Three aspects of investment decisions under terminal wealth constraints

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Declaration

This thesis contains no material that has been accepted for the award of any other degree or diploma in any university. To the best of the author's knowledge, it contains no material previously published or written by another person, except where due reference is made in the thesis itself.

Signed: ______________________________

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Abstract

Pension investors in defined-contribution plans are responsible for making investment and withdrawal decisions regarding their pension savings. This thesis focuses on three aspects of investment decisions in the savings phase.

First, we introduce a generalised option-based portfolio insurance strategy (GOPIS) which is an extension of the traditional option-based portfolio insurance (OBPI) and option-based performance participation proposed by Zagst et al. [J Bank Financ, 105 (2019), 44-61]. The investor can exogenously specify both the benchmark portfolio, to which the minimum guarantee is linked, and the venture portfolio, by which the investor participates in potential market gains. We extend the analysis of conditional stochastic dominance developed by Zagst et al. (2019) to enable the comparison of GOPISs with different venture and benchmark portfolios. We find that the venture portfolio, when determined endogenously to maximise the expected utility of an investor with a constant risk aversion utility function, is the Merton portfolio. We demonstrate that GOPIS can be configured to have a better prospect of delivering higher expected utility over traditional OBPIs.

Second, we explore the impact of an investor's perception towards inflation risk on their investment strategy. Although pension investors are particularly exposed to the risk of inflation, few pre-retirement investment strategies incorporate explicit inflation-proofing. It is shown that ignoring inflation is costly in terms of a retiree's welfare, with reductions of up to 25% possible for the average retiree. More risk averse investors face even larger reductions. When wealth constraints (e.g. minimum guarantee) on the amount of pension savings at retirement are considered, we find that ignoring inflation by using nominal constraints gives a potential reduction in welfare of up to 36% for the average retiree. The results illustrate that nominal constraints are ineffective at reducing the risk of inflation. The conclusion is that investor ignore inflation at their peril. It must be included explicitly in retirement savings targets to improve retirement outcomes. Consequently, there should be greater investment in an index-linked bond or a similar asset.

Finally, we investigate how well different investment strategies can give pension investors more certainty about their income in retirement, whilst allowing them to benefit from taking investment risks. Our model considers ongoing pension contributions to savings, prohibits short-selling and borrowings, and, when applicable, includes wealth constraints. We assume the investor may adopt a risk averse or a loss averse utility function, and income target for later evolves according to the stochastic labour income of the investor. Using a numerical dynamic programming approach under an expected utility maximising framework, we find that a loss aversion utility function gives a high degree of certainty about its desired wealth target and is robust to different market models. Imposing terminal wealth constraints does not improve the certainty of achieving the desired target enough to counter-balance the increased chance of obtaining a lower income. The power utility function is not robust to different market models and becomes too risk-averse with wealth constraints.

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Chapter 1

Introduction

Defined-contribution (DC) pension plan members are responsible for deciding how to invest their pension savings. Typically, DC plan members save and invest to build up a pension fund that can be converted into an income source during retirement. The value of the member's DC fund at retirement may fluctuate due to various factors in the saving phase, such as their salary, contribution amounts and frequency, and the asset allocation of their fund. Asset allocation, also known as portfolio selection, is a decision made by the member (investor) to allocate their pension wealth across risky assets in seeking uncertain returns while taking into account the member's preference towards risks.

To find the optimal asset allocation, we need to model people's preference towards risks (uncertain outcomes). von Neumann and Morgenstern (1947) introduces the Expected Utility Theory (EUT) framework to model investor behaviour towards risk. Under EUT, any uncertain outcome, e.g. wealth at retirement, W_1 is represented by its expected utility $\mathbb{E}[U(W_1)]$ under an objective probability measure $\mathbb{P}$. The investor prefers an outcome W_1 to another outcome W_2 if and only if $\mathbb{E}[U(W_1)] > \mathbb{E}[U(W_2)]$. Therefore, a rational investor seeks to maximise the expected utility $\mathbb{E}[U(W)]$ over all available outcomes. According to EUT, any investor whose behaviour towards risk satisfies the Regularity, Independence and Continuity axioms has a utility function u and always prefers outcomes that maximise the expected utility. Since then, the EUT has been one of the most widely used approaches to model an investor's preference towards risk for portfolio optimisation problems. The pioneering works include the seminal paper of Markowitz (1952) that solves the optimal static portfolio in the mean-variance framework, which is linked to quadratic utility functions. Merton (1969, 1971) derives the optimal portfolio and consumption strategies for an investor in a continuous time setting. Assuming a time-separable utility, Samuelson (1969) solves the portfolio choice problem in a discrete-time setting. The portfolio problems in the pension savings area have been further extended in various directions, taking into account for example of financial market incompleteness (Karatzas et al., 1991), of specific constraints on portfolio weights (Xu and Shreve, 1992), of consumption or income streams (Korn and Krekel, 2002), and of terminal wealth constraints (Grossman and Zhou, 1996; El Karoui et al., 2005; Korn, 2005; Kraft and Steffensen, 2013; Donnelly et al., 2015, 2018).

Portfolio insurance (PI) is one of the extensions that has emerged from the literature on portfolio optimisation and has gained interest in both research and practice from the perspectives of individuals as well as institutional investors. The goal of PI strategies is to limit downside risk while allowing the investor to participate in the potential gains of a specified investment portfolio (Basak, 2002). Therefore, PI strategies are particularly attractive to private and institutional investors who are risk-averse and want to protect their capital while seeking potential growth from market exposure. The two most prominent PI methods are Constant Proportion Portfolio Insurance (CPPI) and Option Based Portfolio Insurance (OBPI). The CPPI was introduced by Black and Jones (1987) and Perold and Sharpe (1988). In this method, the investor specifies a floor as the minimum guarantee of their portfolio. The amount of their current portfolio value above the discounted floor is called the cushion. To ensure a minimum guarantee at maturity, the CPPI strategy is a dynamic asset allocation that invests a constant proportion m (multiplier) of the cushion in the risky asset S_1 and the remainder in the risk-free asset. Both the floor and the multiplier are exogenous to the model. The OBPI method was introduced by Leland and Rubinstein (1988) and involves a combination of a risky asset S_1 covered by a put written on it. The OBPI can be either a static or dynamic asset allocation depending on whether a listed or synthetic put option is used. The strike k is determined exogenously, typically as a proportion of the initial investment. It acts as the minimum guarantee at maturity above which the portfolio will always end up, regardless of the value of S_1 at maturity.

While the dynamic asset allocations of CPPI and OBPI can prevent the portfolio value from falling below the (discounted) floor in a continuous time market, investors may not have enough time to adjust their asset allocations in response to sudden market movements in practice. This limitation means that PI strategies may not always protect the investor from sharp market drops during financial crises, such as those in 1987, 2008 and at the onset of the COVID-19 pandemic. In such cases, the portfolio remains locked in cash for the rest of the investment horizon, earning only the risk-free interest rate and missing out on any potential market recovery. To address this issue, Ben Ameur and Prigent (2014) and Ben Ameur and Prigent (2018) extended the CPPI with time-varying multiple and time-varying floors. For OBPI, Zagst et al. (2019) proposes a modification to the traditional strategy by replacing the risk-free asset with another risky investment S_2. This new approach, named Option-Based Portfolio Participation (OBPP) strategy, enables the investor to participate in the potential gains of S_1, even during critical market conditions where standard PI strategies may fail and become cash-locked.

In Chapter 2 of this thesis, a generalisation of the OBPI and OBPP strategy is provided, named Generalised Option-based Portfolio Insurance (GOPIS). The GOPIS allows the investor to explicitly specify both the benchmark portfolio, to which the minimum guarantee is linked, and the venture portfolio, which enables the investor to participate in potential market gains. First, we provide the replicating strategy for GOPIS and derive the general analytic expression for the moments and conditional moments of GOPIS in a financial market with a finite number of risky assets. Next, we extend the analysis of conditional stochastic dominance in Zagst et al. (2019) to compare the portfolio payoffs for GOPIPs with different benchmark and venture portfolios. The concept of stochastic dominance (SD) introduced by Hadar and Russell (1969) and Hanoch and Levy (1969) allows comparison between random variables to be made, taking into account the preference of decision-makers. One of its main features is that SD does not require precise knowledge of preferences. Let W_i denote the random wealth of the investor at maturity following π_i investment strategy. If investment strategy π_1 dominates π_2 by first-order SD we can safely assert that $\mathbb{E}[U_1(W_1)] \geq \mathbb{E}[U_1(W_2)]$ for all non-decreasing utility functions U_1. If investment strategy π_1 dominates π_2 by second-order SD, we have $\mathbb{E}[U_2(W_1)] \geq \mathbb{E}[U_2(W_2)]$ for all non-decreasing concave utility functions U_2 (e.g. a risk averse utility function). In our numerical analysis, we demonstrate that GOPIS with a minimum variance portfolio as its benchmark has a probability of dominating OBPI and GOPIS with an equally-weighted benchmark portfolio in the first- and second-order SD over a wider range of strike levels k. Finally, assuming that the investor's preference follows a constant risk aversion (CRRA) utility under the EUT framework, we solve for the optimal venture portfolio for a GOPIS with an exogenously specified benchmark portfolio. We show that their optimal venture portfolio is always the Merton portfolio (see Merton, 1969). By setting the Merton portfolio as the venture portfolio, the investor maximises their expected utility maximisation for any exogenously specified benchmark portfolio.

The flexibility of GOPIS allows it to serve as a tool for hedging inflation risks by using an inflation-linked benchmark portfolio. Investors are constantly exposed to the risk of inflation eroding the purchasing power of their savings, and members of DC plans are particularly vulnerable due to the long-term nature of their savings plan. According to a recent survey, about two-thirds of pre-retiree investors are concerned that the value of their savings may not keep up with inflation (Greenwald Research, 2022). Many desire a consistent stream of income in retirement (State Street Global Advisors, 2022). However, these concerns and desires are often ignored in practice. Retirement goals in defined contribution pension plans are usually expressed in nominal terms, with a focus on maximising the absolute size of wealth at retirement. Moreover, over 90% of UK annuitants choose to purchase a level life annuity despite expressing a preference for an inflation-indexed annuity before purchase (Finkelstein and Poterba, 2004). The term "money illusion" refers to the cognitive bias where individuals tend to think in nominal dollars rather than real dollars. This bias stems from the fact that nominal values are more cognitively accessible than real values (Shafir et al., 1997). It is no surprise that many investors tend to think in terms of today's money and may not take into account how inflation may erode the purchasing power of their savings over time. Financial professionals around the world regard underestimating the impact of inflation as the top mistake investors make in their retirement planning, according to a recent survey (Natixis Investment Managers, 2022). Therefore, investors and pension product providers must take into account the impact of inflation on savings to ensure the financial well-being of investors in the long-term. In light of this, Chapter 3 of the thesis looks at a class of problems which revolves around maximizing the expected power utility of wealth at retirement. Specifically, we consider two types of investors: (i) an investor who maximises their expected utility of *real* (inflation-adjusted) wealth without terminal wealth constraints; (ii) an investor who maximises their expected utility of real wealth subject to *real* terminal wealth constraints (e.g., PI strategies and its variants). For each case, we derive an optimal, dynamic investment strategy that maximises the expected utility for the investor. Then, we compare the investment strategies and retirement outcomes of each of these two types of investors with those who erroneously aim to maximise nominal wealth.

The earliest classical optimisation problems in the pension savings area excluded inflation (Merton, 1969, 1971). However, many researchers have since examined the inclusion of inflation into these problems. For example, Menoncin (2002) derived the solution for an investor maximizing the expected exponential utility under inflation risk in a complete market. Brennan and Xia (2002), Battocchio and Menoncin (2004) and Zhang and Guo (2018) studied the investor's problem in an incomplete market without an inflation-linked bond. The former two showed that the inflation risk can instead be partially hedged via a cash-dominated portfolio with a high correlation with inflation. When the correlation between the cash-dominated portfolio and inflation is sufficiently high, the cost imposed by unhedgeable inflation risk is surprisingly low. In these studies, the cost of mis-specifying inflation risk is not explicitly studied. To focus on the level of suboptimality of not considering the inflation risk in the investment strategy, our model includes a tradable inflation-linked bond which completes the financial market in the presence of inflation risks (see also Zhang and Ewald, 2010; Han and Hung, 2012; Chen et al., 2023). Contrary to the findings of Zhang (2012), we show that adopting an inflation-adjusted risk perception yields a different optimal strategy to the Merton portfolio by having an additional inflation risk hedging component. Based on our parameterisation, failing to account for inflation risk reduces the welfare of the investor by 4%, 15%, 25% for low, medium, and high risk averse investors of type (i), respectively.

Extending from Donnelly et al. (2018), we consider a type (ii) investor who has terminal wealth constraints (e.g., the minimum guarantee using a PI strategy) under inflation risk. As alluded to in Chapter 2, imposing terminal wealth constraints produces an optimal dynamic asset allocation that adapts to the fluctuations in the prices of the risk assets. Specifically, when the terminal wealth constraints are likely to be binding, the strategy allocates a greater portion of wealth to the inflation-hedging asset. We find that if an investor "mistakenly" imposes nominal wealth constraints instead of real wealth constraints, their welfare can be significantly reduced by 27% to 36% (depending on their level of risk aversion). More importantly, nominal wealth constraints fail to hedge against inflation-linked wealth targets, leaving the investor vulnerable to having low inflation-adjusted terminal wealth.

The decision of how to invest one's pension savings before retirement is critical for DC pension plan members, and it can have a significant impact on their financial well-being in post-retirement years. To improve their retirement outcomes, investors can take steps such as utilising minimum guarantees and ensuring adequate inflation hedging in their investment strategy. To cite Merton (2014, pp. 1402), the primary concern of the investor is 'Will I have sufficient income in retirement to live comfortably?'. In this thesis, we address this problem by first introducing the generalised option-based portfolio insurance that protects the worst of retirement outcomes using a lower terminal wealth constraint in Chapter 2. In Chapter 3, the portfolio insurance is then extended to also include an upper wealth constraint with which we examine the cost of mis-specifying inflation risks. Chapter 4 studies whether the investor is better off with investment strategies that constrain their retirement income, or equivalently accumulated fund value at retirement, so as to protect them against extreme (negative) scenarios of income in retirement. Through the maximisation of expected utility, we compare several ways of formulating the investor's problem, by allowing for different utility functions and terminal wealth constraints and deriving optimal investment strategies.

The EUT framework and concave (i.e, risk averse) utility functions have had a long-standing influence on research in optimising personal savings plans. However, Kahneman and Tversky (1979) claim that most investors are loss averse and make decisions relative to some reference levels. They observe that individuals evaluate wealth in comparison to a reference rather than in their absolute values, and behave differently over gains and losses relative to the reference level, in particular, risk averse over gains and risk seeking over losses. Moreover, they are significantly more sensitive to losses than to gain. Based on this, Kahneman and Tversky (1979) propose an S-shaped utility function, which we refer to as the loss aversion (LA) utility function in this thesis. This forms the basis of the Prospect Theory (PT) proposed by Kahneman and Tversky (1979) and was later elaborated by Tversky and Kahneman (1992) into Cumulative Prospect Theory. The LA utility function has been studied in investment problems (Berkelaar et al., 2004; Blake et al., 2013; Guan and Liang, 2016; Dong et al., 2020), and in conjunction with a lower terminal wealth (Chen et al., 2017).

In Chapter 4, we consider a model with realistic components of pension savings. Firstly, we model a stochastic labour income process for the investor. The investor earns a stochastic labour income, a portion of which is regularly contributed to their pension savings and ultimately determines the desired or the minimum level of their retirement income. Secondly, our model considers the trading constraints faced by individual investors by prohibiting the investor from short-selling or borrowing against their future incomes. In terms of utilising options, the investors are allowed to only trade in short-term option contracts instead of long-term contracts or invest in a replicating strategy. Thirdly, we consider the investment problems under a constant relative risk aversion utility function and a loss aversion utility function. To solve for the optimal investment asset allocations in Chapter 4, we employ a numerical stochastic dynamic programming model.

We derive the optimal investment strategies for investors with different utility functions (CRRA and LA), and consider the impact of terminal wealth constraints on their investment. We consider a model in which the wealth constraints evolve according to the stochastic labour income of the investor which helps the investor maintains their pre-retirement lifestyle into retirement. This differentiates our model from most literature in which a tradable labour income process is assumed and continuously-traded replicating strategies are devised to replicate the option-like payoffs for the investors. Our findings show that a LA utility function naturally results in a distribution that peaked at the investor's chosen income goal with some level of robustness when tested against the bootstrapped historical market return data. On the other hand, the CRRA utility function yields a more dispersed income distribution, providing the investor less certainty of achieving a sufficient level of income in retirement. Imposing terminal wealth boundary constraints, in both the utility function, result in strategies that provide certainty of achieving the lower boundary but at the cost of a significant reduction in the overall retirement outcome. We conclude that the investor can benefit from adopting a loss aversion-derived optimal investment strategy to target a sufficient level of income at retirement.

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Chapter 2

Generalised Option-based Portfolio Insurance Strategy

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2.1 Abstract

We introduce a generalised option-based portfolio insurance strategy (GOPIS) which is an extension of the traditional option-based portfolio insurance (OBPI) and option-based performance participation (OBPP) proposed by Zagst et al. [Journal of Banking & Finance, 105 (2019), 44-61]. The investor can exogenously specify both the benchmark portfolio, to which the minimum guarantee is linked, and the venture portfolio, by which the investor participates in potential market gains. We extend the analysis of conditional stochastic dominance developed by Zagst et al. (2019) to enable the comparison of GOPISs with different venture and benchmark portfolios. We find that the venture portfolio, when determined endogenously to maximise the expected utility of an investor with a constant risk aversion utility function, is the Merton portfolio.

We demonstrate that GOPIS can be configured to have a better prospect of delivering higher expected utility over traditional OBPIs.

2.2 Introduction

In this paper, we introduce and analyse the generalised option-based portfolio insurance strategy (GOPIS). GOPIS is a generalisation to both the traditional option-based portfolio insurance (OBPI) strategy and its extension option-based performance participation (OBPP) strategy introduced by Zagst et al. (2019). GOPIS is designed to provide the investor with a minimum guarantee linked to a given fraction of a benchmark portfolio while retaining the potential gains resulting from the outperformance of the investor's specified investment portfolio, which we call the venture portfolio. We extend the analysis of GOPIS in the framework of Expected Utility Theory to account for investors with the constant relative risk aversion (CRRA) utility function. We find that the Merton (1969) portfolio is always the optimal venture portfolio for such investors irrespective of their chosen benchmark portfolio.

The most prominent examples of portfolio insurance strategies are CPPI and OBPI. The CPPI was introduced by Black and Jones (1987) and has been extensively studied since (Branger et al., 2010; Ben Ameur and Prigent, 2014; Escobar-Anel et al., 2020). Our paper is closely related to the OBPI strategy that was first introduced by Leland and Rubinstein (1988). The goal of portfolio insurance strategies is to limit portfolio losses in the adverse market environment while allowing the investor to benefit from a favourable market environment. The guarantee is typically a fixed value and is hedged using the risk-free asset. Following a sudden market drop, the risk-free exposure demanded by the guarantee can crowd out the exposure to other risky assets, prohibiting the strategy from participating in a potential market recovery until the end of the investment horizon for CPPI (or the next reallocation date for OBPI). To circumvent this shortcoming of OBPI, Zagst et al. (2019) introduce the option-based performance participation (OBPP) as an extension to the OBPI strategy that substitutes the risk-free guarantee with a risky asset alternative, which they call the reserve asset.

This paper expands upon the work by Zagst et al. (2019) into a generalised option-based portfolio insurance strategy (GOPIS). Instead of parameterising the desired benchmark portfolio as a distinct asset class in the financial market, we formulate GOPIS such that it allows the investors to explicitly specify (i) a benchmark portfolio as a distinct combination of all investable assets in the financial market, to which their minimum guarantee is tied, and (ii) their venture portfolio as another distinct different combination of all investable assets. The investor may choose any venture portfolio that provides them with desired potential gains by participating in the market. This explicit formulation we propose is a generalisation of the OBPP introduced by Zagst et al. (2019). While they implicitly formulate the financial assets to be the venture and benchmark portfolios, our generalised formulation allows for direct comparisons of GOPIS with varying configurations. Specifically, we contribute to the literature by extending the analysis of conditional stochastic dominance to include the comparison of multiple GOPISs with different venture portfolios. This was not considered in Zagst et al. (2019) and we believe is a significant and natural extension of the analysis of stochastic dominance of GOPIS. In our numerical illustration, we show that by using the Merton portfolio as the benchmark (which we discuss next), GOPIS has a high probability of achieving second-order stochastic dominance over the traditional OBPI, when the levels of minimum guarantees are similar between both strategies.

Next, we consider GOPIS in the framework of Expected Utility Theory in which the preference of the investor is known to the constant relative risk aversion (CRRA) utility preference. The problem of finding the optimal asset allocations in the expected utility maximisation framework was pioneered by Merton (1969, 1971). It has been extended to introduce constraints to the optimization problem (Grossman and Zhou, 1996; Kraft and Steffensen, 2013; Donnelly et al., 2015; Chen et al., 2018). For example, Grossman and Zhou (1996) consider a minimum wealth constraint and Kraft and Steffensen (2013) considered a VaR-type constraint where the minimum constraint can be breached with a (small) probability. Chen et al. (2018) studied the problem with a minimum constraint and a VaR constraint simultaneously. Donnelly et al. (2015) proposed an upper constraint on the investor's wealth which leads to better outcomes in the lower quantiles. Jensen and Sørensen (2001) and Boyle and Tian (2007) study the problems of stochastic terminal wealth and VaR constraint. We show that for an investor with CRRA utility preferences and an exogenous guarantee requirement, the optimal GOPIS always has the Merton portfolio as its venture portfolio irrespective of their exogenously specified benchmark portfolio. Using wealth equivalents as a measure of utility loss (see Jensen and Sørensen, 2001), we show that using GOPIS with venture Merton portfolio results in lower utility loss compared to the traditional OBPIs.

The remainder of the paper is structured as follows: in Section 2.3, we introduce notation, financial market, and the GOPIS. We provide the valuation formula of GOPIS which allows us to determine the cost associated with the guarantee. We examine the replicating strategy. We examine its payoff distribution at maturity and provide the analytical expressions of the moments and selected conditional moments of the payoff distributions. Finally, we discuss the criteria in which the first- and second-order stochastic dominance favouring one GOPIS over another can be achieved. In Section 2.4, we show that GOPIS can be the optimal portfolio insurance strategy for risk averse investors with a power utility function. The advantages of GOPIS are exemplified in the numerical illustrations in Section 2.5. Section 2.6 concludes the paper.

2.3 Definition

2.3.1 The financial market

We assume investment in a continuous-time market model over a finite time horizon $[0, T]$ for $T > 0$. We refer to T as the terminal time.

The financial market consists of a risk-free asset, denoted by S_0 and d risky assets, denoted by $\mathbf{S}$. Following Merton (1971), we assume that the risk-free asset earns a constant rate of interest $r > 0$, and that the risky assets are geometric Brownian motions:

$$\mathrm{d}S_0(t) = rS_0(t)\,\mathrm{d}t \tag{2.1}$$

$$\mathrm{d}\mathbf{S}(t) = \mathbf{I_S}(t)\boldsymbol{\mu}\,\mathrm{d}t + \mathbf{I_S}(t)\boldsymbol{\sigma}\,\mathrm{d}\mathbf{W}(t) \tag{2.2}$$

where $\mathbf{W}(t) := (W_1(t), W_2(t), \dots, W_d(t))'$, $0 \le t \le T$ denotes an d-dimensional standard Brownian motion as defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with respect to the real-world measure $\mathbb{P}$ and the Brownian filtration $\mathcal{F}$. Here $\mathbf{I_S}(t)$ denotes the d-dimensional diagonal matrix with the risky asset prices as entries. The constant vector $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_d)'$ describes the drifts of the asset prices and the constant matrix $\boldsymbol{\sigma} = (\sigma_{i,j})_{i,j=1,2,\dots,d}$ describes the volatilities and the correlations of the asset prices. To avoid any arbitrage opportunities, the matrix $\boldsymbol{\sigma}$ is assumed to be non-singular. Then, there exists a unique market price of risk, $\boldsymbol{\theta}$, satisfying

$$\boldsymbol{\theta} = \boldsymbol{\sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1}_d) \tag{2.3}$$

in which $\mathbf{1}_d$ represents a d-column vector of ones. We assume that $\|\boldsymbol{\theta}\| > 0$, where $\|.\|$ denotes the 2-norm of a matrix, i.e. $\|\boldsymbol{x}\| = \sqrt{\sum_i x_i^2}$.

2.3.2 Wealth dynamics

Consider a portfolio with $\boldsymbol{u}(t) = (u_1(t), u_2(t), \dots, u_d(t))'$, where $u_i(t)$ for $i = 1, 2, \dots, d$ are the proportions invested in the i-th risky asset, and $1 - \sum_{i=1}^d u_i(t)$ the risk-free asset, respectively. Suppose that the initial portfolio value is x_0. Then, the wealth process of the portfolio $X^{\boldsymbol{u}}(t)$ follow

$$\begin{aligned} \mathrm{d}X^{\boldsymbol{u}}(t) &= rX^{\boldsymbol{u}}(t) + \boldsymbol{u}'(t)X^{\boldsymbol{u}}(t)\boldsymbol{\sigma}\boldsymbol{\theta}\,\mathrm{d}t + \boldsymbol{u}'(t)X^{\boldsymbol{u}}(t)\boldsymbol{\sigma}\,\mathrm{d}\mathbf{W}(t), \\ X^{\boldsymbol{u}}(0) &= x_0. \end{aligned} \tag{2.4}$$

Define the state price density process H as $H(t) := \exp(-(r + \frac{1}{2}\|\boldsymbol{\theta}\|^2)t - \boldsymbol{\theta}'\mathbf{W}(t))$, for each $t \in [0, T]$.[1] Then, a portfolio process $X^{\boldsymbol{u}}(t)$ must satisfy the budget constraint that

$$\mathbb{E}(H(T)X^{\boldsymbol{u}}(T)) \le x_0. \tag{2.5}$$

The budget constraint is a static characterisation of feasible portfolio process which are financeable from an initial wealth x_0.

[1] In this context, the state price density specifies the price of a security that pays off one dollar in one particular state of the wolrd and zero in all others.

For a portfolio with constant proportions invested in each asset over time, i.e. $\boldsymbol{u} = (u_1, u_2, \ldots, u_d)'$ asset allocation for $t \in [0, T]$, we use shorthand notations $\alpha_{\boldsymbol{u}} = r + \boldsymbol{u}'\boldsymbol{\sigma}\boldsymbol{\theta}$ to denote its drift parameter and $\boldsymbol{\beta}'_{\boldsymbol{u}} = \boldsymbol{u}'\boldsymbol{\sigma}$ the diffusion vector of the portfolio. Then, its wealth process $X^{\boldsymbol{u}}(t)$ follows

$$\begin{aligned} \mathrm{d}X^{\boldsymbol{u}}(t) &= X^{\boldsymbol{u}}(t)\left(\alpha_{\boldsymbol{u}}\,\mathrm{d}t + \boldsymbol{\beta}'_{\boldsymbol{u}}\,\mathrm{d}\mathbf{W}(t)\right) \\ X^{\boldsymbol{u}}(0) &= x_0. \end{aligned}$$

2.3.3 The portfolio insurance strategy

The generalised option-based portfolio insurance strategy (GOPIS) is built on two portfolios, Y and Z. The insured portfolio strategy provides a minimum performance in terms of a constant fraction, k, of the outcomes of the benchmark portfolio Y while keeping the potential for profits resulting from the outperformance of the venture portfolio Z.

At maturity, the portfolio insurance strategy has payoff

$$V(T) = \max\left(pZ(T), kY(T)\right) \tag{2.6}$$

in which p is the participation factor of the venture portfolio. $p < 1$ reflects the cost of acquiring the guarantee linked to the benchmark portfolio. The value of p is calculated in Eq. 2.12. We assume that both the venture and the benchmark portfolios have constant proportions invested in each asset over time, $\boldsymbol{u}_Z$ and $\boldsymbol{u}_Y$, respectively. For brevity, let $a := r + \boldsymbol{u}'_Z\boldsymbol{\sigma}\boldsymbol{\theta}$, $\boldsymbol{b} := \boldsymbol{\sigma}'\boldsymbol{u}_Z$, $\alpha := r + \boldsymbol{u}'_Y\boldsymbol{\sigma}\boldsymbol{\theta}$, $\boldsymbol{\beta} := \boldsymbol{\sigma}'\boldsymbol{u}_Y$. Then, under the real-world probability measure $\mathbb{P}$ we have:

$$\mathrm{d}Z(t) = Z(t)\left(a\,\mathrm{d}t + \boldsymbol{b}'\,\mathrm{d}\mathbf{W}(t)\right), \quad Z(0) = Z_0 \tag{2.7}$$

and

$$\mathrm{d}Y(t) = Y(t)\left(\alpha\,\mathrm{d}t + \boldsymbol{\beta}'\,\mathrm{d}\mathbf{W}(t)\right), \quad Y(0) = Y_0 \tag{2.8}$$

Theorem 2.1. *Suppose that the portfolios Z and Y have constant asset allocations throughout the investment horizon, the value of the portfolio insurance at time t is*

$$V(t) = Y(t)\,[k + \mathit{call}(t, p\cdot Z(t)/Y(t); T, k, 0, \nu)] \tag{2.9}$$

in which ν, the volatility of the embedded exchange option, is defined as

$$\nu := \|\boldsymbol{b} - \boldsymbol{\beta}\|, \tag{2.10}$$

and $\mathit{call}(t, p\cdot X; T, k, r, \sigma)$ denoting the Black-Scholes value of a vanilla call option at time t written on p shares of the asset X, with maturity T, risk-free interest rate r, strike ke^{rT} and volatility σ of the asset X.

Proof. **Theorem 2.1**. Margrabe (1978) was the first to employ the numeraire change in pricing exchange options. We reproduce the proof here in our context for the sake of completeness.

To compute the value of $V(t)$, we exploit the results of equivalent martingale measure (see, e.g Hull, 2012). Let $\mathbb{Q}$ denote the equivalent martingale measure for Y, i.e. the market price of risk under $\mathbb{Q}$-measure is the volatility β of Y. Under $\mathbb{Q}$-measure we have

$$\begin{aligned} \mathrm{d}Z(t) &= Z(t)\left((r + \boldsymbol{\beta}'\boldsymbol{b})\,\mathrm{d}t + \boldsymbol{b}'\,\mathrm{d}\mathbf{W}(t)\right) \\ Z(0) &= Z_0 \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}Y(t) &= Y(t)\left((r + \boldsymbol{\beta}'\boldsymbol{\beta})\,\mathrm{d}t + \boldsymbol{\beta}'\,\mathrm{d}\mathbf{W}(t)\right) \\ Y(0) &= Y_0 \end{aligned}$$

Let $\hat{Z}(t) := Z(t)/Y(t)$, then under $\mathbb{Q}$-measure we have

$$\begin{aligned} \mathrm{d}\hat{Z}(t) &= \hat{Z}(t)\left((\boldsymbol{b} - \boldsymbol{\beta})'\,\mathrm{d}\mathbf{W}(t)\right) \\ \hat{Z}(0) &= \hat{Z}_0 \end{aligned} \tag{2.11}$$

Therefore, $\log(\hat{Z}(t))$ is normally distributed with mean of 0 and variance of $\nu^2 t$ with ν as define in Eq. 2.10.

Applying the results of equivalent martingale measure,[2] the value of the portfolio insurance at time t is given by

$$\begin{aligned} V(t) &= Y(t)\mathbb{E}_{\mathbb{Q}}\left[Y^{-1}(T)V(T)\right] \\ &= Y(t)\mathbb{E}_{\mathbb{Q}}\left[\max\left(p\hat{Z}(T), k\right)\right] \\ &= Y(t)\mathbb{E}_{\mathbb{Q}}\left[k + \left(p\hat{Z}(T) - k\right)^+\right] \\ &= Y(t)\left[k + \text{call}\left(t, p\cdot\hat{Z}(t); T, k, r=0, \sigma=\nu\right)\right]. \end{aligned}$$

Since $\log(\hat{Z}(t))$ is normally distributed under $\mathbb{Q}$, we applied the log-normal call formula to arrive at the last equation.

□

Remark 2.2. To facilitate a return perspective, now and in the following we assume that the initial values of the portfolios equal the initial value of the portfolio insurance V_0, i.e. $Z_0 = Y_0 = V_0$. Then, the participation factor p satisfies

$$k + \text{call}\left(0, p; T, k, 0, \nu\right) = 1. \tag{2.12}$$

The participation factor p can be solved numerically. As shown in Eq. 2.12, the participation factor p is dependent on the investor-defined level of guarantee k, the volatility of the embedded option ν and the maturity T. To examine their relationship, we apply implicit differentiation on Eq. 2.12 with respect to the other three variables. Using Φ and ϕ to denote the cumulative and probability density function of the standard normal distribution, respectively, we obtain:

$$\frac{\mathrm{d}p}{\mathrm{d}k} = \frac{\Phi(d_-^*) - 1}{\Phi(d_+^*)} < 0$$

[2]See, e.g. Hull (2012) Chapter 27.3 or Bjork (2009) Theorem 26.2.

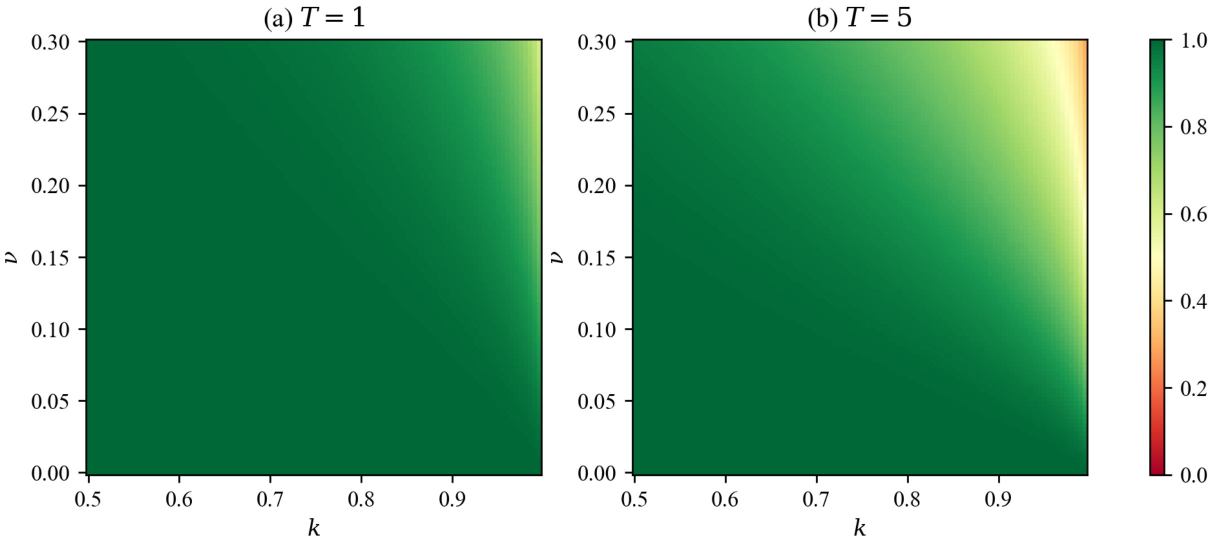

FIGURE 2.1: The participation factor p as a function of the investor-defined level of guarantee k, the volatility of the embedded option ν at two different maturities: (a) $T = 1$ and (b) $T = 5$.

$$\frac{\mathrm{d}p}{\mathrm{d}\nu} = -p\sqrt{T}\frac{\phi(d_+^*)}{\Phi(d_+^*)} < 0$$

$$\frac{\mathrm{d}p}{\mathrm{d}T} = -\frac{1}{2}\frac{p\nu}{\sqrt{T}}\frac{\phi(d_+^*)}{\Phi(d_+^*)} < 0$$

where $d_\pm^* := (\log(\frac{p}{k}) \pm \frac{1}{2}\nu^2 T)/(\nu\sqrt{T}) = d_\pm(0, 1, p, \nu; k)$, as defined in Eq. 2.13.

The derivative $\frac{\mathrm{d}p}{\mathrm{d}k}$ is negative, meaning the increase in the guarantee level is accompanied by an increasing fraction of initial wealth allocated for the exchange option. The derivatives $\frac{\mathrm{d}p}{\mathrm{d}\nu} < 0$ imply that as the volatility of the embedded option increases, a higher cost of the option must be borne by the investor. Similarly, $\frac{\mathrm{d}p}{\mathrm{d}T} < 0$ indicates that as the time horizon increases, the cost of option increases. Figure 2.1 illustrates p as a function of k and ν, with two maturity levels.

Without loss of generality, we further assume that the initial values of the portfolio equal 1 such that

$$Z_0 = Y_0 = V_0 = 1.$$

2.3.3.1 The replicating strategy

We now turn our attention to finding the investment strategy that results in the corresponding portfolio insurance payoff $V(T) = \max(pZ(T), kY(T))$. Theorem 2.3 also serves as the foundation for the optimal investment strategy to Problem 1 which we discuss later in Section 2.4.

Using Φ to denote the cumulative density function of the standard normal distribution, then from Eq. 2.9, we have

$$V(t) = pZ(t)\Phi\left(\frac{\log\frac{pZ(t)}{kY(t)} + \frac{1}{2}\nu^2(T-t)}{\nu\sqrt{T-t}}\right) + kY(t)\left[1 - \Phi\left(\frac{\log\frac{pZ(t)}{kY(t)} - \frac{1}{2}\nu^2(T-t)}{\nu\sqrt{T-t}}\right)\right].$$

Theorem 2.3. *Let $V(t) \equiv v(t, Y(t), X(t); k)$, in which*

$$X(t) := pZ(t)$$

and

$$v(t, y, x; k) := x\Phi(d_+(t)) + ky(1 - \Phi(d_-(t))), \quad y, x > 0,$$

where for any constant $k > 0$,

$$d_\pm(t) := d_\pm(t, y, x, \nu; k) := \frac{\log\frac{x}{ky} \pm \frac{1}{2}\nu^2(T-t)}{\nu\sqrt{T-t}} \tag{2.13}$$

The replicating portfolio for $V(t)$ is to hold the amount $\boldsymbol{\pi}(t)$ in the risky assets

$$\boldsymbol{\pi}(t) := pZ(t)\Phi(d_+(t))\boldsymbol{u}_Z + kY(t)\left(1 - \Phi(d_-(t))\right)\boldsymbol{u}_Y \tag{2.14}$$

in which $\boldsymbol{u}_Z$ and $\boldsymbol{u}_Y$ are the proportion invested in each asset for the venture portfolio Z and the benchmark portfolio Y, respectively.

Proof. **Theorem 2.3**. Differentiating the portfolio insurance function $v(t, y, x; k)$, and using ϕ to denote the probability density function of the standard normal distribution, we get the partial derivatives

$$v_t(t,y,x) = -\frac{1}{2}\frac{x\phi(d_+(t))\nu}{\sqrt{T-t}} = -\frac{1}{2}\frac{x\phi(d_+(t))\nu^2}{\nu\sqrt{T-t}},$$

$$\begin{aligned}
v_y(t,y,x) &= k\left(1-\Phi(d_-(t))\right), & v_{yy}(t,y,x) &= \frac{x\phi(d_+(t))}{y^2\nu\sqrt{T-t}},\\
v_x(t,y,x) &= \Phi(d_+(t)), & v_{xx}(t,y,x) &= \frac{\phi(d_+(t))}{x\nu\sqrt{T-t}},\\
v_{yx}(t,y,x) &= -\frac{\phi(d_+(t))}{y\nu\sqrt{T-t}}, & v_{xy}(t,y,x) &= -\frac{\phi(d_+(t))}{y\nu\sqrt{T-t}}
\end{aligned} \tag{2.15}$$

Using Itô's formula, we have

$$\begin{aligned}
\mathrm{d}v(t) =& v_t(t,y,x)\,\mathrm{d}t+\\
& v_y(t,y,x)\,\mathrm{d}y + \frac{1}{2}v_{yy}(t,y,x)\,\mathrm{d}\langle y,y\rangle+\\
& v_x(t,y,x)\,\mathrm{d}x + \frac{1}{2}v_{xx}(t,y,x)\,\mathrm{d}\langle x,x\rangle+\\
& \frac{1}{2}v_{yx}(t,y,x)\,\mathrm{d}\langle y,x\rangle + \frac{1}{2}v_{xy}(t,y,x)\,\mathrm{d}\langle x,y\rangle
\end{aligned} \tag{2.16}$$

in which

$$\mathrm{d}x = x\left((r+\boldsymbol{u}_Z'\boldsymbol{\sigma}\boldsymbol{\theta})\,\mathrm{d}t + \boldsymbol{u}_Z'\boldsymbol{\sigma}\,\mathrm{d}\mathbf{W}(t)\right), \quad x_0 = p, \tag{2.17}$$

and

$$\mathrm{d}y = y\left((r+\boldsymbol{u}_Y'\boldsymbol{\sigma}\boldsymbol{\theta})\,\mathrm{d}t + \boldsymbol{u}_Y'\boldsymbol{\sigma}\,\mathrm{d}\mathbf{W}(t)\right), \quad y_0 = Y_0. \tag{2.18}$$

Substituting for the derivatives of the portfolio insurance function v, the dynamics of x and y, we find that the dynamics of the portfolio insurance function v satisfy the wealth equation:

$$\mathrm{d}V(t) = rV(t)\,\mathrm{d}t + \boldsymbol{\pi}'(t)\boldsymbol{\sigma}\boldsymbol{\theta}\,\mathrm{d}t + \boldsymbol{\pi}'(t)\boldsymbol{\sigma}\,\mathrm{d}\mathbf{W}(t) \tag{2.19}$$

Hence, $\boldsymbol{\pi}(t)$ is the amount to be invested in the risky assets at time t in order to replicate the payoff of the portfolio insurance strategy. □

Eq. 2.14 shows the dynamic asset allocation of GOPIS that oscillates between the investor's chosen venture portfolio $\boldsymbol{u}_Z$ and the benchmark portfolio $\boldsymbol{u}_Y$ (see also Figure 2.2b). In Section

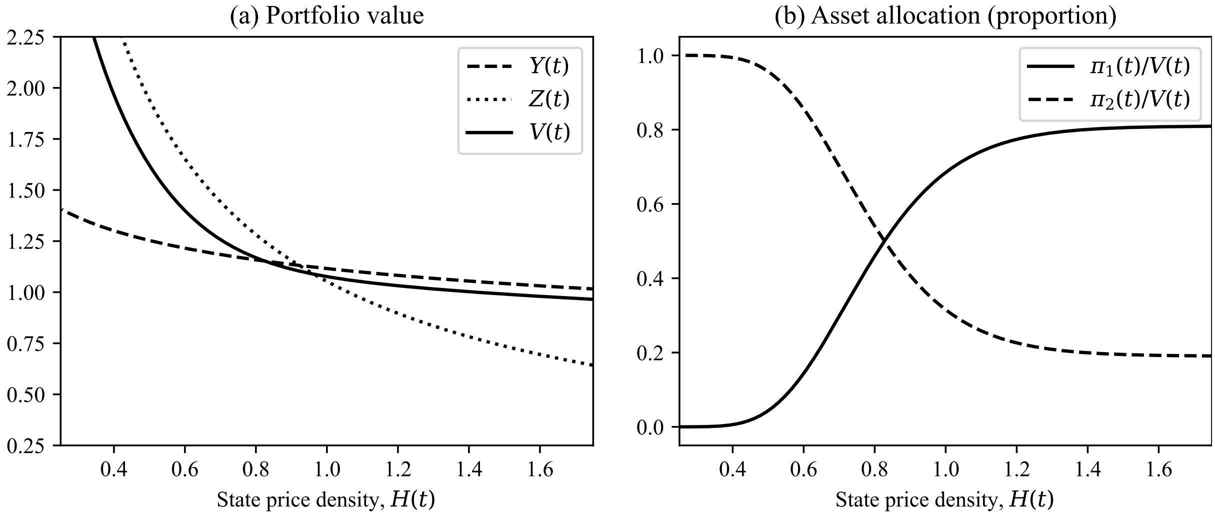

FIGURE 2.2: (a) The interim portfolio values of a GOPIS and its components across a range of state price density values at time $t = 0.5$ with maturity of $T = 1$. (b) The asset allocation of the GOPIS across a range of state price density. The chosen parameters are $r = 0.026$, $\mu_1 = 0.050$, $\mu_2 = 0.068$, $\sigma_{11} = 0.078$, $\sigma_{21} = 0.020$, $\sigma_{22} = 0.142$. The chosen GOPIS has $\boldsymbol{u}_Z = (0\ \ 1)'$, $\boldsymbol{u}_Y = (0.81\ \ 0.19)'$, $k = 0.95$, which has $p = 0.83$.

2.4, we show that selecting the Merton portfolio as the venture portfolio results in the investor maximising their expected utility, regardless of the chosen benchmark portfolio.

Figure 2.2a presents the interim values of GOPIS, $V(t)$ and its components, $Z(t)$ and $Y(t)$, as a function of the state price density $H(t)$ at $t = 2.5$ with maturity $T = 5$. The riskier venture portfolio $Z(t)$, has a constant asset allocation of $\boldsymbol{u}_Z = (0\ \ 1)'$, is a decreasing function of $H(t)$ with a steeper gradient than that of the less risky benchmark portfolio, $Y(t)$ that has a constant asset allocation of $\boldsymbol{u}_Y = (0.81\ \ 0.19)'$. In good market states (low $H(t)$ values), the value of GOPIS follows the value of $Z(t)$, albeit discounted by the participation factor p. As market states worsen (higher $H(t)$ values), the protection mechanism kicks in and $V(t)$ is guaranteed at k fraction of $Y(t)$. Figure 2.2b shows the changes in the asset allocation of $V(t)$ from good market states to poor market states, showing the mechanism by which $V(t)$'s payoff is achieved. As the market states deteriorate, it switches from following the asset allocations of $Z(t)$ to the ones of $Y(t)$. We note that by choosing a risky portfolio as the benchmark, the investor can prevent the cash-lock situations when the asset allocation degenerates to a fully risk-free allocation in the case of the dynamic version of OBPI in poor market states (see Zagst et al., 2019; Ben Ameur and Prigent, 2018).

2.3.3.2 Moments

In this section, we set out the moment functions of the GOPIS by deriving its mth moment for the payoff at maturity specified in Eq. 2.6. Note, the derivation of the moments for OBPI and OBPP was already proceeded in Zagst et al. (2019). Theorem 2.4 extends Zagst et al. (2019) to accommodate the generalised structure of the GOPIS, i.e. allowing for generic $\boldsymbol{u}_Z$ and $\boldsymbol{u}_Y$ in a financial market with a finite $d > 1$ risky assets. Note that all moments discussed are calculated under the real-world measure $\mathbb{P}$.

Theorem 2.4. *The mth moment, $m \in \mathbb{N}$, of the payoff of a GOPIS with venture portfolio $Z(t)$ and benchmark portfolio $Y(t)$ at maturity T is given by*

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[V(T)^m\right] =& k^m \mathbb{E}_{\mathbb{P}}\left[Y(T)^m\right] \\ & \cdot \left(1 + \sum_{i=1}^{m} \sum_{l=0}^{i} \begin{pmatrix} m \\ i \end{pmatrix} \cdot \begin{pmatrix} i \\ l \end{pmatrix} \cdot (-1)^{i-l} \cdot \left(\frac{pZ_0}{kY_0}\right)^l \cdot e^{l[a_m + \frac{1}{2}(l-1)\nu^2]T} \cdot \Phi(d_{+;m,l})\right)\end{aligned}$$

where

$$a_m = a - \alpha + (m-1)(\boldsymbol{b} - \boldsymbol{\beta})'\boldsymbol{\beta},$$

$$\begin{aligned} d_{-;m,l} :&= d_-(0, Y_0, pZ_0 e^{(a_m + l\nu^2)T}, \nu; k) \\ &= \frac{\log \frac{pZ_0}{kY_0} + (a_m + (l - \frac{1}{2})\nu^2)T}{\nu\sqrt{T}} \end{aligned}$$

and k is the constant fraction determining the minimum guarantee, and p is calculated from Eq. 2.12.

Proof. **Theorem 2.4**.

Equivalent Probability Measure. The key step in obtaining the moment of the portfolio insurance strategy is to apply change of numeraire. Let $\mathbb{P}_m$, $m \in \mathbb{N}$, be the equivalent probability measure defined via the Radon-Nikodym derivative:

$$\psi(t) := \frac{\mathrm{d}\mathbb{P}_m}{\mathrm{d}\mathbb{P}} = \frac{Y(t)^m}{\mathbb{E}_{\mathbb{P}}\left[Y(t)^m\right]} = \exp\left(-\frac{1}{2}m^2\boldsymbol{\beta}'\boldsymbol{\beta}t + m\boldsymbol{\beta}'\mathbf{W}(t)\right). \tag{2.20}$$

Following from the Girsanov theorem, the stochastic process $\mathbf{W}^{\mathbb{P}_m}(t)$, is defined by

$$\mathbf{W}^{\mathbb{P}_m}(t) := \mathbf{W}(t) - m \cdot \boldsymbol{\beta} \cdot t, \tag{2.21}$$

$0 \leq t \leq T$, is a n-dimensional Brownian motion under the equivalent probability measure $\mathbb{P}_m$, $m \in \mathbb{N}$.

Next, using Itô's formula, we compute the dynamics of the venture portfolio and the benchmark portfolio under the alternate probability measure $\mathbb{P}_m$. The m-power of the benchmark portfolio has the dynamics under $\mathbb{P}$-measure

$$\begin{aligned} \mathrm{d}Y(t)^m &= Y(t)^m \left((m\alpha + \frac{1}{2}m(m-1)\boldsymbol{\beta}'\boldsymbol{\beta})\,\mathrm{d}t + m\boldsymbol{\beta}'\,\mathrm{d}\mathbf{W}(t) \right) \\ Y(0)^m &= Y_0^m \end{aligned} \tag{2.22}$$

Under $\mathbb{P}$-measure, we know that

$$\mathrm{d}\hat{Z}(t) = \hat{Z}(t) \left((a - \alpha + \boldsymbol{\beta}'\boldsymbol{\beta} - \boldsymbol{\beta}'\boldsymbol{b})\,\mathrm{d}t + (\boldsymbol{b} - \boldsymbol{\beta})'\,\mathrm{d}\mathbf{W}(t) \right) \tag{2.23}$$

Using Eq. 2.21, we find that under $\mathbb{P}_m$-measure, the dynamics of $\hat{Z}(t)$ is

$$\mathrm{d}\hat{Z}(t) = \hat{Z}(t) \left(a_m\,\mathrm{d}t + (\boldsymbol{b} - \boldsymbol{\beta})'\,\mathrm{d}\mathbf{W}^{\mathbb{P}_m}(t) \right) \tag{2.24}$$

where $a_m = a - \alpha + (m-1)(\boldsymbol{b} - \boldsymbol{\beta})'\boldsymbol{\beta}$. Then, the m-power of $\hat{Z}(t)$ under $\mathbb{P}_m$-measure is

$$\begin{aligned} \mathrm{d}\hat{Z}(t)^m &= \hat{Z}(t)^m \left((ma_m + \frac{1}{2}m(m-1)\nu^2)\,\mathrm{d}t + m(\boldsymbol{b} - \boldsymbol{\beta})'\,\mathrm{d}\mathbf{W}^{\mathbb{P}_m}(t) \right) \\ \hat{Z}(0)^m &= \hat{Z}(0)^m \end{aligned} \tag{2.25}$$

The payoff of the portfolio insurance strategy at maturity T is given by

$$V(T) = kY(T) + (pZ(T) - kY(T))^+ .$$

Applying the binomial theorem, we decompose the mth moment of the payoff function into

$$\mathbb{E}_{\mathbb{P}}\left[V(T)^m\right] = \mathbb{E}_{\mathbb{P}}\left[(kY(T))^m\right] + \sum_{i=1}^{m} \begin{pmatrix} m \\ i \end{pmatrix} \mathbb{E}_{\mathbb{P}}\left[(kY(T))^{m-i}\left((pZ(T)-kY(T))^+\right)^i\right],$$

where $\mathbb{E}_{\mathbb{P}}$ denotes the (unconditional) expectation with respect to the real-world measure $\mathbb{P}$.

Applying the change of probability measures, $\mathbb{E}_{\mathbb{P}}[(kY(T))^{m-i}((pZ(T)-kY(T))^+)^i]$ can be expressed as

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[(kY(T))^{m-i}\left((pZ(T)-kY(T))^+\right)^i\right] &= \mathbb{E}_{\mathbb{P}}\left[k^{m-i}\left(Y(T)\right)^m\left(\left(p\hat{Z}(T)-k\right)^+\right)^i\right] \\
&= k^{m-i}\mathbb{E}_{\mathbb{P}}\left[Y(t)^m\right]\cdot\mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{P}_m}{\mathrm{d}\mathbb{P}}\left(\left(p\hat{Z}(T)-k\right)^+\right)^i\right], \\
&= k^{m-i}\mathbb{E}_{\mathbb{P}}\left[Y(t)^m\right]\cdot\mathbb{E}_{\mathbb{P}_m}\left[\left(\left(p\hat{Z}(T)-k\right)^+\right)^i\right]
\end{aligned}$$

where $\mathbb{E}_{\mathbb{P}_m}$ denotes the (unconditional) expectation with respect to the alternate measure $\mathbb{P}_m$ as define in Eq. (2.20).

Applying the binomial theorem on $\mathbb{E}_{\mathbb{P}_m}[((p\hat{Z}(T)-k)^+)^i]$, we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_m}\left[\left(\left(p\hat{Z}(T)-k\right)^+\right)^i\right] &= \mathbb{E}_{\mathbb{P}_m}\left[\left(p\hat{Z}(T)\mathbb{1}(p\hat{Z}(T)>k) - k\mathbb{1}(p\hat{Z}(T)>k)\right)^i\right] \\
&= \mathbb{E}_{\mathbb{P}_m}\left[\sum_{l=0}^{i}\begin{pmatrix} i \\ l \end{pmatrix}\left(-k\mathbb{1}(p\hat{Z}(T)>k)\right)^{i-l}\left(p\hat{Z}(T)\mathbb{1}(p\hat{Z}(T)>k)\right)^l\right] \\
&= \mathbb{E}_{\mathbb{P}_m}\left[\sum_{l=0}^{i}\begin{pmatrix} i \\ l \end{pmatrix}(-1)^{i-l}k^{i-l}\left(p\hat{Z}(T)\right)^l\mathbb{1}(p\hat{Z}(T)>k)\right] \\
&= \sum_{l=0}^{i}\begin{pmatrix} i \\ l \end{pmatrix}(-1)^{i-l}k^{i-l}\mathbb{E}_{\mathbb{P}_m}\left[\left(p\hat{Z}(T)\right)^l\mathbb{1}(p\hat{Z}(T)>k)\right]
\end{aligned}$$

Applying Gaussian Shift Theorem[3] and the dynamics of $\hat{Z}(t)$ and $\hat{Z}(t)^m$ under the probability measures $\mathbb{P}_m$ (see Eqs 2.24, 2.25), we have

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}_m}\left[\left(p\hat{Z}(T)\right)^l \mathbb{1}(p\hat{Z}(T) > k)\right] \\
&= \left(p\hat{Z}(0)\right)^l \mathbb{E}_{\mathbb{P}_m}\left[e^{(la_m+\frac{1}{2}l(l-1)\nu^2-\frac{1}{2}l^2\nu^2)T+l\nu\sqrt{T}\mathcal{Z}}\mathbb{1}(p\hat{Z}(0)e^{(a_m-\frac{1}{2}\nu^2)T+\nu\sqrt{T}\mathcal{Z}} > k)\right] \\
&= \left(p\hat{Z}(0)\right)^l e^{(la_m+\frac{1}{2}l(l-1)\nu^2-\frac{1}{2}l^2\nu^2)T}\cdot \mathbb{E}_{\mathbb{P}_m}\left[e^{\nu\sqrt{T}\mathcal{Z}}\mathbb{1}\left(\mathcal{Z} > -\frac{\log\frac{p\hat{Z}(0)}{k} + (a_m - \frac{1}{2}\nu^2)T}{\nu\sqrt{T}}\right)\right] \\
&= p^l\hat{Z}(0)^l e^{(la_m+\frac{1}{2}l(l-1)\nu^2-\frac{1}{2}l^2\nu^2)T}\cdot e^{\frac{1}{2}l^2\nu^2T}\mathbb{E}_{\mathbb{P}_m}\left[\mathbb{1}\left(\mathcal{Z} + l\nu\sqrt{T} > -\frac{\log\frac{p\hat{Z}(0)}{k} + (a_m - \frac{1}{2}\nu^2)T}{\nu\sqrt{T}}\right)\right] \\
&= p^l\hat{Z}(0)^l e^{(la_m+\frac{1}{2}l(l-1)\nu^2)T}\mathbb{E}_{\mathbb{P}_m}\left[\mathbb{1}\left(\mathcal{Z} > -\frac{\log\frac{p\hat{Z}(0)}{k} + (a_m + (l-\frac{1}{2})\nu^2)T}{\nu\sqrt{T}}\right)\right] \\
&= p^l\hat{Z}(0)^l e^{[la_m+\frac{1}{2}l(l-1)\nu^2]T}\Phi(d_{-;m,l})
\end{aligned}
$$

in which $\mathcal{Z}$ is a standard normal random variable and $d_{-;m,l}$ is given in Eq. (2.4).

Combining the results, we conclude

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[V(T)^m\right] =& \mathbb{E}_{\mathbb{P}}\left[(kY(T))^m\right] \\
&+ \sum_{i=1}^{m}\binom{m}{i}k^{m-i}\mathbb{E}_{\mathbb{P}}\left[Y(t)^m\right]\sum_{l=0}^{i}\binom{i}{l}(-1)^{i-l}k^{i-l}p^l\hat{Z}(0)^l e^{[la_m+\frac{1}{2}l(l-1)\nu^2]T}\Phi(d_{-;m,l}) \\
=& k^m\mathbb{E}_{\mathbb{P}}\left[Y(T)^m\right] \\
&+ k^m\mathbb{E}_{\mathbb{P}}\left[Y(T)^m\right]\cdot\sum_{i=1}^{m}\binom{m}{i}k^{-i}\cdot\sum_{l=0}^{i}\binom{i}{l}(-1)^{i-l}k^i\left(\frac{p\hat{Z}(0)}{k}\right)^l e^{[la_m+\frac{1}{2}l(l-1)\nu^2]T}\Phi(d_{-;m,l}) \\
=& k^m\mathbb{E}_{\mathbb{P}}\left[Y(T)^m\right]\cdot\left(1 + \sum_{i=1}^{m}\sum_{l=0}^{i}\binom{m}{i}\cdot\binom{i}{l}\cdot(-1)^{i-l}\cdot\left(\frac{p\hat{Z}(0)}{k}\right)^l\cdot e^{l[a_m+\frac{1}{2}(l-1)\nu^2]T}\cdot\Phi(d_{-;m,l})\right)
\end{aligned}
$$

□

[3] citation to follow.

To investigate the prospect of mean-variance dominance and stochastic dominance, we will need the first and second moments of the GOPIS (Remark 2.5). As in Eq. 2.9, we adopt the Black-Scholes call option pricing formula to simplify the expressions.

Remark 2.5. Suppose a portfolio insurance strategy with the payoff $V(T) = \max(pZ(T), kY(T))$ at maturity T. Under $\mathbb{P}$, the benchmark portfolio Y has drift parameter α and diffusion vector $\boldsymbol{\beta}$, and the venture portfolio Z has drift parameter a and diffusion vector $\boldsymbol{b}$. Then, we have $\nu = \|\boldsymbol{b} - \boldsymbol{\beta}\|$. Setting $m = 1$ and $m = 2$ in Theorem 2.4, we receive the first and second moments of the portfolio insurance strategy under $\mathbb{P}$ as

$$\mathbb{E}_{\mathbb{P}}[V(T)] = e^{\alpha T} \cdot \left[k + e^{(a-\alpha)T} \cdot \text{call}(0, p; T, k, a - \alpha, \nu) \right], \tag{2.26}$$

and

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[V(T)^2] =& e^{(2\alpha + \boldsymbol{\beta}'\boldsymbol{\beta})T} \cdot \Big[k^2 + p \cdot e^{2[a-\alpha+(\boldsymbol{b}-\boldsymbol{\beta})'\boldsymbol{\beta}]T} \\ & \cdot \text{call}(0, pe^{\nu^2 T}; T, k, a - \alpha + (\boldsymbol{b} - \boldsymbol{\beta})'\boldsymbol{\beta}, \nu) \\ & + k \cdot e^{[a-\alpha+(\boldsymbol{b}-\boldsymbol{\beta})'\boldsymbol{\beta}]T} \cdot \text{call}(0, p; T, k, a - \alpha + (\boldsymbol{b} - \boldsymbol{\beta})'\boldsymbol{\beta}, \nu) \Big]. \end{aligned} \tag{2.27}$$

In a financial market with $d = 2$ risky assets, the moments of a traditional OBPI strategy can be recovered by setting the venture asset allocation to be $\boldsymbol{u}_Z = (0\ \ 1)'$ and the benchmark asset allocation to be $\boldsymbol{u}_Y = (0\ \ 0)'$. Likewise, the moments of OBPP (which has one of the risky assets as its benchmark portfolio) is recovered by setting the benchmark asset allocation $\boldsymbol{u}_Y = (1\ \ 0)'$ and the venture asset allocation $\boldsymbol{u}_Z = (0\ \ 1)'$.

Next, we provide the tools to derive the conditional moments of GOPIS needed for the subsequent analysis. Lemma 2.7 calculates the moment functions for special cases of GOPIS when the benchmark portfolio is a constant instead of a stochastic value. For example, the first and second moments of a traditional OBPI that uses the risk-free asset as its benchmark portfolio can be obtained by setting $\delta = r$ and replacing μ, σ with the mean rate of return and volatility of the corresponding venture portfolio. Lemma 2.7 is also profitable in retrieving the conditional moments of GOPIS when its benchmark portfolio's value is given. Since $\log(Z(T))$ and $\log(Y(T))$ are both linear combinations of $\mathbf{W}(T)$, a d-dimensional independent normal random variable, then the vector $(\log(Z(T)) \;\; \log(Y(T)))'$ follows a multivariate normal distribution. Conditional on $Y(T) = y$, equivalently $\log(Y(T)) = \log(y)$, the random variable $\log(Z(T))|\log(Y(T)) = \log(y)$ is a normal random variable (see, e.g., Greene, 2003, Theorem B.7) whose parameters are given in Remark 2.6.

Remark 2.6. Let $Y(T) = y$ be given. Then, we have

$$(\log(Z(T))\,|\,\log(Y(T)) = y) \sim \mathcal{N}\left(\left(a(y) - \frac{1}{2}(b(y))^2\right)T, (b(y))^2 T\right)$$

with

$$a(y) := a + \rho\frac{\|\boldsymbol{b}\|}{\|\boldsymbol{\beta}\|}\left(\frac{1}{T}\log(y) - \alpha\right) - \frac{1}{2}\rho\|\boldsymbol{b}\|(\rho\|\boldsymbol{b}\| - \|\boldsymbol{\beta}\|),$$

and

$$b(y) := \|\boldsymbol{b}\|\sqrt{1-\rho^2}.$$

where α and a are drift parameters of the benchmark and the venture portfolios, respectively; $\|\boldsymbol{\beta}\|$ and $\|\boldsymbol{b}\|$ are the 2-norm of the diffusion vectors of the benchmark portfolio $\boldsymbol{\beta}$ and the venture portfolio $\boldsymbol{b}$, respectively; and $\rho = \boldsymbol{\beta}'\boldsymbol{b}/(\|\boldsymbol{\beta}\|\cdot\|\boldsymbol{b}\|)$ is the correlation between the two portfolios.

Lemma 2.7. *Suppose at maturity the portfolio insurance strategy $V(T)$ has a payoff*

$$V(T) = \max\left(pZ(T), ke^{\delta T}\right),$$

where $\log(Z(T))$ is normally distributed with mean $(\mu - \frac{1}{2}\sigma^2)T$ and variance $\sigma^2 T$ under $\mathbb{P}$; and p, k, δ are constants. We obtain the first and second moments of the portfolio insurance strategy under $\mathbb{P}$ as

$$\mathbb{E}_{\mathbb{P}}[V(T)] = ke^{\delta T} + e^{\mu T}\cdot call(0, p; T, k, \mu - \delta, \sigma), \tag{2.28}$$

and

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[V(T)^2] =& k^2 e^{2\delta T} + p\cdot e^{2\mu T}\cdot call(0, pe^{\sigma^2 T}; T, k, \mu - \delta, \sigma) \\ &+ ke^{\delta T}\cdot e^{\mu T}\cdot call(0, p; T, k, \mu - \delta, \sigma).\end{aligned} \tag{2.29}$$

Proof. It is straightforward to obtain the results by computing the first and second moments under $\mathbb{P}$. Alternatively, we utilise the results derived from Theorem 2.4. Suppose there exists, under $\mathbb{P}$, a portfolio $Y(t)$, that is a geometric Brownian motion process with a constant drift δ such that $Y(T) = e^{\delta T}$ and zero diffusion. Then, the portfolio yields $e^{\delta T}$ with certainty at time T. Similarly, suppose there exists, under $\mathbb{P}$, a portfolio $Z(t)$, that is a geometric Brownian process with drift parameter μ and volatility parameter σ. The volatility of the embedded option $\nu = \sigma$ as there is no correlation between the deterministic process $Y(t)$ and the stochastic process $Z(t)$. Applying Remark 2.5 with $\alpha = \delta$, $\boldsymbol{\beta} = 0$, $a = \mu$, $\boldsymbol{b} = \sigma$ and $\nu = \sigma$, we obtain the expression for $\mathbb{E}_{\mathbb{P}}[V(T)^m]$. □

2.3.3.3 Stochastic Dominance

Suppose an investor is interested in setting up a portfolio insurance strategy. Which GOPIS configurations should they choose as there exists an infinite number of combinations of venture and benchmark portfolios. Traditionally, a mean-variance analysis is used in evaluating and comparing portfolio performance. The mean-variance of GOPIS can be calculated using the explicit formulas for the moments of the payoff distributions of GOPIS derived in the previous section. The main disadvantage of using mean-variance analysis in comparing GOPIS is that these strategies result in asymmetrical or truncated payoff distributions whose statistical properties are not fully captured in the first two moments. Stochastic dominance (SD), on the other hand, is a more robust comparison method as it makes no assumptions about the distribution of returns. A random payoff X is said to have the first-order SD over a payoff Y if and only if X has a higher expected utility than Y for all non-decreasing utility functions (i.e. non-satiated preference). A random payoff X is said to have the second-order SD over a payoff Y if and only if X has a higher expected utility than Y for all non-decreasing, concave utility functions (i.e., risk averse preference).[4] This makes SD a useful comparison method when only partial information on the investor's preference is available.

[4]We refer readers to Zagst and Kraus (2009) for further details concerning the concept of stochastic dominance in the context of portfolio management.

In our setting, we compare different GOPIS given a condition that the benchmark value of the dominating GOPIS is higher than the benchmark value of the dominated GOPIS at maturity T. The concept of conditional stochastic dominance has been given different interpretations in the literature. An earlier application of conditional stochastic dominance is introduced by Shalit and Yitzhaki (1994) where they studied the probabilistic condition under which all risk-averse investors prefer one risky asset over another, conditional on their existing portfolio of assets. Following Zagst et al. (2019), we use conditional stochastic dominance to first identify the condition under which one strategy dominates the other with respect to the first- and second-order SD. The advantage of using conditional stochastic dominance is that it allows us to first identify the condition under which stochastic dominance is attainable and subsequently calculate the probability of stochastic dominance.

Consider the following two GOPISs:

$$V_1 := V_1(T) = \max(p_1 Z_1(T), k_1 Y_1(T)),$$

and

$$V_2 := V_2(T) = \max(p_2 Z_2(T), k_2 Y_2(T)).$$

We consider V_2 as the candidate portfolio to stochastically dominate V_1 given the relative value of the benchmark portfolios at maturity, $\tilde{Y} := Y_2(T)/Y_1(T)$. It is straightforward to show that it must satisfy $\tilde{Y} > k_1/k_2$ for V_2 to dominate V_1 in the first- and second-order conditional stochastic dominance.[5] Hence, we compare different GOPISs under the condition that $\tilde{Y}$ is in a set $\mathcal{D} := [k_1/k_2, \infty)$. Taking advantage of the fact that $\log(\tilde{Y})$ is a normal random variable (Remark 2.8), we can then evaluate the probability of V_2 stochastically dominate V_1 once we identify the specific condition of stochastic dominance (in Theorems 2.10-2.12).

Remark 2.8. $\log(\tilde{Y}) = \log(Y_2(T)) - \log(Y_1(T))$ follows a normal distribution because it is a linear combination of $\mathbf{W}(T)$, which by definition is a d-dimensional independent normal random

[5] Suppose that $\tilde{Y} < k_1/k_2$. Then, there exists a constant x_1 such that $k_1 Y_1(T) \geq x_1 > k_2 Y_2(T)$ and that $H(x) := F_{V_1(T)|\tilde{Y}=y}(x) - F_{V_2(T)|\tilde{Y}=y}(x) \leq 0$ for $x \in (-\infty, x_1)$. As $F_{V_2(T)|\tilde{Y}=y}$ is above $F_{V_1(T)|\tilde{Y}=y}$ for $x < x_1$, $F_{V_2(T)|\tilde{Y}=y}$ can never dominate $F_{V_1(T)|\tilde{Y}=y}$ in the first- and second-order stochastic dominance (see Mosler (1982)).

variable (since d is assumed to be finite). It is straightforward to show that $\log(\tilde{Y})$ has mean $\mu_y = (\alpha_2 - \frac{1}{2}\|\boldsymbol{\beta}_2\|^2)T - (\alpha_1 - \frac{1}{2}\|\boldsymbol{\beta}_1\|^2)T$ and variance $\sigma_y^2 = (\boldsymbol{\beta}_2 - \boldsymbol{\beta}_1)'(\boldsymbol{\beta}_2 - \boldsymbol{\beta}_1)T$. Then, we have $\log(\tilde{Y}) \sim N(\mu_y, \sigma_y^2)$. Let $\mathcal{D} := [k_1/k_2, \infty)$. When $\tilde{Y} \in \mathcal{D}$, we have $k_1 Y_1(T) \leq k_2 Y_2(T)$.

The viability of stochastic dominance of a random variable X over Y depends on the number of intersections between their cumulative density function $F_X(x)$ and $F_Y(x)$ (see e.g., Levy, 2015, Chapter 3). In our setting, we analyse the conditional cumulative probability density function of V_1 and V_2. Given $\tilde{Y} \in \mathcal{D}$, the number of intersections between the conditional cumulative probability density function of V_1 and V_2 depends on the conditional volatilities of $Z_1(T)|\tilde{Y}$ and $Z_2|\tilde{Y}$. Now, let us consider the vector $\mathbf{x} = (\log(Y_1(T)), \log(Z_1(T)), \log(Y_2(T)), \log(Z_2(T)), \log(\tilde{Y}))'$ which contains the logarithm value of the five processes at time T. Since these processes are driven by d Brownian motions, $\mathbf{x}$ have a joint multivariate normal distribution with mean vector $\mathbf{m}$ and covariance matrix $\boldsymbol{\Sigma} = \mathbf{v}'\mathbf{I}_d\mathbf{v} \cdot T$ where

$$\mathbf{m} = \begin{bmatrix} \alpha_1 - \frac{1}{2}\|\boldsymbol{\beta}_1\|^2 \\ a_1 - \frac{1}{2}\|\boldsymbol{b}_1\|^2 \\ \alpha_2 - \frac{1}{2}\|\boldsymbol{\beta}_2\|^2 \\ a_2 - \frac{1}{2}\|\boldsymbol{b}_2\|^2 \\ (\alpha_2 - \frac{1}{2}\|\boldsymbol{\beta}_2\|^2) - (\alpha_1 - \frac{1}{2}\|\boldsymbol{\beta}_1\|^2) \end{bmatrix} \cdot T, \quad \mathbf{v} = \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{b}_1 \\ \boldsymbol{\beta}_2 \\ \boldsymbol{b}_2 \\ \boldsymbol{\beta}_2 - \boldsymbol{\beta}_1 \end{bmatrix},$$

and $\mathbf{I}_d$ is a d-dimensional identity matrix. Since the conditional distribution of a multivariate normal distribution is a multivariate normal distribution, we can compute the distribution of $\log(p_i Z_i(T))|\tilde{Y} = y)$ for $i = 1, 2$ (see Remark 2.9). Different configurations of venture portfolios in V_1 and V_2 results in different relative values of $\sigma_1(y)$ and $\sigma_2(y)$, and hence different numbers of potential intersections. In the following, we analyse the prospect of the first- and second-order conditional stochastic dominance separately for the three cases: (i) $\sigma_1(y) = \sigma_2(y)$, (ii) $\sigma_1(y) > \sigma_2(y)$, and (iii) $\sigma_1(y) < \sigma_2(y)$.

Remark 2.9. For $i = 1, 2$, the conditional random variable $\log(p_i Z_i(T))|\tilde{Y} = y)$ is normally distributed with mean $\mu_i(y)$ and variance $\sigma_i^2(y)$ where

$$\mu_i(y) = \log(p_i) + \left(a_i(y) - \frac{1}{2}b_i^2(y)\right)T,$$
$$\sigma_i^2(y) = b_i^2(y)T,$$

and

$$a_i(y) = a_i - \frac{1}{2}\|\boldsymbol{b}_i\|^2\tilde{\rho}_i^2 + \tilde{\rho}_i\frac{\|\boldsymbol{b}_i\|}{\|\boldsymbol{\beta}_2 - \boldsymbol{\beta}_1\|}\left(\frac{1}{T}\log(y) - \left((\alpha_2 - \frac{1}{2}\|\boldsymbol{\beta}_2\|^2) - (\alpha_1 - \frac{1}{2}\|\boldsymbol{\beta}_1\|^2)\right)\right),$$
$$b_i^2(y) = \|\boldsymbol{b}_i\|^2\left(1 - \tilde{\rho}_i^2\right),$$

in which $\tilde{\rho}_i := \mathrm{Cor}(\log(Z_i(T)), \log(\tilde{Y}))$ is the correlation between the logarithm of the venture portfolio V_i and $\log(\tilde{Y})$. We note that the conditional volatility $b_i(y)$ is independent of the realised value of $\tilde{Y}$ and hence $\sigma_i(y)$ is known at $t = 0$. We exploit this property to investigate the potentials for conditional stochastic dominance of V_2 over V_1.

Given $\tilde{Y} \in \mathcal{D}$ and $\sigma_1(y) = \sigma_2(y)$, the number of intersections between the conditional cumulative density functions is determined by the relative value of $\mu_1(y)$ and $\mu_2(y)$. Theorem 2.10 outlines the conditions under which V_2 dominates V_1 by first- and second-order stochastic dominance, in situations where the venture portfolios have the same conditional volatility ($\sigma_1(y) = \sigma_2(y)$). A special case of $\sigma_1(y) = \sigma_2(y)$ is when both GOPISs have an identical venture portfolio. Then, the condition $\mu_1(y) \lesseqgtr \mu_2(y)$ is simplified into $p_1 \lesseqgtr p_2$ and we reach the same conclusion as Zagst et al. (2019).

Theorem 2.10. *Let $\sigma_1(y) = \sigma_2(y)$ and $\tilde{Y} := y$ be given. Furthermore, let $y \in \mathcal{D}$. Then:*
(i) We have a first-order stochastic dominance favouring V_2

$$V_1(T) \prec_{1,\mathcal{D}} V_2(T)$$

when $\mu_1(y) \leq \mu_2(y)$, i.e. for all increasing functions U and for all $y \in \mathcal{D}$ it holds that $\mathbb{E}[U(V_1(T))|\tilde{Y} = y] \leq \mathbb{E}[U(V_2(T))|\tilde{Y} = y]$.

(ii) Let

$$\mathcal{D}_* := \{y \in \mathcal{D} \mid \mathbb{E}[V_1(T)|\tilde{Y} = y] \le \mathbb{E}[V_2(T)|\tilde{Y} = y]\}.$$

We have a second-order conditional stochastic dominance favouring V_2

$$V_1(T) \prec_{2,\mathcal{D}_*} V_2(T)$$

when $\mu_1(y) > \mu_2(y)$, i.e. for all increasing, concave functions U and for all $y \in \mathcal{D}_*$ it holds that $\mathbb{E}[U(V_1(T))|\tilde{Y} = y] \le \mathbb{E}[U(V_2(T))|\tilde{Y} = y]$.

Proof. (i) Let $F_X(x)$ denote the cumulative density function of a random variable X. Since $\tilde{Y} \in \mathcal{D}$, we have $k_1 Y_1(T) \le k_2 Y_2(T)$. Hence,

$$F_{V_1(T)|\tilde{Y}=y}(x) = \begin{cases} 0 & \text{if } x < k_1 Y_1(T) \\ F_{p_1 Z_1(T)|\tilde{Y}=y}(x) & \text{if } k_1 Y_1(T) \le x < k_2 Y_2(T) \\ F_{p_1 Z_1(T)|\tilde{Y}=y}(x) & \text{if } k_2 Y_2(T) \le x. \end{cases} \tag{2.30}$$

and

$$F_{V_2(T)|\tilde{Y}=y}(x) = \begin{cases} 0 & \text{if } x < k_1 Y_1(T) \\ 0 & \text{if } k_1 Y_1(T) \le x < k_2 Y_2(T) \\ F_{p_2 Z_2(T)|\tilde{Y}=y}(x) & \text{if } k_2 Y_2(T) \le x. \end{cases} \tag{2.31}$$

From Remark 2.9, we deduce $F_{p_i Z_i(T)|\tilde{Y}=y}(x) = \Phi(\frac{\log(x)-\mu_i(y)}{\sigma_i(y)})$.

When $\mu_1(y) \le \mu_2(y)$, we learn that $\frac{\log(x)-\mu_1(y)}{\sigma_1(y)} \ge \frac{\log(x)-\mu_2(y)}{\sigma_2(y)}$ and hence $F_{V_1(T)|\tilde{Y}=y}(x) \ge F_{V_2(T)|\tilde{Y}=y}(x) \quad \forall x \in \mathbb{R}$. Using Mosler (1982), we conclude $V_1(T) \prec_{1,\mathcal{D}} V_2(T)$.

(ii) Let $\mathcal{D}_*$ be the set of all $y \in \mathcal{D}$ where additionally the conditional expectation of $V_1(T)$ is smaller than that of $V_2(T)$. Since $\mathcal{D}_* \subset \mathcal{D}$, the conditional cumulative distribution functions of $V_1(T)$ and $V_2(T)$ follow Eq. 2.30 and Eq. 2.31 respectively.

When $\mu_1(y) > \mu_2(y)$, we learn that $\frac{\log(x)-\mu_1(y)}{\sigma_1(y)} < \frac{\log(x)-\mu_2(y)}{\sigma_2(y)}$ and hence

$$F_{V_1(T)|\tilde{Y}=y}(x) \begin{cases} \geq F_{V_2(T)|\tilde{Y}=y}(x) & \text{if } x < k_2 Y_2(T) \\ \leq F_{V_2(T)|\tilde{Y}=y}(x) & \text{if } k_2 Y_2(T) \leq x. \end{cases}$$

Thus, there exists a x^* such that $H(x) \geq (\leq)\ 0\ \forall x \leq (\geq)\ x^*$, i.e., $H \in \mathbb{S}_1$, with

$$\mathbb{S}_1 = \{H : \mathbb{R} \to \mathbb{R} : \exists x \in \mathbb{R}, H(x) \geq 0, \forall x \in (-\infty, x_1),$$
$$H(x) \leq 0, \forall x \in (x_1, \infty), H \neq 0\}.$$

Using Mosler (1982), we conclude that

$$V_1(T) \prec_{2,\mathcal{D}_*} V_2(T).$$

□

Given $\tilde{Y} \in \mathcal{D}$ and $\sigma_1(y) > \sigma_2(y)$, it is guaranteed that the conditional cumulative density functions intersect exactly once. This negates the possibility of first-order stochastic dominance for both strategies. Theorem 2.11 outlines the condition under which V_2 stochastically dominates V_1 in the second-order sense. Given $\tilde{Y} \in \mathcal{D}$ and $\sigma_1(y) < \sigma_2(y)$, the conditional cumulative density functions either do not intersect or intersect twice. Theorem 2.12 outlines the condition under which V_2 stochastically dominates V_1 in the first-order sense.

Theorem 2.11. *Let $\sigma_1(y) > \sigma_2(y)$ and $\tilde{Y} := y$ be given. Furthermore, let $y \in \mathcal{D}_*$ where $\mathcal{D}_*$ is given as in Theorem 2.10. Then:*
We have a second-order conditional stochastic dominance favouring V_2

$$V_1(T) \prec_{2,\mathcal{D}_*} V_2(T).$$

Proof. We denote the quantiles of order q of distribution $pZ(T)|\tilde{Y} = y$ by x_q and the quantiles of order of the standard normal distribution $\mathcal{N}(0,1)$ by $\mathcal{Z}_q$. Then, for $\sigma_1(y) \neq \sigma_2(y)$ the quantiles are given by,

$$x_q = e^{\mu(y)+\mathcal{Z}_q\sigma(y)}.$$

According to Levy (1973), when $\sigma_1(y) > \sigma_2(y)$, there is some positive value $\mathcal{Z}_{q_0}$ such that $\mu_1(y) + \mathcal{Z}_{q_0}\sigma_1(y) = \mu_2(y) + \mathcal{Z}_{q_0}\sigma_2(y)$. Hence,

$$\mathcal{Z}_{q_0} = \frac{\mu_2(y) - \mu_1(y)}{\sigma_1(y) - \sigma_2(y)}.$$

For any value $x < x_{q_0}$, $F_{p_1Z_1(T)|\tilde{Y}=y}(x)$ is above $F_{p_2Z_2(T)|\tilde{Y}=y}(x)$ and for any value $x > x_{q_0}$, $F_{p_1Z_1(T)|\tilde{Y}=y}(x)$ is below $F_{p_2Z_2(T)|\tilde{Y}=y}(x)$, i.e.

$$F_{p_1Z_1(T)|\tilde{Y}=y}(x) \begin{cases} \geq F_{p_2Z_2(T)|\tilde{Y}=y}(x) & \text{if } x < x_{q_0} \\ \leq F_{p_2Z_2(T)|\tilde{Y}=y}(x) & \text{if } x > x_{q_0}. \end{cases}$$

When $x_{q_0} > k_2Y_2(T)$, we have

$$F_{V_1(T)|\tilde{Y}=y}(x) \begin{cases} = F_{V_2(T)|\tilde{Y}=y}(x) = 0 & \text{if } x < k_1Y_1(T) \\ \geq F_{V_2(T)|\tilde{Y}=y}(x) = 0 & \text{if } k_1Y_1(T) \leq x < k_2Y_2(T) \\ \geq F_{V_2(T)|\tilde{Y}=y}(x) & \text{if } k_2Y_2(T) \leq x < x_{q_0} \\ \leq F_{V_2(T)|\tilde{Y}=y}(x) & \text{if } x_{q_0} \leq x. \end{cases}$$

When $k_1Y_1(T) < x_{q_0} < k_2Y_2(T)$, we have

$$F_{V_1(T)|\tilde{Y}=y}(x) \begin{cases} = F_{V_2(T)|\tilde{Y}=y}(x) = 0 & \text{if } x < k_1Y_1(T) \\ \geq F_{V_2(T)|\tilde{Y}=y}(x) = 0 & \text{if } k_1Y_1(T) \leq x < x_{q_0} \\ \geq F_{V_2(T)|\tilde{Y}=y}(x) = 0 & \text{if } x_{q_0} \leq x < k_2Y_2(T) \\ \leq F_{V_2(T)|\tilde{Y}=y}(x) & \text{if } k_2Y_2(T) \leq x. \end{cases}$$

When $x_{q_0} < k_1 Y_1(T)$, we have

$$F_{V_1(T)|\tilde{Y}=y}(x) \begin{cases} = F_{V_2(T)|\tilde{Y}=y}(x) = 0 & \text{if } x < x_{q_0} \\ = F_{V_2(T)|\tilde{Y}=y}(x) = 0 & \text{if } x_{q_0} \leq x < k_1 Y_1(T) \\ \geq F_{V_2(T)|\tilde{Y}=y}(x) = 0 & \text{if } k_1 Y_1(T) \leq x < k_2 Y_2(T) \\ \leq F_{V_2(T)|\tilde{Y}=y}(x) & \text{if } k_2 Y_2(T) \leq x. \end{cases}$$

Thus, there exists a x^* such that $H(x) \geq (\leq)\ 0\ \forall x \leq (\geq)\ x^*$, i.e., $H \in \mathbb{S}_1$. In addition, we have $\mathbb{E}[V_1(T)|\tilde{Y} = y] \leq \mathbb{E}[V_2(T)|\tilde{Y} = y]$ for $y \in \mathcal{D}$. Using Mosler (1982), we conclude that

$$V_1(T) \prec_{2,\mathcal{D}_*} V_2(T).$$

□

Theorem 2.12. *Let $\sigma_1(y) < \sigma_2(y)$ and $\tilde{Y} := y$ be given. Furthermore, let $y \in \mathcal{D}$ where $\mathcal{D}$ is given as in Theorem 2.10. Let*

$$\mathcal{D}_1 := \{y \in \mathcal{D} \mid x_{q_0} < k_2 Y_2(T)\},$$

where x_{q_0} is the q_0-quantile such that $x_{q_0} = e^{\mu_1(y) + \mathcal{Z}_{q_0}\sigma_1(y)} = e^{\mu_2(y) + \mathcal{Z}_{q_0}\sigma_2(y)}$.
Then, we have a first-order conditional stochastic dominance favouring V_2

$$V_1(T) \prec_{1,\mathcal{D}_1} V_2(T).$$

Proof. According to Levy (1973), there is some negative value $\mathcal{Z}_{q_0}$ such that $\mu_1(y) + \mathcal{Z}_{q_0}\sigma_1(y) = \mu_2(y) + \mathcal{Z}_{q_0}\sigma_2(y)$ if $\sigma_1(y) < \sigma_2(y)$. Hence,

$$\mathcal{Z}_{q_0} = \frac{\mu_2(y) - \mu_1(y)}{\sigma_1(y) - \sigma_2(y)}.$$

For any value $x < x_{q_0}$, $F_{p_1 Z_1(T)|\tilde{Y}=y}(x)$ is below $F_{p_2 Z_2(T)|\tilde{Y}=y}(x)$ and for any value $x > x_{q_0}$, $F_{p_1 Z_1(T)|\tilde{Y}=y}(x)$ is above $F_{p_2 Z_2(T)|\tilde{Y}=y}(x)$, i.e.

$$F_{p_1 Z_1(T)|\tilde{Y}=y}(x) \begin{cases} \leq F_{p_2 Z_2(T)|\tilde{Y}=y}(x) & \text{if } x < x_{q_0} \\ \geq F_{p_2 Z_2(T)|\tilde{Y}=y}(x) & \text{if } x > x_{q_0}. \end{cases}$$

When $x_{q_0} > k_2 Y_2(T)$, we have

$$F_{V_1(T)|\tilde{Y}=y}(x) \begin{cases} = F_{V_2(T)|\tilde{Y}=y}(x) = 0 & \text{if } x < k_1 Y_1(T) \\ \geq F_{V_2(T)|\tilde{Y}=y}(x) = 0 & \text{if } k_1 Y_1(T) \leq x < k_2 Y_2(T) \\ \leq F_{V_2(T)|\tilde{Y}=y}(x) & \text{if } k_2 Y_2(T) \leq x < x_{q_0} \\ \geq F_{V_2(T)|\tilde{Y}=y}(x) & \text{if } x_{q_0} \leq x. \end{cases}$$

Hence, $H \in \mathbb{S}_2$ where

$$\mathbb{S}_2 = \left\{ \begin{array}{ll} H : \mathbb{R} \to \mathbb{R} : \exists x_1, x_2 \in \mathbb{R}, & \\ H(x) \begin{cases} \geq 0 & \text{if } x \in (-\infty, x_1), \\ \leq 0 & \text{if } x \in (x_1, x_2) \\ \geq 0, & \text{if } x \in (x_2, \infty), \end{cases} & H \neq 0 \end{array} \right\}.$$

Then, the first- and second-order stochastic dominance favouring $V_2(T)$ is not attainable.

When $k_1 Y_1(T) < x_{q_0} < k_2 Y_2(T)$, we have

$$F_{V_1(T)|\tilde{Y}=y}(x) \begin{cases} = F_{V_2(T)|\tilde{Y}=y}(x) = 0 & \text{if } x < k_1 Y_1(T) \\ \geq F_{V_2(T)|\tilde{Y}=y}(x) = 0 & \text{if } k_1 Y_1(T) \leq x < x_{q_0} \\ \geq F_{V_2(T)|\tilde{Y}=y}(x) = 0 & \text{if } x_{q_0} \leq x < k_2 Y_2(T) \\ \geq F_{V_2(T)|\tilde{Y}=y}(x) & \text{if } k_2 Y_2(T) \leq x. \end{cases}$$

When $x_{q_0} < k_1 Y_1(T)$, we have

$$F_{V_1(T)|\tilde{Y}=y}(x) \begin{cases} = F_{V_2(T)|\tilde{Y}=y}(x) = 0 & \text{if } x < x_{q_0} \\ = F_{V_2(T)|\tilde{Y}=y}(x) = 0 & \text{if } x_{q_0} \leq x < k_1 Y_1(T) \\ \geq F_{V_2(T)|\tilde{Y}=y}(x) = 0 & \text{if } k_1 Y_1(T) \leq x < k_2 Y_2(T) \\ \geq F_{V_2(T)|\tilde{Y}=y}(x) & \text{if } k_2 Y_2(T) \leq x. \end{cases}$$

Thus, when $x_{q_0} < k_2 Y_2(T)$, $H(x) \geq 0 \ \forall x \in \mathbb{R}$. Using Mosler (1982), we conclude that

$$V_1(T) \prec_{1,\mathcal{D}_1} V_2(T).$$

□

2.4 Expected utility maximisation with GOPIS for CRRA

The GOPIS provides a convenient way for investors to incorporate some minimum performance guarantee based on a benchmark, i.e. to employ portfolio insurance in managing downside investment risk. The flexibility of GOPIS means it can accommodate a bespoke mix of asset allocations for the venture portfolio $\boldsymbol{u}_Z$. In the previous section, we discussed the prospect of stochastic dominance of a GOPIS over the other. In this section, we show that GOPIS can be the optimal investment strategy that maximises the expected utility whilst meeting a minimum performance constraint.

Suppose the risk preferences of the investor can be modelled using the constant relative risk aversion (CRRA) utility function:

$$U(x) := \frac{1}{\gamma} x^{\gamma}, \quad x > 0, \quad \text{for constant } \gamma \in (-\infty, 1)\backslash\{0\}.$$

Consider an investor who wants his insured portfolio value $V(T)$ to be at least k of a pre-specified benchmark portfolio $Y(T)$ at maturity. Then, the investor can choose from the set of admissible portfolios $\mathcal{A}$ a portfolio process $\boldsymbol{\pi}$ (the amounts to invest in each asset) such that the insured portfolio yields

$$V^{\boldsymbol{\pi}}(T) = \max(pZ(T), kY(T)),$$

with portfolio dynamics satisfies

$$\mathrm{d}V^{\boldsymbol{\pi}}(t) = (rV^{\boldsymbol{\pi}}(t) + \boldsymbol{\pi}(t)\boldsymbol{\sigma\theta})\,\mathrm{d}t + \boldsymbol{\pi}(t)\boldsymbol{\sigma\theta}\,\mathrm{d}\mathbf{W}(t).$$

The set of admissible portfolios for the investor's initial wealth $V_0 > 0$ is defined to be

$$\begin{aligned}\mathcal{A} :=& \{\boldsymbol{\pi} : \Omega \times [0,T] \rightarrow \mathbb{R} : V^{\boldsymbol{\pi}}(0) = V_0, \ \text{ a.s. and} \\ & V^{\boldsymbol{\pi}}(T) \geq kY(T), \ \forall t \in (0,T] \ \text{ a.s.}\}\end{aligned}$$

We say that a portfolio process $\boldsymbol{\pi}$ is admissible if $\boldsymbol{\pi} \in \mathcal{A}$. As discussed in Eq. 2.5, the portfolio process $\boldsymbol{\pi}$ must satisfy

$$\mathbb{E}(H(T)V^{\boldsymbol{\pi}}(T)) \leq V_0.$$

Problem 1. To maximise the expected utility of the portfolio insurance strategy, the investor needs to find $\boldsymbol{\pi} \in \mathcal{A}$ such that

$$\mathbb{E}\left(U(V^{\star}(T))\right) = \sup_{\boldsymbol{\pi} \in \mathcal{A}} \mathbb{E}[U(V(T))]$$

and $V^{\star}(T) \in [kY(T), \infty)$, a.s.

Theorem 2.13 characterises the investor's optimally invested insured portfolio and Lemma 2.14 the asset allocation strategy which generates this wealth process.

Theorem 2.13. *Define*

$$V^{\star}(T) := p_0 Z^{\star}(T) + \max(kY(T) - p_0 Z^{\star}(T), 0) \tag{2.32}$$

in which

$$\mathrm{d}Z^{\star}(t) = Z^{\star}(t)((r + \frac{1}{1-\gamma}\|\boldsymbol{\theta}\|^2)\,\mathrm{d}t + \frac{1}{1-\gamma}\boldsymbol{\theta}'\mathbf{W}(t)), \ \ Z^{\star}(0) = 1,$$

with the participation factor $p_0 > 0$ *chosen to satisfy the budget constraint (Eq. 2.5). Then,* $\sup_{\boldsymbol{u}\in\mathcal{A}} \mathbb{E}[U(V(T))] \leq \mathbb{E}[U(V^\star(T))]$.

Proof. **Theorem 2.13**. The proof is adapted from Donnelly et al. (2015) and Donnelly et al. (2018).

Assume that there exists p_0 so that the budget constraint (2.5) is satisfied with equality by $V^\star(T)$.

For the investor's utility function, the first derivative $U'(x) = x^{\gamma-1}$, which is a strictly decreasing function, has a strictly decreasing inverse I with

$$I(x) := x^{\frac{1}{\gamma-1}}, \qquad x > 0.$$

We can show that for the constant

$$\lambda := {p_0}^{\gamma-1} e^{(\gamma r + \frac{1}{2}\frac{\gamma}{1-\gamma}\theta^2)T},$$

we have $p_0 Z^\star(T) = I(\lambda H(T))$.

We work with $I(\lambda(p_0)H(T))$ in the proof, rather than with $p_0 Z^\star(T)$ due to the properties of $I(x)$ and $U'(x)$: they are both strictly decreasing functions of x.

Let $V(T) \in [kY(T), \infty)$, a.s. be any attainable terminal wealth so that $\mathbb{E}[H(T)V(T)] \leq V_0$. We show that

$$\mathbb{E}(U(V(T))) \leq \mathbb{E}(U(V^\star(T))), \tag{2.33}$$

Then, by arbitrary choice of V, $\sup_{\boldsymbol{u}\in\mathcal{A}} \mathbb{E}(U(V(T))) \leq \mathbb{E}(U(V^\star(T)))$.

From Eq. 2.32 and using the fact that U' is a strictly decreasing function,

$$V^\star(T) = \begin{cases} I(\lambda H(T)) & \text{if } I(\lambda H(T)) \geq kY(T) \\ kY(T) & \text{if } I(\lambda H(T)) < kY(T) \end{cases}$$
$$= \begin{cases} I(\lambda H(T)) & \text{if } \lambda H(T) \leq U'(kY(T)) \\ kY(T) & \text{if } \lambda H(T) > U'(kY(T)) \end{cases}$$

As U is a concave function, then for any $a, b \in \mathbb{R}$, $U(a) - U(b) \leq U'(b) \cdot (a - b)$. In particular,

$$U(V(T)) - U(V^\star(T)) \leq U'(V^\star(T)) \cdot (V(T) - V^\star(T)), \quad \text{a.s.}$$

Take expectations in the above inequality to get

$$\begin{aligned}
&\mathbb{E}\left(U(V(T)) - U(V^\star(T))\right) \\
&\quad \leq \mathbb{E}\left(U'(V^\star(T)) \cdot (V(T) - V^\star(T))\right) \\
&\quad \leq \mathbb{E}\left(U'(V^\star(T)) \cdot (V(T) - V^\star(T)) | \lambda H(T) \leq U'(kY(T))\right) \cdot \mathbb{P}\left[\lambda H(T) \leq U'(kY(T))\right] \\
&\quad\ + \mathbb{E}\left(U'(V^\star(T)) \cdot (V(T) - V^\star(T)) | \lambda H(T) > U'(kY(T))\right) \cdot \mathbb{P}\left[\lambda H(T) > U'(kY(T))\right]
\end{aligned}$$

Observe that on the event $\lambda H(T) \leq U'(kY(T))$,

$$U'(V^\star(T)) = U'(I(\lambda H(T))) = \lambda H(T)$$

so that

$$\begin{aligned}
&\mathbb{E}\left(U'(V^\star(T)) \cdot (V(T) - V^\star(T)) | \lambda H(T) \leq U'(kY(T))\right) \\
&\quad = \mathbb{E}\left(\lambda H(T) \cdot (V(T) - V^\star(T)) | \lambda H(T) \leq U'(kY(T))\right).
\end{aligned}$$

Next, observe that on the event $\lambda H(T) > U'(kY(T))$, since $V(T) \in [kY(T), \infty)$ a.s., then

$$V(T) - V^\star(T) = V(T) - kY(T) \geq 0$$

and

$$U'(V^\star(T)) = U'(kY(T)) < \lambda H(T).$$

Upon multiplying both sides of the inequality $U'(V^\star(T)) < \lambda H(T)$ by the positive random variable $V(T) - V^\star(T)$ and taking expectation, we find that

$$\begin{aligned}&\mathbb{E}\left(U'(V^\star(T)) \cdot (V(T) - V^\star(T)) | \lambda H(T) > U'(kY(T))\right)\\ &\quad \leq \mathbb{E}\left(\lambda H(T) \cdot (V(T) - V^\star(T)) | \lambda H(T) > U'(kY(T))\right).\end{aligned}$$

In summary, we find that

$$\mathbb{E}\left(U'(V^\star(T)) \cdot (V(T) - V^\star(T))\right) \leq \mathbb{E}\left(\lambda H(T) \cdot (V(T) - V^\star(T))\right).$$

As both $V(T)$ and $V^\star(T)$ satisfy the budget constraint (2.5), the last line of the above inequality can be evaluated as

$$\mathbb{E}\left(\lambda H(T) \cdot (V(T) - V^\star(T))\right) \leq \lambda \cdot (V_0 - V_0) = 0,$$

which means $\mathbb{E}\left(U(V(T)) - U(V^\star(T))\right) \leq 0$, as required. □

Lemma 2.14. *Define the constant*

$$\boldsymbol{v} := \frac{1}{1-\gamma} (\boldsymbol{\sigma}')^{-1} \boldsymbol{\theta}. \tag{2.34}$$

The optimal investment strategy for Problem 1 is to invest in the risky assets the amount of

$$\boldsymbol{\pi}^\star(t) := p_0 Z^\star(t) \Phi(d_+(t)) \boldsymbol{v} + kY(t) \left(1 - \Phi(d_-(t))\right) \boldsymbol{u}_Y, \tag{2.35}$$

where p_0 is calculated as Eq. 2.12.

Proof. **Theorem 2.14**. First, we note that the process $Z^\star(t)$ is a wealth process with constant asset allocation $\boldsymbol{v}$ with an initial wealth of 1. Using Theorem 2.3 one can then show that the replicating strategy for payoff $V^\star(T)$ (Eq. 2.32) is given by Eq. 2.35.

□

2.4.1 Expected utility

Next, we turn our attention to computing the expected utility of the GOPIS. We start with calculating the conditional expected utility of GOPIS given a value of the benchmark portfolio. Theorem 2.15 sets out the expected utility of the special case of GOPIS where the benchmark portfolio is given by a constant value, as opposed to a random variable. Then, the unconditional expected utility is calculated using the law of iterated expectations.

Theorem 2.15. *Suppose at maturity the portfolio insurance strategy* $V(T)$ *has payoff*

$$V(T) = \max\left(pZ(T), ky\right),$$

where $\log(Z(T))$ *is normally distributed with mean* $(\mu - \frac{1}{2}\sigma^2)T$ *and variance* $\sigma^2 T$ *under* $\mathbb{P}$*; and* p*,* k*,* y *are constants. Then, the expected utility of the portfolio insurance strategy is*

$$\begin{aligned}\mathbb{E}[U(V(T))] =& \frac{1}{\gamma}(ky)^\gamma \Phi(-d_-(0, y, pe^{\mu T}, \sigma; k)) \\ &+ \frac{1}{\gamma} p^\gamma e^{\gamma(\mu - \frac{1}{2}(1-\gamma)\sigma^2)T} \Phi(d_-(0, y, pe^{\mu T}, \sigma; k) + \gamma\sigma\sqrt{T})\end{aligned}$$

where $d_-(t, y, x, \nu; k)$ *is given in Eq. 2.13.*

Proof. **Theorem 2.15**.

Since $\log(Z(T))$ is normally distributed with mean $(\mu - \frac{1}{2}\sigma^2)T$ and variance $\sigma^2 T$ under $\mathbb{P}$, it follows that $Z(T) = \exp((\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\mathcal{Z})$ where $\mathcal{Z}$ is a standard normal random variable $\mathcal{N}(0, 1)$. The event $pZ(T) < ky$ is equivalent to $\sigma\sqrt{T}\mathcal{Z} < -\log(pe^{\mu T}/ky) + \frac{1}{2}\sigma^2 T$. Then, we have

$$\begin{aligned}
&\mathbb{E}[U(V(T))] = \mathbb{E}[U(ky)\mathbb{1}(pZ(T) < ky)] + \mathbb{E}[U(pZ(T))\mathbb{1}(pZ(T) \geq ky)] \\
=&\mathbb{E}\left[\frac{1}{\gamma}(ky)^\gamma \mathbb{1}\left(\sigma\sqrt{T}\mathcal{Z} < -\log(pe^{\mu T}/ky) + \frac{1}{2}\sigma^2 T\right)\right] \\
&+ \mathbb{E}\left[\frac{1}{\gamma}p^\gamma e^{\gamma(\mu-\frac{1}{2}\sigma^2)T+\gamma\sigma\sqrt{T}\mathcal{Z}}\mathbb{1}\left(\sigma\sqrt{T}\mathcal{Z} \geq -\log(pe^{\mu T}/ky) + \frac{1}{2}\sigma^2 T\right)\right] \\
=&\frac{1}{\gamma}(ky)^\gamma \mathbb{E}\left[\mathbb{1}\left(\sigma\sqrt{T}\mathcal{Z} < -\log(pe^{\mu T}/ky) + \frac{1}{2}\sigma^2 T\right)\right] \\
&+ \frac{1}{\gamma}p^\gamma e^{\gamma(\mu-\frac{1}{2}\sigma^2)T}\mathbb{E}\left[e^{\frac{1}{2}\gamma^2\sigma^2 T}\mathbb{1}\left(\sigma\sqrt{T}(\mathcal{Z}+\gamma\sigma\sqrt{T}) \geq -\log(pe^{\mu T}/ky) + \frac{1}{2}\sigma^2 T\right)\right] \qquad (2.36) \\
=&\frac{1}{\gamma}(ky)^\gamma \mathbb{P}(\mathcal{Z} < -d_-(0, y, pe^{\mu T}, \sigma; k)) \\
&+ \frac{1}{\gamma}p^\gamma e^{\gamma(\mu-\frac{1}{2}(1-\gamma)\sigma^2)T}\mathbb{P}(\mathcal{Z} \geq -d_-(0, y, pe^{\mu T}, \sigma; k) - \gamma\sigma\sqrt{T})) \\
=&\frac{1}{\gamma}(ky)^\gamma \Phi(-d_-(0, y, pe^{\mu T}, \sigma; k)) \\
&+ \frac{1}{\gamma}p^\gamma e^{\gamma(\mu-\frac{1}{2}(1-\gamma)\sigma^2)T}\Phi(d_-(0, y, pe^{\mu T}, \sigma; k) + \gamma\sigma\sqrt{T})
\end{aligned}$$

□

Remark 2.16. For traditional OBPI strategy, we have $V(T) = \max(pS_2(T), ke^{rT})$. Setting $y = e^{rT}$, $\mu = \mu_2$ and $\sigma = \sigma_2$ in Theorem 2.15, we obtain the expected utility of the OBPI strategy as

$$\begin{aligned}
\mathbb{E}[U(V(T))] =&\frac{1}{\gamma}k^\gamma e^{\gamma r T}\Phi(-d_-(0, e^{rT}, pe^{\mu_2 T}, \sigma_2; k)) + \\
&\frac{1}{\gamma}p^\gamma e^{\gamma(\mu_2-\frac{1}{2}(1-\gamma)\sigma_2^2)T}\Phi(d_-(0, e^{rT}, pe^{\mu_2 T}, \sigma_2; k) + \gamma\sigma_2\sqrt{T}).
\end{aligned}$$

Remark 2.17. Let $Y(T) =: y$ be given. From Remark 2.6, we gather that $\log(Z(T))$ follows a normal distribution with mean $(a(y) - \frac{1}{2}(b(y))^2)T$ and variance $(b(y))^2 T$. For brevity, we denote g as the conditional expected utility of the portfolio insurance strategy:

$$g(y) := \mathbb{E}[U(V(T))|Y(T) = y].$$

Applying Theorem 2.15, we obtain

$$\begin{aligned} g(y) =& \frac{1}{\gamma}(ky)^{\gamma}\Phi(-d_{-}(0, y, pe^{a(y)T}, b(y); k))+ \\ & \frac{1}{\gamma}p^{\gamma}e^{\gamma(a(y)-\frac{1}{2}(1-\gamma)(b(y))^2)T}\Phi(d_{-}(0, y, pe^{a(y)T}, b(y); k) + \gamma(b(y))\sqrt{T}). \end{aligned} \tag{2.37}$$

Remark 2.18. Since $\log(Y(T))$ follows a normal distribution with mean $(\alpha - \frac{1}{2}\|\boldsymbol{\beta}\|^2)T$ and variance $\|\boldsymbol{\beta}\|^2 T$, y is a log-normal distribution. The unconditional expected utility of a GOPIS is then given by

$$\mathbb{E}[U(V(T))] = \int_{u=0}^{\infty} g(u)\phi\left(\frac{\log(u) - (\alpha - \frac{1}{2}\|\boldsymbol{\beta}\|^2)T}{\|\boldsymbol{\beta}\|\sqrt{T}}\right)\frac{1}{u\|\boldsymbol{\beta}\|}\,\mathrm{d}u. \tag{2.38}$$

where ϕ denotes the probability density function of the standard normal distribution. The integral can be evaluated numerically with quadrature integration techniques.[6]

2.5 Numerical Illustration

In this section, we turn to numerical examples to compare different configurations of GOPIS. We begin by comparing the mean-variance dominance and stochastic dominance (SD) of the GOPIS with the minimum variance portfolio as its benchmark over the GOPIS with a risk-free benchmark (i.e., a traditional OBPI) and the GOPIS with an equally-weighted portfolio as its benchmark. Next, we examine the utility loss associated with the minimum performance guarantee.

For the numerical analysis, we assume a market consists of $d = 2$ risky assets with $r = 0.026$, $\mu_1 = 0.050$, $\mu_2 = 0.068$, $\sigma_{11} = 0.078$, $\sigma_{21} = 0.020$, $\sigma_{22} = 0.142$.[7] These parameters were estimated from historical monthly real returns of the UK financial market data. We proxy the nominal bond S_1 through the Thomson Reuters UK 10 Years Government Benchmark Index and for equities S_2 we use the FTSE 100 Total Return Index, over the time period from January 1980

[6] We first use a change of variable to transform the log-normal distribution into a standard normal distribution and then apply a single-dimension Gauss-Hermite quadrature integration (see, e.g. Khemka and Butt (2017) and Donnelly et al. (2022) for applications of quadrature integration).

[7] This implies that S_2 has volatility $\sigma_2 = 0.143$ and correlation $\rho = 0.138$ with S_1.

to December 2019. For the real risk-free asset, we used the real rate of returns of the inflation-linked bonds FTSE British Government Index Linked (All Maturities) Index from January 1994 (earliest available) to December 2019. We fix the investment horizon to $T = 5$ (years). The risk aversion parameter of the investor is assumed to be $\gamma = -1.5$.[8]

To demonstrate the flexibility of GOPIS, we consider the following benchmark portfolios: the risk-free asset S_0; the naive equally-weighted portfolio; and the minimum-variance portfolio. We pair these benchmark strategies with a venture portfolio fully invested in equity S_2. The GOPIS with the risk-free asset as its benchmark is a traditional OBPI strategy. We denote the GOPIS with an equally-weighted benchmark portfolio as GOPIS-EW. The naive equally-weighted portfolio assigns $1/N$ share to each of the N available assets. Due to its simple construction that does not require estimations of financial parameters or risk preferences, the equally-weighted portfolio is commonly used by pension funds in practice (Benartzi and Thaler, 2001) and sometimes serves as the benchmark portfolio for comparison in literature (Duchin and Levy, 2009; Fletcher, 2009; Jiang et al., 2019). The GOPIS with the minimum variance portfolio as its benchmark is given by GOPIS-MV.[9] Additionally, we introduce GOPIS-MertonMV as the GOPIS that uses the Merton portfolio as its venture and uses the minimum variance portfolio as its benchmark. GOPIS-MertonMV is the optimal GOPIS for a risk averse investor whose risk aversion parameter is known. We fix the investment horizon to $T = 5$ (years). Table 2.1 shows the GOPIS's configuration and the volatility of their embedded options.

Table 2.1: The asset allocations of the venture and the benchmark portfolio of GOPIS and ν, the volatility of the embedded option. $\boldsymbol{v}$ is the Merton portfolio, i.e. the optimal investment strategy that maximises the expected utility function (Eq. 2.34).

	$\boldsymbol{u}_Z$	$\boldsymbol{u}_Y$	ν
OBPI	$(0\ \ 1)'$	$(0\ \ 0)'$	0.143
GOPIS-EW	$(0\ \ 1)'$	$(0.33\ \ 0.33)'$	0.094
GOPIS-MV	$(0\ \ 1)'$	$(0.81\ \ 0.19)'$	0.124
GOPIS-MertonMV	$\boldsymbol{v}$	$(0.81\ \ 0.19)'$	0.093

[8]Khemka et al. (2021) estimated that life-cycle funds offered in the US, UK, Australia and Denmark have an underlying risk aversion ranging from -4 to -1.

[9]This is similar to the OBPP strategy introduced by Zagst et al. (2019) who parameterise their financial market such that the minimum-variance portfolio is one of their risky assets; and equity as the other risky asset. In this paper, we explicitly specify the minimum-variance benchmark portfolio as a combination of all risky assets using the generalised structure of GOPIS.

First, we compare whether one GOPIS can dominate another in mean-variance dominance. Following Zagst et al. (2019), we define mean-variance dominance favouring a random variable X against a random variable Y, $Y \prec_{MVE} X$, iff X has a higher expectation but lower variance than Y. Figure 2.3 shows the satisfaction of mean-variance dominance favouring GOPIS-MV against (a) OBPI and (b) GOPIS-EW as a function of guarantee levels $k \in [0.5, 0.995]$. To create this figure, the expected values and variances of the GOPISs at their maturity are calculated and compared for a range of guarantee values. The green shades identify the coordinates of k where GOPIS-MV dominate its competitors. We find that the GOPIS-MV dominates the two competitors. This is not surprising given that is not surprising as GOPIS-MV is a hybrid of S_2 and the minimum-variance portfolio. The OBPI strategies are dominated by GOPIS-MV for a greater number of k combinations as indicated by the area of the green shade compared to GOPIS-EW. The combinations of k that result in mean-variance dominance favouring GOPIS-MV loosely track the equal participation line $p_1 = p_2$. This signals that, for same participation factors p, the investor can achieve better mean-variance trade-offs than the traditional OBPI by choosing the equally-weighted portfolio or the minimum variance portfolio as the benchmark. We notice that none of the two strategies dominates the GOPIS-MV in the mean-variance sense.

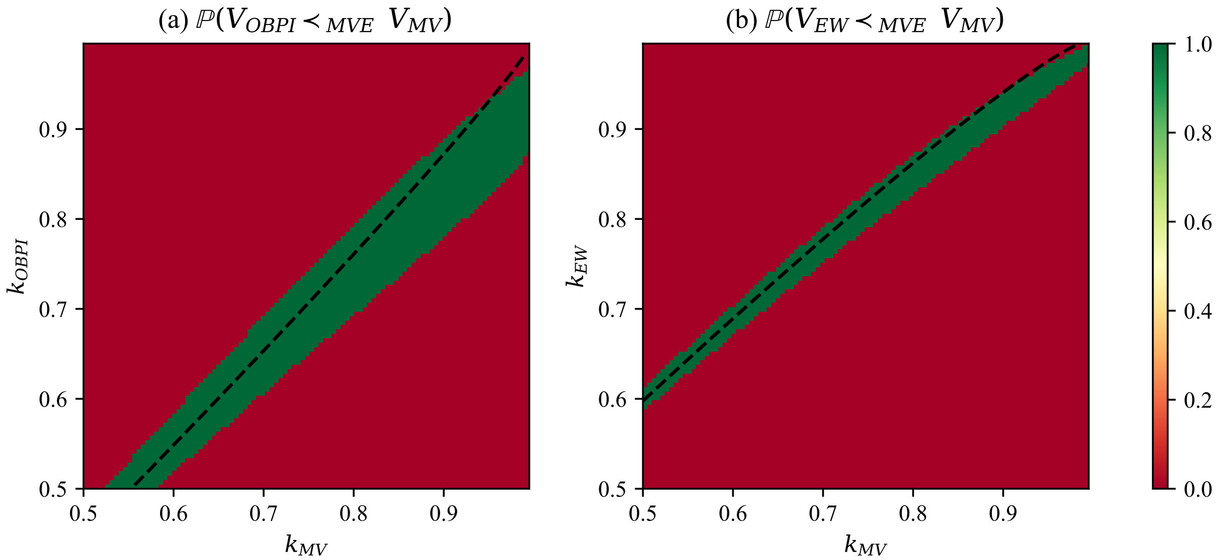

FIGURE 2.3: Satisfaction of the mean-variance dominance favouring GOPIS-MV as a function of the guarantee levels $k \in [0.5, 0.995]$ against: (a) OBPI and (b) GOPIS-EW. The black dashed line plots the k coordinates that result in equal participation $p_1 = p_2$ where $i = 1$ refers to (a) OBPI or (b) GOPIS-EW and $i = 2$ refers to GOPIS-MV.

Next, we examine whether GOPIS-MV dominates the other strategies in the first- and second-order SD. Since the strategies in comparison have an identical venture portfolio, conditioned on $\tilde{Y} = y$, we have $\sigma_1(y) = \sigma_2(y)$ for each pair where $i = 1$ refers to one of the two competitors and $i = 2$ refers to GOPIS-MV. In addition, the condition $\mu_1(y) \lesseqqgtr \mu_2(y)$ is simplified into $p_1 \lesseqqgtr p_2$ for each pair. Figure 2.4 visualises the probabilities of first-order SD favouring GOPIS-MV against (a) OBPI and (b) GOPIS-EW as a function of guarantee levels $k \in [0.5, 0.995]$. According to Theorem 2.10(i), the first-order SD favouring GOPIS-MV is achievable when $p_1 < p_2$. This corresponds to the region on the left of the line of equal participation $p_1 = p_2$. By comparing the sizes of regions to the left the line of equal participation and the shades of probability values in Figure 2.4, we conclude that GOPIS-MV is more likely to dominate OBPI than GOPIS-EW in the first-order sense.

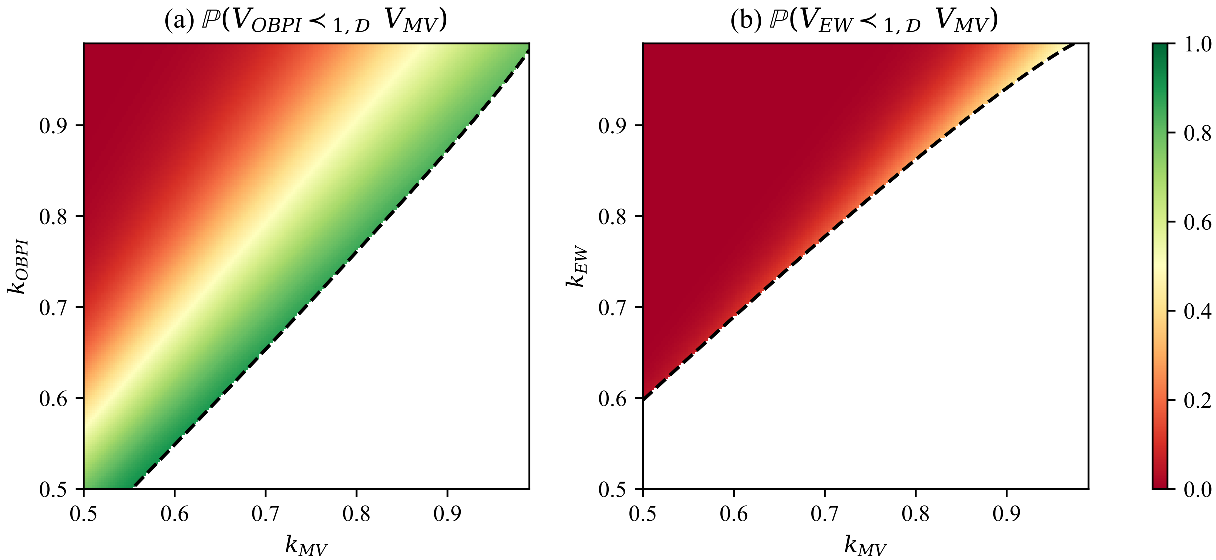

FIGURE 2.4: Probability of the first-order stochastic dominance favouring GOPIS-MV against as a function of the guarantee levels $k \in [0.5, 0.995]$ against: (a) OBPI and (b) GOPIS-EW. The black dashed line plots the k coordinates that result in equal participation $p_1 = p_2$ where $i = 1$ refers to (a) OBPI or (b) GOPIS-EW and $i = 2$ refers to GOPIS-MV.

The area to the right of the line presents the case when $p_1 > p_2$ in which a second-order SD favouring GOPIS-MV is feasible. The second-order SD requires the $y \in \mathcal{D}_* \subseteq \mathcal{D}$, i.e., y that simultaneously satisfies $\mathbb{E}[V_1(T)|\tilde{Y} = y] < \mathbb{E}[V_2(T)|\tilde{Y} = y]$ and $y \in \mathcal{D}$ (see Theorem 2.10(ii)). Figure 2.5 visualises the probabilities of second-order SD favouring GOPIS-MV against (a) OBPI and (b) GOPIS-EW as a function of guarantee levels $k \in [0.5, 0.995]$. GOPIS-MV has high probabilities (above 75%) of dominance over OBPI in the second-order sense for k coordinates right adjacent to the line of equal participation. In contrast, GOPIS-MV has relatively low probabilities of second-order dominance over GOPIS-EW over the entire region right to the line of equal participation (below 50% for vast majority). Based on Figures 2.4 and 2.5, GOPIS-MV is likely the GOPIS that yields higher expected utility than OBPI for non-satiated investors (first-order SD) and for risk averse investors (second-order SD), especially when the participation factors are similar for both strategies.

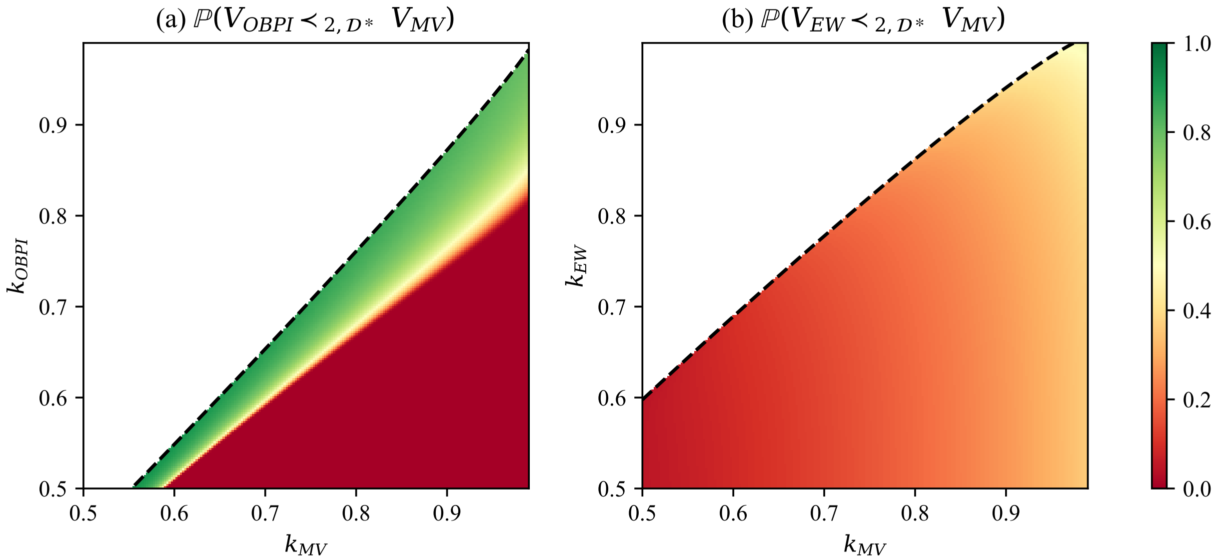

Figure 2.5: Probability of the second-order stochastic dominance favouring GOPIS-MV against as a function of the guarantee levels $k \in [0.5, 0.995]$ against: (a) OBPI and (b) GOPIS-EW. The black dashed line plots the k coordinates that result in equal participation $p_1 = p_2$ where $i = 1$ refers to (a) OBPI or (b) GOPIS-EW and $i = 2$ refers to GOPIS-MV.

As demonstrated in Section 2.4, for a CRRA investor whose risk aversion parameter is known, the optimal GOPIS is achieved by setting $\boldsymbol{u}_Z = \boldsymbol{v}$, i.e. the Merton portfolio. We now consider GOPIS-MertonMV as a GOPIS that uses the Merton portfolio as its venture portfolio and the minimum variance portfolio as its benchmark. Figure 2.6 shows the satisfaction of mean-variance dominance favouring GOPIS-MertonMV against (a) OBPI, and (b) GOPIS-EW as a function of guarantee levels $k \in [0.5, 0.995]$. The mean-variance dominance that favours GOPIS-MertonMV is limited to the region where $k_{MertonMV}$ is relatively high ($k = 0.95$), a region in which the benchmark portfolio is exerting a strong influence on GOPIS.

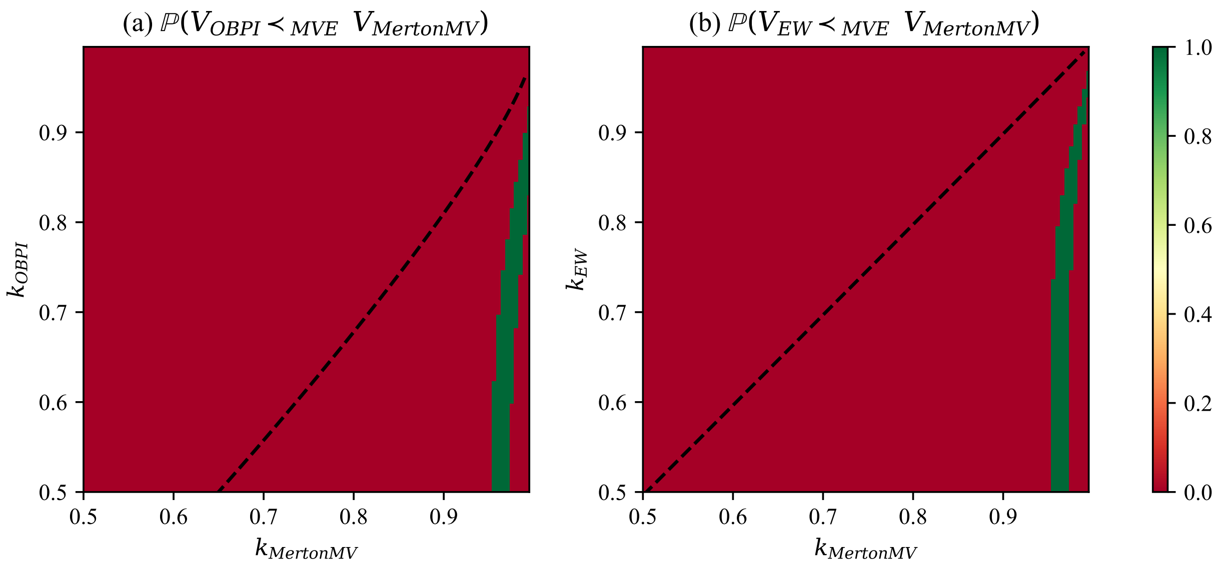

FIGURE 2.6: Satisfaction of the mean-variance dominance favouring GOPIS-MertonMV against as a function of the guarantee levels $k \in [0.5, 0.995]$ against: (a) OBPI and (b) GOPIS-EW. The black dashed line plots the k coordinates that result in equal participation $p_1 = p_2$ where $i = 1$ refers to (a) OBPI and (b) GOPIS-EW and $i = 2$ refers to GOPIS-MertonMV.

Conditioned on $\tilde{Y} = y$, the conditional volatility of $\log(pZ(T))$ for GOPIS-MertonMV is lower than that of OBPI, which corresponds to the situation discussed in Theorem 2.11. Unlike the case with equal venture portfolios, only second-order SD is attainable. The probability of second-order SD is evaluated numerically and is presented in Figure 2.7a. The GOPIS-MertonMV has high probabilities (about 80%) of stochastically dominating OBPI in the second-order sense in the diagonal region where k levels are similar. This indicates that, in addition to achieving the highest expected utility for the CRRA investors, GOPIS-MertonMV has great probabilities of achieving higher expected utility for all risk averse investors.

We refer to Theorem 2.12 in order to compare GOPIS-MertonMV and GOPIS-EW because the conditional volatility of the venture portfolio in GOPIS-MertonMV is higher than that in GOPIS-EW. The probability of first-order SD favouring GOPIS-MertonMV against GOPIS-EW is evaluated numerically and is presented in Figure 2.7b. The GOPIS-MertonMV strategy has moderate probabilities (about 50%) to achieve first-order SD over GOPIS-EW it has a higher k level. This signals that compared to GOPIS-EW, GOPIS-MertonMV is likely to produce higher expected utility for all non-satiated investors (which include risk averse investors) when $k_{MertonMV} > k_{EW}$.

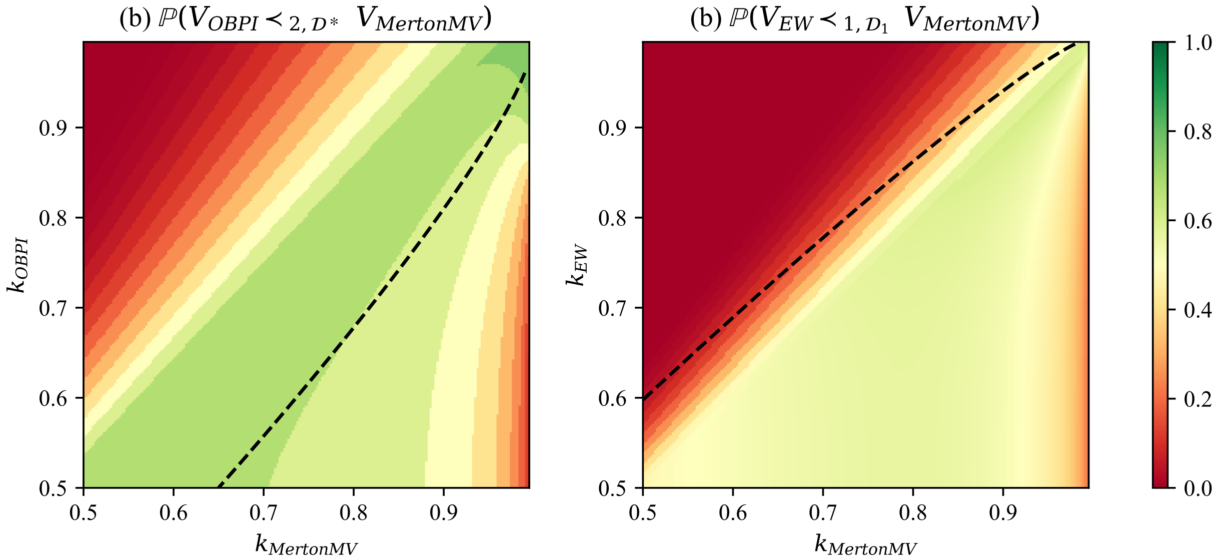

FIGURE 2.7: Probability of the stochastic dominance favouring GOPIS-MertonMV against as a function of the guarantee levels $k \in [0.5, 0.995]$ against: (a) OBPI and (b) GOPIS-EW. The black dashed line plots the k coordinates that result in equal participation $p_1 = p_2$ where $i = 1$ refers to (a) OBPI or (b) GOPIS-EW and $i = 2$ refers to GOPIS-MertonMV.

Introducing a minimum guarantee constraint inevitably results in an optimal strategy that is different to its unconstrained counterpart. This can be interpreted as a compromise between preferences of expected utility maximisation and having a minimum guarantee. From the perspective of expected utility maximisation, GOPIS induces a loss in utility. We measure the utility loss using wealth equivalent (see Jensen and Sørensen, 2001). This is defined as the initial wealth level that is required for an investor to achieve the same level of expected utility following the optimal Merton portfolio, which maximises the investor's expected utility without the minimum guarantee constraint under a CRRA utility function. The optimal level of expected utility for an unconstrained Merton portfolio with an initial wealth of x_0 over a horizon of T for $\gamma \neq 0$ is given by

$$J(x_0; T, \gamma) = \frac{1}{\gamma}(x_0)^\gamma e^{\gamma(r + \frac{1}{2(1-\gamma)}\|\boldsymbol{\theta}\|^2)T}.$$

The wealth equivalent is determined as the $\tilde{x}_0$ that solves the following equation:

$$J(\tilde{x}_0; T, \gamma) = \mathbb{E}[U(V(T))]. \tag{2.39}$$

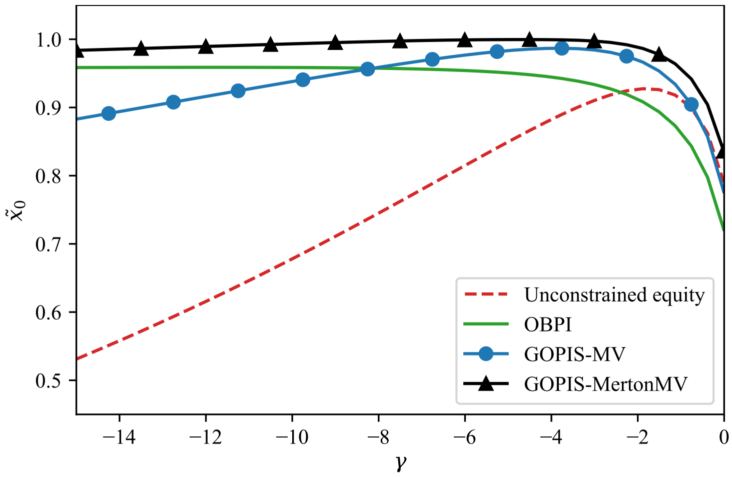

FIGURE 2.8: Wealth equivalents of unconstrained equity, OBPI, GOPIS-MV and GOPIS-MertonMV as a function of relative risk aversion γ of the investors. We assume that $k = 0.95$ for all GOPISs. The GOPIS-MertonMV has a Merton portfolio matching the γ of the investors as its venture portfolio.

Figure 2.8 presents the wealth equivalents of different GOPISs as a function of relative risk aversion γ of the investors. When investing solely in equity (S_2) without a minimum guarantee, very risk averse investors attain low wealth equivalents at low values of γ, indicating the equity investment is overly risky. The wealth equivalent of equity investment increases as the risk aversion of the investor decreases until $\gamma = -1.8$, after which it decreases – indicating that investing solely in S_2 is not enough risky for investors with low risk aversion.

Adding a minimum benchmark to the equity investment lifts the wealth equivalents for investors who are more risk aversion. In the case of a risk-free benchmark, OBPI produces higher wealth equivalents than the unconstrained equity strategy for $\gamma < -2.4$. In the case of a minimum variance benchmark, GOPIS-MV produces higher wealth equivalents than the unconstrained equity strategy when $\gamma < -0.6$. When comparing GOPIS with different benchmarks, OBPI has higher wealth equivalents than GOPIS-MV when $\gamma < -8.1$. This is because as $\gamma \to -\infty$, the optimal Merton strategy converges to the risk-free investment. Therefore, GOPIS with the risk-free benchmark (i.e., OBPI) delivers higher wealth equivalents than GOPIS-MV for low values of γ. It is only the very risk averse investors would prefer using a risk-free benchmark to a stochastic benchmark such as the minimum variance portfolio.

On the other hand, setting the optimal Merton strategy as the venture portfolio results in a

GOPIS that has the highest wealth equivalents and suffers the least in utility loss (see e.g. GOPIS-MertonMV). Since its venture is the optimal Merton strategy, the utility loss is fully attributed to the use of a benchmark guarantee. For high levels of risk aversion, the stochastic benchmark introduces "excess risk" and the overall strategy is penalised with utility loss. For low levels of risk aversion, it rendered the overall strategy "too conservative" and is penalised with utility loss.

2.6 Conclusion

In this paper we have introduced the generalised option-based portfolio insurance strategy (GOPIS) as an extension to the widely studied traditional option-based portfolio insurance strategy and the option-based performance participation strategy by Zagst et al. (2019). GOPIS is a versatile investment strategy with which the investor can explicitly specify a venture portfolio seeking investment gains whilst insuring against a fraction level of a benchmark portfolio. First, we provide the replicating strategy for GOPIS and derive the general analytic expression for the moments and conditional moments of GOPIS in a financial market with finite $d > 1$ risky assets. Next, we extended the analysis of conditional stochastic dominance in Zagst et al. (2019) to enable the comparison of GOPISs with different venture portfolios. The analysis of stochastic dominance is useful in comparing and choosing a GOPIS when only partial information on the investor's preference is available (e.g., risk aversion with an unknown parameter). In our numerical analysis, we demonstrate that GOPIS with a minimum variance portfolio as its benchmark dominates other GOPISs in the mean-variance sense, as well as in the first- and second-order stochastic dominance sense.

For an investor with known utility preference (constant relative risk aversion), we show that their optimal GOPIS is one with the Merton portfolio, the optimal portfolio obtained from an unconstrained expected utility maximisation framework, as its venture portfolio, regardless of its benchmark portfolio. Hence, by adopting a GOPIS structure, the investor can simultaneously optimise their risk aversion utility whilst insuring against a benchmark portfolio.

Our analysis leaves several aspects for future research. For instance, we can extend the underlying Markov process from a Black-Scholes setting to allow for stochastic interest rate or regime switching that better reflects the uncertainty faced by investors in real-life. It would also be interesting to extend the utility function beyond power utility functions, such as the cumulative prospect utility function (Tversky and Kahneman, 1992).

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Chapter 3

Aligning retirement saving goals with inflation

William Lim: 80%, Catherine Donnelly: 10%, Gaurav Khemka: 10%

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Abstract

Pension investors are exposed to the risk of inflation eroding the purchasing power of their savings. Their risk is particularly high due to saving for retirement taking place over many decades. However, few pre-retirement investment strategies incorporate explicit inflation-proofing. It is shown that ignoring inflation is costly in terms of a retiree's welfare, with reductions of up to 25% possible for the average retiree. More risk averse investors face even larger reductions.

In a second study, constraints on the amount of pension savings at retirement are imposed. Constraints are used to give the retiree more certainty about the level of pension savings at the date of their retirement. Such constraints may be expressed in either real or nominal terms. However, ignoring inflation by using nominal constraints gives a potential reduction in welfare of

up to 36% for the average retiree. The results illustrate that nominal constraints are ineffective at reducing the risk of inflation.

The conclusion is that pre-retirees ignore inflation at their peril. It must be included explicitly in retirement savings targets to improve retirement outcomes. Consequently, there should be greater investment in an index-linked bond or a similar asset.

3.1 Introduction

Inflation eating away at the value of pension savings is a concern for many pre-retirees. A 2021 survey of American pre-retirees found that 66% of them were worried that the value of their savings and investments might not keep up with inflation (Greenwald Research, 2022, Table 4). A recent survey of savers from Australia, US, UK and Ireland in 2022 found that many people say they wish for a consistent stream of income in retirement (State Street Global Advisors, 2022).

Yet reality shows us that these concerns, worries and desires are ignored in practice. Retirement goals in defined contribution pension plans are usually expressed in nominal terms as maximizing the absolute size of wealth at retirement. Over 90% of UK annuitants choose to buy a level life annuity despite expressing a preference for an inflation-indexed annuity before purchase (Finkelstein and Poterba, 2004). People do not naturally think in inflation terms - future inflation is, after all, an abstract, nebulous idea. It is not natural, when asked about your retirement savings goals, to allow for inflation. Instead, we think in today's money terms - not yesterday's or tomorrow's economic world. It is no surprise that a recent survey of financial professionals indicates that financial professionals around the world regard underestimating the impact of inflation as the number one mistake investors make in their retirement planning (Natixis Investment Managers, 2022).

If left unchecked, inflation can erode the purchasing power of retirement savings over a long period of time. It is important to examine how the design of investment strategy can address this vulnerability. In light of this, we look at a class of problems which revolves around maximizing the expected power utility of wealth at retirement. Specifically, we consider two types of investors: (i) an investor who maximizes their expected utility of *real* (inflation-adjusted) wealth without terminal wealth constraints; (ii) an investor who maximizes their expected utility of real wealth subject to *real* terminal wealth constraints. The solution in each case is an optimal, dynamic investment strategy which maximizes the problem at hand. Then, we compare the investment strategies and retirement outcomes of each of these two types of investors with those who "mistakenly" target nominal wealth.

Classical optimal control problems in the pension savings area first excluded inflation (Merton, 1969, 1971a). The inclusion of inflation into these classical problems has been examined by many authors. For example, Menoncin (2002) derived the solution for an investor maximizing the expected exponential utility under inflation risk in a complete market. Brennan and Xia (2002), Battocchio and Menoncin (2004) and Zhang and Guo (2018) studied the investor's problem in an incomplete market without an inflation-linked bond. The former two showed that the inflation risk can instead be partially hedged via a cash-dominated portfolio with a high correlation with inflation and the cost imposed by unhedgeable inflation risk is surprisingly low. Our setup involves a tradable inflation-linked bond which completes the financial market in the presence of inflation risks (see also Zhang and Ewald, 2010; Han and Hung, 2012; Chen et al., 2023). This allows us to focus on the level of suboptimality of not considering the inflation risk in the investment strategy. Contrary to the findings of Zhang (2012), we show that adopting an inflation-adjusted risk perception yields a different optimal strategy to the Merton strategy by having an additional inflation risk hedging component. We find that failing to account for inflation risk reduces the welfare of the investor by 4%, 15%, 25% for low, medium, and high risk averse investors, respectively. According to our baseline parameterization, the investor gets a higher value of terminal wealth by incorporating inflation risk in their investment strategy in 27% of the times, compared to ignoring inflation risk.

Next, we consider the investment problem for an investor who has terminal wealth constraints. Being able to guarantee the investor a minimum level of wealth, the lower terminal wealth constraint is a practical and attractive tool in planning for retirement (Alles, 2011). Various forms of the lower wealth constraint are studied, e.g. a static minimum constraint (Korn, 2005; Kraft and Steffensen, 2013) and a minimum performance relative to a benchmark strategy (Teplá, 2001; Boulier et al., 2001; Han and Hung, 2012; Chen et al., 2017). On the other hand, an upper constraint on terminal wealth constraint gives up the possibility of higher returns in exchange for a reduction in the risk of poor retirement outcomes (Donnelly et al., 2015). Our problem is closest to Donnelly et al. (2018) where we extended the problem with both a lower and an upper constraint under inflation risk.

We find that imposing terminal wealth constraints produces a dynamic investment strategy that changes as the prices of the risk assets fluctuate. In particular, when the terminal wealth constraints are likely to be binding the strategy prescribes a high fraction of wealth invested in ILB. The investor suffers a loss in welfare ranging from 27% to 36% (subject to risk aversion coefficient) when they "mistakenly" impose nominal wealth constraints instead of real wealth constraints. More importantly, nominal wealth constraints do not hedge against inflation-linked wealth targets – the investor is still susceptible to having low inflation-adjusted terminal wealth.

When a problem is expressed in nominal terms, the nominal bond is the risk-free asset. However, when it is expressed in real terms, the inflation-linked bond becomes the risk-free asset. While this result is well known (as a consequence of a change-of-numeraire), the impact of expressing these problems in both perceptives have not been explored to the best of our knowledge. We contribute to the literature by first solving investment problems expressed in in nominal vs real for two types of risk averse investors (with and without terminal wealth constraints). Then, we analyze the impact of failing to align the investment problem with inflation risk. For investors who evaluate their retirement goals in real terms, we find that the investment strategy arising from a nominal risk perceptive results in a non-trivial loss of welfare. It does not provide an inflation-linked guarantee, for the constrained case.

The paper is organized as follows. Section 3.2 presents the market model and the investor's preferences. Section 3.3 provides the solution to the unconstrained case which serves as the foundation for the solution to the problem with terminal wealth constraints in Section 3.4. Section 3.5 provides numerical illustrations and discussions on the optimal strategies, the terminal wealth distributions, and the loss in utility when inflation risk are not accounted for. Section 3.6 concludes the paper.

3.2 The market model

3.2.1 Description of the markets

In the classical life-cycle literature, the investor seeks to maximize the expected utility of their nominal wealth at retirement. In contrast, in our model the investor seeks to maximize real wealth, and not nominal wealth. To do this, while keeping a nominal asset in the model, requires a model of inflation. We model inflation through a price index. The value of the price index is $I(t)$ at time $t \geq 0$, with dynamics

$$dI(t) = \mu_I I(t)dt + \sigma_I I(t)dW_1(t), \quad I(0) = I_0 > 0, \text{ constant a.s.,} \tag{3.1}$$

in which μ_I is the expected increase in the index, $\sigma_I > 0$ is the constant volatility of the index and W_1 is a standard Brownian motion. Changes in the value of the price index represent inflation in our model. Note that the index is not an asset and is not traded in the market.

Following Karatzas and Shreve (1998), we assume that the investor trades $D + 1$ assets in a continuous-time market without transaction costs over a finite time horizon $[0, T]$ for an integer $T > 0$. We refer to T as the terminal time or retirement time. The zeroth asset is a nominal bond which increases at a constant rate, with dynamics

$$dS_0(t) = r_{\mathrm{N}} S_0(t)dt, \quad S_0(0) = 1 \text{ a.s.,}$$

for a constant real number r_{N} that represents the nominal annual risk-free interest rate. The prices of the remaining traded risky assets S_n for $n = 1, \ldots, D$ follow the dynamics

$$dS_n(t) = \mu_n S_n(t)dt + \sum_{j=1}^{n} \sigma_{nj} S_n(t) dW_j(t), \quad S_n(0) = S_{n,0} > 0, \tag{3.2}$$

with a constant mean rate of return μ_n and volatility coefficients $\sigma_{nj} > 0$ with W_j for $j = 1, \ldots, n$.

When the investor seeks to maximize nominal wealth at retirement, the nominal bond is the risk-free asset. However, the risk perception of the nominal bond changes when the investor seeks to maximize the expected utility of their real wealth. The nominal bond becomes a risky asset, as it provides no protection against unexpected inflation. To enable the investor to hedge against the inflation risk, an index-linked bond (ILB), S_1, is introduced into the financial market (see e.g. Zhang and Ewald, 2010; Han and Hung, 2012; Donnelly et al., 2022; Chen et al., 2023). For simplicity, we assume that the real return on the ILB is constant, at the annual rate r_{R}, which yields a nominal rate of return of $\mu_1 = \mu_I + r_{\mathrm{R}}$. The dynamics of the price of the ILB are

$$\frac{dS_1(t)}{S_1(t)} = (r_{\mathrm{R}} + \mu_I)dt + \sigma_I dW_1(t), \quad S_1(0) = S_{1,0} > 0, \text{ constant a.s.} \tag{3.3}$$

It is seen by comparing the dynamics in (3.1) and (3.3), that the price of the ILB is perfectly correlated with the value of the price index since they are both driven by the same Brownian motion.

Note that the nominal return of a ILB is not known until the bond is redeemed at maturity or sold. This is because the value of the price index is a random process. However, the real return is known in our model[1]. Our approach here is to fix the real return, and from the real return and the price index model above we can infer the price of the ILB at any time.

For the technical conditions, the vector process $\mathbf{W} := (W_1, \ldots, W_D)'$ is a D-dimensional Brownian motion under a real-world probability measure $\mathbb{P}$. The information $\mathcal{F}_t$ available to market

[1] In practice, the real return is known only approximately, since the indexation of the payments with the price index is not perfect: for practical reasons, the values of the price index used for indexation of real life inflation-linked bonds are generally lagged by at least three months.

investors at each time $t \in [0, T]$ is the natural filtration generated by $\mathbf{W}$, augmented by its $\mathbb{P}$-null sets. For a process η, we write $\eta \in \mathcal{F}^{\star}$ to indicate that it is $\mathcal{F}_t$-progressively measurable. The expectation operator under $\mathbb{P}$ is denoted $\mathbb{E}$.

3.2.2 Investor

We consider two types of pre-retiree investors. The first investor aims to maximize their expected utility of real amount of money at retirement. The second investor, in addition to expected utility maximization, targets their real amount of money at retirement to lie between a pre-specified bound. The idea is that they both can use the accumulated lump sum to buy an inflation-proofed life annuity at retirement to protect their purchasing power post-retirement. We choose this interpretation as it provides financial security for the pre-retiree – protecting them against longevity and investment risk – and allows them to plan for their retirement. For the second investor, we follow broadly the idea of Merton (2014) such that, in targeting a specific amount of money at retirement, the investor chooses a sum of money they would like to have at their retirement date (the targeted upper amount) and a lower value that represents the minimum sum they would be happy with at retirement. At their retirement date, they will have a sum of money that lies between these two investor-chosen extremes, and has a non-zero probability of being at either extreme.

The decisions made by the pre-retiree investor today (i.e. at time 0) with a fixed non-random initial wealth $x_0 > 0$ result in:

- an integer $T > 0$ corresponding to their planned retirement date.
- (if applicable) an upper limit of retirement amount $\$K_U$, that is expressed in today's purchasing power, representing the ideal amount of money the investor would like to have at retirement. The upper limit could be calculated as the amount of money needed to secure an inflation-indexed income paid for life to the investor. With this approach, we avoid the investor having to guess what the inflation index value will be at their retirement date.

- (if applicable) a minimum retirement amount $\$K_L$ that is expressed in today's purchasing power, representing the minimum amount of money the investor would like to have at retirement. Again, this amount can be calculated as the amount of money required to secure the minimum inflation-indexed income required by the investor for their retirement.

We ignore future savings made by the investor. Our results can be adjusted to allow for them, and they are not the focus of the paper.

To set the investment strategy, the investor seeks to maximize the expected value of their real wealth at retirement subject to their real wealth lying between the maximum $\$K_U$ and the minimum $\$K_L$ (when applicable). The optimal investment strategy is calculated in this paper.

The value of the investor's fund at time t is represented by $X^{\boldsymbol{\pi}}(t)$. The superscript $\boldsymbol{\pi}$ refers to an investment strategy which is an investment strategy is a vector process $\boldsymbol{\pi} = (\pi_1 \ldots, \pi_D)' \in \mathcal{F}^\star$ in which:

- an dollar amount $\pi_1(t)$ is invested in the ILB at time t, and
- an dollar amount $\pi_n(t)$ is invested in the nth risky stock at time t.

The remaining amount of the investor's fund

$$\pi_0(t) := X^{\boldsymbol{\pi}}(t) - \sum_{n=1}^{D} \pi_n(t) = X^{\boldsymbol{\pi}}(t) - \boldsymbol{\pi}'\mathbf{1}_D$$

is invested in the nominal bond where $\mathbf{1}_D$ represent a D-column vector of ones.

Define

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_D \end{pmatrix}, \qquad \boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & 0 & \cdots & 0 \\ \sigma_{21} & \sigma_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \sigma_{D1} & \sigma_{D2} & \cdots & \sigma_D \end{pmatrix},$$

in which it is assumed that $\boldsymbol{\sigma}$ is non-singular. Within the considered market, we have the market price of risk $\boldsymbol{\theta} := \boldsymbol{\sigma}^{-1}(\boldsymbol{\mu} - r_{\mathrm{N}}\mathbf{1}_D)$.

The dynamics of the investor's fund value are

$$dX^{\boldsymbol{\pi}}(t) = \left(r_{\mathrm{N}}X^{\boldsymbol{\pi}}(t) + \boldsymbol{\pi}'(t)\boldsymbol{\sigma\theta}\right)dt + \boldsymbol{\pi}'(t)\boldsymbol{\sigma}d\mathbf{W}(t), \tag{3.4}$$

with the initial condition that $X^{\boldsymbol{\pi}}(0) := x_0 > 0$, a.s.

The set of admissible portfolios for the investor's initial wealth x_0 is defined to be

$$\mathcal{A}(x_0) := \left\{ \boldsymbol{\pi} : \Omega \times [0,T] \to \mathbb{R}^D \,\middle|\, \boldsymbol{\pi} \in \mathcal{F}^\star, X^{\boldsymbol{\pi}}(0) = x_0, \text{ a.s. and } X^{\boldsymbol{\pi}}(t) \geq 0, \forall t \in (0,T] \text{ a.s.} \right\}.$$

Define the stochastic exponential

$$\mathcal{E}(-\boldsymbol{\theta} \bullet \mathbf{W})(t) := \exp\left(-\frac{1}{2}\|\boldsymbol{\theta}\|^2 t - \boldsymbol{\theta}'\mathbf{W}(t)\right), \quad \text{for each } t \in [0,T].$$

A portfolio $\boldsymbol{\pi}$ must satisfy the *budget constraint* that the initial wealth is sufficient to attain the retirement wealth that results from following portfolio $\boldsymbol{\pi}$. This means any terminal wealth has to statisfy the budget constraint in order to be admissible. Mathematically, we write the budget constraint as

$$\mathbb{E}\left(\mathcal{E}(-\boldsymbol{\theta} \bullet \mathbf{W})(T)S_0^{-1}(T)X^{\boldsymbol{\pi}}(T)\right) \leq x_0. \tag{3.5}$$

For mathematical simplicity, the utility function of the investor is assumed to be the power utility function

$$U(x) := \begin{cases} \frac{1}{\gamma}x^\gamma, & \gamma \in (-\infty, 1) \setminus \{0\}, \\ \log(x), & \gamma = 0, \end{cases}$$

for a constant γ and $x > 0$.

This choice of utility function allows us to control the investment risk taken by the investor through the parameter γ. We do not recommend that the investor is asked to choose the value of γ; it would be a confusing thing to ask for those unfamiliar with utility functions. Rather, it should be chosen by the manager who implements the investment strategy on behalf of the pre-retiree (see Khemka et al., 2021, for discussion on the risk aversion parameter). The investor seeks to maximize the expected utility of their real retirement wealth subject to the real wealth lying between the maximum $\$K_U$ and minimum $\$K_L$ at the retirement time T.

3.3 Unconstrained problem

We begin by solving the problem for the first investor, i.e. maximizing the expected utility of real retirement wealth without the terminal wealth constraints. The unconstrained solution provides the foundation for the solution to the constrained problem for the second investor. For comparison, we state the results of the classical life-cycle model that considers the unconstrained problem of maximizing the expected utility of nominal wealth. To avoid confusion caused by the subtle difference in notation, we use $\mathbf{u}^\star$ to denote the optimal portfolio for the unconstrained problem that maximizes the expected utility of real retirement wealth, and $\mathbf{u}^\dagger$ the unconstrained problem that maximizes the expected utility of nominal retirement wealth.

Problem 2. Determine $\mathbf{u}^\dagger \in \mathcal{A}(x_0)$ such that

$$\mathbb{E}\left(U\left(X^{\mathbf{u}^\dagger}(T)\right)\right) = \sup_{\mathbf{u}\in\mathcal{A}(x_0)} \left\{\mathbb{E}\left(U\left(X^{\mathbf{u}}(T)\right)\right)\right\}$$

and the budget constraint is satisfied.

The solution to the unconstrained problem that maximizes the expected utility of the nominal retirement wealth (Problem 2) is widely explored in the literature (see, e.g. Merton, 1971b; Korn and Krekel, 2002; Gerrard et al., 2014). We simply state it here. The optimal investment strategy for this problem is

$$\mathbf{u}^\dagger(t) := \frac{1}{1-\gamma}\left(\boldsymbol{\sigma}'\right)^{-1}\boldsymbol{\theta}\, X^{\mathbf{u}^\dagger}(t), \tag{3.6}$$

and $u_0^\dagger(t) = X^{\mathbf{u}^\dagger}(t) - \mathbf{u}^\dagger(t)'\mathbf{1}_D$ where $u_i^\dagger(t)$ denotes the monetary amount allocated to asset i at time t. The optimal investment strategy allocates a constant fraction into each of the assets. This is the so called *Merton strategy.* The more risk averse the investor, the smaller the amount (and the fraction) of wealth the investor invest in the risky assets $i = 1, \ldots, D$.

Next, we consider the unconstrained problem that maximizes the expected utility of the real retirement wealth. The solution is detailed in Appendix 3.A.1.

Problem 3. Determine $\mathbf{u}^\star \in \mathcal{A}(x_0)$ such that

$$\mathbb{E}\left(U\left(\frac{I_0}{I(T)}X^{\mathbf{u}^\star}(T)\right)\right) = \sup_{\mathbf{u}\in\mathcal{A}(x_0)}\left\{\mathbb{E}\left(U\left(\frac{I_0}{I(T)}X^{\mathbf{u}}(T)\right)\right)\right\}$$

and the budget constraint is satisfied.

The optimal investment strategy for Problem 3 is

$$\mathbf{u}^\star(t) := \frac{1}{1-\gamma}\left[(\boldsymbol{\sigma}')^{-1}\boldsymbol{\theta} - \gamma e_1\right] X^{\mathbf{u}^\star}(t), \tag{3.7}$$

where $e_1 = \begin{pmatrix} 1 & 0 & \cdots & 0\end{pmatrix}'$ is a D-vector with 1 in the first component and 0 elsewhere. Similar to the solution to Problem 2, the optimal investment strategy prescribes a constant fraction of wealth in each asset.

Dividing Equation 3.6 by the wealth level we obtain the fraction invested in each risky asset as

$$\frac{1}{1-\gamma}(\boldsymbol{\sigma}')^{-1}\boldsymbol{\theta} + \left(1 - \frac{1}{1-\gamma}\right)e_1. \tag{3.8}$$

This expresses the optimal investment strategy as a portfolio of two components: a tangency component which depends on the mean returns and covariances of the risky assets, and an inflation risk hedging component (see also Brennan and Xia, 2002). The weight in each of these components is controlled by the risk aversion coefficient γ. In the classical problem where the expected utility is calculated from nominal wealth, the inflation risk hedging component is replaced by a "minimum risk" component in which the investor holds the (nominal) risk-free asset. When a problem is expressed in real term, the risk perception of the investor changes and the inflation-linked bond becomes the risk-free asset and the hedging component needs to be adjusted accordingly. This produces different solutions between one that is obtained by maximizing using real wealth and one that is obtained by maximizing using nominal wealth[2].

[2] Zhang 2012 makes incorrectly the opposite conclusion due to a trivial error in Zhang 2012, equation (28); private communication between C. Donnelly and A. Zhang.

For an investor maximizing the expected utility of real wealth, they should optimally increase the fraction of wealth invested in ILB by a factor of $1 - \frac{1}{1-\gamma}$, and reduce the same factor in the money account, compared to the Merton strategy. For $\gamma = 0$, the solutions to both problems are identical, meaning for an investor with log utility the Merton strategy simultaneously maximizes the expected utility of both nominal and real retirement wealth. In all cases, the fractions allocated to other risky assets $i = 2, \cdots, D$ remain the same as there is no change in risk aversion between nominal utility maximization and real utility maximization.

3.4 Constrained problem

Next we introduce the constrained problem for the second investor. This investor seeks to maximize the expected utility of their real retirement wealth subject to the real wealth lying between the maximum $\$K_U$ and minimum $\$K_L$ at the retirement time T. As the retirement wealth of the investor is $X^{\boldsymbol{\pi}}(T)$, its real value is $\frac{I_0}{I(T)} X^{\boldsymbol{\pi}}(T)$. Since the values $\$K_U$ and $\$K_L$ represent real values at time 0, the constraint on retirement wealth is that $\frac{I_0}{I(T)} X^{\boldsymbol{\pi}}(T)$ lies between K_L and K_U.

The investor seeks to find the investment strategy that will maximize the expected utility of their real wealth at retirement. Mathematically, the problem is expressed as follows.

Problem 4. Find $\boldsymbol{\pi}^\star \in \mathcal{A}(x_0)$ such that

$$\mathbb{E}\left(U\left(\frac{I_0}{I(T)} X^{\boldsymbol{\pi}^\star}(T)\right)\right) = \sup_{\boldsymbol{\pi} \in \mathcal{A}(x_0)} \left\{\mathbb{E}\left(U\left(\frac{I_0}{I(T)} X^{\boldsymbol{\pi}}(T)\right)\right)\right\},$$

subject to the terminal constraints that $\frac{I_0}{I(T)} X^{\boldsymbol{\pi}^\star}(T) \in [K_L, K_U]$ a.s. and the budget constraint is satisfied.

To avoid the uninteresting case that the investor can immediately secure the upper value K_U by investing solely in the ILB, and to allow the lower constraint K_L to be satisfied, we assume the following condition on the initial wealth x_0 of the investor.

Assumption 5. $K_L < x_0 I_0^{-1} e^{r_{\mathrm{R}} T} < K_U$.

As proved in Proposition 3.9, the optimal investment strategy that solves Problem 4 is to hold at time t, the amount

- $\pi_0^\star(t) := \Psi(t;(K_L,K_U))A_1Y(t)$ in the nominal bond,
- $\pi_n^\star(t) := \Psi(t;(K_L,K_U))A_nY(t)$ in the nth risky asset for $n = 2,\ldots,D$, and
- $\pi_1^\star(t) := X^\star(t) - \Psi(t;(K_L,K_U))\sum_{n=1}^D A_nY(t)$ in the ILB,

where $\Psi(t;(K_L,K_U))$ is the moderation factor (equation 3.29), $Y(t) = \frac{y_0}{x_0}X^{\mathbf{u}^\star}(t)$, is a constant multiple of the optimal unconstrained wealth process (see Corollary 3.3), and its initial value y_0 that is the solution to

$$S_{1,0}^{-1}x_0 = S_{1,0}^{-1}y_0 - c(0,S_{1,0},y_0;K_U) + p(0,S_{1,0},y_0;K_L),$$

where for $x,y>0$,

$$c(t,x,y;K_U) := y\Phi(d_+(t;K_U)) - xS_{1,0}^{-1}e^{-r_{\mathrm{R}}T}K_U\Phi(d_-(t;K_U)),$$

$$p(t,x,y;K_L) := xS_{1,0}^{-1}e^{-r_{\mathrm{R}}T}K_L\Phi(-d_-(t;K_L)) - y\Phi(-d_+(t;K_L)),$$

$$d_\pm(t;K) := \frac{\ln\left(\frac{Y(t)S_{1,0}}{S_1(t)e^{-r_{\mathrm{R}}T}K}\right) \pm \frac{1}{2}\frac{\|\tilde{\boldsymbol{\theta}}\|^2}{(1-\gamma)^2}(T-t)}{\frac{\|\tilde{\boldsymbol{\theta}}\|}{1-\gamma}\sqrt{T-t}},$$

$A_n := \frac{1}{1-\gamma}\left[(\tilde{\boldsymbol{\sigma}}')^{-1}\,\tilde{\boldsymbol{\theta}}\right]_n$ for $n = 1,2,\ldots,D$, for $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\theta}}$ given by equations (3.13) and (3.14), respectively.

Although not easy to see when expressed in terms of amounts to invest in each asset, the optimal investment strategy has three elements that can be broadly described as:

- Securing at least the worst case value K_L by buying a synthetic European put option with strike price K_L,
- Securing at most the maximum value K_U by selling a synthetic European call option with strike price K_U, and

- Investing the remaining wealth as if the investor did not have any constraints on their retirement wealth, i.e. as if the investor was maximizing the expected utility of the real value of retirement wealth.

The options are written on a specified portfolio rather than on the risky stock. For these reasons, the options are referred to as synthetic options.

3.5 Numerical illustration

Here we provide illustrations of the optimal strategy for the two investors seeking to maximize the expected utility of their real retirement: the first investor without terminal wealth constraints and the second investor with terminal wealth constraints. We adopt a $D = 2$ market model parameterized in Table 3.1 and set the initial wealth of the investors at $x_0 = 1$. We follow the financial market parameters used in Donnelly et al. (2022) which are calibrated to the UK market data from January 1981 to December 2019. We assume the baseline investor has medium risk aversion with parameter of $\gamma = -2.5$. We also compare the results to the high and low risk averse investor which we assume to have $\gamma = -4$ and $\gamma = -1$, respectively[3].

TABLE 3.1: The parameters of the market model.

Parameter	Value	Description
μ_I	0.038	Average increase in inflation index
σ_I	0.078	Volatility of inflation index
r_{N}	0.073	Nominal return on bank account
r_{R}	0.026	Real return on ILB
μ_2	0.091	Mean return on risky stock
σ_{21}	0.096	Volatility coefficient of risky stock with W_1
σ_{22}	0.142	Volatility coefficient of risky stock with W_2

As discussed in Section 3.3, the unconstrained strategy results in a constant fraction of wealth invested in each asset irrespective of the price fluctuation of the risky assets. An investor with

[3] We choose the risk aversions parameters based on the empirical findings of Khemka et al. (2021). According to Khemka et al. (2021), the underlying risk aversion of life-cycle glide path funds offered in the US, UK, Australia and Denmark range between −1 and −4.

risk aversion coefficient $\gamma < 0$ should optimally increase the fraction of wealth allocated the ILB by a constant fraction $1 - 1/(1-\gamma)$ when they maximize the expected utility of real wealth instead of nominal wealth. This increased investment is wholly funded via a reduction in the nominal bond, leaving the allocations to other risky assets unchanged. For a high risk averse investor, this results in an additional 80% fraction of wealth invested in the ILB instead of the nominal bond. For the medium and the low risk averse investors, the additional fraction invested in the ILB is 72% and 50%, respectively.

Comparing the terminal wealth distributions of the two unconstrained strategies $\mathbf{u}^\star(t)$ and $\mathbf{u}^\dagger(t)$, we find that the probability of $X^{\mathbf{u}^\star}(T)$ exceeding $X^{\mathbf{u}^\dagger}(T)$ is 27% in our baseline scenario. This is because compared to the nominal strategy ($\mathbf{u}^\dagger$), the real strategy ($\mathbf{u}^\star$) prescribes a higher fraction of wealth in ILB and a lower fraction of wealth in the nominal bond. Note that in our parameterization, ILB has a lower overall mean return rate ($\mu_I + r_\mathrm{R}$) than the nominal bond return rate (r_N). When $\mu_I + r_\mathrm{R} > r_\mathrm{N} + (1 - \frac{1}{1-\gamma})\sigma_I^2$, the terminal wealths resulted from the real strategies tend to outperform the terminal wealths resulted from the nominal strategy: $\mathbb{P}[X^{\mathbf{u}^\star}(T) < X^{\mathbf{u}^\dagger}(T)] \geq 50\%$. However, following the $\mathbf{u}^\star$ strategy produces an inflation-adjusted terminal wealth distribution that has a narrower range than the distribution produced by the corresponding nominal strategy because of its higher allocation in ILB than the corresponding nominal strategy (see Table 3.2). This makes $\mathbf{u}^\star$ the more appealing strategy for an investor who evaluates their retirement wealth in real terms instead of nominal terms.

For an investor who wishes to maximize their expected utility using real wealth, the naïve Merton strategy inevitably leads to utility loss because it is no longer optimal. The extent of utility loss is unsurprisingly dependent on their risk aversion coefficient γ which drives the difference in the optimal investment strategy. To measure the loss of expected utility[4] of real wealth (for brevity we call this *welfare* of the investor), we compare the certainty equivalent of real wealth, CEW. CEW is defined as the non-random real wealth yielding the same level of expected utility as the terminal wealth of the portfolio:

$$CEW(X(T)) := U^{-1}(\mathbb{E}[U(I_0 I^{-1}(T) X(T))]).$$

[4]See Appendix 3.A.3 for calculation of the expected utility.

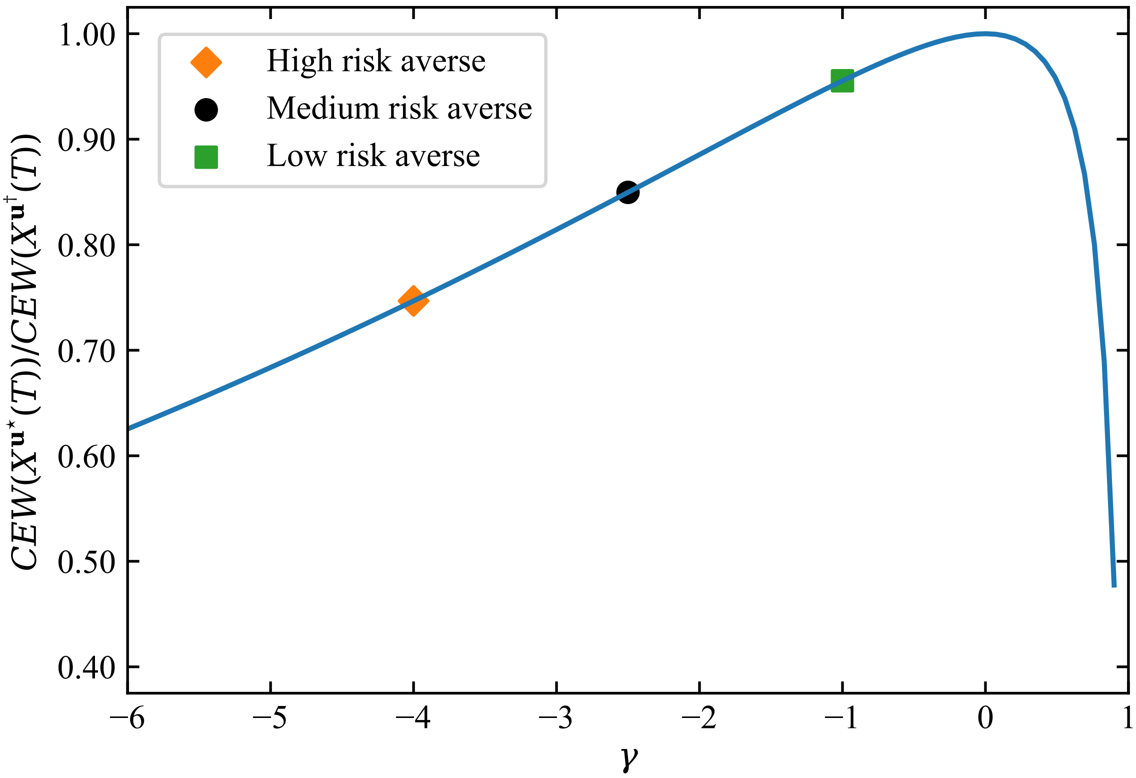

FIGURE 3.1: The ratio $CEW(X^{\mathbf{u}^\star}(T))$ and $CEW(X^{\mathbf{u}^\dagger}(T))$ as a function of the risk aversion coefficient γ. The markers highlight the utility loss for high, medium, and low risk averse investors.

By definition, the optimal unconstrained strategy $\mathbf{u}^\star(t)$ yields the highest expected utility and hence the highest CEW. Figure 3.1 illustrates the ratio of $CEW(X^{\mathbf{u}^\star}(T))$ and $CEW(X^{\mathbf{u}^\dagger}(T))$ as a function of the risk aversion coefficient. For risk aversion coefficients $\gamma < (>)\, 0$, the extent of suboptimality as a result of following $\mathbf{u}^\dagger(t)$ increases as γ decreases (increases). For an investor with γ close to zero, the reduction in the welfare is marginal because the difference in the investment strategy is minimal and both strategies converge as $\gamma \to 0$. By, following the naïve Merton strategy, the medium risk averse investor faces a reduction in welfare of 15%, and the high risk averse and the low risk averse investor faces a reduction of 25% and 4% in welfare, respectively.

As terminal wealth constraints are taken into account, the resultant optimal strategy becomes a dynamic asset allocation that changes as the prices of the risky assets fluctuate. Figure 3.2

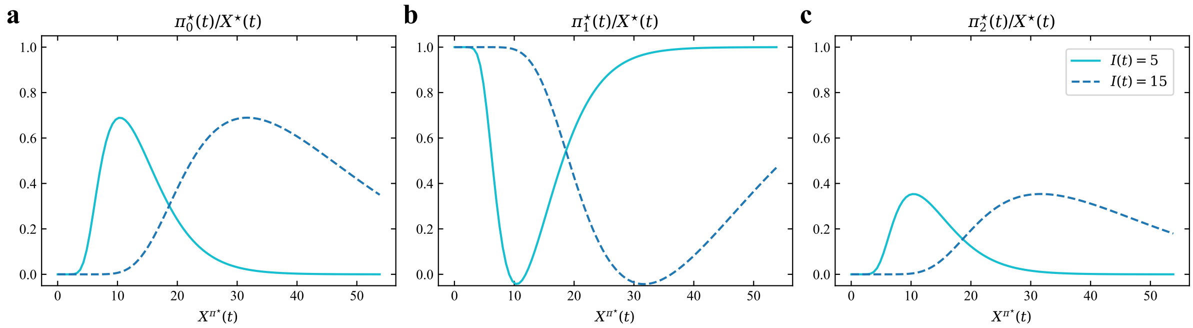

FIGURE 3.2: The optimal constrained strategy for an investor with an inflation-adjusted risk perception: the optimal fraction of wealth invested in each asset as a function of $Y(t)$ given values of $I(t)$ at time $t = 15$ for a $T = 30$-year strategy. The inflation-linked wealth constraints are $K_L = 2$, $K_U = 5$. The risk aversion parameter is $\gamma = -2.5$. (a) shows the fraction of wealth invested in nominal risk-free bond, (b) inflation-linked bond and (c) equity.

shows the fraction of wealth invested in each of the assets as a function of the unconstrained optimal wealth value Y at $t = 15$ for a 30-year strategy, at two given values of $I(t)$. When $Y(t)$ is low relative to $I(t)$, the optimal strategy prescribes a high fraction of the investor's wealth in ILB to secure the worst case value $I_0^{-1}I(T)K_L$. Similarly, when $Y(t)$ is high relative to $I(t)$, the investor should optimally invest a high fraction of their wealth in ILB in order to secure the maximum value at $I_0^{-1}I(T)K_U$. In other situations where both the upper and the lower constraints are less likely to be binding, the constrained optimal strategy allocates the same fractions of wealth in each asset as the unconstrained optimal strategy. The trade-off between the unconstrained optimal strategy and the ILB is moderated by $\Psi(t;(K_L, K_U))$ as defined in Eq. 3.29. The sensitivity of $\Psi(t;(K_L, K_U))$ with respect to $Y(t)$ is higher for smaller values of $I(t)$. This results in a more gradual change in asset allocation over a range of $Y(t)$ values when $I(t) = 15$ compared when to $I(t) = 5$ as seen in Figure 3.2. In addition, as the time remaining until retirement decreases, $\Psi(t;(K_L, K_U))$ becomes more sensitive to the values of $Y(t)$ and $I(t)$. Thus the asset allocation becomes more polarized to either following the unconstrained optimal strategy or to invest wholly in ILB depending on the relative values of $Y(t)$ and $I(t)$. Figure 3.3 depicts the fraction of wealth invested in each of the assets as a function of the unconstrained optimal wealth value Y and the price index I, at $t = 15$ for a 30-year strategy.

An investor's risk perception becomes of great importance when terminal wealth constraints are imposed. To see that, we compare the optimal investment strategy of an investor who

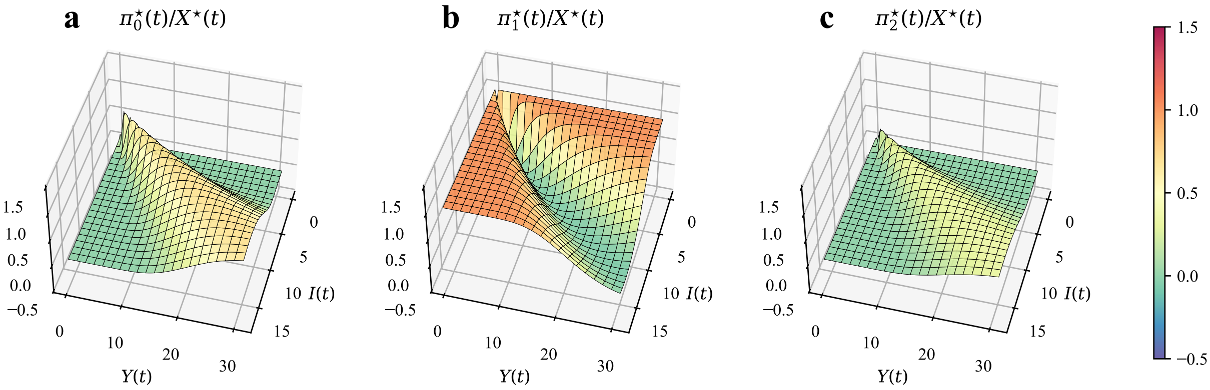

FIGURE 3.3: The optimal constrained strategy for an investor with an inflation-adjusted risk perception: the optimal fraction of wealth invested in each asset as a function of $Y(t)$ and $I(t)$ at time $t = 15$ for a $T = 30$-year strategy. The inflation-linked wealth constraints are $K_L = 2$, $K_U = 5$. The risk aversion parameter is $\gamma = -2.5$. (a) shows the fraction of wealth invested in nominal bond, (b) inflation-linked bond and (c) equity.

expresses all their goals in nominal terms: maximizing utility using nominal wealth and setting up terminal wealth constraints based on nominal wealth values. The terminal wealth of this strategy is given by

$$X^\dagger(T) := \min(\max(y_0^\dagger x_0^{-1} X^{\mathbf{u}^\dagger}(T), K_L^\dagger, K_U^\dagger), \tag{3.9}$$

in which $y_0^\dagger$ is chosen to satisfy the budget constraint (Eq. 3.5). This problem has been studied by Donnelly et al. (2018). Here, we simply compare our optimal constrained strategy maximizing using real wealth, with real wealth terminal constraints (Eq. 3.27) against their constrained strategy that maximizes using nominal wealth, with nominal wealth constraints (Eq. 3.9). Their optimal constrained optimal strategy is illustrated in Figure 3.4. To ensure the results are comparable, we set the lower and the upper nominal wealth constraints to be $K_L e^{(r_\mathrm{N}-r_\mathrm{R})T}$ and $K_U e^{(r_\mathrm{N}-r_\mathrm{R})T}$, respectively[5]. The optimal strategy that maximizes nominal wealth subject to a set of nominal wealth constraints prescribes a dynamic asset allocation that depends primarily on the corresponding optimal unconstrained wealth. Once $X^{\mathbf{u}^\star}(t)$ is taken into account, the price index plays no role in determining the optimal constrained strategy (see Figure 3.4). The investor optimally invest a high fraction of wealth in the nominal bond when

[5]Without loss of generality, we assume that $x_0 = I_0 = 1$. Then, Assumption 5 simplifies into $K_L < e^{r_\mathrm{R}T} < K_U$. We set $K_L = \epsilon e^{r_\mathrm{R}T}$ where $\epsilon < 1$ is a constant reflecting the value of the lower bound relative to the maximum attainable lower bound. For a constrained problem with nominal wealth constraint, the equivalent assumption is $K_L^\dagger < e^{r_\mathrm{N}T} < K_U^\dagger$. To achieve comparable results, we use the same ϵ values in setting the nominal lower bound: $K_L^\dagger = \epsilon e^{r_\mathrm{N}T} = K_L e^{(r_\mathrm{N}-r_\mathrm{R})T}$. Applying a similar procedure with $\epsilon > 1$ to the upper bound, we obtain $K_U^\dagger = K_U e^{(r_\mathrm{N}-r_\mathrm{R})T}$.

$y_0^\dagger x_0^{-1} X^{\mathbf{u}^\dagger}(t)$ approaches the present value of the nominal wealth constraints at both ends. In other situations, the nominal constrained optimal strategy allocates the same fractions of wealth in each asset as the Merton strategy.

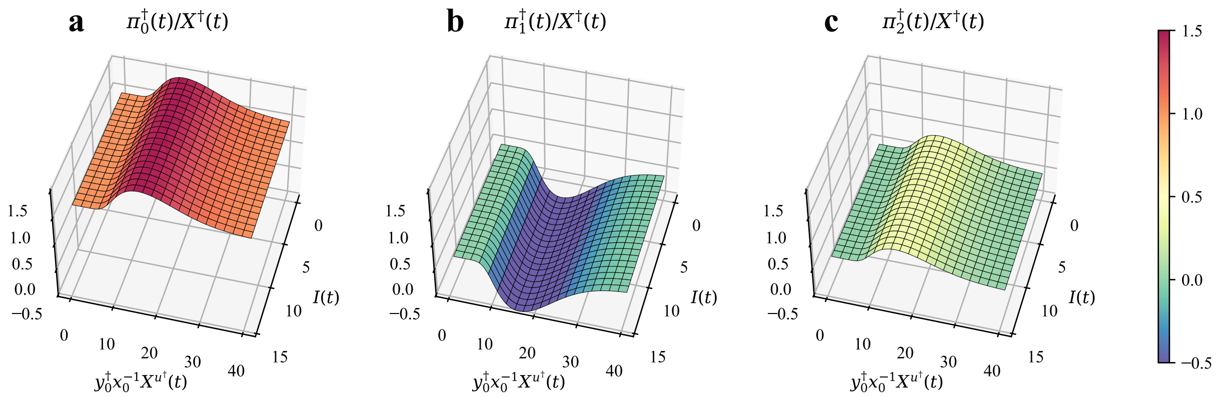

FIGURE 3.4: The optimal constrained strategy for an investor with a nominal risk perception: the optimal fraction of wealth invested in each asset as a function of $y_0^\dagger x_0^{-1} X^{\mathbf{u}^\star}(t)$ and $I(t)$ at time $t = 15$ for a $T = 30$-year strategy. The lower and upper nominal wealth constraint at time T are $2e^{(r_\mathrm{N}-r_\mathrm{R})T}$, $5e^{(r_\mathrm{N}-r_\mathrm{R})T}$, respectively. The risk aversion parameter is $\gamma = -2.5$. (a) shows the fraction of wealth invested in nominal bond, (b) inflation-linked bond and (c) equity.

Next, we discuss the loss in expected utility for the second investor if they were to follow the nominal strategy. The overall loss in expected utility can be attributed to (i) adopting an underlying unconstrained strategy that maximizes the expected utility using nominal wealth instead of real wealth, and (ii) imposing nominal wealth constraints instead of real wealth constraints. To tease out the loss in welfare contributed by the first factor, we first calculate the CEW of a constrained strategy that maximizes the expected utility of *nominal* wealth with *real* terminal wealth constraints (see the dashed line in Figure 3.5). The terminal wealth of this strategy is given by

$$X_1(T) := \min(\max(y_{1,0} x_0^{-1} X^{\mathbf{u}^\dagger}(T), I_0^{-1} I(T) K_L), I_0^{-1} I(T) K_U),$$

in which $y_{1,0}$ is chosen to satisfy the budget constraint (Eq. 3.5). For the investor with real terminal wealth constraints, the loss in welfare attributed to following the nominal utility maximization is less compared to their unconstrained counterparts (see Figure 3.1). This is because the hedging mechanism necessitated by the real wealth constraints ensures that the terminal wealth of both strategies lies between $[I_0^{-1} I(T) K_L, I_0^{-1} I(T) K_U]$, resulting in relatively similar terminal wealth distributions between the constrained investors, compared to unconstrained investors. Now let us consider an investor who specifies their investment problem entirely in

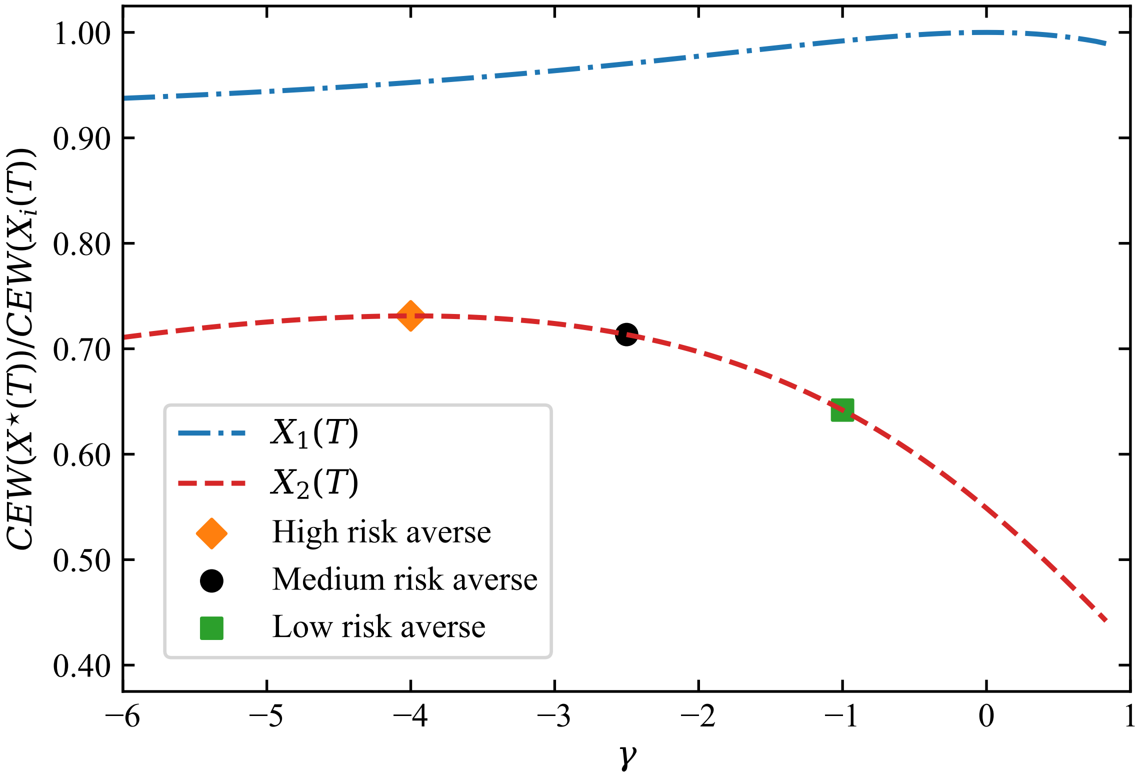

FIGURE 3.5: The ratio CEW of the optimal constrained strategy maximizing the real expected utility subject to real wealth constraints $X^{\star}(T)$ and $X_i(T)$ as a function of the risk aversion coefficient γ. $X_1(T) := \min(\max(y_{1,0}x_0^{-1}X^{\mathbf{u}^{\dagger}}(T), I_0^{-1}I(T)K_L), I_0^{-1}I(T)K_U)$ is an investment strategy that maximizes the expected utility of the nominal wealth subject to real wealth constraints. $X_2(T) := X^{\dagger}(T) = \min(\max(y_0^{\dagger}x_0^{-1}X^{\mathbf{u}^{\dagger}}(T), K_L e^{(r_N - r_R)T}), K_U e^{(r_N - r_R)T})$ is an investment strategy that maximizes the expected utility of nominal wealth subject to nominal wealth constraints. The markers highlight the utility loss of $X_2(T)$ for high, medium, and low risk averse investors.

nominal risk perceptive, i.e. the investor maximizes the expected utility using nominal wealth and imposes nominal wealth constraints (Eq. 3.9). Using nominal wealth constraints instead of real wealth constraints does not provide inflation risk hedging and it results in a profound loss in welfare. Figure 3.5 shows that with nominal wealth constraint, the loss in CEW is further exacerbated by a sizeable margin for all investors irrespective of their risk aversion parameters. The cost of mis-specifying investment strategy in nominal terms rather than real terms reduces the welfare of the medium risk investor by 29%. Similarly, the high risk averse investor the low risk averse investor suffers a loss of 27% and 36% in welfare, respectively. Unlike the unconstrained investors, constrained investors with lower risk aversion tend to suffer larger losses in utility. This is because the nominal wealth constraints are more likely to come into effect for strategies with lower risk aversion. This then leads to a terminal wealth distribution that is significantly different from the distribution produced in the presence of real wealth constraints.

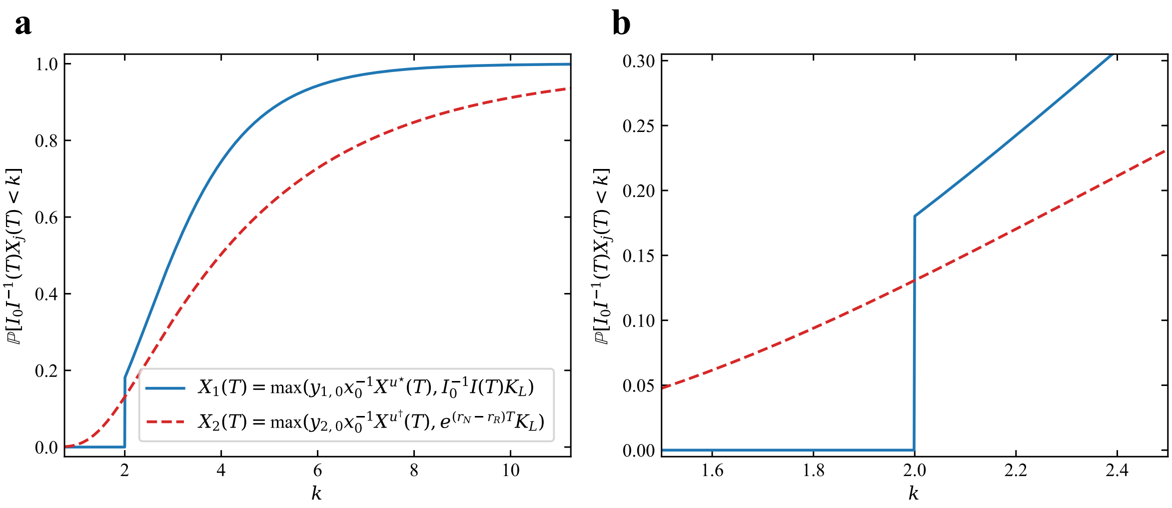

FIGURE 3.6: (a) shows the cumulative probability density function of the optimal constrained strategies for investors with a lower wealth constraint. The solid line refers to the optimal retirement wealth in Eq. 3.22 with $K_U = \infty$. The dashed line refers to the optimal investment strategy maximizing the expected utility with a lower wealth constraint (see Donnelly et al., 2018). (b) shows the same function for a tighter range of k around the real lower wealth constraint.

In addition to the loss in expected utility, using the nominal strategy does not prevent the retirement wealth from falling below an inflation-linked constraint. To demonstrate this, we consider a variant of the second investor who imposes only a lower real wealth constraints. This refers to someone who desires a minimum guarantee and wishes to keep all upside potential. Figure 3.6 shows the cumulative probability density function of the optimal constrained strategies for investors with a lower wealth constraint. The solid line represents the investor who maximizes expected utility using real wealth with a real lower constraint, and the dashed line represents the investor maximizes expected utility using nominal wealth with a nominal lower constraint. The figure shows that the worst outcome of the former investor, when it is expressed in real term, is secured at the lower constraint value at $K_L = 2$. In contrast, the latter investor does not have meaningful protection against the worst outcomes when inflation is taken into account with 13.1% chance falling below the inflation-linked lower bound.

Table 3.2 shows part of the distribution of the real terminal wealth under the unconstrained optimal strategies $X^{\mathbf{u}^\dagger}$, $X^{\mathbf{u}^\star}$, and the constrained strategy for various choices of upper and lower constraints (for quantile definitions see Appendix 3.A.4). The lower and the upper constraint levels are set around the 5% and 95% quantiles of $X^{\mathbf{u}^\star}(T)$, respectively[6]. Adopting the optimal investment strategy that maximizes the expected utility using real wealth produces slightly less risky outcomes when the wealth is measured in today's money terms, compared to the traditional Merton strategy. Adding in real terminal wealth constraints truncates the distribution at the respective upper and lower boundaries, resulting in a more compact distribution of inflation-adjusted outcomes. Using both a lower and an upper constraint simultaneously results in a compact distribution of outcomes where 70% of the outcomes lie between the lower and upper bounds.

TABLE 3.2: Table showing the quantiles of the distribution of the terminal wealth for the unconstrained strategy and the constrained strategy.

p	Unconstrained p-quantile		Constrained p-quantile		
	$\mathcal{Q}_p(\mathcal{X}(x_0, \mathbf{u}^\dagger, 0, \infty))$	$\mathcal{Q}_p(\mathcal{X}(x_0, \mathbf{u}^\star, 0, \infty))$	$\mathcal{Q}_p(\mathcal{X}(x_0, \boldsymbol{\pi}^\star, 2, \infty))$	$\mathcal{Q}_p(\mathcal{X}(x_0, \boldsymbol{\pi}^\star, 0, 5))$	$\mathcal{Q}_p(\mathcal{X}(x_0, \boldsymbol{\pi}^\star, 2, 5))$
0.025	1.222	1.647	2.000	1.660	2.000
0.05	1.517	1.892	2.000	1.907	2.000
0.25	2.956	2.902	2.223	2.926	2.232
0.5	4.698	3.907	2.993	3.938	3.004
0.75	7.468	5.260	4.029	5.000	4.045
0.95	14.546	8.067	6.180	5.000	5.000
0.975	18.062	9.269	7.101	5.000	5.000

3.6 Conclusion

Pre-retiree investors face myriad of investment decisions in planning for retirement in managing investment risk and inflation risk. The problem of finding optimal investment strategies has been traditionally dominated by the maximization of expected utility in a nominal market or a real

[6]Note that the lower and the upper constraint must satisfy Assumption 5.

market entirely. The cost of mis-specifying inflation risk is not explicitly studied. We contribute to the literature by investigating the impact of expressing the investment problem in nominal vs real for two types of risk averse investors (with and without terminal wealth constraints). We model an inflation process explicitly and consider a market model with a nominal bond, an inflation-linked bond and a risky stock. We assume that the investor starts with an initial wealth, i.e. a lump-sum saving at the beginning of the investment horizon. For simplicity, we do not consider an ongoing contribution of savings into the retirement fund, although our results can be adjusted for it.

We find that the risk perception of the investor leads to a different investment strategy for a risk averse investor without terminal wealth constraints, namely to invest more in ILB that hedges inflation risks, compared to the traditional Merton strategy. Failing to account for inflation risk, the investor loses 4% to 25% in welfare depending on the extent of their risk aversion. High risk averse investors suffer larger losses. When wealth constraints are considered, it results in significantly different optimal constrained strategies as the inflation-linked constraints and nominal constraints require different hedging assets. The welfare loss is about 27% to 36% for investors who "mistakenly" specifies their investment strategy according to nominal risk perceptive rather than real risk perceptive. Contrary to the unconstrained investor, low risk averse investors tend to suffer larger losses when nominal wealth constraints are imposed. Moreover, setting up nominal terminal wealth constraints results in an investment strategy that does not guarantee meeting inflation-linked wealth targets. We conclude that pre-retirees preference towards inflation risk (whether to protect against it) can have a significant impact on their investment strategy and outcomes at retirement. This should be catered for in the design of retirement saving products by providers.

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3.A Proofs

3.A.1 The optimal unconstrained investment strategy

We begin by solving the problem without the terminal wealth constraints (Problem 3) which serves as the foundation for the solution to the constraint problem. The key step in solving this unconstrained problem is to do a change of variable, i.e. change of numeraire. We rewrite prices in terms of units of the ILB. The equivalence of solutions between the unconstrained problem and it expressed after the change of numeraire (Problem 6) is shown in Proposition 3.1. This means that we can solve Problem 6 (Proposition 3.2) and then apply the change of variable backwards to get the solution to the original unconstrained problem (Corollary 3.3).

Han and Hung (2012) solve a similar (albeit more complicated) problem to this unconstrained one presented below, within a more complicated setting than ours. They allow for continuously-paid random contributions up to the retirement date, with specified dynamics for the contribution process so that a solution can be found using the Hamilton-Jacobi-Bellman equation. It is possible to obtain the unconstrained solution in our setting from the solution of Han and Hung 2012, by the appropriate parameter choice, but it is not clear how to extend their approach to allow for the terminal wealth constraints.

We do a change of numeraire because the solution is much easier to see, using the ILB as numeraire. The steps are to: re-express the wealth and portfolios as units of the price of the ILB. Then we prove that solving the problem under the change of numeraire gives us the desired solution (albeit expressed in terms of the ILB price). Finally, we solve the unconstrained problem.

We choose the ILB as numeraire instead of the price index, because the ILB is a traded asset and we are pricing an option later on (see Bjork 2009, Remark 26.2.2 for a discussion on the choice of numeraire). The change-of-numeraire technique we employ is also done in Zhang (2012), although she has a slightly different model than ours and uses the price index as numeraire. The other technique presented in Zhang (2012) is, after the change of numeraire, to re-write an investment portfolio in terms of the "new" risky assets. With the ILB as numeraire, the nominal bond becomes a risky asset while the ILB becomes a risk-free asset. The $D-1$ risky stocks are still classified as risky assets. This trick allows us to clearly see that we can utilize well-known results to solve the unconstrained problem that is presented next.

3.A.1.1 Change of numeraire

By expressing all prices and wealth in terms of the ILB price, the solution to Problem 3 can be easily found, using well-known results. This is a change-of-numeraire technique, a good introduction to which can be found in Bjork (2009).

For a portfolio process $\mathbf{u}$, consider $S_1^{-1}(t)X^{\mathbf{u}}(t)$. Applying Ito's Formula to the product and eliminating the amount invested in the ILB, $u_1(t)$, by the substitution $u_1(t) = X^{\mathbf{u}}(t) - u_0(t) - \sum_{n=2}^{D} u_n(t)$, after some algebra we find the dynamics are

$$
\begin{aligned}
d\left(S_1^{-1}X^{\mathbf{u}}\right)(t) = &\left(r_{\mathrm{N}} - \mu_1 + \sigma_I^2\right) S_1^{-1}(t)u_0(t)dt \\
&+ \sum_{n=2}^{D} \left(\mu_n - \mu_1 + \sigma_I^2 - \sigma_I\sigma_{n1}\right) S_1^{-1}(t)u_n(t)dt \\
&+ \left(-\sigma_I S_1^{-1}(t)u_0(t) + \sum_{n=2}^{D} \left(\sigma_{n1} - \sigma_I\right) S_1^{-1}(t)u_n(t)\right) dW_1(t) \\
&+ \sum_{m=2}^{D}\sum_{n=2}^{D} S_1^{-1}(t)u_n(t)\sigma_{nm}dW_m(t).
\end{aligned}
\tag{3.10}
$$

Now define the portfolio showing the normalized amount in each asset, excluding the ILB, as

$$
\tilde{\mathbf{u}}(t) = \begin{pmatrix} \tilde{u}_0(t) \\ \tilde{u}_-(t) \\ \vdots \\ \tilde{u}_D(t) \end{pmatrix} = \begin{pmatrix} S_1^{-1}(t)u_0(t) \\ S_1^{-1}(t)u_2(t) \\ \vdots \\ S_1^{-1}(t)u_D(t) \end{pmatrix}. \tag{3.11}
$$

Then the normalized amount in the ILB is

$$\tilde{u}_+(t) = S_1^{-1}(t)X^{\mathbf{u}}(t) - \tilde{u}_0(t) - \sum_{k=2}^{D} u_k(t).$$

We denominate the normalized wealth process in terms of the normalized portfolio $\tilde{u}$ rather than u. Define the normalized wealth at time t as

$$\tilde{X}^{\tilde{\mathbf{u}}}(t) := S_1^{-1}(t)X^{\mathbf{u}}(t), \tag{3.12}$$

To write down the dynamics (3.10) of $\tilde{X}^{\tilde{\mathbf{u}}}$ in a compact form, define

$$\tilde{\boldsymbol{\mu}} = \begin{pmatrix} r_{\mathrm{N}} + \sigma_I^2 \\ \mu_2 + \sigma_I^2 - \sigma_I\sigma_{21} \\ \vdots \\ \mu_D + \sigma_I^2 - \sigma_I\sigma_{D1} \end{pmatrix}, \quad \tilde{\boldsymbol{\sigma}} = \begin{pmatrix} -\sigma_I & 0 & \cdots & 0 \\ \sigma_{21} - \sigma_I & \sigma_{22} & \cdots & \sigma_{2D} \\ \vdots & \vdots & & \vdots \\ \sigma_{D1} - \sigma_I & \sigma_{D2} & \cdots & \sigma_{DD} \end{pmatrix}, \tag{3.13}$$

in which it is assumed that $\tilde{\boldsymbol{\sigma}}$ is non-singular. Letting $\mathbf{1}$ denote a D-column vector of ones, define

$$\tilde{\boldsymbol{\theta}} := \tilde{\boldsymbol{\sigma}}^{-1}\left(\tilde{\boldsymbol{\mu}} - \mu_1\mathbf{1}\right). \tag{3.14}$$

It can be shown by algebra that $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \boldsymbol{\sigma}_1$, where $\boldsymbol{\sigma}_1 = \begin{pmatrix} \sigma_I & 0 & \cdots & 0 \end{pmatrix}'$ is the transpose of the first row of $\boldsymbol{\sigma}$.

Substituting for $\tilde{X}^{\tilde{\mathbf{u}}}$, $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\theta}}$ in (3.10), we get

$$d\tilde{X}^{\tilde{\mathbf{u}}}(t) = \tilde{\mathbf{u}}'(t)\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\theta}}dt + \tilde{\mathbf{u}}'(t)\tilde{\boldsymbol{\sigma}}d\mathbf{W}(t). \tag{3.15}$$

3.A.1.2 Martingale measures

When the nominal bond S_0 is the numeraire, denote by $\mathbb{Q}_0$ the corresponding martingale measure - that is, the measure under which the discounted asset prices are martingales. By "discounted asset prices", we mean the asset prices expressed in terms of the nominal bond: $S_0^{-1}S_1$, $S_0^{-1}S_1$, $S_0^{-1}S_2$ and so on. The "risk-free" asset is the nominal bond.

The Radon-Nikodym derivative $\frac{d\mathbb{Q}_0}{d\mathbb{P}} = \mathcal{E}(-\boldsymbol{\theta} \bullet \mathbf{W})(T)$ on $\mathcal{F}_T$ and the expectation operator under $\mathbb{Q}_0$ is denoted $\mathbb{E}_{\mathbb{Q}_0}$.

With the ILB S_1 as the numeraire, denote by $\mathbb{Q}$ the corresponding martingale measure under which the normalized asset prices are martingales. By "normalized asset prices" we mean the asset prices expressed in terms of the price of the ILB numeraire: $S_1^{-1}S_0$, $S_1^{-1}S_1$, $S_1^{-1}S_2$ and so on. The "risk-free" asset in this case is the ILB. The nominal bond now becomes a risky asset. The Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(-\tilde{\boldsymbol{\theta}} \bullet \mathbf{W})(T) \quad \text{on } \mathcal{F}_T$$

and a D-dimensional Brownian motion under $\mathbb{Q}$ is

$$\mathbf{W}^{\mathbb{Q}}(t) := \mathbf{W}(t) + \tilde{\boldsymbol{\theta}}t,$$

The expectation operator under $\mathbb{Q}$ is denoted $\mathbb{E}_{\mathbb{Q}}$. Further note that, as $\frac{d\mathbb{Q}}{d\mathbb{Q}_0} = S_{1,0}^{-1}S_0^{-1}(T)S_1(T)$ ((Bjork, 2009, Proposition 26.4)), it follows from $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{d\mathbb{Q}}{d\mathbb{Q}_0}\frac{d\mathbb{Q}_0}{d\mathbb{P}}$ that

$$\mathcal{E}(-\tilde{\boldsymbol{\theta}} \bullet \mathbf{W})(T) = S_{1,0}^{-1}S_0^{-1}(T)S_1(T)\mathcal{E}(-\boldsymbol{\theta} \bullet \mathbf{W})(T). \tag{3.16}$$

3.A.1.3 Equivalence of unconstrained problems

Here we show that we can find the optimal solution for the unconstrained Problem 3 through optimizing in the normalized market. Define the set of admissible portfolios appropriate under the change-of-numeraire as

$$\tilde{\mathcal{A}}(y) := \Big\{ \tilde{\mathbf{u}} : \Omega \times [0,T] \to \mathbb{R} : \tilde{\mathbf{u}} \in \mathcal{F}^{\star}, \tilde{X}^{\tilde{\mathbf{u}}}(0) = y, \text{ a.s. and } \tilde{X}^{\tilde{\mathbf{u}}}(t) \geq 0, \forall t \in (0,T] \text{ a.s.} \Big\}$$

Next the Problem 3 is re-stated in terms of the normalized prices and wealth.

Problem 6. Determine $\tilde{\mathbf{u}}^\star \in \tilde{\mathcal{A}}(S_{1,0}^{-1}x_0)$ such that

$$\mathbb{E}\left(U\left(\tilde{X}^{\tilde{\mathbf{u}}^\star}(T)\right)\right) = \sup_{\tilde{\mathbf{u}} \in \tilde{\mathcal{A}}(S_{1,0}^{-1}x_0)} \left\{\mathbb{E}\left(U\left(\tilde{X}^{\tilde{\mathbf{u}}}(T)\right)\right)\right\}$$

and $\mathbb{E}\left(\mathcal{E}(-\tilde{\boldsymbol{\theta}} \bullet \mathbf{W})(T)\tilde{X}^{\tilde{\mathbf{u}}^\star}(T)\right) \leq S_{1,0}^{-1}x_0$, i.e. the budget constraint is satisfied when the ILB is the numeraire.

The next proposition shows that from the solution to Problem 6, we can obtain the solution to Problem 3.

Proposition 3.1. *Given an optimal investment strategy* $\tilde{\mathbf{u}}^\star = \left(\tilde{u}_1^\star \tilde{u}_2^\star \cdots \tilde{u}_D^\star\right)'$ *for Problem 6, an optimal investment strategy* $\mathbf{u}^\star = \left(u_1^\star u_2^\star \cdots u_D^\star\right)'$ *is defined component-wise as*

$$u_1^\star(t) = S_1(t)\left(\tilde{X}^{\tilde{\mathbf{u}}}(t) - \sum_{n=1}^{D} \tilde{u}_n^\star(t)\right), \quad u_n^\star(t) = S_1(t)\tilde{u}_n^\star(t), \quad \text{for } n = 2, \ldots, D, \tag{3.17}$$

with $u_0^\star(t) = S_1(t)\tilde{u}_1^\star(t)$.

Proof. Pick a portfolio strategy $\tilde{\mathbf{u}}$ and define the corresponding $\mathbf{u}$ through rearranging equation (3.11). Since $I_0 I^{-1}(T) = S_{1,0}e^{r_{\mathrm{R}}T}S_1^{-1}(T)$ and $U(x) = x^\gamma/\gamma$,

$$U\left(\frac{I_0}{I(T)}X^{\mathbf{u}}(T)\right) = \left(e^{r_{\mathrm{R}}T}S_{1,0}\right)^\gamma U\left(\tilde{X}^{\tilde{\mathbf{u}}}(T)\right).$$

It is straightforward to show that $\tilde{\mathbf{u}} \in \tilde{\mathcal{A}}(S_{1,0}^{-1}x_0)$ if and only if $\mathbf{u} \in \mathcal{A}(x_0)$. Then

$$\sup_{\mathbf{u} \in \mathcal{A}(x_0)} \mathbb{E}\left(U\left(\frac{I_0}{I(T)}X^{\mathbf{u}}(T)\right)\right) = \left(e^{r_{\mathrm{R}}T}S_{1,0}\right)^\gamma \sup_{\tilde{\mathbf{u}} \in \tilde{\mathcal{A}}(S_{1,0}^{-1}x_0)} \mathbb{E}\left(U\left(\tilde{X}^{\tilde{\mathbf{u}}}(T)\right)\right).$$

Let $\tilde{\mathbf{u}}^\star \in \tilde{\mathcal{A}}(S_{1,0}^{-1}x_0)$ be a solution to Problem 6 and let $\mathbf{u}^\star \in \mathcal{A}(x_0)$ be the corresponding portfolio strategy defined by equation (3.17). Then

$$\begin{aligned}\mathbb{E}\left(U\left(\frac{I_0}{I(T)}X^{\mathbf{u}^\star}(T)\right)\right) &= \left(e^{r_{\mathrm{R}}T}S_{1,0}\right)^\gamma\,\mathbb{E}\left(U\left(\tilde{X}^{\tilde{\mathbf{u}}^\star}(T)\right)\right)\\ &= \left(e^{r_{\mathrm{R}}T}S_{1,0}\right)^\gamma \sup_{\tilde{\mathbf{u}}\in\tilde{\mathcal{A}}(S_{1,0}^{-1}x_0)}\mathbb{E}\left(U\left(\tilde{X}^{\tilde{\mathbf{u}}}(T)\right)\right)\\ &= \sup_{\mathbf{u}\in\mathcal{A}(x_0)}\mathbb{E}\left(U\left(\frac{I_0}{I(T)}X^{\mathbf{u}}(T)\right)\right).\end{aligned}$$

so that $\mathbf{u}^\star$ attains the supremum of Problem 3. Lastly, using the budget constraint (3.5) and substituting from equation (3.16),

$$\begin{aligned}x_0 \geq& \mathbb{E}\left(\mathcal{E}(-\boldsymbol{\theta}\bullet\mathbf{W})(T)S_0^{-1}(T)X^{\mathbf{u}^\star}(T)\right)\\ \geq& S_{1,0}\mathbb{E}\left(\mathcal{E}(-\tilde{\boldsymbol{\theta}}\bullet\mathbf{W})(T)\,S_1^{-1}(T)X^{\mathbf{u}^\star}(T)\right)\\ \geq& S_{1,0}\mathbb{E}\left(\mathcal{E}(-\tilde{\boldsymbol{\theta}}\bullet\mathbf{W})(T)\,\tilde{X}^{\tilde{\mathbf{u}}^\star}(T)\right).\end{aligned}$$

Rearranging, we get the budget constraint when the ILB is the numeraire. Thus $\mathbf{u}^\star \in \mathcal{A}(x_0)$ is a solution to Problem 3. □

It remains to determine a solution to Problem 6. With it, we can use Proposition 3.1 to obtain the solution to Problem 3.

From Proposition 3.1, the fraction of wealth in each of the traded assets – the nominal bond, the ILB and risky stocks numbered $2,\ldots,D$ – is unchanged after changing the numeraire (recall that $u_1^\star$ and u_0 represent investment in the nominal bond, whereas $u_0^\star$ and u_1 represent investment in the ILB. For $n = 2, 3, \ldots$, both $u_n^\star$ and u_n represent investment in the nth risky asset).

3.A.1.4 Solution to the unconstrained problems

Proposition 3.2. *Define*

$$Z(t) := \exp\left\{\frac{1-2\gamma}{2(1-\gamma)^2}\|\tilde{\boldsymbol{\theta}}\|^2 t + \frac{1}{1-\gamma}\tilde{\boldsymbol{\theta}}'\mathbf{W}(t)\right\}. \tag{3.18}$$

A solution to Problem 6 is $\tilde{\mathbf{u}}^\star = \begin{pmatrix} \tilde{u}^\star_+ & \tilde{u}^\star_- & \cdots & \tilde{u}^\star_D \end{pmatrix}'$ *with*

$$\tilde{\mathbf{u}}^\star(t) := \frac{1}{1-\gamma} \left(\tilde{\boldsymbol{\sigma}}'\right)^{-1} \tilde{\boldsymbol{\theta}}\, \tilde{X}^{\tilde{\mathbf{u}}^\star}(t), \tag{3.19}$$

in which the wealth $\tilde{X}^{\tilde{\mathbf{u}}^\star}(t) = x_0 S_{1,0}^{-1} Z(t)$.

Proof. See Karatzas and Shreve (1998, Example 6.7) for the derivation of the above solution. Substituting into (3.15) for $\tilde{\mathbf{u}}^\star$ given by equation (3.19), we get the dynamics

$$d\tilde{X}^{\tilde{\mathbf{u}}^\star}(t) = \tilde{X}^{\tilde{\mathbf{u}}^\star}(t) \left(\frac{1}{1-\gamma} \|\tilde{\boldsymbol{\theta}}\|^2 dt + \frac{1}{1-\gamma} \tilde{\boldsymbol{\theta}}'(t) d\mathbf{W}(t) \right), \quad \tilde{X}^{\tilde{\mathbf{u}}^\star}(0) = x_0 S_{1,0}^{-1}, \text{ a.s.}$$

The normalized wealth is $\tilde{X}^{\tilde{\mathbf{u}}^\star}(t) = x_0 S_{1,0}^{-1} Z(t)$. The budget constraint when the ILB is the numeraire is satisfied with equality:

$$\mathbb{E}\left(\mathcal{E}(-\tilde{\boldsymbol{\theta}} \bullet \mathbf{W})(T) \tilde{X}^{\tilde{\mathbf{u}}^\star}(T)\right) = x_0 S_{1,0}^{-1} \mathbb{E}_{\mathbb{Q}}\left(Z(T)\right) = x_0 S_{1,0}^{-1}.$$

□

Corollary 3.3. *A solution to Problem 3 is* $\mathbf{u}^\star = \begin{pmatrix} u^\star_1 & u^\star_2 & \cdots & u^\star_D \end{pmatrix}'$ *with*

$$u^\star_1(t) = X^{\mathbf{u}^\star}(t) - \sum_{n=1}^{D} S_1(t) \tilde{u}^\star_n(t), \quad u^\star_n(t) = S_1(t) \tilde{u}^\star_n(t), \quad \textit{for } n = 2, \ldots, D,$$

for each $t \in [0, T]$, *in which* $\tilde{\mathbf{u}}^\star$ *is defined by equation* (3.19) *and the wealth* $X^{\mathbf{u}^\star}(t) = x_0 S_{1,0}^{-1} S_1(t) Z(t)$.

Proof. Apply Proposition 3.1 to define the optimal portfolio for Problem 3, as given in the statement of this proposition. Substitute for $X^{\mathbf{u}^\star}(t) = S_1(t) \tilde{X}^{\tilde{\mathbf{u}}^\star}(t) = x_0 S_{1,0}^{-1} S_1(t) Z(t)$ to get the expressions shown. □

Corollary 3.4. *The solution to Problem 3,* $\mathbf{u}^\star$ *is given by*

$$\mathbf{u}^\star(t) := \frac{1}{1-\gamma} \left[\left(\boldsymbol{\sigma}'\right)^{-1} \boldsymbol{\theta} - \gamma e_1 \right] X^{\mathbf{u}^\star}(t),$$

for each $t \in [0, T]$.

Proof. Substituting into equation (3.4) for the investment strategy $\mathbf{u}^\star$ given by equation (3.7). We obtain the following wealth dynamics

$$\mathrm{d}X^{\mathbf{u}^\star}(t) = X^{\mathbf{u}^\star}(t)\left[r_\mathrm{N} + \frac{\|\boldsymbol{\theta}\|^2}{1-\gamma} - \frac{\gamma}{1-\gamma}(\mu_1 - r_\mathrm{N})\right]\mathrm{d}t + X^{\mathbf{u}^\star}(t)\left[\frac{1}{1-\gamma}\boldsymbol{\theta}' - \frac{\gamma}{1-\gamma}\boldsymbol{\sigma}_1'\right]\mathrm{d}\mathbf{W}(t)$$

From equation (3.12), $X^{\mathbf{u}^\star}(t) = S_1(t)\tilde{X}^{\tilde{\mathbf{u}}}(t)$. Using Ito calculus and substituting into it for $\tilde{\mathbf{u}}^\star$ given by equation (3.19), we get the dynamics

$$\begin{aligned}
\mathrm{d}(S_1\tilde{X}^{\tilde{\mathbf{u}}})(t) &= \tilde{X}^{\tilde{\mathbf{u}}}(t)\,\mathrm{d}S_1(t) + S_1(t)\,\mathrm{d}\tilde{X}^{\tilde{\mathbf{u}}}(t) + S_1(t)\tilde{\mathbf{u}}'(t)\tilde{\boldsymbol{\sigma}}\boldsymbol{\sigma}_1\,\mathrm{d}t \\
&= X^{\mathbf{u}^\star}(t)\left[\mu_1 + \frac{\|\tilde{\boldsymbol{\theta}}\|^2}{1-\gamma} + \frac{1}{1-\gamma}\tilde{\boldsymbol{\theta}}'\boldsymbol{\sigma}_1\right]\mathrm{d}t + X^{\mathbf{u}^\star}(t)\left[\frac{1}{1-\gamma}\tilde{\boldsymbol{\theta}}' + \boldsymbol{\sigma}_1'\right]\mathrm{d}\mathbf{W}(t) \\
&= X^{\mathbf{u}^\star}(t)\left[r_\mathrm{N} + \frac{\|\boldsymbol{\theta}\|^2}{1-\gamma} - \frac{\gamma}{1-\gamma}(\mu_1 - r_\mathrm{N})\right]\mathrm{d}t + X^{\mathbf{u}^\star}(t)\left[\frac{1}{1-\gamma}\boldsymbol{\theta}' - \frac{\gamma}{1-\gamma}\boldsymbol{\sigma}_1'\right]\mathrm{d}\mathbf{W}(t)
\end{aligned}$$

Given the uniqueness of stochastic differential equations, we conclude that the investment strategy $\mathbf{u}^\star(t)$ given in equation (3.7) is one that solves Problem 3. □

3.A.2 The optimal constrained investment strategy

Now we turn back to solving the constrained problem, Problem 4, where the optimal real terminal wealth is constrained to lie in the range $[K_L, K_U]$ a.s. The idea is to determine a wealth (rather than a portfolio) at time T that is at least greater than the optimal wealth for the constrained problem. The replicating portfolio for the candidate terminal wealth can be determined by standard techniques. It is straightforward to check that the candidate replicating portfolio is an optimal solution to Problem 4.

However, again we use an indirect method to get to the solution. We determine the optimal wealth in terms of units of the ILB (Proposition 3.6). Lemma 3.5 shows that expressing Problem 4 using the ILB as numeraire (Problem 7) gives an equivalent problem. This allows us to use the candidate terminal wealth expressed in terms of the ILB price (Proposition 3.6), to determine the corresponding replicating portfolio and hence the solution to Problem 4 (Proposition 3.9).

3.A.2.1 Constrained problem using the normalized wealth

Re-writing Problem 4 in terms of the normalized wealth, the constraint

$$\frac{I_0}{I(T)} X^{\boldsymbol{\pi}^\star}(T) \in [K_L, K_U], \text{ a.s.}$$

is converted to

$$\tilde{X}^{\tilde{\boldsymbol{\pi}}^\star}(T) \in \left[\frac{I(T)K_L}{I_0 S_1(T)}, \frac{I(T)K_U}{I_0 S_1(T)}\right], \text{ a.s.}$$

Problem 7. Determine $\tilde{\boldsymbol{\pi}}^\star \in \tilde{\mathcal{A}}(S_{1,0}^{-1}x_0)$ such that

$$\mathbb{E}\left(U\left(\tilde{X}^{\tilde{\boldsymbol{\pi}}^\star}(T)\right)\right) = \sup_{\tilde{\boldsymbol{\pi}}\in\tilde{\mathcal{A}}(S_{1,0}^{-1}x_0)} \left\{\mathbb{E}\left(U\left(\tilde{X}^{\tilde{\boldsymbol{\pi}}}(T)\right)\right)\right\},$$

$\tilde{X}^{\tilde{\boldsymbol{\pi}}^\star}(T) \in [S_{1,0}^{-1}e^{-r_{\mathrm{R}}T}K_L, S_{1,0}^{-1}e^{-r_{\mathrm{R}}T}K_U]$, a.s. and

$$\mathbb{E}\left(\mathcal{E}(-\tilde{\boldsymbol{\theta}} \bullet \mathbf{W})(T)\tilde{X}^{\tilde{\boldsymbol{\pi}}^\star}(T)\right) \leq S_{1,0}^{-1}x_0. \tag{3.20}$$

Lemma 3.5. *If* $\tilde{\boldsymbol{\pi}}^\star = \left(\tilde{\pi}_1^\star \tilde{\pi}_2^\star \cdots \tilde{\pi}_D^\star\right)'$ *is an optimal investment strategy for Problem 7, then an optimal investment strategy* $\boldsymbol{\pi}^\star = \left(\pi_1^\star \pi_2^\star \cdots \pi_D^\star\right)'$ *for Problem 4 is defined component-wise as*

$$\pi_1^\star(t) = S_1(t)\left(\tilde{X}^{\tilde{\boldsymbol{\pi}}^\star}(t) - \sum_{n=1}^{D}\tilde{\pi}_n^\star(t)\right), \quad \pi_n^\star(t) = S_1(t)\tilde{\pi}_n^\star(t), \quad \text{for } n = 2, \ldots, D, \tag{3.21}$$

with $\pi_0^\star(t) = S_1(t)\tilde{\pi}_1^\star(t)$.

Proof. The proof is exactly the same as that of Proposition 3.1, with the additional step of checking that

$$\tilde{X}^{\tilde{\boldsymbol{\pi}}^\star}(T) \in [S_{1,0}^{-1}e^{-r_{\mathrm{R}}T}K_L, S_{1,0}^{-1}e^{-r_{\mathrm{R}}T}K_U] \quad \text{a.s.}$$

is equivalent to $I_0 I^{-1}(T) X^{\boldsymbol{\pi}^\star}(T) \in [K_L, K_U]$ a.s. Since

$$\frac{I_0}{I(T)} = \frac{S_{1,0}}{S_1(T)} e^{r_{\mathrm{R}} T},$$

the desired inequality follows from rearranging $K_L \leq I_0 I^{-1}(T) X^{\boldsymbol{\pi}^\star}(T) \leq K_U$ and substituting for $\tilde{X}^{\tilde{\boldsymbol{\pi}}^\star}(T) = S_1^{-1}(T) X^{\boldsymbol{\pi}^\star}(T)$. □

By Lemma 3.5, we can use the solution to the normalized, constrained problem to solve the original constrained problem, Problem 4.

3.A.2.2 Candidate terminal wealth for the constrained problems

The next proposition is an intermediate step, which gives an expression for the optimal constrained terminal wealth (Problem 7). The solution to the unconstrained Problem 6, which has optimal wealth process $\tilde{X}^{\tilde{\mathbf{u}}^\star}(t) = x_0 S_{1,0}^{-1} Z(t)$ is used to determine the retirement wealth value for the constrained Problem 7.

The next proposition proposes a process that is the candidate normalized optimal wealth for Problem 7. Define $[x]_+ := \max\{0, x\}$.

Proposition 3.6. *Define the random variable*

$$\tilde{X}^\star(T) := S_{1,0}^{-1} \left(y_0 Z(T) - \left[y_0 Z(T) - e^{-r_R T} K_U \right]_+ + \left[e^{-r_R T} K_L - y_0 Z(T) \right]_+ \right), \tag{3.22}$$

with the value $y_0 > 0$ *chosen so that the budget constraint* (3.20) *is satisfied with equality by* $\tilde{X}^\star(T)$. *Then* $\sup_{\tilde{\boldsymbol{\pi}} \in \tilde{\mathcal{A}}(S_{1,0}^{-1} x_0)} \mathbb{E}\left(U(\tilde{X}^{\tilde{\boldsymbol{\pi}}}(T)) \right) \leq \mathbb{E}\left(U(\tilde{X}^\star(T)) \right)$.

Proof. The proof is an adaptation of Donnelly et al. (2015, Proposition 4.3). For the investor's utility function, the first derivative $U'(x) = x^{\gamma - 1}$, which is a strictly decreasing function, has a strictly decreasing inverse I with

$$I(y) := y^{\frac{1}{\gamma - 1}}, \quad y > 0.$$

For ease of notation, define the random variable

$$H := \mathcal{E}(-\tilde{\boldsymbol{\theta}} \bullet \mathbf{W})(T).$$

After some algebra, we find that the value of $y^\star$ that satisfies

$$S_{1,0}^{-1} y_0 Z(T) = I(y^\star H)$$

is the constant

$$y^\star := \left(y_0 S_{1,0}^{-1}\right)^{\gamma-1} e^{\frac{1}{2}\frac{\gamma}{1-\gamma}\|\tilde{\boldsymbol{\theta}}\|^2 T}$$

We work with $I(y^\star H)$ in the proof, rather than with $S_{1,0}^{-1} y_0 Z(T)$ since both $I(x)$ and $U'(x)$ are strictly decreasing functions.

Define the constants $\tilde{K}_L := S_{1,0}^{-1} e^{-r_{\mathrm{R}}T} K_L$ and $\tilde{K}_U := S_{1,0}^{-1} e^{-r_{\mathrm{R}}T} K_U$ and let $\tilde{X}(T) \in [\tilde{K}_L, \tilde{K}_U]$, a.s. be any attainable terminal wealth (i.e. there exists a portfolio process $\tilde{\boldsymbol{\pi}} \in \mathcal{A}(x_0)$ that replicates $\tilde{X}(T)$) with $\mathbb{E}\left(H\tilde{X}(T)\right) \leq x_0 S_{1,0}^{-1}$. We show that

$$\mathbb{E}\left(U(\tilde{X}(T))\right) \leq \mathbb{E}\left(U(\tilde{X}^\star(T))\right).$$

Then by arbitrary choice of $\tilde{X}$, $\sup_{\tilde{\boldsymbol{\pi}} \in \tilde{\mathcal{A}}(S_{1,0}^{-1} x_0)} \mathbb{E}\left(U(\tilde{X}^{\tilde{\boldsymbol{\pi}}}(T))\right) \leq \mathbb{E}\left(U(\tilde{X}^\star(T))\right)$.

From equation (3.22) and using the fact that U' is a strictly decreasing function,

$$\begin{aligned} \tilde{X}^\star(T) &= \begin{cases} \tilde{K}_U & \text{if } I(y^\star H) > \tilde{K}_U \\ I(y^\star H) & \text{if } I(y^\star H) \in [\tilde{K}_L, \tilde{K}_U] \\ \tilde{K}_L & \text{if } I(y^\star H) < \tilde{K}_L \end{cases} \\ &= \begin{cases} \tilde{K}_U & \text{if } y^\star H < U'(\tilde{K}_U) \\ I(y^\star H) & \text{if } y^\star H \in [U'(\tilde{K}_U), U'(\tilde{K}_L)] \\ \tilde{K}_L & \text{if } y^\star H > U'(\tilde{K}_L). \end{cases} \end{aligned}$$

First, as U is a concave function then for any $a, b \in \mathbb{R}$, $U(a) - U(b) \leq U'(b) \cdot (a - b)$. In particular,

$$U(\tilde{X}(T)) - U(\tilde{X}^\star(T)) \leq U'(\tilde{X}^\star(T)) \cdot (\tilde{X}(T) - \tilde{X}^\star(T)), \quad \text{a.s.}$$

Take expectations in the above inequality to get

$$\begin{aligned}
&\mathbb{E}\left(U(\tilde{X}(T)) - U(\tilde{X}^\star(T))\right) \\
\leq &\mathbb{E}\left(U'(\tilde{X}^\star(T)) \cdot (\tilde{X}(T) - \tilde{X}^\star(T))\right) \\
\leq &\mathbb{E}\left(U'(\tilde{X}^\star(T)) \cdot (\tilde{X}(T) - \tilde{X}^\star(T)) \,|\, y^\star H < U'(\tilde{K}_U)\right) \cdot \mathbb{P}\left[y^\star H < U'(\tilde{K}_U)\right] \\
&+ \mathbb{E}\left(U'(\tilde{X}^\star(T)) \cdot (\tilde{X}(T) - \tilde{X}^\star(T)) \,|\, y^\star H \in [U'(\tilde{K}_U), U'(\tilde{K}_L)]\right) \\
&\quad \cdot \mathbb{P}\left[y^\star H \in [U'(\tilde{K}_U), U'(\tilde{K}_L)]\right] \\
&+ \mathbb{E}\left(U'(\tilde{X}^\star(T)) \cdot (\tilde{X}(T) - \tilde{X}^\star(T)) \,|\, y^\star H > U'(\tilde{K}_L)\right) \cdot \mathbb{P}\left[y^\star H > U'(\tilde{K}_L)\right]
\end{aligned} \tag{3.23}$$

Observe that on the event $\left[y^\star H \in [U'(\tilde{K}_U), U'(\tilde{K}_L)]\right]$,

$$U'(\tilde{X}^\star(T)) = U'(I(y^\star H)) = y^\star H$$

so that

$$\begin{aligned}
&\mathbb{E}\left(U'(\tilde{X}^\star(T)) \cdot (\tilde{X}(T) - \tilde{X}^\star(T)) \,|\, y^\star H \in [U'(\tilde{K}_U), U'(\tilde{K}_L)]\right) \\
= &\mathbb{E}\left(y^\star H \cdot (\tilde{X}(T) - \tilde{X}^\star(T)) \,|\, y^\star H \in [U'(\tilde{K}_U), U'(\tilde{K}_L)]\right).
\end{aligned} \tag{3.24}$$

Next observe that on the event $\left[y^\star H < U'(\tilde{K}_U)\right]$, as $\tilde{X}(T) \in [\tilde{K}_L, \tilde{K}_U]$ a.s. and $\tilde{X}^\star(T) = \tilde{K}_U$ a.s, then

$$\tilde{X}(T) - \tilde{X}^\star(T) = \tilde{X}(T) - \tilde{K}_U \leq 0$$

and

$$U'(\tilde{X}^\star(T)) = U'(\tilde{K}_U) > y^\star H.$$

Upon multiplying both sides of the inequality $U'(\tilde{X}^\star(T)) > y^\star H$ by the negative random variable $\tilde{X}(T) - \tilde{X}^\star(T)$, we find that on the event $\left[y^\star H < U'(\tilde{K}_U)\right]$,

$$U'(\tilde{X}^\star(T)) \cdot (\tilde{X}(T) - \tilde{X}^\star(T)) \leq y^\star H \cdot (\tilde{X}(T) - \tilde{X}^\star(T)).$$

Thus

$$\begin{aligned} &\mathbb{E}\left(U'(\tilde{X}^\star(T)) \cdot (\tilde{X}(T) - \tilde{X}^\star(T)) \,|\, y^\star H < U'(\tilde{K}_U)\right) \\ \leq &\mathbb{E}\left(y^\star H \cdot (\tilde{X}(T) - \tilde{X}^\star(T)) \,|\, y^\star H < U'(\tilde{K}_U)\right). \end{aligned} \tag{3.25}$$

Similarly, on the event $\left[y^\star H > U'(\tilde{K}_L)\right]$,

$$\tilde{X}(T) - \tilde{X}^\star(T) = \tilde{X}(T) - \tilde{K}_L \geq 0$$

and

$$U'(\tilde{X}^\star(T)) = U'(\tilde{K}_L) < y^\star H.$$

Upon multiplying both sides of the inequality $U'(\tilde{X}^\star(T)) < y^\star H$ by the positive random variable $\tilde{X}(T) - \tilde{X}^\star(T)$, we find that on the event $\left[y^\star H < U'(\tilde{K}_L)\right]$,

$$U'(\tilde{X}^\star(T)) \cdot (\tilde{X}(T) - \tilde{X}^\star(T)) \leq y^\star H \cdot (\tilde{X}(T) - \tilde{X}^\star(T)).$$

Thus

$$\begin{aligned} &\mathbb{E}\left(U'(\tilde{X}^\star(T)) \cdot (\tilde{X}(T) - \tilde{X}^\star(T)) \,|\, y^\star H > U'(\tilde{K}_L)\right) \\ \leq &\mathbb{E}\left(y^\star H \cdot (\tilde{X}(T) - \tilde{X}^\star(T)) \,|\, y^\star H > U'(\tilde{K}_L)\right). \end{aligned} \tag{3.26}$$

Substituting equations (3.24), (3.25) and (3.26) into inequality (3.23), it follows that

$$\mathbb{E}\left(U(\tilde{X}(T)) - U(\tilde{X}^\star(T))\right) \leq \mathbb{E}\left(y^\star H \cdot (\tilde{X}(T) - \tilde{X}^\star(T))\right).$$

As both $\tilde{X}(T)$ and $\tilde{X}^\star(T)$ satisfy the budget constraint (3.20), from the last inequality

$$\mathbb{E}\left(y^\star H \cdot (\tilde{X}(T) - \tilde{X}^\star(T))\right) \leq y^\star \cdot (S_{1,0}^{-1}x_0 - S_{1,0}^{-1}x_0) = 0,$$

which means $\mathbb{E}\left(U(\tilde{X}(T)) - U(\tilde{X}^\star(T))\right) \leq 0$, as required.

□

3.A.2.3 Replicating portfolio for the candidate terminal wealth

We turn our attention now to finding the investment strategy that results in the corresponding optimal wealth

$$X^\star(T) = S_1(T)\tilde{X}^\star(T). \tag{3.27}$$

We begin by determining the price of $X^\star(T)$ at any time $t \in [0,T]$. Applying Bjork (2009, Theorem 26.2),

$$X^\star(t) := S_1(t)\mathbb{E}_{\mathbb{Q}}\left(S_1^{-1}(T)X^\star(T) \,|\, \mathcal{F}_t\right) = S_1(t)\mathbb{E}_{\mathbb{Q}}\left(\tilde{X}^\star(T) \,|\, \mathcal{F}_t\right).$$

Substituting for $\tilde{X}^\star(T)$ from equation (3.22),

$$X^\star(t) = \frac{S_1(t)}{S_{1,0}}\mathbb{E}_{\mathbb{Q}}\left(y_0 Z(T) - \left[y_0 Z(T) - e^{-r_{\mathrm{R}}T}K_U\right]_+ + \left[e^{-r_{\mathrm{R}}T}K_L - y_0 Z(T)\right]_+ \,\middle|\, \mathcal{F}_t\right). \tag{3.28}$$

In this section, we determine the price at any time t for the right-hand side of 3.28, by considering each conditional expectation separately.

The next proposition values the payoff $\left[y_0 Z(T) - e^{-r_{\mathrm{R}}T}K_U\right]_+$ as the payoff of a European call option written on a specific portfolio.

Proposition 3.7. *Define*

$$c(t) := \frac{S_1(t)}{S_{1,0}}\mathbb{E}_{\mathbb{Q}}\left(\left[y_0 Z(T) - e^{-r_R T}K_U\right]_+ \,\middle|\, \mathcal{F}_t\right).$$

Then $c(t) \equiv c(t, S_1(t), Y(t); K_U)$, *in which*

$$Y(t) := y_0 \frac{S_1(t)}{S_{1,0}} Z(t),$$

and

$$c(t,x,y;K_U) := y\Phi(d_+(t;K_U)) - xS_{1,0}^{-1} e^{-r_R T} K_U \Phi(d_-(t;K_U)), \quad x, y > 0,$$

where for any constant $K > 0$,

$$d_\pm(t;K) := d_\pm(t,x,y;K) := \frac{\ln\left(\frac{yS_{1,0}}{xe^{-r_R T}K}\right) \pm \frac{1}{2}\frac{\|\tilde{\boldsymbol{\theta}}\|^2}{(1-\gamma)^2}(T-t)}{\frac{\|\tilde{\boldsymbol{\theta}}\|}{1-\gamma}\sqrt{T-t}},$$

For each $n = 1, 2, \ldots, D$, *define*

$$A_n := \frac{1}{1-\gamma}\left[\left(\tilde{\boldsymbol{\sigma}}'\right)^{-1} \tilde{\boldsymbol{\theta}}\right]_n.$$

The replicating portfolio for $c(t)$ *is to hold the amounts*

- $A_1 \cdot Y(t)\Phi(d_+(t;K_U))$ *in the nominal bond,*
- $A_n \cdot Y(t)\Phi(d_+(t;K_U))$ *in nth risky stock, for* $n = 2, \ldots, D$, *and*
- $c(t) - Y(t)\Phi(d_+(t;K_U))\sum_{n=1}^{D} A_n$ *in the ILB.*

Proof. Fix $t \in [0,T)$ and, to ease notation, define $\alpha := \|\tilde{\boldsymbol{\theta}}\|\sqrt{T-t}/(1-\gamma)$. First, as $\mathbf{W}^{\mathbb{Q}}(t) := \mathbf{W}(t) + \tilde{\boldsymbol{\theta}} t$, is a Brownian motion under $\mathbb{E}_{\mathbb{Q}}$,

$$\begin{aligned} c(t) =& \frac{S_1(t)}{S_{1,0}} \mathbb{E}_{\mathbb{Q}}\left(\left[y_0 Z(T) - e^{-r_R T} K_U\right]_+ \middle| \mathcal{F}_t\right) \\ \Rightarrow \left(\frac{S_1(t)}{S_{1,0}}\right)^{-1} c(t) =& \mathbb{E}_{\mathbb{Q}}\left(\left[y_0 Z(t)\frac{Z(T)}{Z(t)} - e^{-r_R T} K_U\right]_+ \middle| \mathcal{F}_t\right) \\ =& \mathbb{E}_{\mathbb{Q}}\left(\left[y_0 Z(t) e^{\frac{1}{1-\gamma}\tilde{\boldsymbol{\theta}}'(\mathbf{W}^{\mathbb{Q}}(T) - \mathbf{W}^{\mathbb{Q}}(t)) - \frac{1}{2}\alpha^2} - e^{-r_R T} K_U\right]_+ \middle| \mathcal{F}_t\right). \end{aligned}$$

Define

$$\mathcal{Z} := \frac{\tilde{\boldsymbol{\theta}}'(\mathbf{W}^{\mathbb{Q}}(T) - \mathbf{W}^{\mathbb{Q}}(t))}{\|\tilde{\boldsymbol{\theta}}\|\sqrt{T-t}} \sim \mathcal{N}(0,1) \text{ under } \mathbb{Q}.$$

Then

$$\left(\frac{S_1(t)}{S_{1,0}}\right)^{-1} c(t) = \mathbb{E}_{\mathbb{Q}}\left(\left[y_0 Z(t) e^{\alpha\mathcal{Z} - \frac{1}{2}\alpha^2} - e^{-r_{\mathrm{R}}T} K_U\right]_+ \,\middle|\, \mathcal{F}_t\right).$$

After some algebra, we find that

$$y_0 Z(t) e^{\alpha\mathcal{Z} - \frac{1}{2}\alpha^2} - e^{-r_{\mathrm{R}}T} K_U > 0 \quad \Leftrightarrow \quad \mathcal{Z} > -d_-$$

where $d_- := d_-(t, S_1(t), y_0 Z(t); K_U)$. Using $\mathbb{1}[A]$ to denote the zero-one indicator function on the set $A \subset \Omega$,

$$\begin{aligned}\left(\frac{S_1(t)}{S_{1,0}}\right)^{-1} c(t) =& \mathbb{E}_{\mathbb{Q}}\left(\left(y_0 Z(t) e^{\alpha\mathcal{Z} - \frac{1}{2}\alpha^2} - e^{-r_{\mathrm{R}}T} K_U\right) \cdot \mathbb{1}[\mathcal{Z} > -d_-] \,\middle|\, \mathcal{F}_t\right) \\ =& y_0 Z(t)\, \mathbb{E}_{\mathbb{Q}}\left(\left(e^{\alpha\mathcal{Z} - \frac{1}{2}\alpha^2}\right) \cdot \mathbb{1}[\mathcal{Z} > -d_-]\right) \\ & - e^{-r_{\mathrm{R}}T} K_U \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}[\mathcal{Z} > -d_-]\right).\end{aligned}$$

By integration over the probability density function of the standard normally-distributed random variable $\mathcal{Z}$ and substituting for $Y(t) = y_0 S_1(t) Z(t)$, we obtain

$$c(t) = Y(t)\Phi(d_+) - \frac{S_1(t)}{S_{1,0}} e^{-r_{\mathrm{R}}T} K_U \Phi(d_-).$$

It can be verified that $c(T) = \frac{S_1(T)}{S_{1,0}}\left[y_0 Z(T) - e^{-r_{\mathrm{R}}T} K_U\right]_+$.

To determine the replicating portfolio, first note t To replicate the payoff $Y(T)$, the fraction of $Y(t)$ in the nominal bond is A_1, the fraction in the nth risky stock is A_n for $n = 2, \ldots, D$ and in the ILB the fraction of $Y(t)$ is $1 - \sum_{n=1}^{D} A_n$. Standard arguments show that the portfolio detailed in the statement of the proposition gives a self-financing portfolio that replicates the payoff at time T. □

The next lemma values the payoff $\left[e^{-r_{\mathrm{R}}T} K_L - y_0 Z(T)\right]_+$ as the payoff on a European put option written on a specific portfolio.

Lemma 3.8. *Define*

$$p(t) := \frac{S_1(t)}{S_{1,0}} \mathbb{E}_{\mathbb{Q}} \left(\left[e^{-r_R T} K_L - y_0 Z(T) \right]_+ \middle| \mathcal{F}_t \right)$$

Then $p(t) \equiv p(t, S_1(t), Y(t); K_L)$ *with*

$$p(t,x,y;K_L) := x S_{1,0}^{-1} e^{-r_R T} K_L \Phi(-d_-(t;K_L)) - y\Phi(-d_+(t;K_L)), \quad x,y > 0 \text{ and } t \in [0,T],$$

where $Y(t)$ *and* $d_\pm(t;K_L) := d_\pm(t, S_1(t), Y(t); K_L)$ *are defined in Proposition 3.7.*

The replicating portfolio is to hold the amounts

- $-A_1 \cdot Y(t)\Phi(-d_+(t;K_L))$ *in the nominal bond,*
- $-A_n \cdot Y(t)\Phi(-d_+(t;K_L))$ *in nth risky stock, for* $n = 2, \ldots, D$, *and*
- $p(t) + Y(t)\Phi(-d_+(t;K_L)) \sum_{n=1}^{D} A_n$ *in the ILB,*

where $\{A_n\}_{n=1}^{D}$ *is defined in Proposition 3.7.*

Proof. Since evaluating the pricing function and determining the replicating portfolio follows the same argument as the proof of Proposition 3.7, the proof is omitted. □

3.A.2.4 Solution to the constrained problem

In this section, we determine the solution to Problem 4.

Proposition 3.9. *The wealth process corresponding to the optimal investment strategy is* $X^\star(t) = X^\star(t, S_1(t), Y(t); K_L, K_U)$ *for each* $t \in [0,T]$, *with*

$$\begin{aligned} X^\star(t) :=& X^\star(t, S_1(t), Y(t); K_L, K_U) \\ =& Y(t) - c\left(t, S_1(t), Y(t); K_U\right) + p\left(t, S_1(t), Y(t); K_L\right). \end{aligned}$$

with the functions c and p defined in Proposition 3.7 and Lemma 3.8, respectively, and the process $Y(t) = S_{1,0}^{-1} y_0 S_1(t) Z(t)$.

The optimal investment strategy that solves Problem 4 is to hold at time t, the amount

- $\pi_0^\star(t) := \Psi(t; (K_L, K_U)) A_1 Y(t)$ *in the nominal bond,*
- $\pi_n^\star(t) := \Psi(t; (K_L, K_U)) A_n Y(t)$ *in the nth risky asset for* $n = 2, \ldots, D$, *and*
- $\pi_1^\star(t) := X^\star(t) - \Psi(t; (K_L, K_U)) \sum_{n=1}^{D} A_n Y(t)$ *in the ILB,*

where

$$\Psi(t; (K_L, K_U)) := \Phi(-d_+(t; K_U)) - \Phi(-d_+(t; K_L)) \tag{3.29}$$

is an option moderation factor at time t, in which $d_\pm(t; K) = d_\pm(t, S_1(t), Y(t); K)$ *are defined in Proposition 3.7 and* $A_n := \frac{1}{1-\gamma} \left[(\tilde{\boldsymbol{\sigma}}')^{-1} \tilde{\boldsymbol{\theta}} \right]_n$ *for* $n = 1, 2, \ldots, D$, *for* $\tilde{\boldsymbol{\sigma}}$ *and* $\tilde{\boldsymbol{\theta}}$ *given by equations* (3.13) *and* (3.14), *respectively.*

Proof. The expression to evaluate for $X^\star(t)$ is equation (3.28). The first term,

$$\frac{S_1(t)}{S_{1,0}} \mathbb{E}_{\mathbb{Q}} \left(y_0 Z(T) \,|\, \mathcal{F}_t \right) = Y(t) \mathbb{E}_{\mathbb{Q}} \left(\frac{Z(T)}{Z(t)} \,\middle|\, \mathcal{F}_t \right) = Y(t),$$

the last equality holding since Z is an exponential martingale under $\mathbb{Q}$. The replicating portfolio for $Y(t)$ is as detailed in the proof of Proposition 3.7.

We have also found the values and replicating portfolios at time t (and the hedging portfolios) for the remaining two terms on the right-hand side of equation (3.28) in Proposition 3.7 and Lemma 3.8. Thus, since $S_1^{-1}(T) X^\star(T) = \tilde{X}^\star(T)$, a.s., as defined in Proposition 3.6, it is straightforward to check that the portfolio $\tilde{\mathbf{u}}^\star$ which results in $\tilde{X}^\star(T)$ is an admissible portfolio, $\tilde{\mathbf{u}}^\star \in \tilde{\mathcal{A}}(S_{1,0}^{-1} x_0)$. Further, as $\tilde{X}^{\tilde{\boldsymbol{\pi}}^\star}(T) \in [S_{1,0}^{-1} e^{-r_{\mathrm{R}} T} K_L, S_{1,0}^{-1} e^{-r_{\mathrm{R}} T} K_U]$, a.s., the portfolio $\tilde{\mathbf{u}}^\star$ is a solution to Problem 6. Hence by Lemma 3.5, $\boldsymbol{\pi}^\star$ is an optimal solution for Problem 4. $\square$

3.A.3 Expected utility

Lemma 3.10. *The expected utility of the optimal real retirement wealth without terminal wealth constraints is given by*

$$\mathbb{E}\left(U\left(\frac{I_0}{I(T)}X^{\mathbf{u}^\star}(T)\right)\right) = \frac{1}{\gamma}x_0^\gamma \exp\left(\gamma\left(r_R + \frac{1}{2}\frac{1}{1-\gamma}\|\tilde{\boldsymbol{\theta}}\|^2\right)T\right), \tag{3.30}$$

and the expected utility of the optimal real retirement wealth with terminal wealth constraints is given by

$$\begin{aligned}
&\mathbb{E}\left(U\left(\frac{I_0}{I(T)}X^\star(T)\right)\right) = \frac{1}{\gamma}K_U^\gamma\Phi\left(\eta(K_U)\right) \\
&+ \frac{1}{\gamma}y_0^\gamma e^{\gamma\left(r_R+\frac{1}{2}\frac{1}{1-\gamma}\|\tilde{\boldsymbol{\theta}}\|^2\right)T}\left[\Phi\left(\eta(K_L) + \frac{\gamma\|\tilde{\boldsymbol{\theta}}\|}{1-\gamma}\sqrt{T}\right) - \Phi\left(\eta(K_U) + \frac{\gamma\|\tilde{\boldsymbol{\theta}}\|}{1-\gamma}\sqrt{T}\right)\right] \\
&+ \frac{1}{\gamma}K_L^\gamma\Phi\left(-\eta(K_L)\right).
\end{aligned} \tag{3.31}$$

Proof. From Proposition 3.2, $\tilde{X}^{\tilde{\mathbf{u}}^\star}(T) = x_0 S_{1,0}^{-1}Z(T)$. As $X^{\mathbf{u}^\star}(T) = S_1(T)\tilde{X}^{\tilde{\mathbf{u}}^\star}(T)$ and $I_0 I^{-1}(T) = S_{1,0}e^{r_{\mathrm{R}}T}S_1^{-1}(T)$,

$$\mathbb{E}\left(U\left(\frac{I_0}{I(T)}X^{\mathbf{u}^\star}(T)\right)\right) = \mathbb{E}\left(U\left(S_{1,0}e^{r_{\mathrm{R}}T}S_1^{-1}(T)S_1(T)\tilde{X}^{\tilde{\mathbf{u}}^\star}(T)\right)\right) = \mathbb{E}\left(U\left(x_0 e^{r_{\mathrm{R}}T}Z(T)\right)\right)$$

Applying the definition of $Z(T)$ from equation (3.18) results in

$$\mathbb{E}\left(U\left(\frac{I_0}{I(T)}X^{\mathbf{u}^\star}(T)\right)\right) = \mathbb{E}\left(U\left(x_0 e^{r_{\mathrm{R}}T}e^{\ln(Z(T))}\right)\right) = \mathbb{E}\left(U\left(x_0 e^{r_{\mathrm{R}}T}e^{\lambda T+\delta\sqrt{T}\Upsilon}\right)\right)$$

in which

$$\Upsilon := \frac{\tilde{\boldsymbol{\theta}}'\mathbf{W}(T)}{\|\tilde{\boldsymbol{\theta}}\|\sqrt{T}} \sim \mathcal{N}(0,1) \text{ under } \mathbb{P},$$
$$\lambda := \frac{1-2\gamma}{2(1-\gamma)^2}\|\tilde{\boldsymbol{\theta}}\|^2, \text{ and } \delta^2 := \frac{1}{(1-\gamma)^2}\|\tilde{\boldsymbol{\theta}}\|^2.$$

Hence,

$$\begin{aligned}\mathbb{E}\left(U\left(\frac{I_0}{I(T)}X^{\mathbf{u}^\star}(T)\right)\right) &= \frac{1}{\gamma}x_0^\gamma e^{\gamma(r_{\mathrm{R}}+\lambda)T}\mathbb{E}\left(\left(e^{\gamma\delta\sqrt{T}\Upsilon}\right)\right)\\ &= \frac{1}{\gamma}x_0^\gamma e^{\gamma(r_{\mathrm{R}}+\lambda)T+\frac{1}{2}\gamma^2\delta^2 T}\\ &= \frac{1}{\gamma}x_0^\gamma e^{\gamma\left(r_{\mathrm{R}}+\frac{1}{2}\frac{1}{1-\gamma}\|\tilde{\boldsymbol{\theta}}\|^2\right)T}.\end{aligned}$$

Similarly, from Proposition 3.6,

$$\begin{aligned}\frac{I_0}{I(T)}X^\star(T) &= \begin{cases} K_U & \text{if } y_0 Z(T) \geq e^{-r_{\mathrm{R}}T}K_U \\ y_0 e^{r_{\mathrm{R}}T}Z(T) & \text{if } y_0 Z(T) \in [e^{-r_{\mathrm{R}}T}K_L, e^{-r_{\mathrm{R}}T}K_U] \\ K_L & \text{if } y_0 Z(T) \leq e^{-r_{\mathrm{R}}T}K_L. \end{cases}\\ &= \begin{cases} K_U & \text{if } \Upsilon \geq -\eta(K_U) \\ y_0 e^{r_{\mathrm{R}}T}e^{\lambda+\delta\Upsilon} & \text{if } \Upsilon \in [-\eta(K_L), -\eta(K_U)] \\ K_L & \text{if } \Upsilon \leq -\eta(K_L). \end{cases}\end{aligned}$$

in which η is a real-valued function

$$\eta(K) := \frac{\ln(y_0/K) + \left(r_{\mathrm{R}} + \frac{1-2\gamma}{2(1-\gamma)^2}\|\tilde{\boldsymbol{\theta}}\|^2\right)T}{\|\tilde{\boldsymbol{\theta}}\|\sqrt{T}/(1-\gamma)}, \quad \forall K > 0.$$

Hence,

$$
\begin{aligned}
&\mathbb{E}\left(U\left(\frac{I_0}{I(T)}X^\star(T)\right)\right)\\
=&\mathbb{E}\left(U\left(K_U\right)\cdot\mathbb{1}\left[\Upsilon\geq-\eta(K_U)\right]\right)+\mathbb{E}\left(U\left(K_L\right)\cdot\mathbb{1}\left[\Upsilon\leq-\eta(K_L)\right]\right)\\
&+\mathbb{E}\left(U\left(y_0e^{r_{\mathrm{R}}T}e^{\lambda T+\delta\sqrt{T}\Upsilon}\right)\cdot\mathbb{1}\left[-\eta(K_L)<\Upsilon<-\eta(K_U)\right]\right)\\
=&U(K_U)\mathbb{P}(\Upsilon\geq-\eta(K_U))+U(K_L)\mathbb{P}(\Upsilon\leq-\eta(K_L))\\
&+U\left(y_0e^{(r_{\mathrm{R}}+\lambda)T}\right)\mathbb{E}\left(e^{\gamma\delta\sqrt{T}\Upsilon}\cdot\mathbb{1}\left[-\eta(K_L)<\Upsilon<-\eta(K_U)\right]\right)\\
=&\frac{1}{\gamma}K_U^\gamma\mathbb{P}(\Upsilon\geq-\eta(K_U))+\frac{1}{\gamma}K_L^\gamma\mathbb{P}(\Upsilon\leq-\eta(K_L))\\
&+\frac{1}{\gamma}y_0^\gamma e^{\gamma(r_{\mathrm{R}}+\lambda)T}e^{\frac{1}{2}\gamma^2\delta^2T}\mathbb{E}\left(\mathbb{1}\left[-\eta(K_L)<\Upsilon+\gamma\delta\sqrt{T}<-\eta(K_U)\right]\right)\\
=&\frac{1}{\gamma}K_U^\gamma\Phi\left(\eta(K_U)\right)+\frac{1}{\gamma}K_L^\gamma\Phi\left(-\eta(K_L)\right)\\
&+\frac{1}{\gamma}y_0^\gamma e^{\gamma\left(r_{\mathrm{R}}+\frac{1}{2}\frac{1}{1-\gamma}\|\tilde{\boldsymbol{\theta}}\|^2\right)T}\left[\Phi\left(\eta(K_L)+\frac{\gamma\|\tilde{\boldsymbol{\theta}}\|}{1-\gamma}\sqrt{T}\right)-\Phi\left(\eta(K_U)+\frac{\gamma\|\tilde{\boldsymbol{\theta}}\|}{1-\gamma}\sqrt{T}\right)\right].
\end{aligned}
$$

□

3.A.4 Real quantiles

Definition 3.11. The p-quantile of a real terminal wealth at time T is

$$\mathcal{Q}_p(\mathcal{X})=\inf\{y\in\mathbb{R}:\mathbb{P}[I_0I^{-1}(T)X(T)\leq y]\geq p\},$$

with the convention that $\inf\{\emptyset\}=\infty$.

Let $\mathcal{X}(x_0,\boldsymbol{\pi},K_L,K_U)$ denote the real terminal wealth for an investor with initial wealth x_0, investment strategy $\boldsymbol{\pi}$, a real lower constraint K_L and a real upper constraint K_U (Problem 4). Then, $\mathcal{X}(x_0,\mathbf{u}^\star,0,\infty)$ is the optimal wealth for the unconstrained investor (Problem 3).

Proposition 3.12. *Suppose an investor has initial wealth $x_0>0$. Fix $p\in(0,1)$ and define the constants:*

$$\beta_p:=\frac{1}{1-\gamma}\|\tilde{\boldsymbol{\theta}}\|\sqrt{T}\Phi^{-1}(p)+\left(r_R+\frac{1-2\gamma}{2(1-\gamma)^2}\|\tilde{\boldsymbol{\theta}}\|^2\right)T, \tag{3.32}$$

and

$$\beta_p^\dagger := \|\boldsymbol{\nu}\|\sqrt{T}\Phi^{-1}(p) + \left(\frac{\gamma(\mu_1 - r_N)}{1-\gamma} - r_R + \sigma_I^2 + \frac{\|\tilde{\boldsymbol{\theta}}\|^2}{1-\gamma}\right)T, \tag{3.33}$$

in which $\boldsymbol{\nu} := \frac{1}{1-\gamma}\boldsymbol{\theta} - \boldsymbol{\sigma}_1$.

If the investor follows the optimal unconstrained strategy ($\mathbf{u}^\star$)*, then the p-quantile of the investor's real terminal wealth* $\mathcal{X}(x_0, \mathbf{u}^\star, 0, \infty)$ *is*

$$\mathcal{Q}_p(\mathcal{X}(x_0, \mathbf{u}^\star, 0, \infty)) = x_0 e^{\beta_p}. \tag{3.34}$$

If the investor follows the optimal constrained strategy ($\boldsymbol{\pi}^\star$) *with a lower constraint* (K_L) *and an upper constraint* (K_U) *then the p-quantile of the investor's real terminal wealth* $\mathcal{X}(x_0, \boldsymbol{\pi}^\star, K_L, K_U)$ *is*

$$\mathcal{Q}_p(\mathcal{X}(x_0, \boldsymbol{\pi}^\star, K_L, K_U)) = \max\{K_L, \min\{K_U, y_0 e^{\beta_p}\}\}. \tag{3.35}$$

If the investor follows the optimal nominal unconstrained strategy ($\mathbf{u}^\dagger$)*, then the p-quantile of the investor's real terminal wealth* $\mathcal{X}(x_0, \mathbf{u}^\dagger, 0, \infty)$ *is*

$$\mathcal{Q}_p(\mathcal{X}(x_0, \mathbf{u}^\dagger, 0, \infty)) = x_0 e^{\beta_p^\dagger}. \tag{3.36}$$

Proof. The proof is an adaptation of Donnelly et al. (2015, Lemma 4.9). We begin by considering the first investor who does not have any terminal wealth constraints. Since there is no probability mass for the unconstrained terminal wealth,

$$p = \mathbb{P}\left[y_0 e^{r_{\mathrm{R}}T} Z(T) \leq \mathcal{Q}_q(\mathcal{X}(x_0, \mathbf{u}^\star, 0, \infty))\right].$$

Substituting for $Z(T)$ from equation (3.18), using the fact that $\tilde{\boldsymbol{\theta}}'\mathbf{W}(T)/(\|\tilde{\boldsymbol{\theta}}\|\sqrt{T}) \sim \mathcal{N}(0,1)$ under $\mathbb{P}$, we obtain the desired expression (3.34).

Next, the second investor with a lower and an upper wealth constraint has a real terminal wealth $\mathcal{X}(x_0, \boldsymbol{\pi}^\star, K_L, K_U)$. The q-quantile of their real terminal wealth is determined as

$$\mathcal{Q}_p(\mathcal{X}(x_0, \boldsymbol{\pi}^\star, K_L, K_U)) = \inf\{y \in \mathbb{R} : \mathbb{P}[\mathcal{X}(x_0, \boldsymbol{\pi}^\star, K_L, K_U) \leq y] \geq p\}.$$

From Proposition 3.6,

$$\mathcal{X}(x_0, \boldsymbol{\pi}^\star, K_L, K_U) = \begin{cases} K_U & \text{if } \mathcal{X}(y_0, \mathbf{u}^\star, 0, \infty) \geq K_U \\ \mathcal{X}(y_0, \mathbf{u}^\star, 0, \infty) & \text{if } \mathcal{X}(y_0, \mathbf{u}^\star, 0, \infty) \in [K_L, K_U] \\ K_L & \text{if } \mathcal{X}(y_0, \mathbf{u}^\star, 0, \infty) \leq K_L. \end{cases}$$

The desired expression (3.35) follows by consideration of the last expression.

Lastly, substituting into equation (3.4) for the investment strategy $\mathbf{u}^\dagger$ given by equation (3.6) we obtain the following wealth dynamics

$$\mathrm{d}(I^{-1}X^{\mathbf{u}^\dagger})(t) = (I^{-1}X^{\mathbf{u}^\dagger})(t) \left(\left[\frac{\gamma(\mu_1 - r_\mathrm{N})}{1-\gamma} - r_\mathrm{R} + \sigma_I^2 + \frac{\|\tilde{\boldsymbol{\theta}}\|^2}{1-\gamma} \right] \mathrm{d}t + \boldsymbol{\nu}' \, \mathrm{d}\mathbf{W}(t) \right).$$

Using the fact that $\boldsymbol{\nu}'\mathbf{W}(T)/(\|\boldsymbol{\nu}\|\sqrt{T}) \sim \mathcal{N}(0,1)$ under $\mathbb{P}$, we obtained the desired expression (3.36). □

Chapter 4

Investing for retirement: Terminal wealth constraints or a desired wealth target?

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Abstract

We investigate how well different investment strategies can give pre-retirees more certainty about their income in retirement, whilst allowing them to benefit from taking investment risk. Under an expected utility-maximizing framework, we find that a loss aversion utility function gives a high degree of certainty about its desired wealth target and is robust to different market models. Imposing terminal wealth constraints does not improve the certainty of achieving the

desired target enough to counter-balance the increased chance of obtaining a lower income. The power utility function is not robust to different market models and becomes too risk-averse with wealth constraints.

4.1 Introduction

How to invest one's pension savings before retirement is a decision faced by defined-contribution (DC) pension plan members. A typical DC pension plan member both saves and invests to build up a pension fund before retirement, from which they withdraw an income in retirement. The member's DC fund value at retirement can be volatile as it depends on several factors in the savings phase, such as their salary, the amount and frequency of contributions, and, how the funds are invested. To cite Merton (2014, pp. 1402), the primary concern of the pre-retiree is 'Will I have sufficient income in retirement to live comfortably?'. This paper studies whether the investor is better off with investment strategies that constrain their retirement income, or equivalently fund value accumulated after the savings phase, so as to protect them against extreme (negative) scenarios of income in retirement.

We compare several ways of formulating the investor's problem, by allowing for different utility functions and terminal wealth boundary constraints and deriving optimal investment strategies through the maximization of expected utility. Using these optimal investment strategies, we calculate, analyze and compare the distributions of income at retirement. Our findings show that a loss aversion utility function gives a very attractive retirement income distribution, with the distribution peaked at the investor's chosen income goal with some level of robustness. In contrast, risk preferences expressed via a constant relative risk aversion (CRRA) utility function give a much more spread out income distribution, providing the investor less certainty on achieving a sufficient level of income in retirement. Imposing terminal wealth boundary constraints, in both the utility function settings, result in strategies that provide certainty of achieving the lower boundary but at the expense of significant reduction in the overall retirement outcome. We conclude that the investor can benefit from following a loss aversion-derived optimal investment strategy to target a sufficient level of income at retirement.

The investment problem with terminal wealth constraints at retirement has been extensively studied in the utility literature, in particular, a lower boundary constraint that guarantees the investor a minimum level of wealth at the cost of reducing the possible upside. For example, (Korn, 2005) and Kraft and Steffensen (2013) consider a static lower constraint while Teplá (2001) and Han and Hung (2012) consider minimum performance relative to a benchmark strategy. On the other hand, Donnelly et al. (2015) demonstrate the benefit of an upper constraint on terminal wealth in reducing the risk of poor retirement outcomes at the expense of giving up the possibility of higher returns. The merits of an upside constraint, in the form of capped investment earning rates, is also discussed by Mahayni and Schneider (2015). Schütte (2017) and Donnelly et al. (2018) consider outcomes with both upper and lower constraints on the terminal wealth based on an exponential utility function and a power utility function, respectively.

As evidenced by Kahneman and Tversky (1979) and Tversky and Kahneman (1992), investors tend to evaluate outcomes relative to a reference level. Several authors use loss aversion utility functions in investment problems pertinent to either DC pension plan members or plan managers (Berkelaar et al., 2004; Jin and Zhou, 2008; He and Zhou, 2011; Blake et al., 2013; Guan and Liang, 2016; Lin et al., 2017; Dong et al., 2020), and in conjuction with a lower terminal wealth constraint (Chen et al., 2017; Dong and Zheng, 2019).

However, the existing research has a number of limitations. There is a tendency to assume a wealth constraint that is either deterministic or a stochastic variable pegged to the labor income process. The literature with terminal constraints assume the labor income process to be either be a deterministic process or a tradable stochastic process where both can be replicated by a portfolio of risk-free and risky assets. Moreover, solving terminal wealth constraints often necessitate option-like payoffs as the wealth constraints are occasionally binding. To this end, the literature prescribes the investor to adopt option replicating strategies. While the above assumptions are required to construct an explicit mathematical solution, they may not be practical. Most investors have the ability to neither short-sell assets, invest more than their total wealth in the risky asset nor ability to trade continuously, which are all critical requirements in implementing these solutions.

This paper contributes to the literature in a number of ways. First, we extend the literature by incorporating a stochastic non-tradable labor income process which governs how both the pre-retirement investor's salary and the terminal wealth constraints evolve. The investor makes contributions during their working years which are invested in line with an optimal investment strategy. The optimal strategy is one which maximizes the expected value of the utility of retirement income. Two forms of the utility function are considered: power utility (i.e. constant reltive risk aversion, or CRRA risk preferences) and loss aversion, and these are further considered in the presence of upper and lower terminal wealth constraints. To the best of our knowledge, this comparison has not been done before.

Second, we model inflation explicitly and study an investment market comprising of a nominal bond, an inflation-linked bond and a risky stock. Inflation is an important consideration for retirement planning and investment, since the time scales are generally over decades. Third, for greater realism, we allow the investors to only trade in short-term option contracts instead of long-term contracts or invest in a replicating strategy. It is not likely that long-term traded option contracts would be available in the market, and neither is it realistic to expect investors to construct an over-the-counter version or to do a replicating portfolio version. Finally, we compare the performance of the investment strategies both under the standard financial market model (geometric Brownian motion driven) from which the strategies are devised, and under historical simulations based on the financial market returns in the UK from December 1918 to December 2019.

We find that for both utility functions, adding in boundary constraints on the fund levels does not materially impact the asset allocation. The exception is when the fund level approaches the boundary constraints at either end. In that case the resultant strategies shift allocations from risky assets to the risk-free asset, to keep the fund level within the boundary constraints. Boundary constraints help improve the certainty of retirement income but do so at the expense of reducing the average retirement income. This is a consequence of reduced exposure to risky stock across all the possible paths of the market prices.

Second, we find that, under the standard financial market model, there is only a weak incentive for a CRRA investor to include boundary constraints. This is because the downside protection rarely comes into effect and the resultant strategies provide little increased certainty to the level of retirement income. Nonetheless, imposing boundary constraints remain a viable approach in preventing investor from achieving very poor retirement outcomes. It enables the CRRA investor to specify and maintain a minimum sufficient level of income at retirement that cannot be easily taken into account alone by the risk aversion parameter.

Compared to the CRRA risk preference, we find that there is an even weaker incentive for a loss aversion investor to include boundary constraints. With a reference level, the loss aversion risk preference enables the investor to target a sufficient level of retirement income without needing to impose boundary constraints. By adjusting the risk exposure throughout the investment phase, which is a feature of the loss aversion utility function, the investor may achieve their target retirement income. However, adding in the boundary constraints induces competing objectives, reduces allocation to the risky stock throughout the savings phase, and ultimately lowers the investor's chance of achieving their target.

Finally, we find that the optimal investment strategy derived from the loss aversion utility function appears more robust to a misspecification of the market model, compared to the that derived from CRRA preferences. The general shape of the retirement income distribution is largely preserved under the loss aversion derived strategy while the CRRA derived distribution changes, to the detriment of the investor.

Overall, our results suggest that an investor planning for their retirement, is better off following the loss aversion-derived optimal investment strategy without constraints on the utility function value at retirement. The remainder of the paper is structured as follows: in Section 4.2 the formulation and solution of the investor's problem are detailed; Section 4.3 discusses the potential outcomes resulting from the various investment strategies from Section 4.2; and, Section 4.4 provides further discussion and concludes the paper.

4.2 The investor's problems

The focus of our study is the distribution of an investor's real replacement ratio (RRR) at retirement. The RRR is the ratio of the annual real income bought at retirement to the annual real salary in the year before retirement. The advantage of using the replacement ratio is that it is more meaningful to the investor, as it tells them the percentage of the pre-retirement salary that they can expect post-retirement and helps them in deciding their desired level of retirement income sufficiency. All real values are with respect to the purchasing power at time 0 when the investor is age 25. This avoids the investor having to do an inflation projection from age 25 to age 65 to determine the future purchasing power of their retirement income.

The constraints of no-short-selling of any asset, no borrowing against future income and discrete, annual re-balancing are included. Their inclusion necessitates the use of numerical methods to determine an investment strategy that maximizes the expected utility of retirement income. Additionally, we use interchangeably the terms "real fund value at retirement" and "real retirement income". In our model, the accumulated fund value at retirement is used to purchase a life annuity. The life annuity has the same value in all future states of the world; see Eq. (4.6).

4.2.1 The financial market model

There are three asset classes available for investment. The uncertainty in the market is represented by a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ that supports a two-dimensional Brownian motion $W(t) = (W_1(t), W_2(t))'$, where $W_1(t)$ is independent of $W_2(t)$ and the prime denotes transposition. Asset 0 is a nominal bond with price process S_0 satisfying

$$dS_0(t)/S_0(t) = r_{\rm N}dt, \qquad S_0(0) = 1, a.s. \tag{4.1}$$

The constant $r_{\rm N}$ is the annual nominal interest rate. The price of Asset 1, an inflation-linked bond, is driven by a price inflation index I with dynamics

$$dI(t)/I(t) = \mu_I dt + \sigma_I dW_1(t), \qquad I(0) = I_0 = 1, a.s, \tag{4.2}$$

with the constant μ_I representing the mean return on the index. The price process of Asset 1, S_1, satisfies

$$dS_1(t)/S_1(t) = (r_{\mathrm{R}} + \mu_I)dt + \sigma_I dW_1(t), \qquad S_1(0) = 1, a.s, \tag{4.3}$$

with the constant r_{R} denoting the real rate of return on the inflation-linked bond.

Asset 2 is a risky stock with price process S_2 with dynamics

$$dS_2(t)/S_2(t) = \mu_2 dt + \sigma_{21} dW_1(t) + \sigma_{22} dW_2(t), \qquad S_2(0) = 1, a.s, \tag{4.4}$$

with the constant μ_2 is the mean return on the stock, $\sigma_{21} > 0$ is its correlation with the price inflation index and $\sigma_{22} > 0$ is its correlation with other, non-modelled, random factors.

The expectation operator under $\mathbb{P}$ is denoted $\mathbb{E}$.

4.2.2 Labor income

We model the labor income of the investor using the labor income structure seen in Cairns et al. (2006) and Blake et al. (2013) in a discretized setting. Specifically, we use a trinomial tree lattice to model the growth in labor income from integer time t to $t+1$ (Boyle, 1988; Hull, 2019). Figure 4.1 illustrates the discretized income labor process.

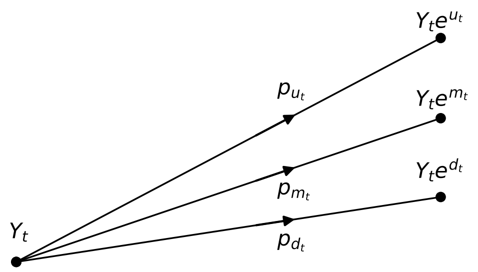

FIGURE 4.1: **Illustrative example of a labor income process**
This figure illustrates of the labor income process modelled using a trinomial tree (not to scale). For each time step, the labor income moves from Y_t to one of the three values, $Y_t e^{u_t}$, $Y_t e^{m_t}$ and $Y_t e^{d_t}$ with probabilities p_{u_t}, p_{m_t} and p_{d_t}, respectively.

Consider an investor, aged 25 at time zero, who is saving for their retirement at age 65 (i.e. at time $T = 40$). As with Blake et al. (2013), the labor income of the investor has an age-dependent mean growth rate of $\mu_{L,t}$, whose components we detail shortly, and a constant volatility of σ_L. His labor income at the integer time t, when he is age $t+25$, is given by Y_t. In our discretized setting, the labor income is determined every year so that there is a 1-year time step in the trinomial tree, i.e. $\Delta t = 1$. From time t to $t+1$, the labor income moves from Y_t to one of the three values, $Y_t e^{u_t}$, $Y_t e^{m_t}$, and $Y_t e^{d_t}$. The movement from Y_t to $Y_t e^{u_t}$, $Y_t e^{m_t}$ and $Y_t e^{d_t}$ are "up", "middle" and "down" movements, respectively. We choose the age-dependent movement factors $u_t = \mu_{L,t} + \sigma_L\sqrt{3\Delta t}$, $m_t = \mu_{L,t}$, and $d_t = \mu_{L,t} - \sigma_L\sqrt{3\Delta t}$. The mean growth rate is given by:

$$\mu_{L,t} = r_L + (P_{t+25+1} - P_{t+25})/P_{t+25},$$

in which r_L represents the long-term annual real growth rate in national earnings, P_{t+25} is the career salary profile at age $t+25$ such that $(P_{t+25+1} - P_{t+25})/P_{t+25}$ reflects the promotional salary increase from time t to time $t+1$. Let p_{u_t}, p_{m_t} and p_{d_t} denote the probabilities of up, middle and down movements at each time, we choose the value of these probabilities so that the mean and standard deviation of the change in labor income in the discrete process matches $\mu_{L,t}$ and σ_L, respectively. Finally, we adopt the following quadratic function to model the salary profile P_x, at integer age $x = 25, \ldots, 65$:

$$P_x = 1 + h_1\left[\frac{(x-20)}{65-20} - 1\right] + h_2\left[-1 + \frac{4(x-20)}{65-20} - \left(\frac{\sqrt{3}(x-20)}{65-20}\right)^2\right],$$

where the parameters h_1 and h_2 are estimated from the historical data on promotional salary increases of UK male salary data from ages 20 to 65 (see Blake et al. 2007, 2013).

4.2.3 Fund accumulation

The fund value of the investor at time t is denoted by the random variable F_t. Starting at age 25 ($t = 0$), the investor puts a constant proportion $\pi \in (0, 1]$ of their real labor income into the retirement fund at each integer time. For each integer year, the value F_{t-} represents the fund value at time t just *before* a lump-sum contribution is paid into the fund, and the value F_t

represents the fund value at time t just *after* a lump-sum contribution is paid with

$$F_t = F_{t-} + \pi Y_t.$$

The investor decides upon their investment strategy once a year, immediately after their annual lump-sum contribution to their pension fund. The strategy is effectively a one-year buy-and-hold strategy until the following year's new contribution and new investment strategy are implemented. Short-selling of any assets and borrowing against future labor income are prohibited.

The investment decision made at time t is represented by the random vector $\boldsymbol{\theta}_t = \left(\theta_t^0, \theta_t^1, \theta_t^2\right)'$, in which the random variable θ_t^i represents the proportion of the fund value F_t invested in asset i, for $i \in \{0, 1, 2\}$, and the sum of the proportions satisfies $\sum_{i=0}^{2} \theta_t^i = 1$, for every $t \in \{0, 1, \ldots, T-1\}$. Furthermore, due to the no-short-selling constraint on all assets, $\theta_t^i \in [0, 1]$ for every $t \in \{0, 1, \ldots, T-1\}$ and $i \in \{0, 1, 2\}$.

At each integer time $t < T$, the investor makes a lump-sum contribution to the fund. As the investor implements a one-year buy-and-hold strategy at the start of each year, the real fund value of the investor increases from the value F_t at time $t \in \{0, 1, \ldots, T-1\}$ to

$$F_{t+1} = (F_{t-} + \pi Y_t) \times \sum_{i=0}^{2} \theta_t^i \frac{S_i(t+1)}{S_i(t)} \frac{I(t)}{I(t+1)} \tag{4.5}$$

at integer time $t+1$.

At the terminal time T, the investor uses their accumulated retirement fund value, F_T, to buy a single life annuity whose income increases in line with the price inflation index. We assume that the underlying investment for the inflation-indexed life annuity is the inflation-linked bond. The single life annuity pays an annual income in advance, with the first income payment due at time T, until the investor dies. The expected cost of purchasing at time T (i.e., age 65) an inflation-indexed annuity income paid annually in advance for life, with the first payment being

£1 per annum at age 65, is

$$\ddot{a}_{65} = \mathbb{E}\left(\sum_{k=0}^{x_{\max}} {}_kp_{65}\frac{S_1(T)}{S_1(T+k)}\frac{I(T+k)}{I(T)}\right) = \sum_{k=0}^{x_{\max}} {}_kp_{65}e^{-r_{\mathrm{R}}k}, \tag{4.6}$$

in which $x_{\max} < \infty$ is the maximum attainable age by the investor and ${}_kp_{65}$ is the real-world probability that the investor survives from age 65 to age $65 + k$. Thus, the investor can buy, at age 65, an annual real annuity income of $F_T/\ddot{a}_{65}$ to last for their retirement, with the annuity income increasing in line with the inflation index I.

4.2.4 Risk preferences and the problems to solve

The risk preferences of the investor are modelled by a utility function υ. The utility function is either a power utility one, based on Expected Utility Theory, or a loss aversion utility function, based on Cumulative Prospect Theory. The general goal of the investor is to find an investment strategy that maximizes the expected utility of their real retirement income.

4.2.4.1 Power utility investor with no retirement fund value constraints

First, assume the investor's risk preferences follows a power utility function

$$\upsilon_1(x) = x^\gamma/\gamma, \qquad \text{for } x > 0. \tag{4.7}$$

The constant $\gamma \in (-\infty, 1) \setminus \{0\}$ represents the relative risk aversion of the investor.

The investor's problem is to determine the one-year buy-and-hold investment strategies $\{\boldsymbol{\theta}_t\}_{t=0,1,\ldots,T-1}$ which solve

$$\max_{\{\boldsymbol{\theta}_t\}_{t=0}^{T-1}} \mathbb{E}\left[\beta^{T-t}\upsilon_1(F_T)\right], \tag{4.8}$$

in which $\beta \in (0, 1]$ is the investor's personal discount factor.

Numerical dynamic programming is used to derive an optimal investment strategy. With $V_t(\cdot)$ denoting the value function at time t, these optimal proportions are chosen to achieve the maximum in the Bellman equation

$$V_t(F_t, Y_t) = \max_{\boldsymbol{\theta}_t \in [0,1]^3} \left(\beta \, \mathbb{E}\left[V_{t+1}(F_{t+1}, Y_{t+1})\right] | \mathcal{F}_t\right), \quad \text{for } t = 0, 1, \ldots, T-1, \tag{4.9}$$
$$\text{with terminal value } V_T(F_T, Y_T) = \upsilon_1(F_T),$$

subject to budget constraint (4.5). From the terminal value, we work backwards to find an optimal investment strategy.

4.2.4.2 Loss aversion utility investor with no retirement fund value constraints

The next utility function considered is a loss aversion utility function. Such utility functions arise from the prospect theory framework, which was proposed and developed in Kahneman and Tversky (1979) and Tversky and Kahneman (1992). The framework features an S-shape loss aversion utility function and a probability weighting function. The former models the preferences of investors who measure losses and gains relative to a reference point and are risk-seeking with respect to losses but risk-averse with respect to gains. The latter applies a non-linear transformation to probability measure which inflates a small probability and deflates a large probability. The loss aversion utility is arguably more realistic than an Expected Utility Theory-based utility function as the loss aversion utility reflects the risk-seeking behaviour of investors when they have made a loss with respect to their reference point (Mitchell et al., 2004). The problem of maximizing terminal wealth from the perspective of individual investors under a loss aversion utility function has been studied by various authors (for example, Berkelaar et al., 2004; Blake et al., 2013; Chen et al., 2017; Dong and Zheng, 2019). Jin and Zhou (2008), He and Zhou (2011), and van Bilsen and Laeven (2020) study the implications of loss aversion with a probability weighting function. In this study, we adopt the modelling framework of Blake et al. (2013) as they tackled the investor's problem in a discrete-time rebalancing setting that is similar to ours.

In the framework of Blake et al. (2013), which we apply here, the investor has a target value that they would like their fund value to achieve at their retirement date, i.e., the level sufficient to support their desired retirement. The investor maximizes the weighted sum of the expected utility gained from gains and losses about a series of interim targets as well as the retirement date target. The idea of the interim targets is to keep the investor on track for the retirement date (final) target. The loss aversion utility function is given by

$$\upsilon_2(F,K) = \begin{cases} \frac{1}{\nu_1}(F-K)^{\nu_1}\,, & \text{if } F \geq K, \\ -\frac{\lambda}{\nu_2}(K-F)^{\nu_2}\,, & \text{if } F < K, \end{cases} \tag{4.10}$$

where F is the fund level, K is the target level, ν_1 is the degree of risk-aversion to gains, ν_2 is the degree of risk-seeking with respect to losses, and λ is the relative weighting of losses compared to gain.

Given an income of Y_t at time t, the final target, $K_k(T, Y_t)$, is the expected fund level at time T required to achieve a replacement ratio of k; and the interim target at time $s \geq t$, $K_k(s, Y_t)$, is the final target less the expected values of all future contributions to the fund from time s to time T, discounted to time s. Mathematically,

$$K_k(s,Y_t) = \begin{cases} \mathbb{E}[Y_T|\mathcal{F}_t] \times \ddot{a}_{65} \times k, & \text{for } s = T, \\ K_k(T,Y_t)e^{-r_{\mathrm{R}}(T-s)} - \pi \sum_{j=s}^{T-1} \mathbb{E}[Y_j|\mathcal{F}_t]e^{-r_{\mathrm{R}}(T-1-j)}, & \text{for } s = t, t+1, \ldots, T-1. \end{cases} \tag{4.11}$$

The investor's problem is to determine the one-year buy-and-hold investment strategies $\{\boldsymbol{\theta}_t\}_{t=0,1,\ldots,T-1}$ which solve

$$\max_{\{\boldsymbol{\theta}_t\}_{t=0}^{T-1}} \mathbb{E}\left[\omega\left(\sum_{s=0}^{T-1}\beta^s\upsilon_2\big(F_t, K_k(s,Y_t)\big)\right) + \beta^{T-t}\upsilon_2\big(F_T, K_k(t,Y_T)\big)\right], \tag{4.12}$$

in which $\omega \in [0,1)$ is the weighting factor applied to the utility derived from the interim targets, and $\beta \in (0,1]$ is the investor's personal discount factor. The choice of $\omega < 1$ reflects the lower weight given to the interim targets compared to the retirement date target, which has an implicit weight of unity.

Denoting by $V_t(\cdot)$ the value function at time t, the investor's problem is solved using the Bellman equation

$$V_t(F_t, Y_t) = \max_{\boldsymbol{\theta}_t \in [0,1]^3} \left(\omega\, \upsilon_2\big(F_t, K_k(t, Y_t)\big) + \beta\, \mathbb{E}\left[V_{t+1}(F_{t+1}, Y_{t+1})|\mathcal{F}_t\right]\right), \quad \text{for } t = 0, 1, \ldots, T-1,$$
$$\text{with terminal value } V_T(F_T, Y_T) = \upsilon_2\big(F_T, K_k(T, Y_T)\big), \tag{4.13}$$

subject to budget constraint (4.5). From the terminal value, we work backwards to find the optimal investment strategies.

4.2.5 Adding in both lower and upper fund value boundary constraints

Here, we describe how fund value boundary constraints that apply only at retirement are incorporated into the problems detailed in Section 4.2.4 in order to answer one of our research questions: how does applying constraints on the fund value at retirement change the distribution of the real replacement ratio? Note, they are in addition to the no-short-selling and no borrowing constraints. We implement them using options.

First, the investor chooses two values, $l \geq 0$ and $u \geq l$, guided by their minimum and maximum levels of income sufficiency, between which they would like their replacement ratio at retirement to lie. Then the corresponding fund value boundary constraints at the time of retirement, $K_l(T, Y_t)$ and $K_u(T, Y_t)$, respectively (see Eqs 4.15-4.14), are calculated. The fund value boundary constraints captures additional risk preferences of the investors that are not accounted for in the utility functions. The upper bound represents the level beyond which the investor forgoes to increase the lower quantiles of the retirement fund value (Donnelly et al., 2015). The lower bound is the minimum that the investor needs at retirement, to secure a minimal sufficient income in retirement. This is the same interpretation as in Donnelly et al. (2018).

The implementation of the investment strategy in the presence of the lower and upper bounds involves either trading in very long-term options or investing in the underlying replicating portfolio, both of which may be unrealistic for an individual investor. In our discrete-time framework where trading is done once annually, we try to achieve a realistic solution to the problem that may be attractive to investors. The solution has two parts.

One part involves the imposition of interim bounds to constrain the investor's pre-retirement fund value, in a similar fashion to the interim target values for the loss aversion utility. The idea is to ensure the investor's fund value is on track to be between the lower and upper bounds at retirement date.

The other part of the solution is that the investor trades in one-year options at each integer time. In addition, this is a more realistic possibility than trading in 40-year options to meet the problem's solution as long-dated options are relatively rare and prohibitively expensive for individual investors. The one-year options are chosen to keep the fund value between the interim bounds at the end of each year. The net fund value, after the cost of option trading has been deducted, is then fed into the appropriate system to solve, either Eq. (4.9) or Eq. (4.13). The procedure by which the one-year options-based strategy is carried out is described next.

4.2.5.1 One-year options-based strategy

Here we impose a state-dependent constraint that the investor maintains his fund level at or above the level of $K_l(s, Y_t)$ – the lower fund level bound for the entire remaining investment period, $s \in [t, T]$, given his current state at time t with an income of Y_t. The calculation of $K_l(s, Y_t)$ is detailed below. The intuition for this is that if the investor were to allow their fund level to fall below the value of $K_l(s, Y_t)$, no subsequent trading strategy could guarantee that they would meet the boundary constraints at retirement (Teplá, 2001; Korn, 2005).

Consider a situation where the investor can buy and sell European options, which are one-year options written annually on the investor's one-year buy-and-hold strategies. As described in the previous section, the investor chooses values for the lower replacement ratio l and upper replacement ratio u such that the upper fund level bound is higher than the lower fund level bound $K_u(T, Y_t) > K_l(T, Y_t)$. The investor would like the real value of their fund value at retirement to lie between these two boundary values.

The final and interim lower fund level bounds are the fund level required to achieve the replacement ratio of l at retirement. The fund level bounds evolve in accordance to the states of the investor:

$$K_l(s, Y_t) = \begin{cases} Y_M(s, Y_t) \times \ddot{a}_{65} \times l, & \text{for } s = T, \\ K_l(T, Y_t)e^{-r_{\mathrm{R}}(T-s)} - \pi \sum_{j=s}^{T-1} Y_M(j, Y_t)e^{-r_{\mathrm{R}}(T-1-j)}, & \text{for } s = t, t+1, \ldots, T-1, \end{cases} \tag{4.14}$$

in which

$$Y_M(s, Y_t) = \begin{cases} Y_t \exp(\sum_{j=t}^{s-1} u_j), & \text{for } s = t+1, \ldots, T, \\ Y_t, & \text{for } s = t \end{cases}$$

is the maximum income attainable at time s given the investor has an income of Y_t at time t.

The interim fund level bounds are constructed to ensure that the investor can achieve a minimum replacement ratio of l at retirement even when he is at the maximum labor income path throughout the investment period.

Similarly, the upper bounds, beyond which the investor forgoes the upside risk, also evolve in accordance to the state of the investor:

$$K_u(s, Y_t) = \begin{cases} Y_N(s, Y_t) \times \ddot{a}_{65} \times u, & \text{for } s = T, \\ K_u(T, Y_t)e^{-r_{\mathrm{R}}(T-s)} - \pi \sum_{j=s}^{T-1} Y_N(j, Y_t)e^{-r_{\mathrm{R}}(T-1-j)}, & \text{for } s = t, t+1, \ldots, T-1, \end{cases} \tag{4.15}$$

in which

$$Y_N(s, Y_t) = \begin{cases} Y_t \exp(\sum_{j=t}^{s-1} d_j), & \text{for } s = t+1, \ldots, T, \\ Y_t, & \text{for } s = t \end{cases}$$

is the minimum income attainable at time s given the investor has an income of Y_t at time t.

Given the current state with an income of Y_t at time t, the investor deals in one-year options in order to keep their fund value at integer time $s > t$ between the interim bounds $[K_l(s, Y_t), K_u(s, Y_t)]$. The investor's resultant annual investment strategy at each integer time t has three elements that can be broadly described as:

- Securing at least the interim lower bound $K_l(t+1, Y_t)$ at time $t+1$ by buying a one-year put option with strike price $K_l(t+1, Y_t)$;
- Securing at most the interim upper bound $K_u(t+1, Y_t)$ at time $t+1$ by selling a one-year call option with strike price $K_u(t+1, Y_t)$; and,
- Investing the remaining fund value to maximize the expected utility of the real fund value.

Next we describe the evolution of the investor's fund value process $\{F_t^{\text{opt}}\}_{t\in[0_-,T]}$ under this strategy. First, consider a portfolio which (i) has a value Z_t at time t, whose calculation we detail shortly; and (ii) invests at time t the amount $Z_t \times \theta_t^{\text{opt},i}$ in asset i, for $i = 0, 1, 2$. The proportions $\{\theta_t^{\text{opt},i}\}_{i=0,1,2}$ are obtained by the maximization of either (4.8) or (4.12), according to which utility function is being applied.

At integer time $t \in \{0, 1, \ldots, T-1\}$, the investor buys a put option, with its strike price $K_l(t+1, Y_t)$ calculated according to Eq. (4.14). Simultaneously, the investor sells a call option, written on the same portfolio as the put option, with its strike price $K_u(t+1, Y_t)$ calculated according to Eq. (4.15). The fund value at integer time $t \in \{0, 1, \ldots, T-1\}$ is calculated recursively as follows. The starting fund value is defined to be $F_0^{\text{opt}} = F_{0_-} + \pi Y_0$. For time $t = 0, 1, \ldots, T-1$, the value Z_t is the solution to the equation

$$F_t^{\text{opt}} = Z_t - \text{call}(Z_t, K_u(t+1, Y_t), r_{\text{R}}, \sigma_{Z_t}, 1) + \text{put}(Z_t, K_l(t+1, Y_t), r_{\text{R}}, \sigma_{Z_t}, 1), \tag{4.16}$$

where call(put)(S, K, r, σ, T) denotes the Black-Scholes value of a European call (put) option at inception with S refers to the value of the underlying asset at inception, K the strike price, r the risk-free rate of return[1], σ_S the volatility of the underlying asset, and T the time to maturity. The underlying asset of the one-year options is the portfolio with value Z_t and its volatility σ_{Z_t}, determined by the volatilities of the assets i and the proportions invested in each of the assets $\theta_t^{\text{opt},i}$ for $i = 0, 1, 2$:

$$\sigma_{Z_t} = \sqrt{(\theta_t^{\text{opt},0})^2\sigma_I^2 + (\theta_t^{\text{opt},2})^2\sigma_{22}^2 + 2\theta_t^{\text{opt},0}\theta_t^{\text{opt},2}(\sigma_{21} - \sigma_I)^2}. \tag{4.17}$$

[1]We use the real rate of returns, r_{R}, because the fund levels and prices are specified in real terms (Fischer, 1978).

After one year, the fund value just after the lump-sum contribution made at integer age $t+1$ is calculated as

$$F_{t+1}^{\text{opt}} = \pi Y_{t+1} + \min\left(\max\left(Z_t \times \sum_{i=0}^{2} \theta_t^{\text{opt},i} \frac{S_i(t+1)}{S_i(t)} \frac{I(t)}{I(t+1)}, K_l(t+1, Y_t)\right), K_u(t+1, Y_t)\right). \tag{4.18}$$

As no contribution is made at retirement $t = T$, we set

$$F_T^{\text{opt}} = \min\left(\max\left(Z_{T-1} \times \sum_{i=0}^{2} \theta_{T-1}^{\text{opt},i} \frac{S_i(T)}{S_i(T-1)} \frac{I(T-1)}{I(T)}, K_l(T, Y_t)\right), K_u(T, Y_t)\right). \tag{4.19}$$

We remark that this strategy cannot be implemented directly from individual exchange-traded options. The strike price applies to the value of a basket of assets. This means that the investor is unable to buy the equivalent of each one-year put option through the purchase of a combination of one-year European put options written separately on each of the three assets. There is no strike price to apply to each of the three options separately. Instead, the strike price operates across the three underlying assets in each option, rather than being applied individually[2].

The investor's problems with fund value constraints are to maximize Eq. (4.8) or Eq. (4.12) with the change that the fund value evolves as F_t^{opt} instead of F_t. The adjusted problems are solved using the Bellman equations (Eq. 4.9 or Eq. 4.13) subject to a new set of budget constraints (Eqs. 4.16 - 4.19).

4.2.6 Numerical methods used

The numerical results for the discrete-time optimization problems detailed in Section 4.2.4 and 4.2.5 are obtained by a numerical stochastic dynamic programming approach.

First, we discretize the state of the problems into three state variables, which include the investor's fund level, F_t, his labor income Y_t, and his age (or time until retirement t). The fund level is discretized into 201 equidistant grid points that spans the range of 0 and $50 \times Y_t$

[2]We perform sensitivity analysis to allow for transaction costs by following the Leland (1985) approach to adjust volatility given in Eq. 4.17, which reduces the (already weak) incentive of using the option-based strategies (see Appendix 4.A)

for problems with no constraints on fund values and the range of $K_l(t, Y_t)$ and $K_u(t, Y_t)$ for problems with constraints on fund values. The labor income is discretized into up to 9 grid points that spans the range of income at each age and coincides with the nodes in the lattice (see Figure 4.1). Assuming a step in time corresponds to exactly one year, we solves the problem for each integer time.

For each grid point in the discretized state space, we solve the Bellman equations for value functions V_t recursively with backward induction, by finding the optimal asset allocations θ_t^i for $i \in \{0, 1, 2\}$ using a multi-dimensional optimization with constraints such that $\theta_t^i \in [0, 1]$ at time $T - 1$, $T - 2$ and so on. Once the backward induction has reached $t = 0$, the optimal investment strategy for an investor can be derived by following the optimal asset allocations that corresponds with the fund level grid point for each integer time $t = 0, \ldots, T - 1$. For the problem with constraints on fund values, an additional root-finding procedure is embedded in the optimization process in order to obtain Z_t, which is necessitated by the option payoffs (see Eq. 4.18).

Since the value function is only available at pre-defined grid points, the value function V_t between grid points is interpolated. The expectation with respect to the stochastic return in the Bellman equations is calculated with two-dimensional Gauss-Hermite quadrature with total number of nodes $10,000$. The large number of nodes ensures the shape of value functions in the presence of the option payoffs is adequately captured. For recent applications of quadrature integration and interpolation in lifecycle modelling, see Horneff et al. (2015), Andréasson et al. (2017) and Khemka and Butt (2017).

In our numerical procedure, we use the investor's normalized fund level (defined as the fund level divided by the labor income at time t) in place of the fund level. This numerical trick enables us to take advantage of the scale independence of the problems under the CRRA risk preference, allowing the problems to be solved with just two state variables (see Gomes and Michaelides, 2003).

4.3 Numerical analysis

We evaluate the potential retirement outcomes numerically from the perspective of an investor who starts investing at age 25 and retires at age 65. First, in Section 4.3.1, we discuss our baseline parameterizations under which we derive the optimal investment strategies implied by the problems considered in Section 4.2. Next, we provide some economic explanations to the asset allocations of these strategies in Section 4.3.2. Finally in Section 4.3.3, we compare the potential retirement outcomes following each of the investment strategies using simulated results.

4.3.1 Parameterization of the model variables

We calibrate the financial market model detailed in Section 4.2.1 to UK market data[3]. The risky stock is represented by the FTSE 100 Total Return Index. We proxy for nominal bonds through the Thomson Reuters UK 10 Years Government Benchmark Index and for inflation we use the Retail Price Index (RPI). Using the data from January 1980 to December 2019 and adjusting for inflation, 12-month real rolling returns were calculated, resulting in 468 annual returns that were used to calibrate the parameters for the nominal bond and the risky stock. The real risk free rate is calibrated to the real rate of returns of the inflation-linked bonds (called index-linked gilts) issued by the UK government, which were first introduced in 1981. Specifically, we use the real returns of the FTSE British Government Index Linked (All Maturities) Index from January 1994 (earliest available) to December 2019. The parameter values are shown in Table 4.1.

The investor, who is exactly age 25 at $t = 0$, has a starting labor income of $Y_0 = 1$ unit per annum. His real labor income (i.e. the income expressed in terms of purchasing power at age 25) gets updated on each integer year following the tree process described in Section 4.2.2. The mean growth rate of the labor income has parameters $r_L = 0.019$, $h_1 = -0.1865$ and $h_2 = 0.7537$, which are obtained from Blake et al. (2013) who fitted the equations to UK male national income

[3] The data is obtained from DataStream.

Table 4.1: **Parameters for the stochastic financial market model.**
This table reports the parameters for the stochastic financial market model as described in Section 4.2.1, in which r_{N} is the annual nominal interest rate; r_{R} is the real rate of return; μ_I are σ_I are the mean return and volatility on the price inflation index, respectively; μ_2 is the mean return on the risky stock, σ_{21} is its correlation with the price inflation index and σ_{22} is its correlation with non-modelled risks. The parameters are calibrated to the UK market using data from January 1981 to December 2019.

r_{N}	r_{R}	μ_I	σ_I	μ_2	σ_{21}	σ_{22}
0.073	0.026	0.038	0.078	0.091	0.096	0.142

data. Finally, we assume a relatively low income volatility of $\sigma_L = 0.01$ such that the labor income progression fluctuates within a reasonable range as seen in well-functioning economies.

The investor contributes a constant fraction $\pi = 11.5\%$ of his income as a lump-sum annually on each integer year for 40 years (i.e. $T = 40$). This is between (i) the UK minimum mandatory contribution of 8% (Vanguard and Nest Insight, 2019) and (ii) the contribution rate of 16% required to achieve a two-thirds replacement ratio after 40 years of savings following a life-style investment strategy (Byrne et al., 2007). We assume that the investor starts with initial savings of $F_{0-} = 0$ units.

The goals and bounds are expressed as RRRs. To translate them into a fund value target, we use an annuity rate of $\ddot{a}_{65} := 14.3779$, based on a continuously-compounded, annual real rate of return $r_{\mathrm{R}} = 0.026$ (consistent with Table 4.1) and the male annuitant mortality table S1PMA[4]. Note that the annuity rate is constant across all simulations and across all the studied problems.

The CRRA utility function is parameterized with $\gamma = -2$. For the loss aversion utility function, we adopt the same parameter values of $\lambda = 4.5$, $\nu_1 = 0.44$ and $\nu_1 = 0.88$ as those in Blake et al. (2013) who calibrate the parameters to UK data which is context of our investigation as well. The investor targets a RRR of $k = 2/3$. The personal discount factors of both utility functions (Eqs. 4.8 and 4.12) are assumed to be $\beta = 0.96$; and the weighting factor for interim utility derived under the loss aversion risk reference in Eq. 4.12 is $\omega = 0.5$.

For the problems with constraints on the fund value at retirement time, we set the lower bound of the replacement ratio at $l = 0.3$, so as to satisfy the initial budget constraints (Eq. 4.18) at

[4]S1PMA is a UK, male mortality model which can be downloaded from S1 series table, CMI mortality and morbidity tables [accessed on 10 November 2020].

time 0 for the baseline parameterization and our sensitivity analyses. The upper bound of the replacement ratio is assumed to be $u = 1.5$.

TABLE 4.2: **Parameters for labor income and other investment specification.** This table reports the parameters for labor income and other investment specification. For labor income, we follow Blake et al. (2013)'s model specification, in which r_L and σ_L are the mean growth rate and volatility of the labor income, respectively; h_1 and h_2 are the curvature of parameters of the salary profile; Y_0 is the initial labor income. As with Blake et al. (2013), the parameters are fitted to the UK male national income data. The investor has an initial savings of F_{0-} and contributes a constant fraction π of his income annually. The annuity rate $\ddot{a}_{65}$ is calculated from the male annuitant mortality table S1PMA by the Institute and Faculty of Actuaries. l and u are the lower and upper bound of the replacement ratio, respectively.

Labor income		**Investment specification**	
r_L	0.019	F_{0-}	0
σ_L	0.1	π	11.5%
h_1	-0.1865	$\ddot{a}_{65}$	14.3779
h_2	0.7537	l	0.3
Y_0	1	u	1.5

4.3.2 Investment strategies

Here we present the optimal asset allocations of the four different strategies:

- CRRA preferences with no lower and upper bounds on the RRR;
- CRRA preferences with the RRR at retirement constrained to lie in the range $[l, u]$;
- Loss aversion preferences with no lower and upper bounds on the RRR; and
- Loss aversion preferences with the RRR at retirement constrained to lie in the range $[l, u]$.

These optimal strategies are calculated from the problems described in Sections 4.2. The investment strategies depends on three state variables of the investor: his fund level, his labor income, and his age (or time until retirement). We simplify the discussion by focusing on the normalized fund level of the investor, which is defined as the fund level divided by the labor income at time t. Taking advantage of the scale-independence property of CRRA utility function, this enables us to fully characterize the investment strategies under the CRRA risk preference with just two state variables: the investor's normalized fund level and his age. Although the problems under the loss aversion risk preference are not scale-independent and its state space cannot be reduced, using the normalized fund level better reveals the asset allocation decisions implied by the problem.

Given the investor's normalized fund level and his age, we note that the asset allocation decisions remain largely unchanged across different levels of labor income. In particular, a slightly higher allocation to the risky stock is observed when the labor income is lower, and vice versa.

For brevity we assume that the investor's labor income matches the expected value across all ages. In addition, we focus our discussion on the optimal allocation to the risky stock because it best illustrates the risk appetite of the investor prescribed by the respective investment strategies. Figure 4.2 shows the allocations to the risky stock against the normalized fund level for each of the four strategies over three different points in the investment period with varying time until retirement[5].

Under the CRRA utility function, at time $T-1$, the allocation to the risky stock is a constant value at about 45% without the boundary constraints. For time $t < T$, the allocation tends asymptotically to about 45% as the fund level increases. This is because an investor with CRRA risk preferences maintains a constant optimal allocation proportion to the risky stock after accounting for his future contributions (Korn and Krekel, 2002).

Under the loss aversion utility function, the allocation to the risky stock are V-shaped curves. The investor tends to take risks when their fund level is lower (i.e. they are in the risk seeking domain) to try and achieve the target. Conversely, when their fund is higher than their target (i.e., when they are in the risk averse domain), the likelihood of falling below the target is low and the investor can afford to take additional risks to generate extra utility. Moreover, the V-shapes are narrower and shallower for younger ages. This indicates a higher overall risk appetite for a larger range of balances when the investors are further away from retirement. As the investor approaches retirement, the propensity to 'lock-in' the target imposes a significant penalty on falling below the target. In turn, this leads to the wider and deeper V-shaped allocation to the

[5] For allocation to nominal bonds and additional visualizations of the risky stock allocation, please see Appendix 4.B.

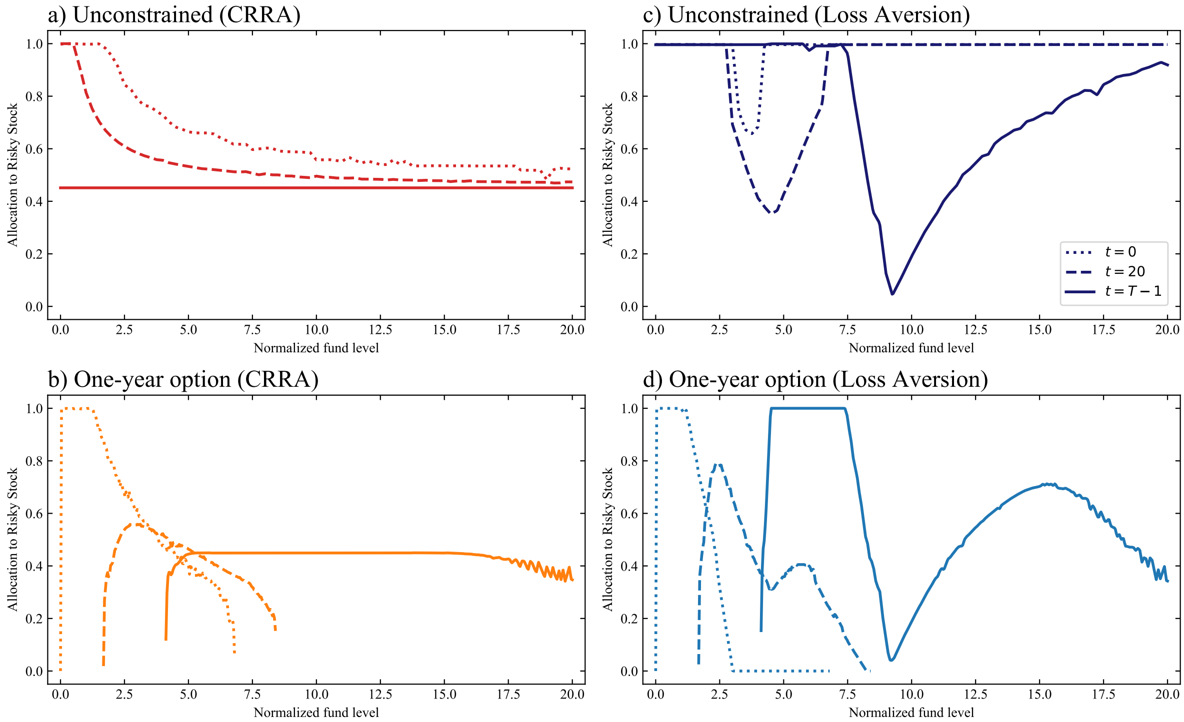

FIGURE 4.2: **Optimal allocations to the risky stock.**
These figures show the optimal allocation to the risky stock for the four strategies in three different points of the investment period, assuming the investor's labor income is at its expected value. Due to the time-specific budget constraints, the constrained strategies are only applicable for a specific range of normalized wealth for each point in time.

risky stock (Warren, 2019). Our findings are consistent with the results of Blake et al. (2013) and Butt et al. (2019).

For both utility functions, adding in boundary constraints on the fund levels does not materially impact the asset allocation except when the fund level approaches the boundary constraints. Under the one-year option strategy, as the fund level approaches the boundary, the optimal allocation to the risky assets (stock and nominal bond) reduce towards zero (and thus volatility tends to zero) to reduce the cost of options and ensure that the fund is able to satisfy the boundary constraints. A closer inspection reveals that the reduction in the allocation to the risky stock is offset by increasing the allocation first to the nominal bond, which is a safer risky asset, and later, to the inflation-indexed bond, which is the real risk-free asset. As fund level reaches the boundary, the investor invests entirely in the inflation-indexed bond. Our findings are consistent with the results of Donnelly et al. (2018) albeit they study only the problem with power utility in a continuous time framework with only one risky asset.

Finally, we note that optimal allocation to the nominal bond is the complement of the optimal allocation to the risky stock whenever the strategy prescribes no allocation to the inflation-indexed bond. This is the case for most fund levels at most ages, with a few exceptions when a risk-free investment is desired. The exceptions happen in two instances. First, when the fund level of the one-year option strategies approaches their bounds at both ends. In other words, as discussed earlier, when the investor has no other choice but to invest fully in the risk-free asset to meet the boundary constraints. Second, when a loss averse investor's fund level is near his target level in the last 10 years of his working life, due to the 'lock-in' nature of the loss aversion utility function.

4.3.3 Distribution of retirement outcomes

Here we present the distributions of the RRR under the four different strategies. For each strategy, we report the distributions of the RRR at retirement time T, or equivalently age 65. The outcomes under these strategies are simulated first, under the stochastic financial market model of Section 4.2.1 with parameter values shown in Table 4.1, and second, under historical market return data in the UK from December 1918 to December 2019.

4.3.3.1 Distribution under the stochastic financial market model

For each strategy, $10,000$ simulations were generated to obtain the distribution of the RRR. Histograms of the distribution of the RRR are shown in Figure 4.3 and the corresponding summary statistics are reported in Table 4.3 .

Without the boundary constraints, the loss aversion utility function leads to a distribution of retirement outcomes that centers around the investor's target (Figure 4.3c). Following the optimal investment strategy under the unconstrained loss aversion preferences, the investor achieves the highest mean RRR and a sufficiently high probability of achieving the target.

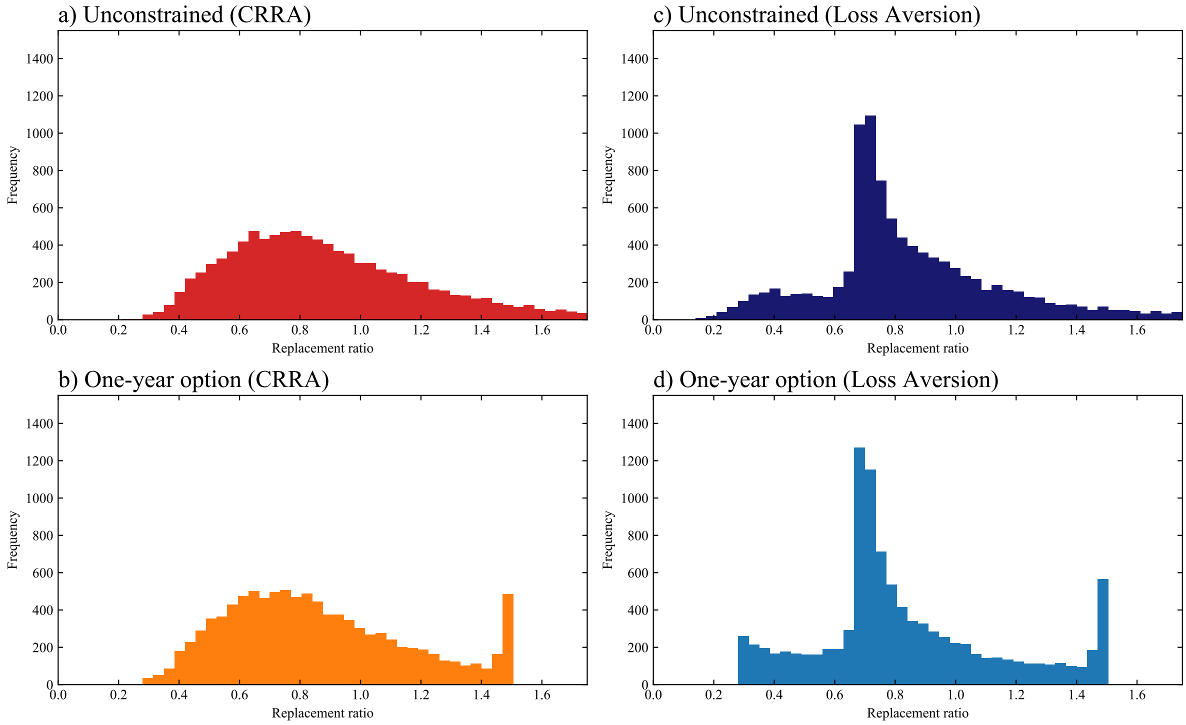

FIGURE 4.3: **Histrograms of simulated real replacement ratios under the stochastic financial market model.**
These figures show the histograms of $10,000$ simulated real replacement ratios at time T for an investor following either one of the four strategies under the stochastic financial market model (the figures are right-truncated at the value of 1.75).

However, this exposes the investor to have a relatively high chance of failing to meet the lower boundary at 2% probability. The investor also faces relatively high variability (as measured by standard deviation) in their outcome distribution and is most prone to achieving low RRRs (as measured by Conditional Value at Risk (CVaR), which we define as the probability-weighted average of the lowest 1% of the RRRs). Although the loss aversion strategy leads to highest chance of achieving their retirement income level, it offers the least protection against the worst outcomes.

Adding in the boundary constraints to the loss aversion risk preference problem cuts off the distribution below l and above u (Figure 4.3d) with a probability mass at each end-point. The one-year option strategy provides a downside protection against the worst outcomes by having a higher CVaR value at 0.304, eliminates the outcomes below l and above u, and reins in the standard deviation. However, there is a concomitant reduction in both the mean outcomes and

Table 4.3: **Comparison of outcomes of investment strategies under the stochastic financial market model.**
This table reports the summary statistics of the outcomes distributions following each of the four investment strategies under the stochastic financial market model specified in Section 4.2.1. We report the mean, the standard deviation (SD), CVaR (Conditional Value at Risk, a risk measure which we define as the probability-weighted average of the lowest 1% of the RRRs), the probability of failing to achieve the lower boundary, and the probability of achieving the target RRR.

	Mean	SD	CVaR	$\mathbb{P}[\text{RRR} < l]$	$\mathbb{P}[\text{RRR} \geq k]$
CRRA, unconstrained	0.916	0.374	0.324	0.2%	73.2%[a]
CRRA with one-year options	0.856	0.295	0.327	-	69.9%[a]
Loss aversion, unconstrained	0.925	0.487	0.222	2.0%	82.0%
Loss aversion with one-year options	0.832	0.302	0.304	-	78.1%

[a]Since there is no explicit target for the CRRA investor, we report the probability of achieving the same target as the loss averse investor, namely $k = 2/3$.

the chance of attaining a RRR near the target level. This is attributed to the interference of boundary constraints on the natural targeting mechanism of the loss aversion risk preference. The boundary constraints limit the risk taking tendency of a loss averse investor when the fund level is lower than the target (Figure 4.2d). Adding in the boundary constraints results in a more risk averse strategy that shifts an average of about 20% of allocation from the risky stock to the nominal bond each year across the investment period (Figure 4.4). This impairs the investor's ability to catch up, eventually costing them a reduction of 0.093 in mean RRR and a reduction of 3.9% in the probability of achieving the target replacement ratio k at retirement.

On the other hand, the simulated outcomes show that the use of the well-studied CRRA utility function to express the investor's risk preferences give a much more dispersed range of outcomes (Figure 4.3a). There is no strong concentration of outcomes about a particular value. Instead, the distribution peaks gently in the range of $[0.7, 0.9]$ of RRRs. Without the boundary constraints, the CRRA utility function also results in a distribution with relative high values of mean and standard deviation that are not dissimilar to the unconstrained loss aversion strategy. Compared to the unconstrained loss aversion strategy, the unconstrained CRRA investor is less vulnerable to the risk of achieving poor retirement outcomes with $\mathbb{P}[\text{RRR} < l]$ at 0.2% and a CVaR value of 0.324. This is because when there are significant investment losses (relative to the target), the loss aversion risk preference, in search of higher potential gains, prescribes a high

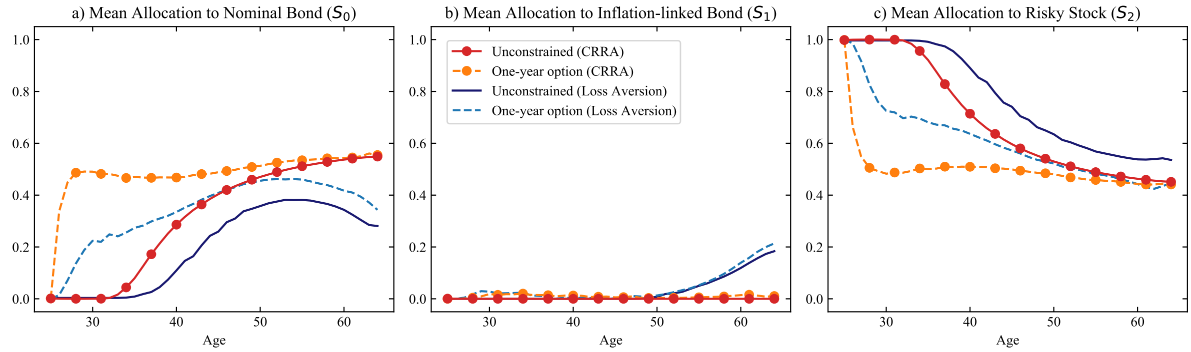

FIGURE 4.4: **Mean allocations over the investment phase.**
These figures show the mean allocations to each asset over the investment phase of the investor for the four investment strategies. The initial increase in allocation to the nominal bond and reduction in allocation to the risky stock observed in the one-year option strategies (for both CRRA and loss aversion utility preferences) is a result of our simulation parameter of $F_{0-} = 0$ which places the investor closer to the lower boundary constraint at the left-side of Figure 4.2. The mean allocations of the constrained CRRA strategy moves closer towards its unconstrained counterpart compared to the loss aversion strategies because over time fewer of the CRRA simulations are bounded by the constraints (see Section 4.2.5.1).

allocation to risky assets which also increases the chance of incurring greater investment losses. Contrarily, the CRRA risk preference seeks an optimal allocation to risky assets independent of the fund level.

Constraining the RRRs to lie in the range $[l, u]$ at age 65 implies that the investor is more risk averse than the risk preference implied by the CRRA utility function. By completely eliminating the chance of outcomes below l and above u, the one-year option strategy also reduces the standard deviation and marginally improves the lower quantiles (e.g. CVaR) of the RRR at retirement for the CRRA investor (see Donnelly et al., 2015). However, this is accompanied by an increase in the allocation to the nominal bond over the investment period and costs the investor a reduction of 0.060 in mean RRR at retirement.

Based on the simulation results, the incentive of including the boundary constraints is not strong for investors under either risk preference structure. The downside protection rarely comes into effect and the resultant strategies provide little improvement in the lowest percentiles of RRRs at retirement, especially under the CRRA risk preference. We also conduct sensitivity analyses to examine the impact of boundary constraints have on the retirement outcomes with different levels of the boundary constraints and different utility parameters (see Appendix 4.A). The boundary constraints have impact on the shape as well as the key statistics of the simulated distributions. A smaller distance between the boundary constraints results in larger changes in the retirement outcomes, and vice versa. Furthermore, adding in constraints has a greater effect on less risk averse investors implied by the utility function parameters alone, since it limits their otherwise higher exposure to the risky asset.

4.3.3.2 Distribution under historical UK market return data

This section analyzes the performance of the optimized strategies when the market returns do not conform to the model set out in Section 4.2.1. Specifically, we use the moving block bootstrap method to simulate a series of retirement outcomes using the historical UK return data. This method preserves the cross sectional correlation and the serial dependence of the original data. To begin, we randomly draw a starting date with replacement. Next, for each investment strategy, we simulate the retirement outcomes (RRRs) after 40 years of investment following the starting date, with a starting fund level of $F_{0-} = 0$ units and starting labor income of $Y_0 = 1$ unit at age 25. This procedure is then repeated $10,000$ times. Histograms of the distribution of the RRR are shown in Figure 4.5 and in Table 4.4 their summary statistics.

TABLE 4.4: **Comparison of outcomes of investment strategies under historical simulations.**
This table reports the summary statistics of the outcomes distributions following each of the four investment strategies under historical simulations. We report the mean, the standard deviation (SD), CVaR (Conditional Value at Risk, a risk measure which we define as the weighted average of the lowest 1% of the RRRs), the probability of failing to achieve the lower boundary, and the probability of achieving the target RRR.

	Mean	SD	CVaR	$\mathbb{P}[\text{RRR} < l]$	$\mathbb{P}[\text{RRR} \geq k]$
CRRA, unconstrained	0.729	0.280	0.239	3.6%	55.2%[a]
CRRA with one-year options	0.705	0.301	0.300	-	54.5%[a]
Loss aversion, unconstrained	0.770	0.214	0.277	0.6%	76.0%
Loss aversion with one-year options	0.754	0.243	0.304	-	70.3%

[a]Since there is no explicit target for the CRRA investor, we report the probability of achieving the same target as the loss averse investor, namely $k = 2/3$.

Since there were no index-linked gilts issued prior to 1981, we have assumed the existence of inflation-indexed bonds prior to 1981 which have a constant real rate of return $r_{\text{R}} = 0.026$ as discussed in Section 4.3.1. Furthermore, for consistency with this assumption and with the

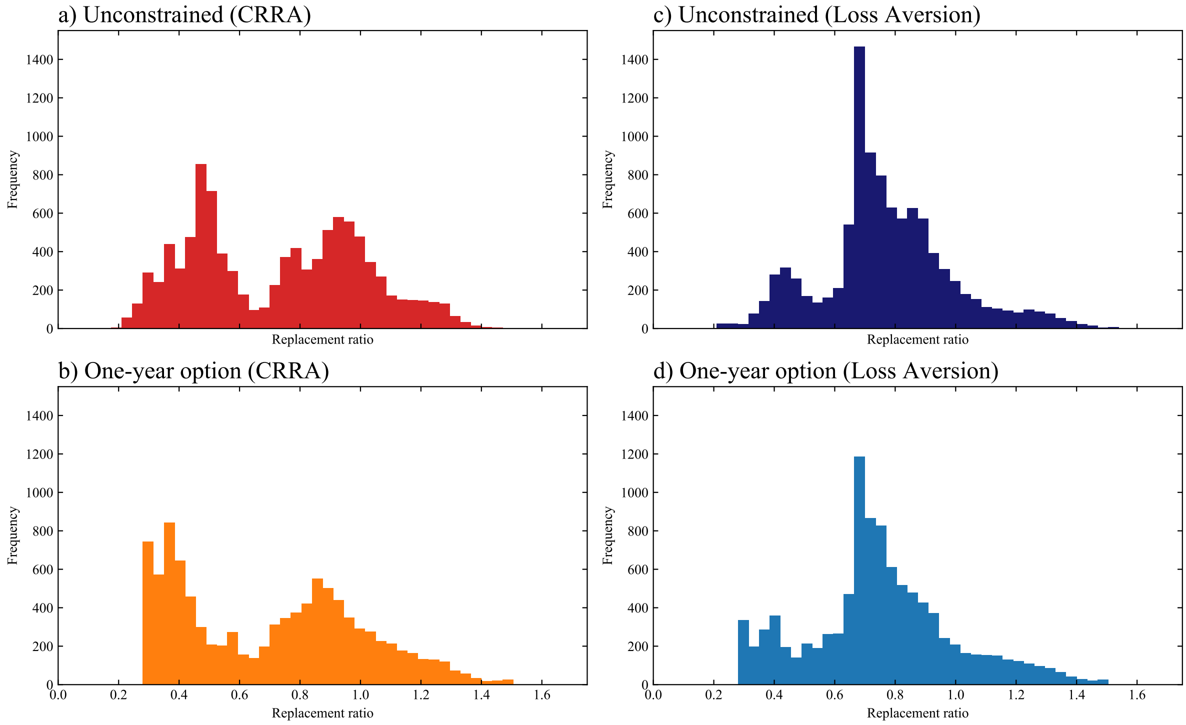

FIGURE 4.5: **Histrograms of simulated real replacement ratios under historical simulations.**
These figures show the histograms of 10,000 simulated real replacement ratios at time T for an investor following either one of the four strategies under historical simulations (the figures are right-truncated at the value of 1.75).

constant annuity rate assumption (which we detail in the next paragraph), we assume that the real rate of return for inflation-indexed bonds issued since 1981 is also r_{R}. This is a critical assumption that allows us to do the historical simulation, as otherwise we do not have a single set of 40-year data which rely on actual inflation-indexed bond returns. However, we acknowledge that this limits the strength of the conclusions drawn from the historical simulation exercise.

The second assumption is that the annuity rate is a constant value $\ddot{a}_{65}$ of 14.3779 at all times (see Section 4.3.1). We were not able to obtain data on the cost of inflation-indexed annuities over any reasonable time period. Keeping the same annuity rate for all simulations imply that the resultant real replacement ratio (RRR) is a constant multiple of the investor's real fund value. With these two assumptions, we use the historical UK returns data for the period from December 1918 to December 2019 that encompasses a wide range of market conditions, including periods with different levels of interest rates as well as the financial market crisis of 2008[6].

Under the historical simulations, the CRRA risk preferences lead to bimodal distributions. The bimodality is is largely attributed to the investment period of the simulations. Specifically, the simulations whose retirement date is before mid-1980s typically achieve lower RRRs and cluster around the first peak of the distribution because of the low real returns of the nominal bonds over the period. The investor with CRRA risk preferences is susceptible to the risk of achieving a low RRR at retirement, with a probability of 3.6% failing to achieve at least the lower bound l and a low value of CVaR at 0.239. Adding in the boundary constraints provides some extent of downside risk protection by removing the chance of outcomes below l and lifting the CVaR to the value of l. It shifts the distribution slightly towards the left (as seen in a reduction of 0.024 in the mean) and pushes it against the lower boundary of l.

On the other hand, the outcomes distributions of loss aversion risk preferences show mild sign of bimodality whilst retaining much resemblance of their counterparts as seen in Section 4.3.3.1, with bunching of outcomes around the investor's target. This demonstrates that the loss aversion investment strategy successfully keeps the majority of the outcomes around the target, including simulations that retire prior to mid-1980s. Having the highest mean, the lowest standard deviation and the lowest $\mathbb{P}[\text{RRR} < l]$, the unconstrained loss aversion strategy appears to be the best performing strategy among the four strategies. The relative low risk of achieving RRRs below l can be eliminated by imposing the boundary constraints, which also increases the CVaR to a value slightly above l.

In a market model that does not conform to the model set out in Section 4.2.1, the risk averse investor is more susceptible to achieving very poor RRRs[7]. This may incentivize a risk averse investors to include the boundary constraints – the downside protection is more likely to come

[6] The historical simulation data is created by splicing the DataStream data from 1980 onwards with long-term monthly indices supplied by David Wilkie, of whom the authors are very appreciative, from 1918 to 1980. We do this for the inflation data, the nominal bond data and the risky stock data. The data is available from the authors upon request.

[7] We perform a robustness check by drawing the starting dates from a smaller range of dates. Under CRRA risk preference, the risks of achieving very poor RRRs are related to the (cumulative) asset returns during the simulation period – the risks are particularly high for simulations that end before mid-1980s. This relationship is not apparent under the loss aversion risk preference.

into effect preventing the worst of retirement outcomes. In contrast, the loss averse risk preference prescribes asset allocation depending on the cumulative investment gain and loss (relative to the target). This investment strategy leads to largely robust retirement outcomes even when the realized market returns deviate from the assumptions under which the strategy is derived (comparing Figure 4.3c and Figure 4.5c). In both cases, it gives the investor a higher chance of achieving the target RRR. As a result, the need of downside protection is not nearly as material for the investor following a loss aversion utility function compared to the CRRA investor.

4.4 Conclusion

In this paper, we have studied the investment strategies that allow the investor to constrain their retirement incomes, or equivalently financial wealth accumulated after the savings phase, within a framework of expected utility maximization. We contribute to the literature by extending Donnelly et al. (2018) to allow the investor to have a non-tradable stochastic labor income, to have wealth boundary constraints that evolve according to the labor income process, and to use short term option contracts in managing their investment risks. This differentiates our study from most literature in which a tradable labor income process is assumed and continuously-traded replicating strategies are devised to replicate the option-like payoffs for the investors. In addition to the traditional risk aversion utility preferences, we also investigate the investment behaviour implied by the loss aversion risk preference that has been attracting the interest of both practitioners and academic researchers.

We find that there is no strong incentive for CRRA investor to include terminal wealth constraints under the standard market model. Constraints are useful for the very risk averse investor, since they protect them from extremely adverse outcomes. Thus they give more certainty in the retirement outcome. Yet this comes at a potential cost of investing in low return bonds for long periods of time, to avoid breeching the constraints.

On the other hand, the loss aversion risk preferences naturally lead to a high chance of achieving the investment target, when the latter evolves according to the labor income of the investor.

Our analysis shows that the investor's retirement outcomes are robustly centered around the investment target even when the realized market returns do not conform to the assumptions under which the strategies are derived. The terminal wealth constraints do little to increase the certainty of income in retirement. Indeed, similarly to the CRRA framework, they result in a strategy that is too risk averse. Instead, the investor can benefit from following a loss aversion-derived strategy in conjunction with an appropriate target income. This leads to an optimized strategy which naturally focuses on the desired income level and keeps the retirement income above the lower constraint in the vast majority of scenarios.

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4.A Sensitivity analysis

In this section we conduct additional analyses to gauge the sensitivity of our findings to a range of alternate model specifications. We find that our results are consistent across the various scenarios with minor differences caused by the differences in model specification[8].

[8] We also perform sensitivity analysis on the time discount factor β. Adjusting the time discount factor to 0.98 and 1, we find minimal impact on the results under loss aversion risk preferences, and no impact under the CRRA risk preferences.

4.A.1 Option transaction cost, levels of constraints, and utility parameters

The boundary constraints have impact on the shape as well as the key statistics of the simulated distributions. We find that a smaller distance between the boundary constraints results in larger changes in the retirement outcomes, and vice versa. Moreover, imposing constraints has a greater effect on less risk averse investors implied by the utility function parameters alone, since it limits their otherwise higher exposure to the risky asset.

The baseline analysis assumes the investor pays for the options at the (no-arbitrage) prices calculated using the Black-Scholes formulae. This means the investor pays no premium in buying options and incur no cost in selling options. To test the impact of transaction costs, we follow the Leland (1985) apporach to allow for a relative transaction cost of 10%. This further reduces the attractiveness of the boundary constraints as the investor needs to absorb the additional costs as they implement to option strategies, resulting in greater loss in mean RRRs at retirement.

Next we examine the impact of the width of the boundary constraints. A wider band (lower l or higher u) means the wealth constraints are less likely to be binding. Therefore, the impact of the constraints on the simulated outcomes are lessened compared to our baseline set-up. In the same way, a narrower band (higher l or lower u) increases the chances of constraints coming into effect and the impact of constraints on simulated retirement outcomes are more pronounced[9]. Our analysis indicates that the changes due to the boundary constraints are modest.

Finally, we investigate how the boundary constraints affect investors with different utility preferences. As discussed in Section 4 of the main article, the boundary constraints induce competing objectives and cause the investor to invest more in the nominal bond and less in the risky stock. The extent of the impact depends on the investor's utility preferences. For CRRA investors, the impact is more pronounced (larger extent of reduction in means and standard deviations) for a less risk averse investor compared to a more risk averse investor. This is because the former

[9] We do not include the sensitivity results of a higher l here because $l = 0.3$ is the maximum attainable value in order to satisfy the fund dynamic budget constraints according our baseline set-up, beyond which a higher annual contribution rate is required. The results of additional sensitivity analyses are available from the authors upon request.

would adhere to a higher allocation to the risky stock in the absence of boundary constraints. Similar effect is observed for investors with loss aversion utility preference. In most cases, a less loss averse investor tends to invest more aggressively in the risky stock than a more loss averse investor. Therefore, the boundary constraints have greater impact on the less loss averse investor. Lastly, an investor who sets a higher target level needs to invest more in the risky stock throughout the savings phase and hence experiences greater "distraction" induced by the boundary constraints, compared to someone with a lower target.

TABLE 4.5: **Sensitivity analysis under CRRA utility preferences.**
This table reports the key statistics of the simulated outcomes following the unconstrained strategy and the differences caused by the boundary constraints (in parentheses) under CRRA utility preferences. Each scenario differs from the baseline parameterization in one aspect: with transaction cost, lower $l = 0.267$, lower $u = 1.467$, higher $u = 1.533$, more risk averse $\gamma = -3$, and less risk averse $\gamma = -1$.

	Mean	SD	CVaR	$\mathbb{P}[\text{RRR} < l]$
Base	0.916 (-0.060)	0.374 (-0.079)	0.324 (0.003)	0.2% (-0.2%)
With transaction cost	0.916 (-0.080)	0.374 (-0.083)	0.324 (0.012)	0.2% (-0.2%)
Lower l	0.916 (-0.032)	0.374 (-0.070)	0.324 (0.001)	0.2% (-0.2%)
Lower u	0.916 (-0.064)	0.374 (-0.084)	0.324 (0.004)	0.2% (-0.2%)
Higher u	0.916 (-0.060)	0.374 (-0.076)	0.324 (0.003)	0.2% (-0.2%)
More risk averse	0.886 (-0.046)	0.334 (-0.055)	0.337 (-0.003)	0.1% (-0.1%)
Less risk averse	0.977 (-0.097)	0.474 (-0.152)	0.289 (0.026)	0.5% (-0.5%)

TABLE 4.6: **Sensitivity analysis under loss aversion utility preferences.**
This table reports the key statistics of the simulated outcomes following the unconstrained strategy and the differences caused by the boundary constraints (in parentheses) under loss aversion utility preferences. Each scenario differs from the baseline parameterization in one aspect: with transaction cost, lower $l = 0.267$, lower $u = 1.467$, higher $u = 1.533$, more loss averse $\lambda = 9$, less loss averse $\lambda = 2.25$, lower $k = 0.633$, and upper $k = 0.700$.

	Mean	SD	CVaR	$\mathbb{P}[\text{RRR} < l]$	$\mathbb{P}[\text{RRR} \geq k]$
Base	0.925 (-0.093)	0.487 (-0.184)	0.222 (0.081)	2.1% (-2.1%)	82.0% (-3.9%)
With transaction cost	0.925 (-0.107)	0.487 (-0.193)	0.222 (0.084)	2.1% (-2.1%)	82.0% (-5.3%)
Lower l	0.925 (-0.060)	0.487 (-0.175)	0.222 (0.048)	2.1% (-2.1%)	82.0% (-0.7%)
Lower u	0.925 (-0.098)	0.487 (-0.191)	0.222 (0.082)	2.1% (-2.1%)	82.0% (-4.3%)
Higher u	0.925 (-0.091)	0.487 (-0.183)	0.222 (0.082)	2.1% (-2.1%)	82.0% (-3.8%)
More loss averse	0.848 (-0.073)	0.399 (-0.141)	0.224 (0.079)	1.9% (-1.9%)	82.4% (-3.9%)
Less loss averse	0.977 (-0.109)	0.550 (-0.221)	0.219 (0.085)	2.4% (-2.4%)	80.4% (-4.2%)
Lower k	0.923 (-0.093)	0.494 (-0.191)	0.224 (0.080)	1.8% (-1.8%)	84.9% (-3.4%)
Higher k	0.925 (-0.091)	0.477 (-0.175)	0.220 (0.084)	2.3% (-2.3%)	78.8% (-4.6%)

4.A.2 A low interest environment

In this section we study the retirement outcomes distribution of the four strategies in a low interest environment scenario such as the one experienced by the global economy in the last decade. To stipulate a low interest market environment, we first calibrate the financial market using UK market data from January 2009 to December 2019. Then, using the Black–Litterman model (1992) we adjust the real risk-free rate towards a forward looking equilibrium of 0.5% p.a. to reflect the low real rate of return in a low interest scenario (see also Butt et al., 2019). Finally, we assume that the labor income of investors has a real growth of $r_L = 0.5\%$, a consistent assumption in a low interest/inflation environment, and contributes $\pi = 16\%$ of his labor income into his retirement fund every year, which is the minimum amount required to secure the lower constraint l in the low interest environment.

Table 4.7: **Parameters for the stochastic financial market in a low interest scenario.** This table reports the financial parameters assumption in the stipulated low interest environment.

r_{N}	r_{R}	μ_I	σ_I	μ_2	σ_{21}	σ_{22}
0.052	0.005	0.030	0.058	0.086	0.023	0.083

The summary statistics of the outcomes distributions is given in Table 4.8. The low average rate of returns and low volatilities of asset returns in this scenario results in the relatively low means and standard deviations of the outcome distributions. Low asset volatilities, coupled with a relatively high contribution rate, keep the investor above the lower constraints in nearly all of the simulated outcomes. Adding in the boundary constraints in this environment does investors a disservice by prescribing them overly risk averse strategies, lowering the mean outcomes significantly. In a low interest scenario, investors needs to take risks strategically to increase their investment earning potential. The unconstrained loss aversion strategy does that by encouraging investors to take risks (especially when they are behind or ahead of their target) and leads them to a favorable outcomes distribution with a high mean and a high probability of achieving their target. This scenario analysis supports our main conclusion that the loss aversion-derived strategy naturally focuses on the desired income level and keeps the retirement income above the lower constraint.

Table 4.8: **Comparison of outcomes of investment strategies in a low interest scenario.**
This table reports the summary statistics of the outcomes distributions in a low interest scenario. We report the mean, the standard deviation (SD), CVaR (Conditional Value at Risk, a risk measure which we define as the probability-weighted average of the lowest 1% of the RRRs), the probability of failing to achieve the lower boundary, and the probability of achieving the target RRR.

	Mean	SD	CVaR	$\mathbb{P}[\text{RRR} < l]$	$\mathbb{P}[\text{RRR} \geq k]$
CRRA, unconstrained	0.709	0.144	0.416	0.0%	58.0%[a]
CRRA with one-year options	0.495	0.087	0.316	-	3.8%[a]
Loss aversion, unconstrained	0.974	0.400	0.345	0.1%	83.3%
Loss aversion with one-year options	0.534	0.167	0.302	-	25.5%

[a]Since there is no explicit target for the CRRA investor, we report the probability of achieving the same target as the loss averse investor, namely $k = 2/3$.

4.B Supplementary visualization

This section presents supplementary visualizations to the optimal investment strategies implied by the utility preferences, with and without boundary constraints. Figure 4.6 and 4.7 show the allocations to the nominal bond and inflation-linked bond, respectively, against the normalized fund level for each of the four strategies over three different points in the investment period with varying time until retirement. Figure 4.8 and 4.9 show the allocation to the risky stock against the normalized fund level over the entire investment periods.

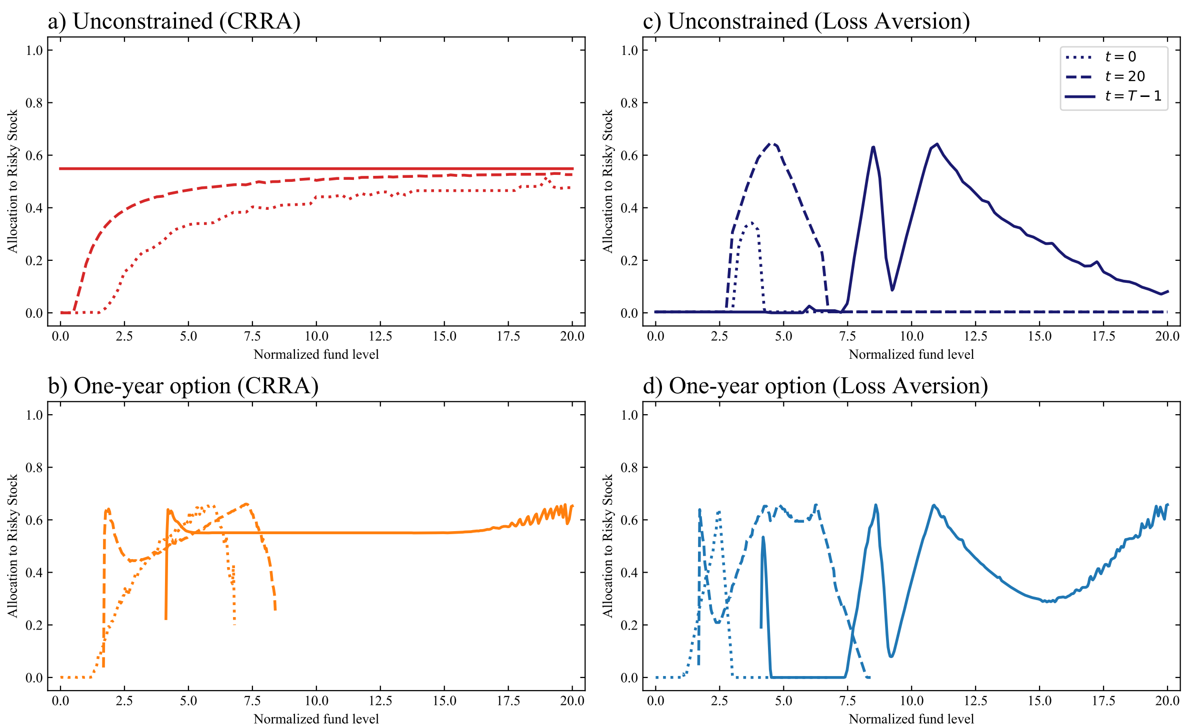

FIGURE 4.6: **Optimal allocations to the nominal bond.**
These figures show the optimal allocation to the nominal bond for the four strategies in three different points of the investment period, assuming the investor's labor income is at its expected value. Due to the time-specific budget constraints, the constrained strategies are only applicable for a specific range of normalized wealth for each point in time.

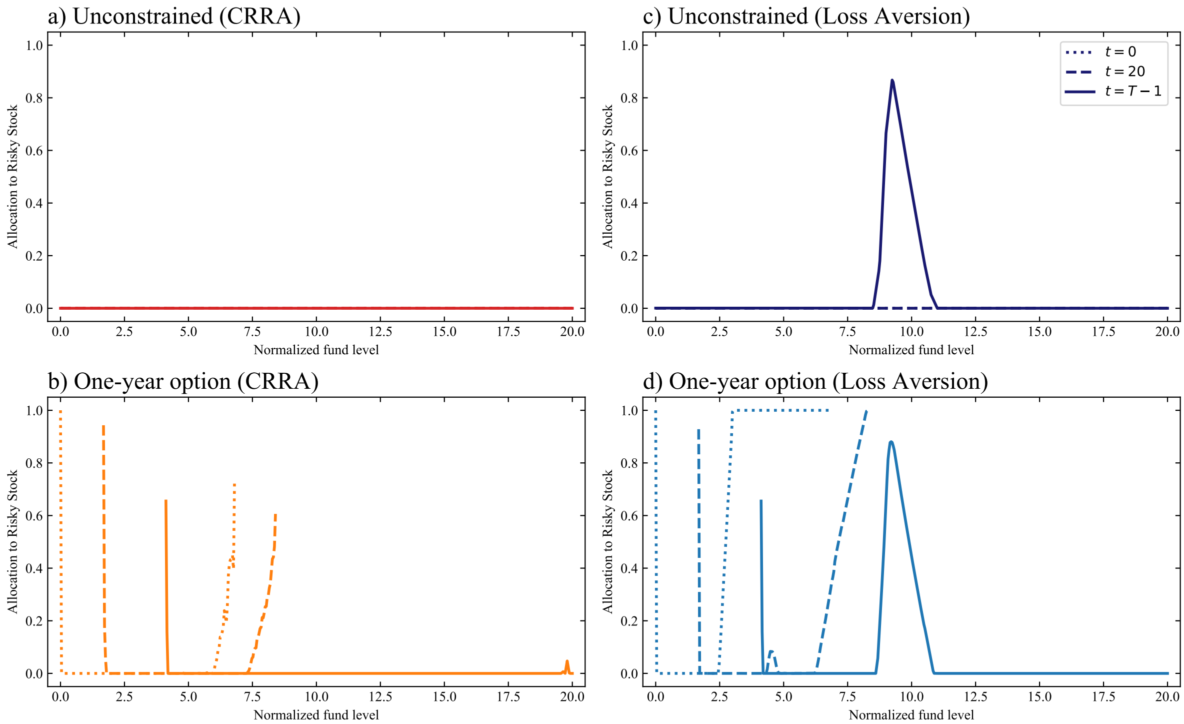

FIGURE 4.7: **Optimal allocations to the inflation-linked bond.**
These figures show the optimal allocation to the inflation-linked bond for the four strategies in three different points of the investment period, assuming the investor's labor income is at its expected value. Due to the time-specific budget constraints, the constrained strategies are only applicable for a specific range of normalized wealth for each point in time.

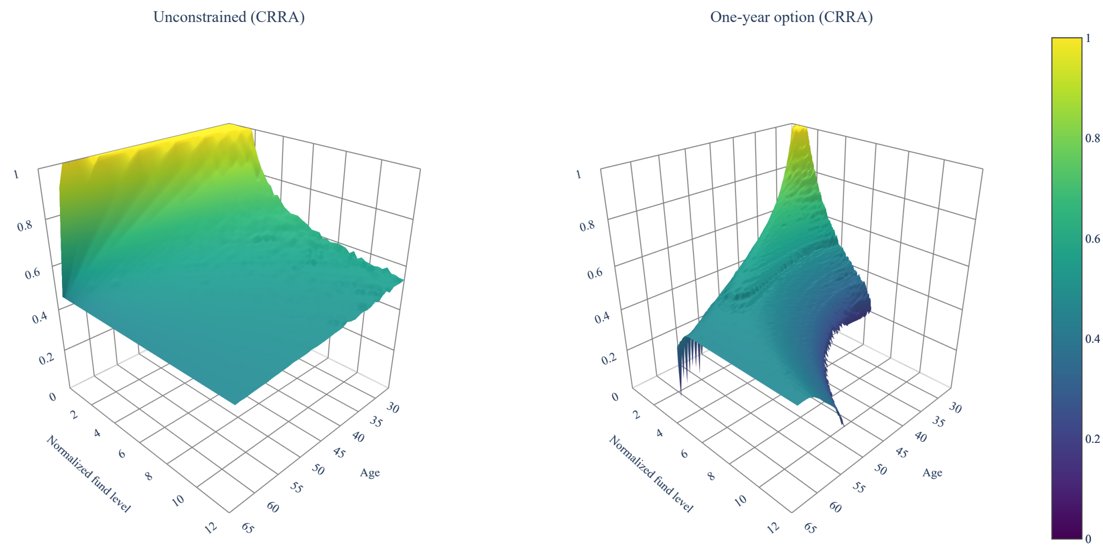

FIGURE 4.8: **Optimal allocations to the risky stock under the CRRA risk preference.** These figures show the optimal allocation to the risky stock under the CRRA risk preference across normalized fund level and age, assuming the investor's labor income is at its expected value. Due to the time-specific budget constraints, the constrained strategies are only applicable for a specific range of normalized wealth for each point in time (see Section 2.5.1 of the main article).

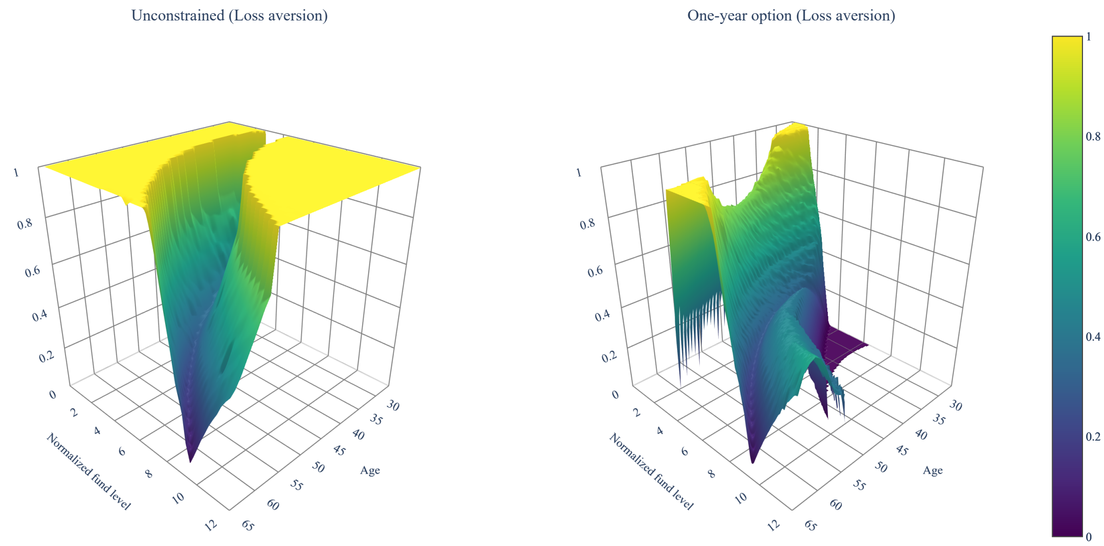

FIGURE 4.9: **Optimal allocations to the risky stock under the loss aversion risk preference.**

These figures show the optimal allocation to the risky stock under the loss aversion risk preference across normalized fund level and age, assuming the investor's labor income is at its expected value. Due to the time-specific budget constraints, the constrained strategies are only applicable for a specific range of normalized wealth for each point in time (see Section 2.5.1 of the main article).

Chapter 5

Conclusion

The shift from a defined-benefit pension system to a defined-contribution (DC) system has led to the transfer of risk to plan members, who are now responsible for making investment and withdrawal decisions regarding their pension savings. The primary concern for pre-retiree investors is to ensure a sufficient income in retirement to live comfortably (Merton, 2014). In light of this, this thesis focuses on three aspects of making optimal investment decisions for investors who are in the savings phase. In particular, all three chapters touch upon the use of terminal wealth constraints in the context of pension savings. The first aspect concerns the use of portfolio insurance. In Chapter 2, we extend the literature on option-based portfolio insurance and introduce generalised option-based portfolio insurance strategy. It is a versatile portfolio insurance strategy with which the investor can incorporate a deterministic or a stochastic minimum guarantee to their pension savings. The second aspect concerns the risk perceptions towards inflation, which can significantly impact investment strategies. In Chapter 3, We find that the mis-specifying inflation risk reduces the expected utility of the risk averse investors, and more importantly, ignoring inflation in terminal wealth constraints does not ensure real savings adhering to the real constraints. The third aspect concerns the types of utility functions, i.e., the risk averse and loss averse utility functions. In Chapter 4, we derive the optimal investment strategies for investors with different utility functions, and consider the impact of terminal wealth constraints on their savings. Our model takes into account ongoing investor contributions to savings, prohibits short-selling, and, when applicable, includes wealth constraints that evolve according to the stochastic labour income of the investor. We find that the loss averse utility function, without wealth constraints, naturally results in a more favourable retirement income distribution that peaks at the investor's chosen income goal.

Portfolio insurance can be an attractive strategy for investors saving for retirement as it safeguards their savings while still allowing them to participate in favourable market movements over a long period. In Chapter 2, we introduced the generalised option-based portfolio insurance strategy (GOPIS) as an extension of the widely studied option-based portfolio insurance (OBPI) strategy, and the option-based performance participation strategy (OBPP) by Zagst et al. (2019), which itself is an extension of the former. The investor specifies both the benchmark portfolio, to which the minimum guarantee is linked, and the venture portfolio, by which the investor participates in potential market gains. We provide the replicating strategy for GOPIS and derive the general analytic expression for the moments and conditional moments of GOPIS in a financial market with a finite number of risky assets. More importantly, we extended the analysis of conditional stochastic dominance developed by Zagst et al. (2019) to enable the comparison of GOPISs with different venture and benchmark portfolios. The first- and second-order stochastic dominance provides a decision-making rule valid for all non-satiated and risk-averse investors, respectively. Our analysis shows that GOPIS can be configured such that it has a better prospect of delivering higher expected utility over traditional OBPIs, while only knowing partial information on the investor's preferences. Next, we studied the investment optimisation for an investor whose utility preference is known to be a constant relative risk aversion (CRRA) function under the framework of Expected Utility Theory. We show that, by setting the Merton portfolio 1969 as the venture portfolio, GOPIS is the optimal investment strategy that maximises expected utility for all exogenously specified benchmark portfolios.

DC plan members are particularly vulnerable to the risk of inflation eroding the purchasing power of their savings due to the long-term nature of retirement savings plans. However, investors tend to think in terms of nominal terms rather than real terms, which may influence the investment strategies they adopt for their saving plans. Chapter 3 contributes to the literature by investigating the cost of mis-specifying inflation risk for two types of risk averse investors (with and without terminal wealth constraints). Contrary to the findings of Zhang (2012), we show that by adopting real risk perception the investment strategy should have an additional inflation risk hedging component, which leads to a different solution to the Merton portfolio which maximises the expected utility using nominal risk perception. We estimate that ignoring inflation can lead to a significant reduction in the investor's utility, with reduction of up to 25% possible for the average investors. Extending the work by Donnelly et al. (2018) to incorporate inflation risk, we find that mis-specifying inflation in the presence of wealth constraints leads to a vastly different optimal constrained strategies because the inflation-linked constraints and nominal constraints require different hedging assets. Therefore, in addition to reduction in utility, the investor's real wealth is not secured within the inflation-linked constraints had they 'mistakenly' specifies investment strategy according to nominal risk perceptive rather than real risk perceptive. The retirement saving products providers should communicate inflation risk and provide inflation-hedging options to their pension members.

The stochastic optimal problems in Chapter 2 and 3 are formulated in continuous time markets and are solved analytically using the martingale method (see, e.g., Karatzas et al., 1987; Cox and Huang, 1989). The analytical results reveal insights about the prospect of first- and second-order stochastic dominance of GOPIS, and the optimality investment strategy under different risk perspectives towards inflation risk. In Chapter 4, we expand our models to incorporate additional components of pension savings. For example, the investor earns a stochastic labour income, a portion of which is regularly contributed to their pension savings and ultimately determines the desired or the minimum level of their retirement income. We also prohibit the investors from short-selling or borrowing against future incomes. For greater realism, we allow the investors to only trade in short-term option contracts instead of long-term contracts or invest in a replicating strategy (which requires frequent adjustment on asset allocation). Finally, in addition to the risk averse utility function, we consider an S-shape loss aversion utility function proposed by (Kahneman and Tversky, 1979). To solve for the optimal investment asset allocations, we employ a numerical stochastic dynamic programming model.

We find that a loss aversion utility function naturally results in a more favourable retirement income distribution that peaks at the investor's chosen income goal. On the other hand, the constant risk aversion utility function yields a more dispersed income distribution, providing the investor less certainty of achieving a sufficient level of income in retirement. The strategy derived from loss aversion utility function appears more robust to a misspecification of the market model, compared to that derived from the risk averse utility function. The shape of the retirement outcome distribution is largely preserved for the former strategy when tested against the bootstrapped historical market return data, while the latter changes, to the detriment of the investor. Imposing terminal wealth boundary constraints, in both the utility function, result in strategies that provide certainty of achieving the lower boundary but at the cost of a significant reduction in the overall retirement outcome. We conclude that the investor can benefit from adopting a loss aversion-derived optimal investment strategy to target a sufficient level of income at retirement.

This thesis can be extended in several directions. While our thesis uses the Black-Scholes market model that follows geometric Brownian motions, a natural extension would be to move beyond the assumption of normally distributed returns and model the stock processes with a more general stochastic volatility model, e.g. such as one proposed by Heston (1993). This would likely result in an investment strategy with an additional term that hedges the stochastic volatility risk. It may be of an interest to investigate how a strategy balances its components that hedge the inflation risk, stochastic volatility risk, and minimum guarantees. Besides, our works fall under the framework the expected utility theory that may not always accurately predict the observed behaviour of investors in real-world scenarios or experiments settings. One potential extension is to employ the cumulative prospect theory (CPT) proposed by Tversky and Kahneman (1992), which features an S-shape loss aversion utility function, which was also used in Chapter 4. In addition, the CPT accounts for the tendency of individuals to overweight the extreme outcomes with low probabilities through a probability distortion function. Such a framework offers a more realistic representation of investor behavior than the expected utility theory framework used in our thesis. For instance, a recent paper by Escobar-Anel et al. (2020) introduced the Behavioral Portfolio Insurance, an extension to the widely studied constant proportion portfolio insurance that is optimal within the CPT framework. Exploring such extensions could help design better investment strategies for individuals whose preferences align with the assumptions of CPT.

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