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Calculus of Variations

Estimates and existence results for a fully nonlinear Yamabe problem on manifolds with boundary

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Abstract In this paper we consider a fully nonlinear version of the Yamabe problem on compact Riemannian manifold with boundary. Under various conditions we derive local estimates for solutions and establish some existence results.

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1 Introduction

Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold of dimension $n \geq 3$. The Yamabe problem is to find metrics conformal to g with constant scalar curvature. The problem has been solved through the work of Yamabe [50], Trudinger [42], Aubin [2] and Schoen [38]. See [25] for a survey. The Yamabe and related problems have attracted much attention in the last 30 years or so, see, e.g., [40], [3] and the references therein. Analogues of the Yamabe problem on compact manifolds $(\mathcal{M}, g)$ with boundary have been studied by Cherrier [9], Escobar [11–13], Han and Li [23,24], Ambrosetti et al. [1] and others.

In recent years, fully nonlinear versions of the Yamabe problem have received much attention since the work of Viaclovsky [48]. Let $(\mathcal{M}^n, g)$ be a smooth compact Riemannian manifold of dimension $n \geq 3$. The Schouten tensor is defined as

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$$A_g := \frac{1}{n-2}\left(Ric_g - \frac{1}{2(n-1)}R_g g\right),$$

where Ric_g and R_g are the Ricci tensor and the scalar curvature of g respectively. Let $\lambda(A_g) = (\lambda_1(A_g), \ldots, \lambda_n(A_g))$ denote the eigenvalues of A_g with respect to g. One interesting problem is to find conformal metrics on $(\mathcal{M}, g)$ with a prescribed symmetric function of the eigenvalues of the Schouten tensors.

To be more precise, let

$$\Gamma_1 := \left\{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \sum \lambda_i > 0\right\}$$

and

$$\Gamma_n := \left\{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \lambda_i > 0 \text{ for } 1 \le i \le n\right\}.$$

We assume that

$$\Gamma \subset \mathbb{R}^n \text{ is an open convex symmetric cone with vertex at the origin} \tag{1.1}$$

satisfying

$$\Gamma_n \subset \Gamma \subset \Gamma_1 \tag{1.2}$$

and assume that

$$f \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma}) \text{ is a symmetric function}, \quad f > 0 \text{ in } \Gamma \tag{1.3}$$

verifying some of the following properties which will be specified in each situation:

$$f \text{ is homogeneous of degree one on } \Gamma, \tag{1.4}$$

$$f_i := \frac{\partial f}{\partial \lambda_i} > 0 \quad \text{on } \Gamma, \tag{1.5}$$

and

$$f \text{ is concave on } \Gamma. \tag{1.6}$$

Conditions (1.4), (1.5) and (1.6) imply that (see [45, Lemma 3.2])

$$\mathcal{T} := \sum_i f_i(\lambda) \ge c_0 > 0, \quad \forall \lambda \in \Gamma \tag{1.7}$$

for some positive number $c_0 > 0$.

The fully nonlinear Yamabe problem on a closed manifold $(\mathcal{M}, g)$ is to find a metric $\tilde{g}$ conformal to g such that

$$F(A_{\tilde{g}}) := f(\lambda(A_{\tilde{g}})) = 1 \quad \text{and} \quad \lambda(A_{\tilde{g}}) \in \Gamma \qquad \text{on } \mathcal{M}. \tag{1.8}$$

When $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$, this problem is known as the σ_k-Yamabe problem in the literature, where, for each $1 \le k \le n$, σ_k is the k-th elementary symmetric function defined by

$$\sigma_k(\lambda) := \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k} \quad \text{for all } \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$$

and

$$\Gamma_k := \{\lambda \in \mathbb{R}^n : \sigma_l(\lambda) > 0, 1 \le l \le k\}$$

which is an open convex symmetric cone with vertex at the origin.

For problem (1.8) a number of existence results have been available in the literature. In [47] Viaclovsky established the existence result for (1.8) with $(f, \Gamma) = (\sigma_n^{1/n}, \Gamma_n)$ for a class of manifolds. In [5,6], Chang et al. obtained the existence result on 4-manifolds for (1.8) with $(f, \Gamma) = (\sigma_2^{1/2}, \Gamma_2)$. For $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ with $k = 3, 4$ on 4-manifolds and with $k = 2, 3$ on 3-manifolds that are not simply connected, the existence result was established by Gursky and Viaclovsky in [21]. When $(\mathcal{M}, g)$ is locally conformally flat and $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ for $1 \leq k \leq n$, Guan and Wang [18] and Li and Li [26] independently proved the existence of solutions of (1.8). Li and Li [26,29] also established a general result that (1.8) is still solvable if $(\mathcal{M}, g)$ is locally conformally flat and if (f, Γ) satisfies (1.1)–(1.6) with $f|_{\partial\Gamma} = 0$. In [19] Guan and Wang proved local interior C^1 and C^2 estimates for solutions of (1.8) with $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$, such estimates were also studied in [20,26,41] and was extended to a general class of (f, Γ) in [7]. This latter result, where concavity of f is assumed, can be deduced as a corollary of the results in [26] and [32]. Local interior C^1 estimates for a general class of (f, Γ), without the concavity assumption on f, are established in [33]. Using such local estimates and the algebraic fact found in [17] that $\lambda(A_g) \in \Gamma_k$ for $k > \frac{n}{2}$ implies the positivity of the Ricci tensor, Gursky and Viaclovsky [22] solved (1.8) on general manifolds if (f, Γ) satisfies (1.1)–(1.6) with $f|_{\partial\Gamma} = 0$ and if $\Gamma \subset \Gamma_k$ for some $k > \frac{n}{2}$. In [44], Trudinger and Wang proved a Harnack inequality for the set of metrics $\tilde{g}$ conformal to g with $\lambda(A_{\tilde{g}}) \in \Gamma_k$ for some $k > \frac{n}{2}$ and gave a different proof of the existence result in [22]. For $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ with $k \leq \frac{n}{2}$ on general manifolds, Sheng et al. [41] established the existence result under a variational structure condition which includes the case $k = 2$ and $n \geq 4$, while Ge and Wang [14] independently obtained a proof for $k = 2$ and $n > 8$.

In the rest of this paper we will assume that $(\mathcal{M}^n, g)$, $n \geq 3$, is a smooth compact Riemannian manifold with nonempty smooth boundary $\partial\mathcal{M}$. For a given constant $c \in \mathbb{R}$, we are interested in finding a metric $\tilde{g}$ conformal to g such that

$$\begin{cases} F(A_{\tilde{g}}) := f(\lambda(A_{\tilde{g}})) = 1, & \lambda(A_{\tilde{g}}) \in \Gamma \ \text{ on } \mathcal{M}, \\ h_{\tilde{g}} = c & \text{on } \partial\mathcal{M}, \end{cases} \tag{1.9}$$

where $h_{\tilde{g}}$ denotes the mean curvature of $\partial\mathcal{M}$ with respect to the outer normal (A Euclidean ball has positive boundary mean curvature). Note that when $(f, \Gamma) = (\sigma_1, \Gamma_1)$, this is the Yamabe problem on a compact manifold with boundary. Therefore (1.9) is a fully nonlinear version of the Yamabe problem with boundary.

This problem was proposed by Li and Li in [28,30] in which they considered the corresponding blow-up problem of (1.9) and obtained some Liouville type theorems and a Harnack type inequality. These results indicate positively that it should be possible to establish some existence results for (1.9) under suitable conditions.

Writing $\tilde{g} = e^{-2u}g$ for some smooth function u on $\mathcal{M}$. Using the transformation laws for the Schouten tensor and mean curvature, (1.9) is equivalent to the fully nonlinear elliptic equations

$$\begin{cases} F(U) := f(\lambda_g(U)) = e^{-2u}, & \lambda_g(U) \in \Gamma \ \text{ on } \mathcal{M}, \\ \frac{\partial u}{\partial \nu} = ce^{-u} - h_g & \text{on } \partial\mathcal{M} \end{cases} \tag{1.10}$$

with

$$U = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|_g^2 g + A_g, \tag{1.11}$$

where $\lambda_g(U)$ denotes the eigenvalues of U with respect to g, ν is the unit inward normal vector field to $\partial\mathcal{M}$ in $(\mathcal{M}, g)$ and ∇ denotes the Levi-Civita connection with respect to g.

Recall that the second fundamental form Π of $\partial\mathcal{M}$ with respect to g is defined as

$$\Pi(X, Y) := -g(\nabla_X \nu, Y), \quad \text{for any } X, Y \in T(\partial\mathcal{M}),$$

where $T(\partial\mathcal{M})$ denotes the tangent bundle over $\partial\mathcal{M}$. A point $x \in \partial\mathcal{M}$ is called an umbilic point in $(\mathcal{M}, g)$ if

$$\Pi(X, Y) = h_g(x) g(X, Y) \quad \text{for all } X, Y \in T_x(\partial\mathcal{M}).$$

The boundary $\partial\mathcal{M}$ is called umbilic if every point of $\partial\mathcal{M}$ is an umbilic point. The notion of umbilic point is conformally invariant, i.e. a point is umbilic with respect to g is still umbilic with respect to the metric $\tilde{g} := e^{-2u} g$ for any function $u \in C^2(\mathcal{M})$ (see [12]).

Our first existence result is for locally conformally flat manifolds with umbilic boundary.

Theorem 1.1 *Assume that* (f, Γ) *satisfies* (1.1)–(1.6) *with* $f|_{\partial\Gamma} = 0$ *and that* $(\mathcal{M}, g)$ *is a smooth compact locally conformally flat Riemannian manifold with smooth umbilic boundary* $\partial\mathcal{M}$. *Suppose that* $\lambda(A_g) \in \Gamma$ *on* $\mathcal{M}$ *and* $h_g \geq 0$ *on* $\partial\mathcal{M}$. *Then problem* (1.10) *with* $c = 0$ *has a solution* $u \in C^\infty(\mathcal{M})$.

The existence of solutions of (1.8) with (f, Γ) satisfying (1.1)–(1.6) and $f|_{\partial\Gamma} = 0$ has been proved in [27,29] on compact locally conformally flat manifolds without boundary. The proof of Theorem 1.1 is based on [27,29]. By making use of the double of a compact manifold, the problem in Theorem 1.1 reduces to a corresponding problem on compact locally conformally flat manifold without boundary. In order to establish C^0 estimates, via the Harnack inequality obtained in [27], we need to assume $h_g \geq 0$ on $\partial\mathcal{M}$.

Recall that the Yamabe problem on compact manifolds $(\mathcal{M}, g)$ with boundary $\partial\mathcal{M}$ is to find a conformally related metric $\tilde{g}$ of constant scalar curvature on $\mathcal{M}$ and constant mean curvature on $\partial\mathcal{M}$. By writing $\tilde{g} = u^{\frac{4}{n-2}} g$ for some positive smooth function on $\mathcal{M}$, this problem is equivalent to finding a smooth positive solution u to the boundary value problem

$$\begin{cases} -\dfrac{4(n-1)}{n-2} \Delta_g u + R_g u = a u^{\frac{n+2}{n-2}} & \text{on } \mathcal{M}, \\ -\dfrac{2}{n-2} \dfrac{\partial u}{\partial \nu} + h_g u = c u^{\frac{n}{n-2}} & \text{on } \partial\mathcal{M}, \end{cases} \tag{1.12}$$

where a and c are constants. For any $a > 0$ and any c, the existence of a solution of (1.12) has been proved in [23,24] under the assumption that $(\mathcal{M}, g)$ is of positive type and satisfies one of the following assumptions:

(i) $(\mathcal{M}^n, g)$, $n \geq 3$, is locally conformally flat with umbilic boundary;
(ii) $n \geq 5$ and $\partial\mathcal{M}$ is not umbilic.

For any $a > 0$, it was proved earlier in [12,13] that (1.12) is solvable for $c = 0$ and for at least one $c_+(a) > 0$ and one $c_-(a) < 0$ under the same assumptions. Here we call a manifold $(\mathcal{M}, g)$ of positive type if

$$\lambda_1(\mathcal{M}) := \min_{\varphi \in H^1(\mathcal{M})\setminus\{0\}} \frac{\int_{\mathcal{M}} \left(|\nabla\varphi|_g^2 + c(n)R_g\varphi^2\right) + \frac{n-2}{2}\int_{\partial\mathcal{M}} h_g\varphi^2}{\int_{\mathcal{M}}\varphi^2} > 0,$$

where $c(n) = \frac{n-2}{4(n-1)}$.

An interesting question for (1.9) is to identify good conditions which guarantee the existence of a solution. Theorem 1.1 is such an attempt, and it shows that $h_g \geq 0$ on $\partial\mathcal{M}$ is a sufficient condition. Unlike the Yamabe problem with boundary, we tend to believe that the hypothesis "$h_g \geq 0$ on $\partial\mathcal{M}$" in Theorem 1.1 can not be replaced by "$\lambda_1(\mathcal{M}) > 0$".

Our next result concerns (1.10) with $c > 0$. We consider more general equation

$$\begin{cases} F(U) := f(\lambda_g(U)) = \varphi_0 e^{-2u}, & \lambda_g(U) \in \Gamma \ \text{ on } \mathcal{M}, \\ \frac{\partial u}{\partial \nu} = h_0 e^{-u} - h_g & \text{on } \partial\mathcal{M}, \end{cases} \tag{1.13}$$

where $\varphi_0 \in C^\infty(\mathcal{M})$ and $h_0 \in C^\infty(\partial\mathcal{M})$ are positive functions. This problem is equivalent to finding a metric $\tilde{g}$ conformal to g such that $f(\lambda(A_{\tilde{g}})) = \varphi_0$ on $\mathcal{M}$ and $h_{\tilde{g}} = h_0$ on $\partial\mathcal{M}$.

Theorem 1.2 *Assume that* (f, Γ) *satisfies* (1.1)–(1.6) *with* $f|_{\partial\Gamma} = 0$ *and* $\Gamma \subset \Gamma_k$ *for some* $k > \frac{n}{2}$. *Let* $(\mathcal{M}, g)$ *be a smooth compact Riemannian manifold with smooth boundary* $\partial\mathcal{M}$. *Suppose that* $\lambda(A_g) \in \Gamma$ *on* $\mathcal{M}$, $h_g \geq 0$ *on* $\partial\mathcal{M}$, $\partial\mathcal{M}$ *is umbilic, and* $(\mathcal{M}, g)$ *is locally conformally flat near* $\partial\mathcal{M}$. *Then for any positive functions* $\varphi_0 \in C^\infty(\mathcal{M})$ *and* $h_0 \in C^\infty(\partial\mathcal{M})$ *problem* (1.13) *has a solution* $u \in C^\infty(\mathcal{M})$.

In [22,44] more general equations than (1.8) on general closed manifolds have been solved when $\Gamma \subset \Gamma_k$ for some $k > \frac{n}{2}$. By using the double of a manifold we adapt the Harnack inequality of Trudinger and Wang [44] to our situation. This, together with the C^1 and C^2 estimates in Sects. 2–4, allows us to obtain the existence result by modifying the degree argument in [44].

In Sects. 2, 3 we establish under suitable conditions on (f, Γ) some local C^1 and C^2 estimates for solutions of the following more general equation

$$\begin{cases} F(U) := f(\lambda_g(U)) = \psi(x, u), & \lambda_g(U) \in \Gamma \ \text{ on } \mathcal{O}_1, \\ \frac{\partial u}{\partial \nu} = \eta(x, u) - h_g & \text{on } \mathcal{O}_1 \cap \partial\mathcal{M}, \end{cases} \tag{1.14}$$

where $\mathcal{O}_1$ is an open set of $\mathcal{M}$, U is defined by (1.11), $\psi \in C^2(\mathcal{O}_1 \times \mathbb{R})$ and $\eta \in C^2((\mathcal{O}_1 \cap \partial\mathcal{M}) \times \mathbb{R})$. By extension we can always assume that $\eta \in C^2(\mathcal{O}_1 \times \mathbb{R})$.

C^1 and C^2 estimates have been studied extensively on closed manifolds, see [7, 19,20,26,41] for local interior estimates and [47] for global estimates. Global estimates have also been studied in [16] on compact manifolds under Dirichlet boundary condition.

The first result on gradient estimates is the following.

Theorem 1.3 *Assume that* (f, Γ) *satisfies* (1.1)–(1.5), (1.7) *and that* $(\mathcal{M}, g)$ *is a smooth compact Riemannian manifold with smooth boundary* $\partial\mathcal{M}$. *Let* $\mathcal{O}_1$ *be an open set of* $\mathcal{M}$ *and let* $u \in C^3(\mathcal{O}_1)$ *be a solution of* (1.14). *If*

$$a \leq u \leq b \quad \text{on } \mathcal{O}_1 \tag{1.15}$$

for some constants a and b, then, for any open set $\mathcal{O}_2$ *of* $\mathcal{M}$ *satisfying* $\overline{\mathcal{O}}_2 \subset \mathcal{O}_1$,

$$|\nabla u|_g \leq C \quad \text{on } \mathcal{O}_2$$

for some positive constant C depending only on n, (f, Γ), g, ψ, η, a, b, $\mathcal{O}_1$ *and* $\mathcal{O}_2$.

Note that the gradient estimate given in Theorem 1.3 depends on the bound of $|u|$ on $\mathcal{O}_1$. If we only know $u \geq -C_0$ on $\mathcal{O}_1$, the next result gives the gradient estimates for solutions of the equation

$$\begin{cases} F(U) := f(\lambda_g(U)) = e^{-2u}, & \lambda_g(U) \in \Gamma \text{ on } \mathcal{O}_1, \\ \frac{\partial u}{\partial \nu} = ce^{-u} - h_g & \text{on } \mathcal{O}_1 \cap \partial\mathcal{M}, \end{cases} \tag{1.16}$$

where $\mathcal{O}_1$ is an open set of $\mathcal{M}$, if (f, Γ) further satisfies the condition (H_α) introduced in [26], see Definition 2.1 in Sect. 2.

Theorem 1.4 *Assume that* (f, Γ) *satisfies* (1.1)–(1.5), (1.7) *and the condition* (H_1), *and that* $(\mathcal{M}, g)$ *is a smooth compact Riemannian manifold with smooth boundary* $\partial\mathcal{M}$. *Let* $\mathcal{O}_1$ *be an open set of* $\mathcal{M}$ *and let* $u \in C^3(\mathcal{O}_1)$ *be a solution of* (1.16). *If*

$$u \geq -C_0 \quad \text{on } \mathcal{O}_1 \tag{1.17}$$

for some constant C_0, *then, for any open set* $\mathcal{O}_2$ *of* $\mathcal{M}$ *satisfying* $\overline{\mathcal{O}}_2 \subset \mathcal{O}_1$,

$$|\nabla u|_g \leq C \quad \text{on } \mathcal{O}_2 \tag{1.18}$$

for some positive constant C *depending only on* n, c, (f, Γ), g, C_0, $\mathcal{O}_1$ *and* $\mathcal{O}_2$.

Remark 1.1 The same result in Theorem 1.4 holds true for more general equation than (1.16). In fact, if $u \in C^3(\mathcal{O}_1)$ is a solution of (1.14) with $\psi(x, u) := \psi_0(x)e^{-pu}$ and $\eta(x, u) := \eta_0(x)e^{-qu}$ for some numbers $p > 0, q > 0$ and some functions $\psi_0 \in C^2(\mathcal{M})$, $\eta_0 \in C^2(\partial\mathcal{M})$, and if u satisfies (1.17), then u satisfies (1.18) with the constant C depending only on $n, p, q, (f, \Gamma), g, C_0, \psi_0, \eta_0, \mathcal{O}_1$ and $\mathcal{O}_2$. One can see this easily from the proof of Theorem 1.4.

By assuming that $\partial\mathcal{M}$ is umbilic and that $(\mathcal{M}, g)$ is locally conformally flat near $\partial\mathcal{M}$, we derive in section 3 the following C^2 estimates for solutions of (1.14) under suitable conditions on ψ and η.

Theorem 1.5 *Assume that* (f, Γ) *satisfies* (1.1)–(1.6) *and that* $(\mathcal{M}, g)$ *is a smooth compact Riemannian manifold with smooth boundary* $\partial\mathcal{M}$. *Suppose that* $\partial\mathcal{M}$ *is umbilic and* $(\mathcal{M}, g)$ *is locally conformally flat near* $\partial\mathcal{M}$. *Let* $\mathcal{O}_1$ *be an open set of* $\mathcal{M}$ *and let* $u \in C^4(\mathcal{O}_1)$ *be a solution of* (1.14). *Assume that* ψ *and* η *satisfy one of the following conditions:*

(i) $\eta \equiv 0$, $\frac{\partial \psi}{\partial \nu} \equiv 0$ *on* $(\mathcal{O}_1 \cap \partial\mathcal{M}) \times \mathbb{R}$ *and* $\psi \in C^3(\mathcal{O}_1 \times \mathbb{R})$;
(ii) η *is positive on* $(\mathcal{O}_1 \times \partial\mathcal{M}) \times \mathbb{R}$ *and* ψ *is any function on* $\mathcal{O}_1 \times \mathbb{R}$.

If

$$|u| \leq C_0 \quad \text{on } \mathcal{O}_1 \tag{1.19}$$

for some constant C_0, *then, for any open set* $\mathcal{O}_2$ *of* $\mathcal{M}$ *satisfying* $\overline{\mathcal{O}}_2 \subset \mathcal{O}_1$,

$$|\nabla u|_g + |\nabla^2 u|_g \leq C \quad \text{on } \mathcal{O}_2$$

for some constant C *depending only on* n, C_0, g, (f, Γ), ψ, η, $\mathcal{O}_1$ *and* $\mathcal{O}_2$.

Both Theorems 1.3 and 1.5 are used in the proof of Theorem 1.2. In Sect. 4 we establish C^2 estimates on general manifolds with umbilic boundary, without any locally conformally flat assumption, for the following Monge-Ampére type problem

$$\begin{cases} \det(g^{-1}\cdot U) = e^{-2nu}, & \lambda_g(U) \in \Gamma_n \text{ on } \mathcal{O}_1, \\ \frac{\partial u}{\partial \nu} = ce^{-u} - h_g & \text{on } \mathcal{O}_1 \cap \partial\mathcal{M}, \end{cases} \tag{1.20}$$

where U is defined by (1.11).

Theorem 1.6 *Assume that $(\mathcal{M}, g)$ is a smooth compact Riemannian manifold with smooth umbilic boundary $\partial\mathcal{M}$. Let $\mathcal{O}_1$ be an open set of $\mathcal{M}$ and let $u \in C^4(\mathcal{O}_1)$ be a solution of* (1.20) *with $c > 0$. If*

$$|u| \le C_0 \quad \text{on } \mathcal{O}_1 \tag{1.21}$$

for some constant C_0, then, for any open set $\mathcal{O}_2$ of $\mathcal{M}$ satisfying $\overline{\mathcal{O}}_2 \subset \mathcal{O}_1$,

$$|\nabla u|_g + |\nabla^2 u|_g \le C \quad \text{on } \mathcal{O}_2 \tag{1.22}$$

for some positive constant C depending only on n, c, g, C_0, $\mathcal{O}_1$ and $\mathcal{O}_2$.

The Dirichlet problem for (1.8) has been studied in [16] for $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ and the existence of solutions is established whenever there exists an admissible supersolution. A similar problem for $(f, \Gamma) = (\sigma_n^{1/n}, \Gamma_n)$ was studied in [36]. The Neumann problem for Hessian equations has been studied in [10,35,37,43,46], most of the works are for Monge-Ampere equations. The results in [43] and [46] concern, respectively, general Hessian equations on Euclidean balls and on general domains in dimension two.

We draw readers' attention to some closely related independent work of Sophie Chen in [8].

2 Gradient estimates

In this section we will prove Theorems 1.3 and 1.4 concerning gradient estimates for solutions of (1.14) and (1.16). We use the distance function $d_g(x, \partial\mathcal{M})$ in $(\mathcal{M}, g)$ to the boundary $\partial\mathcal{M}$. Clearly there is a suitable small constant $\delta_0 > 0$ such that $d_g(x, \partial\mathcal{M})$ is smooth in $\{x \in \mathcal{M} : d_g(x, \partial\mathcal{M}) \le 2\delta_0\}$. Moreover

$$\frac{\partial}{\partial \nu} d_g(x, \partial\mathcal{M}) = 1 \quad \text{on } \partial\mathcal{M}.$$

We will fix a positive constant C_1 such that

$$\Pi(X, X) \ge -C_1 g(X, X) \quad \text{for } X \in T(\partial\mathcal{M}). \tag{2.1}$$

It is well-known that we can always find a metric conformal to g with vanishing mean curvature on $\partial\mathcal{M}$. Since a conformal change of metrics does not affect our C^1 and C^2 estimates, without loss of generality, in sections 2–4 we always assume that $h_g = 0$ on $\partial\mathcal{M}$ in the arguments.

Proof of Theorem 1.3 By shrinking $\mathcal{O}_2$ if necessary, we can always choose a cut-off function $\rho \in C_0^\infty(\mathcal{O}_1)$ such that

$$0 \le \rho \le 1 \quad \text{in } \mathcal{O}_1, \quad |\nabla \rho|_g \le C\sqrt{\rho} \quad \text{in } \mathcal{O}_1, \qquad \rho = 1 \quad \text{on } \mathcal{O}_2 \tag{2.2}$$

and

$$\frac{\partial \rho}{\partial \nu} = 0 \quad \text{on } \mathcal{O}_1 \cap \partial\mathcal{M} \ \text{ if } \mathcal{O}_1 \cap \partial\mathcal{M} \ne \emptyset, \tag{2.3}$$

where C is a constant depending only on n, g, $\mathcal{O}_1$ and $\mathcal{O}_2$.

Let

$$\gamma(t) := \frac{1}{\Lambda}(1+t-a)^{-\Lambda}, \quad t \in [a,b],$$

where the number Λ is large enough so that $\Lambda \geq 8(1+b-a)$. Then we choose a function $\varphi \in C^\infty(\mathcal{M})$ such that $\varphi(x) = d_g(x, \partial\mathcal{M})$ when $d_g(x, \partial\mathcal{M}) \leq \varepsilon_0$, where $0 < \varepsilon_0 \leq \delta_0$ is sufficiently small. Since u satisfies (1.15) and $\varphi = 0$ on $\partial\mathcal{M}$, we may further assume that φ is chosen in a way so that

$$|\eta_u \varphi| \leq \frac{1}{2} \quad \text{and} \quad |\eta_{uu}\varphi| \leq \frac{1}{2} \qquad \text{on } \mathcal{M} \times [a,b]. \tag{2.4}$$

We now consider the function

$$G := \frac{1}{2}\rho e^{\alpha} |\nabla u - \beta \nabla\varphi|_g^2 \quad \text{on } \mathcal{O}_1,$$

where

$$\alpha(x) := B\varphi(x) + \gamma(u - \eta(x,u)\varphi), \quad \beta(x) := \eta(x,u)$$

and B is a large number to be determined later. In order to derive the desired bound on $|\nabla u|_g$ over $\mathcal{O}_2$, it suffices to show that G can be bounded in $\mathcal{O}_1$ by some constant C depending only on n, (f,Γ), g, ψ, η, a, b, $\mathcal{O}_1$ and $\mathcal{O}_2$. Suppose the maximum of G over $\mathcal{O}_1$ is attained at some point $x_0 \in \mathcal{O}_1$. In the following we will always assume that $G(x_0) \geq 1$; otherwise we are done.

We first claim that x_0 must be an interior point of $\mathcal{O}_1$, i.e. $x_0 \in \mathcal{O}_1 \backslash \partial\mathcal{M}$. To this end, suppose $x_0 \in \partial\mathcal{M}$ and choose an orthonormal frame field $\{e_1, \ldots, e_n\}$ around x_0 such that $e_n = \nu$ on $\partial\mathcal{M}$. In the following for any smooth function ϕ we use $\phi_i, \phi_{ij}, \ldots$ to denote the covariant derivatives of ϕ of all orders. It is easy to see that on $\partial\mathcal{M}$ there hold

$$\varphi_n = 1, \quad \varphi_l = 0 \quad \text{and} \quad \varphi_{ln} = \varphi_{nl} = 0 \qquad \text{for } 1 \leq l \leq n-1.$$

By using the boundary condition $u_n = \eta(x,u)$ we thus have on $\partial\mathcal{M}$ that

$$u_n - \beta\varphi_n = 0 \quad \text{and} \quad G = \frac{1}{2}\rho e^{\gamma(u)}|\nabla u - \beta\nabla\varphi|_g^2 = \frac{1}{2}\rho e^{\gamma(u)}\sum_{l=1}^{n-1} u_l^2.$$

Therefore, since $\rho_n = 0$ on $\mathcal{O}_1 \cap \partial\mathcal{M}$,

$$\begin{aligned} G_n &= \rho e^{\gamma(u)} \sum_{l=1}^{n} (u_l - \beta\varphi_l)(u_{ln} - \beta_n\varphi_l - \beta\varphi_{ln}) + G\alpha_n \\ &= \rho e^{\gamma(u)} \sum_{l=1}^{n-1} u_l u_{nl} + BG. \end{aligned}$$

By using again the boundary condition $u_n = \eta(x,u)$ on $\partial\mathcal{M}$ we have

$$\begin{aligned} u_{nl} &= e_l(u_n) - du\left(\nabla_{e_l} e_n\right) = e_l(\eta) - \sum_{k=1}^{n-1} \langle \nabla_{e_l} e_n, e_k\rangle_g u_k \\ &= \eta_l + \eta_u u_l + \sum_{k=1}^{n-1} \Pi(e_k, e_l) u_k. \end{aligned}$$

Consequently, by using (2.1), (1.15) and the fact $G(x_0) \geq 1$ we can choose a large number B such that

$$\begin{aligned} G_n(x_0) &= \rho e^{\gamma(u)} \sum_{k,l=1}^{n-1} \Pi(e_k, e_l) u_k u_l + \rho e^{\gamma(u)} \sum_{l=1}^{n-1} (\eta_l + \eta_u u_l) u_l + BG \\ &\geq (B - 2C_1 - 2|\eta_u|)\, G + \rho e^{\gamma(u)} \sum_{l=1}^{n-1} \eta_l u_l \\ &> 0. \end{aligned}$$

However, by the maximality of $G(x_0)$ we have $G_n(x_0) \leq 0$. We thus derive a contradiction.

Therefore $x_0 \in \mathcal{O}_1 \backslash \partial\mathcal{M}$. Choose normal coordinates around x_0 such that

$$(U_{ij}) = \left(u_{ij} + u_i u_j - \frac{1}{2} |\nabla u|_g^2 g_{ij} + (A_g)_{ij} \right) \tag{2.5}$$

is diagonal at x_0. Then, by setting $\xi_l = u_l - \beta\varphi_l$, we have at x_0 that

$$0 = G_i = \rho e^{\alpha} \sum_l (u_{li} - \beta_i \varphi_l - \beta \varphi_{li})\, \xi_l + \alpha_i G + \frac{\rho_i}{\rho} G \tag{2.6}$$

and

$$\begin{aligned} 0 \geq (G_{ij}) = \Bigg(&(\alpha_{ij} - \alpha_i \alpha_j) G + \frac{\rho \rho_{ij} - 2\rho_i \rho_j}{\rho^2} G - \frac{\alpha_i \rho_j + \alpha_j \rho_i}{\rho} G \\ &+ \rho e^{\alpha} \sum_l \left(u_{lij} - \beta_{ij} \varphi_l - \beta_i \varphi_{lj} - \beta_j \varphi_{li} - \beta \varphi_{lij} \right) \xi_l \\ &+ \rho e^{\alpha} \sum_l (u_{li} - \beta_i \varphi_l - \beta \varphi_{li}) \left(u_{lj} - \beta_j \varphi_l - \beta \varphi_{lj} \right) \Bigg). \end{aligned} \tag{2.7}$$

Let $F^{ij} := \frac{\partial F}{\partial U_{ij}}(U)$. Since (U_{ij}) is diagonal at x_0, we have $F^{ij} = f_i \delta_{ij}$ at x_0; moreover (F^{ij}) is positive definite by (1.5) (see e.g. [4]). It then follows from (2.7) that

$$\begin{aligned} 0 &\geq e^{-\alpha} F^{ij} G_{ij} \\ &= e^{-\alpha} G \sum_i f_i (\alpha_{ii} - \alpha_i^2) + e^{-\alpha} G \sum_i f_i \frac{\rho \rho_{ii} - 2\rho_i^2}{\rho^2} - 2 e^{-\alpha} G \sum_i f_i \alpha_i \frac{\rho_i}{\rho} \\ &\quad + \rho \sum_{i,l} f_i \, (u_{lii} - \beta_{ii} \varphi_l - 2\beta_i \varphi_{li} - \beta \varphi_{lii})\, \xi_l \\ &\quad + \rho \sum_{i,l} f_i \, (u_{li} - \beta_i \varphi_l - \beta \varphi_{li})^2 \\ &\geq \frac{1}{2} \rho |\xi|^2 \sum_i f_i (\alpha_{ii} - \alpha_i^2) - |\xi|^2 \sum_i f_i \alpha_i \rho_i - C\mathcal{T} |\xi|^2 + \mathcal{E}, \end{aligned} \tag{2.8}$$

where $\mathcal{T} := \sum_i f_i \geq c_0 > 0$ by (1.8) and

$$\mathcal{E} := \rho \sum_{i,l} f_i \, (u_{lii} - \beta_{ii} \varphi_l - 2\beta_i \varphi_{li} - \beta \varphi_{lii})\, \xi_l.$$

Since $G(x_0) \geq 1$, we have $|\nabla u| \leq C|\xi|$ for some universal constant C. Note that

$$\beta_i = \eta_i + \eta_u u_i \quad \text{and} \quad \beta_{ii} = \eta_{ii} + 2\eta_{iu}u_i + \eta_{uu}u_i^2 + \eta_u u_{ii}.$$

By using Ricci identity $u_{lii} = u_{iil} + R_{rili}u_r$, where R_{ijkl} denotes the Riemann curvature tensor of g, we have

$$\mathcal{E} \geq \rho \sum_{i,l} f_i u_{iil}\xi_l - \rho\eta_u \sum_{i,l} f_i u_{ii}\varphi_l\xi_l - C\rho\mathcal{T}|\xi|^3.$$

By the degree one homogeneity of f and (2.5) we have

$$-\rho\eta_u \sum_{i,l} f_i u_{ii}\varphi_l\xi_l \geq -C\rho\mathcal{T}|\xi|^3.$$

Therefore by using (1.14), (2.6) and the fact $|\nabla\rho| \leq C\sqrt{\rho}$ we have

$$\begin{aligned}
\mathcal{E} &\geq \rho \sum_{i,l} f_i u_{iil}\xi_l - C\rho\mathcal{T}|\xi|^3 \\
&= \rho \sum_{i,l} f_i \left(U_{ii} - u_i^2 + \frac{1}{2}|\nabla u|^2 g_{ii} - (A_g)_{ii} \right)_l \xi_l - C\rho\mathcal{T}|\xi|^3 \\
&\geq -2\rho \sum_{i,l} f_i u_{il}u_i\xi_l + \rho\mathcal{T} \sum_{k,l} u_{kl}u_k\xi_l - C\rho\mathcal{T}|\xi|^3 \\
&\geq -2\rho \sum_{i,l} f_i \left(u_{li} - \beta_i\varphi_l - \beta\varphi_{il}\right) \xi_l u_i \\
&\quad + \rho\mathcal{T} \sum_{k,l} \left(u_{lk} - \beta_k\varphi_l - \beta\varphi_{lk}\right) \xi_l u_k - C\rho\mathcal{T}|\xi|^3 \\
&= |\xi|^2 \sum_i f_i u_i(\rho\alpha_i + \rho_i) - \frac{1}{2}\mathcal{T}|\xi|^2 \sum_k u_k(\rho\alpha_k + \rho_k) - C\rho\mathcal{T}|\xi|^3 \\
&\geq \rho|\xi|^2 \sum_i f_i u_i\alpha_i - \frac{1}{2}\rho\mathcal{T}|\xi|^2 \sum_k u_k\alpha_k - C\sqrt{\rho}\mathcal{T}|\xi|^3.
\end{aligned}$$

Plugging this estimate into (2.8) we have

$$\begin{aligned}
0 \geq\ & \frac{1}{2}\rho|\xi|^2 \sum_i f_i(\alpha_{ii} - \alpha_i^2) + \rho|\xi|^2 \sum_i f_i u_i\alpha_i - |\xi|^2 \sum_i f_i\alpha_i\rho_i \\
& - \frac{1}{2}\rho|\xi|^2\mathcal{T} \sum_k u_k\alpha_k - C\sqrt{\rho}\mathcal{T}|\xi|^3.
\end{aligned}$$

Note that

$$\alpha_i = B\varphi_i + \gamma' \left((1 - \eta_u\varphi)u_i - \eta_i\varphi - \eta\varphi_i\right)$$

and

$$\begin{aligned}
\alpha_{ii} = B\varphi_{ii} &+ \gamma'' \left((1 - \eta_u\varphi)u_i - \eta_i\varphi - \eta\varphi_i\right)^2 \\
&+ \gamma' \Big((1 - \eta_u\varphi)u_{ii} - \eta_{uu}\varphi u_i^2 - 2\eta_{iu}u_i\varphi - 2\eta_u u_i\varphi_i - \eta_{ii}\varphi - 2\eta_i\varphi_i - \eta\varphi_{ii} \Big).
\end{aligned}$$

It follows that

$$0 \geq \frac{1}{2}\rho|\xi|^2\sum_i f_i u_i^2 \left\{\left(\gamma'' - (\gamma')^2\right)(1-\eta_u\varphi)^2 + 2\gamma'(1-\eta_u\varphi) - \gamma'\eta_{uu}\varphi\right\}$$
$$+\frac{1}{2}\rho|\xi|^2\gamma'(1-\eta_u\varphi)\sum_i f_i u_{ii} - \frac{1}{2}\rho\gamma'\mathcal{T}|\xi|^2|\nabla u|^2(1-\eta_u\varphi) - C\sqrt{\rho}\mathcal{T}|\xi|^3.$$

Note that $u_{ii} = U_{ii} - u_i^2 + \frac{1}{2}|\nabla u|^2 g_{ii} - (A_g)_{ii}$. Using the degree one homogeneity of f we obtain

$$0 \geq \frac{1}{2}\rho|\xi|^2\sum_i f_i u_i^2 \left\{\left(\gamma'' - (\gamma')^2\right)(1-\eta_u\varphi)^2 + \gamma'(1-\eta_u\varphi) - \gamma'\eta_{uu}\varphi\right\}$$
$$-\frac{1}{4}\rho\gamma'\mathcal{T}|\xi|^2|\nabla u|^2(1-\eta_u\varphi) - C\sqrt{\rho}\mathcal{T}|\xi|^3.$$

From the definition of γ and (2.4) one can verify that

$$\left(\gamma'' - (\gamma')^2\right)(1-\eta_u\varphi)^2 + \gamma'(1-\eta_u\varphi) - \gamma'\eta_{uu}\varphi \geq 0 \quad \text{and} \quad \gamma' \leq -c_1 < 0.$$

Thus

$$0 \geq c_1\rho\mathcal{T}|\xi|^2|\nabla u|^2 - C\sqrt{\rho}\mathcal{T}|\xi|^3 \geq c_1\rho\mathcal{T}|\xi|^4 - C\sqrt{\rho}\mathcal{T}|\xi|^3.$$

This gives the desired estimate. □

Before giving the proof of Theorem 1.4, let us recall the condition (H_α) on (f,Γ) introduced in [26].

Definition 2.1 *We say* (f,Γ) *satisfies condition* (H_α) *for some* $\alpha > 0$ *if there exists some positive constants* ε_1 *and* c_1 *such that for any* $(\lambda,\xi) \in \Gamma \times \mathbb{R}^n$ *satisfying*

$$f(\lambda) \leq \alpha, \quad |\xi| \geq \varepsilon_1^{-1} \quad \textit{and} \quad \left|\xi_i\left(\lambda_i - \frac{1}{2}|\xi|^2\right)\right| \leq \varepsilon_1|\xi|^3 \qquad \textit{for } 1 \leq i \leq n,$$

there holds

$$\sum_i f_i(\lambda)\left\{\left(\lambda_i + \frac{1}{2}|\xi|^2 - \xi_i^2\right)^2 + \xi_i^2(|\xi|^2 - \xi_i^2)\right\} \geq c_1|\xi|^4\sum_i f_i(\lambda).$$

Some discussions have been given in [26] on the condition (H_α) for (f,Γ). Here are two remarks.

Remark 2.1 (i) It is easy to check that if f is homogeneous on Γ, then (f,Γ) satisfies the condition (H_1) if and only if (f,Γ) satisfies the condition (H_α) for each $\alpha > 0$.

(ii) From the proof of [19, Lemma 2.4] and [20, Theorem 3], one can see that the following two classes of (f,Γ) satisfy the condition (H_α):

$$(f,\Gamma) = (\sigma_k^{1/k}, \Gamma_k) \quad \text{with } 1 \leq k \leq n$$

and

$$(f,\Gamma) = \left((\sigma_k/\sigma_l)^{1/(k-l)}, \Gamma_k\right)$$

with $0 \leq l < k \leq n$ and $(n-k+1)(n-l+1) > 2(n+1)$.

Proof of Theorem 1.4 Consider the function

$$G := \frac{1}{2}\rho e^{B\varphi}|\nabla u - ce^{-u}\nabla\varphi|_g^2 \quad \text{on } \mathcal{O}_1,$$

where φ and ρ are as in the proof of Theorem 1.3, and B is a fixed positive constant satisfying

$$B > 2C_1 + 2|c|e^{C_0}.$$

We need to show that G can be bounded by some constant C depending only on n, c, (f, Γ), g, C_0, $\mathcal{O}_1$ and $\mathcal{O}_2$. Suppose the maximum of G over $\mathcal{O}_1$ is attained at some point $x_0 \in \mathcal{O}_1$. In the following we will always assume that $G(x_0) \geq 1$.

Similar to the proof of Theorem 1.3 we have $x_0 \in \mathcal{O}_1 \backslash \partial\mathcal{M}$. As before we choose normal coordinates around x_0 such that (U_{ij}) is diagonal at x_0. Then, by setting $\xi_l = u_l - ce^{-u}\varphi_l$, we have at x_0 that

$$0 = G_i = \rho e^{B\varphi}\sum_l \left(u_{li} - ce^{-u}(\varphi_{li} - \varphi_l u_i)\right)\xi_l + BG\varphi_i + \frac{\rho_i}{\rho}G \tag{2.9}$$

and

$$\begin{aligned} 0 \geq (G_{ij}) = \Big(&(B\varphi_{ij} - B^2\varphi_i\varphi_j)G + \frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2}G - \frac{\rho_i\varphi_j + \rho_j\varphi_i}{\rho}BG \\ &+ \rho e^{B\varphi}\sum_l \left(u_{lij} - ce^{-u}(\varphi_{lij} - \varphi_{li}u_j - \varphi_{lj}u_i - \varphi_l u_{ij} + \varphi_l u_i u_j)\right)\xi_l \\ &+ \rho e^{B\varphi}\sum_l \left(u_{li} - ce^{-u}(\varphi_{li} - \varphi_l u_i)\right)\left(u_{lj} - ce^{-u}(\varphi_{lj} - \varphi_l u_j)\right)\Big). \end{aligned} \tag{2.10}$$

It then follows from (2.10) that

$$0 \geq e^{-B\varphi}F^{ij}G_{ij} = I + II - \frac{C}{\rho}\mathcal{T}G. \tag{2.11}$$

where

$$\begin{aligned} I &:= \rho\sum_{i,l} f_i\left(u_{lii} - ce^{-u}(\varphi_{lii} - 2\varphi_{li}u_i - \varphi_l u_{ii} + \varphi_l u_i^2)\right)\xi_l, \\ II &:= \rho\sum_{i,l} f_i\left(u_{li} - ce^{-u}(\varphi_{li} - \varphi_l u_i)\right)^2. \end{aligned}$$

Similar to the estimate for $\mathcal{E}$ in the proof of Theorem 1.3, we have by (2.9) that

$$I \geq \rho\sum_{i,l} f_i u_{iil}\xi_l - \frac{C}{\sqrt{\rho}}\mathcal{T}G^{\frac{3}{2}} \geq -\frac{C}{\sqrt{\rho}}\mathcal{T}G^{\frac{3}{2}}. \tag{2.12}$$

For the term II, by using the elementary inequality

$$(a+b)^2 \geq \frac{1}{2}a^2 - b^2 \qquad \text{for any } a, b \in \mathbb{R} \tag{2.13}$$

we have

$$\begin{aligned} II &\geq \frac{1}{2}\rho\sum_{i,l} f_i(u_{li}+(A_g)_{li})^2 - \rho\sum_{i,l} f_i\left((A_g)_{il}+ce^{-u}(\varphi_{li}-\varphi_l u_i)\right)^2 \\ &\geq \frac{1}{2}\rho\sum_{i,l} f_i(u_{li}+(A_g)_{li})^2 - C\mathcal{T}G. \end{aligned}$$

Using (2.5) we then obtain

$$\begin{aligned} II &\geq \frac{1}{2}\rho\sum_{i,l} f_i\left(U_{li}+\frac{1}{2}|\nabla u|_g^2 g_{li} - u_l u_i\right)^2 - C\mathcal{T}G \\ &= \frac{1}{2}\rho\sum_{i} f_i\left\{\left(U_{ii}+\frac{1}{2}|\nabla u|_g^2 - u_i^2\right)^2 + u_i^2\left(|\nabla u|_g^2 - u_i^2\right)\right\} - C\mathcal{T}G. \end{aligned} \tag{2.14}$$

Let $\xi = (\xi_1, \ldots, \xi_n)$. By using the elementary inequality (2.13) once again, we have

$$\begin{aligned} &\sum_i f_i\left\{\left(U_{ii}+\frac{1}{2}|\nabla u|_g^2 - u_i^2\right)^2 + u_i^2(|\nabla u|_g^2 - u_i^2)\right\} \\ &\qquad \geq \frac{1}{2}\sum_i f_i\left\{\left(U_{ii}+\frac{1}{2}|\xi|^2-\xi_i^2\right)^2 + \xi_i^2(|\xi|^2-\xi_i^2)\right\} \\ &\qquad\quad - \sum_i f_i\left(\frac{1}{2}|\nabla u|_g^2 - \frac{1}{2}|\xi|^2 - u_i^2 + \xi_i^2\right)^2 \\ &\qquad\quad + \sum_i f_i\left(u_i^2(|\nabla u|_g^2 - u_i^2) - \xi_i^2(|\xi|^2-\xi_i^2)\right). \end{aligned}$$

It is easy to see that

$$\left|\frac{1}{2}|\nabla u|_g^2 - \frac{1}{2}|\xi|^2 - u_i^2 + \xi_i^2\right| \leq C(1+|\nabla u|_g) \leq C|\xi|$$

and

$$\left|u_i^2(|\nabla u|_g^2 - u_i^2) - \xi_i^2(|\xi|^2-\xi_i^2)\right| \leq C(1+|\nabla u|_g^3) \leq C|\xi|^3.$$

Consequently

$$\begin{aligned} &\sum_i f_i\left\{\left(U_{ii}+\frac{1}{2}|\nabla u|_g^2 - u_i^2\right)^2 + u_i^2(|\nabla u|_g^2 - u_i^2)\right\} \\ &\qquad \geq \frac{1}{2}\sum_i f_i\left\{\left(U_{ii}+\frac{1}{2}|\xi|^2-\xi_i^2\right)^2 + \xi_i^2(|\xi|^2-\xi_i^2)\right\} - C\mathcal{T}|\xi|^3. \end{aligned}$$

This together with (2.14) implies that

$$II \geq \frac{1}{4}\rho\sum_i f_i\left\{\left(U_{ii}+\frac{1}{2}|\xi|^2-\xi_i^2\right)^2 + \xi_i^2(|\xi|^2-\xi_i^2)\right\} - \frac{C}{\sqrt{\rho}}\mathcal{T}G^{\frac{3}{2}}. \tag{2.15}$$

Combining (2.11), (2.12) and (2.15) yields

$$\rho\sum_i f_i\left\{\left(U_{ii}+\frac{1}{2}|\xi|^2-\xi_i^2\right)^2+\xi_i^2(|\xi|^2-\xi_i^2)\right\}\le\frac{C}{\sqrt{\rho}}\mathcal{T}G^{\frac{3}{2}}+\frac{C}{\rho}\mathcal{T}G. \tag{2.16}$$

In order to apply the (H_α) condition on (f,Γ) with $\lambda_i=U_{ii}$ and $\xi_i=u_i-ce^{-u}\varphi_i$, we need to check

$$\left|\xi_i\left(U_{ii}-\frac{1}{2}|\xi|^2\right)\right|\le\varepsilon_1|\xi|^3. \tag{2.17}$$

To see this, recall that (U_{ij}) is diagonal, one has

$$\begin{aligned}\xi_i\left(U_{ii}-\frac{1}{2}|\xi|^2\right)&=\sum_l\xi_l U_{il}-\frac{1}{2}\xi_i|\xi|^2\\&=\sum_l\xi_l\left(u_{il}+u_iu_l-\frac{1}{2}|\nabla u|^2\delta_{il}+(A_g)_{il}\right)-\frac{1}{2}\xi_i|\xi|^2\end{aligned}$$

By using (2.9) we have

$$\left|\sum_l\xi_l u_{il}\right|\le\left|\sum_l\xi_l\left(u_{li}-ce^{-u}(\varphi_{li}-\varphi_l u_i)\right)\right|+C|\xi|^2\le\frac{C}{\sqrt{\rho}}|\xi|^2.$$

Moreover, by direct calculation one can see that

$$\left|\sum_l\xi_l\left(u_iu_l-\frac{1}{2}|\nabla u|_g^2\delta_{il}\right)-\frac{1}{2}\xi_i|\xi|^2\right|\le C(1+|\nabla u|_g^2)\le C|\xi|^2.$$

Consequently we have

$$\left|\xi_i\left(U_{ii}-\frac{1}{2}|\xi|^2\right)\right|\le\frac{C}{\sqrt{\rho}}|\xi|^2\le\frac{C}{\sqrt{G}}|\xi|^3\le\varepsilon_1|\xi|^3$$

if we further assume that $G(x_0)\ge C^2/\varepsilon_1^2$. Therefore we may apply the (H_α) condition on (f,Γ) to (2.16) to get

$$c_1\rho\mathcal{T}|\xi|^4\le\frac{C}{\sqrt{\rho}}\mathcal{T}G^{\frac{3}{2}}+\frac{C}{\rho}\mathcal{T}G\le\frac{C}{\rho}\mathcal{T}\left(G^{\frac{3}{2}}+G\right).$$

Consequently $G^2\le C(G^{\frac{3}{2}}+G)$ which implies $G(x_0)\le C$. □

3 C^2 estimates: general equations

The aim of this section is to show Theorem 1.5. The gradient bound there is a direct consequence of Theorem 1.3. In what follows we will concentrate on the derivation of Hessian estimates by assuming C^0 and C^1 bounds.

We may assume $\mathcal{O}_1\cap\partial\mathcal{M}\neq\emptyset$ since otherwise the results follow from the well-known local interior estimates (see [19,26]). Without loss of generality, we may also assume that g is conformally flat on $\mathcal{O}_1$, i.e. there exists a function $\varphi\in C^\infty(\mathcal{O}_1)$ such that $e^{2\varphi}g$ is a flat metric on $\mathcal{O}_1$. Since $\mathcal{O}_1\cap\partial\mathcal{M}$ is umbilic in g, it is also umbilic in the flat metric. Therefore $\mathcal{O}_1\cap\partial\mathcal{M}$ is either a part of a hyperplane or a part of a sphere in $\mathbb{R}^n$.

By shrinking $\mathcal{O}_1$ and using the conformal diffeomorphism in $\mathbb{R}^n$ if necessary we may assume that

$$\mathcal{O}_1 = B_4^+ := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x| < 4 \text{ and } x_n \geq 0\}$$

and

$$\mathcal{O}_1 \cap \partial\mathcal{M} := \partial B_4^+ \cap \{x_n = 0\}.$$

Observe that $e^{2(u+\varphi)}\tilde{g}$ is also a flat metric, By using the conformal invariance the function $v := u + \varphi$ satisfies the following equation

$$\begin{cases} F(V) := f(\lambda(V)) = \tilde{\psi}(x, v), & \lambda(V) \in \Gamma \ \text{ in } B_4^+, \\ v_n = \tilde{\eta}(x, v) & \text{on } \partial B_4^+ \cap \{x_n = 0\}, \end{cases} \tag{3.1}$$

where $\tilde{\psi}(x, v) = \psi(x, v - \varphi)e^{-2\varphi}$, $\tilde{\eta}(x, v) = \eta(x, v - \varphi)e^{-\varphi}$, V denotes the matrix function

$$V := D^2 v + dv \otimes dv - \frac{1}{2}|Dv|^2 I,$$

I is the $n \times n$ identity matrix, D is the standard connection on $\mathbb{R}^n$, Dv and $D^2 v$ denote the gradient and Hessian of v respectively, and $\lambda(V)$ denote the eigenvalues of V.

Recall that we can assume $h_g = 0$ on $\partial\mathcal{M}$. So $\varphi_n = 0$ on $\partial B_4^+ \cap \{x_n = 0\}$. Therefore when $\eta \equiv 0$ and $\frac{\partial \psi}{\partial v} \equiv 0$ on $\partial\mathcal{M} \times \mathbb{R}$, we have $\tilde{\eta} \equiv 0$ and $\tilde{\psi}_n \equiv 0$ on $(\partial B_4^+ \cap \{x_n = 0\}) \times \mathbb{R}$. Thus, in order to show Theorem 1.5, it suffices to prove the following result.

Theorem 3.1 *Assume that* (f, Γ) *satisfies* (1.1)–(1.6) *and that* $\tilde{\psi}$, $\tilde{\eta}$ *satisfy one of the following conditions:*

(i) $\tilde{\eta} \equiv 0$ *and* $\tilde{\psi}_n \equiv 0$ *on* $(\partial B_4^+ \cap \{x_n = 0\}) \times \mathbb{R}$ *and* $\tilde{\psi} \in C^3(B_4^+ \times \mathbb{R})$*;*
(ii) $\tilde{\eta}$ *is positive and* $\tilde{\psi}$ *is any function.*

If $v \in C^4(B_4^+)$ *is a solution of* (3.1) *satisfying*

$$|v| \leq C_0 \quad \textit{and} \quad |Dv| \leq C_0 \qquad \textit{in } B_4^+ \tag{3.2}$$

for some positive constant C_0*, then*

$$|D^2 v| \leq C \quad \textit{in } B_1^+ \tag{3.3}$$

for some positive constant C *depending only on* n, C_0, (f, Γ), $\tilde{\psi}$ *and* $\tilde{\eta}$*.*

Proof of Theorem 3.1 under condition (i) This case can be reduced to the local interior estimates. To this end, we define

$$\bar{v}(x', x_n) = \begin{cases} v(x', x_n), & \text{when } x_n \geq 0, \\ v(x', -x_n), & \text{when } x_n \leq 0 \end{cases} \tag{3.4}$$

and

$$\bar{\psi}((x', x_n), t) = \begin{cases} \tilde{\psi}((x', x_n), t), & \text{when } x_n \geq 0, \ \ t \in \mathbb{R}, \\ \tilde{\psi}((x', -x_n), t), & \text{when } x_n \leq 0, \ \ t \in \mathbb{R}. \end{cases}$$

Since $v_n(x', 0) = 0$ and $\tilde{\psi}_n((x', 0), t) = 0$, it is easy to see $\bar{v} \in C^2(B_4)$ and $\bar{\psi} \in C^{2,1}(B_4 \times \mathbb{R})$. Let

$$\overline{V} := D^2 \bar{v} + d\bar{v} \otimes d\bar{v} - \frac{1}{2}|D\bar{v}|^2 I.$$

By direct calculation one can see that

$$\overline{V}(x', x_n) = \begin{cases} V(x', x_n), & \text{when } x_n \geq 0, \\ \mathcal{Q}^t V(x', -x_n)\mathcal{Q}, & \text{when } x_n \leq 0, \end{cases}$$

where $\mathcal{Q}$ is the orthogonal matrix $\mathcal{Q} := \operatorname{diag}[1, \dots, 1, -1]$. Therefore

$$F(\overline{V}) := f(\lambda(\overline{V})) = \bar{\psi}(x, \bar{v}), \quad \lambda(\overline{V}) \in \Gamma \qquad \text{in } B_4. \tag{3.5}$$

It then follows from [15, Lemma 17.16] that $\bar{v} \in C^{4,\alpha}(B_2)$ for any $\alpha \in (0,1)$. Now the local interior estimates in [26] can be applied to obtain $|D^2\bar{v}| \leq C$ in B_1 for some constant C depending only on n, C_0, (f, Γ) and $\tilde{\psi}$. This in particular implies (3.3). □

Remark 3.1 Note that the function $\bar{v}$ defined by (3.4) satisfies (3.5) and has uniform C^2 estimates. Since f is concave, by Evans-Krylov theory and Schauder theory we can obtain uniform estimates on $\|\bar{v}\|_{C^{4,\alpha}(B_{1/2})}$ for any $\alpha \in (0,1)$. Therefore, for a solution $u \in C^2(\mathcal{M})$ of (1.14) with ψ and η satisfying (i) in Theorem 1.5, if (1.19) holds then

$$\|u\|_{C^{4,\alpha}(\mathcal{O}_2)} \leq C$$

for some constant C depending only on n, C_0, α, (f, Γ), ψ, $\mathcal{O}_1$ and $\mathcal{O}_2$.

Next we prove Theorem 3.1 under condition (ii). We will use $\{e_1, \dots, e_n\}$ to denote the standard orthonormal basis in $\mathbb{R}^n$, i.e. $e_i = (0, \dots, 1, \dots, 0)$, where 1 is in the ith spot and 0 elsewhere. The following result gives the double tangential derivative estimates without any restrictions on $\tilde{\psi}$ and $\tilde{\eta}$.

Lemma 3.1 *Assume that* (f, Γ) *satisfies* (1.1)–(1.6). *If* $v \in C^4(B_4^+)$ *is a solution of* (3.1) *satisfying*

$$|v| \leq C_0 \quad \text{and} \quad |Dv| \leq C_0 \qquad \text{in } B_4^+ \tag{3.6}$$

for some positive constant C_0*, then there exists a positive constant* C_1 *depending only on* n*,* C_0*,* (f, Γ)*,* $\tilde{\psi}$ *and* $\tilde{\eta}$ *such that*

$$v_{\tau\tau} \leq C_1 \quad \text{in } B_3^+ \tag{3.7}$$

for any vector $\tau \in \operatorname{span}\{e_1, \dots, e_{n-1}\}$ *with* $|\tau| = 1$.

Proof By rotation it suffices to establish (3.7) for $\tau = e_1$. Let $\rho \in C_0^\infty(B_4)$ be a radial cut-off function such that $0 \leq \rho \leq 1$ in $\mathbb{R}^n$, $\rho = 1$ in B_3, and $|D\rho| \leq C\rho^{\frac{1}{2}}$ in B_4. Consider the function

$$H = \rho e^{\beta x_n}\left(v_{11} + v_1^2\right) \quad \text{on } B_4^+,$$

where β is a large positive constant to be determined later. Suppose the maximum of H over B_4^+ is attained at some point x_0, then either $x_0 \in B_4^+ \cap \{x_n > 0\}$ or $x_0 \in B_4 \cap \{x_n = 0\}$. In the following we will always assume that $H(x_0) \geq 1$ and $v_{11}(x_0) \geq 1$; otherwise we are done.

Let us first calculate H_n on $B_4 \cap \{x_n = 0\}$. Since ρ is radially symmetric, we have $\rho_n = 0$ on $x_n = 0$. Thus, by using the boundary condition $v_n = \tilde{\eta}(x, v)$, it is easy to see that

$$\begin{aligned} H_n &= \rho\left(v_{11n} + 2v_1 v_{1n} + \beta(v_{11} + v_1^2)\right) \\ &= \rho\left((\beta + \tilde{\eta}_v)v_{11} + (\beta + \tilde{\eta}_{vv} + 2\tilde{\eta}_v)v_1^2 + (2\tilde{\eta}_1 + 2\tilde{\eta}_{1v})v_1 + \tilde{\eta}_{11}\right). \end{aligned}$$

If $x_0 \in B_4 \cap \{x_n = 0\}$, then, using (3.6) and $v_{11}(x_0) \geq 1$, we have $H_n(x_0) > 0$ by choosing β large enough. But by the maximality of $H(x_0)$ we have $H_n(x_0) \leq 0$. We thus derive a contradiction. Therefore $x_0 \in B_4^+ \cap \{x_n > 0\}$.

Now at x_0 we have

$$0 = H_i = \left(\frac{\rho_i}{\rho} + \beta\delta_{in}\right) H + \rho e^{\beta x_n} \left(v_{11i} + 2v_1 v_{1i}\right), \quad 1 \leq i \leq n \tag{3.8}$$

and

$$\begin{aligned} 0 \geq (H_{ij}) = \Big(& \frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2} H - \beta^2 \delta_{in}\delta_{jn} H - \frac{\rho_i\delta_{jn} + \rho_j\delta_{in}}{\rho}\beta H \\ & + \rho e^{\beta x_n}(v_{11ij} + 2v_1 v_{1ij} + 2v_{1i}v_{1j})\Big). \end{aligned}$$

Let $F^{ij} := \frac{\partial F}{\partial V_{ij}}(V)$ and $\mathcal{T} := \sum_i F^{ii}$. We know that (F^{ij}) is positive definite and $\mathcal{T} \geq f(1, \ldots, 1) > 0$, Thus we have at x_0 that

$$\begin{aligned} 0 &\geq e^{-\beta x_n} F^{ij} H_{ij} \\ &= e^{-\beta x_n} H F^{ij} \frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2} - \beta^2 e^{-\beta x_n} H F^{nn} - 2\beta e^{-\beta x_n} H F^{in} \frac{\rho_i}{\rho} \\ &\quad + \rho F^{ij}\left(v_{11ij} + 2v_1 v_{1ij} + 2v_{1i}v_{1j}\right) \\ &\geq -\frac{C}{\rho}\mathcal{T} H + \rho F^{ij}\left(v_{11ij} + 2v_1 v_{1ij} + 2v_{1i}v_{1j}\right). \end{aligned} \tag{3.9}$$

By differentiating (3.1) twice and using the concavity of F we have

$$F^{ij}\Big(v_{ijk} + 2v_{ik}v_j - \sum_l v_l v_{lk}\delta_{ij}\Big) = \tilde{\psi}_k + \tilde{\psi}_v v_k, \quad 1 \leq k \leq n$$

and

$$\begin{aligned} & F^{ij}\Big(v_{ij11} + 2v_{i11}v_j + 2v_{i1}v_{j1} - \sum_l (v_l v_{l11} + v_{l1}^2)\delta_{ij}\Big) \\ & \quad \geq \tilde{\psi}_{11} + 2\tilde{\psi}_{1v} v_1 + \tilde{\psi}_{vv} v_1^2 + \tilde{\psi}_v v_{11}. \end{aligned}$$

Therefore, by using (3.8) and (3.6),

$$\begin{aligned} \rho F^{ij} v_{ij11} &\geq -2\rho F^{ij}\left(v_{i11}v_j + v_{i1}v_{j1}\right) + \rho\mathcal{T}\sum_l \Big(v_l v_{l11} + v_{l1}^2\Big) - \frac{C}{\rho}\mathcal{T}H \\ &\geq 4\rho v_1 F^{ij} v_{1i} v_j - 2\rho\mathcal{T} v_1 \sum_l v_l v_{l1} - 2\rho F^{ij} v_{i1} v_{j1} \\ &\quad + \rho\mathcal{T}\sum_l v_{l1}^2 - \frac{C}{\rho}\mathcal{T}H. \end{aligned} \tag{3.10}$$

By using (3.1) and (3.6) we have

$$\begin{aligned}2\rho F^{ij}v_1v_{1ij} &= 2\rho v_1 F^{ij}\Big(V_{ij} - v_iv_j + \frac{1}{2}|Dv|^2\delta_{ij}\Big)_1\\ &= 2\rho v_1(\tilde{\psi}_1 + \tilde{\psi}_v v_1) - 4\rho v_1 F^{ij}v_{i1}v_j + 2\rho v_1\mathcal{T}\sum_l v_{l1}v_l\\ &\geq -4\rho v_1F^{ij}v_{i1}v_j + 2\rho v_1\mathcal{T}\sum_l v_{l1}v_l - \frac{C}{\rho}\mathcal{T}H. \qquad (3.11)\end{aligned}$$

Combining (3.9)–(3.11) yields

$$0 \geq -\frac{C}{\rho}\mathcal{T}H + \rho\mathcal{T}\sum_l v_{l1}^2 \geq -\frac{C}{\rho}\mathcal{T}H + \rho\mathcal{T}v_{11}^2 \geq -\frac{C}{\rho}\mathcal{T}H + \frac{1}{\rho}\mathcal{T}H^2.$$

Consequently, we have $H(x_0) \leq C$, and the proof is complete. □

Lemma 3.2 *Under the hypotheses of Theorem* 3.1 *with (ii) satisfied, there exists a positive constant C depending only on n, C_0, (f,Γ), $\tilde{\psi}$ and $\tilde{\eta}$ such that*

$$|v_{nn}(x',0)| \leq C \quad \textit{whenever } |x'| \leq 2. \qquad (3.12)$$

Proof It is convenient to consider the function $w := e^v$. By direct calculation and the degree one homogeneity of F it follows from (3.1) that w satisfies

$$\begin{cases} F(W) := f(\lambda(W)) = \hat{\psi}(x,w), & \lambda(W)\in\Gamma \text{ in } B_4^+\\ w_n = \hat{\eta}(x,w) & \text{on } \partial B_4^+\cap\{x_n=0\},\end{cases} \qquad (3.13)$$

where $\hat{\psi}(x,w) = w\tilde{\psi}(x,\log w)$, $\hat{\eta}(x,w) = w\tilde{\eta}(x,\log w)$ and

$$W := D^2w - \frac{1}{2w}|Dw|^2 I.$$

Moreover, by using (3.2) one can see that there is a positive constant C_2 depending only on C_0 such that

$$\frac{1}{C_2} \leq w \leq C_2 \quad \text{and} \quad |Dw| \leq C_2 \qquad \text{in } B_4^+. \qquad (3.14)$$

We now introduce a linear elliptic differential operator $\mathcal{L}$ on B_4^+ by

$$\mathcal{L}\phi = F^{ij}\phi_{ij} - w^{-1}\mathcal{T}\sum_l w_l\phi_l, \quad \forall\, \phi\in C^2(B_4^+),$$

where $F^{ij} := \frac{\partial F}{\partial W_{ij}}(W)$ and $\mathcal{T} := \sum_i F^{ii}$. Recall that (F^{ij}) is positive definite and $\mathcal{T} \geq f(1,\ldots,1) > 0$. By differentiating (3.13) with respect to x_n we obtain

$$F^{ij}\left(w_{ijn} - w^{-1}w_lw_{ln}\delta_{ij} + \frac{1}{2}w^{-2}w_n|Dw|^2\delta_{ij}\right) = \hat{\psi}_n + \hat{\psi}_w w_n.$$

Thus

$$\mathcal{L}w_n = \hat{\psi}_n + \hat{\psi}_w w_n - \frac{1}{2}w^{-2}w_n\mathcal{T}|Dw|^2.$$

By using (3.14) and the degree one homogeneity of f we have

$$|\mathcal{L}\hat{\eta}| \leq |\hat{\eta}_w F^{ij}w_{ij}| + C\mathcal{T} \leq C\mathcal{T} \quad \text{in } B_4^+.$$

Consequently

$$\left|\mathcal{L}\left(w_n - \hat{\eta}(x,w)\right)\right| \le C\mathcal{T} \quad \text{in } B_4^+. \tag{3.15}$$

Next we consider the function w_n. Since $\lambda(W) \in \Gamma \subset \Gamma_1$, we have $\Delta w \ge 0$ in B_4^+. Therefore from Lemma 3.1 it follows that

$$w_{nn} \ge -\sum_{i=1}^{n-1} w_{ii} \ge -(n-1)C_1 \quad \text{in } B_3^+.$$

Since $\hat{\eta}$ is positive, we have $\hat{\eta} \ge 2\alpha_0$ on $\partial B_4^+ \cap \{x_n = 0\}$ for some uniform constant $\alpha_0 > 0$. Therefore

$$w_n(x', x_n) + (n-1)C_1 x_n \ge w_n(x', 0) \ge 2\alpha_0 \quad \text{for } (x', x_n) \in B_3^+.$$

Thus there exists a universal constant $0 < \varepsilon_0 \le 1$ such that

$$w_n(x', x_n) \ge \alpha_0 > 0 \quad \text{for } (x', x_n) \in B_3^+ \cap \{x_n \le \varepsilon_0\}. \tag{3.16}$$

In order to establish (3.12), it suffices to show that $|w_{nn}(0)| \le C$ for some constant C depending only on n, C_0, (f, Γ), $\tilde{\psi}$ and $\tilde{\eta}$. Consider the function

$$\phi = Ax_n + B|x|^2 \pm \left(w_n - \hat{\eta}(x,w)\right) \quad \text{on } \overline{B_{\varepsilon_0}^+},$$

where A and B are sufficiently large positive constants to be chosen below. Clearly on $\partial B_{\varepsilon_0}^+ \cap \{x_n = 0\}$ we have $\phi \ge 0$ since $w_n - \hat{\eta}(x,w) = 0$ there. Also, by using (3.14) we may choose B large enough so that $\phi \ge 0$ on $\partial B_{\varepsilon_0}^+ \cap \{x_n > 0\}$. Thus

$$\phi \ge 0 \quad \text{on } \partial B_{\varepsilon_0}^+. \tag{3.17}$$

With the number B chosen above, by using (3.14), (3.15) and (3.16) we have

$$\begin{aligned}\mathcal{L}\phi &= -Aw^{-1}w_n\mathcal{T} + B\mathcal{L}(|x|^2) \pm \mathcal{L}\left(w_n - \hat{\eta}(x,w)\right)\\ &\le -AC_1^{-1}\alpha_0\mathcal{T} + C\mathcal{T}\\ &< 0\end{aligned} \tag{3.18}$$

in $B_{\varepsilon_0}^+$ if we choose A large enough.

By the maximum principle, it follows from (3.17) and (3.18) that $\phi \ge 0$ in $B_{\varepsilon_0}^+$. Since $\phi(0) = 0$, we therefore have $\phi_n(0) \ge 0$. consequently $|w_{nn}(0)| \le C$ for some uniform constant C. □

Proof of Theorem 3.1 under condition (ii) Consider the function

$$\widetilde{G}(x,\xi) = \rho(x)\left(v_{\xi\xi}(x) + \langle Dv(x), \xi\rangle^2\right), \quad \forall x \in \overline{B_2^+} \text{ and } \xi \in \mathbb{S}^n,$$

where $\rho \in C_0^\infty(B_2)$ is a radial cut-off function such that $0 \le \rho \le 1$ in $\mathbb{R}^n$, $\rho = 1$ on B_1, and $|D\rho| \le C\rho^{\frac{1}{2}}$ in B_2. Suppose the maximum of $\widetilde{G}$ over $\overline{B_2^+} \times \mathbb{S}^n$ is attained at $(\bar{x}, \bar{\xi})$, then either $\bar{x} \in B_2^+ \cap \{x_n > 0\}$ or $\bar{x} \in B_2 \cap \{x_n = 0\}$.

If $\bar{x} \in B_2^+ \cap \{x_n > 0\}$, then similar to the proof of [26, Theorem 1.20] one can show that $\widetilde{G}(\bar{x}, \bar{\xi}) \le C$ for some universal constant C. If $\bar{x} \in B_2 \cap \{x_n = 0\}$, we may write $\bar{\xi} = \alpha e_n + \beta\tau$, where τ is a unit vector in span$\{e_1, \dots, e_{n-1}\}$, and α and β are two numbers satisfying $\alpha^2 + \beta^2 = 1$. Then

$$v_{\bar{\xi}\bar{\xi}} = \alpha^2 v_{nn} + \beta^2 v_{\tau\tau} + 2\alpha\beta v_{n\tau}.$$

Since $v_n = \tilde{\eta}(x, v)$ on $x_n = 0$, we have $v_{n\tau} = \tilde{\eta}_\tau + \tilde{\eta}_v v_\tau$ which is bounded. Therefore it follows from Lemmas 3.1 and 3.2 that $v_{\bar{\xi}\bar{\xi}} \le C$ at $\bar{x}$ and hence $\widetilde{G}(\bar{x}, \bar{\xi}) \le C$.

The above argument shows that $v_{\xi\xi} \le C$ in B_1^+ for any unit vector $\xi \in \mathbb{S}^n$. Since $\Delta v \ge 0$, we also have $v_{\xi\xi} \ge -C$ in B_1^+ for any $\xi \in \mathbb{S}^n$. Therefore $|D^2 v| \le C$ in B_1^+. □

4 C^2 estimates: Monge–Ampere type equations

In this section we prove Theorem 1.6. The gradient bounds in (1.22) follows from Theorem 1.3 directly. It remains only to prove the Hessian estimates. We first have the following double normal derivative estimates.

Lemma 4.1 *Under the hypotheses of Theorem* 1.6, *there holds*

$$|\nabla^2 u(\nu, \nu)| \le C \quad \text{on } \partial\mathcal{O}_2 \cap \mathcal{M}$$

for some positive constant C depending only on n, c, g, C_0, $\mathcal{O}_1$ and $\mathcal{O}_2$.

Proof Recall that we assume $h_g = 0$ on $\partial\mathcal{M}$ and that $d_g(x, \partial\mathcal{M})$ is smooth in $\mathcal{M}_{\delta_0} := \{x \in \mathcal{M} : d_g(x, \partial\mathcal{M}) \le \delta_0\}$. We may extend the unit inward normal vector field ν to a smooth vector field in $\mathcal{M}_{\delta_0}$, still denoted it as ν, by parallel translating along the unit-speed geodesics perpendicular to $\partial\mathcal{M}$. Clearly $\nabla_\nu \nu = 0$ in $\mathcal{M}_{\delta_0}$. Thus along any such geodesic γ starting from a point $\gamma(0) \in \mathcal{O}_1 \cap \partial\mathcal{M}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_\nu(\gamma(t))) = \nu(u_\nu) = \nabla^2 u(\nu, \nu) + du(\nabla_\nu \nu) = \nabla^2 u(\nu, \nu).$$

Since $\lambda_g(U) \in \Gamma_n$, we have

$$\nabla^2 u(\nu, \nu) + u_\nu^2 - \frac{1}{2}|\nabla u|_g^2 + A_g(\nu, \nu) > 0.$$

Therefore it follows from (1.21) that there is a uniform constant C_1 such that $\frac{\mathrm{d}}{\mathrm{d}t}(u_\nu(\gamma(t))) \ge -C_1$ as long as γ is in $\mathcal{O}_1$. Consequently

$$u_\nu(\gamma(t)) \ge u_\nu(\gamma(0)) - C_1 d_g(\gamma(t), \partial\mathcal{M}) = ce^{-u(\gamma(0))} - C_1 d_g(\gamma(t), \partial\mathcal{M})$$

as along as γ is in $\mathcal{O}_1$. Fix an open set $\mathcal{O}_3$ of $\mathcal{M}$ such that $\overline{\mathcal{O}}_2 \subset \mathcal{O}_3$ and $\overline{\mathcal{O}}_3 \subset \mathcal{O}_1$. Since $c > 0$ and $u \le C_0$ in $\mathcal{O}_1$, there exist uniform constants $0 < \delta_1 \le \delta_0$ and $\alpha_0 > 0$ such that

$$u_\nu \ge \alpha_0 \quad \text{in } \mathcal{O}_3 \cap \mathcal{M}_{\delta_1}. \tag{4.1}$$

Now we are going to introduce a linear elliptic differential operator $\mathcal{L}_u$ on $\mathcal{M}$. Since $\lambda_g(U) \in \Gamma_n$, we can define a tensor U^{-1} on $\mathcal{M}$, which in local frame has the representation $U^{-1} = \{U^{ij}\}$, where $U^{ij}U_{jk} = \delta_k^i$ and $\{U_{ij}\}$ denotes the local representation of U. We define

$$\mathcal{L}_u \psi := U^{ij}\psi_{ij} - \mathrm{tr}_g(U^{-1})\langle \nabla u, \nabla \psi \rangle_g, \quad \forall\, \psi \in C^2(\mathcal{M}).$$

At the end of the proof we will show that for the local function $u_\nu - ce^{-u}$ there holds

$$|\mathcal{L}_u(u_\nu - ce^{-u})| \le C\left(1 + \mathrm{tr}_g(U^{-1})\right) \quad \text{in } \mathcal{M}_{\delta_1} \cap \mathcal{O}_1 \tag{4.2}$$

for some uniform constant C.

We now fix a point $x_0 \in \mathcal{O}_2 \cap \partial\mathcal{M}$ and consider the function

$$\psi = A\varphi + B\eta \pm \left(u_\nu - ce^{-u}\right),$$

where A and B are two large positive constants to be chosen below, $\varphi(x) = d_g(x, \partial\mathcal{M})$, and $\eta := d_g(x, x_0)^2$. We can choose a uniform constant $0 < \delta_2 \leq \delta_1$ such that $\mathcal{O}_{\delta_2}(x_0) := \{x \in \mathcal{M} : d_g(x, x_0) < \delta_2\} \subset \mathcal{O}_3 \cap \mathcal{M}_{\delta_1}$ and η is smooth in $\mathcal{O}_{\delta_2}(x_0)$ with

$$|\eta|_{C^2(\mathcal{O}_{\delta_2}(x_0))} \leq C$$

for some uniform constant C independent of x_0.

Let us do some calculation first on φ. Choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ around x_0 such that $e_n = \nu$ in $\mathcal{M}_{\delta_2}$. Then $\varphi_n = 1$ in $\mathcal{M}_{\delta_2}$. Since φ satisfies the Hamilton–Jacobi equation $|\nabla\varphi| = 1$ in $\mathcal{M}_{\delta_2}$ we have $\varphi_i = 0$ in $\mathcal{M}_{\delta_2}$ for $1 \leq i \leq n-1$. Thus it follows from (4.1) that

$$\langle \nabla u, \nabla\varphi\rangle_g = u_n \geq \alpha_0 \quad \text{in } \mathcal{O}_3 \cap \mathcal{M}_{\delta_2}. \tag{4.3}$$

By direct calculation we can see that on $\partial\mathcal{M}$ there holds

$$\varphi_{ij} = \begin{cases} -\Pi(e_i, e_j), & 1 \leq i, j \leq n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\partial\mathcal{M}$ is totally geodesic in $(\mathcal{M}, g)$, we have $\nabla^2\varphi = 0$ on $\partial\mathcal{M}$. Therefore, one may choose a uniform constant $0 < \delta_3 \leq \delta_2$ such that

$$\nabla^2\varphi \leq \frac{1}{2}\alpha_0 g \quad \text{in } \mathcal{M}_{\delta_3}. \tag{4.4}$$

Returning to the function ψ. Since $u_\nu - ce^{-u} = 0$ on $\partial\mathcal{M}$, we have $\psi \geq 0$ on $\partial\mathcal{M} \cap \mathcal{O}_{\delta_3}(x_0)$. Since $\eta \geq \delta_3^2$ on $\partial\mathcal{O}_{\delta_3}(x_0) \cap (\mathcal{M}\backslash\partial\mathcal{M})$ and since u and $|\nabla u|$ are bounded, we also have $\psi \geq 0$ on $\partial\mathcal{O}_{\delta_3}(x_0) \cap (\mathcal{M}\backslash\partial\mathcal{M})$ by choosing B large enough. Therefore

$$\psi \geq 0 \quad \text{on } \partial\mathcal{O}_{\delta_3}(x_0). \tag{4.5}$$

In the following we will show that

$$\mathcal{L}_u\psi \leq 0 \quad \text{in } \mathcal{O}_{\delta_3}(x_0) \tag{4.6}$$

if A is chosen large enough. It is clear that $\mathcal{L}_u(\eta) \leq C\left(1 + \mathrm{tr}_g(U^{-1})\right)$. This together with (4.2) gives

$$\begin{aligned} \mathcal{L}_u\psi &= A\mathcal{L}_u\varphi + B\mathcal{L}_u\eta \pm \mathcal{L}_u(u_\nu - ce^{-u}) \\ &\leq A\mathcal{L}_u\varphi + C\left(1 + \mathrm{tr}_g(U^{-1})\right). \end{aligned}$$

But from (4.3) and (4.4) it follows that

$$\begin{aligned} \mathcal{L}_u\varphi &= U^{ij}\varphi_{ij} - \mathrm{tr}_g(U^{-1})\langle \nabla u, \nabla\varphi\rangle_g \\ &\leq \frac{1}{2}\alpha_0 \mathrm{tr}_g(U^{-1}) - \alpha_0 \mathrm{tr}_g(U^{-1}) \\ &\leq -\frac{1}{2}\alpha_0 \mathrm{tr}_g(U^{-1}). \end{aligned}$$

Therefore

$$\mathcal{L}_u\psi \leq -\frac{1}{2}A\alpha_0 \mathrm{tr}_g(U^{-1}) + C\left(1 + \mathrm{tr}_g(U^{-1})\right) \quad \text{in } \mathcal{O}_{\delta_3}(x_0).$$

Note that

$$\mathrm{tr}_g(U^{-1}) \geq n \det(g \cdot U^{-1})^{\frac{1}{n}} = n \det(g^{-1} \cdot U)^{-\frac{1}{n}} = ne^{2u} \geq \beta_0 > 0$$

for some uniform constant $\beta_0 > 0$. Thus

$$\mathcal{L}_u \psi \leq -\frac{1}{2} A\alpha_0 \mathrm{tr}_g(U^{-1}) + C\mathrm{tr}_g(U^{-1}) \leq 0 \quad \text{in } \mathcal{O}_{\delta_3}(x_0)$$

if we choose A large enough.

By the maximum principle it follows from (4.5) and (4.6) that $\psi \geq 0$ in $\mathcal{O}_{\delta_3}(x_0)$. Since $\psi(x_0) = 0$, we therefore have $\psi_\nu(x_0) \geq 0$. Consequently $|\nabla^2 u(\nu, \nu)|(x_0) \leq C$ for some uniform constant C.

In order to complete the proof, we still need to prove claim (4.2). Fix a local orthonormal frame field $\{e_1, \ldots, e_n\}$ with $e_n = \nu$. For the local function u_ν it is easy to check

$$(u_\nu)_{ij} - u_{nij} = \Gamma_{in}^k u_{kj} + \Gamma_{jn}^k u_{ki} + b_{ij}^k u_k,$$

where

$$\Gamma_{ij}^k = \langle \nabla_{e_i} e_j, e_k \rangle_g \quad \text{and} \quad b_{ij}^k = e_j(\Gamma_{in}^k) + \Gamma_{jl}^k \Gamma_{in}^l - \Gamma_{ji}^l \Gamma_{ln}^k,$$

they are all bounded functions. Recall the commutation formula $u_{nij} - u_{ijn} = R_{ikjn} u_k$, we therefore have

$$(u_\nu)_{ij} - u_{ijn} = \Gamma_{in}^k u_{kj} + \Gamma_{jn}^k u_{ki} + (b_{ij}^k + R_{ikjn}) u_k$$

Noting that $(u_\nu)_l = u_{ln} + \Gamma_{ln}^k u_k$, we thus have

$$\begin{aligned} \mathcal{L}_u(u_\nu) = & \left(U^{ij} u_{ijn} - \mathrm{tr}_g(U^{-1}) u_l u_{ln} \right) + 2\Gamma_{in}^k U^{ij} u_{kj} \\ & + \left(b_{ij}^k + R_{ikjn} \right) U^{ij} u_k - \mathrm{tr}_g(U^{-1}) \Gamma_{ln}^k u_l u_k. \end{aligned}$$

By taking logarithm on (1.20) and then taking covariant differentiation, we have

$$U^{ij} \left(u_{ijn} + 2u_{in} u_j - u_l u_{ln} g_{ij} + (A_g)_{ij,n} \right) = -2nu_n.$$

Consequently

$$\begin{aligned} \mathcal{L}_u(u_\nu) = & -2nu_n + U^{ij} \left(2\Gamma_{in}^k u_{kj} - 2u_{in} u_j - (A_g)_{ij,n} \right) \\ & + \left(b_{ij}^k + R_{ikjn} \right) U^{ij} u_k - \mathrm{tr}_g(U^{-1}) \Gamma_{ln}^k u_l u_k. \end{aligned}$$

Since (U^{ij}) is positive definite, we have $|U^{ij}| \leq C\mathrm{tr}_g(U^{-1})$. Using the fact $U^{ij} U_{jk} = \delta_k^i$, the relation between U_{ij} and u_{ij} and the boundedness of $|\nabla u|_g$, one can see that

$$|U^{ij} u_{kj}| + |U^{ij} u_{in}| \leq C \left(1 + \mathrm{tr}_g(U^{-1}) \right)$$

Therefore

$$|\mathcal{L}_u(u_\nu)| \leq C \left(1 + \mathrm{tr}_g(U^{-1}) \right) \quad \text{in } \mathcal{M}_{\delta_1} \cap \mathcal{O}_1.$$

By direct calculation we also have

$$\mathcal{L}_u(e^{-u}) = -e^{-u} U^{ij} u_{ij} + e^{-u} U^{ij} u_i u_j + e^{-u} \mathrm{tr}_g(U^{-1}) |\nabla u|^2$$

Hence

$$|\mathcal{L}_u(e^{-u})| \le C\left(1 + \mathrm{tr}_g(U^{-1})\right) \quad \text{in } \mathcal{M}_{\delta_1} \cap \mathcal{O}_1.$$

Putting the above two estimates together, we therefore obtain (4.2). □

Theorem 1.6 now follows from the combination of Lemma 4.1 and the following result which provides more information than what we really need to complete the proof of Theorem 1.6.

Lemma 4.2 *Assume that* (f, Γ) *satisfies* (1.1)–(1.6) *and that* $(\mathcal{M}, g)$ *is a smooth compact Riemannian manifold with smooth umbilic boundary* $\partial\mathcal{M}$. *Let* $\mathcal{O}_1$ *be an open set of* $\mathcal{M}$ *and let* $u \in C^4(\mathcal{O}_1)$ *be a solution of* (1.16). *If*

$$|u| \le C_0 \quad \textit{and} \quad |\nabla u|_g \le C_0 \qquad \textit{in } \mathcal{O}_1 \tag{4.7}$$

and

$$\nabla^2 u(\nu, \nu) \le C_0 \quad \textit{on } \mathcal{O}_1 \cap \partial\mathcal{M} \tag{4.8}$$

for some constant C_0, *then, for any open set* $\mathcal{O}_2$ *of* $\mathcal{M}$ *satisfying* $\overline{\mathcal{O}}_2 \subset \mathcal{O}_1$,

$$|\nabla^2 u|_g \le C \quad \textit{in } \mathcal{O}_2$$

for some constant C *depending only on* n, c, C_0, g, (f, Γ), $\mathcal{O}_1$ *and* $\mathcal{O}_2$.

Proof Consider the function $w := e^u$. Recall that we can assume $h_g = 0$ on $\partial\mathcal{M}$, from (1.16) it is easy to check that

$$\begin{cases} F(W) := f(\lambda_g(W)) = w^{-1}, & \lambda_g(W) \in \Gamma \ \text{ in } \mathcal{O}_1, \\ \frac{\partial w}{\partial \nu} = c & \text{on } \mathcal{O}_1 \cap \partial\mathcal{M}, \end{cases} \tag{4.9}$$

where

$$W := \nabla^2 w - \frac{1}{2w}|\nabla w|_g^2 g + w A_g.$$

Moreover, it follows from (4.7) and (4.8) that

$$C_1^{-1} \le w \le C_1 \quad \text{and} \quad |\nabla w|_g \le C_1 \qquad \text{in } \mathcal{O}_1 \tag{4.10}$$

and

$$\nabla^2 w(\nu, \nu) \le C_1 \quad \text{on } \mathcal{O}_1 \cap \partial\mathcal{M} \tag{4.11}$$

for some positive constant C_1 depending only on C_0. Note that $\partial\mathcal{M}$ is totally geodesic in $(\mathcal{M}, g)$. Thus $\nabla_\tau \nu = 0$ on $\partial\mathcal{M}$ for any $\tau \in T(\partial\mathcal{M})$. Since $w_\nu = c$ on $\partial\mathcal{M}$, for any $\tau \in T(\partial\mathcal{M})$ there holds

$$\nabla^2 w(\nu, \tau) = \tau(w_\nu) - dw(\nabla_\tau \nu) = 0. \tag{4.12}$$

Now let $U(\mathcal{M})$ denote the unit tangent bundle over $\mathcal{M}$ and consider the function

$$Q(\xi) := \rho e^{\beta\varphi\circ\pi(\xi)} \nabla^2 w(\xi, \xi), \quad \xi \in U(\mathcal{M}),$$

where $\pi : T\mathcal{M} \to \mathcal{M}$ is the canonical projection, $\beta > 0$ is a positive constant to be chosen below, $\varphi \in C^\infty(\mathcal{M})$ is a fixed function satisfying $\varphi(x) = d_g(x, \partial\mathcal{M})$ in $\mathcal{M}_{\delta_0} := \{x \in \mathcal{M} : d_g(x, \partial\mathcal{M}) \le \delta_0\}$, and $\rho \in C_0^\infty(\mathcal{O}_1)$ is a cut-off function satisfying (2.2) and (2.3). Suppose the maximum of Q over $U(\mathcal{M})$ is attained at $\bar{\xi} \in T_{\bar{x}}\mathcal{M}$ for some $\bar{x} \in \mathcal{O}_1$. In the following we will assume that $\nabla^2 w(\bar{\xi}, \bar{\xi}) \geq 1$ since otherwise we are done. We have to consider two cases: either $\bar{x} \in \mathcal{O}_1 \cap \partial\mathcal{M}$ or $\bar{x} \in \mathcal{O}_1 \backslash \partial\mathcal{M}$.

Case 1. $\bar{x} \in \mathcal{O}_1 \cap \partial\mathcal{M}$. We write $\bar{\xi} = \alpha\nu + \delta\tau$, where τ is a unit vector in $T_{\bar{x}}(\partial\mathcal{M})$ and α and δ are two numbers satisfying $\alpha^2 + \delta^2 = 1$. Then by using the maximality of $\nabla^2 w(\bar{\xi}, \bar{\xi})$ and (4.12) one can see that at $\bar{x}$ there holds

$$\begin{aligned}\nabla^2 w(\bar{\xi}, \bar{\xi}) &= \alpha^2 \nabla^2 w(\nu, \nu) + \delta^2 \nabla^2 w(\tau, \tau) + 2\alpha\delta\nabla^2 w(\nu, \tau)\\ &\leq \left(\alpha^2 + \delta^2\right) \nabla^2 w(\bar{\xi}, \bar{\xi})\\ &= \nabla^2 w(\bar{\xi}, \bar{\xi}).\end{aligned}$$

This implies that we can take $\bar{\xi}$ so that either $\bar{\xi} = \nu$ or $\bar{\xi} \in T_{\bar{x}}(\partial\mathcal{M})$.

If $\bar{\xi} = \nu$, then (4.11) implies that $Q(\bar{\xi}) \leq C$ for some uniform constant C. So we may assume that $\bar{\xi}$ is a unit vector in $T_{\bar{x}}(\partial\mathcal{M})$. Choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ around $\bar{x}$ so that $e_n = \nu$ on $\partial\mathcal{M}$, and write $\bar{\xi} = \bar{\xi}^i e_i$ at $\bar{x}$. We then define a vector field ξ near $\bar{x}$ by $\xi = \xi^i e_i$, where $\xi^i(x) = \bar{\xi}^i$ for x near $\bar{x}$. Note that $\xi^n = 0$ near $\bar{x}$. It is clear that ξ is a smooth local section of $U(\mathcal{M})$. Thus $Q := Q(\xi)$ has a local maximum at $\bar{x}$. This implies that

$$Q_n \leq 0 \quad \text{at } \bar{x}. \tag{4.13}$$

Set $E = \nabla^2 w(\xi, \xi)$. Then, since $\rho_n = 0$ and $\varphi_n = 1$ on $\mathcal{O}_1 \cap \partial\mathcal{M}$, it follows from (4.13) that

$$E_n + \beta E \leq 0 \quad \text{at } \bar{x}. \tag{4.14}$$

Observe that

$$\begin{aligned}E_n = \nabla_n(\xi^i\xi^j w_{ij}) &= \xi^i\xi^j w_{ijn} + 2\xi^i\nabla_n\xi^j w_{ij}\\ &= \xi^i\xi^j\left(w_{nij} + R_{kijn}w_k\right) + 2\xi^i\nabla_n\xi^j w_{ij}.\end{aligned}$$

As calculated in the proof of Lemma 4.1, with the same notations as there we have for $1 \leq i, j \leq n-1$

$$\begin{aligned}w_{nij} &= (w_\nu)_{ij} - \Gamma_{in}^k w_{kj} - \Gamma_{jn}^k w_{ki} - b_{ij}^k w_k\\ &= -\Gamma_{in}^k w_{kj} - \Gamma_{jn}^k w_{ki} - b_{ij}^k w_k.\end{aligned}$$

By the maximality of $Q(\bar{\xi})$ we have $w_{ii} \leq CE$ at $\bar{x}$ for $1 \leq i \leq n$. Since $\Delta w \geq 0$ in $\mathcal{M}$, we further have $|\nabla^2 w| \leq CE$ at $\bar{x}$. Thus $E_n \geq -C_3 - C_4 E$ for some uniform constants C_3 and C_4. This together with (4.14) implies that

$$(\beta - C_4)E \leq C_3 \quad \text{at } \bar{x}.$$

Therefore $E \leq C$ if we choose $\beta > C_4$. Consequently $Q(\bar{\xi}) \leq C$.

Case 2. $\bar{x} \in \mathcal{O}_1 \backslash \partial\mathcal{M}$. Choose normal coordinates $x^1, \ldots, x^n$ around $\bar{x}$ such that

$$g_{ij} = \delta_{ij} \quad \text{and} \quad \frac{\partial g_{ij}}{\partial x^k} = 0 \qquad \text{at } \bar{x}.$$

Moreover, such normal coordinates can be chosen so that $\{w_{ij}\}$ is diagonal at $\bar{x}$ and $w_{11} = \nabla^2 w(\bar{\xi}, \bar{\xi})$.

Consider the local function $Z := w_{11}/g_{11}$. By direct calculation we have

$$Z_i = w_{11i} \quad \text{and} \quad Z_{ij} = w_{11ij} \qquad \text{at } \bar{x}.$$

It is clear that the function

$$\widetilde{Q} := \rho e^{\beta\varphi} Z$$

has a local maximum at $\bar{x}$. Thus at $\bar{x}$ we have

$$0 = \widetilde{Q}_i = \rho e^{\beta\varphi} w_{11i} + \left(\beta\varphi_i + \frac{\rho_i}{\rho}\right)\widetilde{Q} \tag{4.15}$$

and

$$\begin{aligned} 0 &\geq (\widetilde{Q}_{ij}) \\ &= \left((\beta\varphi_{ij} - \beta^2\varphi_i\varphi_j)\widetilde{Q} + \frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2}\widetilde{Q} - \frac{\rho_i\varphi_j + \rho_j\varphi_i}{\rho}\beta\widetilde{Q} + \rho e^{\beta\varphi} w_{11ij}\right). \end{aligned}$$

Let $F^{ij} := \frac{\partial F}{\partial W_{ij}}(W)$ and $\mathcal{T} := \mathrm{tr}_g(F^{ij})$. Then using $|\nabla\rho| \leq C\sqrt{\rho}$ and the inequality

$$|w_{11ij} - w_{ij11}| \leq C|\nabla^2 w| \leq Cw_{11} \leq \frac{C}{\rho}\widetilde{Q},$$

we have

$$0 \geq e^{-\beta\varphi} F^{ij}\widetilde{Q}_{ij} \geq \rho F^{ij} w_{11ij} - \frac{C}{\rho}\mathcal{T}\widetilde{Q} \geq \rho F^{ij} w_{ij11} - \frac{C}{\rho}\mathcal{T}\widetilde{Q}. \tag{4.16}$$

By differentiating (4.9) twice and using the concavity of F we get

$$\begin{aligned} &F^{ij} w_{ij11} - \mathcal{T}\Big(w^{-1}\sum_l w_l w_{l11} + w^{-1}\sum_l w_{l1}^2 - 2w^{-2}\sum_l w_l w_{l1} w_1 \\ &\quad + w^{-3} w_1^2 |\nabla w|^2 - \frac{1}{2} w^{-2}|\nabla w|^2 w_{11}\Big) + F^{ij}\left(w_{11}A_{ij} + 2w_1 A_{ij,1} + wA_{ij,11}\right) \\ &\geq -w^{-2} w_{11} + 2w^{-3} w_1^2. \end{aligned}$$

This together with commutation formula and (4.15) implies that

$$\begin{aligned} \rho F^{ij} w_{ij11} &\geq -C\rho\mathcal{T} w_{11} + \rho w^{-1}\mathcal{T}\sum_l w_l w_{l11} + \rho w^{-1}\mathcal{T} w_{11}^2 \\ &\geq -C\rho\mathcal{T} w_{11} + \rho w^{-1}\mathcal{T}\sum_l w_l w_{11l} + \rho w^{-1}\mathcal{T} w_{11}^2 \\ &\geq -\frac{C}{\rho}\mathcal{T}\tilde{Q} + \rho w^{-1}\mathcal{T} w_{11}^2. \end{aligned}$$

Thus, it follows from (4.16) that $\widetilde{Q}^2 \leq C\widetilde{Q}$. Consequently $Q(\bar{\xi}) = \widetilde{Q}(\bar{x}) \leq C$. The proof is complete. □

5 Some existence results

Let $\mathcal{M}$ be a smooth compact manifold with smooth boundary $\partial\mathcal{M}$. We make use of the double $\widehat{\mathcal{M}}$ of $\mathcal{M}$ which is obtained by gluing two copies of $\mathcal{M}$ along the boundary $\partial\mathcal{M}$. There is a canonical way to make $\widehat{\mathcal{M}}$ into a smooth compact manifold without boundary [49]. Given a smooth Riemannian metric g on $\mathcal{M}$, there is a standard metric $\hat{g}$ on $\widehat{\mathcal{M}}$ induced from g. In general $\hat{g}$ is only continuous on $\widehat{\mathcal{M}}$. However, if $\partial\mathcal{M}$ is totally geodesic in $(\mathcal{M}, g)$, then $\hat{g}$ is $C^{2,1}$ on $\widehat{\mathcal{M}}$, see [12, Appendix] for instance.

5.1 Proof of Theorem 1.1

We may assume that $(\mathcal{M}, g)$ is not conformally equivalent to the standard half sphere $\mathbb{S}^n_+$ since otherwise the existence result is obvious.

First note that we may assume $\lambda(A_g) \in \Gamma$ on $\mathcal{M}$ and $h_g > 0$ on $\partial\mathcal{M}$ in the following argument. To see this, consider the metric $g_\varepsilon := (1-\varepsilon\varphi)^{\frac{4}{n-2}} g$, where $\varepsilon > 0$ is a small number, and $\varphi \in C^\infty(\mathcal{M})$ is a function such that $\varphi(x) = d_g(x, \partial\mathcal{M})$ when $d_g(x, \partial\mathcal{M}) \le \delta_0$. Since $\lambda(A_g) \in \Gamma$ on $\mathcal{M}$, we can fix an $\varepsilon > 0$ small enough so that $\lambda(A_{g_\varepsilon}) \in \Gamma$ on $\mathcal{M}$. Then noting that $\varphi = 0$, $\frac{\partial\varphi}{\partial\nu} = 1$ and $h_g \ge 0$ on $\partial\mathcal{M}$ we have

$$h_{g_\varepsilon} = -\frac{2}{n-2}\frac{\partial}{\partial\nu}(1-\varepsilon\varphi) + h_g = \frac{2\varepsilon}{n-2} + h_g > 0 \quad \text{on } \partial\mathcal{M},$$

Since $R_g > 0$ on $\mathcal{M}$ and $h_g > 0$ on $\partial\mathcal{M}$, one can find a metric g_0 conformal to g such that $R_{g_0} > 0$ on $\mathcal{M}$ and $h_{g_0} = 0$ on $\partial\mathcal{M}$, see [23, Theorem 0.1] for instance. Write $g = e^{-2\varphi} g_0$ for some function $\varphi \in C^\infty(\mathcal{M})$. Let $u \in C^\infty(\mathcal{M})$ be a solution of (1.10) with $c = 0$ and let $\tilde{g} := e^{-2u} g$. Then $\tilde{g} = e^{-2v} g_0$ with $v = u + \varphi$ and $h_{\tilde{g}} = 0$ on $\partial\mathcal{M}$. Let $\widehat{\mathcal{M}}$ denote the double of $\mathcal{M}$, and let $\hat{g}_0$ and $\hat{\tilde{g}}$ denote the standard metrics on $\widehat{\mathcal{M}}$ induced from g_0 and $\tilde{g}$ respectively. Since $\partial\mathcal{M}$ is totally geodesic in both $(\mathcal{M}, g_0)$ and $(\mathcal{M}, \tilde{g})$, $\hat{g}_0$ and $\hat{\tilde{g}}$ are in $C^{2,1}(\widehat{\mathcal{M}})$. Moreover, $\hat{\tilde{g}}$ is still conformal to $\hat{g}_0$ with $\hat{\tilde{g}} = e^{-2\hat{v}} \hat{g}_0$ for some function $\hat{v} \in C^{2,1}(\widehat{\mathcal{M}})$, and

$$f(\lambda(A_{e^{-2\hat{v}}\hat{g}_0})) = 1, \quad \lambda(A_{e^{-2\hat{v}}\hat{g}_0}) \in \Gamma \qquad \text{on } \widehat{\mathcal{M}}.$$

Since $(\mathcal{M}, g_0)$ is locally conformally flat, so is $(\widehat{\mathcal{M}}, \hat{g}_0)$. Note that $R_{\hat{g}_0} > 0$ on $\widehat{\mathcal{M}}$ and $(\widehat{\mathcal{M}}, \hat{g}_0)$ is not conformally equivalent to $\mathbb{S}^n$. Therefore, it follows from the proof of [27, Theorem 1] that

$$|\nabla \hat{v}| \le C \quad \text{and} \quad \hat{v} \ge -C \qquad \text{on } \widehat{\mathcal{M}}$$

for some universal constant C. However, an upper bound for $\hat{v}$ is not yet available since we do not have $\lambda(A_{\hat{g}_0}) \in \Gamma$. Since $\hat{v} = v$ and $v = u + \varphi$ on $\mathcal{M}$, we have

$$|\nabla u| \le C_0 \quad \text{and} \quad u \ge -C_0 \qquad \text{on } \mathcal{M} \tag{5.1}$$

for some universal constant C_0.

Returning to problem (1.10) with $c = 0$. Suppose the minimum of u over $\mathcal{M}$ is attained at some point $x_0 \in \mathcal{M}$. Since $h_g > 0$ on $\partial\mathcal{M}$, we have $x_0 \in \mathcal{M}\backslash\partial\mathcal{M}$. Thus $\nabla u = 0$ and $\nabla^2 u \ge 0$ at x_0. Since $\lambda(A_g) \in \Gamma$, we therefore have $e^{-2u} \ge f(\lambda(A_g)) > 0$. Consequently there is a uniform constant C such that

$$\min_{\mathcal{M}} u \le C. \tag{5.2}$$

Combining (5.1) and (5.2) gives

$$-C \le u \le C \quad \text{and} \quad |\nabla u| \le C \qquad \text{on } \mathcal{M}.$$

Therefore it follows from Remark 3.1 that

$$\|u\|_{C^{4,\alpha}(\mathcal{M})} \le C$$

for some uniform constant C.

Now we will use the degree theory argument to prove the existence. To this end, as in [26], for each $0 \le t \le 1$ let

$$f_t(\lambda) := f(t\lambda + (1-t)\sigma_1(\lambda)e)$$

which is defined on

$$\Gamma_t := \left\{\lambda \in \mathbb{R}^n : t\lambda + (1-t)\sigma_1(\lambda)e \in \Gamma\right\},$$

where $e = (1, 1, \dots, 1)$.

We now consider the problem

$$\begin{cases} f_t(\lambda(A_{g_u})) = 1, & \lambda(A_{g_u}) \in \Gamma_t \text{ on } \mathcal{M}, \\ h_{g_u} = 0 & \text{on } \mathcal{M}, \end{cases}$$

where $g_u = e^{-2u}g$ for some smooth function u on $\mathcal{M}$. From the above argument we have already obtained $\|u\|_{C^{4,\alpha}(\mathcal{M})} \le C$ for some uniform constant independent of t. Now we set

$$\begin{aligned} \mathcal{O}_t^* = \Big\{ & u \in C^{4,\alpha}(\mathcal{M}) : \lambda(A_{g_u}) \in \Gamma_t, \|u\|_{C^{4,\alpha}(\mathcal{M})} < 2C, \\ & \frac{1}{2} < f_t(\lambda(A_{g_u})) < 2 \text{ on } \mathcal{M} \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\mathcal{M} \Big\}. \end{aligned}$$

Define $F_t : \mathcal{O}_t^* \to C^{2,\alpha}(\mathcal{M})$ by $F_t[u] := f_t(\lambda(A_{g_u})) - 1$. It follows from [31] that $\deg(F_t, \mathcal{O}_t^*, 0)$ is well-defined and is independent of t. But when $t = 0$ the corresponding problem is the Yamabe problem with boundary. Based on [39] it was shown in [23] that $\deg(F_0, \mathcal{O}_0^*, 0) = -1$. Therefore $\deg(F_1, \mathcal{O}_1^*, 0) = -1 \neq 0$. The proof is thus complete.

5.2 Proof of Theorem 1.2

The proof of Theorem 1.2 is based on some lemmas in the following.

Lemma 5.1 *Let* (f, Γ) *satisfy* (1.1)–(1.6) *and let* $(\mathcal{M}, g)$ *be a smooth compact Riemannian manifold with smooth boundary* $\partial\mathcal{M}$. *Suppose* $\partial\mathcal{M}$ *is umbilic and* $(\mathcal{M}, g)$ *is locally conformally flat near* $\partial\mathcal{M}$. *Let* $u \in C^4(\mathcal{M})$ *be a solution of* (1.14) *with* η *being positive. If* $|u| \le C_0$ *on* $\mathcal{M}$, *then*

$$|\nabla u|_g + |\nabla^2 u|_g \le C \quad \text{on } \mathcal{M}$$

for some constant C *depending only on* n, C_0, g, ψ, η, *and* (f, Γ).

Proof This is the combination of Theorems 1.3 and 1.5. □

Lemma 5.2 *Let* (f, Γ) *and* $(\mathcal{M}, g)$ *be as in Lemma* 5.1 *with* $\lambda(A_g) \in \Gamma$ *on* $\mathcal{M}$ *and* $h_g \ge 0$ *on* $\partial\mathcal{M}$. *Then problem* (1.14) *with* $\mathcal{O}_1 = \mathcal{M}$, $\psi(x,z) = \psi_0(x)e^{az}$ *and* $\eta(x,z) = \eta_0(x)e^{bz}$ *has a unique solution, where* a *and* b *are positive constants, and* $\psi_0 \in C^2(\mathcal{M})$ *and* $\eta_0 \in C^2(\partial\mathcal{M})$ *are positive functions.*

Proof By perturbing g as in the proof of Theorem 1.1, we may assume that $\lambda(A_g) \in \Gamma$ on $\mathcal{M}$ and $h_g > 0$ on $\partial\mathcal{M}$. From the maximum principle, it is easy to check that there is a positive uniform constant C such that $-C \le u \le C$ on $\mathcal{M}$ for any solution u of (1.14) with $\psi(x,z) = \psi_0(x)e^{az}$ and $\eta(x,z) = \eta_0(x)e^{bz}$. Therefore, it follows from Lemma 5.1 and the result of Lieberman and Trudinger [34] that we have uniform $C^{2,\alpha}(\mathcal{M})$

estimates on u. Since $a > 0$, $b > 0$, ψ_0 and η_0 are positive, the linearized problem is uniquely solvable. Therefore, the method of continuity concludes the existence and uniqueness. □

Next we will use the recent results of Trudinger and Wang in [44] to establish a Harnack type inequality.

Lemma 5.3 *Let* $(\mathcal{M}, g)$ *be a smooth compact Riemannian manifold with smooth boundary* $\partial\mathcal{M}$. *Suppose* $\partial\mathcal{M}$ *is umbilic and* $(\mathcal{M}, g)$ *is locally conformally flat near* $\partial\mathcal{M}$. *For* $k > \frac{n}{2}$, *let* $[g]_k^+$ *denote the set of* C^∞ *metrics* $\tilde{g}$ *conformal to* g *such that* $\lambda(A_{\tilde{g}}) \in \Gamma_k$ *on* $\mathcal{M}$ *and* $h_{\tilde{g}} \geq 0$ *on* $\partial\mathcal{M}$. *If* $(\mathcal{M}, g)$ *is not conformally equivalent to the standard half sphere* $\mathbb{S}_+^n$, *then there is a positive constant* C *depending only on* k *and* $(\mathcal{M}, g)$ *such that for any metric* $\tilde{g} := \chi g \in [g]_k^+$ *there holds*

$$\max_{\mathcal{M}} \chi \leq C \min_{\mathcal{M}} \chi. \tag{5.3}$$

Proof Let $[g]_k^*$ denote the set of metrics $\tilde{g}$ conformal to g such that $\lambda(A_{\tilde{g}}) \in \Gamma_k$ on $\mathcal{M}$ and $h_{\tilde{g}} > 0$ on $\partial\mathcal{M}$. We remark that it suffices to establish the Harnack inequality (5.3) for $\tilde{g} \in [g]_k^*$. Indeed, for any metric $\tilde{g} = \chi g \in [g]_k^+$, as in the proof of Theorem 1.1 we can find a function $\varphi \in C^\infty(\mathcal{M})$ with $\frac{1}{2} \leq \varphi \leq 1$ on $\mathcal{M}$ such that $\varphi\tilde{g} = (\varphi\chi)g \in [g]_k^*$. Thus we have

$$\frac{1}{2} \max_{\mathcal{M}} \chi \leq \max_{\mathcal{M}} (\varphi\chi) \leq C \min_{\mathcal{M}} (\varphi\chi) \leq C \min_{\mathcal{M}} \chi,$$

which gives the desired inequality.

By a conformal deformation of g, without loss of generality, we may assume that $h_g = 0$ on $\partial\mathcal{M}$. Since $\partial\mathcal{M}$ is umbilic, it must be totally geodesic in $(\mathcal{M}, g)$. Let $\widehat{\mathcal{M}}$ be the double of $\mathcal{M}$. For any metric $\tilde{g}$ on $\mathcal{M}$, there is a standard metric $\hat{\tilde{g}}$ on $\widehat{\mathcal{M}}$ induced by $\tilde{g}$. In general $\hat{\tilde{g}}$ is only continuous on $\widehat{\mathcal{M}}$. However, since $\partial\mathcal{M}$ is totally geodesic in $(\mathcal{M}, g)$, it follows from [12, Appendix] that $\hat{g}$ is $C^{2,1}$ on $\widehat{\mathcal{M}}$.

For any metric $\tilde{g} \in [g]_k^*$, note that $\hat{\tilde{g}}$ is conformal to $\hat{g}$, we may write $\hat{\tilde{g}} = e^{-2\hat{w}}\hat{g}$ for some function $\hat{w} \in C^0(\widehat{\mathcal{M}})$ which is C^∞ in $\widehat{\mathcal{M}} \backslash \partial\mathcal{M}$. Let $w_1 := \hat{w}|_{\mathcal{M}}$ and $w_2 := w|_{\widehat{\mathcal{M}} \backslash (\mathcal{M} \backslash \partial\mathcal{M})}$. Since $h_{\tilde{g}} > 0$ on $\partial\mathcal{M}$, we have $\frac{\partial w_1}{\partial \nu_1} > 0$ and $\frac{\partial w_2}{\partial \nu_2} > 0$ on $\partial\mathcal{M}$, where ν_1 and ν_2 denote the inward unit normal vector fields to $\partial\mathcal{M}$ in $(\mathcal{M}, g)$ and $(\widehat{\mathcal{M}} \backslash (\mathcal{M} \backslash \partial\mathcal{M}), \hat{g})$ respectively. Thus we may smoothly extend w_1 and w_2 to a neighborhood $\mathcal{U}$ of $\partial\mathcal{M}$ in $\widehat{\mathcal{M}}$ so that $w_i \leq w$ and $\lambda(A_{\hat{g}_{w_i}}) \in \Gamma_k$ on $\mathcal{U}$ for $i = 1, 2$, where $\hat{g}_{w_i} := e^{-2w_i}\hat{g}$. Therefore, noting that our background metric $\hat{g}$ is $C^{2,1}$ on $\widehat{\mathcal{M}}$ and $\hat{w} = \max\{w_1, w_2\}$ on $\mathcal{U}$, we can apply [44, Lemma 3.7] to conclude that $\hat{\tilde{g}} = e^{-2\hat{w}}\hat{g}$ is k-admissible in the sense of Trudinger and Wang (this will be simply called k-admissible in the sequel).

In the following we will follow the idea in [44] to give the proof of Lemma 5.3. Suppose the Harnack inequality (5.3) does not hold. Then there is a sequence of smooth metrics $g_j := e^{-2w_j}g \in [g]_k^*$ such that

$$\max_{\mathcal{M}} w_j - \min_{\mathcal{M}} w_j \geq j, \quad j = 1, 2, \ldots.$$

By subtracting a constant if necessary, we may assume $\max_{\mathcal{M}} w_j = 0$. Then $\min_{\mathcal{M}} w_j \to -\infty$ as $j \to \infty$. Consider the function $\hat{w}_j$ on $\widehat{\mathcal{M}}$ induced by w_j through $\hat{g}_j = e^{-2\hat{w}_j}\hat{g}$. From the above argument we know $\hat{g}_j$ is k-admissible. Thus [44, Lemma 3.1] shows that for $\hat{u}_j := e^{\hat{w}_j}$ there holds the Hölder estimate

$$\frac{|\hat{u}_j(x)-\hat{u}_j(y)|}{d_{\hat{g}}(x,y)^{\alpha}} \le C \int\limits_{\widehat{\mathcal{M}}} \hat{u}_j \mathrm{d}\mu_{\hat{g}}$$

for some $0 < \alpha \le 2 - \frac{n}{k}$, where α and C are independent of j. Note that $0 < \hat{u}_j \le 1$. By Arzela–Ascoli theorem we may assume that $\hat{u}_j \to \hat{u}$ uniformly on $\widehat{\mathcal{M}}$ for some function $\hat{u} \in C^{\alpha}(\widehat{\mathcal{M}})$ with $0 \le \hat{u} \le 1$. Since $\max_{\mathcal{M}} \hat{u}_j = 1$, we have $\hat{u} \not\equiv 0$. Define $\hat{w} := \log \hat{u}$, and let

$$S_{\hat{w}} := \bigcap_{\beta<0} \left\{x \in \widehat{\mathcal{M}} : \hat{w}(x) < -\beta\right\}$$

which is called the set of singularity points of $\hat{w}$. Since $\min_{\mathcal{M}} \hat{u}_j \to 0$ as $j \to \infty$, we know $S_{\hat{w}} \neq \emptyset$. Moreover, since each $\hat{g}_j$ is k-admissible, as the limit $\hat{g}_{\hat{w}} := e^{-2\hat{w}}\hat{g}$ is also k-admissible on $\widehat{\mathcal{M}}$.

Since $\hat{g}$ is smooth away from $\partial\mathcal{M}$ and is locally conformally flat near $\partial\mathcal{M}$, by the k-admissibility of $\hat{g}_{\hat{w}}$, the argument in [44] shows that near any singularity point x_0 of $\hat{w}$ there holds

$$\hat{w}(x) = 2\,\log|x - x_0| + o(1) \tag{5.4}$$

in a normal neighborhood of x_0; moreover, the singularity points are isolated. For a fixed point $y \in \widehat{\mathcal{M}}\backslash S_{\hat{w}}$, by using the Bishop volume comparison theorem and an approximation argument it was shown in [44, Lemma 3.4] that the ratio

$$Q(r) := \frac{\mathrm{Vol}_{\hat{g}_{\hat{w}}}\left(B_{y,r}[\hat{g}_{\hat{w}}]\right)}{r^n} \le \omega_n, \quad 0 < r < \infty,$$

where $B_{y,r}[\hat{g}_{\hat{w}}]$ denotes the geodesic ball in $\mathcal{M}$ of radius r with center at y, and ω_n is the volume of the unit ball in $\mathbb{R}^n$. But by (5.4) it was shown in [44, Lemma 3.4] that each singularity point of $\hat{w}$ contributes a factor ω_n to the ratio $Q(r)$. Therefore $S_{\hat{w}}$ must consists of a single point, say $S_{\hat{w}} = \{x_0\}$, and $Q(r) \equiv \omega_n$. Moreover, noting the symmetry of $\hat{w}$ with respect to $\partial\mathcal{M}$, we must have $x_0 \in \partial\mathcal{M}$.

Next we are going to show that $\hat{w}$ is $C^{2,1}$ away from x_0. Since $\hat{g}$ is smooth in $\widehat{\mathcal{M}}\backslash\partial\mathcal{M}$, the argument of [44, Lemma 3.5] can be applied directly to show that $\hat{w} \in C^{1,1}(\widehat{\mathcal{M}}\backslash\partial\mathcal{M})$. However, since $\hat{g}$ is only $C^{2,1}$ across $\partial\mathcal{M}$, when we consider the regularity at a point $y_0 \in \partial\mathcal{M}\backslash\{x_0\}$, we need to check carefully the proof of [44, Lemma 3.5] when using the existence result on a Dirichlet problem in [16]. We may choose a neighborhood $\mathcal{O}_1$ of y_0 in $\widehat{\mathcal{M}}\backslash\{x_0\}$ on which $\hat{g}$ is conformally flat, i.e. $\hat{g} = e^{-2\eta}\hat{g}_0$ for some function $\eta \in C^{2,1}(\mathcal{O}_1)$, where $\hat{g}_0$ is the flat metric. Then $\hat{g}_{\hat{w}} = e^{-2\hat{v}}\hat{g}_0$ with $\hat{v} = \hat{w} + \eta$. Since $\hat{g}_{\hat{w}}$ is k-admissible, there is a sequence of k-admissible metrics $\hat{g}_j := e^{-2\hat{v}_j}\hat{g}_0$ with $\hat{v}_j$ smooth on $\mathcal{O}_1$ and $\hat{v}_j \to \hat{v}$ uniformly on $\mathcal{O}_2$ for some neighborhood $\mathcal{O}_2$ of y_0 satisfying $\overline{\mathcal{O}}_2 \subset \mathcal{O}_1$. Let $\{\varepsilon_j\}$ be a sequence of positive numbers such that $\varepsilon_j \searrow 0$ and

$$0 < \varepsilon_j < \sigma_k\left(\lambda_{\hat{g}_0}\left(\nabla_0^2\hat{v}_j + d\hat{v}_j \otimes d\hat{v}_j - \frac{1}{2}|\nabla_0\hat{v}_j|^2\hat{g}_0\right)\right)$$

where ∇_0 denotes the Levi–Civita connection of $\hat{g}_0$. Consider the problem

$$\begin{cases} \sigma_k\left(\lambda_{\hat{g}_0}\left(\nabla_0^2\hat{\varphi}_j + d\hat{\varphi}_j \otimes d\hat{\varphi}_j - \frac{1}{2}|\nabla_0\hat{\varphi}_j|^2\hat{g}_0\right)\right) = \varepsilon_j & \text{on } \mathcal{O}_2, \\ \hat{\varphi}_j = \hat{v}_j & \text{on } \partial\mathcal{O}_2. \end{cases}$$

It follows from [16] that such $\hat{\varphi}_j$ exists, $\|\hat{\varphi}_j\|_{C^2(\mathcal{O}_2)} \le C$ and $\{\hat{\varphi}_j\}$ is monotone increasing. Define $\hat{\varphi} := \lim_{j\to\infty} \hat{\varphi}_j$, then $\hat{\varphi} \in C^{1,1}(\mathcal{O}_2)$. As shown in [44, Lemma 3.5] $\hat{\varphi} = \hat{v}$ on $\mathcal{O}_2$.

Therefore $\hat{w} = \hat{v} - \eta \in C^{1,1}(\mathcal{O}_2)$. Combining the above we obtain $\hat{w} \in C^{1,1}(\widehat{\mathcal{M}}\backslash\{x_0\})$. By the symmetry of $\hat{w}$ with respect to $\partial\mathcal{M}$ we have $\frac{\partial \hat{w}}{\partial \nu} = 0$ on $\partial\mathcal{M}\backslash\{x_0\}$. Moreover the argument in [44, Lemma 3.5] gives $R_{\hat{g}_{\hat{w}}} = 0$ a.e. on $\widehat{\mathcal{M}}$, where $R_{\hat{g}_{\hat{w}}}$ denotes the scalar curvature of $\hat{g}_{\hat{w}}$. Thus by the Yamabe equation we know $\hat{v} := e^{-\frac{n-2}{2}\hat{w}}$ satisfies

$$\begin{cases} -\Delta_{\hat{g}} \hat{v} + \frac{n-2}{4(n-1)} R_{\hat{g}} \hat{v} = 0 & \text{in } \mathcal{M}\backslash\{x_0\}, \\ \frac{\partial \hat{v}}{\partial \nu} = 0 & \text{on } \partial\mathcal{M}\backslash\{x_0\}. \end{cases}$$

Note that $\hat{g}$ and $R_{\hat{g}}$ are smooth on $\mathcal{M}$, it follows from the regularity theory for uniformly elliptic equation with Neumann boundary condition that $\hat{v} \in C^\infty(\mathcal{M}\backslash\{x_0\})$. Thus $\hat{w} \in C^\infty(\mathcal{M}\backslash\{x_0\})$. By symmetry we correspondingly have $\hat{w} \in C^\infty((\widehat{\mathcal{M}}\backslash(\mathcal{M}\backslash\partial\mathcal{M}))\backslash\{x_0\})$. Therefore $\hat{w} \in C^{2,1}(\widehat{\mathcal{M}}\backslash\{x_0\})$.

By the k-admissibility of $\hat{g}_{\hat{w}}$ and the result in [17] we know $\hat{g}_{\hat{w}}$ has nonnegative Ricci curvature. The asymptotic formula (5.4) implies that $(\widehat{\mathcal{M}}\backslash\{x_0\}, \hat{g}_{\hat{w}})$ is a complete manifold with $Q(r) = \omega_n$. Hence $(\widehat{\mathcal{M}}\backslash\{x_0\}, \hat{g}_{\hat{w}})$ is isometric to the Euclidean space. Consequently $(\widehat{\mathcal{M}}, \hat{g})$ is conformally equivalent to $\mathbb{S}^n$. This contradicts the assumption that $(\mathcal{M}, g)$ is not conformally equivalent to $\mathbb{S}^n_+$. □

Proof of Theorem 1.2 Without loss of generality, we may assume that $(\mathcal{M}, g)$ is not conformally equivalent to $\mathbb{S}^n_+$. As indicated in the proof of Theorem 1.1, we may assume $\lambda(A_g) \in \Gamma$ on $\mathcal{M}$ and $h_g > 0$ on $\partial\mathcal{M}$. We will adapt the idea in the proof of [44, Theorem C] to complete the argument.

For a positive function $v \in C^2(\mathcal{M})$ we will use the notation

$$V[v] = -\nabla^2 v + \frac{n}{n-2}\frac{\nabla v \otimes \nabla v}{v} - \frac{1}{n-2}\frac{|\nabla v|^2}{v} g + \frac{n-2}{2} v A_g. \tag{5.5}$$

Note that for the metric $g_v := v^{\frac{4}{n-2}} g$ we have

$$A_{g_v} = \frac{2}{n-2} v^{-1} V[v] \quad \text{on } \mathcal{M} \quad \text{and} \quad h_{g_v} = v^{-\frac{n}{n-2}}\left(-\frac{2}{n-2}\frac{\partial v}{\partial \nu} + h_g v\right) \quad \text{on } \partial\mathcal{M}.$$

Thus, if we can prove the existence of a positive function $v \in C^2(\mathcal{M})$ such that

$$\begin{cases} f(\lambda(V[v])) = \frac{n-2}{2}\varphi_0 v^p, & \lambda(V[v]) \in \Gamma \text{ on } \mathcal{M}, \\ \frac{\partial v}{\partial \nu} - \frac{n-2}{2} h_g v = -\frac{n-2}{2} h_0 v^q & \text{on } \partial\mathcal{M}, \end{cases} \tag{5.6}$$

where $p = \frac{n+2}{n-2}$, $q = \frac{n}{n-2}$, and $\lambda(V[v])$ denote the eigenvalues of $V[v]$ with respect to g, then $u = -\frac{2}{n-2}\log v$ is a solution of (1.13).

Since $\lambda(A_g) \in \Gamma$ on $\mathcal{M}$, $h_g > 0$ on $\partial\mathcal{M}$, $p > 1$ and $q > 1$, it is always possible to find positive numbers α_0 and ε_0 such that

$$\begin{cases} f(\lambda(V[\alpha_0])) > \frac{n-2}{2}\varphi_0\left(\alpha_0^2 + \varepsilon_0^2\right)^{p/2} & \text{on } \mathcal{M}, \\ -h_g \alpha_0 < -h_0(\alpha_0^2 + \varepsilon_0^2)^{q/2} & \text{on } \partial\mathcal{M}. \end{cases} \tag{5.7}$$

Let $\varepsilon \in (0, \varepsilon_0]$ be any number. We consider the auxiliary problem $(P_{t,\varepsilon})$ as follows

$$\begin{cases} f(\lambda(V[v])) = \frac{n-2}{2} t\varphi_0\left(v^2 + \varepsilon^2\right)^{p/2}, & \lambda(V[v]) \in \Gamma \quad \text{on } \mathcal{M}, \\ \frac{\partial v}{\partial \nu} - \frac{n-2}{2} h_g v = -\frac{n-2}{2} t h_0 \left(v^2 + \varepsilon^2\right)^{q/2} & \text{on } \partial\mathcal{M}, \end{cases}$$

where $t > 0$ is a parameter.

Claim 1. For any $t_0 > 0$ there exists a positive constant C independent of ε such that any solution v of $(P_{t,\varepsilon})$ with $t \geq t_0$ and $0 < \varepsilon \leq \varepsilon_0$ satisfies $v \leq C$ on $\mathcal{M}$. Moreover, for each $0 < \varepsilon \leq \varepsilon_0$ there exists $\bar{t} > 1$ such that $(P_{t,\varepsilon})$ has no solution for $t \geq \bar{t}$.

Indeed, suppose there exist two sequences $\{t_j\}$ and $\{\varepsilon_j\}$ satisfying $t_j \geq t_0$ and $0 < \varepsilon_j \leq \varepsilon_0$ and a solution v_j of (P_{t_j,ε_j}) such that $\sup_{\mathcal{M}} v_j \to \infty$. Then by Lemma 5.3 we have $m_j := \inf_{\mathcal{M}} v_j \to \infty$. Note that for the function $\tilde{v}_j := v_j/m_j$

$$\begin{cases} f(\lambda(V[\tilde{v}_j])) \geq \frac{n-2}{2} t_0 \varphi_0 m_j^{p-1} \to \infty & \text{on } \mathcal{M}, \\ \frac{\partial \tilde{v}_j}{\partial \nu} - \frac{n-2}{2} h_g \tilde{v}_j \leq -\frac{n-2}{2} t_0 h_0 m_j^{q-1} \to -\infty & \text{on } \partial\mathcal{M}. \end{cases}$$

By Lemma 6.1 and compare with constant functions we have $\inf_{\mathcal{M}} \tilde{v}_j \to \infty$. This is a contradiction since $\inf_{\mathcal{M}} \tilde{v}_j = 1$. For the second assertion, let $t > 1$ and let v be a solution of $(P_{t,\varepsilon})$. Then

$$\begin{cases} f(\lambda(V[v])) \geq \frac{n-2}{2} t \varphi_0 \varepsilon^p & \text{on } \mathcal{M}, \\ \frac{\partial v}{\partial \nu} - \frac{n-2}{2} h_g v \leq -\frac{n-2}{2} t h_0 \varepsilon^q & \text{on } \partial\mathcal{M}. \end{cases}$$

By Lemma 6.1 again this implies that $v \geq c_0 t$ for some positive constant c_0 independent of t. Thus $(P_{t,\varepsilon})$ has no solution if t is large enough since v_t is uniformly bounded from above.

It is important to note that the constant C in Claim 1 does not depend on ε. Unless stated otherwise, constants appeared below allow ε-dependence. From now on we denote $(P_{t,\varepsilon})$ simply by (P_t).

We now define the mapping $T_t : C^2(\mathcal{M}) \to C^2(\mathcal{M})$ so that for any $v_1 \in C^2(\mathcal{M})$, $T_t(v_1)$ is the solution of

$$\begin{cases} f(\lambda(V[v])) = \frac{n-2}{2} t \varphi_0 \left(v_1^2 + \varepsilon^2\right)^{p/2}, & \lambda(V[v]) \in \Gamma \text{ on } \mathcal{M}, \\ \frac{\partial v}{\partial \nu} - \frac{n-2}{2} h_g v = -\frac{n-2}{2} t h_0 \left(v_1^2 + \varepsilon^2\right)^{q/2} & \text{on } \partial\mathcal{M}. \end{cases}$$

From Lemma 5.2 it follows that T_t is well-defined. Moreover $T_t(v_1)$ is a positive function. By a priori estimates one can see that T_t is a compact operator for each $t > 0$. Note that $T_t = tT_1$ for $t > 0$, we may continuously extend T_t to $t = 0$ by setting $T_0 = 0$.

For the number $\alpha_0 > 0$ satisfying (5.7) we set

$$\Phi := \left\{ v \in C^2(\mathcal{M}) : |v| < \alpha_0 \text{ on } \mathcal{M} \right\}.$$

Claim 2. There exists a large number R_0 independent of t such that for any $R \geq R_0$

$$(I - T_t)^{-1}(0) \cap \partial(\Phi \cap B_R) = \emptyset$$

for $0 \leq t \leq 1$, where $B_R := \{\varphi \in C^2(\mathcal{M}) : \|\varphi\|_{C^2(\mathcal{M})} < R\}$.

To see this, let $v \in (I - T_t)^{-1}(0) \cap \partial(\Phi \cap B_R)$ for some $0 < t \leq 1$. This implies that v is a solution of (P_t) and $0 < v \leq \alpha_0$ on $\mathcal{M}$. By using (5.7) we have

$$\begin{cases} f(\lambda(V[v])) \leq \frac{n-2}{2} \varphi_0 \left(\alpha_0^2 + \varepsilon^2\right)^{p/2} < f(\lambda(V[\alpha_0])) & \text{on } \mathcal{M}, \\ \frac{\partial v}{\partial \nu} - \frac{n-2}{2} h_g v \geq -\frac{n-2}{2} h_0 \left(\alpha_0^2 + \varepsilon^2\right)^{q/2} > \frac{\partial \alpha_0}{\partial \nu} - \frac{n-2}{2} h_g \alpha_0 & \text{on } \partial\mathcal{M}. \end{cases}$$

Therefore Lemma 6.1 implies that $0 < v < \alpha_0$. Consequently $v \in \partial B_R$, i.e.

$$\|v\|_{C^2(\mathcal{M})} = R. \tag{5.8}$$

However, note that the function $\tilde{v} := t^{-1}v$ satisfies

$$\begin{cases} \frac{n-2}{2}\varphi_0\varepsilon^p \le f(\lambda(V[\tilde{v}])) = \frac{n-2}{2}\varphi_0\left(t^2\tilde{v}^2+\varepsilon^2\right)^{p/2} \le C_0 & \text{on } \mathcal{M}, \\ -C_1 \le \frac{\partial\tilde{v}}{\partial\nu} - \frac{n-2}{2}h_g\tilde{v} = -\frac{n-2}{2}h_0\left(t^2\tilde{v}^2+\varepsilon^2\right)^{q/2} \le -\frac{n-2}{2}h_0\varepsilon^q & \text{on } \partial\mathcal{M}, \end{cases}$$

for some positive constants C_0 and C_1. By Lemma 6.1 we have $1/C_3 \le \tilde{v} \le C_3$ and hence Lemma 5.1 gives $\|\tilde{v}\|_{C^2(\mathcal{M})} < R_0$ for some number R_0 independent of t. Hence $\|v\|_{C^2(\mathcal{M})} < R_0$. Thus, in view of (5.8), Claim 2 holds with this R_0.

From Claim 2 it follows that the Leray–Schauder degree $\deg(I - T_t, \Phi \cap B_{R_0}, 0)$ is well-defined for $0 \le t \le 1$ and is independent of t. Since $T_0 = 0$ we have

$$\deg(I - T_1, \Phi \cap B_{R_0}, 0) = \deg(I, \Phi \cap B_{R_0}, 0) = 1.$$

On the other hand, from Claim 1, Lemma 6.1, and Lemma 5.1 it follows that there exists $R' \ge R_0$ such that $\|v\|_{C^2(\mathcal{M})} < R'$ for any solution v of (P_t) with $1 \le t \le \bar{t}$. Therefore $\deg(I - T_t, B_{R'}, 0)$ is well-defined for $t \in [1, \bar{t}]$ and is independent of t. Since $(P_{\bar{t}})$ has no solution, we therefore have

$$\deg(I - T_1, B_{R'}, 0) = \deg(I - T_{\bar{t}}, B_{R'}, 0) = 0.$$

Let $K := \bar{B}_{R'} \backslash \Phi$ which is closed in $C^2(\mathcal{M})$. If $0 \notin (I - T_1)(K)$, then the excision property of Leray–Schauder degree implies that

$$0 = \deg(I - T_1, B_{R'}, 0) = \deg(I - T_1, \Phi \cap B_{R'}, 0) = 1$$

which is absurd. Therefore T_1 has a fixed point v_ε in K, which is also a solution of (P_1). Moreover, the definition of Φ implies that $\sup_{\mathcal{M}} v_\varepsilon \ge \alpha_0 > 0$. It then follows from Claim 1, Lemma 5.3, Lemma 5.1, and the result in [34] that

$$\frac{1}{C} \le v_\varepsilon \le C \quad \text{and} \quad \|v_\varepsilon\|_{C^{2,\alpha}(\mathcal{M})} \le C$$

for some positive constant C independent of ε, where $\alpha \in (0,1)$. This implies that there is a sequence $\varepsilon_j \searrow 0$ such that v_{ε_j} converges in $C^2(\mathcal{M})$ to a solution v of (5.6). □

6 Appendix: a comparison principle

We include here a comparison principle which is repeatedly used in the proof of Theorem 1.2. For a positive function $v \in C^2(\mathcal{M})$ we still use the notation $V[v]$ defined by (5.5).

Lemma 6.1 *Assume that* (f, Γ) *satisfies* (1.3) *and* (1.5) *and that* $(\mathcal{M}, g)$ *is a smooth compact Riemannian manifold with smooth boundary* $\partial\mathcal{M}$. *Let* $a \in C^0(\partial\mathcal{M})$ *and let* $v, \xi \in C^2(\mathcal{M})$ *be two positive functions satisfying*

$$\begin{cases} f(\lambda(V[v])) \ge f(\lambda(V[\xi])), & \lambda(V[v]), \lambda(V[\xi]) \in \Gamma \ \text{on } \mathcal{M}, \\ \left(\frac{\partial}{\partial\nu} - a\right)v \le \min\left\{\left(\frac{\partial}{\partial\nu} - a\right)\xi, 0\right\} & \text{on } \partial\mathcal{M}. \end{cases} \tag{6.1}$$

Then either $v \equiv \xi$ *on* $\mathcal{M}$ *or* $v > \xi$ *on* $\mathcal{M}$.

Proof We first prove that $v \geq \xi$ on $\mathcal{M}$. Suppose it is not true then by using the positivity of v, we find a number $\beta > 1$ such that $\beta v \geq \xi$ on $\mathcal{M}$ and $\beta v(\bar{x}) = \xi(\bar{x})$ for some $\bar{x} \in \mathcal{M}$. If $\bar{x} \in \mathcal{M}\backslash\partial\mathcal{M}$, then

$$\beta v = \xi, \quad \nabla(\beta v) = \nabla\xi \quad \text{and} \quad \nabla^2(\beta v) \geq \nabla^2\xi \qquad \text{at } \bar{x}$$

and therefore

$$\lambda(V[\beta v]) \leq \lambda(V[\xi]) \quad \text{at } \bar{x}.$$

It follows, using (1.5) and $\beta > 1$, that

$$f(\lambda(V[\xi]) \geq f(\lambda(V[\beta v])) = f(\beta\lambda(V[v])) > f(\lambda(V[v])) \quad \text{at } \bar{x}$$

which is a contradiction.

If $\bar{x} \in \partial\mathcal{M}$, then

$$\beta v = \xi \quad \text{and} \quad \frac{\partial}{\partial\nu}(\beta v - \xi) \geq 0 \qquad \text{at } \bar{x}.$$

One the other hand, using hypothesis in the lemma, the fact $\beta > 1$, and the fact $(\frac{\partial}{\partial\nu} - a)v \leq 0$ on $\partial\mathcal{M}$, we have

$$\left(\frac{\partial}{\partial\nu} - a\right)(\beta v) \leq \left(\frac{\partial}{\partial\nu} - a\right)v \leq \left(\frac{\partial}{\partial\nu} - a\right)\xi \quad \text{on } \partial\mathcal{M}.$$

Since $(\beta v - \xi)(\bar{x}) = 0$, we have $\frac{\partial}{\partial\nu}(\beta v - \xi) \leq 0$ at $\bar{x}$. Therefore

$$\frac{\partial}{\partial\nu}(\beta v - \xi)(\bar{x}) = 0.$$

Using the Hopf Lemma, and the strong maximum principle, we must have $\beta v - \xi \equiv 0$. This implies $f(\lambda(V[\xi])) > f(\lambda(V[v]))$ on $\mathcal{M}$ as before. We get a contradiction again.

Therefore $v \geq \xi$ on $\mathcal{M}$. If $v > \xi$ on $\mathcal{M}$, we are done. Otherwise, there exists $\bar{x} \in \mathcal{M}$ such that $v(\bar{x}) = \xi(\bar{x})$. If $\bar{x} \in \partial M$ then it follows from the boundary condition in (6.1) that $\frac{\partial}{\partial\nu}(v - \xi) \leq 0$ at $\bar{x}$. The Hopf Lemma implies that this can not occur unless $v \equiv \xi$ on $\mathcal{M}$. If $\bar{x} \in \mathcal{M}\backslash\partial\mathcal{M}$, the strong maximum principle implies $v \equiv \xi$ on $\mathcal{M}$. The proof is thus complete. □

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