mathcal{P}-\frac{1}{2}\sigma B^2\right)\delta^{ij}+\sigma B^iB^j,\quad i,j=1,2,3.\tag{9}$$

In the following, we denote with $\boldsymbol{e}_i$, $i=1,2,3$, the tangent vector in the $x^i$ direction. Then, using $\nabla\cdot\boldsymbol{B}=0$, one can verify the identity

$$\nabla\cdot\Pi=(\partial_i\Pi^{ij})\boldsymbol{e}_j=\nabla\left(\mathcal{P}-\frac{1}{2}\sigma B^2\right)+\boldsymbol{B}\cdot\nabla(\sigma\boldsymbol{B}).\tag{10}$$

Hence, force balance now takes the form

$$(1-\sigma)(\nabla\times\boldsymbol{B})\times\boldsymbol{B}=\nabla\mathcal{P}-\frac{1}{2}B^2\nabla\sigma+(\boldsymbol{B}\cdot\nabla\sigma)\boldsymbol{B}.\tag{11}$$

In the presence of symmetry, it is known that system (11) can be reduced to a well-posed Grad–Shafranov type equation under appropriate ellipticity conditions, among which the requirement $1-\sigma>0$.[7,37,39] In this context, the anisotropy $\sigma$ is usually assumed to be a function of the magnetic flux $\Psi$ and the field strength $B$. Isotropic equilibria correspond to the case $\mathcal{P}=P$ and $\sigma=0$. More generally, solutions with constant $\sigma\neq1$ are qualitatively equivalent to isotropic configurations with $P=\mathcal{P}/(1-\sigma)$. Next, let $P_0\in\mathbb{R}$ be a constant. Force balance equation (11) can be trivially satisfied by setting $\sigma=1$ and $\mathcal{P}=P_0/2$ so that

$$P_\perp=\frac{1}{2}(P_0-B^2),\quad P_\parallel=\frac{1}{2}(P_0+B^2).\tag{12}$$

With this choice, the Lorentz force $(\nabla\times\boldsymbol{B})\times\boldsymbol{B}$ is decoupled from the force balance equation, and the quasisymmetry problem can be considered on its own. Indeed, finding a solution of (5) under the quasisymmetry condition (4) now amounts to solving the boundary value problem

$$\begin{aligned}\nabla\cdot\boldsymbol{B}=0,\quad \boldsymbol{B}\times\boldsymbol{u}=\nabla\zeta,\quad \boldsymbol{u}\cdot\nabla B=0,\quad \nabla\cdot\boldsymbol{u}=0\ \text{ in }\Omega,\\ \boldsymbol{B}\cdot\boldsymbol{n}=0\ \text{ on }\partial\Omega.\end{aligned}\tag{13}$$

Notice that these are just the equations satisfied by a quasisymmetric vector field tangential to $\partial\Omega$. We remark that, however, the equilibria resulting from (12) are expected to be unstable to firehose instabilities.[39] Furthermore, observe that in this construction, the pressure fields $P_\perp$ and $P_\parallel$ are not constrained at the boundary. Instead, they are functions of the magnetic energy density. If one further demands that, for example, $P_\perp=0$ on $\partial\Omega$, this results in an additional boundary condition for the magnetic field strength, $\boldsymbol{B}^2=P_0$ on $\partial\Omega$. We will briefly return to this point later on, although the present study will be mainly focused on the case in which the pressure fields are not constrained at the bounding surface. We also remark that, if the quasisymmetry condition (4) is not imposed, any solenoidal magnetic field

satisfying tangential boundary conditions represents a solution of anisotropic magnetohydrodynamics (5) as long as the pressure fields are chosen as in Eq. (12).

It is useful to briefly discuss the physical consistency of Eq. (12). First, observe that both $P_\perp$ and $P_\parallel$ can be made non-negative by taking a sufficiently large constant $P_0$. Now suppose that the curvature $|\boldsymbol{b}\cdot\nabla\boldsymbol{b}|$ is a small quantity and rewrite the force balance equation $(\nabla\times\boldsymbol{B})\times\boldsymbol{B}=\nabla\cdot\Pi$ in the following form:

$$\begin{aligned}-\frac{1}{2}\boldsymbol{b}\times(\nabla B^2\times\boldsymbol{b})+B^2\boldsymbol{b}\cdot\nabla\boldsymbol{b}\\ =\boldsymbol{b}\times(\nabla P_\perp\times\boldsymbol{b})+(P_\parallel-P_\perp)\boldsymbol{b}\cdot\nabla\boldsymbol{b}\\ +\left[\boldsymbol{b}\cdot\nabla P_\parallel+(P_\parallel-P_\perp)(\nabla\cdot\boldsymbol{b})\right]\boldsymbol{b}.\end{aligned}\tag{14}$$

Collecting terms of the same order and assuming that they exactly balance each other, one obtains the equations

$$B^2=P_\parallel-P_\perp,\tag{15a}$$

$$\boldsymbol{b}\times\left[\nabla\left(P_\perp+\frac{B^2}{2}\right)\times\boldsymbol{b}\right]=\boldsymbol{0},\tag{15b}$$

$$\boldsymbol{b}\cdot\nabla P_\parallel+(P_\parallel-P_\perp)(\nabla\cdot\boldsymbol{b})=0.\tag{15c}$$

By eliminating $P_\parallel$, these equations reduce to

$$\boldsymbol{b}\times\left[\nabla\left(P_\perp+\frac{B^2}{2}\right)\times\boldsymbol{b}\right]=\boldsymbol{0},\tag{16a}$$

$$\boldsymbol{b}\cdot\nabla\left(P_\perp+\frac{B^2}{2}\right)=0,\tag{16b}$$

which imply Eq. (12) as desired. A similar ordering argument would apply for the case in which the curvature is large compared to the other terms in (14).

Now consider system (13). We must distinguish two cases.

1. The vector fields $\boldsymbol{B}$ and $\boldsymbol{u}$ are linearly dependent. Then, $\nabla\zeta=\boldsymbol{0}$, and the magnetic field $\boldsymbol{B}$ is self-quasisymmetric. Indeed, system (13) reduces to the magnetic differential equation

$$\begin{aligned}\nabla\cdot\boldsymbol{B}=0,\quad \boldsymbol{B}\cdot\nabla B=0\ \text{ in }\Omega,\\ \boldsymbol{B}\cdot\boldsymbol{n}=0\ \text{ on }\partial\Omega.\end{aligned}\tag{17}$$

Note that self-quasisymmetry implies that field strength does not change along field lines. Hence, guiding center motion is not accelerated in the direction of the magnetic field (there are no bouncing particles), and the conserved momentum is the momentum parallel to $\boldsymbol{B}$. We remark that, in the context of isotropic ideal magnetohydrodynamics, equilibria such that $\boldsymbol{B}\cdot\nabla B=0$ are known as isodynamic fields.[40] Isodynamic configurations are free of neoclassical transport, but in a symmetric setting with closed flux surfaces, they also exhibit a vanishing field on any curved magnetic axis.[41] Due to the regularity of the boundary $\partial\Omega$, the unit outward normal $\boldsymbol{n}$ can be expressed as

$$\boldsymbol{n}=\frac{\nabla\Psi}{|\nabla\Psi|},\tag{18}$$

where $\Psi$ is a single-valued function corresponding to the usual flux function. For the magnetic field $\boldsymbol{B}$ to possess nested flux surfaces, we demand that

$$\boldsymbol{B}\cdot\nabla\Psi=0\ \text{ in }\Omega,\tag{19}$$

or, denoting with $\Theta$ a possibly multivalued (angle) variable,

$$\boldsymbol{B}=\nabla\Psi\times\nabla\Theta\ \text{ in }\Omega.\tag{20}$$

Restricted to a sufficiently small open (simply connected) set $U\subset\Omega$, the variable $\Theta$ can always be chosen as a single-valued function (Lie–Darboux theorem[42]). In the following, we shall refer to functions associated with the representation of vector fields, such as $\Psi$ and $\Theta$, as Clebsch potentials. It should be noted that when the Clebsh representation (20) holds globally on $\Omega$, the vector potential $\boldsymbol{A}$ associated with the magnetic field $\boldsymbol{B}=\nabla\times\boldsymbol{A}$ is $\boldsymbol{A}=\Psi\nabla\Theta+\nabla g$ for some possibly multivalued function $g$. The total magnetic helicity is then $H=\int_\Omega\boldsymbol{A}\cdot\boldsymbol{B}dV=\int_\Omega\nabla g\cdot(\nabla\Psi\times\nabla\Theta)dV=\int_\Omega\nabla\cdot(\Psi\nabla\Theta\times\nabla g)dV=\int_{\partial\Omega}\Psi(\nabla\Theta\times\nabla g)\cdot\boldsymbol{n}dS$. In particular, if the domain $\Omega$ is a solid torus (or any manifold with a connected boundary) and since $\Psi|_{\partial\Omega}=\Psi_e$ is constant on the boundary by construction, the total magnetic helicity, $H=\Psi_e\int_{\partial\Omega}(\nabla\Theta\times\nabla g)\cdot\boldsymbol{n}dS=\Psi_e\int_\Omega\nabla\cdot(\nabla\Theta\times\nabla g)dV=0$, vanishes regardless of the gauge. The possibility of generating non-zero helicity using Clebsch potentials solely depends on the (first) homology of the domain with respect to that of the boundary and has to be examined on a case by case basis. Alternatively, a relative helicity formula can be used to compare vector potentials.[45] Using (20), system (17) reduces to a single nonlinear first-order partial differential equation for the variable $\Theta$,

$$|\nabla\Psi\times\nabla\Theta|^2=E(\Psi,\Theta)\ \text{ in }\Omega,\tag{21}$$

where $E=E(\Psi,\Theta)$ is some non-negative single-valued function of the variables $\Psi$ and $\Theta$. Observe that in Eq. (21), $\Psi$ is given as an input to the problem. Such pre-assignment of the flux function will be used to construct solutions, although it is not necessary in general. Hence, both the bounding surface $\partial\Omega$ and the foliation of the torus in the volume $\Omega$ are fixed. Assume a non-vanishing magnetic field, $E\neq0$. Then, by defining the function

$$\Theta'=\int\frac{d\Theta}{\sqrt{E(\Psi,\Theta)}},\tag{22}$$

Eq. (21) can be written as

$$|\nabla\Psi\times\nabla\Theta'|^2=1\ \text{ in }\Omega.\tag{23}$$

Notice that Eq. (23) resembles the eikonal equation of geometric optics, $|\nabla\Theta'|=1/f$ with $f$ a positive function.[43] Indeed, it is equivalent to

$$|\nabla\Theta'|^2=\frac{1}{|\nabla\Psi|^2}+\frac{(\nabla\Psi\cdot\nabla\Theta')^2}{|\nabla\Psi|^2}\ \text{ in }\Omega.\tag{24}$$

In general, the solution of a first-order partial differential equation, such as (24), involves the integration of the associated characteristic system of ordinary differential equations. Even if such solution is obtained, it usually has a local nature. Therefore, the existence of a global self-quasisymmetric magnetic field in $\Omega$ is contingent upon the possibility of extending and/or patching local solutions consistently within $\Omega$. For a given set of flux

surfaces $\Psi$, the characteristic surfaces of Eq. (24) can be examined by writing the equation in the form $F(\Theta'_x, \Theta'_y, \Theta'_z) = 0$ where the lower index denotes partial differentiation and by determining solutions $\Lambda$ of the characteristic equation

$$\frac{\partial F}{\partial \Theta'_x}\frac{\partial \Lambda}{\partial x} + \frac{\partial F}{\partial \Theta'_y}\frac{\partial \Lambda}{\partial y} + \frac{\partial F}{\partial \Theta'_z}\frac{\partial \Lambda}{\partial z} = 0. \tag{25}$$

One finds that surfaces $\Lambda = \Psi$ are characteristic, and therefore, their properties depend on the choice of the flux function. We also remark that, if one does not enforce boundary conditions, constructing self-quasisymmetric fields becomes a simpler task. For example, denoting with $(r, \phi, z)$ cylindrical coordinates, the vector field

$$\boldsymbol{B} = f\left(r, \frac{z}{r} - \phi\right)\nabla r \times \nabla\left(\frac{z}{r} - \phi\right), \tag{26}$$

with $f$ an arbitrary function of $r$ and $z/r - \phi$, satisfies $\nabla \cdot \boldsymbol{B} = 0$, $\boldsymbol{B} \cdot \nabla B = 0$, and it does not possess continuous Euclidean symmetries in general because the equation $\mathfrak{L}_{\boldsymbol{u}}\boldsymbol{B} = \boldsymbol{0}$ with $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$ does not have solutions such that $\boldsymbol{u} \neq \boldsymbol{0}$. In particular, the quantity $z/r - \phi$ does not correspond to an Euclidean helical symmetry due to the radial dependence. In addition, the vector field (26) is tangential to cylindrical surfaces [see the example of Fig. 1, which corresponds to the choice $f = e^{-r}\sin(z/r - \phi)$]. Finally, note that in order to validate the asymmetry of a solution in most cases it is easier to verify the violation of the condition $\mathfrak{L}_{\boldsymbol{u}}B^2 = \boldsymbol{u} \cdot \nabla B^2 = 0$ for $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$. Indeed, choosing a curvilinear coordinate system $(y^1, y^2, y^3)$ such that $\boldsymbol{u} = \boldsymbol{e}_3$ and denoting with $B^k$ and $g_{k\ell}$, $k, \ell = 1, 2, 3$, the components of magnetic field and the Euclidean metric tensor with respect to the new coordinates, it follows that $\boldsymbol{u} \cdot \nabla B^2\Theta' = 2g_{k\ell}B^k\partial B^\ell/\partial y^3$ since by hypothesis, $\boldsymbol{u}$ is an Euclidean isometry such that $\partial g_{k\ell}/\partial y^3 = 0$, $k, \ell = 1, 2, 3$. On the other hand,

$$\mathfrak{L}_{\boldsymbol{u}}\boldsymbol{B} = \mathfrak{L}_{\boldsymbol{e}_3}(B^i\boldsymbol{e}_i) = (\mathfrak{L}_{\boldsymbol{e}_3}B^i)\boldsymbol{e}_i + B^i\mathfrak{L}_{\boldsymbol{e}_3}\boldsymbol{e}_i = \frac{\partial B^i}{\partial y^3}\boldsymbol{e}_i + B^i[\boldsymbol{e}_3, \boldsymbol{e}_i], \tag{27}$$

by the Leibniz rule. The commutator $\mathfrak{L}_{\boldsymbol{e}_i}\boldsymbol{e}_j = [\boldsymbol{e}_i, \boldsymbol{e}_j]$ between basis vectors is always zero since second partial derivatives are symmetric (Clairaut's theorem), $\mathfrak{L}_{[\boldsymbol{e}_i, \boldsymbol{e}_j]}f = \frac{\partial^2 f}{\partial y^i \partial y^j} - \frac{\partial^2 f}{\partial y^j \partial y^i} = 0$ for any function $f(y^1, y^2, y^3)$. This implies $\mathfrak{L}_{\boldsymbol{u}}\boldsymbol{B} = (\partial B^i/\partial y^3)\boldsymbol{e}_i$, and therefore, $\mathfrak{L}_{\boldsymbol{u}}\boldsymbol{B}$ cannot vanish as long as $\mathfrak{L}_{\boldsymbol{u}}B^2$ is different from zero. Nevertheless, if $\mathfrak{L}_{\boldsymbol{u}}B^2 = 0$, one still needs to compute $\mathfrak{L}_{\boldsymbol{u}}\boldsymbol{B}$ to determine whether the magnetic field is symmetric or not.

2. The vector fields $\boldsymbol{B}$ and $\boldsymbol{u}$ are linearly independent, implying that $\nabla\zeta \neq \boldsymbol{0}$. From the second equation in (13), one sees that both $\boldsymbol{B}$ and $\boldsymbol{u}$ must be orthogonal to $\nabla\zeta$. However, notice that on the boundary, $\nabla\zeta$ is not necessarily aligned with the unit outward normal $\boldsymbol{n}$. Hence, in general, $\zeta$ should not be identified with the

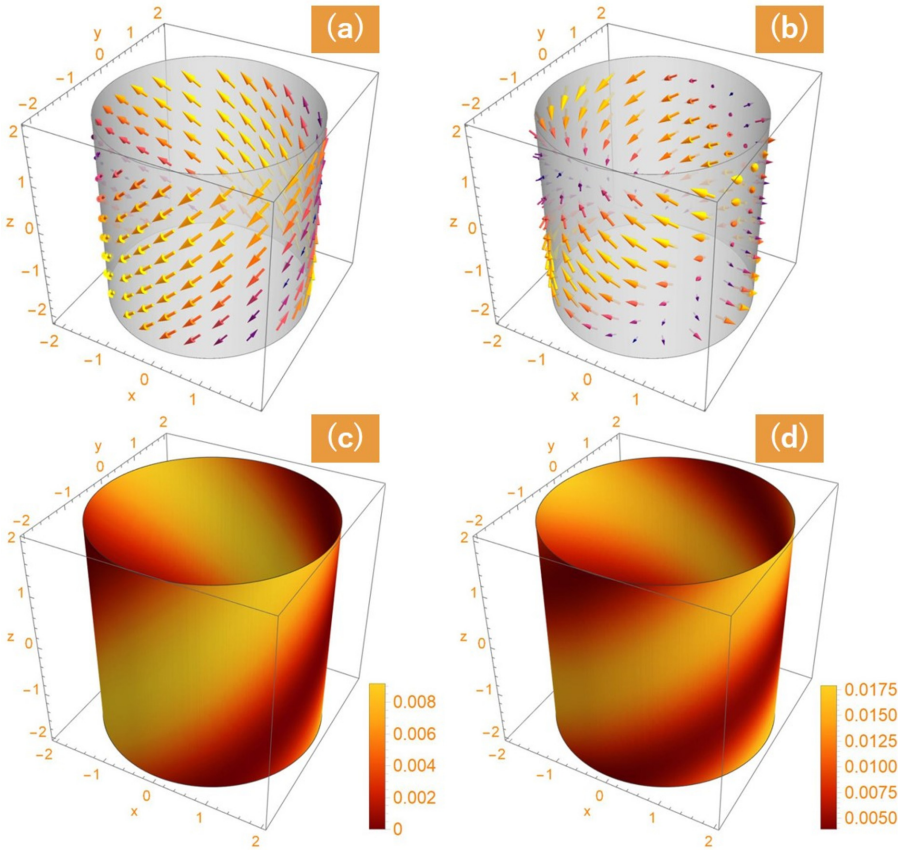


**FIG. 1.** (a) Plot of the self-quasisymmetric magnetic field $\boldsymbol{B} = e^{-r}\sin(z/r - \phi)\nabla r \times \nabla(z/r - \phi)$ on a level set of $r$. (b) Plot of the current field $\nabla \times \boldsymbol{B}$. (c) Plot of the magnetic field strength $B^2$. Notice that $B^2$ does not change along $\boldsymbol{B}$ [compare with (a)]. (d) Plot of the current field strength $|\nabla \times \boldsymbol{B}|^2$.

flux function $\Psi$ taking constant values on $\partial\Omega$. Nonetheless, as mentioned in the introduction, configurations with $\zeta = \Psi$ are desirable in stellarator applications. Denoting with $\vartheta$ and $\rho$ two possibly multivalued (angle) variables and recalling that $\boldsymbol{B}$ and $\boldsymbol{u}$ are solenoidal vector fields, the space of solutions of system (13) can be restricted to

$$\boldsymbol{B} = \nabla\zeta \times \nabla\vartheta, \quad \boldsymbol{u} = \nabla\zeta \times \nabla\rho. \tag{28}$$

Since $\boldsymbol{B} \cdot \nabla\Psi = 0$ in $\Omega$, from (28), one has

$$\Psi = \Psi(\zeta, \vartheta). \tag{29}$$

For a given flux function $\Psi = \Psi(\boldsymbol{x})$, we assume that Eq. (29) can be inverted to obtain

$$\zeta = \zeta(\vartheta, \boldsymbol{x}). \tag{30}$$

Then, upon substituting Eqs. (28) and (30) into Eq. (13), system (13) reduces to the following coupled nonlinear first order partial differential equations for the Clebsch potentials $\vartheta$ and $\rho$,

$$\nabla\zeta \times \nabla\vartheta \cdot \nabla\rho = 1, \quad |\nabla\zeta \times \nabla\vartheta|^2 = F(\zeta, \rho) \quad \text{in } \Omega, \tag{31}$$

where $F$ is some (unknown) non-negative single-valued function of the Clebsch potentials $\zeta(\vartheta, \boldsymbol{x})$ and $\rho$. Notice that, since Eq. (31) is a first-order system, the main hurdle toward finding an integral is again represented by the possibility of extending a local solution to the whole $\Omega$. If Clebsch potentials $\vartheta$ and $\rho$ can be determined so that (31) is satisfied, the corresponding vector fields (28) define a quasisymmetric solution of anisotropic magnetohydrodynamics. Finally, observe that system (31) can be further reduced to a single nonlinear second-order partial differential equation in regions $V \subset \Omega$ where $\nabla\zeta \times \nabla B \neq \boldsymbol{0}$. Indeed, the condition $\boldsymbol{u} \cdot \nabla B = 0$ then implies that $\rho = \rho(\zeta, B^2)$ with $B^2 = |\nabla\zeta \times \nabla\vartheta|^2$. Therefore, the remaining equation $\boldsymbol{B} \times \boldsymbol{u} = \nabla\zeta$ can be written as

$$\nabla\zeta \times \nabla\vartheta \cdot \nabla|\nabla\zeta \times \nabla\vartheta|^2 = \frac{1}{\dfrac{\partial\rho}{\partial B^2}} \quad \text{in } V. \tag{32}$$

For a given $\rho = \rho(\zeta, B^2)$, a quasisymmetric solution in $V$ can, thus, be obtained by solving the equation above for the unknown variable $\vartheta$. Observe that Eq. (32) can be studied by setting the right-hand side to 1 provided that a function $\lambda = \lambda(\zeta, B^2)$ satisfying $\lambda(\lambda^2 + B^2\frac{\partial\lambda^2}{\partial B^2})\frac{\partial\rho}{\partial B^2} = 1$ can be found. Indeed, if a solution $\boldsymbol{B} = \nabla\zeta \times \nabla\vartheta$ of the equation with the right-hand side set to unity is obtained, then the vector field $\boldsymbol{B}' = \lambda\nabla\zeta \times \nabla\vartheta$ solves (32). We remark that Eq. (32) is degenerate because it is invariant under the transformation $\vartheta \to \vartheta + \xi$, with $\xi = \xi(\zeta)$ an arbitrary function of $\zeta$. Unfortunately, Eq. (32) cannot hold in the whole $\Omega$ (that is we never have $V = \Omega$). To see this, consider again system (13) and suppose that $\nabla\zeta \times \nabla B \neq \boldsymbol{0}$ in $\Omega$. Since both $\boldsymbol{B}$ and $\boldsymbol{u}$ lie on surfaces of constant $\zeta$, the conditions $\nabla \cdot \boldsymbol{u} = 0$ and $\boldsymbol{u} \cdot \nabla B = 0$ imply that the quasisymmetry must satisfy

$$\boldsymbol{u} = \omega(\zeta, B)\nabla\zeta \times \nabla B, \tag{33}$$

where $\omega = \omega(\zeta, B)$ is some function of $\zeta$ and $B$. Using the property $\boldsymbol{B} \cdot \nabla\zeta = 0$, the equation $\boldsymbol{B} \times \boldsymbol{u} = \nabla\zeta$, therefore, reduces to

$$\omega\boldsymbol{B} \cdot \nabla B = 1. \tag{34}$$

Due to the regularity of the magnetic field and its derivatives, we must have $\omega \neq 0$ for (34) to hold. Using $\nabla \cdot \boldsymbol{B} = 0$ and $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, the equation above implies that

$$0 = \int_\Omega \nabla \cdot (B\boldsymbol{B})dV = \int_\Omega \frac{dV}{\omega}. \tag{35}$$

For the quasisymmetry $\boldsymbol{u}$ to be a continuous function, the fraction $1/\omega$ must be continuous. Hence, Eq. (35) can be satisfied only if there exists at least one point $\boldsymbol{x}^* \in \Omega$ such that $1/\omega(\boldsymbol{x}^*) = 0$, implying $|\boldsymbol{u}(\boldsymbol{x}^*)| = \infty$. This contradicts the regularity of $\boldsymbol{u}$. Therefore, a global regular quasisymmetric configuration cannot exist in the case $\nabla\zeta \times \nabla B \neq \boldsymbol{0}$. In other words, if a global regular quasisymmetric configuration exists, there must be regions/points where $\nabla\zeta \times \nabla B = \boldsymbol{0}$. Evidently, the property $\rho = \rho(\zeta, B^2)$ breaks down at those points where $\nabla\zeta \times \nabla B = \boldsymbol{0}$, and Eq. (32) does not hold there. When $\zeta = \Psi$ is the flux function, the fact that $\nabla\Psi \times \nabla B = \boldsymbol{0}$ somewhere is a known property of omnigeneous fields (see for example, Ref. 23) It is also worth noticing that from the perspective of confinement, the only physically significant component of $\boldsymbol{u}$ is $\boldsymbol{u} \cdot \boldsymbol{B}/B$. Hence, in principle one may allow configurations such that $|\boldsymbol{u}|$ is divergent but the projection $\boldsymbol{u} \cdot \boldsymbol{B}/B$ is not.

## III. HARMONIC ORTHOGONAL COORDINATES AND ASYMMETRIC TORI

In this section, we are concerned with the representation of asymmetric tori through special sets of coordinates. Such representation will be used in Secs. IV and V to construct examples of symmetric and quasisymmetric anisotropic magnetohydrodynamic equilibria in asymmetric tori. Let $(r, \phi, z)$ denote cylindrical coordinates. First, consider an axially symmetric torus with circular cross section. Such surface is defined as a level set of the function

$$\Psi_{\text{ax}} = \frac{1}{2}\left[(r - r_0)^2 + z^2\right], \tag{36}$$

where $r_0$ is a positive real constant representing the major radius of the torus (the distance from the origin to the center of the torus). The axial symmetry is described by the vector field $\boldsymbol{u} = \partial_\phi = r^2\nabla\phi$. Indeed, $\mathfrak{L}_{\partial_\phi}\Psi_{\text{ax}} = 0$. The axially symmetric torus (36) can be thought as a special case of a more general family of toroidal surfaces in $\mathbb{R}^3$, which correspond to level sets of the function

$$\Psi_{\text{T}} = \frac{1}{2}\left[(\mu - \mu_0)^2 + \mathcal{E}(z - h)^2\right]. \tag{37}$$

Here, $\mu = \mu(x, y)$ is a single-valued function in the $(x, y)$ plane measuring the distance from the origin in $\mathbb{R}^2$ and $\mu_0$ is a positive real constant corresponding to the major radius of the torus. (More generally $\mu_0$ can be a single-valued function, although this case is not considered here.) The more the level sets of $\mu$ differ from circles, the more the torus $\Psi_{\text{T}}$ departs from axial symmetry in the $(x, y)$ plane. Similarly, the single-valued function $h = h(\boldsymbol{x})$ expresses the deviation of the toroidal axis from the $(x, y)$ plane, while the positive function $\mathcal{E} = \mathcal{E}(\boldsymbol{x})$ represents the departure of cross sections from circles. In particular, tori with non-circular cross sections can be represented by appropriate choice of $\mu$, $h$, and $\mathcal{E}$. For example, a torus with elliptic

cross section can be obtained by setting $\mu = r$ and $h = 0$, and by taking $\mathcal{E}$ to be a positive constant different from unity. Also notice that without loss of generality one can set $\mathcal{E} = 1$ in Eq. (37), although depending on the sought toroidal surface, this may result in more complicated expressions for $h$. The choice $\mathcal{E} = 1$ will be used in the remaining part of the paper. Figure 2 shows an axially symmetric torus (36) and three asymmetric tori (37) obtained from different choices of $\mu$ and $h$. The asymmetry of the tori (b), (c), and (d) in Fig. 2 can be verified by observing that these surfaces are not Lie-invariant with respect to the generator of continuous Euclidean isometries, that is the equation $\mathfrak{L}_{\boldsymbol{u}}\Psi_{\mathrm{T}} = 0$ with $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$, does not have solution for any nontrivial choice of the constant vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$.

If quasisymmetry is not imposed, solutions of (5) in asymmetric tori (37) can be easily obtained by enforcing (12) and setting

$$\boldsymbol{B} = \nabla\Psi_{\mathrm{T}} \times \nabla\Phi, \tag{38}$$

with $\Phi$ an arbitrary function whose gradient field is linearly independent of $\nabla\Psi_{\mathrm{T}}$. In this case, a sufficient condition for the perpendicular pressure $P_\perp = (P_0 - B^2)/2$ to be constant on $\partial\Omega$ (which corresponds to a level set of $\Psi_{\mathrm{T}}$ by construction) is that $\Phi$ is chosen by requiring that $\nabla\Psi_{\mathrm{T}} \times \nabla P_\perp = \boldsymbol{0}$, or, substituting the expression for $P_\perp$,

$$|\nabla\Psi_{\mathrm{T}} \times \nabla\Phi|^2 = E(\Psi_T) \quad \text{in } \Omega, \tag{39}$$

where $E = E(\Psi_{\mathrm{T}})$ is a non-negative function of $\Psi_{\mathrm{T}}$. Of course, if $P_\perp$ is constant on $\partial\Omega$, so is $P_\parallel$ due to (12). Regarding the solvability of (39), considerations analogous to those pertaining to Eq. (21) apply. Furthermore, by comparison with Eq. (21), it is clear that a magnetic field fulfilling (39) is also self-quasisymmetric since $\boldsymbol{B} \cdot \nabla B = 0$ [recall Eq. (17)]. Solutions of (39) can be obtained easily in axially symmetric tori (36). Indeed, first observe that the vector fields

$$\boldsymbol{B} = \lambda\nabla\phi, \tag{40}$$

with $\lambda = \lambda(r, z)$, are self-quasisymmetric because they satisfy (17) when $\Omega$ is an axially symmetric torus (36). Then, Eq. (39) holds provided that $\lambda^2 = r^2 E(\Psi_{\mathrm{ax}})$. Of course, the self-quasisymmetry of (40) corresponds to the usual rotational symmetry because $\mathfrak{L}_{\partial_\phi}\boldsymbol{B} = \boldsymbol{0}$. However, the construction leading to (40) can be generalized to a certain class of asymmetric tori and, in particular, to derive symmetric solutions in asymmetric toroidal domains. To see this, notice that the orthogonal coordinates $(\log r, \phi, z)$ represent a special case of a larger family of orthogonal harmonic coordinates $(x^1, x^2, x^3) = (\mu, \nu, z)$ satisfying the following properties in $\Omega$:

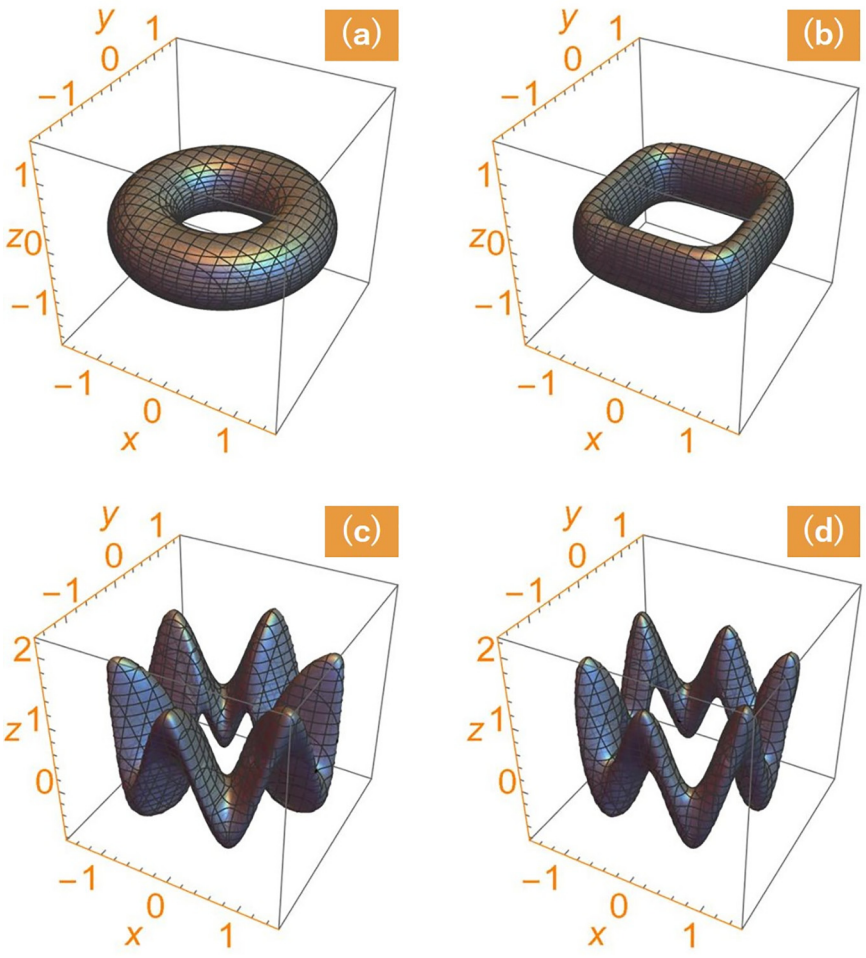


**FIG. 2.** (a) Axially symmetric torus $\Psi_{\mathrm{ax}} = 0.1$ with $r_0 = 1$. (b) Asymmetric torus $\Psi_{\mathrm{T}} = 0.1$ with $\mu = \sqrt{x^4 + y^4}$, $\mu_0 = 1$, and $h = 0$. (c) Asymmetric torus $\Psi_{\mathrm{T}} = 0.1$ with $\mu = r$, $\mu_0 = 1$, and $h = r\sin^2(3\phi)$. (d) Asymmetric torus $\Psi_{\mathrm{T}} = 0.1$ with $\mu = \sqrt{x^4 + y^4}$, $\mu_0 = 1$, and $h = r\sin^2(3\phi)$.

$$\nabla x^i \cdot \nabla x^j \propto \delta^{ij}, \quad \Delta x^i = 0, \quad |\nabla x^1| = |\nabla x^2|, \quad |\nabla x^3| = 1, \quad i,j = 1,2,3. \tag{41}$$

Here, the point $r = 0$ is not included in the coordinates domain to avoid singularities. Since these coordinates share the same geometric properties of cylindrical coordinates, they simplify vector calculus and provide a starting point to attempt a generalization of symmetric configurations to an asymmetric setting. For these reasons, they will be used in the analysis of quasisymmetric magnetic fields of Secs. IV and V. Nonetheless, it should be noted that other choices of coordinates are possible as well.

The coordinates $\mu$ and $\nu$ can be constructed as follows. Let $\Gamma_1$ and $\Gamma_2$ denote two smooth non-intersecting Jordan (simple and closed) curves in $\mathbb{R}^2$. Let $\Sigma_1$ and $\Sigma_2$ be the areas enclosed by $\Gamma_1$ and $\Gamma_2$, respectively. Assume that $\Sigma_2 \subset \Sigma_1$ and define $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Sigma = \Sigma_1 - \bar{\Sigma}_2$, where the bar denotes the closure of a set. Then, the coordinate $\mu = \mu(x,y)$ is obtained as solution of the following Dirichlet boundary value problem for the Laplace equation,

$$\Delta\mu = 0 \ \text{ in } \Sigma, \quad \mu = c_1 \ \text{ on } \Gamma_1, \quad \mu = c_2 \ \text{ on } \Gamma_2, \quad c_1, c_2 \in \mathbb{R}, \quad c_1 \neq c_2. \tag{42}$$

Observe that $\mu$ can be thought of as a measure of the distance from the origin in $\mathbb{R}^2$. Next, the coordinate $\nu = \nu(x,y)$ is chosen as the harmonic conjugate of $\mu$. [Two functions $\mu(x,y)$ and $\nu(x,y)$ are harmonic conjugates if they satisfy the Cauchy–Riemann equations $\partial\mu/\partial x = \partial\nu/\partial y$ and $\partial\mu/\partial y = -\partial\nu/\partial x$.] In particular, denoting with $\tilde{\boldsymbol{n}}$ the unit outward normal to the boundary $\Gamma$, the coordinate $\nu$ is a multivalued (angle) variable whose gradient $\nabla\nu$ is a harmonic vector field in $\Sigma$, i.e.,

$$\Delta\nu = 0 \ \text{ in } \Sigma, \quad \nabla\nu \cdot \tilde{\boldsymbol{n}} = 0 \ \text{ on } \Gamma. \tag{43}$$

Finally, Eq. (41) is satisfied by the coordinates $(x^1, x^2, x^3) = (\mu, \nu, z)$ in virtue of the Cauchy–Riemann equations mentioned above.

As an example, suppose that $\Gamma_1$ and $\Gamma_2$ are the boundaries of the horizontal sections of two confocal elliptic cylinders centered at the origin in $\mathbb{R}^3$. Then, the orthogonal harmonic coordinates $(\mu, \nu, z)$ correspond to the so-called elliptic cylindrical coordinates, which are related to the Cartesian coordinates $(x, y, z)$ by the transformation

$$x = a\cosh\mu\cos\nu, \quad y = a\sinh\mu\sin\nu, \tag{44}$$

where $a$ is a positive real constant such that the foci of the elliptic sections of the cylinders are located at $\boldsymbol{x} = (a, 0, z)$ and $\boldsymbol{x} = (-a, 0, z)$. Introducing the quantity

$$\delta = \frac{1}{a^2|\nabla\mu|^2} = \frac{1}{a^2|\nabla\nu|^2} = \sin^2\nu + \sinh^2\mu = \sqrt{\left(\frac{x^2+y^2}{a^2} - 1\right)^2 + \frac{4y^2}{a^2}}, \tag{45}$$

one can derive the following expressions for the quadrant $x, y \geq 0$,

$$\mu = \operatorname{arcsinh}\left[\frac{1}{\sqrt{2}}\sqrt{-1 + \frac{x^2+y^2}{a^2} + \delta}\right], \tag{46a}$$

$$\nu = \arcsin\left[\frac{1}{\sqrt{2}}\sqrt{1 - \frac{x^2+y^2}{a^2} + \delta}\right], \tag{46b}$$

and also

$$\nabla\mu = \frac{\sinh\mu\cos\nu\nabla x + \cosh\mu\sin\nu\nabla y}{a(\sin^2\nu + \sinh^2\mu)} = \frac{\frac{\sinh\mu}{\cosh\mu}x\nabla x + \frac{\cosh\mu}{\sinh\mu}y\nabla y}{a^2(\sin^2\nu + \sinh^2\mu)}, \tag{47a}$$

$$\nabla\nu = \frac{\sinh\mu\cos\nu\nabla y - \cosh\mu\sin\nu\nabla x}{a(\sin^2\nu + \sinh^2\mu)} = \frac{\frac{\sinh\mu}{\cosh\mu}x\nabla y - \frac{\cosh\mu}{\sinh\mu}y\nabla x}{a^2(\sin^2\nu + \sinh^2\mu)}. \tag{47b}$$

In the other quadrants, the coordinate $\nu$ can be defined so that $\nu \in [0, 2\pi)$ when moving around a $\mu$ contour, such as $\Gamma_1$ or $\Gamma_2$. In particular, $\pi - \nu$ for $x < 0$ and $y \geq 0$, $2\pi - \nu$ for $x \geq 0$ and $y < 0$, and $\pi + \nu$ for $x, y < 0$ with $\nu$ given by (46b). In Fig. 3, level sets of cylindrical coordinates $\log r$, $\phi$ and elliptic cylindrical coordinates $\mu$, $\nu$ as defined in (46) are shown together with the respective gradient fields. By substituting (46a) into the expression (37), one obtains elliptic toroidal surfaces. In Secs. IV and V, we will see how to embed symmetric and quasisymmetric magnetic fields within such domains.

## IV. SYMMETRIC AND SELF-QUASISYMMETRIC MAGNETIC FIELDS IN ASYMMETRIC TORI

Let $\Omega$ denote the volume enclosed by the elliptic torus

$$\Psi_{\text{el}} = \frac{1}{2}\left[(\mu - \mu_0)^2 + z^2\right], \tag{48}$$

with $\mu$ given by (46a). Observe that level sets of $\Psi_{\text{el}}$ consist of tori that have no twist in the cross section. Again, the surface $\Psi_{\text{el}}$ does not possess continuous Euclidean symmetries because the equation $\mathfrak{L}_{\boldsymbol{u}}\Psi_{\text{el}} = 0$ with $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$ does not have nontrivial solutions $\boldsymbol{u} \neq \boldsymbol{0}$. In particular, note that horizontal cuts of $\Psi_{\text{el}}$ are ellipsoidal. As discussed in the previous section, in the absence of quasisymmetry and boundary conditions on the pressure fields, anisotropic magnetohydrodynamic equilibria within $\Omega$ can be obtained by enforcing (12) and by setting $\boldsymbol{B} = \nabla\Psi_{\text{el}} \times \nabla\Phi$ with $\Phi$ an arbitrary function whose gradient field is linearly independent of $\nabla\Psi_{\text{el}}$. Observing that $\nabla\Psi_{\text{el}} \cdot \nabla\nu = 0$, a family of solutions with a more familiar form is

$$\boldsymbol{B} = \nabla\Psi_{\text{el}} \times \nabla\nu + \lambda\nabla\nu, \tag{49}$$

with $\lambda = \lambda(\mu, z)$. Next, we seek for a translationally symmetric solution such that the direction of symmetry is $\boldsymbol{u} = \nabla z$. For simplicity, assume that $\boldsymbol{B} = \lambda\nabla\nu$. Evidently, $\nabla \cdot \boldsymbol{B} = \nabla \cdot \boldsymbol{u} = 0$. Hence, Eq. (13) is satisfied provided that

$$\lambda\nabla\nu \times \nabla z = \nabla\zeta, \quad \frac{\partial}{\partial z}\lambda^2|\nabla\nu|^2 = 0. \tag{50}$$

Since by construction $\nabla\nu \times \nabla z = \nabla\mu$ and $\partial|\nabla\nu|/\partial z = 0$, these conditions are identically satisfied for any $\lambda = \lambda(\mu)$. Then, $\zeta = \int \lambda d\mu$. This example shows that a symmetric solution can be fitted within an asymmetric domain. In Fig. 4(a), plot of a symmetric magnetic field constructed as described above is given.

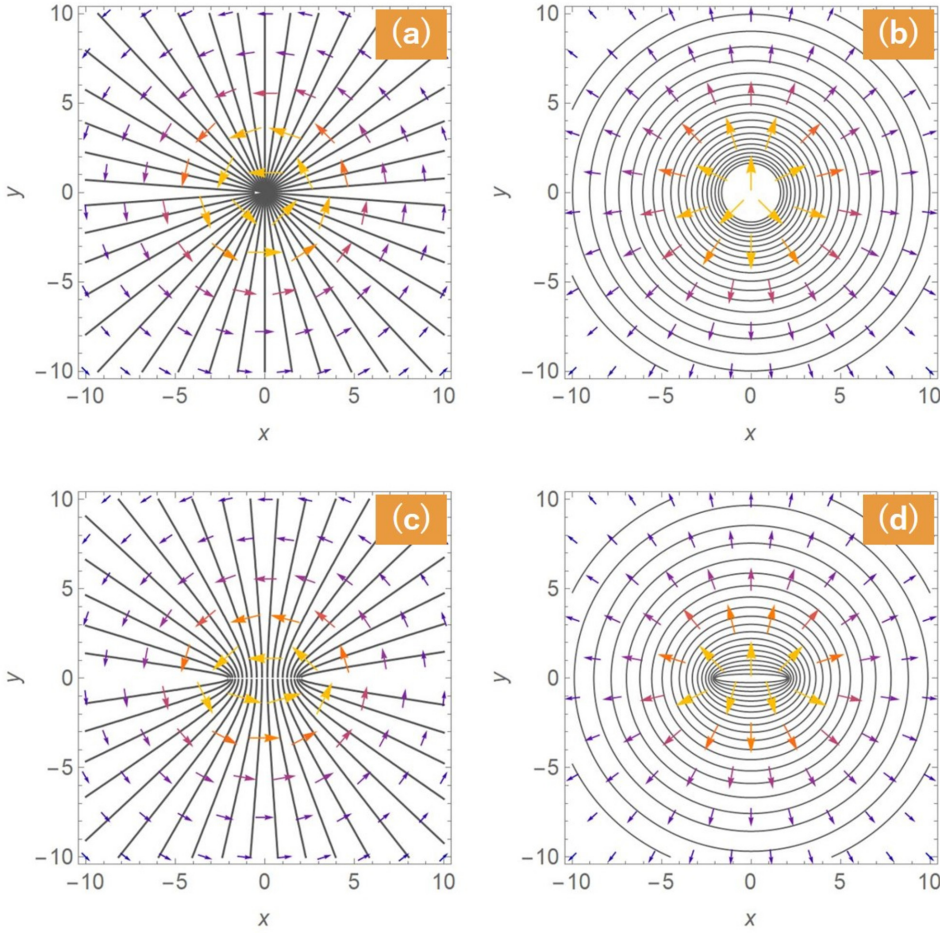


**FIG. 3.** (a) Contours of $\phi$ and gradient field $\nabla\phi$. (b) Contours of $\log r$ and gradient field $\nabla \log r$. (c) Contours of $\nu$ as defined in (46b) and gradient field $\nabla\nu$. For each contour $\nu_0$ in the upper half plane, the specular contour $2\pi - \nu_0$ is shown in the lower half. (d) Contours of $\mu$ as defined in (46a) and gradient field $\nabla\mu$. In these plots, $a = 2$.

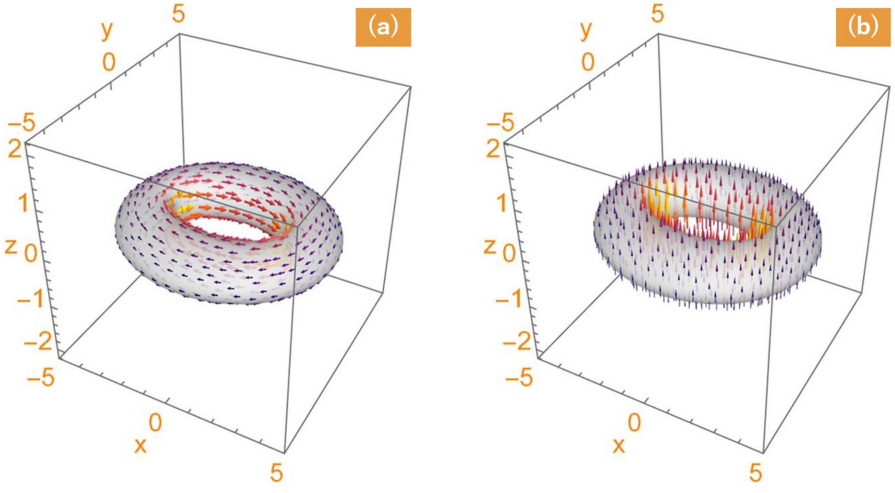


**FIG. 4.** (a) Plot of the magnetic field $\boldsymbol{B} = -e^{-\mu}\nabla\nu$ with translational symmetry $\boldsymbol{u} = \nabla z$ over the surface $\Psi_{el} = 0.1$. Here, $\mu$ and $\nu$ are elliptic cylindrical coordinates as given in (46). This magnetic field is a symmetric solution of anisotropic magnetohydrodynamics (13) with pressure fields (12) in an asymmetric domain whose boundary is a level set of $\Psi_{el}$. (b) Plot of the corresponding current $\nabla \times \boldsymbol{B} = e^{-\mu}|\nabla\mu|^2\nabla z$. In these plots, $\mu_0 = 1$ and $a = 2$.

Let us now consider the existence of self-quasisymmetric magnetic fields (17) within $\Omega$ such that the perpendicular pressure $P_\perp$ is constant on the boundary. Such solution can be obtained by solving (39) with $\Psi_T = \Psi_{el}$ for the Clebsch potential $\Phi$. Assuming that $B \neq 0$, it is convenient to introduce the quantity $\Phi' = \Phi/\sqrt{E(\Psi_{el})}$ so that the equation to be solved becomes

$$|\nabla\Psi_{el} \times \nabla\Phi'|^2 = 1 \quad \text{in } \Omega. \tag{51}$$

Substituting the expression for $\Psi_{el}$, one arrives at the following nonlinear first-order partial differential equation

$$\left(\frac{\partial\Phi'}{\partial\nu}\right)^2\left[z^2 + \frac{(\mu-\mu_0)^2}{a^2\delta}\right] + \left[z\frac{\partial\Phi'}{\partial\mu} - (\mu-\mu_0)\frac{\partial\Phi'}{\partial z}\right]^2 = a^2\delta \quad \text{in } \Omega. \tag{52}$$

To further simplify the equation, it is convenient to introduce the change of variables $(\mu, z) \to (\rho, \theta)$ defined by

$$\mu - \mu_0 = \rho\cos\theta, \quad z = \rho\sin\theta. \tag{53}$$

Note that $\rho^2 = 2\Psi_{el}$. Then, Eq. (52) reads as

$$\left(\frac{\partial\Phi'}{\partial\nu}\right)^2\rho^2\left(\sin^2\theta + \frac{\cos^2\theta}{a^2\delta}\right) + \left(\frac{\partial\Phi'}{\partial\theta}\right)^2 = a^2\delta \quad \text{in } \Omega, \tag{54}$$

where $\delta = \sin^2\nu + \sinh^2(\mu_0 + \rho\cos\theta)$. For given $\rho$, (54) is a nonlinear first-order partial differential equation with two independent variables quadratic in the derivatives on a flux surface $\Psi_{el}$. Indeed, it has the form

$$F(p, q, \nu, \theta) = fp^2 + q^2 - g = 0, \tag{55}$$

where

$$p = \frac{\partial\Phi'}{\partial\nu}, \quad q = \frac{\partial\Phi'}{\partial\theta}, \quad f = \rho^2\left(\sin^2\theta + \frac{\cos^2\theta}{a^2\delta}\right), \quad g = a^2\delta. \tag{56}$$

A local solution of (54) can be found by considering the Cauchy problem for the characteristic system of ordinary differential equations associated with (54) (see, e.g., Ref. 44) To this end, denote with $\xi$ a parameter and assign initial conditions as below:

$$\nu = \nu_0(\xi), \quad \theta = \theta_0(\xi), \quad \Phi' = \Phi'_0(\xi), \quad p = p_0(\xi), \quad q = q_0(\xi), \tag{57}$$

where the functions $\nu_0$, $\theta_0$, $\Phi'_0$, $p_0$, and $q_0$ are determined so that the following non-degeneracy, compatibility, and transversality conditions are fulfilled:

$$\left(\frac{\partial F}{\partial p}\right)^2 + \left(\frac{\partial F}{\partial q}\right)^2 \neq 0, \tag{58a}$$

$$\left(\frac{d\nu_0}{d\xi}\right)^2 + \left(\frac{d\theta_0}{d\xi}\right)^2 \neq 0, \tag{58b}$$

$$F(p_0, q_0, \nu_0, \theta_0) = 0, \tag{58c}$$

$$\frac{d\Phi'_0}{d\xi} = p_0\frac{d\nu_0}{d\xi} + q_0\frac{d\theta_0}{d\xi}, \tag{58d}$$

$$\frac{\partial F}{\partial p}\frac{d\theta_0}{d\xi} - \frac{\partial F}{\partial q}\frac{d\nu_0}{d\xi} \neq 0. \tag{58e}$$

The non-degeneracy conditions (58a) and (58b) ensure that the equation $F = 0$ is nontrivial (it contains first order derivatives of the unknown) and that the initial data are not restricted to a single point. The compatibility conditions (58c) and (58d) guarantee that the initial data are consistent with the equation $F = 0$ and with the requirement that $\Phi' = \Phi'(\nu, \theta)$. Finally, the transversality condition (58e) is needed to avoid characteristic curves on the flux surface so that solutions of the characteristic system of ordinary differential equations leave the curve where initial data are assigned. Equation (58a) is identically satisfied as long as $f$ and $p$, and/or $q$ are different from zero. Indeed,

$$\left(\frac{\partial F}{\partial p}\right)^2 + \left(\frac{\partial F}{\partial q}\right)^2 = 4f^2p^2 + 4q^2 \neq 0. \tag{59}$$

Equation (58c) can be solved for $q_0$. We have

$$q_0^2 = g_0 - f_0p_0^2. \tag{60}$$

Here, $g_0 = g(\nu_0, \theta_0)$ and $f_0 = f(\nu_0, \theta_0)$. The other Eqs. (58b), (58d), and (58e) can be satisfied, for example, by demanding that $\nu_0 = \xi$, $\theta_0 = c_\theta \in \mathbb{R}$, and $p_0 = 0$. In this case (58b), (58d), and (58e) become

$$\left(\frac{d\nu_0}{d\xi}\right)^2 + \left(\frac{d\theta_0}{d\xi}\right)^2 = 1 \neq 0, \tag{61a}$$

$$\frac{d\Phi'_0}{d\xi} = p_0\frac{d\nu_0}{d\xi} + q_0\frac{d\theta_0}{d\xi} = 0, \tag{61b}$$

$$\frac{\partial F}{\partial p}\frac{d\theta_0}{d\xi} - \frac{\partial F}{\partial q}\frac{d\nu_0}{d\xi} = -2q_0 \neq 0. \tag{61c}$$

Hence, a set of consistent initial conditions is

$$\nu_0 = \xi, \quad \theta_0 = c_\theta, \quad \Phi'_0 = c_{\Phi'}, \quad p_0 = 0, \quad q_0 = \sqrt{g_0}. \tag{62}$$

with $c_{\Phi'} \in \mathbb{R}$. This choice implies that the value of $\Phi'$ is prescribed on the toroidal curve $\nu_0 = \xi$, and the solution of the characteristic equations will extend in the poloidal direction $\theta$. Other consistent choices of initial data are possible. For example, one can assign initial data on a poloidal curve as $\nu_0 = c_\nu$, $\theta_0 = \xi$, $\Phi'_0 = c_{\Phi'}$, $p_0 = \sqrt{g_0/f_0}$, and $q_0 = 0$ with $c_\nu \in \mathbb{R}$. Then, solving the characteristic Lagrange–Charpit system[44]

$$\frac{d\nu}{\frac{\partial F}{\partial p}} = \frac{d\theta}{\frac{\partial F}{\partial q}} = \frac{d\Phi'}{p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q}} = -\frac{dp}{\frac{\partial F}{\partial \nu}} = -\frac{dq}{\frac{\partial F}{\partial \theta}} = d\tau, \tag{63}$$

one obtains a parametric solution $\nu = \nu(\xi, \tau)$, $\theta(\xi, \tau)$, and $\Phi'(\xi, \tau)$ in a neighborhood of the line defined by the initial conditions (62). This result implies that, for a given function $E(\Psi_{el}) > 0$, a self-quasisymmetric solution $\boldsymbol{B} = \sqrt{E(\Psi_{el})}\nabla\Psi_{el} \times \nabla\Phi'$ of anisotropic magnetohydrodynamics with constant perpendicular pressure on the boundary can be constructed in a neighborhood $U \subset \Omega$ such that the boundary condition $\boldsymbol{B}\cdot\boldsymbol{n} = 0$ is satisfied on $\partial U \cap \partial\Omega \neq \varnothing$ (note that boundary conditions are satisfied by construction, the limitation on the size of $U$ coming from the locality of the solution $\Phi'$ on each flux surface). As shown by the example above, in the context of anisotropic magnetohydrodynamics, the existence of quasisymmetric solutions ultimately translates into the global consistency of locally obtained solutions, i.e., whether the solution defined in $U$ can be prolonged to cover the whole $\Omega$. In particular, if a set of flux surfaces is assigned, it fails to accommodate a global quasisymmetric solution

because the characteristic curves are not compatible with periodic solutions. This behavior is intrinsic, and it unavoidably leads to the study of the properties of characteristic curves. We, therefore, conjecture that rigorous results concerning the existence of quasisymmetric vector fields cannot be obtained unless the issues of global extension of solutions and existence of periodic solutions of first-order nonlinear partial differential equations are carefully addressed. Finally, we observe that the results discussed in the present section apply to asymmetric tori generated through general orthogonal harmonic coordinates, i.e., they are not restricted to elliptic geometry.

## V. QUASISYMMETRIC MAGNETIC FIELDS IN ASYMMETRIC TORI

The purpose of the present section is to offer constructive examples of quasisymmetric magnetic fields. We will see that these vector fields can be interpreted as local solutions of anisotropic magnetohydrodynamics (13) in an asymmetric torus with perpendicular and parallel pressure fields given by (12). In particular, these solutions hold in a neighborhood $U \subset \Omega$ with $\partial U \cap \partial\Omega \neq \varnothing$, they do not have continuous Euclidean symmetries, no boundary conditions are imposed on the pressure fields, and $\boldsymbol{B} \times \boldsymbol{u} = \nabla\zeta \neq \boldsymbol{0}$. (The fields are not self-quasisymmetric.) To achieve this goal, we solve equation (31) in a neighborhood $U$ of a toroidal domain $\Omega$ whose boundary $\partial\Omega$ corresponds to a level set of a function $\Psi_T$.

First, we prescribe the toroidal domain $\Omega$. Instead of breaking axial symmetry by modifying the distance function $r$ with a new variable $\mu$, it is convenient to deform the torus by introducing a non-zero $h$ in $\Psi_T$ of (37). To simplify calculations, we postulate that $h = h(r, \phi)$. [This assumption implies that the reference axis with respect to which the vertical distance of a point on a flux surface is measured is located at $z = h(r, \phi)$, and that this reference axis is the same for all $z$.] We, thus, have

$$\Psi_T = \frac{1}{2}\left[(r - r_0)^2 + (z - h)^2\right]. \tag{64}$$

If level sets of $\Psi_T$ define toroidal surfaces, we may regard $x^1 = r - r_0$ and $x^2 = z - h$ as generalized radial and vertical coordinates. Let $x^3$ be a third generalized toroidal coordinate. Then, the magnetic field $\boldsymbol{B} = \nabla\Psi_T \times \nabla\Theta$ can be written as

$$\boldsymbol{B} = \left(\frac{\partial\Psi}{\partial x^1}\frac{\partial\Theta}{\partial x^2} - \frac{\partial\Psi}{\partial x^2}\frac{\partial\Theta}{\partial x^1}\right)\nabla x^1 \times \nabla x^2 + \frac{\partial\Theta}{\partial x^3}\nabla\Psi_T \times \nabla x^3. \tag{65}$$

In the following, we look for a toroidal quasisymmetric magnetic field and assume that $\partial\Theta/\partial x^3 = 0$. Then, Eq. (65) becomes

$$\boldsymbol{B} = f(r, z - h)\nabla r \times \nabla(z - h). \tag{66}$$

Here, $f$ is a function of $r$ and $z$–$h$ to be determined together with $h$ by imposing the quasisymmetry condition. Observe that this magnetic field reduces to the axially symmetric magnetic field $-f(r, z)r\nabla\phi$ when $h = 0$. Next, notice that $\boldsymbol{B} \cdot \nabla\Psi_T = 0$ implying $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$. The magnetic field also satisfies $\nabla \cdot \boldsymbol{B} = 0$. Hence, the remaining equations in (13) are

$$\boldsymbol{B} \times \boldsymbol{u} = \frac{dg}{d\zeta}\nabla\zeta, \quad \boldsymbol{u} \cdot \nabla B = 0, \quad \nabla \cdot \boldsymbol{u} = 0 \ \text{ in } \Omega. \tag{67}$$

In the equation above, we have replaced $\zeta$ with a function of $\zeta$, $g = g(\zeta)$, for later convenience. Assuming the Clebsch representation $\boldsymbol{u} = \nabla\zeta \times \nabla\rho$ with $\zeta = z - h$, from (67), we, thus, arrive at (31), which now reads as a system of equations that must be solved for the Clebsch potential $\rho$ and the displacement $h$,

$$\nabla r \times \nabla(z - h) \cdot \nabla\rho = 1, \tag{68a}$$

$$\nabla(z - h) \times \nabla\rho \cdot \nabla\left[\frac{1}{r^2}\left(\frac{\partial h}{\partial\phi}\right)^2\right] = 0. \tag{68b}$$

Observe that in deriving (68a) and (68b), we set $dg/d\zeta = f$ and eliminated the dependence of $f$ on $r$. Next, define the quantity

$$\eta = \frac{1}{r}\frac{\partial h}{\partial\phi}. \tag{69}$$

Then, assuming that $\nabla(z - h)$ and $\nabla(r^{-1}\partial h/\partial\phi)^2$ are linearly independent, Eq. (68b) implies that

$$\rho = \rho(z - h, \eta). \tag{70}$$

Substituting (70) into (68a), we arrive at

$$\frac{\partial\rho}{\partial\eta}\nabla r \times \nabla(z - h) \cdot \nabla\eta = 1, \tag{71}$$

which, after some manipulations, becomes

$$-\frac{\partial\rho}{\partial\eta}\frac{1}{r^2}\frac{\partial^2 h}{\partial\phi^2} = 1. \tag{72}$$

Since $h$ does not depend on $z$, the equation above implies that $\rho = \rho_1 + \rho_2$ for some functions $\rho_1 = \rho_1(\eta)$ and $\rho_2 = \rho_2(z - h)$. However, by construction $\boldsymbol{u} = \nabla(z - h) \times \nabla\rho$, implying that $\rho_2$ does not contribute to the representation of $\boldsymbol{u}$. We can, therefore, simply put $\rho = \rho(\eta)$. Then, for given $\rho(\eta)$, Eq. (72) must be solved for $h$. Now recall that $\partial h/\partial\phi = r\eta$. Therefore, Eq. (71) is equivalent to

$$\frac{\partial\rho}{\partial\phi} = -r. \tag{73}$$

Assuming that the function $\rho = \rho(\eta)$ can be inverted with inverse $\rho^{-1}$ and integrating with respect to $\phi$ gives

$$\frac{\partial h}{\partial\phi} = r\rho^{-1}(\alpha - r\phi), \tag{74}$$

where $\alpha = \alpha(r)$ is an arbitrary integration factor. Once $\rho^{-1}$ is assigned, a further integration with respect to $\phi$ gives the desired solution. However, due to the radial dependence of $\alpha - r\phi$, the quantity $\partial h/\partial\phi$ is not periodic in $\phi$. Therefore, the components of the corresponding quasisymmetric magnetic field (66) and the flux function (64) become multivalued functions, and the obtained quasisymmetric solution is well-defined only locally. In particular, the solution is defined in some region $U \subset \Omega$, and the irregularities of the bounding surface $\partial\Omega$ arising from the multivalued nature of $\Psi_T$ can be removed by repairing (smoothing) the flux function $\Psi_T$ outside $U$. To see this, set $\rho(\eta) = k^{-1}\eta$, with $k > 0$ a real constant, and choose $f = \sin(z - h)$. Then, we have

$$\frac{\partial h}{\partial \phi} = kr(\alpha - r\phi), \quad h = kr\phi\left(\alpha - \frac{1}{2}r\phi\right) + \beta, \tag{75}$$

with $\beta = \beta(r)$ an arbitrary integration factor, and also

$$\boldsymbol{B} = -\sin\left[z - \beta - kr\phi\left(\alpha - \frac{1}{2}r\phi\right)\right][r\nabla\phi + k(\alpha - r\phi)\nabla z], \tag{76a}$$

$$\boldsymbol{u} = r\left(\frac{\partial \alpha}{\partial r} - \phi\right)\nabla\phi + \nabla r + \frac{\partial}{\partial r}\left(k\frac{\alpha^2}{2} + \beta\right)\nabla z. \tag{76b}$$

For the component of the magnetic field to be single-valued, one must determine $\alpha$ and $\beta$ so that

$$\boldsymbol{B}|_{\phi=0} = \boldsymbol{B}|_{\phi=2\pi}, \tag{77}$$

which implies

$$z - \beta = z - \beta - 2\pi kr(\alpha - \pi r) + 2\pi m, \quad m \in \mathbb{Z}, \tag{78a}$$

$$k\alpha = k(\alpha - 2\pi r). \tag{78b}$$

However, excluding the case $k = 0$, which corresponds to axial symmetry, these conditions cannot be satisfied. Hence, the local nature of the solution.

As an example of local quasisymmetric configuration, consider the case $\alpha = \beta = 0$. Then, one can verify the quasisymmetry of the derived solution as below:

$$\boldsymbol{B} = -\sin\left(z + \frac{1}{2}kr^2\phi^2\right)(r\nabla\phi - kr\phi\nabla z), \tag{79a}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{79b}$$

$$B^2 = \sin^2\left(z + \frac{1}{2}kr^2\phi^2\right)\left(1 + k^2r^2\phi^2\right), \tag{79c}$$

$$\boldsymbol{u} = \nabla r - r\phi\nabla\phi, \tag{79d}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{79e}$$

$$\boldsymbol{B} \times \boldsymbol{u} = -\nabla\cos\left(z + \frac{1}{2}kr^2\phi^2\right), \tag{79f}$$

$$\boldsymbol{u} \cdot \nabla B^2 = 0, \tag{79g}$$

$$\boldsymbol{B} \cdot \nabla\Psi_T = 0. \tag{79h}$$

We also remark that this solution does not have continuous Euclidean symmetries because the equation $\mathfrak{L}_u \boldsymbol{B} = \boldsymbol{0}$ with $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$ does not have solutions with $\boldsymbol{a}, \boldsymbol{b} \neq \boldsymbol{0}$, and that, it is discontinuous

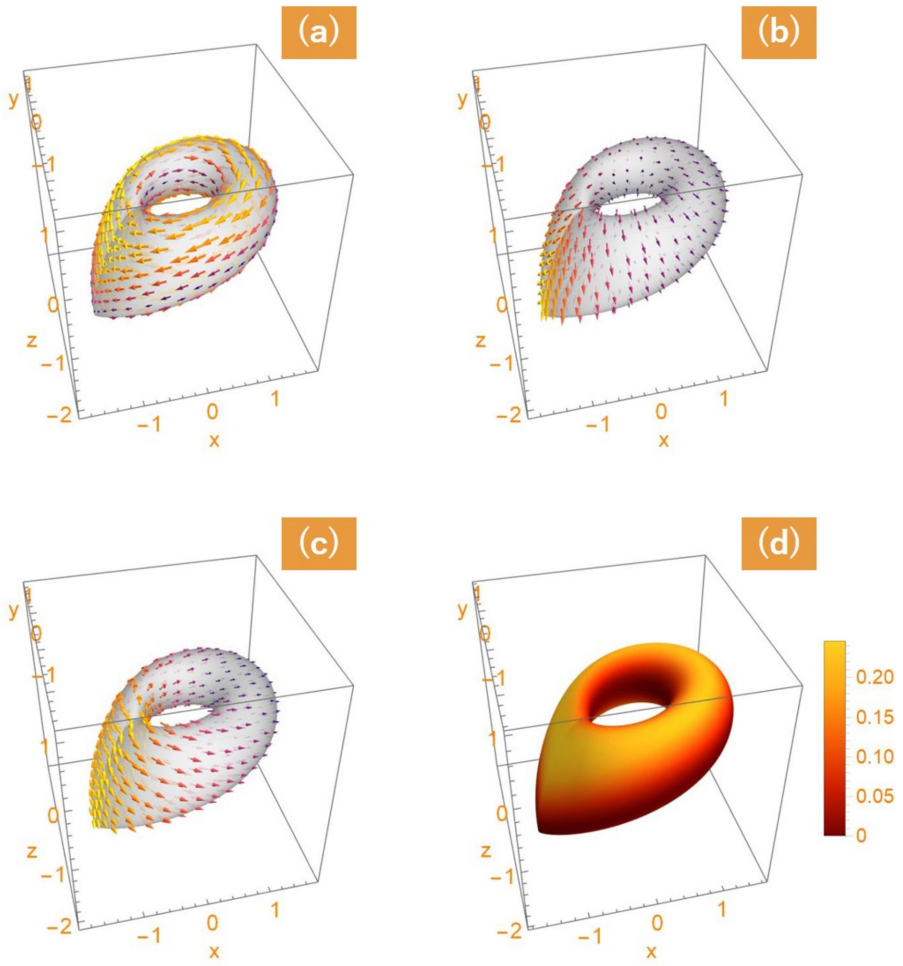


**FIG. 5.** (a) Plot of the quasisymmetric magnetic field $\boldsymbol{B}$ of (79a) on a level set of (64). (b) Plot of the current $\nabla \times \boldsymbol{B}$. (c) Plot of the quasisymmetry $\boldsymbol{u}$. (d) Plot of the field strength $B^2$. Observe that $\boldsymbol{B}$, $\nabla \times \boldsymbol{B}$, $\boldsymbol{u}$, and $\Psi_T$ exhibit a discontinuity at $\phi = 0$ due to the multivalued nature of $\phi$. In these plots, $k = 0.18$.

(non-periodic) at $\phi = 0$. A plot of the obtained magnetic field and related quantities is given in Fig. 5.

The quasisymmetric magnetic field of Eq. (79) is such that the quasisymmetry $\boldsymbol{u}$ is not tangential to flux surfaces $\Psi_T$ since $\boldsymbol{u} \cdot \nabla \Psi_T \neq 0$. However, in the design of a stellarator, it is customary to demand that $\boldsymbol{u}$ lies on flux surfaces to ensure that the conserved momentum associated with guiding center motion is well approximated by the flux function. This requirement amounts to identifying $\zeta$ with the flux function $\Psi_T$ in quasisymmetry equation (4). Such configuration can be achieved by repeating the procedure discussed above while enforcing the Clebsch representations

$$\boldsymbol{B} = f(\Psi_T)\nabla r \times \nabla(z-h), \quad \boldsymbol{u} = \nabla\Psi_T \times \nabla\rho, \tag{80}$$

where $f$ is an arbitrary function of $\Psi_T$. In this case, system (31) reduces to

$$\nabla r \times \nabla(z-h) \cdot \nabla\rho = 1, \tag{81a}$$

$$\nabla\Psi_T \times \nabla\rho \cdot \nabla\left[\frac{1}{r^2}\left(\frac{\partial h}{\partial \phi}\right)^2\right] = 0. \tag{81b}$$

Here, we set $dg/d\zeta = dg/d\Psi_T = f$. Again, defining $\eta = r^{-1}\partial h/\partial\phi$ and requiring that $\rho = \rho(\eta)$, one arrives at Eq. (72) with solution (74).

For example, if $\rho = k^{-1}\eta$ and $\alpha = \beta = 0$, one obtains the family of quasisymmetry magnetic fields

$$\boldsymbol{B} = -f(r\nabla\phi - kr\phi\nabla z), \tag{82a}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{82b}$$

$$B^2 = f^2\left(1 + k^2r^2\phi^2\right), \tag{82c}$$

$$\boldsymbol{u} = -(r - r_0)\nabla z + \left(z + \frac{1}{2}kr^2\phi^2\right)(\nabla r - r\phi\nabla\phi), \tag{82d}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{82e}$$

$$\boldsymbol{B} \times \boldsymbol{u} = f\nabla\Psi_T = \nabla g, \tag{82f}$$

$$\boldsymbol{u} \cdot \nabla B^2 = 0, \tag{82g}$$

$$\boldsymbol{B} \cdot \nabla\Psi_T = 0, \tag{82h}$$

$$\boldsymbol{u} \cdot \nabla\Psi_T = 0. \tag{82i}$$

Observe that both $\boldsymbol{B}$ and $\boldsymbol{u}$ are tangential to flux surfaces. A plot of this solution for $f = -\sin(\Psi_T)$ is given in Fig. 6.

The construction of quasisymmetric magnetic fields presented above can be generalized to the case of surfaces defined through harmonic orthogonal coordinates $(\mu, \nu, z)$. Setting $h = h(\mu, \nu)$, the relevant Clebsch parameterization is

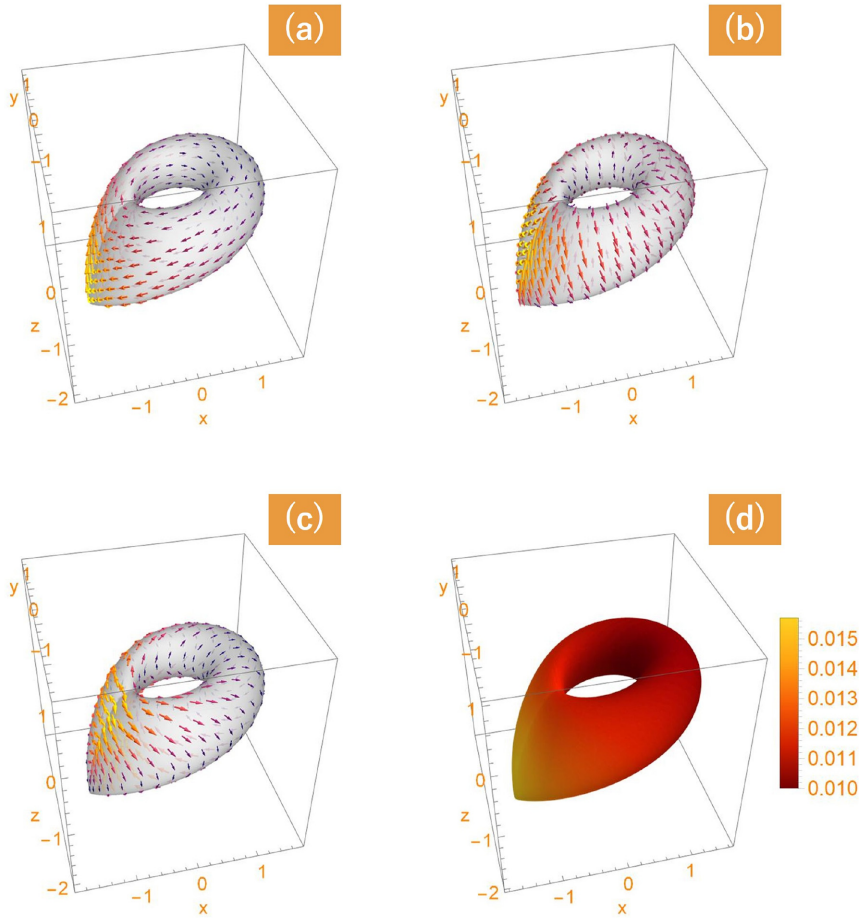


**FIG. 6.** (a) Plot of the quasisymmetric magnetic field $\boldsymbol{B}$ of (82) for $f = -\sin(\Psi_T)$ on a level set of (64). (b) Plot of the current $\nabla \times \boldsymbol{B}$. (c) Plot of the quasisymmetry $\boldsymbol{u}$. (d) Plot of the field strength $B^2$. Observe that $\boldsymbol{B}$, $\nabla \times \boldsymbol{B}$, $\boldsymbol{u}$, and $\Psi_T$ exhibit a discontinuity at $\phi = 0$ due to the multivalued nature of $\phi$. In these plots, $k = 0.18$.

$$\Psi_T = \frac{1}{2}\left[(\mu-\mu_0)^2 + (z-h)^2\right], \qquad (83)$$
$$\boldsymbol{B} = f(\Psi_T)\nabla\mu \times \nabla(z-h), \quad \boldsymbol{u} = \nabla\Psi_T \times \nabla\rho.$$

Then, putting $dg/d\Psi_T = f$, quasisymmetry equation (31) now read

$$\nabla\Psi_T \times \nabla\rho \cdot \nabla\left\{|\nabla\mu|^2\left[1 + |\nabla\mu|^2\left(\frac{\partial h}{\partial \nu}\right)^2\right]\right\} = 0, \qquad (84a)$$
$$\nabla\mu \times \nabla(z-h) \cdot \nabla\rho = 1. \qquad (84b)$$

Again, requiring that $\rho = \rho(\eta)$ with

$$\eta = |\nabla\mu|^2\left[1 + |\nabla\mu|^2\left(\frac{\partial h}{\partial \nu}\right)^2\right], \qquad (85)$$

we arrive at one equation, which determines the displacement $h$:

$$-\frac{d\rho}{d\eta}\frac{\partial\eta}{\partial\nu}|\nabla\mu|^2 = 1. \qquad (86)$$

For a given $\rho$, the corresponding solution $h$ of the equation above assigns a family of quasisymmetric magnetic fields parametrized by the function $f$ as defined in equation (83). Nonetheless, as in the previous cases, the globality of the solution is not guaranteed.

## VI. CONCLUDING REMARKS

In this work, we studied the existence of quasisymmetric magnetic fields in asymmetric toroidal domains through particular constructive examples. If the perpendicular and parallel pressure fields are chosen as in Eq. (12), this problem is equivalent to finding quasisymmetric magnetohydrodynamic equilibria within the framework of anisotropic magentohydrodynamics. In particular, constructing a quasisymmetric configuration amounts to solving system (13) for the vector fields $\boldsymbol{B}$ and $\boldsymbol{u}$. This system can be written in terms of two coupled nonlinear first-order partial differential equations (31), or a single nonlinear second-order partial differential equation (32).

In the approach developed in the present study, the flux function was given as input to the derived equations in order to enforce the toroidal shape of the boundary and the existence of nested flux surfaces. By using harmonic orthogonal coordinates, we first devised a method to obtain symmetric solutions of (31) in asymmetric toroidal domains. Then, we obtained regular local self-quasisymmetric configurations by fixing the level sets of the flux function to elliptic tori, and by reducing the self-quasisymmetry equation (23) to a nonlinear first-order partial differential equation with two independent variables on each flux surface. Finally, we constructed regular local quasisymmetric magnetic fields in asymmetric tori by adjusting the shape of flux surfaces to satisfy the quasisymmetry condition (31), which was reduced to a single equation for the displacement $h$. These solutions are local in the sense that they solve (31) in a region $U \subset \Omega$ and satisfy boundary conditions on a portion of the boundary, $\partial U \cap \partial\Omega \neq \varnothing$. In particular, they are not periodic in the toroidal angle $\phi$.

The obtained results highlight two aspects that characterize quasisymmetric vector fields. On the one hand, the symmetry of the boundary should be treated separately from the symmetry of the solution, since we have shown that a symmetric magnetic field can be fitted within an asymmetric domain. On the other hand, due to the first-order nature of the governing equations, the mathematical challenge posed by quasisymmetry can be ascribed to the local nature of the solutions obtained by integrating the characteristic system of ordinary differential equations. In particular, for a given set of flux surfaces, a global quasisymmetric configuration can be achieved only if characteristic curves are compatible with periodic solutions in the toroidal and poloidal angles. A possible strategy to obtain periodic solutions is to allow additional freedom in the shape of flux surfaces by considering more general displacements $\mu_0$ and $h$, and to generalize the ansatz (83) on the Clebsch form of the magnetic field. This work also suggests that the analysis of existence and regularity of periodic solutions to quadratic first order partial differential equations may provide the conditions on the shape of flux surfaces that allow global quasisymmetry. Finally, we note that the possibility that the obstruction encountered in the derivation of global solutions is intrinsic to three-dimensional equilibria cannot be ruled out, in the sense that the absence of Euclidean isometry may prevent the existence of such fields.

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## AUTHOR DECLARATIONS

### Conflict of Interest

The authors have no conflicts to disclose.

## DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

## APPENDIX: CHARACTERISTIC SURFACES OF ISOTROPIC IDEAL MAGNETOHYDRODYNAMIC EQUILIBRIA

In this appendix, we review the derivation of the characteristic surfaces associated with isotropic ideal magnetohydrodynamic equilibria (1a). If such surfaces do not exist, a first-order system of partial differential equations is elliptic and the value of the solution $u$ on any given surface $S$ always determines the normal derivative of $u$ on $S$ through the governing differential operator $L$. When characteristic surfaces are present, this property is broken, and hyperbolicity is introduced into the equations. First, observe that the nonlinear first-order system (1a) can be written in the form

$$Lu = \sum_{i=1}^{3} A^i \frac{\partial u}{\partial x^i} = 0, \qquad (A1)$$

where $u = \{B_x, B_y, B_z, P\}$ denotes the unknown variables and the $A^i(\boldsymbol{x}, u)$, $i = 1, 2, 3$, are $4 \times 4$ matrices with expressions

$$A^1 = \begin{bmatrix} 0 & -B_y & -B_z & -1 \\ 0 & B_x & 0 & 0 \\ 0 & 0 & B_x & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} B_y & 0 & 0 & 0 \\ -B_x & 0 & -B_z & -1 \\ 0 & 0 & B_y & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$
$$A^3 = \begin{bmatrix} B_z & 0 & 0 & 0 \\ 0 & B_z & 0 & 0 \\ -B_x & -B_y & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \qquad (A2)$$

The characteristic surfaces $S$ associated with (1a) are determined as solutions of the characteristic equation

$$\det\left(\sum_{i=1}^{3} A^i \frac{\partial S}{\partial x^i}\right) = (\nabla S)^2 (\nabla S \cdot \boldsymbol{B})^2 = 0. \tag{A3}$$

The quadratic term $(\nabla S)^2$ corresponds to a trivial characteristic surface $\nabla S = \mathbf{0}$ with double multiplicity, implying that system (1a) has a twice elliptic component. On the other hand, the term $(\nabla S \cdot \boldsymbol{B})^2$ results in the nontrivial characteristic surface $\nabla S \cdot \boldsymbol{B} = 0$. The multiplicity of this surface is again two, and system (1a), therefore, qualifies as twice elliptic and twice hyperbolic. Finally, notice that the characteristic surface $\nabla S \cdot \boldsymbol{B} = 0$ depends on the unknown $\boldsymbol{B}$, a fact that is related to the mathematical difficulty encountered in developing a general theory for the existence of solutions of the tangential boundary value problem (1).

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