Local Hardy Spaces and Quadratic Estimates for Dirac Type Operators on Riemannian Manifolds

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Declaration

I hereby declare that except where otherwise indicated the work in this thesis is my own. The material in Chapter 2 is from my published paper [56] entitled "Local quadratic estimates and holomorphic functional calculi." The material in Chapter 3 is from my collaboration [21] with Andrea Carbonaro and Alan McIntosh that has been submitted under the title "Local Hardy spaces of differential forms on Riemannian manifolds." I intend to submit the material in Chapter 4 as a separate paper.

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Abstract

The connection between quadratic estimates and the existence of a bounded holomorphic functional calculus of an operator provides a framework for applying harmonic analysis to the theory of differential operators. This is a generalization of the connection between Littlewood–Paley–Stein estimates and the functional calculus provided by the Fourier transform. We use the former approach in this thesis to study first-order differential operators on Riemannian manifolds. The theory developed is local in the sense that it does not depend on the spectrum of the operator in a neighbourhood of the origin. When we apply harmonic analysis to obtain estimates, the local theory only requires that we do so up to a finite scale. This allows us to consider manifolds with exponential volume growth in situations where the global theory requires polynomial volume growth.

A holomorphic functional calculus is constructed for operators on a reflexive Banach space that are bisectorial except possibly in a neighbourhood of the origin. We prove that this functional calculus is bounded if and only if certain local quadratic estimates hold. For operators with spectrum in a neighbourhood of the origin, the results are weaker than those for bisectorial operators. For operators with a spectral gap in a neighbourhood of the origin, the results are stronger. In each case, however, local quadratic estimates are a more appropriate tool than standard quadratic estimates for establishing that the functional calculus is bounded.

This theory allows us to define local Hardy spaces of differential forms that are adapted to a class of first-order differential operators on a complete Riemannian manifold with at most exponential volume growth. The local geometric Riesz transform associated with the Hodge–Dirac operator is bounded on these spaces provided that a certain condition on the exponential growth of the manifold is satisfied. A characterisation of these spaces in terms of local molecules is also obtained. These results can be viewed as the localisation of those for the Hardy spaces of differential forms introduced by Auscher, McIntosh and Russ.

Finally, we introduce a class of first-order differential operators that act on the trivial bundle over a complete Riemannian manifold with at most exponential volume growth and on which a local Poincaré inequality holds. A local quadratic estimate is established for certain perturbations of these operators. As an application, we solve the Kato square root problem for divergence form operators on complete Riemannian manifolds with Ricci curvature bounded below that are embedded in Euclidean space with a uniformly bounded second fundamental form. This is based on the framework for Dirac type operators that was introduced by Axelsson, Keith and McIntosh.

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