Local Hardy Spaces and Quadratic Estimates for Dirac Type Operators on Riemannian Manifolds

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June 2010

A thesis submitted for the degree of Doctor of Philosophy
of The Australian National University.

Declaration

I hereby declare that except where otherwise indicated the work in this thesis is my own. The material in Chapter 2 is from my published paper [56] entitled "Local quadratic estimates and holomorphic functional calculi." The material in Chapter 3 is from my collaboration [21] with Andrea Carbonaro and Alan McIntosh that has been submitted under the title "Local Hardy spaces of differential forms on Riemannian manifolds." I intend to submit the material in Chapter 4 as a separate paper.

Andrew J. Morris

Acknowledgements

It has been an honour and a pleasure to have had Alan McIntosh as my supervisor. I found his resolve for solving difficult problems inspiring. This thesis is a result of his dedication and patience as a mentor and his kindness as a friend.

I was supported by the Australian Government through an Australian Postgraduate Award. I was also supported by the Mathematical Sciences Institute at the Australian National University and I thank the staff and faculty there for providing a friendly and encouraging environment. In particular, I would like to mention Ben Andrews, Andrew Hassell, Adam Rennie and Robert Taggart who always made time to answer my questions and provide advice. I also learnt a great deal from my fellow students and I am especially grateful to Charles Baker, Lashi Bandara, Jiakun Liu and Robert Scealy.

A part of this work was conducted at the Dipartimento di Matematica dell' Università degli Studi di Genova. I would like to thank Andrea Carbonaro for the wonderful memories I have of that time and for making our collaboration so successful. It is also a pleasure to thank Pascal Auscher, Andreas Axelsson, Chema Martell and Pierre Portal whose helpful conversations and suggestions have helped improve this work enormously.

I thank my family for their enduring belief in me. To my dearest Georgia, thank you for supporting me and bringing happiness into my day.

Abstract

The connection between quadratic estimates and the existence of a bounded holomorphic functional calculus of an operator provides a framework for applying harmonic analysis to the theory of differential operators. This is a generalization of the connection between Littlewood–Paley–Stein estimates and the functional calculus provided by the Fourier transform. We use the former approach in this thesis to study first-order differential operators on Riemannian manifolds. The theory developed is local in the sense that it does not depend on the spectrum of the operator in a neighbourhood of the origin. When we apply harmonic analysis to obtain estimates, the local theory only requires that we do so up to a finite scale. This allows us to consider manifolds with exponential volume growth in situations where the global theory requires polynomial volume growth.

A holomorphic functional calculus is constructed for operators on a reflexive Banach space that are bisectorial except possibly in a neighbourhood of the origin. We prove that this functional calculus is bounded if and only if certain local quadratic estimates hold. For operators with spectrum in a neighbourhood of the origin, the results are weaker than those for bisectorial operators. For operators with a spectral gap in a neighbourhood of the origin, the results are stronger. In each case, however, local quadratic estimates are a more appropriate tool than standard quadratic estimates for establishing that the functional calculus is bounded.

This theory allows us to define local Hardy spaces of differential forms that are adapted to a class of first-order differential operators on a complete Riemannian manifold with at most exponential volume growth. The local geometric Riesz transform associated with the Hodge–Dirac operator is bounded on these spaces provided that a certain condition on the exponential growth of the manifold is satisfied. A characterisation of these spaces in terms of local molecules is also obtained. These results can be viewed as the localisation of those for the Hardy spaces of differential forms introduced by Auscher, McIntosh and Russ.

Finally, we introduce a class of first-order differential operators that act on the trivial bundle over a complete Riemannian manifold with at most exponential volume growth and on which a local Poincaré inequality holds. A local quadratic estimate is established for certain perturbations of these operators. As an application, we solve the Kato square root problem for divergence form operators on complete Riemannian manifolds with Ricci curvature bounded below that are embedded in Euclidean space with a uniformly bounded second fundamental form. This is based on the framework for Dirac type operators that was introduced by Axelsson, Keith and McIntosh.

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Chapter 1

Introduction

A functional calculus associates a linear operator T on a Banach space $\mathcal{X}$ and a space of functions $\mathcal{F}$ with a mapping from $\mathcal{F}$ into the space of linear operators on $\mathcal{X}$ that is canonical in a certain sense. It is usual to denote this mapping by $f \mapsto f(T)$ for all f in $\mathcal{F}$. In applications, such as those discussed below, it is often desirable to know that $f(T)$ is bounded and that its operator norm is controlled by some property of f in $\mathcal{F}$. This is the important notion of a bounded functional calculus.

The Dunford–Riesz–Taylor functional calculus is defined for closed operators T with nonempty resolvent sets and the space of functions $H(\Omega)$ that are holomorphic on a domain Ω in $\mathbb{C}$ that contains a neighbourhood of the spectrum of T. We postpone the definition of this functional calculus until the beginning of Chapter 2. The idea of McIntosh in [53] was to instead design a functional calculus suited to operators of type S_ω. These are closed operators satisfying certain resolvent bounds and having spectrum contained in the bisector S_ω centered at the origin in the complex plane of angle ω in $(0, \pi/2)$. The advantage of the resulting functional calculus is that it is defined for functions that need not be holomorphic in a neighbourhood of the origin nor the point at infinity.

It is shown in [53] that the McIntosh functional calculus for an injective operator T of type S_ω on a Hilbert space $\mathcal{H}$ is bounded if and only if quadratic estimates of the form

$$\int_0^\infty \|tT(I + t^2T^2)^{-1}u\|_{\mathcal{H}}^2 \, \frac{\mathrm{d}t}{t} \lesssim \|u\|_{\mathcal{H}}^2$$

hold for all $u \in \mathcal{H}$. Establishing these types of quadratic estimates in order to have a bounded holomorphic functional calculus has been used with great effect in many applications. Most notable is the proof of the Kato Conjecture by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian in [41, 6] and it's many extensions, including that by Axelsson, Keith and McIntosh in [11] and that by Hytönen, McIntosh and Portal in [43].

More generally, given an operator T of type S_ω and a domain Ω that touches the spectrum of T nontangentially at a point, the functional calculus on the space of functions that are holomorphic in Ω depends on quadratic estimates approaching the point of contact. Indeed, the lower and upper limits in the quadratic estimates above correspond to the spectral points at infinity and at the origin, respectively. The case of several points of contact has also been considered by Franks and McIntosh in [34].

In Chapter 2, we replicate the construction by McIntosh in [53] for operators on a Banach space $\mathcal{X}$ that satisfy resolvent bounds and have spectrum contained in either the set $S_{\omega\cup R}$ or the set $S_{\omega\setminus R}\cup\{0\}$, as depicted in Figure 1.1. This also builds on work by Cowling, Doust, McIntosh and Yagi in [31]. For operators of type $S_{\omega\cup R}$, the functional calculus that we construct is defined for functions that must be holomorphic in a neighbourhood of the origin but need not be holomorphic in a neighbourhood of the point at infinity. For operators of type $S_{\omega\setminus R}$, our functional calculus is defined for functions that need not even be defined in a neighbourhood of the origin. As a result, the functional calculus in both cases only depends on quadratic estimates near the spectral point at infinity. We refer to these as local quadratic estimates, since they are of the form

$$\int_0^1 \|tT(I+t^2T^2)^{-1}u\|_{\mathcal{X}}^2\,\frac{\mathrm{d}t}{t} + \|(I+T^2)^{-1}u\|_{\mathcal{X}}^2 \lesssim \|u\|_{\mathcal{X}}^2$$

for all u in a suitable subspace of $\mathcal{X}$. This also builds on work by Albrecht, Duong and McIntosh in [2].

The advantage of only having to establish local quadratic estimates is that techniques from harmonic analysis that usually require at most polynomial volume growth can then be applied on metric measure spaces with exponential volume growth. The theory of type $S_{\omega\setminus R}$ operators is a special case of the theory of type S_ω operators and the results are stronger. In particular, self-adjoint operators with a spectral gap at the origin are of type $S_{\omega\setminus R}$, where $R>0$ is equal to the spectral gap. Our results show that the existence of a bounded functional calculus for such operators requires only local quadratic estimates.

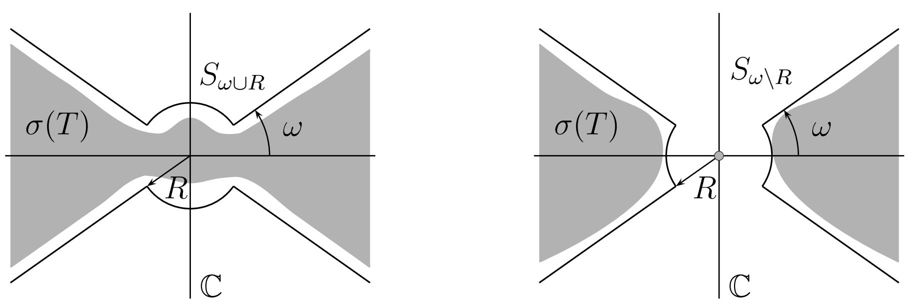

FIGURE 1.1: The sets $S_{\omega\cup R}$ and $S_{\omega\setminus R}$ for $\omega\in(0,\pi/2)$ and $R>0$. The shaded areas depict the spectrums of an operator of type $S_{\omega\cup R}$ and an operator of type $S_{\omega\setminus R}$. In both cases, the origin may be in the spectrum.

The theory of type $S_{\omega\cup R}$ operators, which is a weak version of the theory in [53], is also more suited to certain applications. For example, consider the operator $-i\mathrm{d}/\mathrm{d}x$ on the Sobolev space $W^{1,2}(\mathbb{R})$. The connection between singular convolution operators and the functional calculus of $-i\mathrm{d}/\mathrm{d}x$ is well-understood. In particular, local Hilbert transforms h are defined for $a>0$ as the Fourier multiplier

$$(hu)\widehat{\ }(\xi) = i\xi(|\xi|^2+a)^{-1/2}\widehat{u}(\xi).$$

These correspond to the operator $h(-i\mathrm{d}/\mathrm{d}x)$ under our new functional calculus, where $h(z) = z(z^2 + a)^{-1/2}$ is holomorphic at the origin but not at the point at infinity.

Local Hilbert and Riesz transforms motivate the definition of local Hardy spaces. The local Hardy space $h^1(\mathbb{R}^n)$ introduced by Goldberg in [36] is an intermediate space $H^1(\mathbb{R}^n) \subset h^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. The Hardy space $H^1(\mathbb{R}^n)$ is suited to quasi-homogenous multipliers, and indeed the boundedness of the Riesz transforms defined for $j = 1, \dots, n$ by

$$(R_j u)\widehat{\ }(\xi) = i\xi_j|\xi|^{-1}\widehat{u}(\xi)$$

is built into its definition. The local Hardy space $h^1(\mathbb{R}^n)$, however, is suited to *smooth* quasi-homogenous multipliers, and the boundedness of local Riesz transforms such as those defined for $a > 0$ and $j = 1, \dots, n$ by

$$(r_j u)\widehat{\ }(\xi) = i\xi_j(|\xi|^2 + a)^{-1/2}\widehat{u}(\xi)$$

is built into its definition.

In Chapter 3, we define local Hardy spaces of differential forms $h^p_{\mathcal{D}}(\wedge T^*M)$ that are adapted to a class of first-order differential operators $\mathcal{D}$ that are of type $S_{\omega\cup R}$ and satisfy local quadratic estimates on a Riemannian manifold M with exponential volume growth. This is an extension of the work by Auscher, McIntosh and Russ in [9] to construct Hardy spaces of differential forms $H^p_D(\wedge T^*M)$ for the Hodge–Dirac operator D on Riemannian manifolds with polynomial volume growth. To be precise, let M denote a complete Riemannian manifold with geodesic distance ρ and Riemannian measure μ. We adopt the convention that such a manifold M is smooth and connected. Let $L^2(\wedge T^*M)$ denote the Hilbert space of square-integrable differential forms on M. Let d and d^* denote the exterior derivative and its adjoint on $L^2(\wedge T^*M)$. The Hodge–Dirac operator is $D = d + d^*$ and the Hodge–Laplacian is $\Delta = D^2$. The geometric Riesz transform $D\Delta^{-1/2}$ is bounded on $L^2(\wedge T^*M)$ because $\mathcal{D}$ is self-adjoint. This led the authors of [9] to construct Hardy spaces of differential forms $H^p_D(\wedge T^*M)$, or simply H^p_D, for all $p \in [1, \infty]$. Amongst other things, they show that the geometric Riesz transform is bounded on H^p_D and that H^1_D has a molecular characterisation.

The atomic characterisation of $H^1(\mathbb{R}^n)$, due to Coifman [28] and Latter [49], was used by Coifman and Weiss in [30] to define a Hardy space of functions on a space of homogeneous type. A requirement in the definition of Hardy space atoms a is that they satisfy the moment condition $\int a = 0$. The approach taken in [9] is instead based on the connection between the tent spaces $T^p(\mathbb{R}^{n+1}_+)$ and $H^p(\mathbb{R}^n)$. This connection was first recognised by Coifman, Meyer and Stein who showed in Section 9B of [27] that $H^p(\mathbb{R}^n)$ is isomorphic to a complemented subspace of $T^p(\mathbb{R}^{n+1}_+)$ for all $p \in [1, \infty]$. More precisely, there exist two bounded operators $P : H^p(\mathbb{R}^n) \to T^p(\mathbb{R}^{n+1}_+)$ and $\pi : T^p(\mathbb{R}^{n+1}_+) \to H^p(\mathbb{R}^n)$ such that $P\pi$ is a projection and $H^p(\mathbb{R}^n)$ is isomorphic to $P\pi(T^p(\mathbb{R}^{n+1}_+))$.

The definition of the tent space $T^1(\mathbb{R}^{n+1}_+)$ and its atoms, which are not required to satisfy a moment condition, admit natural generalisations to differential forms. Also, both P and π are convolution-type operators, which can be interpreted as functions of $-i\nabla := (-i\partial/\partial x_1, \dots, -i\partial/\partial x_n)$. The idea in [9] was to define H^p_D in terms of the tent space of differential forms $T^p(\wedge T^*M \times (0,\infty))$ and operators $\mathcal{Q}$ and $\mathcal{S}$, which are adapted to D in the same way that P and π are adapted to $-i\nabla$. The main requirement for the construction was that operators such as the projection $\mathcal{QS}$ be bounded on $T^p(\wedge T^*M \times (0,\infty))$. The authors of [9] prove this by using off-diagonal estimates for the resolvents of D to establish uniform bounds on tent space atoms.

We adapt the definition of H^p_D to define local Hardy spaces of differential forms $h^p_D(\wedge T^*M)$, or simply h^p_D, for all $p \in [1,\infty]$. We first consider a general locally doubling metric measure space X, and define a local tent space $t^p(X \times (0,1])$ and a new function space $L^p_{\mathcal{Q}}(X)$, both of which have an atomic characterisation when $p=1$ and admit a natural generalisation to differential forms. Classically, it can be shown that the local Hardy space $h^p(\mathbb{R}^n)$ is isomorphic to a complemented subspace of $t^p(\mathbb{R}^n \times (0,1]) \oplus L^p_{\mathcal{Q}}(\mathbb{R}^n)$. Whilst square function characterisations for $h^p(\mathbb{R}^n)$ are certainly known, this characterisation appears to be new.

The atomic characterisation of $h^1(\mathbb{R}^n)$, due to Goldberg in [36], consists of two types of atoms. The first kind are supported on balls of radius less than one and satisfy a moment condition, whilst the second kind are supported on balls with radius larger than one but are not required to satisfy a moment condition. In our new characterisation, we can associate the first kind of atoms with elements of $t^1(\mathbb{R}^n \times (0,1])$ and the second kind with elements of $L^1_{\mathcal{Q}}(\mathbb{R}^n)$.

The definition of H^p_D in [9] is limited to Riemannian manifolds that are doubling, which we define below using the following notation. Given $x \in M$ and $r > 0$, let $B(x,r)$ denote the open geodesic ball in M with centre x and radius r, and let $V(x,r)$ denote the Riemannian measure $\mu(B(x,r))$.

Definition 1.1.1. A complete Riemannian manifold M is *doubling* if there exists $A \geq 1$ such that

$$0 < V(x,2r) \leq AV(x,r) < \infty \tag{D}$$

for all $x \in X$ and $r > 0$.

The doubling condition is equivalent to the requirement that there exist $A \geq 1$ and $\kappa \geq 0$ such that

$$0 < V(x,\alpha r) \leq A\alpha^\kappa V(x,r) < \infty$$

for all $x \in X$, $r > 0$ and $\alpha \geq 1$. This condition is imposed to define H^p_D because the Hardy space norm incorporates global geometry. The nature of the local Hardy space, however, allows us to define h^p_D on manifolds that are only locally doubling. Specifically, we define h^p_D on the following class of manifolds.

Definition 1.1.2. A complete Riemannian manifold M is *exponentially locally doubling* if there exist $A \geq 1$ and $\kappa, \lambda \geq 0$ such that

$$0 < V(x,\alpha r) \leq A\alpha^\kappa e^{\lambda(\alpha-1)r} V(x,r) < \infty \tag{$\mathrm{E}_{\kappa,\lambda}$}$$

for all $x \in M$, $r > 0$ and $\alpha \geq 1$. The constants κ and λ are referred to as the *polynomial* and *exponential growth parameters*, respectively.

The class of doubling Riemannian manifolds includes $\mathbb{R}^n$ with the Euclidean distance and the standard Lebesgue measure, as well as Lie groups with polynomial volume growth; other examples are listed in [9]. The class of exponentially locally doubling Riemannian manifolds is larger and includes hyperbolic space (see Section 3.H.3 of [35]), Lie groups with exponential volume growth (see Section II.4 of [33]) and thus all Lie groups. More generally, by Gromov's variant of the Bishop comparison theorem (see [16, 37]), all Riemannian manifolds with Ricci curvature bounded from below are exponentially locally doubling. This includes Riemannian manifolds with bounded geometry, noncompact symmetric spaces and Damek–Ricci spaces.

Taylor recently defined local Hardy spaces of functions on Riemannian manifolds with bounded geometry in [67]. Hardy spaces of functions have also been considered on some nondoubling metric measure spaces by Carbonaro, Mauceri and Meda in [18, 19, 20] and by Mauceri, Meda and Vallarino in [51, 52]. The theory developed in those papers applies to $\mathbb{R}^n$ with the Euclidean distance and the Gaussian measure, as well as to Riemannian manifolds on which the Ricci curvature is bounded from below and the Laplace–Beltrami operator has a spectral gap.

The Hardy spaces H^p_D in [9] are defined using the holomorphic functional calculus of D in $L^2(\wedge T^*M)$. In particular, the authors consider the class $H^\infty(S^o_\theta)$ of functions that are bounded and holomorphic on the open bisector S^o_θ of angle $\theta \in (0, \pi/2)$ centered at the origin in the complex plane. This is because the function $\operatorname{sgn}(\operatorname{Re}(z)) = z/\sqrt{z^2}$ maps to the geometric Riesz transform $D\Delta^{-1/2}$ under the $H^\infty(S^o_\theta)$ functional calculus. The local Hardy spaces, however, are suited to the local geometric Riesz transforms $D(\Delta + aI)^{-1/2}$ for $a > 0$, so we consider the smaller class $H^\infty(S^o_{\theta\cup r})$ of functions that are bounded and holomorphic on $S^o_{\theta\cup r} = S^o_\theta \cup D^o_r$, where D^o_r is the open disc of radius $r > 0$ centered at the origin in the complex plane.

The space h^1_D has a characterisation in terms of local molecules, which are defined in Section 3.6.1. This is the first main result of Chapter 3.

Theorem 1.1.3. *Let $\kappa, \lambda \geq 0$ and suppose that M is a complete Riemannian manifold satisfying $(\mathrm{E}_{\kappa,\lambda})$. If $N \in \mathbb{N}$, $N > \kappa/2$ and $q \geq \lambda$, then $h^1_D = h^1_{D,mol(N,q)}$.*

The following is the principal result of Chapter 3.

Theorem 1.1.4. *Let $\kappa, \lambda \geq 0$ and suppose that M is a complete Riemannian manifold satisfying $(\mathrm{E}_{\kappa,\lambda})$. Let $\theta \in (0, \pi/2)$ and $r > 0$ such that $r \sin\theta > \lambda/2$. Then for all $f \in H^\infty(S^o_{\theta,r})$, the operator $f(D)$ on L^2 has a bounded extension such that*

$$\|f(D)u\|_{h^p_D} \lesssim \|f\|_\infty \|u\|_{h^p_D}$$

for all $u \in h^p_D$ and $p \in [1, \infty]$.

There is then the following corollary for the local geometric Riesz transforms.

Corollary 1.1.5. Let $\kappa, \lambda \geq 0$ and suppose that M is a complete Riemannian manifold satisfying $(\mathrm{E}_{\kappa,\lambda})$. If $a > \lambda^2/4$, then the local geometric Riesz transform $D(\Delta + aI)^{-1/2}$ has a bounded extension to h^p_D for all $p \in [1, \infty]$.

The theory in Chapter 3 is actually developed for a large class of first-order differential operators $\mathcal{D}$, which we introduce in Section 3.4. Theorems 1.1.3 and 1.1.4

follow from the more general results in Theorems 3.6.13 and 3.6.19 by setting $\mathcal{D} = D$, where D will always denote the Hodge–Dirac operator. Although the space $h^2_{\mathcal{D}}(\wedge T^*M)$ is defined so that it can be identified with $L^2(\wedge T^*M)$, the proof of the embedding $h^p_{\mathcal{D}}(\wedge T^*M) \subseteq L^p(\wedge T^*M)$ for all $p \in [1,2)$ in Section 3.7 requires the additional assumption that $\mathcal{D}$ is a self-adjoint, elliptic differential operator with finite propagation speed and that $\inf_{x\in M} V(x,1) > 0$. The following is then a corollary of the more general results in Theorem 3.7.6 and Corollary 3.7.7.

Corollary 1.1.6. Let $\kappa, \lambda \geq 0$ and suppose that M is a complete Riemannian manifold satisfying $(\mathrm{E}_{\kappa,\lambda})$ and that $\inf_{x\in M} V(x,1) > 0$. If $a > \lambda^2/4$, then the local geometric Riesz transform $D(\Delta + aI)^{-1/2}$ has a bounded extension such that

$$\|D(\Delta + aI)^{-\frac{1}{2}} u\|_p \lesssim \|u\|_{h^p_D}$$

for all $u \in h^p_D$ and $p \in [1,2]$.

Taylor proved in [66] that on a Riemannian manifold with bounded geometry, where Δ_0 denotes the Hodge–Laplacian on functions, a necessary condition for the operator $f(\sqrt{\Delta_0})$ to be bounded on L^p for all $p \in (1,\infty)$ is that f be holomorphic and satisfy inhomogeneous Mihlin boundary conditions on an open strip of width $W \geq \lambda/2$ in the complex plane, where $\lambda \geq 0$ is such that $(\mathrm{E}_{\kappa,\lambda})$ holds. This result was improved by Mauceri, Meda and Vallarino in [50]. The need for f to be holomorphic on a strip was originally noted by Clerc and Stein in the setting of noncompact symmetric spaces in [25], and that work was extended by Stanton and Tomas in [62], by Cheeger, Gromov and Taylor in [23] and by Anker and Lohoué in [3]. In Chapter 3, we do not assume bounded geometry. Theorem 1.1.4 represents the beginning of the development of an approach to this theory based on first-order operators. Moreover, given that $r \sin\theta$ is the width of the largest open strip contained in $S^o_{\theta\cup r}$, Taylor's result suggests that the bound $r\sin\theta > \lambda/2$ in Theorem 1.1.4 may be the best possible.

Let us now briefly recall the Kato square root problem on $\mathbb{R}^n$. Given a strictly accretive matrix valued function A on $\mathbb{R}^n$ with bounded measurable coefficients, the Kato square root problem is to determine the domain of the square root $\sqrt{\mathrm{div}(A\nabla)}$ of the divergence form operator $\mathrm{div}(A\nabla)$. The original questions posed by Kato can be found in [46, 47] and are discussed further in [54]. The problem was solved completely in the case $n = 1$ by Coifman, McIntosh and Meyer in [26], in the case $n = 2$ by Hofmann and McIntosh in [42] and finally for all $n \in \mathbb{N}$ by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian in [6]. The reader is referred to the references within those works for the full list of attributes that lead to those results, since it is not possible to include them all here.

In [8], prior to the solution of the Kato problem in all dimensions, Auscher, McIntosh and Nahmod reduced the one dimensional problem to proving quadratic estimates for a related first-order elliptic system. Subsequently, Axelsson, Keith and McIntosh in [11] developed a general framework for proving quadratic estimates for perturbations of Dirac type operators on $\mathbb{R}^n$. In this unifying approach, the solution of the Kato problem in all dimensions, as well as many results in the Calderón program such as the boundedness of the Cauchy singular integral operator on Lipschitz curves, follow as immediate corollaries. Their results also have applications to compact Riemannian manifolds, and it is these applications that we extend to certain noncompact manifolds in this thesis.

In Chapter 4, we introduce a class of first-order differential operators that act on the trivial bundle over a complete Riemannian manifold. This is based on the framework that was introduced in [11], although it resembles more closely the subsequent development by the same authors in [10]. The main result is a local quadratic estimate for certain L^∞ perturbations of these operators. The statement of the result requires some technical preliminaries and we postpone it until Theorem 4.1.5. As an application, we solve the Kato square root problem for divergence form operators on complete Riemannian manifolds M with Ricci curvature bounded below that are embedded in $\mathbb{R}^n$ with a uniformly bounded second fundamental form. These manifolds are not required to be compact.

To state the result, we require the following setup, which will be made more precise in Section 4.1. Let TM denote the tangent bundle over M and let T_xM denote the tangent space at each $x \in M$. Let $L^2(TM)$ denote the Hilbert space of square-integrable vector fields on M and let $W^{1,2}(M)$ denote the Sobolev space of functions on M. The gradient and divergence on M are closed operators

$$\begin{aligned} \text{grad} &: L^2(M) \to L^2(TM) \\ \text{div} &: L^2(TM) \to L^2(M) \end{aligned}$$

with domain $\mathsf{D}(\text{grad}) = W^{1,2}(M)$ and $-\text{div}$ being formally adjoint to grad. Let I denote the identity operator on $L^2(M)$ and following [10] define the operator

$$S = \begin{bmatrix} I \\ \text{grad} \end{bmatrix} : \mathsf{D}(S) \subseteq L^2(M) \to L^2(M) \oplus L^2(TM)$$

with domain $\mathsf{D}(S) = W^{1,2}(M)$ and adjoint

$$S^* = \begin{bmatrix} I & -\text{div} \end{bmatrix} : \mathsf{D}(S^*) \subseteq L^2(M) \oplus L^2(TM) \to L^2(M).$$

We also require the following notation. For all Banach spaces X and Y, let $\mathcal{L}(X,Y)$ denote the Banach algebra of bounded linear operators from X into Y, and let $\mathcal{L}(X) = \mathcal{L}(X,Y)$. Now let $\mathscr{L}(TM,\mathbb{C})$, $\mathscr{L}(\mathbb{C},TM)$ and $\mathscr{L}(TM)$ denote the vector bundles over M whose fibers at each $x \in M$ are given by $\mathcal{L}(T_xM,\mathbb{C})$, $\mathcal{L}(\mathbb{C},T_xM)$ and $\mathcal{L}(T_xM)$, respectively. Also, let $\mathscr{L}^\infty(TM,\mathbb{C})$, $\mathscr{L}^\infty(\mathbb{C},TM)$ and $\mathscr{L}^\infty(TM)$ denote the spaces of L^∞ sections of the respective bundles.

Given $A_{00} \in L^\infty(M)$ as well as $A_{01} \in \mathscr{L}^\infty(TM,\mathbb{C})$, $A_{10} \in \mathscr{L}^\infty(\mathbb{C},TM)$ and $A_{11} \in \mathscr{L}^\infty(TM)$, define the operator $A : L^2(M) \oplus L^2(TM) \to L^2(M) \oplus L^2(TM)$ by

$$(Au)_x = \begin{bmatrix} (A_{00})_x & (A_{01})_x \\ (A_{10})_x & (A_{11})_x \end{bmatrix} \begin{bmatrix} (u_0)_x \\ (u_1)_x \end{bmatrix}$$

for all $u = (u_0, u_1) \in L^2(M) \oplus L^2(TM)$ and $x \in M$, where $(\cdot)_x$ denotes the value of a function or section at x. Furthermore, given $a \in L^\infty(M)$, suppose that there exists κ_1, $\kappa_2 > 0$ such that the following accretivity conditions are satisfied:

$$\begin{aligned} \operatorname{Re}\langle a\,u, u\rangle_{L^2(TM)} &\geq \kappa_1 \|u\|^2_{L^2(M)} && \text{for all} \quad u \in L^2(M); \\ \operatorname{Re}\langle ASu, Su\rangle_{L^2(M)\oplus L^2(TM)} &\geq \kappa_2 \|u\|^2_{W^{1,2}(M)} && \text{for all} \quad u \in W^{1,2}(M). \end{aligned} \tag{1.1.1}$$

The divergence form operator $L_A : \mathsf{D}(L_A) \to L^2(M)$ is then defined by

$$L_A u = aS^* ASu = -a\,\mathrm{div}(A_{11}\,\mathrm{grad}\,u) - a\,\mathrm{div}(A_{10}u) + aA_{01}\,\mathrm{grad}\,u + aA_{00}u \quad (1.1.2)$$

for all $u \in \mathsf{D}(L_A) = \{u \in W^{1,2}(M) : ASu \in \mathsf{D}(S^*)\}$. The following result is then proved as a corollary of Theorem 4.1.5 in Section 4.2.

Theorem 1.1.7. *Let $n \in \mathbb{N}$ and suppose that M is a complete Riemannian manifold with Ricci curvature bounded below that is embedded in $\mathbb{R}^n$ with a uniformly bounded second fundamental form. If a and A satisfy the accretivity conditions* (1.1.1)*, then the divergence form operator L_A defined by* (1.1.2) *has a square root $\sqrt{L_A}$ with domain $\mathsf{D}(\sqrt{L_A}) = W^{1,2}(M)$ and*

$$\|\sqrt{L_A}u\|_{L^2(M)} \eqsim \|u\|_{W^{1,2}(M)}$$

for all $u \in W^{1,2}(M)$.

The following notational conventions are adopted throughout the thesis. For all $x, y \in \mathbb{R}$, we write $x \lesssim y$ to mean that there exists a constant $c \geq 1$, which may only depend on constants specified in the relevant preceding hypotheses, such that $x \leq cy$. To emphasize that the constant c depends on a specific parameter p, we write $x \lesssim_p y$. Also, we write $x \eqsim y$ to mean that $x \lesssim y \lesssim x$. For all normed spaces X and Y, we write $X \subseteq Y$ to mean both the set theoretical inclusion and the topological inclusion, whereby $\|x\|_Y \lesssim \|x\|_X$ for all $x \in X$. Finally, we write $X = Y$ to mean that X and Y are equal as sets and that they have equivalent norms.

Chapter 2

Local Quadratic Estimates and Functional Calculi

We begin this chapter by fixing notation and recalling the Dunford–Riesz–Taylor functional calculus. The holomorphic functional calculus for operators of type $S_{\omega\cup R}$ is constructed in Section 2.2. Local quadratic estimates are defined in Section 2.2.1 and the equivalence with bounded holomorphic functional calculi is proved in Section 2.2.2. The analogous results for operators of type $S_{\omega\setminus R}$ are in Section 2.3.

2.1 Notation and Preliminaries

Throughout this chapter, let $\mathcal{X}$ denote a nontrivial complex reflexive Banach space with norm $\|\cdot\|_{\mathcal{X}}$. An *operator* T on $\mathcal{X}$ is a linear mapping $T : \mathsf{D}(T) \to \mathcal{X}$, where the *domain* $\mathsf{D}(T)$ is a subspace of $\mathcal{X}$. The *range* $\mathsf{R}(T) = \{Tu : u \in \mathsf{D}(T)\}$ and the *null-space* $\mathsf{N}(T) = \{u \in \mathsf{D}(T) : Tu = 0\}$. Let $\overline{\mathsf{D}(T)}$ and $\overline{\mathsf{R}(T)}$ denote the closure of these subspaces in $\mathcal{X}$. An operator T is *closed* if the graph $\mathsf{G}(T) = \{(u, Tu) : u \in \mathsf{D}(T)\}$ is a closed subspace of $\mathcal{X} \times \mathcal{X}$, and *bounded* if the operator norm

$$\|T\| = \sup\{\|Tu\|_{\mathcal{X}} : u \in \mathsf{D}(T) \text{ and } \|u\|_{\mathcal{X}} \le 1\}$$

is finite. To minimise notation, we also denote the norm on $\mathcal{X}$ by $\|\cdot\|$ when there is no danger of confusion. The unital algebra of bounded operators on $\mathcal{X}$ is denoted by $\mathcal{L}(\mathcal{X})$, where the unit is the identity operator I on $\mathcal{X}$. The *resolvent set* $\rho(T)$ is the set of all $z \in \mathbb{C}$ for which the operator $zI - T$ has a bounded inverse with domain equal to $\mathcal{X}$. The *resolvent* $R_T(z)$ is the operator on $\mathcal{X}$ defined by

$$R_T(z) = (zI - T)^{-1}$$

for all $z \in \rho(T)$. The *spectrum* $\sigma(T)$ is the complement of the resolvent set in the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

Given an open set $\Omega \subseteq \mathbb{C}_\infty$, let $H(\Omega)$ denote the algebra of holomorphic functions on Ω. Note that a function f is holomorphic in a neighbourhood of the point at infinity if $f(1/z)$ is holomorphic in a neighbourhood of the origin. The following functional calculus is usually attributed to N. Dunford, F. Riesz and A. E. Taylor. The precise formulation below is from [65].

Definition 2.1.1 (Dunford–Riesz–Taylor $H(\Omega)$ functional calculus)**.** Let T be a closed operator on $\mathcal{X}$ with nonempty resolvent set. If Ω is a proper open subset of $\mathbb{C}_\infty$ that contains $\sigma(T) \cup \{\infty\}$ and $f \in H(\Omega)$, then define $f(T) \in \mathcal{L}(\mathcal{X})$ by

$$f(T)u = f(\infty)u + \frac{1}{2\pi i}\int_\gamma f(z)R_T(z)u \,\mathrm{d}z \tag{2.1.1}$$

for all $u \in \mathcal{X}$, where $f(\infty) = \lim_{z\to\infty} f(z)$ and γ is the boundary of an unbounded Cauchy domain that is oriented clockwise and envelopes $\sigma(T)$ in Ω.

If T is a bounded operator on $\mathcal{X}$, then Ω in Definition 2.1.1 need not contain the point at infinity, in which case $f(T)u = \frac{1}{2\pi i}\int_\gamma f(z)R_T(z)u \,\mathrm{d}z$. A comprehensive list of attributes and references to the literature on this topic can be found at the end of Chapter VII in [32]. The following theorem, which is set as an exercise in [2], is a consequence of Runge's Theorem.

Theorem 2.1.2. *The mapping given by* (2.1.1) *is the unique algebra homomorphism from $H(\Omega)$ into $\mathcal{L}(\mathcal{X})$ with following properties:*

1. *If $\mathbf{1}(z) = 1$ for all $z \in \Omega$, then $\mathbf{1}(T) = I$ on $\mathcal{X}$;*
2. *If $\lambda \in \rho(T) \setminus \Omega$ and $f(z) = (\lambda - z)^{-1}$ for all $z \in \Omega$, then $f(T) = R_T(\lambda)$;*
3. *If $(f_n)_n$ is a sequence in $H(\Omega)$ that converges uniformly on compact subsets of Ω to $f \in H(\Omega)$, then $f_n(T)$ converges to $f(T)$ in $\mathcal{L}(\mathcal{X})$.*

We now introduce the following setup. Given $0 \le \mu < \theta < \pi/2$, define the closed and open bisectors in the complex plane as follows:

$$\begin{aligned} S_\mu &= \{z \in \mathbb{C} : |\arg z| \le \mu \text{ or } |\pi - \arg z| \le \mu\}; \\ S_\theta^o &= \{z \in \mathbb{C}\setminus\{0\} : |\arg z| < \theta \text{ or } |\pi - \arg z| < \theta\}. \end{aligned}$$

Given $r \ge 0$, define the closed and open discs as follows:

$$\begin{aligned} D_r &= \{z \in \mathbb{C} : |z| \le r\} \\ D_r^o &= \{z \in \mathbb{C} : |z| < r\}. \end{aligned}$$

These are combined together as follows:

$$\begin{aligned} S_{\mu\cup r} &= S_\mu \cup D_r; & S_{\mu\setminus r} &= S_\mu \setminus D_r^o; \\ S_{\theta\cup r}^o &= S_\theta^o \cup D_r^o; & S_{\theta\setminus r}^o &= S_\theta^o \setminus D_r. \end{aligned}$$

Note that $D_0 = \{0\}$ and $D_0^o = \emptyset$ so that $S_{\mu\cup 0} = S_{\mu\setminus 0} = S_\mu$ and $S_{\theta\cup 0}^o = S_{\theta\setminus 0}^o = S_\theta^o$. Let $S_{\theta,r}^o$ denote either $S_{\theta\cup r}^o$ or $S_{\theta\setminus r}^o$. A function on $S_{\theta,r}^o$ is called *nondegenerate* if it is not identically zero on either component of $S_{\theta,r}^o$.

Let $H^\infty(S_{\theta,r}^o)$ denote the algebra of bounded holomorphic functions on $S_{\theta,r}^o$. Given $f \in H^\infty(S_{\theta,r}^o)$ and $t \in (0,1]$, define $f^* \in H^\infty(S_{\theta,r}^o)$ and $f_t \in H^\infty(S_{\theta,r/t}^o)$ as follows:

$$\begin{aligned} f^*(z) &= \overline{f(\bar{z})} \quad \text{for all } \ z \in S_{\theta,r}^o; \\ f_t(z) &= f(tz) \quad \text{for all } \ z \in S_{\theta,r/t}^o. \end{aligned}$$

Given $\alpha, \beta > 0$, define the following sets:

$$\Psi_\alpha^\beta(S_{\theta,r}^o) = \{\psi \in H^\infty(S_{\theta,r}^o) : |\psi(z)| \lesssim \min(|z|^\alpha, |z|^{-\beta})\};$$
$$\Theta^\beta(S_{\theta,r}^o) = \{\phi \in H^\infty(S_{\theta,r}^o) : |\phi(z)| \lesssim |z|^{-\beta}\}.$$

Let $\Psi_\alpha(S_{\theta,r}^o) = \bigcup_{\beta>0} \Psi_\alpha^\beta(S_{\theta,r}^o)$, $\Psi^\beta(S_{\theta,r}^o) = \bigcup_{\alpha>0} \Psi_\alpha^\beta(S_{\theta,r}^o)$, $\Psi(S_{\theta,r}^o) = \bigcup_{\beta>0} \Psi^\beta(S_{\theta,r}^o)$ and $\Theta(S_{\theta,r}^o) = \bigcup_{\beta>0} \Theta^\beta(S_{\theta,r}^o)$.

2.2 Operators of Type $S_{\omega\cup R}$

We construct holomorphic functional calculi for the following class of operators, where $\mathcal{X}$ denotes a nontrivial complex reflexive Banach space.

Definition 2.2.1. Let $\omega \in [0, \pi/2)$ and $R \geq 0$. An operator T on $\mathcal{X}$ is of *type* $S_{\omega\cup R}$ if $\sigma(T) \subseteq S_{\omega\cup R}$, and for each $\theta \in (\omega, \pi/2)$ and $r > R$, there exists $C_{\theta\cup r} > 0$ such that

$$\|R_T(z)\| \leq \frac{C_{\theta\cup r}}{|z|}$$

for all $z \in \mathbb{C} \setminus S_{\theta\cup r}$.

The condition that an operator T is of type $S_{\omega\cup 0}$ is precisely the condition that T is of type S_ω (or ω-sectorial). The theory of type S_ω operators is well-understood and can be found in, for instance, [2, 38, 47, 53].

The following important lemma allows us to obtain stronger results in reflexive Banach spaces. The proof below is derived from the proof of Theorem 3.8 in [31].

Lemma 2.2.2. Let $\omega \in [0, \pi/2)$ and $R \geq 0$. Let T be an operator of type $S_{\omega\cup R}$ on $\mathcal{X}$. If $r > R$, then

$$\overline{\mathsf{D}(T)} = \{u \in \mathcal{X} : \lim_{n\to\infty} (I + \tfrac{i}{rn}T)^{-1}u = u\} = \mathcal{X}.$$

Proof. If $u \in \mathcal{X}$ and $\lim_{n\to\infty}(I + \frac{i}{rn}T)^{-1}u = u$, then $u \in \overline{\mathsf{D}(T)}$ simply because $\mathsf{R}((I + \frac{i}{rn}T)^{-1}) = \mathsf{D}(T)$ for all $n \in \mathbb{N}$.

To prove the converse, first suppose that $u \in \mathsf{D}(T)$. The resolvent bounds in Definition 2.2.1 imply that

$$\|(I + \tfrac{i}{rn}T)^{-1}u - u\| = \|\tfrac{i}{rn}(I + \tfrac{i}{rn}T)^{-1}Tu\| = \|R_T(irn)Tu\| \lesssim (1/rn)\|Tu\|$$

for all $n \geq 1$, which implies that $\lim_{n\to\infty}(I + \frac{i}{rn}T)^{-1}u = u$. Now suppose that $u \in \overline{\mathsf{D}(T)}$. For each $\epsilon > 0$, there exists $v \in \mathsf{D}(T)$ and $N \in \mathbb{N}$ such that $\|u - v\| < \epsilon$ and

$$\begin{aligned}\|(I + \tfrac{i}{rn}T)^{-1}u - u\| &\leq \|(I + \tfrac{i}{rn}T)^{-1}(u - v)\| + \|(I + \tfrac{i}{rn}T)^{-1}v - v\| + \|v - u\| \\ &\lesssim (rn\|R_T(irn)\| + 1)\|u - v\| + (1/rn)\|Tv\| \\ &\lesssim \epsilon\end{aligned}$$

for all $n > N$, as required.

The proof that $\overline{\mathsf{D}(T)} = \mathcal{X}$ uses the fact that $\mathcal{X}$ is reflexive and follows exactly as in the proof of Theorem 3.8 in [31]. $\square$

For the remainder of this section, fix $\omega \in [0, \pi/2)$ and $R \geq 0$, and let T be an operator of type $S_{\omega \cup R}$ on $\mathcal{X}$. An operator of type $S_{\omega \cup R}$ has a nonempty resolvent set, which of course implies that it is closed, so the Dunford–Riesz–Taylor $H(\Omega)$ functional calculus applies. Following the ideas in [53], however, we introduce the following preliminary functional calculus.

Definition 2.2.3 ($\Theta(S^o_{\theta \cup r})$ functional calculus)**.** Given $\theta \in (\omega, \pi/2)$, $r > R$ and $\phi \in \Theta(S^o_{\theta \cup r})$, define $\phi(T) \in \mathcal{L}(\mathcal{X})$ by

$$\phi(T)u = \frac{1}{2\pi i} \int_{+\partial S^o_{\tilde{\theta} \cup \tilde{r}}} \phi(z) R_T(z) u \, \mathrm{d}z := \lim_{\rho \to \infty} \frac{1}{2\pi i} \int_{(+\partial S^o_{\tilde{\theta} \cup \tilde{r}}) \cap D_\rho} \phi(z) R_T(z) u \, \mathrm{d}z \tag{2.2.1}$$

for all $u \in \mathcal{X}$, where $\tilde{\theta} \in (\omega, \theta)$, $\tilde{r} \in (R, r)$ and $+\partial S^o_{\tilde{\theta} \cup \tilde{r}}$ denotes the boundary of $S^o_{\tilde{\theta} \cup \tilde{r}}$ oriented clockwise.

The exceptional feature of (2.2.1) is that the contour of integration is allowed to touch the spectrum of T at infinity. This is made possible by the decay of ϕ and the resolvent bounds in Definition 2.2.1. A standard calculation using the resolvent equation shows that the mapping $\Theta(S^o_{\theta \cup r}) \mapsto \mathcal{L}(\mathcal{X})$ given by (2.2.1) is an algebra homomorphism. There is also no ambiguity in our notation, since if Ω is an open set in $\mathbb{C}_\infty$ that contains $S_{\theta \cup r} \cup \{\infty\}$, then the operators defined by (2.1.1) and (2.2.1) coincide for functions in $\Theta(S^o_{\theta \cup r}) \cap H(\Omega)$. This is because ϕ in $\Theta(S^o_{\theta \cup r}) \cap H(\Omega)$ is holomorphic in a neighbourhood of infinity, so the Θ-class decay implies that $\phi(\infty) = 0$. Cauchy's Theorem, the resolvent bounds and the Θ-class decay then allow us to modify the contour of integration in (2.2.1) to that in (2.1.1). In particular, if $\lambda \in \mathbb{C} \setminus S_{\theta \cup r}$ and $f(z) = (\lambda - z)^{-1}$ for all $z \in S_{\theta \cup r}$, then $f(T) = R_T(\lambda)$.

The proofs of the next two results are based on proofs for operators of type S_ω that were communicated to the author by Alan McIntosh in a graduate course. The first is a convergence lemma for the $\Theta(S^o_{\theta \cup r})$ functional calculus.

Proposition 2.2.4. Let $\theta \in (\omega, \pi/2)$ and $r > R$. If $(\phi_n)_n$ is a sequence in $\Theta(S^o_{\theta \cup r})$ and there exists $c, \delta > 0$ and $\phi \in \Theta(S^o_{\theta \cup r})$ such that the following hold:

1. $\sup_n |\phi_n(z)| \leq c|z|^{-\delta}$ for all $z \in S^o_{\theta \cup r}$;

2. ϕ_n converges to ϕ uniformly on compacts subsets of $S^o_{\theta \cup r}$,

then $\phi_n(T)$ converges to $\phi(T)$ in $\mathcal{L}(\mathcal{X})$.

Proof. Fix $\tilde{\theta} \in (\omega, \theta)$ and $\tilde{r} \in (R, r)$. Let γ denote the boundary of $S_{\tilde{\theta} \cup \tilde{r}}$ oriented clockwise. Given $r_0 \geq \tilde{r}$, divide γ into $\gamma_0 = \gamma \cap D_{r_0}$ and $\gamma_\infty = \gamma \cap (\mathbb{C} \setminus D_{r_0})$, so

$$\phi_n(T)u - \phi(T)u = \frac{1}{2\pi i} \left(\int_{\gamma_0} + \int_{\gamma_\infty} \right) (\phi_n(z) - \phi(z)) R_T(z) u \, \mathrm{d}z = I_1 + I_2$$

for all $u \in \mathcal{X}$. Given $\epsilon > 0$, choose $r_0 > \tilde{r}$ such that

$$\|I_2\| \lesssim \int_{r_0}^\infty (|\phi_n(z)| + |\phi(z)|) \|R_T(z)u\| \frac{\mathrm{d}|z|}{|z|} \lesssim \int_{r_0}^\infty |z|^{-\delta} \frac{\mathrm{d}|z|}{|z|} \|u\| < \epsilon \|u\|$$

for all $n \in \mathbb{N}$ and $u \in \mathcal{X}$. Now, since ϕ_n converges to ϕ uniformly on compact subsets of $S^o_{\theta\cup r}$, there exists $N \in \mathbb{N}$ such that

$$\|I_1\| \lesssim \int_{|z|=\tilde{r}} |\phi_n(z) - \phi(z)| \frac{|\mathrm{d}z|}{|z|} \|u\| + \int_{\tilde{r}}^{r_0} |\phi_n(z) - \phi(z)| \frac{\mathrm{d}|z|}{|z|} \|u\| < \epsilon \|u\|$$

for all $n > N$ and $u \in \mathcal{X}$. The result follows. □

The next lemma allows us to derive an $H^\infty(S^o_{\theta\cup r})$ functional calculus from the $\Theta(S^o_{\theta\cup r})$ functional calculus.

Lemma 2.2.5. Let $\theta \in (\omega, \pi/2)$ and $r > R$. If $(\phi_n)_n$ is a sequence in $\Theta(S^o_{\theta\cup r})$ and there exists $f \in H^\infty(S^o_{\theta\cup r})$ such that the following hold:

1. $\sup_n \|\phi_n\|_\infty < \infty$;
2. $\sup_n \|\phi_n(T)\| < \infty$;
3. ϕ_n converges to f uniformly on compacts subsets of $S^o_{\theta\cup r}$,

then $\lim_n \phi_n(T)u$ exists in $\mathcal{X}$ for all $u \in \mathcal{X}$. Moreover, if $f \in \Theta(S^o_{\theta\cup r})$, then $\lim_n \phi_n(T)u = f(T)u$ for all $u \in \mathcal{X}$.

Proof. Let $\tilde{\phi}_n(z) = (1+\frac{i}{r}z)^{-1}\phi_n(z)$ and $\tilde{\phi}(z) = (1+\frac{i}{r}z)^{-1}f(z)$ for all $z \in S^o_{\theta\cup r}$. There exists $c > 0$ such that the sequence $(\tilde{\phi}_n)_n$ in $\Theta(S^o_{\theta\cup r})$ satisfies $\sup_n |\tilde{\phi}_n(z)| \leq c|z|^{-1}$ for all $z \in S^o_{\theta\cup r}$, and converges to $\tilde{\phi} \in \Theta(S^o_{\theta\cup r})$ uniformly on compact subsets of $S^o_{\theta\cup r}$. Proposition 2.2.4 then implies that

$$\lim_n \|\tilde{\phi}_n(T)u - \tilde{\phi}(T)u\| = 0 \tag{2.2.2}$$

for all $u \in \mathcal{X}$.

If $u \in \mathsf{D}(T)$, then $u = (I + \frac{i}{r}T)^{-1}v$ for some $v \in \mathcal{X}$, so we have

$$\phi_n(T)u = \phi_n(T)(I + \tfrac{i}{r}T)^{-1}v = \tilde{\phi}_n(T)v$$

and (2.2.2) implies that $\lim_n \phi_n(T)u = \tilde{\phi}(T)v$. Note that the second equality above holds because $(1 + \frac{i}{r}z)^{-1}$ is in $\Theta(S^o_{\theta\cup r})$.

If $u \in \mathcal{X}$, then $u \in \overline{\mathsf{D}(T)}$ by Lemma 2.2.2. For each $\epsilon > 0$, there exists $v \in \mathsf{D}(T)$ such that $\|u - v\| < \epsilon$, and it follows from what was just proved that $(\phi_n(T)v)_n$ is a Cauchy sequence in $\mathcal{X}$. Therefore, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned}\|\phi_n(T)u - \phi_m(T)u\| &\leq \|\phi_n(T)(u-v)\| + \|\phi_n(T)v - \phi_m(T)v\| + \|\phi_m(T)(v-u)\| \\ &\lesssim \sup_n \|\phi_n(T)\|\, \epsilon\end{aligned}$$

for all $n > m > N$, and $\lim_n \phi_n(T)u$ exists in $\mathcal{X}$.

Finally, if $f \in \Theta(S^o_{\theta\cup r})$, then $\tilde{\phi}(T) = f(T)(I + \frac{i}{r}T)^{-1}$ and $\lim_n \phi_n(T)u = f(T)u$ for all $u \in \mathsf{D}(T)$ by the above. If $u \in \mathcal{X}$, then for each $\epsilon > 0$, there exists $v \in \mathsf{D}(T)$ and $N \in \mathbb{N}$ such that

$$\begin{aligned}\|\phi_n(T)u - f(T)u\| &\leq \|\phi_n(T)(u-v)\| + \|\phi_n(T)v - f(T)v\| + \|f(T)(v-u)\| \\ &\lesssim (\sup_n \|\phi_n(T)\| + \|f(T)\|)\epsilon\end{aligned}$$

for all $n > N$, and $\lim_n \phi_n(T)u = f(T)u$. □

The usefulness of condition (2) in the preceding lemma suggests the following definition, which allows us to construct an $H^\infty(S^o_{\theta\cup r})$ functional calculus. This is based on the analogous construction for operators of type S_ω that was communicated to the author by Alan McIntosh in a graduate course.

Definition 2.2.6 ($H^\infty(S^o_{\theta\cup r})$ functional calculus)**.** Given $\theta \in (\omega, \pi/2)$ and $r > R$, the operator T has a *bounded* $H^\infty(S^o_{\theta\cup r})$ *functional calculus* if there exists $c > 0$ such that

$$\|\phi(T)\| \leq c\|\phi\|_\infty$$

for all $\phi \in \Theta(S^o_{\theta\cup r})$. If T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus, then given $f \in H^\infty(S^o_{\theta\cup r})$ define $f(T) \in \mathcal{L}(\mathcal{X})$ by

$$f(T)u = \lim_n (f\phi_n)(T)u \tag{2.2.3}$$

for all $u \in \mathcal{X}$, where $(\phi_n)_n$ is a uniformly bounded sequence in $\Theta(S^o_{\theta\cup r})$ that converges to 1 uniformly on compact subsets of $S^o_{\theta\cup r}$.

The operator in (2.2.3) is well-defined by Lemma 2.2.5. In particular, the definition is independent of the choice of sequence $(\phi_n)_n$ in Definition 2.2.6. As an example, consider the sequence defined by $\phi_n(z) = (1 + \frac{i}{rn}z)^{-1}$ for all $z \in S^o_{\theta\cup r}$ and $n \in \mathbb{N}$, which satisfies $\sup_n \|\phi_n\|_\infty = 1$. The requirement that T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus then implies that

$$\|f(T)\| \leq \sup_n \|(f\phi_n)(T)\| \leq c \sup_n \|f\phi_n\|_\infty \leq c\|f\|_\infty$$

for all $f \in H^\infty(S^o_{\theta\cup r})$, where c is the constant from Definition 2.2.6.

Lemma 2.2.5 also shows that the operators defined by (2.2.1) and (2.2.3) coincide for functions in $\Theta(S^o_{\theta\cup r})$. Furthermore, if Ω is an open set in $\mathbb{C}_\infty$ that contains $S_{\theta\cup r} \cup \{\infty\}$, then the operators defined by (2.1.1) and (2.2.3) coincide for functions in $H^\infty(S^o_{\theta\cup r}) \cap H(\Omega)$ by Theorem 2.1.2. There is also the following analogue of Theorem 2.1.2.

Theorem 2.2.7. *The mapping given by* (2.2.3) *is an algebra homomorphism from* $H^\infty(S^o_{\theta\cup r})$ *into* $\mathcal{L}(\mathcal{X})$ *with following properties:*

1. *If* $\mathbf{1}(z) = 1$ *for all* $z \in S^o_{\theta\cup r}$, *then* $\mathbf{1}(T) = I$ *on* $\mathcal{X}$*;*
2. *If* $\lambda \in \mathbb{C} \setminus S_{\omega\cup R}$ *and* $f(z) = (\lambda - z)^{-1}$ *for all* $z \in S^o_{\theta\cup r}$, *then* $f(T) = R_T(\lambda)$*;*
3. *If* $(f_n)_n$ *is a sequence in* $H^\infty(S^o_{\theta\cup r})$ *and there exists* $f \in H^\infty(S^o_{\theta\cup r})$ *such that the following hold:*
   - *(i)* $\sup_n \|f_n\|_\infty < \infty$*;*
   - *(ii)* $\sup_n \|f_n(T)\| < \infty$*;*
   - *(iii)* f_n *converges to* f *uniformly on compacts subsets of* $S^o_{\theta\cup r}$*,*

   then $\|f(T)\| \leq \sup_n \|f_n(T)\|$ *and* $\lim_n f_n(T)u = f(T)u$ *for all* $u \in \mathcal{X}$.

Proof. Let $f, g \in H^\infty(S^o_{\theta\cup r})$. If $(\phi_n)_n$ satisfies the requirements of Definition 2.2.6, then so does $(\phi_n^2)_n$. Therefore, the algebra homomorphism property of the $\Theta(S^o_{\theta\cup r})$ functional calculus implies that

$$(fg)(T)u = \lim_n (fg\phi_n^2)(T)u = \lim_n f_n(T)g_n(T)u$$

for all $u \in \mathcal{X}$, where $f_n = f\phi_n$ and $g_n = g\phi_n$. This shows that for each $\epsilon > 0$ and $u \in \mathcal{X}$, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned}\|(fg)(T)u - f_n(T)g(T)u\| &\le \|(fg)(T)u - f_n(T)g_n(T)u\| + \|f_n(T)[g_n(T)u - g(T)u]\| \\ &\lesssim \sup_n \|f_n(T)\|\,\epsilon\end{aligned}$$

for all $n > N$. Hence, $(fg)(T)u = \lim_n f_n(T)g(T)u = f(T)g(T)u$ for all $u \in \mathcal{X}$.

It remains to prove (1) and (3), since (2) holds by the coincidence of (2.1.1) and (2.2.3). If $\phi_n(z) = (1 + \frac{i}{rn}z)^{-1}$ for all $z \in S^o_{\theta\cup r}$ and $n \in \mathbb{N}$, then by Lemma 2.2.2 we have

$$\mathbf{1}(T)u = \lim_n \phi_n(T)u = \lim_n (I + \tfrac{i}{rn}T)^{-1}u = u$$

for all $u \in \mathcal{X}$. The final part of the theorem follows from the algebra homomorphism property, as in the proof of Lemma 2.2.5. □

2.2.1 Local Quadratic Estimates

Throughout this section, fix $\omega \in [0, \pi/2)$ and $R \ge 0$, and let T be an operator of type $S_{\omega\cup R}$ on $\mathcal{X}$. If $R = 0$, then T is of type S_ω. In that case, given $\theta \in (\omega, \pi/2)$ and $\psi \in \Psi(S^o_\theta)$, it was proved in [53, 2] that T has a bounded $H^\infty(S^o_\theta)$ functional calculus if and only if the quadratic estimate

$$\int_0^\infty \|\psi_t(T)u\|_2^2 \,\frac{\mathrm{d}t}{t} \eqsim \|u\|^2$$

holds for all $u \in \overline{\mathsf{R}(T)}$. In the next section, given $\theta \in (\omega, \pi/2)$ and $r > R$, we prove that T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus if and only if certain local quadratic estimates hold. In this section, we define local quadratic estimates and prove the equivalence of local quadratic norms. This requires that we introduce the Φ-class of holomorphic functions and develop a local version of the McIntosh approximation technique.

Definition 2.2.8. Given $\theta \in (0, \pi/2)$, $r \ge 0$ and $\beta > 0$, define $\Phi^\beta(S^o_{\theta,r})$ to be the set of all $\phi \in \Theta^\beta(S^o_{\theta,r})$ with the following properties:

1. For all $z \in S^o_{\theta,r}$, $\phi(z) \ne 0$;
2. $\inf_{z\in D^o_r} |\phi(z)| \ne 0$;
3. $\sup_{t\ge 1} |\phi_t(z)| \lesssim |\phi(z)|$ for all $z \in S^o_{\theta,r} \setminus D_r$,

where $S^o_{\theta,r}$ denotes either $S^o_{\theta\cup r}$ or $S^o_{\theta\setminus r}$. Note that (2) is obviated in the case of $S^o_{\theta\setminus r}$. Also, let $\Phi(S^o_{\theta,r}) = \bigcup_{\beta>0} \Phi^\beta(S^o_{\theta,r})$.

For example, if $\theta \in (0, \pi/2)$, $0 < r < \sqrt{a}$ and $\beta > 0$, then the functions $e^{-\sqrt{z^2+a}}$, e^{-z^2} and $(z^2+a)^{-\beta}$ are in $\Phi^\beta(S^o_{\theta\cup r})$. The next result is the local version of an exercise in Lecture 3 of [2].

Lemma 2.2.9 (McIntosh approximation)**.** Let $\theta \in (\omega, \pi/2)$ and $r > R$. Given non-degenerate $\psi \in \Psi(S^o_{\theta\cup r})$ and $\phi \in \Phi(S^o_{\theta\cup r})$, there exist $\eta \in \Psi(S^o_{\theta\cup r})$ and $\varphi \in \Theta(S^o_{\theta\cup r})$ such that

$$\int_0^1 \eta_t(z)\psi_t(z)\ \frac{\mathrm{d}t}{t} + \varphi(z)\phi(z) = 1 \tag{2.2.4}$$

for all $z \in S^o_{\theta\cup r}$. Given $0 < \alpha < \beta \leq 1$ and $f \in \Theta(S^o_{\theta\cup r})$, if

$$\Psi_{\alpha,\beta}(z) = f(z)\int_\alpha^\beta \eta_t(z)\psi_t(z)\ \frac{\mathrm{d}t}{t} \quad \text{and} \quad \Phi(z) = f(z)\varphi(z)\phi(z)$$

for all $z \in S^o_{\theta\cup r}$, then

$$\lim_{\alpha\to 0} \|(\Psi_{\alpha,1}(T) + \Phi(T))u - f(T)u\| = 0 \tag{2.2.5}$$

for all $u \in \mathcal{X}$. Moreover, if T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus, then this holds for any $f \in H^\infty(S^o_{\theta\cup r})$.

Proof. Given $f \in H^\infty(S^o_{\theta\cup r})$, let $f_-(z) = f(-z)$ and $f^*(z) = \overline{f(\bar{z})}$ for all $z \in S^o_{\theta\cup r}$. Let $c = \int_0^\infty |\psi(t)\psi(-t)\phi(t)\phi(-t)|^2\ \frac{\mathrm{d}t}{t}$ and define the functions

$$\eta = c^{-1}\psi^*\psi_-\psi^*_-\phi\phi^*\phi_-\phi^*_- \quad \text{and} \quad \varphi = \frac{1}{\phi}\left(1 - \int_0^1 \eta_t\psi_t\frac{\mathrm{d}t}{t}\right),$$

in which case (2.2.4) is immediate and $\eta \in \Psi(S^o_{\theta\cup r})$. The function φ is holomorphic on $S^o_{\theta\cup r}$ by Morera's Theorem, since $\phi(z) \neq 0$ for all $z \in S^o_{\theta\cup r}$, and bounded on D^o_r, since $\inf_{z\in D^o_r}|\phi(z)| \neq 0$. A change of variable shows that $\int_0^\infty \eta_t(x)\psi_t(x)\frac{\mathrm{d}t}{t} = 1$ for all $x \in \mathbb{R}\setminus\{0\}$, and since $z \mapsto \int_0^\infty \eta_t(z)\psi_t(z)\frac{\mathrm{d}t}{t}$ is holomorphic on S^o_θ, we must have $\int_0^\infty \eta_t(z)\psi_t(z)\frac{\mathrm{d}t}{t} = 1$ for all $z \in S^o_\theta$. It then follows from property (3) in Definition 2.2.8 that

$$|\varphi(z)| = \frac{1}{|\phi(z)|}\int_1^\infty |\eta_t(z)\psi_t(z)|\frac{\mathrm{d}t}{t} \lesssim \frac{\sup_{t\geq 1}|\phi_t(z)|}{|\phi(z)|}\int_1^\infty (t|z|)^{-\delta}\frac{\mathrm{d}t}{t} \lesssim |z|^{-\delta}$$

for all $z \in S^o_\theta$ and some $\delta > 0$, so $\varphi \in \Theta(S^o_{\theta\cup r})$.

To prove (2.2.5), let $f \in \Theta(S^o_{\theta\cup r})$ and note that there exists $\delta > 0$ such that

$$\begin{aligned} |\Psi_{\alpha,1}(z)| &\lesssim |f(z)|\int_0^1 \min(|tz|^\delta, |tz|^{-\delta})\ \frac{\mathrm{d}t}{t} \\ &= \min(\|f\|_\infty, |z|^{-\delta})\left(|z|^\delta\int_0^{1/|z|} t^\delta\ \frac{\mathrm{d}t}{t} + |z|^{-\delta}\int_{1/|z|}^\infty t^{-\delta}\ \frac{\mathrm{d}t}{t}\right) \\ &\lesssim \min(\|f\|_\infty, |z|^{-\delta}) \end{aligned} \tag{2.2.6}$$

for all $\alpha \in (0,1)$ and $z \in S^o_{\theta\cup r}$, where the constants associated with each instance of $\lesssim$ do not depend on α. This shows that $\Psi_{\alpha,1} + \Phi$ is in $\Theta(S^o_{\theta\cup r})$ for all $\alpha \in (0,1)$

with $\sup_{\alpha\in(0,1)}|\Psi_{\alpha,1}(z)+\Phi(z)|\le c|z|^{-\delta}$ for some $c>0$. Also, given a compact set $K\subset S^o_{\theta\cup r}$, it follows from (2.2.4) that there exists $c_K>0$ such that

$$|\Psi_{\alpha,1}(z)+\Phi(z)-f(z)|\le\|f\|_\infty\int_0^\alpha|\eta_t(z)\psi_t(z)|\,\frac{\mathrm{d}t}{t}\lesssim|\alpha z|^\delta\le c_K\alpha^\delta$$

for all $\alpha\in(0,1)$ and $z\in K$. Therefore, the sequence $(\Psi_{1/n,1}+\Phi)_n$ converges to f uniformly on compact subsets of $S^o_{\theta\cup r}$, and (2.2.5) follows from the version of the convergence lemma in Proposition 2.2.4.

Now let $f\in H^\infty(S^o_{\theta\cup r})$ and suppose that T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus. It follows as in (2.2.6) that $\sup_{\alpha\in(0,1)}\|\Psi_{\alpha,1}+\Phi\|_\infty\lesssim\|f\|_\infty<\infty$. Also, there exists $\delta>0$ such that

$$\begin{aligned}|\Psi_{\alpha,1}(z)|&\lesssim\|f\|_\infty\int_\alpha^1\min(|tz|^\delta,|tz|^{-\delta})\,\frac{\mathrm{d}t}{t}\\&=\|f\|_\infty\min\left(|z|^\delta\int_\alpha^1 t^\delta\,\frac{\mathrm{d}t}{t},|z|^{-\delta}\int_\alpha^1 t^{-\delta}\,\frac{\mathrm{d}t}{t}\right)\\&\lesssim\min(|z|^\delta,|\alpha z|^{-\delta})\\&\le\alpha^{-\delta}|z|^{-\delta}\end{aligned}$$

for all $\alpha\in(0,1)$ and $z\in S^o_{\theta\cup r}$. This shows that $\Psi_{\alpha,1}+\Phi$ is in $\Theta(S^o_{\theta\cup r})$ for all $\alpha\in(0,1)$, and since T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus, the result follows by Theorem 2.2.7. □

We now introduce local quadratic norms on $\mathcal{X}$ adapted to the operator T and define the notion of local quadratic estimates.

Definition 2.2.10. Let $\theta\in(\omega,\pi/2)$ and $r>R$. Given $\psi\in\Psi(S^o_{\theta\cup r})$ and $\phi\in\Phi(S^o_{\theta\cup r})$, define the *local quadratic norm* $\|\cdot\|_{T,\psi,\phi}$ by

$$\|u\|_{T,\psi,\phi}=\left(\int_0^1\|\psi_t(T)u\|^2\,\frac{\mathrm{d}t}{t}+\|\phi(T)u\|^2\right)^{\frac{1}{2}}$$

for all $u\in\mathcal{X}$. The operator T satisfies (ψ,ϕ) *quadratic estimates* if there exists $c>0$ such that $\|u\|_{T,\psi,\phi}\le c\|u\|$ for all $u\in\mathcal{X}$, and *reverse* (ψ,ϕ) *quadratic estimates* if there exists $c>0$ such that $\|u\|\le c\|u\|_{T,\psi,\phi}$ for all $u\in\mathcal{X}$ satisfying $\|u\|_{T,\psi,\phi}<\infty$.

Given nondegenerate $\psi\in\Psi(S^o_{\theta\cup r})$ and $\phi\in\Phi(S^o_{\theta\cup r})$ in Definition 2.2.10, if T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus, then Lemma 2.2.9 implies that the local quadratic norm $\|\cdot\|_{T,\psi,\phi}$ is indeed a norm on $\mathcal{X}$. We use the next two lemmas to prove that families of local quadratic norms are equivalent for operators that have a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus. These are local analogues of results in [2].

Lemma 2.2.11. Let $\theta\in(\omega,\pi/2)$ and $r>R$. Given $\psi,\tilde{\psi}\in\Psi(S^o_{\theta\cup r})$ and $\phi\in\Theta(S^o_{\theta\cup r})$, there exists $c>0$ and $\delta>0$ such that the following hold:

1. $\|(f\psi_t)(T)\|\le c\|f\|_\infty$;
2. $\|(f\phi)(T)\|\le c\|f\|_\infty$;

3. $\|(f\phi\psi_t)(T)\| \le c\|f\|_\infty t^\delta(1+\log(1/t))$;

4. $\|(f\psi_t\tilde{\psi}_s)(T)\| \le c\|f\|_\infty \times \begin{cases} (s/t)^\delta(1+\log(t/s)) & \text{if} \quad s\in(0,t]; \\ (t/s)^\delta(1+\log(s/t)) & \text{if} \quad s\in(t,1] \end{cases}$

for all $t \in (0,1]$ and $f \in H^\infty(S^0_{\theta\cup r})$.

Proof. Fix $\tilde{\theta} \in (\omega,\theta)$ and $\tilde{r} \in (R,r)$. Let γ denote the boundary of $S_{\tilde{\theta}\cup\tilde{r}}$ oriented clockwise. Choose $\delta > 0$ so that $\psi, \tilde{\psi} \in \Psi^\delta_\delta(S^o_{\theta\cup r})$ and $\phi \in \Theta^\delta(S^o_{\theta\cup r})$. The resolvent bounds then imply that

$$\begin{aligned}
\frac{\|(f\psi_t)(T)\|}{\|f\|_\infty} &\lesssim \frac{1}{\|f\|_\infty}\int_\gamma |f(z)\psi_t(z)|\|R_T(z)\|\ |\mathrm{d}z| \\
&\lesssim \int_\gamma \min(|tz|^\delta, |tz|^{-\delta})\ \frac{|\mathrm{d}z|}{|z|} \\
&\lesssim t^\delta \int_{|z|=\tilde{r}} |z|^{\delta-1}|\mathrm{d}z| + t^\delta \int_{\tilde{r}}^{\tilde{r}/t} |z|^{\delta-1}\ \mathrm{d}|z| + t^{-\delta}\int_{\tilde{r}/t}^\infty |z|^{-\delta-1}\ \mathrm{d}|z| \\
&\lesssim 1
\end{aligned}$$

for all $t \in (0,1]$. Similarly, we obtain

$$\frac{\|(f\phi)(T)\|}{\|f\|_\infty} \lesssim \int_{|z|=\tilde{r}} |z|^{-1}\ |\mathrm{d}z| + \int_{\tilde{r}}^\infty |z|^{-\delta-1}\ \mathrm{d}|z| \lesssim 1$$

and

$$\begin{aligned}
\frac{\|(f\phi\psi_t)(T)\|}{\|f\|_\infty} &\lesssim \int_\gamma \min(1,|z|^{-\delta})\min(|tz|^\delta,|tz|^{-\delta})\ \frac{|\mathrm{d}z|}{|z|} \\
&\lesssim t^\delta \int_{|z|=\tilde{r}} |z|^{\delta-1}\ |\mathrm{d}z| + t^\delta\int_{\tilde{r}}^{\tilde{r}/t} |z|^{-1}\ \mathrm{d}|z| + t^{-\delta}\int_{\tilde{r}/t}^\infty |z|^{-2\delta-1}\ \mathrm{d}|z| \\
&\lesssim t^\delta + t^\delta\log(1/t) + t^{-\delta}(1/t)^{-2\delta} \\
&\lesssim t^\delta(1+\log(1/t)).
\end{aligned}$$

for all $t \in (0,1]$. Also, if $0 < s \le t \le 1$, then

$$\begin{aligned}
\frac{\|(f\psi_t\tilde{\psi}_s)(T)\|}{\|f\|_\infty} &\lesssim \int_\gamma \min(|tz|^\delta,|tz|^{-\delta})\min(|sz|^\delta,|sz|^{-\delta})\ \frac{|\mathrm{d}z|}{|z|} \\
&\lesssim (s/t)^\delta \int_{|z|=\tilde{r}} |z|^{-1}\ |\mathrm{d}z| + (st)^\delta\int_{\tilde{r}}^{\tilde{r}/t} |z|^{2\delta-1}\ \mathrm{d}|z| \\
&\qquad + (s/t)^\delta\int_{\tilde{r}/t}^{\tilde{r}/s} |z|^{-1}\ \mathrm{d}|z| + (st)^{-\delta}\int_{\tilde{r}/s}^\infty |z|^{-2\delta}\ \mathrm{d}|z| \\
&\lesssim (s/t)^\delta + (st)^\delta(1/t)^{2\delta} + (s/t)^\delta\log(t/s) + (st)^{-\delta}(1/s)^{-2\delta} \\
&\lesssim (s/t)^\delta(1+\log(t/s)).
\end{aligned}$$

The same argument applied in the case $0 < t < s \le 1$ completes the proof. $\square$

Lemma 2.2.12. Let $\theta \in (\omega, \pi/2)$ and $r > R$. Let $\psi \in \Psi(S^o_{\theta\cup r})$ and $\phi \in \Phi(S^o_{\theta\cup r})$. If $(u_n)_n$ is sequence in $\mathcal{X}$ and there exists $u \in \mathcal{X}$ such that the following hold:

1. $\|u_n\|_{T,\psi,\phi} < \infty$ for all $n \in \mathbb{N}$;
2. $(u_n)_n$ is a Cauchy sequence under the local quadratic norm $\|\cdot\|_{T,\psi,\phi}$;
3. $\lim_{n\to\infty} \|u_n - u\| = 0$,

then $\|u\|_{T,\psi,\phi} < \infty$ and $\lim_{n\to\infty} \|u_n - u\|_{T,\psi,\phi} = 0$.

Proof. For each $\alpha \in (0,1)$, choose $N(\alpha) \in \mathbb{N}$ so that $\|u_{N(\alpha)} - u\|^2 < 1/(1-\log\alpha)$. Lemma 2.2.11 then implies that

$$\begin{aligned}
&\int_\alpha^1 \|\psi_t(T)u\|^2 \,\frac{\mathrm{d}t}{t} + \|\phi(T)u\|^2 \\
&\le \int_\alpha^1 \|\psi_t(T)(u_{N(\alpha)} - u)\|^2 \,\frac{\mathrm{d}t}{t} + \|\phi(T)(u_{N(\alpha)} - u)\|^2 + \sup_n \|u_n\|^2_{T,\psi,\phi} \\
&\lesssim (1-\log\alpha)\|u_{N(\alpha)} - u\|^2 + \sup_n \|u_n\|^2_{T,\psi,\phi} \\
&\lesssim \sup_n \|u_n\|^2_{T,\psi,\phi}
\end{aligned}$$

for all $\alpha \in (0,1)$. The Cauchy condition guarantees that $\sup_n \|u_n\|_{T,\psi,\phi} < \infty$, so we must have $\|u\|_{T,\psi,\phi} < \infty$.

For each $\epsilon > 0$, conditions (2) and (3) combined with the result just proved guarantee that there exists $\alpha_0 \in (0,1)$ and $N \in \mathbb{N}$ such that

$$\sup_{n>N} \int_0^{\alpha_0} \|\psi_t(T)u_n\|^2 \,\frac{\mathrm{d}t}{t} < \epsilon, \quad \sup_{n>N} \|u_n - u\| < \epsilon \quad \text{and} \quad \int_0^{\alpha_0} \|\psi_t(T)u\|^2 \,\frac{\mathrm{d}t}{t} < \epsilon.$$

Lemma 2.2.11 then implies that

$$\|u_n - u\|^2_{T,\psi,\phi} \le \left(\int_0^{\alpha_0} + \int_{\alpha_0}^1\right) \|\psi_t(T)(u_n - u)\|^2 \,\frac{\mathrm{d}t}{t} + \|\phi(T)(u_n - u)\|^2 L \lesssim \epsilon$$

for all $n > N$, as required. ☐

The following result is essential for establishing the connection between bounded holomorphic functional calculi and quadratic estimates. This is a local analogue of Proposition E in [2].

Proposition 2.2.13. Let $\theta \in (\omega, \pi/2)$ and $r > R$. Given nondegenerate functions ψ, $\tilde{\psi} \in \Psi(S^o_{\theta\cup r})$ and ϕ, $\tilde{\phi} \in \Phi(S^o_{\theta\cup r})$, there exists $c > 0$ such that

$$\|f(T)u\|_{T,\tilde{\psi},\tilde{\phi}} \le c\|f\|_\infty \|u\|_{T,\psi,\phi}$$

for all $f \in \Theta(S^o_{\theta\cup r})$ and $u \in \mathcal{X}$ satisfying $\|u\|_{T,\psi,\phi} < \infty$. Moreover, if T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus, then there exists $c > 0$ such that

$$\|f(T)u\|_{T,\tilde{\psi},\tilde{\phi}} \le c\|f\|_\infty \|u\|_{T,\psi,\phi}$$

for all $f \in H^\infty(S^o_{\theta\cup r})$ and $u \in \mathcal{X}$ satisfying $\|u\|_{T,\psi,\phi} < \infty$.

Proof. Let $f \in \Theta(S^o_{\theta\cup r})$ and let $u \in \mathcal{X}$ satisfying $\|u\|_{T,\psi,\phi} < \infty$. Lemma 2.2.9 gives $\eta \in \Psi(S^o_{\theta\cup r})$ and $\varphi \in \Theta(S^o_{\theta\cup r})$ such that

$$\int_0^1 \eta_t(z)\psi_t(z)\psi_t(z)\ \frac{\mathrm{d}t}{t} + \varphi(z)\phi(z) = 1$$

for all $z \in S^o_{\theta\cup r}$. Given $0 < \alpha < \beta \le 1$, define

$$\Psi_{\alpha,\beta}(z) = f(z)\int_\alpha^\beta \eta_t(z)\psi_t(z)\psi_t(z)\ \frac{\mathrm{d}t}{t} \quad \text{and} \quad \Phi(z) = f(z)\varphi(z)\phi(z)$$

for all $z \in S^o_{\theta\cup r}$, so $\lim_{\alpha\to 0}\|(\Psi_{\alpha,1}(T) + \Phi(T))u - f(T)u\| = 0$. Now write

$$\begin{aligned}\|\Psi_{\alpha,\beta}(T)u + \Phi(T)u\|^2_{T,\tilde{\psi},\tilde{\phi}} &\le \int_0^1 \|\tilde{\psi}_t(T)\Psi_{\alpha,\beta}(T)u\|^2\ \frac{\mathrm{d}t}{t} + \int_0^1 \|\tilde{\psi}_t(T)\Phi(T)u\|^2\ \frac{\mathrm{d}t}{t}\\ &\quad + \|\tilde{\phi}(T)\Psi_{\alpha,\beta}(T)u\|^2 + \|\tilde{\phi}(T)\Phi(T)u\|^2\\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

We use Lemma 2.2.11 to obtain the following Schur-type estimates:

Estimate for I_1:

$$\begin{aligned}I_1 &= \int_0^1 \left\|\int_\alpha^\beta (\tilde{\psi}_t\psi_s)(T)(f\eta_s\psi_s)(T)u\ \frac{\mathrm{d}s}{s}\right\|^2\ \frac{\mathrm{d}t}{t}\\ &\le \int_0^1 \left(\int_\alpha^\beta \|(\tilde{\psi}_t\psi_s)(T)\|\|(f\eta_s\psi_s)(T)u\|\ \frac{\mathrm{d}s}{s}\right)^2\ \frac{\mathrm{d}t}{t}\\ &\le \int_0^1 \left(\int_\alpha^\beta \|(\tilde{\psi}_t\psi_s)(T)u\|\ \frac{\mathrm{d}s}{s}\right)\left(\int_\alpha^\beta \|(\tilde{\psi}_t\psi_s)(T)\|\|(f\eta_s\psi_s)(T)u\|^2\ \frac{\mathrm{d}s}{s}\right)\ \frac{\mathrm{d}t}{t}\\ &\le \sup_{t\in(0,1]}\left(\int_\alpha^\beta \|(\tilde{\psi}_t\psi_s)(T)u\|\ \frac{\mathrm{d}s}{s}\right)\int_0^1\int_\alpha^\beta \|(\tilde{\psi}_t\psi_s)(T)\|\|(f\eta_s\psi_s)(T)u\|^2\ \frac{\mathrm{d}s}{s}\ \frac{\mathrm{d}t}{t}\\ &\lesssim \sup_{s\in(0,1]}\left(\int_0^1 \|(\tilde{\psi}_t\psi_s)(T)\|\ \frac{\mathrm{d}t}{t}\right)\int_\alpha^\beta \|(f\eta_s)(T)\psi_s(T)u\|^2\ \frac{\mathrm{d}s}{s}\\ &\lesssim \|f\|^2_\infty \int_\alpha^\beta \|\psi_t(T)u\|^2\ \frac{\mathrm{d}t}{t};\end{aligned}$$

Estimate for I_2:

$$\begin{aligned}I_2 &= \int_0^1 \|(f\varphi\tilde{\psi}_t)(T)\phi(T)u\|^2\ \frac{\mathrm{d}t}{t}\\ &\lesssim \|f\|^2_\infty \int_0^1 t^{2\eta}(1+\log(1/t))^2\ \frac{\mathrm{d}t}{t}\|\phi(T)u\|^2\\ &\lesssim \|f\|^2_\infty\|\phi(T)u\|^2;\end{aligned}$$

Estimate for I_3:

$$\begin{aligned}
I_3 &= \left\| \int_\alpha^\beta (f\tilde{\phi}\eta_s\psi_s)(T)\psi_s(T)u \ \frac{\mathrm{d}s}{s} \right\|^2 \\
&\le \int_\alpha^\beta \|(f\tilde{\phi}\eta_s\psi_s)(T)\|^2 \ \frac{\mathrm{d}s}{s} \int_\alpha^\beta \|\psi_s(T)u\|^2 \ \frac{\mathrm{d}s}{s} \\
&\lesssim \|f\|_\infty^2 \int_\alpha^\beta t^{2\eta}(1+\log(1/t))^2 \ \frac{\mathrm{d}t}{t} \int_\alpha^\beta \|\psi_t(T)u\|^2 \ \frac{\mathrm{d}t}{t} \\
&\lesssim \|f\|_\infty^2 \int_\alpha^\beta \|\psi_t(T)u\|^2 \ \frac{\mathrm{d}t}{t};
\end{aligned}$$

Estimate for I_4:

$$I_4 = \|(f\tilde{\phi}\varphi)(T)\phi(T)u\|^2 \lesssim \|f\|_\infty^2 \|\phi(T)u\|^2.$$

Therefore, we have

$$\|\Psi_{\alpha,1}(T)u + \Phi(T)u\|_{T,\tilde{\psi},\tilde{\phi}} \lesssim \|f\|_\infty \|u\|_{T,\psi,\phi}$$

for all $\alpha \in (0,1)$, and

$$\|\Psi_{\alpha,\beta}(T)u\|^2_{T,\tilde{\psi},\tilde{\phi}} \le I_1 + I_3 \lesssim \|f\|_\infty^2 \int_\alpha^\beta \|\psi_t(T)u\|^2 \ \frac{\mathrm{d}t}{t}$$

for all $0 < \alpha < \beta \le 1$. Now, since $\|u\|_{T,\psi,\phi} < \infty$, for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\int_{\frac{1}{n}}^{\frac{1}{m}} \|\psi_t(T)u\|^2 \ \frac{\mathrm{d}t}{t} < \epsilon$$

for all $n > m > N$, which implies that

$$\|(\Psi_{1/n,1}(T) + \Phi(T))u - (\Psi_{1/m,1}(T) + \Phi(T))u\|_{T,\tilde{\psi},\tilde{\phi}} = \|\Psi_{1/n,1/m}(T)u\|_{T,\tilde{\psi},\tilde{\phi}} \lesssim \|f\|_\infty \epsilon$$

for all $n > m > N$. This shows that $(\Psi_{1/n,1}(T)u + \Phi(T)u)_n$ is a Cauchy sequence under the local quadratic norm $\|\cdot\|_{T,\tilde{\psi},\tilde{\phi}}$, so by Lemma 2.2.12 we have

$$\lim_{\alpha\to 0} \|(\Psi_{\alpha,1}(T) + \Phi(T))u - f(T)u\|_{T,\tilde{\psi},\tilde{\phi}} = 0$$

and $\|f(T)u\|_{T,\tilde{\psi},\tilde{\phi}} \lesssim \|f\|_\infty \|u\|_{T,\psi,\phi}$, as required.

Finally, if T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus, then the proof above holds for $f \in H^\infty(S^o_{\theta\cup r})$ by Lemma 2.2.9. □

2.2.2 The Main Equivalence

We connect the theory from the previous two sections. The first result is an immediate consequence of Proposition 2.2.13.

Proposition 2.2.14. Let $\omega \in [0, \pi/2)$ and $R \geq 0$. Let T be an operator of type $S_{\omega \cup R}$ on $\mathcal{X}$. If there exists $\theta_0 \in (\omega, \pi/2)$, $r_0 > R$, nondegenerate $\psi, \tilde{\psi} \in \Psi(S^o_{\theta_0 \cup r_0})$ and nondegenerate $\phi, \tilde{\phi} \in \Phi(S^o_{\theta_0 \cup r_0})$ such that T satisfies (ψ, ϕ) quadratic estimates and reverse $(\tilde{\psi}, \tilde{\phi})$ quadratic estimates, then T has a bounded $H^\infty(S^o_{\theta \cup r})$ functional calculus for all $\theta \in (\omega, \pi/2)$ and $r > R$.

Proof. Let $\theta \in (\omega, \pi/2)$ and $r > R$. Given $g \in H^\infty(S^o_{\theta_0 \cup r_0})$, let g_0 denote the restriction of g to $S^o_{\min\{\theta,\theta_0\} \cup \min\{r,r_0\}}$. Using the properties of the $\Theta(S^o_{\theta \cup r})$ functional calculus, Proposition 2.2.13 implies that there exists $c > 0$ such that

$$\|f(T)u\|_{T,\tilde{\psi},\tilde{\phi}} = \|f_0(T)u\|_{T,\tilde{\psi}_0,\tilde{\phi}_0} \leq c\|f_0\|_\infty \|u\|_{T,\psi_0,\phi_0} \leq c\|f\|_\infty \|u\|_{T,\psi,\phi}$$

for all $f \in \Theta(S^o_{\theta \cup r})$ and $u \in \mathcal{X}$ satisfying $\|u\|_{T,\psi,\phi} < \infty$. The quadratic estimates then imply that there exists $\tilde{c} > 0$ such that

$$\|f(T)u\| \leq \tilde{c}\|f\|_\infty \|u\|$$

for all $f \in \Theta(S^o_{\theta \cup r})$ and $u \in \mathcal{X}$, as required. ☐

A converse of the above result holds for dual pairs of operators.

Definition 2.2.15. A *dual pair of Banach spaces* $\langle \mathcal{X}, \mathcal{X}' \rangle$ is a pair of complex Banach spaces $(\mathcal{X}, \mathcal{X}')$ associated with a sesquilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{X} \times \mathcal{X}'$ that satisfies the following properties:

1. $|\langle u, v \rangle| \leq C_0 \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}'}$ for all $u \in \mathcal{X}$ and $v \in \mathcal{X}'$;
2. $\|u\|_{\mathcal{X}} \leq C_1 \sup\limits_{v \in \mathcal{X}'} \dfrac{|\langle u, v \rangle|}{\|v\|_{\mathcal{X}'}}$ for all $u \in \mathcal{X}$;
3. $\|v\|_{\mathcal{X}'} \leq C_2 \sup\limits_{u \in \mathcal{X}} \dfrac{|\langle u, v \rangle|}{\|u\|_{\mathcal{X}}}$ for all $v \in \mathcal{X}'$,

for some constants C_0, C_1 and $C_2 > 0$.

Definition 2.2.16. Given a dual pair of Banach spaces $\langle \mathcal{X}, \mathcal{X}' \rangle$, a *dual pair of operators* $\langle T, T' \rangle$ consists of an operator T on $\mathcal{X}$ and an operator T' on $\mathcal{X}'$ such that

$$\langle Tu, v \rangle = \langle u, T'v \rangle$$

for all $u \in \mathsf{D}(T)$ and $v \in \mathsf{D}(T')$.

If T is an operator of type $S_{\omega \cup R}$ on a Hilbert space, then the adjoint operator T^* provides a dual pair of operators $\langle T, T^* \rangle$ of type $S_{\omega \cup R}$ under the inner-product. We use the next lemma to prove the equivalence of bounded holomorphic functional calculi and quadratic estimates.

Lemma 2.2.17. Let $\omega \in [0, \pi/2)$ and $R \geq 0$. Let $\langle T, T' \rangle$ be a dual pair of operators of type $S_{\omega \cup R}$. If $\theta \in (\omega, \pi/2)$ and $r > R$, then T has a bounded $H^\infty(S^o_{\theta \cup r})$ functional calculus if and only if T' has a bounded $H^\infty(S^o_{\theta \cup r})$ functional calculus. Moreover, if T has a bounded $H^\infty(S^o_{\theta \cup r})$ functional calculus, then

$$\langle f(T)u, v \rangle = \langle u, f^*(T')v \rangle$$

for all $u \in \mathcal{X}$, $v \in \mathcal{X}'$ and $f \in H^\infty(S^o_{\theta \cup r})$, where f^* is defined in Section 2.1.

Proof. Let $\theta \in (\omega, \pi/2)$ and $r > R$. If $z \in \mathbb{C} \setminus S_{\omega\cup R}$, then

$$\begin{aligned}\langle R_T(z)u, v\rangle &= \langle R_T(z)u, (\bar{z}I - T')R_{T'}(\bar{z})v\rangle \\ &= \langle zR_T(z)u, R_{T'}(\bar{z})v\rangle - \langle TR_T(z)u, R_{T'}(\bar{z})v\rangle \\ &= \langle u, R_{T'}(\bar{z})v\rangle\end{aligned}$$

for all $u \in \mathcal{X}$ and $v \in \mathcal{X}'$, since $\mathsf{R}(R_T(z)) \subseteq \mathsf{D}(T)$ and $\mathsf{R}(R_{T'}(\bar{z})) \subseteq \mathsf{D}(T')$. This shows that, for an appropriate contour γ in $\mathbb{C}$, we have

$$\langle \phi(T)u, v\rangle = \frac{1}{2\pi i}\int_\gamma \phi(z)\langle R_T(z)u, v\rangle\frac{\mathrm{d}z}{z} = \frac{1}{2\pi i}\int_\gamma \phi(z)\langle u, R_{T'}(\bar{z})v\rangle\frac{\mathrm{d}z}{z} = \langle u, \phi^*(T')v\rangle$$

for all $u \in \mathcal{X}$, $v \in \mathcal{X}'$ and $\phi \in \Theta(S^o_{\theta\cup r})$. Therefore, we have

$$\frac{\|\phi(T)u\|_{\mathcal{X}}}{\|u\|_{\mathcal{X}}} \lesssim \sup_{v\in\mathcal{X}'}\frac{|\langle\phi(T)u, v\rangle|}{\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}'}} = \sup_{v\in\mathcal{X}'}\frac{|\langle u, \phi^*(T')v\rangle|}{\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}'}} \lesssim \sup_{v\in\mathcal{X}'}\frac{\|\phi^*(T')v\|_{\mathcal{X}'}}{\|v\|_{\mathcal{X}'}}$$

for all $u \in \mathcal{X}$ and $\phi \in \Theta(S^o_{\theta\cup r})$. The dual version of this inequality holds by the same reasoning. Therefore, there exists $c > 0$ such that $\frac{1}{c}\|\phi(T)\| \leq \|\phi^*(T')\| \leq c\|\phi(T)\|$ for all $\phi \in \Theta(S^o_{\theta\cup r})$, which proves that T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus if and only if T' has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus.

Now suppose that T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus. Let $(\phi_n)_n$ be a sequence of functions satisfying the requirements of Definition 2.2.6 so that

$$f(T)u = \lim_n (f\phi_n)(T)u$$

for all $u \in \mathcal{X}$ and $f \in H^\infty(S^o_{\theta\cup r})$. For each $\epsilon > 0$, $u \in \mathcal{X}$ and $v \in \mathcal{X}'$, there exists $N \in \mathbb{N}$ such that

$$|\langle (f\phi_n)(T)u, v\rangle - \langle f(T)u, v\rangle| \lesssim \|(f\phi_n)(T)u - f(T)u\|\|v\| < \epsilon$$

for all $n > N$. The dual version of this statement also holds, so we have

$$\langle f(T)u, v\rangle = \lim_{n\to\infty}\langle (f\phi_n)(T)u, v\rangle = \lim_{n\to\infty}\langle u, (f^*\phi_n^*)(T')v\rangle = \langle u, f^*(T')v\rangle$$

for all $u \in \mathcal{X}$ and $v \in \mathcal{X}'$, as required. □

This brings us to the principal result of this section. The proof is based on the proof of Theorem 7 in [53] and Theorem F in [2].

Theorem 2.2.18. *Let $\omega \in [0, \pi/2)$ and $R \geq 0$. Let $\langle T, T'\rangle$ be a dual pair of operators of type $S_{\omega\cup R}$ on $\langle \mathcal{X}, \mathcal{X}'\rangle$. The following statements are equivalent:*

1. *The operators T and T' satisfy (ψ, ϕ) quadratic estimates for all ψ in $\Psi(S^o_{\theta\cup r})$ and ϕ in $\Phi(S^o_{\theta\cup r})$ and all θ in $(\omega, \pi/2)$ and $r > R$;*

2. *There exists θ in $(\omega, \pi/2)$, $r > R$ and nondegenerate $\psi, \tilde{\psi}$ in $\Psi(S^o_{\theta\cup r})$ and nondegenerate $\phi, \tilde{\phi}$ in $\Phi(S^o_{\theta\cup r})$ such that T satisfies (ψ, ϕ) quadratic estimates and T' satisfies $(\tilde{\psi}, \tilde{\phi})$ quadratic estimates;*

3. *The operator T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus for all θ in $(\omega, \pi/2)$ and $r > R$;*

4. *There exists θ in $(\omega, \pi/2)$ and $r > R$ such that T has a bounded $H^\infty(S^o_{\theta\cup r})$ functional calculus.*

Proof. It suffices to prove that (2) implies (3) and that (4) implies (1). First, suppose that (2) holds. Fix $\theta_0 \in (\omega, \pi/2)$, $r_0 > R$, nondegenerate $\psi, \tilde{\psi} \in \Psi(S^o_{\theta_0\cup r_0})$ and nondegenerate $\phi, \tilde{\phi} \in \Phi(S^o_{\theta_0\cup r_0})$ such that T satisfies (ψ, ϕ) quadratic estimates and T' satisfies $(\tilde{\psi}, \tilde{\phi})$ quadratic estimates. Let $\theta \in (\omega, \pi/2)$ and $r > R$. Lemma 2.2.9 gives $\eta \in \Psi(S^o_{\theta_0\cup r_0})$ and $\varphi \in \Theta(S^o_{\theta_0\cup r_0})$ such that

$$\int_0^1 \eta_t(z)\tilde{\psi}^*_t(z)\psi_t(z)\,\frac{\mathrm{d}t}{t} + \varphi(z)\tilde{\phi}^*(z)\phi(z) = 1$$

for all $z \in S^o_{\theta_0 r_0}$. Given $\alpha \in (0,1)$ and $f \in \Theta(S^o_{\theta\cup r})$, if

$$\Psi_{\alpha,1}(z) = f(z)\int_\alpha^1 \eta_t(z)\tilde{\psi}^*_t(z)\psi_t(z)\,\frac{\mathrm{d}t}{t} \quad \text{and} \quad \Phi(z) = f(z)\varphi(z)\tilde{\phi}^*(z)\phi(z)$$

for all $z \in S^o_{\min\{\theta,\theta_0\}\cup\min\{r,r_0\}}$, then

$$\lim_{\alpha\to 0}\|(\Psi_{\alpha,1}(T) + \Phi(T))u - f(T)u\|_{\mathcal{X}} = \lim_{\alpha\to 0}\|(\Psi_{\alpha,1}(T) + \Phi(T))u - f_0(T)u\|_{\mathcal{X}} = 0$$

for all $u \in \mathcal{X}$, where f_0 denotes the restriction of f to $S^o_{\min\{\theta,\theta_0\}\cup\min\{r,r_0\}}$. The dual pairing and Lemma 2.2.11 imply that

$$\begin{aligned}
&|\langle\Psi_{\alpha,1}(T)u + \Phi(T)u, v\rangle| \\
&\quad\leq \int_\alpha^1 |\langle(f\eta_t)(T)\psi_t(T)u, \psi_t(T')v\rangle|\,\frac{\mathrm{d}t}{t} + |\langle(f\varphi)(T)\phi(T)u, \phi(T')v\rangle| \\
&\quad\lesssim \int_\alpha^1 \|(f\eta_t)(T)\|\|\psi_t(T)u\|_{\mathcal{X}}\|\psi_t(T')v\|_{\mathcal{X}'}\,\frac{\mathrm{d}t}{t} + \|(f\varphi)(T)\|\|\phi(T)u\|_{\mathcal{X}}\|\phi(T')v\|_{\mathcal{X}'} \\
&\quad\lesssim \|f\|_\infty\|u\|_{T,\psi,\phi}\|v\|_{T',\psi,\phi}
\end{aligned}$$

for all $u \in \mathcal{X}$, $v \in \mathcal{X}'$, $\alpha \in (0,1)$ and $f \in \Theta(S^o_{\theta\cup r})$. The quadratic estimates then imply that

$$|\langle f(T)u, v\rangle| \lesssim \|f\|_\infty\|u\|_{T,\psi,\phi}\|v\|_{T',\psi,\phi} \lesssim \|f\|_\infty\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}'}$$

for all $u \in \mathcal{X}$, $v \in \mathcal{X}'$ and $f \in \Theta(S^o_{\theta\cup r})$, which implies (3).

Now, suppose that (4) holds. Fix $\theta_0 \in (\omega, \pi/2)$ and $r_0 > R$ such that T has a bounded $H^\infty(S^o_{\theta_0\cup r_0})$ functional calculus, and choose nondegenerate $\tilde{\psi} \in \Psi(S^o_{\theta_0\cup r_0})$ and nondegenerate $\tilde{\phi} \in \Phi(S^o_{\theta_0\cup r_0})$. Let $\theta \in (\omega, \pi/2)$, $r > R$, $\psi \in \Psi(S^o_{\theta\cup r})$ be nondegenerate and $\phi \in \Phi(S^o_{\theta\cup r})$ be nondegenerate. Given $g \in H^\infty(S^o_{\theta\cup r})$, let g_0 denote the restriction of g to $S^o_{\min\{\theta,\theta_0\}\cup\min\{r,r_0\}}$. A discrete version of Proposition 2.2.13 shows that

$$\|f(T)u\|_{T,\psi,\phi} = \|f_0(T)u\|_{T,\psi_0,\phi_0} \lesssim \|f\|_\infty\left(\sum_{k=0}^\infty \|\tilde{\psi}_{2^{-k}}(T)u\|^2_{\mathcal{X}} + \|\tilde{\phi}(T)u\|^2_{\mathcal{X}}\right)^{\frac{1}{2}}$$

for all $f \in H^\infty(S^o_{\theta\cup r})$ and $u \in \mathcal{X}$ for which the right-hand-side is finite. In particular, since we can take f to be a constant function, this shows that

$$\|u\|^2_{T,\psi,\phi} \lesssim \sum_{k=0}^{\infty} \|\tilde{\psi}_{2^{-k}}(T)u\|^2_{\mathcal{X}} + \|\tilde{\phi}(T)u\|^2_{\mathcal{X}}$$

for all $u \in \mathcal{X}$ for which the right-hand-side is finite. Choose $w \in \mathcal{X}'$ such that $\|w\|_{\mathcal{X}'} = 1$ and $\sup\{|\langle\tilde{\psi}_{2^{-k}}(T)u, v\rangle| : v \in \mathcal{X}',\ \|v\|_{\mathcal{X}'} = 1\} \le 2|\langle\tilde{\psi}_{2^{-k}}(T)u, w\rangle|$. The dual pairing and Lemma 2.2.11 then imply that

$$\begin{aligned}
\sum_{k=0}^{n} \|\tilde{\psi}_{2^{-k}}(T)u\|^2_{\mathcal{X}} + \|\tilde{\phi}(T)u\|^2_{\mathcal{X}} &\lesssim \sum_{k=0}^{n} |\langle\tilde{\psi}_{2^{-k}}(T)u, w\rangle| \|u\|_{\mathcal{X}} + \|u\|^2_{\mathcal{X}} \\
&= \sum_{k=0}^{n} |\langle u, \tilde{\psi}^*_{2^{-k}}(T')w\rangle| \|u\|_{\mathcal{X}} + \|u\|^2_{\mathcal{X}} \\
&= \sum_{k=0}^{n} \operatorname{sgn}\left(\langle u, \tilde{\psi}^*_{2^{-k}}(T')w\rangle\right) \langle u, \tilde{\psi}^*_{2^{-k}}(T')w\rangle \|u\|_{\mathcal{X}} + \|u\|^2_{\mathcal{X}} \\
&\le \sup_{r_k \in \{-1,1\}} \langle u, \textstyle\sum_{k=0}^{n} r_k \tilde{\psi}^*_{2^{-k}}(T')w\rangle \|u\|_{\mathcal{X}} + \|u\|^2_{\mathcal{X}} \\
&\le \sup_{r_k \in \{-1,1\}} \|\left(\textstyle\sum_{k=0}^{n} r_k \tilde{\psi}^*_{2^{-k}}\right)(T')\| \|w\|_{\mathcal{X}'} \|u\|^2_{\mathcal{X}} \\
&\lesssim \|u\|^2_{\mathcal{X}}
\end{aligned}$$

for all $u \in \mathcal{X}$ and $n \in \mathbb{N}$, where the final inequality holds because Lemma 2.2.17 implies that T' has a bounded $H^\infty(S^o_{\theta_0\cup r_0})$ functional calculus, and because $\sum_{k=0}^{n} r_k \tilde{\psi}^*_{2^{-k}}$ is in $\Psi(S^o_{\theta_0\cup r_0})$ for any sequence $(r_k)_k$ taking values in $\{-1, 1\}$ and all $n \in \mathbb{N}$. This shows that T satisfies (ψ, ϕ) quadratic estimates. The same reasoning shows that T' satisfies (ψ, ϕ) quadratic estimates, which implies (1). □

2.3 Operators of Type $S_{\omega\setminus R}$

We develop an analogous theory for the following class of operators, where $\mathcal{X}$ denotes a nontrivial complex reflexive Banach space.

Definition 2.3.1. Let $\omega \in [0, \pi/2)$ and $R > 0$. An operator T on $\mathcal{X}$ is of *type* $S_{\omega\setminus R}$ if $\sigma(T) \subseteq S_{\omega\setminus R} \cup \{0\}$, and for each $\theta \in (\omega, \pi/2)$ and $r \in [0, R)$, there exists $C_{\theta\setminus r} > 0$ such that

$$\|R_T(z)\| \le \frac{C_{\theta\setminus r}}{|z|}$$

for all $z \in \mathbb{C} \setminus (S_{\theta\setminus r} \cup \{0\})$.

The theory of type $S_{\omega\setminus R}$ operators is similar to that of type $S_{\omega\cup R}$ operators. The main difference arises for operators with a nontrivial null space, which means that 0 is in the spectrum. The following specialization of Lemma 2.2.2 allows us to deal with this possibility. The proof is omitted since it is essentially the same as the proof of Theorem 3.8 in [31].

Lemma 2.3.2. Let $\omega \in [0, \pi/2)$ and $R > 0$. Let T be an operator of type $S_{\omega\backslash R}$ on $\mathcal{X}$. If $r \in (0, R)$, then the following hold

$$\overline{\mathsf{D}(T)} = \{u \in \mathcal{X} : \lim_{n\to\infty} (I + \tfrac{i}{rn}T)^{-1}u = u\};$$
$$\overline{\mathsf{R}(T)} = \{u \in \mathcal{X} : \lim_{n\to\infty} (I + \tfrac{in}{r}T)^{-1}u = 0\};$$
$$\mathsf{N}(T) = \{u \in \mathcal{X} : \lim_{n\to\infty} (I + \tfrac{in}{r}T)^{-1}u = u\},$$

and $\overline{\mathsf{D}(T)} = \overline{\mathsf{R}(T)} \oplus \mathsf{N}(T) = \mathcal{X}$.

For the remainder of this section, fix $\omega \in [0, \pi/2)$ and $R > 0$, and let T be an operator of type $S_{\omega\backslash R}$ on $\mathcal{X}$. Also, let $\mathsf{P}_{\overline{\mathsf{R}(T)}}$ and $\mathsf{P}_{\mathsf{N}(T)}$ denote the projections from $\mathcal{X}$ onto $\overline{\mathsf{R}(T)}$ and $\mathsf{N}(T)$, as given by Lemma 2.3.2. We introduce an analogue of Definition 2.2.3.

Definition 2.3.3 ($\Theta(S^o_{\theta\backslash r})$ functional calculus)**.** Given $\theta \in (\omega, \pi/2)$, $r \in [0, R)$ and $\phi \in \Theta(S^o_{\theta\backslash r})$, define $\phi(T_{\overline{\mathsf{R}}}) \in \mathcal{L}(\mathcal{X})$ by

$$\phi(T_{\overline{\mathsf{R}}})u = \frac{1}{2\pi i}\int_{+\partial S^o_{\tilde{\theta}\backslash\tilde{r}}} \phi(z)R_T(z)u \,\mathrm{d}z := \lim_{\rho\to\infty} \frac{1}{2\pi i}\int_{+\partial S^o_{\tilde{\theta}\backslash\tilde{r}}\cap D_\rho} \phi(z)R_T(z)u \,\mathrm{d}z \quad (2.3.1)$$

for all $u \in \mathcal{X}$, where $\tilde{\theta} \in (\omega, \theta)$, $\tilde{r} \in (r, R)$ and $+\partial S^o_{\tilde{\theta}\backslash\tilde{r}}$ denotes the boundary of $S^o_{\tilde{\theta}\backslash\tilde{r}}$ oriented clockwise.

A standard calculation shows that the mapping $\Theta(S^o_{\theta\backslash r}) \mapsto \mathcal{L}(\mathcal{X})$ given by (2.3.1) is an algebra homomorphism. The reason for the notation $\phi(T_{\overline{\mathsf{R}}})$ will become apparent in Lemma 2.3.5. This requires the following convergence lemma for the $\Theta(S^o_{\theta\backslash r})$ functional calculus, which is proved in essentially the same way as Proposition 2.2.4.

Proposition 2.3.4. Let $\theta \in (\omega, \pi/2)$ and $r \in [0, R)$. If $(\phi_n)_n$ is a sequence in $\Theta(S^o_{\theta\backslash r})$ and there exists $c, \delta > 0$ and $\phi \in \Theta(S^o_{\theta\backslash r})$ such that the following hold:

1. $\sup_n |\phi_n(z)| \leq c|z|^{-\delta}$ for all $z \in S^o_{\theta\backslash r}$;
2. ϕ_n converges to ϕ uniformly on compacts subsets of $S^o_{\theta\backslash r}$,

then $\phi_n(T_{\overline{\mathsf{R}}})$ converges to $\phi(T_{\overline{\mathsf{R}}})$ in $\mathcal{L}(\mathcal{X})$.

We now establish the connection between the operators defined by (2.1.1) and (2.3.1).

Lemma 2.3.5. Let $\theta \in (\omega, \pi/2)$ and $r \in [0, R)$. If Ω is an open set in $\mathbb{C}_\infty$ that contains $S^o_{\theta\backslash r} \cup \{0, \infty\}$ and $\phi \in \Theta(S^o_{\theta\backslash r}) \cap H(\Omega)$, then

$$\phi(T)u = \phi(T_{\overline{\mathsf{R}}})\mathsf{P}_{\overline{\mathsf{R}(T)}}u + \phi(0)\mathsf{P}_{N(T)}u$$

for all $u \in \mathcal{X}$. If $\phi \in \Theta(S^o_{\theta\backslash r})$, then

$$\phi(T_{\overline{\mathsf{R}}})u = \phi(T_{\overline{\mathsf{R}}})\mathsf{P}_{\overline{\mathsf{R}(T)}}u = \mathsf{P}_{\overline{\mathsf{R}(T)}}\phi(T_{\overline{\mathsf{R}}})\mathsf{P}_{\overline{\mathsf{R}(T)}}u$$

for all $u \in \mathcal{X}$.

Proof. Let Ω be an open set in $\mathbb{C}_\infty$ containing $S^o_{\theta\backslash r}\cup\{0,\infty\}$. Let $\phi\in\Theta(S^o_{\theta\backslash r})\cap H(\Omega)$. If γ is a contour satisfying the requirements of (2.1.1), then Cauchy's Theorem, the resolvent bounds in Definition 2.3.1 and the Θ-class decay imply that

$$\phi(T)u=\phi(\infty)u+\frac{1}{2\pi i}\int_\gamma \phi(z)R_T(z)u\ \mathrm{d}z=\frac{1}{2\pi i}\left(\int_{+\partial S^o_{\tilde\theta\backslash\tilde r}}+\int_{+\partial D_\delta}\right)\phi(z)R_T(z)u\ \mathrm{d}z$$

for all $u\in\mathcal{X}$, $\tilde\theta\in(\omega,\theta)$, $\tilde r\in(r,R)$ and $\delta\in(0,\tilde r)$ satisfying $D_\delta\subset\Omega$.

If $u\in\mathsf{N}(T)$, then $R_T(z)u=\frac{1}{z}u$ for all $z\in\rho(T)$. The function $z\mapsto\frac{1}{z}\phi(z)$ is holomorphic in $S^o_{\tilde\theta\backslash\tilde r}$ and in a neighbourhood of infinity. Therefore, Cauchy's Theorem and the Θ-class decay imply that

$$\int_{+\partial S^o_{\tilde\theta\backslash\tilde r}}\phi(z)R_T(z)u\ \mathrm{d}z=\int_{+\partial S^o_{\tilde\theta\backslash\tilde r}}\frac{\phi(z)}{z}u\ \mathrm{d}z=0 \tag{2.3.2}$$

for all $u\in\mathsf{N}(T)$. Also, Cauchy's integral formula implies that

$$\int_{+\partial D_\delta}\phi(z)R_T(z)u\ \mathrm{d}z=\int_{+\partial D_\delta}\frac{\phi(z)}{z-0}u\ \mathrm{d}z=2\pi i\,\phi(0)u$$

for all $u\in\mathsf{N}(T)$.

If $u\in\mathsf{R}(T)$, then there exists $v\in\mathcal{X}$ such that $u=Tv$, in which case

$$\|zR_T(z)u\|=\|zR_T(z)Tv\|=\|z(zR_T(z)-I)v\|\le|z|(C_{\theta\backslash r}+1)\|v\|$$

for all $z\in D_\delta\backslash\{0\}$ and $\delta\in(0,r)$. A limiting argument then shows that for each $\epsilon>0$ and $u\in\overline{\mathsf{R}(T)}$, there exists $\eta\in(0,r)$ such that $\|zR_T(z)u\|<\epsilon$ for all $z\in D_\eta\setminus\{0\}$, in which case

$$\left\|\int_{+\partial D_\eta}\phi(z)R_T(z)u\ \mathrm{d}z\right\|\le\|\phi\|_\infty\int_{|z|=\eta}\|zR_T(z)u\|\ \frac{|\mathrm{d}z|}{|z|}<2\pi\|\phi\|_\infty\epsilon.$$

Another application of Cauchy's Theorem allows us to conclude that

$$\int_{+\partial D_\delta}\phi(z)R_T(z)u\ \mathrm{d}z=0$$

for all $u\in\overline{\mathsf{R}(T)}$, which completes the proof of the first part of the theorem.

Now let $\phi\in\Theta(S^o_{\theta\backslash r})$. To complete the proof, it suffices to show that $\phi(T_{\overline{R}})u$ is in $\overline{\mathsf{R}(T)}$ for all $u\in\overline{\mathsf{R}(T)}$, since (2.3.2) implies that $\phi(T_{\overline{R}})u=\phi(T_{\overline{R}})\mathsf{P}_{\overline{R(T)}}u$ for all $u\in\mathcal{X}$. For each $n\in\mathbb{N}$, define

$$\psi_n(z)=\frac{1}{1-\frac{i}{rn}z}-\frac{1}{1-\frac{rn}{i}z}=\frac{-(\frac{i}{rn}+\frac{rn}{i})z}{1-(\frac{i}{rn}+\frac{rn}{i})z+z^2}$$

for all $z\in\mathbb{C}\setminus\{\frac{rn}{i},\frac{i}{rn}\}$. The sequence $(\phi\psi_n)_n$ in $\Theta(S^o_{\theta\backslash r})$ converges to ϕ uniformly on compact subsets of $S^o_{\theta\backslash r}$ and there exists $c,\delta>0$ such that

$$\sup_n|\phi(z)\psi_n(z)|\le\sup_n\|\psi_n\|_{L^\infty(S^o_{\theta\backslash r})}|\phi(z)|\le c|z|^{-\delta}$$

for all $z \in S^o_{\theta\backslash r}$, so Proposition 2.3.4 implies that $\lim_n \|(\phi\psi_n)(T_{\overline{R}})u - \phi(T_{\overline{R}})u\| = 0$ for all $u \in \mathcal{X}$. The first part of this lemma then shows that

$$\begin{aligned}(\phi\psi_n)_n(T_{\overline{R}})u &= \psi_n(T_{\overline{R}})\phi(T_{\overline{R}})u \\ &= \psi_n(T)\mathsf{P}_{\overline{\mathsf{R}(T)}}\phi(T_{\overline{R}})u \\ &= [(I - \tfrac{i}{rn}T)^{-1} - (I - \tfrac{rn}{i}T)^{-1}]\mathsf{P}_{\overline{\mathsf{R}(T)}}\phi(T_{\overline{R}})u \\ &= TR_T(\tfrac{rn}{i})R_T(\tfrac{i}{rn})\mathsf{P}_{\overline{\mathsf{R}(T)}}\phi(T_{\overline{R}})u\end{aligned}$$

for all $u \in \mathcal{X}$ and $n \in \mathbb{N}$, which completes the proof. □

We use the following class of functions to incorporate the null space of T in a holomorphic functional calculus.

Definition 2.3.6. Given $\theta \in [0, \pi/2)$ and $r \geq 0$, define $H^\infty(S^o_{\theta\backslash r}, \{0\})$ to be the algebra of functions that are defined on $S^o_{\theta\backslash r} \cup \{0\}$ and holomorphic on $S^o_{\theta\backslash r}$.

The next lemma, which is proved in the same way as Lemma 2.2.5, allows us to derive an $H^\infty(S^o_{\theta\backslash r}, \{0\})$ functional calculus from the $\Theta(S^o_{\theta\backslash r})$ functional calculus.

Lemma 2.3.7. Let $\theta \in (\omega, \pi/2)$ and $r \in [0, R)$. If $(\phi_n)_n$ is a sequence in $\Theta(S^o_{\theta\backslash r})$ and there exists $f \in H^\infty(S^o_{\theta\backslash r})$ such that the following hold:

1. $\sup_n \|\phi_n\|_\infty < \infty$;
2. $\sup_n \|\phi_n(T_{\overline{R}})\| < \infty$;
3. ϕ_n converges to f uniformly on compacts subsets of $S^o_{\theta\backslash r}$,

then $\lim_n \phi_n(T_{\overline{R}})u$ exists in $\mathcal{X}$ for all $u \in \mathcal{X}$. Moreover, if $f \in \Theta(S^o_{\theta\backslash r})$, then $\lim_n \phi_n(T_{\overline{R}})u = f(T_{\overline{R}})u$ for all $u \in \mathcal{X}$.

This suggests the following definition.

Definition 2.3.8 ($H^\infty(S^o_{\theta\backslash r}, \{0\})$ functional calculus)**.** Given both $\theta \in (\omega, \pi/2)$ and $r \in [0, R)$, the operator T has a *bounded* $H^\infty(S^o_{\theta\backslash r}, \{0\})$ *functional calculus* if there exists $c > 0$ such that

$$\|\phi(T_{\overline{\mathsf{R}}})\| \leq c\|\phi\|_\infty$$

for all $\phi \in \Theta(S^o_{\theta\backslash r})$. If T has a bounded $H^\infty(S^o_{\theta\backslash r}, \{0\})$ functional calculus and $f \in H^\infty(S^o_{\theta\backslash r}, \{0\})$, then define $f(T) \in \mathcal{L}(\mathcal{X})$ by

$$f(T)u = \lim_n (f\phi_n)(T_{\overline{\mathsf{R}}})\mathsf{P}_{\overline{\mathsf{R}(T)}}u + f(0)\mathsf{P}_{N(T)}u \tag{2.3.3}$$

for all $u \in \mathcal{X}$, where $(\phi_n)_n$ is a uniformly bounded sequence in $\Theta(S^o_{\theta\backslash r})$ that converges to 1 uniformly on compact subsets of $S^o_{\theta\backslash r}$.

The operator in (2.3.3) is well-defined by Lemma 2.3.7. The requirement that T has a bounded $H^\infty(S^o_{\theta\backslash r}, \{0\})$ functional calculus implies that

$$\|f(T)\| \leq \sup_n \|(f\phi_n)(T_{\overline{\mathsf{R}}})\| + |f(0)| \leq c \sup_n \|f\phi_n\|_{L^\infty(S^o_{\theta\backslash r})} + |f(0)| \leq c\|f\|_\infty$$

for all $f \in H^\infty(S^o_{\theta\backslash r}, \{0\})$, where c is the constant from Definition 2.3.8.

Lemma 2.3.7 also shows that the operators defined by (2.3.1) and (2.3.3) coincide on $\overline{\mathsf{R}(T)}$ for functions in $\Theta(S^o_{\theta\backslash r}) \cap H^\infty(S^o_{\theta\backslash r}, \{0\})$. Furthermore, if Ω is an open set in $\mathbb{C}_\infty$ that contains $(S^o_{\theta\backslash r}) \cup \{0, \infty\}$, then the operators defined by (2.1.1) and (2.3.3) coincide on $\mathcal{X}$ for functions in $H^\infty(S^o_{\theta\backslash r}, \{0\}) \cap H(\Omega)$ by Theorem 2.1.2 and Lemma 2.3.5. There is also the following analogue of Theorem 2.2.7.

Theorem 2.3.9. *The mapping given by* (2.3.3) *is an algebra homomorphism from* $H^\infty(S^o_{\theta\backslash r}, \{0\})$ *into* $\mathcal{L}(\mathcal{X})$ *with following properties:*

1. *If* $\mathbf{1}(z) = 1$ *for all* $z \in S^o_{\theta\backslash r} \cup \{0\}$, *then* $\mathbf{1}(T) = I$ *on* $\mathcal{X}$*;*
2. *If* $\lambda \in \mathbb{C} \setminus (S_{\omega\backslash R} \cup \{0\})$ *and* $f(z) = (\lambda - z)^{-1}$ *for all* $z \in S^o_{\theta\backslash r} \cup \{0\}$, *then* $f(T) = R_T(\lambda)$*;*
3. *If* $(f_n)_n$ *is a sequence in* $H^\infty(S^o_{\theta\backslash r}, \{0\})$ *and there exists* $f \in H^\infty(S^o_{\theta\backslash r}, \{0\})$ *such that the following hold:*
   - *(i)* $\sup_n \|f_n\|_\infty < \infty$*;*
   - *(ii)* $\sup_n \|f_n(T)\| < \infty$*;*
   - *(iii)* f_n *converges to* f *uniformly on compacts subsets of* $S^o_{\theta\backslash r} \cup \{0\}$,

   then $\|f(T)\| \le \sup_n \|f_n(T)\|$ *and* $\lim_n f_n(T)u = f(T)u$ *for all* $u \in \mathcal{X}$.

Proof. Let $f, g \in H^\infty(S^o_{\theta\backslash r}, \{0\})$. If $u \in \overline{\mathsf{R}(T)}$, then using Lemma 2.3.5 and following the proof of Theorem 2.2.7, we obtain $(fg)(T)u = f(T)g(T)u$. If $u \in \mathsf{N}(T)$, then

$$(fg)(T)u = f(0)g(0)u = f(0)g(T)u = f(T)g(T)u.$$

It remains to prove (1) and (3), since (2) holds by the coincidence of (2.1.1) and (2.3.3). If $\phi_n(z) = (1 + \frac{i}{rn}z)^{-1}$ for all $z \in S^o_{\theta\backslash r}$ and $n \in \mathbb{N}$, then Lemmas 2.3.2 and 2.3.5 imply that

$$\mathbf{1}(T)u = \lim_n \phi_n(T_{\overline{\mathsf{R}}})\mathsf{P}_{\overline{\mathsf{R}(T)}}u + \mathsf{P}_{\mathsf{N}(T)}u = \lim_n (I + \tfrac{i}{rn}T)^{-1}\mathsf{P}_{\overline{\mathsf{R}(T)}}u + \mathsf{P}_{\mathsf{N}(T)}u = u$$

for all $u \in \mathcal{X}$. Now let $(f_n)_n$ be a sequence in $H^\infty(S^o_{\theta\backslash r}, \{0\})$ with the properties listed in the theorem. If $u \in \overline{\mathsf{R}(T)}$, then using Lemma 2.3.7 and following the proof of Theorem 2.2.7, we obtain $\lim_n f_n(T)u = f(T)u$. If $u \in \mathsf{N}(T)$, then

$$\lim_n f_n(T)u = \lim_n f_n(0)u = f(0)u = f(T)u,$$

which completes the proof. ∎

All of the results in Section 2.2.1 have a natural analogue for type $S_{\omega\backslash R}$ operators with restrictions to $\overline{\mathsf{R}(T)}$ where required. The proofs are essentially the same. In particular, the McIntosh approximation technique goes over directly. Local quadratic estimates are then restricted to $\overline{\mathsf{R}(T)}$, as below.

Definition 2.3.10. Let $\theta \in (\omega, \pi/2)$ and $r \in [0, R)$. Given both $\psi \in \Psi(S^o_{\theta\backslash r})$ and $\phi \in \Phi(S^o_{\theta\backslash r})$, define the *local quadratic norm* $\|\cdot\|_{T_{\overline{\mathsf{R}}},\psi,\phi}$ by

$$\|u\|_{T_{\overline{\mathsf{R}}},\psi,\phi} = \left(\int_0^1 \|\psi_t(T_{\overline{\mathsf{R}}})u\|^2 \,\frac{\mathrm{d}t}{t} + \|\phi(T_{\overline{\mathsf{R}}})u\|^2 \right)^{\frac{1}{2}}$$

for all $u \in \mathcal{X}$. The operator T satisfies (ψ, ϕ) *quadratic estimates on* $\overline{\mathsf{R}(T)}$ if there exists $c > 0$ such that $\|u\|_{T_{\overline{\mathsf{R}}},\psi,\phi} \leq c\|u\|$ for all $u \in \overline{\mathsf{R}(T)}$, and *reverse* (ψ, ϕ) *quadratic estimates on* $\overline{\mathsf{R}(T)}$ if there exists $c > 0$ such that $\|u\| \leq c\|u\|_{T_{\overline{\mathsf{R}}},\psi,\phi}$ for all $u \in \overline{\mathsf{R}(T)}$ satisfying $\|u\|_{T_{\overline{\mathsf{R}}},\psi,\phi} < \infty$.

The next result is an immediate consequence of the analogue of Proposition 2.2.13 for type $S_{\omega\backslash R}$ operators.

Proposition 2.3.11. Let $\omega \in [0, \pi/2)$ and $R > 0$. Let T be an operator of type $S_{\omega\backslash R}$ on $\mathcal{X}$. If there exists $\theta_0 \in (\omega, \pi/2)$, $r_0 \in [0, R)$, nondegenerate $\psi, \tilde{\psi} \in \Psi(S^o_{\theta_0\backslash r_0})$ and nondegenerate $\phi, \tilde{\phi} \in \Phi(S^o_{\theta_0\backslash r_0})$ such that T satisfies (ψ, ϕ) quadratic estimates on $\overline{\mathsf{R}(T)}$ and reverse $(\tilde{\psi}, \tilde{\phi})$ quadratic estimates on $\overline{\mathsf{R}(T)}$, then T has a bounded $H^\infty(S^o_{\theta\backslash r}, \{0\})$ functional calculus for all $\theta \in (\omega, \pi/2)$ and $r \in [0, R)$.

The full equivalence also holds for dual pairs of operators of type $S_{\omega\backslash R}$.

Theorem 2.3.12. *Let $\omega \in [0, \pi/2)$ and $R > 0$. Let $\langle T, T'\rangle$ be a dual pair of operators of type $S_{\omega\backslash R}$ on $\langle \mathcal{X}, \mathcal{X}'\rangle$. The following statements are equivalent:*

1. *The operators T and T' satisfy (ψ, ϕ) quadratic estimates on $\overline{\mathsf{R}(T)}$ and $\overline{\mathsf{R}(T')}$ for all ψ in $\Psi(S^o_{\theta\backslash r})$ and ϕ in $\Phi(S^o_{\theta\backslash r})$ and all θ in $(\omega, \pi/2)$ and r in $[0, R)$;*
2. *There exists θ in $(\omega, \pi/2)$, r in $[0, R)$, nondegenerate $\psi, \tilde{\psi}$ in $\Psi(S^o_{\theta\backslash r})$ and nondegenerate $\phi, \tilde{\phi}$ in $\Phi(S^o_{\theta\backslash r})$ such that T satisfies (ψ, ϕ) quadratic estimates on $\overline{\mathsf{R}(T)}$ and T' satisfies $(\tilde{\psi}, \tilde{\phi})$ quadratic estimates on $\overline{\mathsf{R}(T')}$;*
3. *The operator T has a bounded $H^\infty(S^o_{\theta\backslash r}, \{0\})$ functional calculus for all θ in $(\omega, \pi/2)$ and r in $[0, R)$;*
4. *There exists θ in $(\omega, \pi/2)$ and r in $[0, R)$ such that the operator T has a bounded $H^\infty(S^o_{\theta\backslash r}, \{0\})$ functional calculus.*

A dual pair $\langle T, T'\rangle$ of operators of type $S_{\omega\backslash R}$ is also a dual pair of operators of type S_ω, as defined in [31]. Therefore, we conclude that Theorem 2.3.12 and the standard equivalence for operators of type S_ω, as in Theorem 2.4 of [31], show that local quadratic estimates are equivalent to standard quadratic estimates for operators of type $S_{\omega\backslash R}$.

Chapter 3

Local Hardy Spaces of Differential Forms

We begin this chapter by developing local analogues of some basic tools from harmonic analysis in the context of a locally doubling metric measure space X. The local tent spaces $t^p(X \times (0,1])$ and the new spaces $L^p_{\mathscr{Q}}(X)$ are introduced and shown to have atomic characterisations when $p = 1$ in Sections 3.2 and 3.3, respectively. We also obtain duality and interpolation results for these spaces. Next, we introduce a general class of first-order differential operators, which includes the Hodge–Dirac operator. We denote these operators by $\mathcal{D}$ and prove exponential off-diagonal estimates for their resolvents in Section 3.4. These are used to prove the main technical estimate in Section 3.5, which allows us to define the local Hardy spaces of differential forms $h^p_{\mathcal{D}}(\wedge T^*M)$ in Section 3.6. We also obtain duality and interpolation results for these spaces. In Section 3.7, we prove the embedding $h^p_{\mathcal{D}}(\wedge T^*M) \subseteq L^p(\wedge T^*M)$ for all $p \in [1,2]$ by requiring additional properties on both the operator $\mathcal{D}$ and the manifold M. Throughout this chapter we adopt the notation from Section 2.1.

3.1 Localisation

The first three sections of this chapter do not require a differentiable structure. To distinguish these results, it is convenient to let X denote a metric measure space with metric ρ and Borel measure μ.

Notation. A ball in X will always refer to an open metric ball. Given $x \in X$ and $r > 0$, let $B(x,r)$ denote the ball in X with centre x and radius r, and let $V(x,r)$ denote the measure $\mu(B(x,r))$. Given $\alpha, r > 0$ and a ball B of radius r, let αB denote the ball with the same centre as B and radius αr.

The results in this section hold if we assume the following condition.

Definition 3.1.1. A metric measure space X is *locally doubling* if for each $r > 0$, the function $x \mapsto V(x,r)$ is continuous on X, and if for each $b > 0$, there exists $A_b \geq 1$ such that

$$0 < V(x,2r) \leq A_b V(x,r) < \infty \tag{D$_{\mathrm{loc}}$}$$

for all $x \in X$ and $r \in (0,b]$.

The following stronger condition is required in Sections 3.2 and 3.3.

Definition 3.1.2. A locally doubling metric measure space X is *exponentially locally doubling* if there exist $A \geq 1$ and $\kappa, \lambda \geq 0$ such that

$$0 < V(x, \alpha r) \leq A\alpha^{\kappa} e^{\lambda(\alpha-1)r} V(x, r) < \infty \qquad (\mathrm{E}_{\kappa,\lambda})$$

for all $x \in X$, $r > 0$ and $\alpha \geq 1$.

Remark 3.1.3. The continuity of $x \mapsto V(x, r)$ is assured on a complete Riemannian manifold and in most applications, but in general only lower semicontinuity is guaranteed. We require this condition because it implies that the volumes of open balls and closed balls are identical (see also Remark 3.2.2).

If $\sup_b A_b < \infty$, then $(\mathrm{D}_{\mathrm{loc}})$ is equivalent to condition (D) from Definition 1.1.1. In fact, the doubling condition was used by Coifman and Weiss in [29] to give an example of a space of homogeneous type. The results here are a localised version of that work. We begin by proving the following useful consequence of local doubling.

Proposition 3.1.4. If X is locally doubling, then for each $b > 0$ there exists $\kappa_b \geq 0$ such that

$$V(x, \alpha r) \leq A_b \alpha^{\kappa_b} V(x, r)$$

for all $x \in X$, $r \in (0, b]$ and $\alpha \in [1, 2b/r]$.

Proof. Let $N = \lceil \log_2 \alpha \rceil$, which is the smallest integer not less than $\log_2 \alpha$, so that $2^{N-1} < \alpha \leq 2^N$ and $B(x, \frac{\alpha}{2^N} r) \subseteq B(x, r)$. Application of the $(\mathrm{D}_{\mathrm{loc}})$ inequality N times reveals that

$$V(x, \alpha r) \leq A_b^N V(x, \tfrac{\alpha}{2^N} r) \leq A_b \alpha^{\kappa_b} V(x, r),$$

where $\kappa_b = \log_2 A_b$. □

We introduce the local property of homogeneity, which is the local analog of the property of homogeneity from [29], and show that it holds on a locally doubling space. This property allows us to apply harmonic analysis locally on X.

Definition 3.1.5. A metric space (X, ρ) has the *local property of homogeneity* if for each $b > 0$ there exists $N_b \in \mathbb{N}$ such that for all $x \in X$ and $r \in (0, b]$, the ball $B(x, r)$ contains at most N_b points $(x_j)_{j=1,\ldots,N_b}$ satisfying $\rho(x_j, x_k) \geq r/2$ for all $j \neq k$.

Remark 3.1.6. The local property of homogeneity is equivalent to the requirement that if $b > 0$, then for all $x \in X$, $r \in (0, b]$ and $n \in \mathbb{N}$, the ball $B(x, r)$ contains at most N_b^n points $(x_j)_{j=1,\ldots,N_b^n}$ satisfying $\rho(x_j, x_k) \geq r/2^n$ for $j \neq k$. The proof of this is similar to that of Lemma 1.1 in Chapter III of [29]. This property is more suited to applications. It can be used, for instance, to prove the next proposition.

Proposition 3.1.7. If X is a locally doubling metric measure space, then it has the local property of homogeneity.

Proof. This follows the proof of the Remark in Chapter III of [29]. □

Following the scheme of [29], we use the local property of homogeneity to prove local covering lemmas. The next two proofs are adapted from those given by Aimar in [1], which treats the global case.

Proposition 3.1.8 (Vitali-Wiener type covering lemma)**.** Let X be a metric space with the local property of homogeneity. Let $\mathscr{B}$ be a collection of balls in X. If there is a finite upper bound on the radii of the balls in $\mathscr{B}$, then there exists a sequence $(B_j)_j$ of disjoint balls in $\mathscr{B}$ with the property that each $B \in \mathscr{B}$ is contained in some $4B_j$.

Proof. Fix $R > 0$ such that the radii $r(B) \leq R$ for all $B \in \mathscr{B}$. Let $\delta \in (0,1)$ to be fixed later, and for each $k \in \mathbb{N}$ define

$$\mathscr{B}_k = \{B \in \mathscr{B} \mid \delta^k R < r(B) \leq \delta^{k-1} R\}.$$

Proceeding recursively for $k = 1, 2, \ldots$, choose a maximal disjoint subset $\tilde{\mathscr{B}}_k$ of $\mathscr{B}_k$ according to the following requirements:

1. $\tilde{\mathscr{B}}_k \subseteq \mathscr{B}_k$;
2. If $B, B' \in \bigcup_{j=1}^k \tilde{\mathscr{B}}_j$ and $B \neq B'$, then $B \cap B' = \emptyset$;
3. If $B \in \mathscr{B}_k \setminus \tilde{\mathscr{B}}_k$, then there exists $B' \in \bigcup_{j=1}^k \tilde{\mathscr{B}}_j$ such that $B \cap B' \neq \emptyset$.

To show that each $\tilde{\mathscr{B}}_k$ is countable, choose $B_0 \in \tilde{\mathscr{B}}_k$ and write

$$\tilde{\mathscr{B}}_k = \bigcup_{n\in\mathbb{N}} \{B \in \tilde{\mathscr{B}}_k \mid B \subseteq nB_0\}.$$

For each $n \in \mathbb{N}$, the centres of all of the balls in $\{B \in \tilde{\mathscr{B}}_k \mid B \subseteq nB_0\}$ are separated by at a least a distance of $\delta^k R$ and contained in a ball of radius nR, so countability follows by the local property of homogeneity. Therefore, the collection $\tilde{\mathscr{B}} = \bigcup_k \tilde{\mathscr{B}}_k$ is a sequence $(B_j)_j$ of disjoint balls in $\mathscr{B}$.

To complete the proof, let $B \in \mathscr{B} \setminus \tilde{\mathscr{B}}$. For some $k \in \mathbb{N}$, we have $B \in \mathscr{B}_k \setminus \tilde{\mathscr{B}}_k$ and there exists $B' \in \bigcup_{j=1}^k \tilde{\mathscr{B}}_j$ such that $B \cap B' \neq \emptyset$. In particular, we have $B' \in \tilde{\mathscr{B}}_{k'}$ for some $k' \leq k$, so if x' denotes the centre of B', then

$$\rho(y, x') \leq 2r(B) + r(B') \leq 2\delta^{k'-1} R + r(B') \leq (2/\delta + 1) r(B')$$

for all $y \in B$. If we set $\delta = 2/3$, then $B \subseteq 4B'$ and the proof is complete. □

Proposition 3.1.9 (Whitney type covering lemma)**.** Let X be a metric space with the local property of homogeneity. Let O be a nonempty proper open subset of X and let ${}^cO = X \setminus O$. For each $h > 0$, there exists a sequence of disjoint balls $(B_j)_j$ with centre $x_j \in X$ and radius

$$r_j = \tfrac{1}{8} \min(\rho(x_j, {}^cO), h)$$

such that, if $\tilde{B}_j = 4B_j$, then $O = \bigcup_j \tilde{B}_j$ and the following bounded intersection property is satisfied:

$$\sup_j \sharp\big(\{k \mid \tilde{B}_j \cap \tilde{B}_k \neq \emptyset\}\big) < \infty.$$

Furthermore, there exists a sequence $(\phi_j)_j$ of nonnegative functions supported in $\tilde{B}_j$ such that $\inf_{x\in B_j} \phi_j(x) > 0$ and $\sum_j \phi_j = \mathbf{1}_O$, where $\mathbf{1}_O$ denotes the characteristic function of O.

Proof. Let $\mathscr{B}$ denote the collection of all balls with centre $x \in O$ and radius $r = \frac{1}{8}\min(\rho(x, {}^cO), h)$. Proposition 3.1.8 gives a sequence $(B_j)_j = (B(x_j, r_j))_j$ of disjoint balls from $\mathscr{B}$ such that $O \subseteq \bigcup_j \tilde{B}_j$, and since $4r_j < \rho(x_j, {}^cO)$, we actually have $O = \bigcup_j \tilde{B}_j$.

We note some facts to help prove that $(B_j)_j$ has the bounded intersection property. First, if $x \in \tilde{B}_j$, then

$$\rho(x, {}^cO) \geq \rho(x_j, {}^cO) - \rho(x_j, x) \geq 8r_j - 4r_j = 4r_j. \tag{3.1.1}$$

Second, given $c > 0$, if $x \in \tilde{B}_j$ and $\rho(x_j, {}^cO) \leq cr_j$, then

$$\rho(x, {}^cO) \leq \rho(x, x_j) + \rho(x_j, {}^cO) \leq (4 + c)r_j. \tag{3.1.2}$$

Now suppose that $\tilde{B}_j \cap \tilde{B}_k \neq \emptyset$. This implies that

$$\rho(x_j, x_k) \leq 4(r_j + r_k) \leq h. \tag{3.1.3}$$

Consider two cases: (1) If $\rho(x_j, {}^cO) > 2h$, then by (3.1.3) we have

$$\rho(x_k, {}^cO) \geq \rho(x_j, {}^cO) - \rho(x_j, x_k) > h,$$

so $r_k = h/8 = r_j$ and $B_k \subseteq 9B_j$; (2) If $\rho(x_j, {}^cO) \leq 2h$, then by (3.1.3) we have

$$\rho(x_k, {}^cO) \leq \rho(x_k, x_j) + \rho(x_j, {}^cO) \leq 3h,$$

which implies that $\rho(x_k, {}^cO) \leq 24r_k$, since either $\rho(x_k, {}^cO) = 8r_k$ or $h = 8r_k$. In this case, if $x \in \tilde{B}_j \cap \tilde{B}_k$, then by (3.1.1) and (3.1.2) with $c = 24$ we obtain

$$4r_j \leq \rho(x, {}^cO) \leq 28r_j \quad \text{and} \quad 4r_k \leq \rho(x, {}^cO) \leq 28r_k,$$

so $(1/7)r_j \leq r_k \leq 7r_j$ and $B_k \subseteq 39B_j$.

The above shows that for each $j \in \mathbb{N}$, the centres of all balls $\tilde{B}_k$ satisfying $\tilde{B}_j \cap \tilde{B}_k \neq \emptyset$ are separated by at a least a distance of $(1/7)r_j$ and contained in a ball of radius $39r_j \leq 5h$. The bounded intersection property then follows from the local property of homogeneity.

To construct the sequence of functions $(\phi_j)_j$, let η be the function equal to 1 on [0,1) and 0 on $[1, \infty)$. For each $j \in \mathbb{N}$, define

$$\psi_j(x) = \eta\left(\frac{\rho(x, x_j)}{4r_j}\right)$$

for all $x \in X$. These are nonnegative functions supported in $\tilde{B}_j$. We also have $1 \leq \sum_j \psi_j(x) < \infty$ for all $x \in O$, since $O = \bigcup_j \tilde{B}_j$ and the bounded intersection property is satisfied. The required functions are then defined for each $j \in \mathbb{N}$ by

$$\phi_j(x) = \begin{cases} \psi_j(x)/\sum_j \psi_j(x), & \text{if} \quad x \in O; \\ 0, & \text{if} \quad x \in {}^cO. \end{cases}$$

□

We now prove a general version of the fundamental theorem for the (centered) *local maximal operator* $\mathcal{M}_{\mathrm{loc}}$ defined for all measurable functions f on X by

$$\mathcal{M}_{\mathrm{loc}}f(x) = \sup_{r\in(0,1]} \frac{1}{V(x,r)} \int_{B(x,r)} |f(y)| \,\mathrm{d}\mu(y)$$

for all $x \in X$.

Proposition 3.1.10. Let X be a locally doubling metric measure space. If f is a measurable function on X, then $\mathcal{M}_{\mathrm{loc}}f$ is lower semicontinuous, and thus measurable, and the following hold:

1. If $\alpha > 0$, then $\mu(\{x \in X \mid \mathcal{M}_{\mathrm{loc}}f(x) > \alpha\}) \lesssim \|f\|_1/\alpha$ for all $f \in L^1(X)$;
2. If $1 < p \leq \infty$, then $\|\mathcal{M}_{\mathrm{loc}}f\|_p \lesssim_p \|f\|_p$ for all $f \in L^p(X)$.

Proof. The lower semicontinuity of $\mathcal{M}_{\mathrm{loc}}f$ is guaranteed by Fatou's Lemma and the continuity of the mapping $x \mapsto V(x,r)$ from Definition 3.1.1.

To prove (1), let $f \in L^1(X)$ and set $E_\alpha = \{x \in X \mid \mathcal{M}_{\mathrm{loc}}f(x) > \alpha\}$ for each $\alpha > 0$. If $x \in E_\alpha$, then there exists $r_x \in (0,1]$ such that

$$\frac{1}{V(x,r_x)} \int_{B(x,r_x)} |f(y)| \,\mathrm{d}\mu(y) > \alpha.$$

By Proposition 3.1.8, the collection $\mathscr{B} = (B(x,r_x))_{x\in E_\alpha}$ contains a subsequence $(B_j)_j$ of disjoint balls such that, if $\tilde{B}_j = 4B_j$, then $(\tilde{B}_j)_j$ cover E_α. Therefore, by $(\mathrm{D}_{\mathrm{loc}})$ we have

$$\int_X |f(y)| \,\mathrm{d}\mu(y) \geq \sum_j \int_{B_j} |f(y)| \,\mathrm{d}\mu(y) > \alpha \sum_j \mu(B_j) \gtrsim \alpha\mu(E_\alpha).$$

The proof of (2) is then standard (see, for instance, Section I.1.5 of [63]). □

We conclude this section by proving that a locally doubling space is exponentially locally doubling, as in Definition 3.1.2, if and only if it satisfies a certain additional condition on volume growth. Whilst we do not make explicit use of this equivalence, it shows why $(\mathrm{E}_{\kappa,\lambda})$ is often a more useful assumption than $(\mathrm{D}_{\mathrm{loc}})$. In particular, it allows us to obtain the atomic characterisation of the space $L^1_{\mathscr{Q}}(X)$ in Section 3.3.

Proposition 3.1.11. Let X be a locally doubling metric measure space. Then X is exponentially locally doubling if and only if there exist $A_0 \geq 1$ and $b_0, \delta > 0$ such that

$$V(x,r+\delta) \leq A_0 V(x,r) \qquad (\mathrm{D}_{\mathrm{glo}})$$

for all $x \in X$ and $r \geq b_0$.

Proof. If X satisfies $(\mathrm{E}_{\kappa,\lambda})$, then for any $b_0 > 0$ and $\delta > 0$, we have

$$V(x,r+\delta) = V(x,(1+\delta/r)r) \leq A(1+\delta/b_0)^\kappa e^{\lambda\delta} V(x,r)$$

for all $r \geq b_0$ and $x \in X$.

To prove the converse, suppose X satisfies ($\mathrm{D}_{\mathrm{glo}}$) and let $\alpha > 1$. Consider three cases:

If $r > b_0$, choose $N \in \mathbb{N}$ so that $\alpha r - N\delta \in (r, r+\delta]$. Application of the ($\mathrm{D}_{\mathrm{glo}}$) inequality $N+1$ times reveals that

$$V(x, \alpha r) \leq A_0^{N+1} V(x,r) \leq A_0 e^{\lambda(\alpha-1)r} V(x,r), \tag{3.1.4}$$

where $\lambda = (\log A_0)/\delta$;

If $r \in (0, b_0]$ and $\alpha \in (1, 2b_0/r]$, then Proposition 3.1.4 implies that

$$V(x, \alpha r) \leq A_{b_0} \alpha^{\kappa_{b_0}} V(x,r); \tag{3.1.5}$$

If $r \in (0, b_0]$ and $\alpha > 2b_0/r$, then we obtain

$$\begin{aligned} V(x, \alpha r) &= V(x, (\alpha r/2b_0)2b_0) \\ &\leq A_0 e^{\lambda(\alpha r/2b_0 - 1)2b_0} V(x, 2b_0) \\ &\leq A_0 e^{\lambda(\alpha-1)r} V(x, (2b_0/r)r) \\ &\leq A_0 A_{b_0} \alpha^{\kappa_{b_0}} e^{\lambda(\alpha-1)r} V(x,r), \end{aligned}$$

where we used (3.1.4) to obtain the first inequality and (3.1.5) to obtain the final inequality.

These show that X satisfies ($\mathrm{E}_{\kappa,\lambda}$) with $\kappa = \kappa_{b_0}$ and $\lambda = (\log A_0)/\delta$. □

3.2 Local Tent Spaces $t^p(X \times (0,1])$

We introduce the local tent spaces $t^p(X \times (0,1])$, or simply t^p, for all $p \in [1,\infty]$ in the context of a locally doubling metric measure space X. Note that functions on $X \times (0,1]$ are assumed to be complex-valued. There is also the following notation.

Notation. The cone of aperture $\alpha > 0$ and height 1 with vertex at $x \in X$ is

$$\Gamma_\alpha^1(x) = \{(y,t) \in X \times (0,1] \mid \rho(x,y) < \alpha t\}.$$

Let $\Gamma^1(x) = \Gamma_1^1(x)$. For any closed set $F \subseteq X$ and any open set $O \subseteq X$, define

$$R_\alpha^1(F) = \bigcup_{x \in F} \Gamma_\alpha^1(x) \quad \text{and} \quad T_\alpha^1(O) = (X \times (0,1]) \setminus R_\alpha^1({}^cO),$$

where ${}^cO = X \setminus O$. Let $T^1(O) = T_1^1(O)$ and call it the *truncated tent over* O. Note that

$$T_\alpha^1(O) = \{(y,t) \in X \times (0,1] \mid \rho(y, {}^cO) \geq \alpha t\}.$$

For any ball B in X of radius $r(B) > 0$, the *truncated Carleson box over* B is

$$C^1(B) = B \times (0, \min\{r(B), 1\}].$$

Finally, if E is a measurable subset of $X \times (0,1]$, then $\mathbf{1}_E$ denotes the characteristic function of E on $X \times (0,1]$.

The local Lusin operator $\mathcal{A}_{\mathrm{loc}}$ and its dual $\mathcal{C}_{\mathrm{loc}}$ are defined for any measurable function f on $X \times\ (0,1]$ as follows:

$$\mathcal{A}_{\mathrm{loc}} f(x) = \left(\iint_{\Gamma^1(x)} |f(y,t)|^2 \,\frac{\mathrm{d}\mu(y)}{V(x,t)}\,\frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}};$$

$$\mathcal{C}_{\mathrm{loc}} f(x) = \sup_{B \in \mathscr{B}_2(x)} \left(\frac{1}{\mu(B)} \iint_{T^1(B)} |f(y,t)|^2 \,\mathrm{d}\mu(y) \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}}$$

for all $x \in X$, where $\mathscr{B}_2(x)$ denotes the set of all balls B in X of radius $r(B) \leq 2$ such that $x \in B$. We now define the local tent spaces.

Definition 3.2.1. Let X be a locally doubling metric measure space. For each $p \in [1,\infty)$, the *local tent space* $t^p(X \times (0,1])$ consists of all measurable functions f on $X \times\ (0,1]$ with

$$\|f\|_{t^p} = \|\mathcal{A}_{\mathrm{loc}} f\|_p < \infty.$$

The *local tent space* $t^\infty(X \times (0,1])$ consists of all measurable functions f on $X \times\ (0,1]$ with

$$\|f\|_{t^\infty} = \|\mathcal{C}_{\mathrm{loc}} f\|_\infty < \infty.$$

Remark 3.2.2. Recall that in Definition 3.1.1 we required the continuity of the mapping $x \mapsto V(x,r)$ for each $r > 0$. This implies that the volumes of open balls and closed balls are identical, which ensures that $\mathcal{A}_{\mathrm{loc}} f$ and $\mathcal{C}_{\mathrm{loc}} f$ are lower semicontinuous and thus measurable.

The local tent spaces are Banach spaces under the usual identification of functions that are equal almost everywhere. This follows as in the global case in [27]. In particular, completeness holds by dominated convergence upon noting that for each compact set $K \subseteq X \times (0,1]$ and each $p \in [1,\infty]$, we have

$$\|\mathbf{1}_K f\|_{t^p} \lesssim_{K,p} \left(\iint_K |f(y,t)|^2 \,\mathrm{d}\mu(y)\mathrm{d}t \right)^{\frac{1}{2}} \gtrsim_{K,p} \|f\|_{t^p} \tag{3.2.1}$$

for all measurable functions f on $X \times (0,1]$.

Let $L^2_\bullet(X \times (0,1])$, or simply $L^2_\bullet$, denote the Hilbert space of all measurable functions f on $X \times (0,1]$ with

$$\|f\|_{L^2_\bullet} = \left(\iint |f(y,t)|^2 \,\mathrm{d}\mu(y) \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} < \infty.$$

We have $t^2 = L^2_\bullet$, since if $(y,t) \in \Gamma^1(x)$, then $t \leq 1$, $B(y,t) \subseteq B(x,2t)$ and also $B(x,t) \subseteq B(y,2t)$, so by ($\mathrm{D}_{\mathrm{loc}}$) we obtain

$$\|f\|_{t^2}^2 \eqsim \iiint \mathbf{1}_{\Gamma^1(x)}(y,t)\ |f(y,t)|^2 \,\frac{\mathrm{d}\mu(y)}{V(y,t)}\,\frac{\mathrm{d}t}{t} \mathrm{d}\mu(x) = \|f\|_{L^2_\bullet}^2.$$

These observations lead us to the following density result, which is crucial to the extension procedure in Section 3.5.

Proposition 3.2.3. Let X be a locally doubling metric measure space. For all $p \in [1, \infty)$ and $q \in [1, \infty]$, the set $t^p \cap t^q$ is dense in t^p.

Proof. Let $f \in t^p$ and $p \in [1, \infty)$. Fix a ball B in X and define

$$f_k = \mathbf{1}_{kB \times [1/k, 1]} f$$

for each $k \in \mathbb{N}$. The functions f_k belong to $t^p \cap t^q$ for all $q \in [1, \infty]$ by (3.2.1), and $\lim_{k \to \infty} \|f - f_k\|_{t^p} = 0$ by dominated convergence. □

We characterise t^1 in terms of the following atoms.

Definition 3.2.4. Let X be a locally doubling metric measure space. A t^1-*atom* is a measurable function a on $X \times (0, 1]$ supported in the truncated tent $T^1(B)$ over a ball B in X of radius $r(B) \leq 2$ with

$$\|a\|_{L^2_\bullet} = \left(\iint_{T^1(B)} |a(y,t)|^2 \, \mathrm{d}\mu(y) \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \leq \mu(B)^{-\frac{1}{2}}.$$

If a is a t^1-atom corresponding to a ball B as above, then the Cauchy–Schwarz inequality implies that $a \in t^1 \cap t^2$ with $\|a\|_{t^2} \lesssim \|a\|_{L^2_\bullet} \leq \mu(B)^{-1/2}$ and

$$\|a\|_{t^1} \leq \mu(B)^{\frac{1}{2}} \|a\|_{t^2} \lesssim 1. \tag{3.2.2}$$

Remark 3.2.5. If $(\lambda_j)_j$ is a sequence in ℓ^1 and $(a_j)_j$ is a sequence of t^1-atoms, then (3.2.2) implies that $\sum_j \lambda_j a_j$ converges in t^1 with $\| \sum_j \lambda_j a_j \|_{t^1} \lesssim \|(\lambda_j)_j\|_{\ell^1}$. Note that this did not require the condition $r(B) \leq 2$ in Definition 3.2.4.

The atomic characterisation of t^1 asserts the converse of the above remark. This is the content of the following theorem.

Theorem 3.2.6. *Let X be a locally doubling metric measure space. If $f \in t^1$, then there exist a sequence $(\lambda_j)_j$ in ℓ^1 and a sequence $(a_j)_j$ of t^1-atoms such that $\sum_j \lambda_j a_j$ converges to f in t^1 and almost everywhere in $X \times (0, 1]$. Moreover, we have*

$$\|f\|_{t^1} \eqsim \inf\{\|(\lambda_j)_j\|_{\ell^1} : f = \textstyle\sum_j \lambda_j a_j\}.$$

Also, if $p \in (1, \infty)$ and $f \in t^1 \cap t^p$, then $\sum_j \lambda_j a_j$ converges to f in t^p as well.

The proof of Theorem 3.2.6 is an adaptation of the work by Russ in [60], which in turn is based on the original proof by Coifman, Meyer and Stein in [27]. For this, we introduce the notion of local γ-density.

Definition 3.2.7. Let X be a locally doubling metric measure space. Let F be a closed subset of X with $O = {}^cF$ and $\mu(O) < \infty$. For each $\gamma \in (0, 1)$, the points of *local γ-density* with respect to F are the elements of the set

$$F^\gamma_{\mathrm{loc}} = \left\{ x \in X \;\middle|\; \inf_{0 < r \leq 1} \frac{\mu(F \cap B(x,r))}{V(x,r)} \geq \gamma \right\}.$$

The complement of this set is denoted by $O^\gamma_{\mathrm{loc}} = {}^c(F^\gamma_{\mathrm{loc}})$.

Local γ-density can be understood in terms of the local maximal operator $\mathcal{M}_{\mathrm{loc}}$ from Section 3.1. For each $\gamma \in (0,1)$, the following hold:

1. F^γ_{loc} is closed;
2. $F^\gamma_{\mathrm{loc}} \subseteq F$;
3. $O^\gamma_{\mathrm{loc}} = \{x \in X \mid \mathcal{M}_{\mathrm{loc}} \mathbf{1}_O(x) > 1-\gamma\}$;
4. $\mu(O^\gamma_{\mathrm{loc}}) \lesssim \mu(O)$.

The proof of these properties relies on Proposition 3.1.10 and is left to the reader.

The proof of Theorem 3.2.6 also requires the following lemma, which is adapted from Lemma 2.1 in [60].

Lemma 3.2.8. Let X be a locally doubling metric measure space. Let F be a closed subset of X and let Φ be a nonnegative measurable function on $X \times (0,1]$. For each $\eta \in (0,1)$, there exists $\gamma \in (0,1)$ such that

$$\iint_{R^1_{1-\eta}(F^\gamma_{\mathrm{loc}})} \Phi(y,t) V(y,t)\ \mathrm{d}\mu(y)\mathrm{d}t \lesssim \int_F \iint_{\Gamma^1(x)} \Phi(y,t)\ \mathrm{d}\mu(y)\mathrm{d}t\mathrm{d}\mu(x).$$

Proof. Fix $\eta \in (0,1)$ and let $\gamma \in (0,1)$ to be chosen later. For each (y,t) in $R^1_{1-\eta}(F^\gamma_{\mathrm{loc}})$, choose $\xi \in F^\gamma_{\mathrm{loc}}$ such that $(y,t) \in \Gamma^1_{1-\eta}(\xi)$. We then have

$$\mu(F \cap B(\xi,t)) \geq \gamma V(\xi,t).$$

Also, the condition $\rho(\xi,y) < (1-\eta)t$ implies that $B(\xi,\eta t) \subseteq B(y,t)$. Therefore, we have $B(\xi,\eta t) \subseteq B(\xi,t) \cap B(y,t)$ and by Proposition 3.1.4 there exists $c_\eta \in (0,1)$, depending on η, such that

$$c_\eta V(\xi,t) \leq V(\xi,\eta t) \leq \mu(B(\xi,t) \cap B(y,t)).$$

Now choose $\gamma \in (1-c_\eta, 1)$. The above inequalities show that there exists $C_{\eta,\gamma} > 0$, depending on η and the choice of γ, such that

$$\begin{aligned}
\mu(F \cap B(y,t)) &\geq \mu(F \cap B(\xi,t)) - \mu(B(\xi,t) \cap {}^c B(y,t)) \\
&\geq (\gamma - (1-c_\eta)) V(\xi,t) \\
&\geq C_{\eta,\gamma} V(y,t),
\end{aligned}$$

where the final inequality follows from $(\mathrm{D_{loc}})$ and $B(y,t) \subseteq B(\xi,2t)$.

Using the above inequality and Fubini's theorem we obtain

$$\begin{aligned}
&\iint_{R^1_{1-\eta}(F^\gamma_{\mathrm{loc}})} \Phi(y,t) V(y,t)\ \mathrm{d}\mu(y)\mathrm{d}t \\
&\qquad \lesssim \iint_{R^1_{1-\eta}(F^\gamma_{\mathrm{loc}})} \Phi(y,t) \mu(F \cap B(y,t))\ \mathrm{d}\mu(y)\mathrm{d}t \\
&\qquad \leq \iint_{R^1_1(F)} \int_{F \cap B(y,t)} \Phi(y,t)\ \mathrm{d}\mu(x)\mathrm{d}\mu(y)\mathrm{d}t \\
&\qquad \leq \int_F \iint_{\Gamma^1(x)} \Phi(y,t)\ \mathrm{d}\mu(y)\mathrm{d}t\mathrm{d}\mu(x).
\end{aligned}$$

□

We now complete the proof of the atomic characterisation of t^1.

Proof of Theorem 3.2.6. Let $f \in t^1$ and for each $k \in \mathbb{Z}$, define

$$O_k = \{x \in X \mid \mathcal{A}_{\mathrm{loc}} f(x) > 2^k\}$$

and $F_k = {}^c O_k$. The lower semicontinuity of $\mathcal{A}_{\mathrm{loc}} f$ ensures that O_k is open. We also have $\mu(O_k) \leq 2^{-k}\|f\|_{t^1} < \infty$.

Let $\eta \in (0,1)$ to be chosen later and let $\gamma \in (0,1)$ be the constant, which depends on η, from Lemma 3.2.8. Let F_k^* denote the set $(F_k)^\gamma_{\mathrm{loc}}$ from Definition 3.2.7 and let $O_k^* = {}^c(F_k^*)$. We then have $O_k \subseteq O_k^*$ and $\mu(O_k^*) \lesssim \mu(O_k)$.

First, we establish that f is supported in $\bigcup_{k\in\mathbb{Z}} T^1_{1-\eta}(O_k^*)$. For each $k \in \mathbb{Z}$, we apply Lemma 3.2.8 with $\Phi(y,t) = |f(y,t)|^2 (V(y,t)t)^{-1}$ and $F = F_k$ to obtain

$$\begin{aligned}
\iint_{{}^c\left(\bigcup_{j\in\mathbb{Z}} T^1_{1-\eta}(O_j^*)\right)} |f(y,t)|^2 \, \mathrm{d}\mu(y)\frac{\mathrm{d}t}{t} &= \iint_{\bigcap_{j\in\mathbb{Z}} R^1_{1-\eta}(F_j^*)} |f(y,t)|^2 \, \mathrm{d}\mu(y)\frac{\mathrm{d}t}{t} \\
&\leq \iint_{R^1_{1-\eta}(F_k^*)} |f(y,t)|^2 \, \mathrm{d}\mu(y)\frac{\mathrm{d}t}{t} \\
&\lesssim \int_{F_k} \iint_{\Gamma^1(x)} |f(y,t)|^2 \, \frac{\mathrm{d}\mu(y)}{V(y,t)} \frac{\mathrm{d}t}{t} \mathrm{d}\mu(x) \\
&\lesssim \int \mathbf{1}_{F_k}(x) (\mathcal{A}_{\mathrm{loc}} f(x))^2 \, \mathrm{d}\mu(x),
\end{aligned}$$

where the final inequality follows from $(\mathrm{D}_{\mathrm{loc}})$, since if $(y,t) \in \Gamma^1(x)$, then $t \leq 1$ and $B(x,t) \subseteq B(y,2t)$. If k is a negative integer, then pointwise on X we have $\mathbf{1}_{F_k}(\mathcal{A}_{\mathrm{loc}} f)^2 \leq \mathcal{A}_{\mathrm{loc}} f$ and $\lim_{k\to-\infty} \mathbf{1}_{F_k}(\mathcal{A}_{\mathrm{loc}} f)^2 = 0$, where $\mathcal{A}_{\mathrm{loc}} f \in L^1(X)$. Therefore, by dominated convergence we have

$$\lim_{k\to-\infty} \int \mathbf{1}_{F_k}(x)(\mathcal{A}_{\mathrm{loc}} f(x))^2 \, \mathrm{d}\mu(x) = 0,$$

which implies that $f = 0$ almost everywhere on ${}^c\Big(\bigcup_{j\in\mathbb{Z}} T^1_{1-\eta}(O_j^*)\Big)$, as required.

Now we decompose f into t^1-atoms. For each $k \in \mathbb{Z}$, apply Proposition 3.1.9 with $O = O_k^*$ and $h > 0$ to be chosen later. This gives a sequence of disjoint balls $(B_j^k)_{j\in I_k}$, where each ball $B_j^k = B(x_j^k, r_j^k)$ has radius $r_j^k = \frac{1}{8}\min(\rho(x_j^k, {}^cO_k^*), h)$ and I_k is some indexing set. It also gives a sequence of nonnegative functions $(\phi_j^k)_{j\in I_k}$ supported in $\tilde{B}_j^k = 4B_j^k$ such that $\sum_{j\in I_k} \phi_j^k = \mathbf{1}_{O_k^*}$. For each (y,t) in $X \times (0,1]$, we have

$$\mathbf{1}_{T^1_{1-\eta}(O_k^*)\setminus T^1_{1-\eta}(O_{k+1}^*)}(y,t) = \sum_{j\in I_k} \phi_j^k(y) \mathbf{1}_{T^1_{1-\eta}(O_k^*)\setminus T^1_{1-\eta}(O_{k+1}^*)}(y,t),$$

since either $(y,t) \in T^1_{1-\eta}(O_k^*) \setminus T^1_{1-\eta}(O_{k+1}^*)$, in which case $y \in O_k^*$ and we have $\sum_{j\in I_k} \phi_j^k(y) = 1$, or both sides of the equation are zero. Given that f is supported in $\bigcup_{k\in\mathbb{Z}} T^1_{1-\eta}(O^*_k)$, the following holds for almost every $(y,t) \in X \times (0,1]$:

$$\begin{aligned} f(y,t) &= f(y,t)\sum_{k\in\mathbb{Z}} \mathbf{1}_{T^1_{1-\eta}(O^*_k)\setminus T^1_{1-\eta}(O^*_{k+1})}(y,t) \\ &= \sum_{k\in\mathbb{Z}}\sum_{j\in I_k} f(y,t)\phi^k_j(y)\mathbf{1}_{T^1_{1-\eta}(O^*_k)\setminus T^1_{1-\eta}(O^*_{k+1})}(y,t) \\ &= \sum_{k\in\mathbb{Z}}\sum_{j\in I_k} \lambda^k_j a^k_j(y,t), \end{aligned} \tag{3.2.3}$$

where

$$a^k_j(y,t) = \frac{1}{\lambda^k_j} f(y,t)\phi^k_j(y)\mathbf{1}_{T^1_{1-\eta}(O^*_k)\setminus T^1_{1-\eta}(O^*_{k+1})}(y,t),$$

$$\lambda^k_j = \left(\mu(\alpha B^k_j)\iint |f(y,t)|^2\phi^k_j(y)^2\mathbf{1}_{T^1_{1-\eta}(O^*_k)\setminus T^1_{1-\eta}(O^*_{k+1})}(y,t)\,\mathrm{d}\mu(y)\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}$$

and $\alpha > 0$ will be chosen later.

Given that $f \in t^1$, the series in (3.2.3) also converges to f in t^1 by dominated convergence. The same reasoning shows that if $f \in t^1 \cap t^p$ for some $p \in (1,\infty)$, then the series also converges to f in t^p. It remains to choose the constants $\eta \in (0,1)$, $h > 0$ and $\alpha > 0$ so that (3.2.3) is the required atomic decomposition.

First, consider the support of a^k_j. If $(y,t) \in \operatorname{sppt} a^k_j$, then $y \in \operatorname{sppt}\phi^k_j \subseteq 4B^k_j$ and we have

$$\rho(y,z) \geq \rho(x^k_j,z) - \rho(x^k_j,y) \geq (\alpha-4)r^k_j \tag{3.2.4}$$

for all $z \in {}^c(\alpha B^k_j)$. We also have $\rho(y,{}^cO^*_k) \geq (1-\eta)t$, since $(y,t) \in T^1_{1-\eta}(O^*_k)$. Now consider two cases: (1) If $8r^k_j = \min(\rho(x^k_j,{}^cO^*_k),h) = \rho(x^k_j,{}^cO^*_k)$, then

$$(1-\eta)t \leq \rho(y,{}^cO^*_k) \leq \rho(y,x^k_j) + \rho(x^k_j,{}^cO^*_k) \leq 12r^k_j,$$

so by (3.2.4) we have

$$\rho(y,z) \geq (\alpha-4)(1-\eta)t/12 \tag{3.2.5}$$

for all $z \in {}^c(\alpha B^k_j)$; (2) If $8r^k_j = \min(\rho(x^k_j,{}^cO^*_k),h) = h$, then

$$\rho(y,z) \geq (\alpha-4)h/8 \tag{3.2.6}$$

for all $z \in {}^c(\alpha B^k_j)$.

Now choose $\eta \in (0,1)$, $h > 0$ and $\alpha > 0$ such that

$$(\alpha-4)(1-\eta)/12 \geq 1, \quad (\alpha-4)h/8 \geq 1 \quad \text{and} \quad \alpha h/8 \leq 2.$$

For example, set $\eta = 1/4$, $h = 1/2$ and $\alpha = 20$. It then follows from (3.2.5) and (3.2.6) that $\rho(y,{}^c(\alpha B^k_j)) \geq t$ and so $\operatorname{sppt} a^k_j \subseteq T^1(\alpha B^k_j)$, where the radius of αB^k_j is $\alpha r^k_j \leq \alpha h/8 \leq 2$. Also, it is immediate that $\|a^k_j\|_{L^2_\bullet} = \mu(\alpha B^k_j)^{-1/2}$ and thus a^k_j is a t^1-atom.

It remains to prove the norm equivalence. Using the support condition just proved and applying Lemma 3.2.8 with $F = F_k$ and

$$\Phi(y,t) = \mathbf{1}_{T^1(\alpha B^k_j)}(y,t)|f(y,t)|^2(V(y,t)t)^{-1}$$

gives

$$
\begin{aligned}
(\lambda_j^k)^2\mu(\alpha B_j^k)^{-1} &\leq \iint_{T^1(\alpha B_j^k)\cap {}^c[T^1_{1-\eta}(O^*_{k+1})]} |f(y,t)|^2 \, \mathrm{d}\mu(y)\frac{\mathrm{d}t}{t} \\
&= \iint_{R^1_{1-\eta}(F^*_{k+1})} \mathbf{1}_{T^1(\alpha B_j^k)}(y,t)|f(y,t)|^2 \, \mathrm{d}\mu(y)\frac{\mathrm{d}t}{t} \\
&\lesssim \int_{F_{k+1}} \iint_{\Gamma^1(x)} \mathbf{1}_{T^1(\alpha B_j^k)}(y,t)|f(y,t)|^2 \, \frac{\mathrm{d}\mu(y)}{V(y,t)}\frac{\mathrm{d}t}{t}\mathrm{d}\mu(x) \\
&\lesssim \int_{{}^cO_{k+1}\cap\alpha B_j^k} (\mathcal{A}_{\mathrm{loc}}f(x))^2 \, \mathrm{d}\mu(x) \\
&\lesssim 2^{2k}\mu(\alpha B_j^k).
\end{aligned}
$$

Furthermore, by ($\mathrm{D_{loc}}$) we have $\lambda_j^k \lesssim 2^k\mu(B_j^k)$, and since for each $k \in \mathbb{Z}$ the balls $(B_j^k)_j$ are disjoint and contained in O_k^*, we obtain

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}}\sum_{j\in I_k}|\lambda_j^k| &\leq \sum_{k\in\mathbb{Z}} 2^k\mu(O_k^*) \\
&\lesssim \sum_{k\in\mathbb{Z}} 2^k\mu(O_k) \\
&= \sum_{k\in\mathbb{Z}} 2\int_{2^{k-1}}^{2^k} \mu(\{x\in X \mid \mathcal{A}_{\mathrm{loc}}f(x) > 2^k\}) \, \mathrm{d}t \\
&\lesssim \sum_{k\in\mathbb{Z}} \int_{2^{k-1}}^{2^k} \mu(\{x\in X \mid \mathcal{A}_{\mathrm{loc}}f(x) > t\}) \, \mathrm{d}t \\
&= \|f\|_{t^1},
\end{aligned}
$$

which completes the proof. □

Remark 3.2.9. If $b > 1$, then a judicious choice of $\eta \in (0,1)$, $h > 0$ and $\alpha > 0$ in the proof of Theorem 3.2.6 allows us to characterise $f \in t^1$ in terms t^1-atoms supported on truncated tents $T^1(B)$ over balls B with radius $r(B) \leq b$. The constants in the norm equivalence $\eqsim$ then depend on b and, as we may expect, become unbounded as b approaches 1.

It is also possible to characterise t^1 in terms of atoms supported in truncated Carleson boxes.

Definition 3.2.10. Let X be a locally doubling metric measure space. A t^1-*Carleson atom* is a measurable function a on $X \times (0,1]$ supported in the truncated Carleson box $C^1(B)$ over a ball B in X of radius $r(B) > 0$ with $\|a\|_{L^2_\bullet} \leq \mu(B)^{-1/2}$.

It is immediate that Theorem 3.2.6 holds with t^1-Carleson atoms in place of t^1-atoms. As explained in Remark 3.2.5, the converse of Theorem 3.2.6 does not require the upper bound $r(B) \leq 2$ on the radii of the supports of t^1-atoms. This may not be the case for t^1-Carleson atoms on a locally doubling metric measure space. In the following proposition, however, we show that this is the case on an exponentially locally doubling metric measure space. We will need this to prove the molecular characterisation of $h^1_{\mathcal{D}}$ in Lemma 3.6.17. This is the first indication that $(\mathrm{E}_{\kappa,\lambda})$ is more suited to our purposes than $(\mathrm{D}_{\mathrm{loc}})$.

Proposition 3.2.11. Let X be an exponentially locally doubling metric measure space. If $(\lambda_j)_j$ is a sequence in ℓ^1 and $(a_j)_j$ is a sequence of t^1-Carleson atoms, then $\sum_j \lambda_j a_j$ converges in t^1 with $\|\sum_j \lambda_j a_j\|_{t^1} \lesssim \|(\lambda_j)_j\|_{\ell^1}$.

Proof. It is enough to show that $\sup \|a\|_{t^1} \lesssim 1$, where the supremum is taken over all a that are t^1-Carleson atoms.

Let a be a t^1-Carleson atom supported on a ball B in X of radius $r(B) > 0$ with $\|a\|_{L^2_\bullet} \leq \mu(B)^{-1/2}$. First suppose that $r(B) \leq 1$. Property $(\mathrm{D}_{\mathrm{loc}})$ implies that $\mu(2B) \leq c\mu(B)$ for some $c > 0$ that does not depend on B. Also, we have $C^1(B) \subset T^1(2B)$ and the radius $r(2B) \leq 2$. This implies that $a/\sqrt{c}$ is a t^1-atom and the result follows by (3.2.2).

Now suppose that $r(B) > 1$. Let $\mathscr{B}$ be the collection of all balls centered in B with radius equal to $1/4$. Proposition 3.1.8 gives a sequence $(B_j)_j$ of disjoint balls from $\mathscr{B}$ such that $B \subseteq \bigcup_j \tilde{B}_j$, where $\tilde{B}_j = 4B_j$. We also have the following bounded intersection property:

$$\sup_j \sharp\big(\{k \mid \tilde{B}_j \cap \tilde{B}_k \neq \emptyset\}\big) < \infty.$$

This follows from the local property of homogeneity, and in particular Remark 3.1.6, since for each $j \in \mathbb{N}$, the centres of all balls $\tilde{B}_k$ satisfying $\tilde{B}_j \cap \tilde{B}_k \neq \emptyset$ are separated by at least a distance of $1/4$ and contained in $2\tilde{B}_j$. Therefore, the following are well defined for each $j \in \mathbb{N}$:

$$\tilde{a}_j = \frac{a\mathbf{1}_{C^1(\tilde{B}_j)}}{\sum_k \mathbf{1}_{C^1(\tilde{B}_k)}}; \quad a_j = \frac{\tilde{a}_j}{\mu(\tilde{B}_j)^{\frac{1}{2}}\|\tilde{a}_j\|_{L^2_\bullet}}; \quad \lambda_j = \mu(\tilde{B}_j)^{\frac{1}{2}}\|\tilde{a}_j\|_{L^2_\bullet}.$$

Also, we have $C^1(B) = B \times (0,1] \subseteq \bigcup_j C^1(\tilde{B}_j)$, since the radius $r(\tilde{B}_j) = 1$. We can then write $a = \sum_j \lambda_j a_j$, where each a_j is a t^1-atom by the previous paragraph. Therefore, we have

$$\|a\|_{t^1}^2 \lesssim \Big(\sum_j |\lambda_j|\Big)^2 \leq \Big(\sum_j \mu(\tilde{B}_j)\Big)\Big(\sum_j \|\tilde{a}_j\|_2^2\Big) \lesssim \mu\Big(\bigcup_j B_j\Big)\|a\|_2^2,$$

where we used $(\mathrm{D}_{\mathrm{loc}})$ in the final inequality to obtain $\mu(\tilde{B}_j) \lesssim \mu(B_j)$. Each B_j is contained in $(1+\frac{1}{4r(B)})B$, so by $(\mathrm{E}_{\kappa,\lambda})$ we obtain

$$\|a\|_{t^1}^2 \lesssim \mu((1+\tfrac{1}{4r(B)})B)\mu(B)^{-1} \lesssim 1,$$

which completes the proof. □

The following duality and interpolation results for the local tent spaces follow as in the global case.

Theorem 3.2.12. *Let X be a locally doubling metric measure space. If $p \in [1,\infty)$ and $1/p + 1/p' = 1$, then the mapping*

$$g \mapsto \langle f, g\rangle_{L^2_\bullet} = \iint f(x,t)\overline{g(x,t)}\,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t}$$

for all $f \in t^p$ and $g \in t^{p'}$, is an isomorphism from $t^{p'}$ onto the dual space $(t^p)^*$.

Proof. For $p = 1$ and $p' = \infty$, the proof is closely related to the atomic characterisation in Theorem 3.2.6 and follows the proof of Theorem 1 in [27]. The remaining cases follow the proof of Theorem 2 in [27]. □

Theorem 3.2.13. *Let X be a locally doubling metric measure space. If $\theta \in (0,1)$ and $1 \leq p_0 < p_1 \leq \infty$, then*

$$[t^{p_0}, t^{p_1}]_\theta = t^{p_\theta},$$

where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$ and $[\cdot,\cdot]_\theta$ denotes complex interpolation.

Proof. The interpolation space $[t^{p_0}, t^{p_1}]_\theta$ is well-defined because

$$t^p(X \times (0,1]) \subseteq L^2_{\text{loc}}(X \times (0,1])$$

for all $p \in [1,\infty]$ by (3.2.1). This allows us to construct the Banach space $t^{p_0} + t^{p_1}$, which is the smallest ambient space in which t^{p_0} and t^{p_1} are continuously embedded. The proof then follows that given by Bernal in Theorem 3 and Proposition 1 of [14]. □

We conclude this section by dealing with a technicality involving the space t^∞. In contrast with Proposition 3.2.3, the set $t^\infty \cap t^2$ may not be dense in t^∞ when X is not compact. Therefore, we define $\tilde{t}^\infty$ to be the closure of $t^1 \cap t^\infty$ in t^∞, and note the following corollary.

Corollary 3.2.14. Let X be a locally doubling metric measure space. If $\theta \in (0,1)$ and $1 \leq p < \infty$, then

$$[t^p, \tilde{t}^\infty]_\theta = t^{p_\theta},$$

where $1/p_\theta = (1-\theta)/p$ and $[\cdot,\cdot]_\theta$ denotes complex interpolation. Also, the set $\tilde{t}^\infty \cap t^q$ is dense in $\tilde{t}^\infty$ for all $q \in [1,\infty]$, and t^2 is dense in $t^1 + \tilde{t}^\infty$.

Proof. If $\theta \in (0,1)$, then by a standard property of complex interpolation, as in Theorem 1.9.3(g) of [68], and Theorem 3.2.13, we have

$$[t^1, \tilde{t}^\infty]_\theta = [t^1, t^\infty]_\theta = t^{1/(1-\theta)}.$$

If $p \in (1,\infty)$, then by the standard reiteration theorem for complex interpolation, as in Theorem 1.7 in Chapter IV of [48], we have

$$[t^p, \tilde{t}^\infty]_\theta = [t^1, \tilde{t}^\infty]_{(1-\theta)(1-1/p)+\theta} = t^{p_\theta},$$

where the density properties required to apply the reiteration theorem are guaranteed by Proposition 3.2.3.

Finally, the interpolation in Theorem 3.2.13 implies that $t^1 \cap t^\infty \subseteq t^q$ for all $q \in [1,\infty]$. Therefore, the density of $t^1 \cap t^\infty$ in $\tilde{t}^\infty$ implies that $\tilde{t}^\infty \cap t^q$ is dense in $\tilde{t}^\infty$ for all $q \in [1,\infty]$. The density of $t^1 \cap t^2$ in t^1 from Proposition 3.2.3 then implies that t^2 is dense in $t^1 + \tilde{t}^\infty$. □

3.3 Some New Function Spaces $L^p_{\mathscr{Q}}(X)$

We introduce some new function spaces $L^p_{\mathscr{Q}}(X)$, or simply $L^p_{\mathscr{Q}}$, for all $p \in [1, \infty]$ in the context of a locally doubling metric measure space X. Note that functions on X are assumed to be complex-valued. We begin with the following abstraction of the unit cube structure in $\mathbb{R}^n$.

Definition 3.3.1. Let X be a metric measure space. A *unit cube structure* on X is a countable collection $\mathscr{Q} = (Q_j)_j$ of disjoint measurable sets that cover X, for which there exists $\delta \in (0, 1]$ and a sequence of balls $(B_j)_j$ in X of radius equal to 1 such that

$$\delta B_j \subseteq Q_j \subseteq B_j.$$

The sets in $\mathscr{Q}$ are called *unit cubes*.

A unit cube structure exists on a locally doubling space.

Lemma 3.3.2. If X is a locally doubling metric measure space, then it has a unit cube structure.

Proof. The cubes are constructed in the same way that general dyadic cubes are constructed by Stein in Section I.3.2 of [64]. Let $\mathscr{B}$ be the collection of all balls in X with radius equal to $1/4$. Proposition 3.1.8 gives a sequence $(B_j)_j$ of disjoint balls from $\mathscr{B}$ such that $X = \bigcup_j 4B_j$. The unit cubes Q_j are then defined recursively for each $j \in \mathbb{N}$ by

$$Q_j = 4B_j \cap {}^c\Big(\bigcup_{k<j} Q_k\Big) \cap {}^c\Big(\bigcup_{k>j} B_k\Big).$$

We have $\delta = 1/4$ in this unit cube structure. □

In the proof above we could instead use the dyadic cubes constructed by Christ in [24], which we will introduce in Chapter 4. In any case, this brings us to the definition of $L^1_{\mathscr{Q}}(X)$.

Definition 3.3.3. Let X be a locally doubling metric measure space. Let $\mathscr{Q} = (Q_j)_j$ be a unit cube structure on X. For each $p \in [1, \infty)$, the space $L^p_{\mathscr{Q}}(X)$ consists of all measurable functions f on X with

$$\|f\|_{L^p_{\mathscr{Q}}} = \Big(\sum_{Q_j \in \mathscr{Q}} \big(\mu(Q_j)^{\frac{1}{p}-\frac{1}{2}} \|\mathbf{1}_{Q_j} f\|_2\big)^p \Big)^{\frac{1}{p}} < \infty.$$

The space $L^\infty_{\mathscr{Q}}(X)$ consists of all measurable functions f on X with

$$\|f\|_{L^\infty_{\mathscr{Q}}} = \sup_{Q_j \in \mathscr{Q}} \mu(Q_j)^{-\frac{1}{2}} \|\mathbf{1}_{Q_j} f\|_2 < \infty.$$

These are Banach spaces under the usual identification of functions that are equal almost everywhere. The space $L^2_{\mathscr{Q}}(X)$ is exactly the Hilbert space $L^2(X)$. More generally, completeness holds because for each compact set $K \subseteq X$ and each $p \in [1, \infty]$, we have

$$\|\mathbf{1}_K f\|_{L^p_{\mathscr{Q}}} \lesssim_{K,p} \|\mathbf{1}_K f\|_2 \lesssim_{K,p} \|f\|_{L^p_{\mathscr{Q}}} \tag{3.3.1}$$

for all measurable functions f on X.

We will see that the $L^p_{\mathscr{Q}}$ spaces are independent of the unit cube structure $\mathscr{Q}$ used in their definition. First, however, we consider their relationship with the L^p spaces.

Proposition 3.3.4. Let X be a locally doubling metric measure space. The following hold:

1. $L^p_{\mathscr{Q}} \cap L^q_{\mathscr{Q}}$ is dense in $L^p_{\mathscr{Q}}$ for all $p \in [1,\infty)$ and $q \in [1,\infty]$;
2. $L^p_{\mathscr{Q}} \subseteq L^p$ for all $p \in [1,2]$;
3. $L^p \subseteq L^p_{\mathscr{Q}}$ for all $p \in [2,\infty]$.

Proof. Let $p \in [1,\infty)$ and $f \in L^p_{\mathscr{Q}}$. Fix a ball B in X of radius $r(B) \geq 1$ and define $f_k = \mathbf{1}_{kB} f$ for each $k \in \mathbb{N}$. The functions f_k belong to $L^p_{\mathscr{Q}} \cap L^q_{\mathscr{Q}}$ for all $q \in [1,\infty]$ by (3.3.1), and

$$\lim_{k\to\infty} \|f - f_k\|^p_{L^p_{\mathscr{Q}}} = \lim_{k\to\infty} \sum_{Q_j \cap (k-2)B = \emptyset} \left(\mu(Q_j)^{\frac{1}{p}-\frac{1}{2}} \|\mathbf{1}_{Q_j} f\|_2\right)^p = 0$$

because $f \in L^p_{\mathscr{Q}}$, which proves (1).

We use Hölder's inequality to prove (2) and (3). If $p \in [1,2]$, then

$$\|f\|_p^p = \sum_{Q_j \in \mathscr{Q}} \|\mathbf{1}_{Q_j} f^p\|_1 \leq \sum_{Q_j \in \mathscr{Q}} \left(\mu(Q_j)^{\frac{1}{p}-\frac{1}{2}} \|\mathbf{1}_{Q_j} f\|_2\right)^p = \|f\|^p_{L^p_{\mathscr{Q}}}$$

for all $f \in L^p_{\mathscr{Q}}$, which proves (2). If $p \in [2,\infty)$, then

$$\|f\|^p_{L^p_{\mathscr{Q}}} = \sum_{Q_j \in \mathscr{Q}} \left(\mu(Q_j)^{\frac{1}{p}-\frac{1}{2}} \|\mathbf{1}_{Q_j} f^2\|_1^{\frac{1}{2}}\right)^p \leq \sum_{Q_j \in \mathscr{Q}} \|\mathbf{1}_{Q_j} f\|_p^p = \|f\|_p$$

for all $f \in L^p$, whilst

$$\|f\|_{L^\infty_{\mathscr{Q}}} = \sup_{Q_j \in \mathscr{Q}} \mu(Q_j)^{-\frac{1}{2}} \|\mathbf{1}_{Q_j} f^2\|_1^{\frac{1}{2}} \leq \sup_{Q_j \in \mathscr{Q}} \|\mathbf{1}_{Q_j} f^2\|_\infty^{\frac{1}{2}} = \|f\|_\infty$$

for all $f \in L^\infty$, which proves (3). □

Now we turn to the atomic characterisation of $L^1_{\mathscr{Q}}$.

Definition 3.3.5. Let X be a locally doubling metric measure space. An $L^1_{\mathscr{Q}}$-*atom* is a measurable function a on X supported on a ball B in X of radius $r(B) \geq 1$ with $\|a\|_2 \leq \mu(B)^{-1/2}$.

If a is an $L^1_{\mathscr{Q}}$-atom, then a belongs to $L^1_{\mathscr{Q}} \cap L^2$ with $\|a\|_1 \lesssim 1$. If X is exponentially locally doubling, then it is shown in the following theorem that $\|a\|_{L^1_{\mathscr{Q}}} \lesssim 1$. This allows us to prove that $L^1_{\mathscr{Q}}$ is precisely the subspace of L^1 in which functions have an atomic characterisation consisting purely of atoms supported on balls with large radii. The effectiveness of $(\mathrm{E}_{\kappa,\lambda})$ in the proof of the first part of the following theorem can be understood in terms of its equivalence with the condition $(\mathrm{D}_{\mathrm{glo}})$ from Proposition 3.1.11.

Theorem 3.3.6. *Let X be an exponentially locally doubling metric measure space. The following hold:*

1. *If $(\lambda_j)_j$ is a sequence in ℓ^1 and $(a_j)_j$ is a sequence of $L^1_{\mathscr{Q}}$-atoms, then $\sum_j \lambda_j a_j$ converges in $L^1_{\mathscr{Q}}$ with $\| \sum_j \lambda_j a_j \|_{L^1_{\mathscr{Q}}} \lesssim \|(\lambda_j)_j\|_{\ell^1}$;*
2. *If $f \in L^1_{\mathscr{Q}}$, then there exist a sequence $(\lambda_j)_j$ in ℓ^1 and a sequence $(a_j)_j$ of $L^1_{\mathscr{Q}}$-atoms such that $\sum_j \lambda_j a_j$ converges to f in $L^1_{\mathscr{Q}}$ and almost everywhere in X. Moreover, we have*

$$\|f\|_{L^1_{\mathscr{Q}}} \approx \inf\{\|(\lambda_j)_j\|_{\ell^1} : f = \textstyle\sum_j \lambda_j a_j\}.$$

Also, if $p \in (1, \infty)$ and $f \in L^1_{\mathscr{Q}} \cap L^p_{\mathscr{Q}}$, then $\sum_j \lambda_j a_j$ converges to f in $L^p_{\mathscr{Q}}$ as well.

Proof. To prove (1), it is enough to show that $\sup\{\|a\|_{L^1_{\mathscr{Q}}} : a \text{ is an } L^1_{\mathscr{Q}}\text{-atom}\} \lesssim 1$. Let a be an $L^1_{\mathscr{Q}}$-atom supported on a ball B of radius $r(B) \geq 1$. Let

$$\mathscr{Q}_B = \{Q_j \in \mathscr{Q} : Q_j \cap B \neq \emptyset\}.$$

For each $Q_j \in \mathscr{Q}_B$, there exists a ball B_j in X of radius equal to 1 such that

$$\delta B_j \subseteq Q_j \subseteq B_j,$$

where δ is the constant associated with $\mathscr{Q}$ in Definition 3.3.1. The Cauchy–Schwarz inequality and the properties of the unit cube structure imply that

$$\|a\|^2_{L^1_{\mathscr{Q}}} \leq \|a\|_2^2 \sum_{Q_j \in \mathscr{Q}_B} \mu(Q_j) = \|a\|_2^2 \, \mu\Big(\bigcup_{Q_j \in \mathscr{Q}_B} Q_j \Big) \leq \mu(B)^{-1} \mu((1 + \tfrac{2}{r(B)})B).$$

The lower bound on $r(B)$ and $(\mathrm{E}_{\kappa,\lambda})$ then imply that $\|a\|_{L^1_{\mathscr{Q}}} \lesssim 1$, where the constant in $\lesssim$ does not depend on a.

To prove (2), let $f \in L^1_{\mathscr{Q}}$. We can write $f(x) = \sum_{Q_j \in \mathscr{Q}} \lambda_j a_j(x)$ for almost every $x \in X$, where

$$a_j(x) = \frac{\mathbf{1}_{Q_j} f(x)}{\mu(Q_j)^{\frac{1}{2}} \|\mathbf{1}_{Q_j} f\|_2} \quad \text{and} \quad \lambda_j = \mu(Q_j)^{\frac{1}{2}} \|\mathbf{1}_{Q_j} f\|_2.$$

Given that $f \in L^1_{\mathscr{Q}}$, this series also converges to f in $L^1_{\mathscr{Q}}$. The same reasoning shows that if $f \in L^1_{\mathscr{Q}} \cap L^p_{\mathscr{Q}}$ for some $p \in (1, \infty)$, then the series also converges to f in $L^p_{\mathscr{Q}}$. Also, each a_j is supported in $Q_j \subseteq B_j$, so by $(\mathrm{D}_{\mathrm{loc}})$ we obtain

$$\|a_j\|_2 \leq \mu(Q_j)^{-\frac{1}{2}} \leq \mu(\delta B_j)^{-\frac{1}{2}} \lesssim \mu(B_j)^{-\frac{1}{2}}.$$

Therefore, each a_j is a constant multiple of an $L^1_{\mathscr{Q}}$-atom, and this constant does not depend on f or Q_j. The result then follows since $\|(\lambda_j)_j\|_{\ell^1} = \|f\|_{L^1_{\mathscr{Q}}}$. □

Remark 3.3.7. The proof of the second part of Theorem 3.3.6 actually shows that a function in $L^1_{\mathscr{Q}}$ has a characterisation in terms of $L^1_{\mathscr{Q}}$-atoms supported on balls of radius *equal* to 1.

The definition of $L^1_{\mathscr{Q}}$-atoms does not require a unit cube structure. Therefore, the atomic characterisation of $L^1_{\mathscr{Q}}$ shows that, up to an equivalence of norms, $L^1_{\mathscr{Q}}$ is independent of the unit cube structure $\mathscr{Q}$ used in its definition. The atomic characterisation of $L^1_{\mathscr{Q}}$ is also related to the following duality.

Theorem 3.3.8. *Let X be an exponentially locally doubling metric measure space. If $p \in [1,\infty)$ and $1/p + 1/p' = 1$, then the mapping*

$$g \mapsto \langle f, g\rangle_{L^2} = \int f(x)\overline{g(x)}\,\mathrm{d}\mu(x)$$

for all $f \in L^p_{\mathscr{Q}}$ and $g \in L^{p'}_{\mathscr{Q}}$, is an isometric isomorphism from $L^{p'}_{\mathscr{Q}}$ onto the dual space $(L^p_{\mathscr{Q}})^*$.

Proof. Let $p \in [1,\infty)$. If $f \in L^p_{\mathscr{Q}}$ and $g \in L^{p'}_{\mathscr{Q}}$, then Hölder's inequality gives

$$\begin{aligned}|\langle f, g\rangle_{L^2}| &\le \sum_{Q_j\in\mathscr{Q}} |\langle \mathbf{1}_{Q_j} f, \mathbf{1}_{Q_j} g\rangle_{L^2}| \\ &\le \sum_{Q_j\in\mathscr{Q}} \|\mathbf{1}_{Q_j} f\|_2 \|\mathbf{1}_{Q_j} g\|_2\, \mu(Q_j)^{\frac{1}{p}-\frac{1}{2}} \mu(Q_j)^{\frac{1}{2}-\frac{1}{p}} \\ &\le \|f\|_{L^p_{\mathscr{Q}}} \|g\|_{L^{p'}_{\mathscr{Q}}}.\end{aligned}$$

To prove the converse, given p and $q \in [1,\infty)$, let $w_q(Q_j) = \mu(Q_j)^{1-q/2}$ for all $Q_j \in \mathscr{Q}$, and define $\ell^p(w_q)$ to be the space of all sequences $\xi = (\xi_{Q_j})_{Q_j\in\mathscr{Q}}$ with $\xi_{Q_j} \in L^2(Q_j)$ and

$$\|\xi\|_{\ell^p(w_q)} = \Big(\sum_{Q_j\in\mathscr{Q}} \|\mathbf{1}_{Q_j}\xi_{Q_j}\|_2^p\, w_q(Q_j)\Big)^{\frac{1}{p}} < \infty.$$

Let $T \in (L^p_{\mathscr{Q}})^*$ and define $\tilde{T} \in (\ell^p(w_p))^*$ by

$$\tilde{T}(\xi) = T\Big(\sum_{Q_j\in\mathscr{Q}} \mathbf{1}_{Q_j}\xi_{Q_j}\Big)$$

for all $\xi \in \ell^p(w_p)$. It is immediate that $\|\tilde{T}\| \le \|T\|$, and by the standard duality there exists $\eta \in \ell^{p'}(w_p)$ such that $\|\eta\|_{\ell^{p'}(w_p)} \le \|\tilde{T}\|$ and

$$\tilde{T}(\xi) = \sum_{Q_j\in\mathscr{Q}} \langle \mathbf{1}_{Q_j}\xi_{Q_j}, \mathbf{1}_{Q_j}\eta_{Q_j}\rangle_{L^2}\, w_p(Q_j)$$

for all $\xi \in \ell^p(w_p)$. Therefore, we have

$$T(f) = \tilde{T}((\mathbf{1}_{Q_j} f)_{Q_j\in\mathscr{Q}}) = \sum_{Q_j\in\mathscr{Q}} \langle f, \mathbf{1}_{Q_j}\eta_{Q_j}\rangle_{L^2}\, w_p(Q_j) = \langle f, g\rangle_{L^2}$$

for all $f \in L^p_{\mathscr{Q}}$, where $g = \sum_{Q_j\in\mathscr{Q}} \mathbf{1}_{Q_j}\eta_{Q_j} w_p(Q_j)$. Now consider two cases: (1) If $p \in (1,\infty)$, then

$$\|g\|_{L^{p'}_{\mathscr{Q}}} = \Big(\sum_{Q_j\in\mathscr{Q}} \mu(Q_j)^{1-\frac{p'}{2}} \|\eta_{Q_j}\mu(Q_j)^{1-\frac{p}{2}}\|_2^{p'}\Big)^{\frac{1}{p'}} = \|\eta\|_{\ell^{p'}(w_p)} \le \|T\|;$$

(2) If $p = 1$, then

$$\begin{aligned}
\|g\|_{L^\infty_{\mathscr{Q}}} &= \sup_{Q_j\in\mathscr{Q}} \mu(Q_j)^{-\frac{1}{2}}\|\mathbf{1}_{Q_j}g\|_2 \\
&= \sup_{Q_j\in\mathscr{Q}} \mu(Q_j)^{-\frac{1}{2}} \sup_{\substack{\|f\|_2=1,\\ \operatorname{sppt} f\subseteq Q_j}} |\langle f,g\rangle_{L^2}| \\
&= \sup_{Q_j\in\mathscr{Q}} \sup_{\substack{\|f\|_2=1,\\ \operatorname{sppt} f\subseteq Q_j}} \mu(Q_j)^{-\frac{1}{2}}|T(f)| \\
&\leq \sup_{Q_j\in\mathscr{Q}} \sup_{\substack{\|f\|_2=1,\\ \operatorname{sppt} f\subseteq Q_j}} \mu(Q_j)^{-\frac{1}{2}}\|T\|\|f\|_{L^1_{\mathscr{Q}}} \\
&= \|T\|,
\end{aligned}$$

which completes the proof. □

The duality between $L^1_{\mathscr{Q}}$ and $L^\infty_{\mathscr{Q}}$ shows that, up to an equivalence of norms, $L^\infty_{\mathscr{Q}}$ is independent of the unit cube structure $\mathscr{Q}$ used in its definition. This is made explicit by the following corollary.

Corollary 3.3.9. Let X be an exponentially locally doubling metric measure space. Let $\mathscr{B}^1$ denote the set of all balls B in X of radius $r(B) \geq 1$. Then

$$\|f\|_{L^\infty_{\mathscr{Q}}} \eqsim \sup_{B\in\mathscr{B}^1} \mu(B)^{-\frac{1}{2}}\|\mathbf{1}_B f\|_2$$

for all $f \in L^\infty_{\mathscr{Q}}$.

Proof. Let $f \in L^\infty_{\mathscr{Q}}$. Given $Q \in \mathscr{Q}$, let B be a ball in X of radius $r(B) = 1$ such that $\delta B \subseteq Q \subseteq B$, where δ is the constant associated with $\mathscr{Q}$ in Definition 3.3.1. It follows by $(\mathrm{D}_{\mathrm{loc}})$ that $\mu(B) \lesssim \mu(\delta B)$, where the constant in $\lesssim$ does not depend on Q. Therefore, we have

$$\mu(Q)^{-\frac{1}{2}}\|\mathbf{1}_Q f\|_2 \lesssim \mu(B)^{-\frac{1}{2}}\|\mathbf{1}_B f\|_2$$

for all $Q \in \mathscr{Q}$, which implies that

$$\|f\|_{L^\infty_{\mathscr{Q}}} \lesssim \sup_{B\in\mathscr{B}^1} \mu(B)^{-\frac{1}{2}}\|\mathbf{1}_B f\|_2.$$

To show the converse, suppose that $g \in L^2$ is supported in a ball $B \in \mathscr{B}^1$ with radius $r(B) \geq 1$. As in the first part of the proof of Proposition 3.3.6, we find that

$$\|g\|^2_{L^1_{\mathscr{Q}}} \leq \|g\|_2^2\,\mu((1+\tfrac{2}{r(B)})B) \lesssim \|g\|_2^2\,\mu(B),$$

where the second inequality, which follows from $(\mathrm{E}_{\kappa,\lambda})$ since $r(B) \geq 1$, does not depend on g or B. Using this and Theorem 3.3.8, we obtain

$$\begin{aligned}
\sup_{B\in\mathscr{B}^1} \mu(B)^{-\frac{1}{2}}\|\mathbf{1}_B f\|_2 &= \sup_{B\in\mathscr{B}^1} \mu(B)^{-\frac{1}{2}} \sup_{\substack{\|g\|_2=1,\\ \operatorname{sppt} f\subseteq B}} |\langle g,f\rangle_{L^2}| \\
&\leq \sup_{B\in\mathscr{B}^1} \sup_{\substack{\|g\|_2=1,\\ \operatorname{sppt} f\subseteq B}} \mu(B)^{-\frac{1}{2}}\|g\|_{L^1_{\mathscr{Q}}}\|f\|_{L^\infty_{\mathscr{Q}}} \\
&\lesssim \|f\|_{L^\infty_{\mathscr{Q}}},
\end{aligned}$$

which completes the proof. □

Given that $L^1_{\mathscr{Q}}$ and $L^\infty_{\mathscr{Q}}$ are independent of the choice of $\mathscr{Q}$, the following interpolation result shows that, up to an equivalence of norms, the $L^p_{\mathscr{Q}}$ spaces for all $p \in (1,\infty)$ are independent of the unit cube structure $\mathscr{Q}$ used in their definition.

Theorem 3.3.10. *Let X be an exponentially locally doubling metric measure space. If $\theta \in (0,1)$ and $1 \le p_0 < p_1 \le \infty$, then*

$$[L^{p_0}_{\mathscr{Q}}, L^{p_1}_{\mathscr{Q}}]_\theta = L^{p_\theta}_{\mathscr{Q}}$$

isometrically, where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$ and $[\cdot,\cdot]_\theta$ denotes complex interpolation.

Proof. The interpolation space $[L^{p_0}_{\mathscr{Q}}, L^{p_1}_{\mathscr{Q}}]_\theta$ is well-defined because

$$L^p_{\mathscr{Q}}(X) \subseteq L^2_{\mathrm{loc}}(X)$$

for all $p \in [1,\infty]$ by (3.3.1). This allows us to construct the Banach space $L^{p_0}_{\mathscr{Q}} + L^{p_1}_{\mathscr{Q}}$, which is the smallest ambient space in which $L^{p_0}_{\mathscr{Q}}$ and $L^{p_1}_{\mathscr{Q}}$ are continuously embedded.

The space $\ell^p(w_p)$ was defined for all $p \in [1,\infty)$ in the proof of Theorem 3.3.8. Likewise, let

$$w_\infty(Q_j) = \mu(Q_j)^{-\frac{1}{2}}$$

for all $Q_j \in \mathscr{Q}$, and define $\ell^\infty(w_\infty)$ to be the space of all sequences $\xi = (\xi_{Q_j})_{Q_j\in\mathscr{Q}}$ with $\xi_{Q_j} \in L^2(Q_j)$ and

$$\|\xi\|_{\ell^\infty(w_\infty)} = \sup_{Q_j\in\mathscr{Q}} \|\mathbf{1}_{Q_j}\xi_{Q_j}\|_2\, w_\infty(Q_j) < \infty.$$

If $1 \le p_0 < p_1 < \infty$, then $w_{p_0}^{(1-\theta)/p_0} w_{p_1}^{\theta/p_1} = w_{p_\theta}^{1/p_\theta}$, whilst if $p_1 = \infty$, then $w_{p_0}^{(1-\theta)/p_0} w_\infty^\theta = w_{p_\theta}^{1/p_\theta}$. Therefore, by the interpolation of vector-valued ℓ^p spaces, as in Theorem 1.18.1 of [68], and the interpolation of weighted L^2 spaces, as in Theorem 5.5.3 of [13], we obtain

$$[\ell^{p_0}(w_{p_0}), \ell^{p_1}(w_{p_1})]_\theta = \ell^{p_\theta}(w_{p_\theta})$$

isometrically. Note that the isometric equivalence is proved by Triebel in Remark 1 of Section 1.18.1 of [68], and the proof for $p_1 = \infty$ is given in Remark 2 of the same reference.

Define the operators R and S by

$$R\xi = \sum_{Q_j\in\mathscr{Q}} \mathbf{1}_{Q_j}\xi_{Q_j} \quad \text{and} \quad Sf = (\mathbf{1}_{Q_j} f)_{Q_j\in\mathscr{Q}}$$

for all sequences $\xi = (\xi_{Q_j})_{Q_j\in\mathscr{Q}}$ with $\xi_{Q_j} \in L^2(Q_j)$, and all measurable functions f on X. If $p \in [1,\infty]$, then the restricted operators

$$R : \ell^p(w_p) \to L^p_{\mathscr{Q}} \quad \text{and} \quad S : L^p_{\mathscr{Q}} \to \ell^p(w_p)$$

are bounded with operator norms equal to 1. Moreover, we have $RS = I$ on $L^p_{\mathscr{Q}}$ and $R(\ell^p(w_p)) = L^p_{\mathscr{Q}}$. The operator R is a retraction and S is its coretraction. It follows by Theorem 1.2.4 of [68], which concerns the interpolation of spaces related by a retraction, that S is an isometric isomorphism from $[L^{p_0}_{\mathscr{Q}}, L^{p_1}_{\mathscr{Q}}]_\theta$ onto

$$SR([\ell^{p_0}(w_{p_0}), \ell^{p_1}(w_{p_1})]_\theta) = SR(\ell^{p_\theta}(w_{p_\theta})) = S(L^{p_\theta}_{\mathscr{Q}})$$

in $\ell^{p_\theta}(w_{p_\theta})$. Therefore, we have $[L^{p_0}_{\mathscr{Q}}, L^{p_1}_{\mathscr{Q}}]_\theta = L^{p_\theta}_{\mathscr{Q}}$ isometrically. □

We conclude this section by defining $\tilde{L}^\infty_{\mathscr{Q}}$ to be the closure of $L^1_{\mathscr{Q}} \cap L^\infty_{\mathscr{Q}}$ in $L^\infty_{\mathscr{Q}}$, and noting the following corollary.

Corollary 3.3.11. Let X be an exponentially locally doubling metric measure space. If $\theta \in (0,1)$ and $1 \leq p < \infty$, then

$$[L^p_{\mathscr{Q}}, \tilde{L}^\infty_{\mathscr{Q}}]_\theta = L^{p_\theta}_{\mathscr{Q}}$$

isometrically, where $1/p_\theta = (1-\theta)/p$ and $[\cdot,\cdot]_\theta$ denotes complex interpolation. Also, the set $\tilde{L}^\infty_{\mathscr{Q}} \cap L^q_{\mathscr{Q}}$ is dense in $\tilde{L}^\infty_{\mathscr{Q}}$ for all $q \in [1,\infty]$, and L^2 is dense in $L^1_{\mathscr{Q}} + \tilde{L}^\infty_{\mathscr{Q}}$.

Proof. The proof follows that of Corollary 3.2.14 by using Proposition 3.3.4(1) and Theorem 3.3.10. □

3.4 Exponential Off-Diagonal Estimates

We return to the setting of a complete Riemannian manifold M and derive the off-diagonal estimates required to define and characterise our local Hardy spaces. To consider differential forms on M, let us first dispense with some technicalities.

For each $k = 0, ..., \dim M$ and $x \in M$, let $\wedge^k T^*_x M$ denote the kth exterior power of the cotangent space $T^*_x M$. Let $\wedge^k T^* M$ denote the bundle over M whose fibre at x is $\wedge^k T^*_x M$, and let $\wedge T^* M = \oplus_{k=0}^{\dim M} \wedge^k T^* M$. A *differential form* is a section of $\wedge T^* M$. For each $p \in [1,\infty]$, let $L^p(\wedge T^* M)$ denote the Banach space of all measurable differential forms u with

$$\|u\|_{L^p(\wedge T^*M)} = \begin{cases} \big(\int_M |u(x)|^p_{\wedge T^*_x M} \, \mathrm{d}\mu(x) \big)^{\frac{1}{p}}, & \text{if} \quad p \in [1,\infty); \\ \operatorname{ess\,sup}_{x \in M} |u(x)|_{\wedge T^*_x M}, & \text{if} \quad p = \infty, \end{cases}$$

where $|\cdot|_{\wedge T^*_x M}$ is the norm associated with the inner-product $\langle \cdot,\cdot \rangle_{\wedge T^*_x M}$ given by the bundle metric on $\wedge T^* M$ at x.

We apply the functional calculi from Section 2.2 in the case $\mathcal{X} = L^2(\wedge T^* M)$. To do this, recall the notation for operators from Section 2.1. There is also the following additional notation.

Notation. Given a bounded measurable scalar-valued function η on M, let ηI denote the operator on $L^2(\wedge T^* M)$ of pointwise multiplication by η. Square brackets $[\cdot,\cdot]$ denote the commutator operator.

The following are hypotheses that an operator $\mathcal{D}$ on $L^2(\wedge T^* M)$ may satisfy:

(A1) There exists $\omega \in [0,\pi/2)$ and $R \geq 0$ such that the operator $\mathcal{D}$ is of *type* $S_{\omega \cup R}$. In particular, for each $\theta \in (\omega,\pi/2)$ and $r > R$, the constant

$$C_{\theta \cup r} := \sup\{|z| \|R_{\mathcal{D}}(z)\| : z \in \mathbb{C} \setminus S_{\theta \cup r}\}$$

is finite.

(A2) The operator $\mathcal{D}$ satisfies (A1) and has a *bounded* $H^\infty(S^o_{\theta \cup r})$ *functional calculus* in $L^2(\wedge T^* M)$ for all $\theta \in (\omega,\pi/2)$ and $r > R$.

(A3) The operator $\mathcal{D}$ is a first-order differential operator in the following sense. There exists $C_{\mathcal{D}} > 0$ such that for all smooth compactly-supported scalar-valued functions $\eta \in C_0^\infty(M)$, the domain $\mathsf{D}(\mathcal{D}) \subseteq \mathsf{D}(\mathcal{D} \circ \eta I)$ and the commutator $[\mathcal{D}, \eta I]$ is a pointwise multiplication operator such that

$$|[\mathcal{D}, \eta I]u(x)|_{\wedge T_x^* M} \leq C_{\mathcal{D}} |d\eta(x)|_{T_x^* M} |u(x)|_{\wedge T_x^* M}$$

for all $u \in \mathsf{D}(\mathcal{D})$ and almost all $x \in M$, where d is the exterior derivative.

Given an operator $\mathcal{D}$ satisfying (A1), Theorem 2.2.18 shows that (A2) is equivalent to the requirement that $\mathcal{D}$ and its adjoint $\mathcal{D}^*$ satisfy local quadratic estimates.

Note that, as a means of generalizing this theory to other contexts, one could replace the space $C_0^\infty(M)$ in (A3) with the space of bounded scalar-valued Lipschitz functions $\mathrm{Lip}(M)$. This stronger condition is still satisfied by the Hodge–Dirac operator, as in Example 3.4.1 below, and it obviates the need to construct smooth approximations in the proof of Lemma 3.4.2. Moreover, all of the results in this chapter hold under that condition.

Example 3.4.1. The Hodge–Dirac operator $D = d + d^*$ is self-adjoint so it immediately satisfies (A1)–(A2) with $\omega = 0$, $R = 0$ and $C_{\theta \cup r} \leq 1/\sin\theta$ for all $\theta \in (0, \pi/2)$ and $r > 0$. It also satisfies (A3), since it is a first-order differential operator, and $C_D = 1$, since for all $\eta \in C_0^\infty(M)$ we have

$$|[D, \eta I]u(x)|_{\wedge T_x^* M} = |d\eta(x) \wedge u(x) - d\eta(x) \lrcorner u(x)|_{\wedge T_x^* M} = |d\eta(x)|_{T_x^* M} |u(x)|_{\wedge T_x^* M}$$

for all $u \in L^2(\wedge T^* M)$ and almost all $x \in M$, where $\wedge$ and $\lrcorner$ denote the exterior and (left) interior products on $\wedge T_x^* M$, respectively. Note that the second equality above holds because $d\eta(x) \lrcorner$ is an antiderivation on $\wedge T_x^* M$, which implies that

$$d\eta \lrcorner (d\eta \wedge u) = |d\eta|^2_{T^* M} u - d\eta \wedge (d\eta \lrcorner u)$$

pointwise almost everywhere on M.

Off-diagonal estimates, otherwise known as Davies–Gaffney estimates, provide a measure of the decay associated with the action of an operator. Their use as a substitute for pointwise kernel bounds is becoming abundant in the literature. In particular, they are an essential tool used to prove the Kato Conjecture in [6] and the related results in [11], as we will see in Chapter 4. The theory of off-diagonal estimates has also been developed in its own right by Auscher and Martell in [7]. The following notation is suited to these estimates.

Notation. For all $x \geq 0$, let $\langle x \rangle = \min\{1, x\}$. For all closed subsets $E, F \subseteq M$, let $\rho(E, F) = \inf_{x \in E, y \in F} \rho(x, y)$.

We prove off-diagonal estimates for the resolvents $R_{\mathcal{D}}(z)$ and then deduce estimates for more general functions of $\mathcal{D}$ by using holomorphic functional calculus. The following proof utilizes the higher-commutator technique of McIntosh and Nahmod from Section 2 of [55]. Note that we could instead apply the technique for establishing off-diagonal estimates used by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian in [6] and by Auscher, Axelsson and McIntosh in [5].

Lemma 3.4.2. Let $0 \leq \omega < \theta < \pi/2$ and $0 \leq R < r$ and suppose that $\mathcal{D}$ is an operator on $L^2(\wedge T^*M)$ of type $S_{\omega\cup R}$ satisfying (A1) and (A3) with constants $C_{\theta\cup r} > 0$ and $C_{\mathcal{D}} > 0$. For each $a \in (0,1)$ and $b \geq 0$, there exists $c > 0$ such that

$$\|\mathbf{1}_E R_{\mathcal{D}}(z)\mathbf{1}_F\| \leq c\,\frac{C_{\theta\cup r}}{|z|}\left\langle \frac{1}{\rho(E,F)|z|}\right\rangle^b \exp\left(-a\frac{\rho(E,F)|z|}{C_{\mathcal{D}}C_{\theta\cup r}}\right)$$

for all $z \in \mathbb{C}\setminus S_{\theta\cup r}$ and closed subsets E and F of M.

Proof. Let E and F be closed subsets of M with $\rho(E,F) > 0$. For each $\epsilon > 0$, there exists $\eta : M \to [0,1]$ in $C_0^\infty(M)$ such that

$$\eta(x) = \begin{cases} 1, & \text{if} \quad x \in E; \\ 0, & \text{if} \quad \rho(x,E) \geq \rho(E,F) \end{cases}$$

and $\|d\eta\|_\infty = \sup_{x\in M} |d\eta(x)|_{T_x^*M} \leq (1+\epsilon)/\rho(E,F)$. The function η can be constructed from smooth approximations of the Lipschitz function f defined by

$$f(x) = \begin{cases} 1 - \rho(x,E)/\rho(E,F), & \text{if} \quad \rho(x,E) < \rho(E,F); \\ 0, & \text{if} \quad \rho(x,E) \geq \rho(E,F) \end{cases}$$

for all $x \in M$. Note that f is Lipschitz because the geodesic distance ρ is Lipschitz on a Riemannian manifold. For further details see, for instance, [12].

Fix $a \in (0,1)$ and $\delta \in (a,1)$. It suffices to show that

$$\|\mathbf{1}_E R_{\mathcal{D}}(T)\mathbf{1}_F\| \leq \inf_{n\in\mathbb{N}_0} n!\frac{C_{\theta\cup r}}{|z|}\left(\frac{(1+\epsilon)C_{\mathcal{D}}C_{\theta\cup r}}{\rho(E,F)|z|}\right)^n, \tag{3.4.1}$$

where $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. For $b = 0$, the result follows from (3.4.1) by choosing $\epsilon > 0$ such that $\delta/(1+\epsilon) \geq a$, since $e^{\delta x} = \sum_{n\in\mathbb{N}_0}(\delta x)^n/n! \leq \frac{1}{1-\delta}\sup_{n\in\mathbb{N}_0} x^n/n!$ for all $x > 0$. For each $b > 0$, the result follows from (3.4.1) by choosing $\epsilon > 0$ even smaller, since $e^{-\delta x} \lesssim x^{-b}e^{-(\delta-\epsilon)x}$ for all $x > 0$.

We make repeated use, without reference, of the following easily verified identities for operators A, B and C:

$$[A, BC] = [A,B]C + B[A,C]; \quad [A, B^{-1}] = B^{-1}[B,A]B^{-1}$$

on the largest domains for which both sides are defined.

First, we show by induction that

$$[\eta I, ([\mathcal{D},\eta I]R_{\mathcal{D}}(z))^n] = -n([\mathcal{D},\eta I]R_{\mathcal{D}}(z))^{n+1} \tag{3.4.2}$$

for all $n \in \mathbb{N}$. The commutator $[\mathcal{D},\eta I]$ is a pointwise multiplication operator by hypothesis (A3). This implies that $[\eta I, [\mathcal{D},\eta I]] = 0$, so (3.4.2) holds for $n = 1$. If (3.4.2) holds for some $k \in \mathbb{N}$, then

$$\begin{aligned}
&[\eta I, ([\mathcal{D},\eta I]R_{\mathcal{D}}(z))^{k+1}] \\
&\quad = [\eta I, [\mathcal{D},\eta I]R_{\mathcal{D}}(z)]([\mathcal{D},\eta I]R_{\mathcal{D}}(z))^k + [\mathcal{D},\eta I]R_{\mathcal{D}}(z)[\eta I, ([\mathcal{D},\eta I]R_{\mathcal{D}}(z))^k] \\
&\quad = [\mathcal{D},\eta I][\eta I, R_{\mathcal{D}}(z)]([\mathcal{D},\eta I]R_{\mathcal{D}}(z))^k - k([\mathcal{D},\eta I]R_{\mathcal{D}}(z))^{k+2} \\
&\quad = -[\mathcal{D},\eta I]R_{\mathcal{D}}(z)[\mathcal{D},\eta I]R_{\mathcal{D}}(z)([\mathcal{D},\eta I]R_{\mathcal{D}}(z))^k - k([\mathcal{D},\eta I]R_{\mathcal{D}}(z))^{k+2} \\
&\quad = -(k+1)([\mathcal{D},\eta I]R_{\mathcal{D}}(z))^{k+2},
\end{aligned}$$

so (3.4.2) holds for all $n \in \mathbb{N}$. Next, we show by induction that

$$\overbrace{[\eta I, ...[\eta I,}^{n} R_{\mathcal{D}}(z)]...] = (-1)^n n! R_{\mathcal{D}}(z)([\mathcal{D}, \eta I] R_{\mathcal{D}}(z))^n \tag{3.4.3}$$

for all $n \in \mathbb{N}$. This is immediate for $n = 1$. If (3.4.3) holds for some $k \in \mathbb{N}$, then by (3.4.2) we have

$$\begin{aligned}
&\overbrace{[\eta I, ...[\eta I,}^{k+1} R_{\mathcal{D}}(z)]...] \\
&\qquad = (-1)^k k! [\eta I, R_{\mathcal{D}}(z)([\mathcal{D}, \eta I] R_{\mathcal{D}}(z))^k] \\
&\qquad = (-1)^k k! \{[\eta I, R_{\mathcal{D}}(z)]([\mathcal{D}, \eta I] R_{\mathcal{D}}(z))^k + R_{\mathcal{D}}(z)[\eta I, ([\mathcal{D}, \eta I] R_{\mathcal{D}}(z))^k]\} \\
&\qquad = (-1)^k k! \{-R_{\mathcal{D}}(z)([\mathcal{D}, \eta I] R_{\mathcal{D}}(z))^{k+1} - k R_{\mathcal{D}}(z)([\mathcal{D}, \eta I] R_{\mathcal{D}}(z))^{k+1}\} \\
&\qquad = (-1)^{k+1}(k+1)! R_{\mathcal{D}}(z)([\mathcal{D}, \eta I] R_{\mathcal{D}}(z))^{k+1},
\end{aligned}$$

so (3.4.3) holds for all $n \in \mathbb{N}$. Using (3.4.3) with hypotheses (A1) and (A3), we obtain

$$\begin{aligned}
\|\mathbf{1}_E R_{\mathcal{D}}(z) \mathbf{1}_F\| &\leq \|(\eta I)^n R_{\mathcal{D}}(z) \mathbf{1}_F\| \\
&= \|(\eta I)^{n-1} [\eta I, R_{\mathcal{D}}(z)] \mathbf{1}_F\| \\
&= \| \overbrace{[\eta I, ...[\eta I,}^{n} R_{\mathcal{D}}(z)]...] \mathbf{1}_F\| \\
&\leq n! \|R_{\mathcal{D}}(z)([\mathcal{D}, \eta I] R_{\mathcal{D}}(z))^n\| \\
&\leq n! (C_{\mathcal{D}} \|d\eta\|_\infty)^n \|R_{\mathcal{D}}(z)\|^{n+1} \\
&\leq n! \frac{C_{\theta \cup r}}{|z|} \left(\frac{(1+\epsilon) C_{\mathcal{D}} C_{\theta \cup r}}{\rho(E,F)|z|} \right)^n
\end{aligned}$$

for all $n \in \mathbb{N}_0$, which proves (3.4.1). □

The following proof was inspired by the proof of Lemma 7.3 in [44] by Hytönen, van Neerven and Portal.

Lemma 3.4.3. Let $0 \leq \omega < \theta < \pi/2$ and $0 \leq R < r$ and suppose that $\mathcal{D}$ is an operator satisfying the assumptions of Lemma 3.4.2. Let $M \geq 0$ and $\delta > 0$. For each $\psi \in \Psi^{\delta}_{M+\delta}(S^o_{\theta \cup r})$, $\phi \in \Theta^{\delta}(S^o_{\theta \cup r})$ and $a \in (0,1)$, there exists $c > 0$ such that the following hold:

1. $\|\mathbf{1}_E (f\psi_t)(\mathcal{D}) \mathbf{1}_F\| \leq c \|f\|_\infty \left\langle \frac{t}{\rho(E,F)} \right\rangle^M \exp\left(-a \frac{r}{C_{\mathcal{D}} C_{\theta \cup r}} \rho(E,F) \right)$;

2. $\|\mathbf{1}_E (f\phi\,)(\mathcal{D}) \mathbf{1}_F\| \leq c \|f\|_\infty \exp\left(-a \frac{r}{C_{\mathcal{D}} C_{\theta \cup r}} \rho(E,F) \right)$,

for all $t \in (0,1]$, $f \in H^\infty(S^o_{\theta \cup r})$ and closed subsets E and F of M.

Proof. For all $\tilde{\theta} \in (\omega, \theta)$ and $\tilde{r} \in (R, r)$, let $+\partial S^o_{\tilde{\theta}\cup\tilde{r}}$ denote the boundary of $S^o_{\tilde{\theta}\cup\tilde{r}}$ oriented clockwise, and divide this into $\gamma_{\tilde{r}} = +\partial S^o_{\tilde{\theta}\cup\tilde{r}} \cap D_{\tilde{r}}$ and $\gamma_{\tilde{\theta}} = +\partial S^o_{\tilde{\theta}\cup\tilde{r}} \cap S_{\tilde{\theta}}$. Using the Cauchy integral formula from (A1), we have

$$\mathbf{1}_E(f\psi_t)(\mathcal{D})\mathbf{1}_F = \frac{1}{2\pi i}\left(\int_{\gamma_{\tilde{r}}} + \int_{\gamma_{\tilde{\theta}}}\right) f(z)\psi_t(z)\mathbf{1}_E R_{\mathcal{D}}(z)\mathbf{1}_F \,\mathrm{d}z = I_1 + I_2$$

for all $\tilde{\theta} \in (\omega, \theta)$ and $\tilde{r} \in (R, r)$. It follows by Lemma 3.4.2 that for each $a \in (0, 1)$ and $b \geq 0$, we have

$$\begin{aligned}\|I_1\| &\lesssim C_{\tilde{\theta}\cup\tilde{r}}\|f\|_\infty \int_{\gamma_{\tilde{r}}} \min\{|tz|^{M+\delta}, |tz|^{-\delta}\} \left\langle \frac{1}{\rho(E,F)|z|}\right\rangle^M e^{-a\rho(E,F)|z|/C_{\mathcal{D}}C_{\tilde{\theta}\cup\tilde{r}}} \frac{|\mathrm{d}z|}{|z|} \\ &\lesssim_{r,R} C_{\tilde{\theta}\cup\tilde{r}}\|f\|_\infty \langle t/\rho(E,F)\rangle^M e^{-a\rho(E,F)\tilde{r}/C_{\mathcal{D}}C_{\tilde{\theta}\cup\tilde{r}}}\end{aligned}$$

and

$$\begin{aligned}\|I_2\| \lesssim C_{\tilde{\theta}\cup\tilde{r}}\|f\|_\infty \Bigg(&\int_{\tilde{r}}^{\tilde{r}/t} \frac{|tz|^{M+\delta}}{(\rho(E,F)|z|)^b} e^{-a\rho(E,F)|z|/C_{\mathcal{D}}C_{\tilde{\theta}\cup\tilde{r}}} \frac{\mathrm{d}|z|}{|z|} \\ &+ \int_{\tilde{r}/t}^{\infty} \frac{|tz|^{-\delta}}{(\rho(E,F)|z|)^b} e^{-a\rho(E,F)|z|/C_{\mathcal{D}}C_{\tilde{\theta}\cup\tilde{r}}} \frac{\mathrm{d}|z|}{|z|}\Bigg)\end{aligned}$$

for all $\tilde{\theta} \in (\omega, \theta)$ and $\tilde{r} \in (R, r)$. Setting $b = 0$ shows that

$$\|I_2\| \lesssim_{r,R} C_{\tilde{\theta}\cup\tilde{r}}\|f\|_\infty e^{-a\rho(E,F)\tilde{r}/C_{\mathcal{D}}C_{\tilde{\theta},\tilde{r}}},$$

and setting $b = M$ shows that

$$\|I_2\| \lesssim_{r,R} C_{\tilde{\theta}\cup\tilde{r}}\|f\|_\infty (t/\rho(E,F))^M e^{-a\rho(E,F)\tilde{r}/C_{\mathcal{D}}C_{\tilde{\theta},\tilde{r}}}.$$

Altogether, this shows that for each $a \in (0, 1)$, there exists $c > 0$ such that

$$\|\mathbf{1}_E(f\psi_t)(\mathcal{D})\mathbf{1}_F\| \leq c\, C_{\tilde{\theta}\cup\tilde{r}}\|f\|_\infty \langle t/\rho(E,F)\rangle^M e^{-a(\tilde{r}/C_{\mathcal{D}}C_{\tilde{\theta},\tilde{r}})\rho(E,F)}$$

for all $\tilde{\theta} \in (\omega, \theta)$ and $\tilde{r} \in (R, r)$. The first result follows by noting that

$$\sup\{\tilde{r}/C_{\tilde{\theta}\cup\tilde{r}} : \tilde{\theta} \in (\omega, \theta),\, \tilde{r} \in (R, r)\} = r/C_{\theta\cup r}.$$

The proof of the second result is similar. □

We conclude this section with a useful application of this result.

Proposition 3.4.4. Let $0 \leq \omega < \theta < \pi/2$ and $0 \leq R < r$ and suppose that $\mathcal{D}$ is an operator satisfying the assumptions of Lemma 3.4.2. Let $0 < \sigma < \alpha$ and $0 < \tau < \beta$. For each $\psi \in \Psi^\beta_\alpha(S^o_{\theta\cup r})$, $\tilde{\psi} \in \Psi^\alpha_\beta(S^o_{\theta\cup r})$, $\phi \in \Theta^\beta(S^o_{\theta\cup r})$, $\tilde{\phi} \in \Theta^\alpha(S^o_{\theta\cup r})$ and $a \in (0, 1)$, there exists $c > 0$ such that the following hold:

1. $$\|\mathbf{1}_E(\psi_t f\tilde{\psi}_s)(\mathcal{D})\mathbf{1}_F\| \leq c\|f\|_\infty \begin{cases}(s/t)^\tau \langle t/\rho(E,F)\rangle^{\alpha+\tau} e^{-a(r/C_{\mathcal{D}}C_{\theta\cup r})\rho(E,F)} & \text{if } s \leq t;\\ (t/s)^\sigma \langle s/\rho(E,F)\rangle^{\beta+\sigma} e^{-a(r/C_{\mathcal{D}}C_{\theta\cup r})\rho(E,F)} & \text{if } t \leq s;\end{cases}$$

2.$\|\mathbf{1}_E(\phi\ f\tilde{\psi}_s)(\mathcal{D})\mathbf{1}_F\| \leq c\|f\|_\infty\ s^\tau e^{-a(r/C_{\mathcal{D}}C_{\theta\cup r})\rho(E,F)}$;

3.$\|\mathbf{1}_E(\psi_t f\tilde{\phi}\ )(\mathcal{D})\mathbf{1}_F\| \leq c\|f\|_\infty\ t^\sigma e^{-a(r/C_{\mathcal{D}}C_{\theta\cup r})\rho(E,F)}$;

4.$\|\mathbf{1}_E(\phi\ f\tilde{\phi}\ )(\mathcal{D})\mathbf{1}_F\| \leq c\|f\|_\infty\ e^{-a(r/C_{\mathcal{D}}C_{\theta\cup r})\rho(E,F)}$,

for all $s,t \in (0,1]$, $f \in H^\infty(S^o_{\theta\cup r})$ and closed subsets E and F of M.

Proof. To prove (1), first suppose that $0 < s \leq t \leq 1$ and choose $\delta \in (0, \beta-\tau)$. Let $g_{(s)}(z) = (sz)^{-(\tau+\delta)}\tilde{\psi}_s(z)f(z)$ and $\eta(z) = z^{\tau+\delta}\psi(z)$ so that

$$\psi_t f\tilde{\psi}_s = (s/t)^{\tau+\delta}g_{(s)}\eta_t.$$

The function η is in $\Psi^{\beta-\tau-\delta}_{\alpha+\tau+\delta}(S^o_{\theta\cup r})$ and the functions $g_{(s)}$ are in $\Psi(S^o_{\theta\cup r})$ and satisfy $\sup_{s\in(0,1]}\|g_{(s)}\|_\infty \lesssim \|f\|_\infty$. Therefore, Lemma 3.4.3 provides the required off-diagonal estimate. The proof in the case $0 < t \leq s \leq 1$ is analogous.

The results in (2) and (3) follow from Lemma 3.4.3 by writing the following:

$$(\phi f\tilde{\psi}_s)(z) = s^\tau z^\tau \phi(z)f(z)(sz)^{-\tau}\tilde{\psi}(sz);$$
$$(\psi_t f\tilde{\phi})(z) = t^\sigma z^\sigma \tilde{\phi}(z)f(z)(tz)^{-\sigma}\psi(tz).$$

The result in (4) follows immediately from Lemma 3.4.3. □

3.5 The Main Estimate

We consider a complete Riemannian manifold M that is exponentially locally doubling. The spaces $t^p(X\times(0,1])$ and $L^p_{\mathscr{Q}}(X)$ introduced in Sections 3.2 and 3.3 consist of measurable functions. We begin by showing that it is a simple matter to formulate that theory for differential forms.

The local Lusin operator $\mathcal{A}_{\text{loc}}$ is defined for any measurable family of differential forms $U = (U_t)_{t\in(0,1]}$ on M, where each U_t is a section of ΛT^*M, by

$$\mathcal{A}_{\text{loc}}U(x) = \left(\iint_{\Gamma^1(x)} |U_t(y)|^2_{\wedge T^*_y M}\ \frac{\mathrm{d}\mu(y)}{V(x,t)}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}$$

for all $x \in M$. The dual operator $\mathcal{C}_{\text{loc}}$ is defined in the same way. For each $p \in [1,\infty]$, the local tent space $t^p(\wedge T^*M\times(0,1])$ consists of all measurable families of differential forms U on M with

$$\|U\|_{t^p} = \begin{cases} \left(\int_M (\mathcal{A}_{\text{loc}}U(x))^p\ \mathrm{d}\mu(x)\right)^{\frac{1}{p}}, & \text{if} \quad p\in[1,\infty); \\ \text{ess sup}_{x\in M}\,\mathcal{C}_{\text{loc}}U(x), & \text{if} \quad p=\infty. \end{cases}$$

Let $L^2_\bullet(\wedge T^*M\times(0,1])$ denote the space of all measurable families of differential forms U on M with $\|U\|^2_{L^2_\bullet} = \int_0^1 \|U_t\|^2_{L^2(\wedge T^*M)}\frac{\mathrm{d}t}{t}$. As before, this is an equivalent norm on $t^2(\wedge T^*M\times(0,1])$.

Next, fix a unit cube structure $\mathscr{Q} = (Q_j)_j$ on M. For each $p \in [1,\infty]$, the space $L^p_{\mathscr{Q}}(\wedge T^*M)$ consists of all measurable differential forms u on M with

$$\|u\|_{L^p_{\mathscr{Q}}} = \begin{cases} \left(\sum_{Q_j \in Q} (\mu(Q_j)^{\frac{1}{p}-\frac{1}{2}} \|\mathbf{1}_{Q_j} u\|_{L^2(\wedge T^*M)})^p \right)^{\frac{1}{p}}, & \text{if } \quad p \in [1,\infty); \\ \sup_{Q_j \in \mathscr{Q}} \mu(Q_j)^{-\frac{1}{2}} \|\mathbf{1}_{Q_j} f\|_{L^2(\wedge T^*M)}, & \text{if } \quad p = \infty. \end{cases}$$

As before, we have $L^2_{\mathscr{Q}}(\wedge T^*M) = L^2(\wedge T^*M)$.

A $t^1(\wedge T^*M)$-atom is a measurable family of differential forms $A = (A_t)_{t\in(0,1]}$ on M supported in the truncated tent $T^1(B)$ over a ball B in M of radius $r(B) \leq 2$ with $\|A\|_{L^2_\bullet} \leq \mu(B)^{-1/2}$. The atomic characterisation in Theorem 3.2.6 is proved in this context by defining the local maximal operator $\mathcal{M}_{\text{loc}}$ for all measurable differential forms u on M by

$$\mathcal{M}_{\text{loc}} u(x) = \sup_{r\in(0,1]} \frac{1}{V(x,r)} \|\mathbf{1}_{B(x,r)} u\|_{L^1(\wedge T^*M)}$$

for all $x \in M$

An $L^1_{\mathscr{Q}}(\wedge T^*M)$-atom is a measurable differential form a on M supported on a ball B in M of radius $r(B) \geq 1$ with $\|a\|_2 \leq \mu(B)^{-1/2}$. The proof of the atomic characterisation in Theorem 3.3.6 goes over directly.

The duality and interpolation results from Sections 3.2 and 3.3 extend to this setting as well. In what follows, we only consider spaces of differential forms and usually omit writing $\wedge T^*M$ and $\wedge T^*M \times (0,1]$.

Definition 3.5.1. Let M be a complete Riemannian manifold. Let $\omega \in [0,\pi/2)$ and $R \geq 0$ and suppose that $\mathcal{D}$ is an operator on $L^2(\wedge T^*M)$ of type $S_{\omega\cup R}$ satisfying (A1)–(A2). Given $\theta \in (\omega,\pi/2)$, $r > R$, $\psi \in \Psi(S^o_{\theta\cup r})$ and $\phi \in \Theta(S^o_{\theta\cup r})$, define the bounded operators $\mathcal{Q}^{\mathcal{D}}_{\psi,\phi} : L^2 \to L^2_\bullet \oplus L^2$ and $\mathcal{S}^{\mathcal{D}}_{\psi,\phi} : L^2_\bullet \oplus L^2 \to L^2$ by

$$\mathcal{Q}^{\mathcal{D}}_{\psi,\phi} u = (\psi_t(\mathcal{D})u, \phi(\mathcal{D})u)$$

for all $u \in L^2$ and $t \in (0,1]$, and

$$\mathcal{S}^{\mathcal{D}}_{\psi,\phi}(U,u) = \int_0^1 \psi_s(\mathcal{D}) U_s \frac{\mathrm{d}s}{s} + \phi(\mathcal{D})u = \lim_{a\to 0} \int_a^1 \psi_s(\mathcal{D}) U_s \frac{\mathrm{d}s}{s} + \phi(\mathcal{D})u$$

for all $(U,u) \in L^2_\bullet \oplus L^2$.

The operator $\mathcal{Q}^{\mathcal{D}}_{\psi,\phi}$ is bounded because hypothesis (A2) implies that $\mathcal{D}$ satisfies local quadratic estimates by Theorem 2.2.18. The adjoint operator $\mathcal{D}^*$ also satisfies (A1)–(A2) by Lemma 2.2.17. Therefore, the operator $\mathcal{S}^{\mathcal{D}}_{\psi,\phi} = (\mathcal{Q}^{\mathcal{D}^*}_{\psi^*,\phi^*})^*$, where ψ^* and ϕ^* are defined in Section 2.1, is also bounded.

The remainder of this section is dedicated to the proof of the following theorem, which is fundamental to the definition of our local Hardy spaces. It is a local analogue of Theorem 4.9 in [9]. The proof below simplifies some aspects of the original proof.

Theorem 3.5.2. *Let $\kappa,\lambda \geq 0$ and suppose that M is a complete Riemannian manifold satisfying* $(\mathrm{E}_{\kappa,\lambda})$. *Let $\omega \in [0,\pi/2)$ and $R \geq 0$ and suppose that $\mathcal{D}$ is an operator on $L^2(\wedge T^*M)$ of type $S_{\omega\cup R}$ satisfying* (A1) – (A3). *Let $\theta \in (\omega,\frac{\pi}{2})$, $r > R$ and $\beta > \kappa/2$ such that $r/C_{\mathcal{D}}C_{\theta\cup r} > \lambda/2$, where $C_{\theta\cup r}$ is from* (A1) *and $C_{\mathcal{D}}$ is from* (A3).

For each $\psi \in \Psi^\beta(S^o_{\theta\cup r})$, $\tilde{\psi} \in \Psi_\beta(S^o_{\theta\cup r})$, $\phi \in \Theta^\beta(S^o_{\theta\cup r})$ and $\tilde{\phi} \in \Theta(S^o_{\theta\cup r})$, there exists $c > 0$ such that the following hold for all $f \in H^\infty(S^o_{\theta\cup r})$:

1. *The operator* $Q^{\mathcal{D}}_{\psi,\phi} f(\mathcal{D}) \mathcal{S}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}$ *has a bounded extension* $\mathcal{P}_f$ *satisfying*

$$\|\mathcal{P}_f(U,u)\|_{t^p \oplus L^p_{\mathscr{Q}}} \leq c\|f\|_\infty \|(U,u)\|_{t^p \oplus L^p_{\mathscr{Q}}}$$

for all $(U,u) \in t^p \oplus L^p_{\mathscr{Q}}$ *and* $p \in [1,2]$*;*

2. *The operator* $Q^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}} f(\mathcal{D}) \mathcal{S}^{\mathcal{D}}_{\psi,\phi}$ *has a bounded extension* $\tilde{\mathcal{P}}_f$ *satisfying*

$$\|\tilde{\mathcal{P}}_f(U,u)\|_{t^p \oplus L^p_{\mathscr{Q}}} \leq c\|f\|_\infty \|(U,u)\|_{t^p \oplus L^p_{\mathscr{Q}}}$$

for all $(U,u) \in t^p \oplus L^p_{\mathscr{Q}}$ *and* $p \in [2,\infty]$.

Proof. Hypothesis (A2) and the comments in the paragraph after Definition 3.5.1 guarantee that both $Q^{\mathcal{D}}_{\psi,\phi} f(\mathcal{D}) \mathcal{S}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}$ and $Q^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}} f(\mathcal{D}) \mathcal{S}^{\mathcal{D}}_{\psi,\phi}$ satisfy the estimates in (1) and (2) on $t^2 \oplus L^2_{\mathscr{Q}}$.

To prove (1), define the following operators:

$$\begin{aligned} \mathcal{P}^{1,1}_f U &= \int_0^1 \psi_t(\mathcal{D}) f(\mathcal{D}) \tilde{\psi}_s(\mathcal{D}) U_s \frac{\mathrm{d}s}{s}; \quad \mathcal{P}^{1,2}_f u = \psi_t(\mathcal{D}) f(\mathcal{D}) \tilde{\phi}(\mathcal{D}) u; \\ \mathcal{P}^{2,1}_f U &= \int_0^1 \phi(\mathcal{D}) f(\mathcal{D}) \tilde{\psi}_s(\mathcal{D}) U_s \frac{\mathrm{d}s}{s}; \quad \mathcal{P}^{2,2}_f u = \phi(\mathcal{D}) f(\mathcal{D}) \tilde{\phi}(\mathcal{D}) u, \end{aligned}$$

for all $U \in L^2_\bullet$, $u \in L^2$ and $t \in (0,1]$, so we have the system

$$Q^{\mathcal{D}}_{\psi,\phi} f(\mathcal{D}) \mathcal{S}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}(U,u) = \begin{pmatrix} \mathcal{P}^{1,1}_f & \mathcal{P}^{1,2}_f \\ \mathcal{P}^{2,1}_f & \mathcal{P}^{2,2}_f \end{pmatrix} \begin{pmatrix} U \\ u \end{pmatrix}$$

for all $(U,u) \in L^2_\bullet \oplus L^2$.

We claim that there exists $c > 0$ such that

$$\|Q^{\mathcal{D}}_{\psi,\phi} f(\mathcal{D}) \mathcal{S}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}(A,a)\|_{t^1 \oplus L^1_{\mathscr{Q}}} \leq c\|f\|_\infty \tag{3.5.1}$$

for all A that are t^1-atoms and a that are $L^1_{\mathscr{Q}}$-atoms. The proof of (3.5.1) is quite technical, so we postpone it to Lemmas 3.5.3, 3.5.5, 3.5.4 and 3.5.6.

The set $t^1 \cap t^2$ is dense in t^1 by Proposition 3.2.3. Therefore, to prove that there exist bounded extensions $\mathcal{P}^{1,1}_f : t^1 \to t^1$ and $\mathcal{P}^{2,1}_f : t^1 \to L^1_{\mathscr{Q}}$, it suffices to show that

$$\|\mathcal{P}^{1,1}_f U\|_{t^1} \lesssim \|f\|_\infty \|U\|_{t^1} \quad \text{and} \quad \|\mathcal{P}^{2,1}_f U\|_{L^1_{\mathscr{Q}}} \lesssim \|f\|_\infty \|U\|_{t^1} \tag{3.5.2}$$

for all $U \in t^1 \cap t^2$.

If $U \in t^1 \cap t^2$, then by Theorem 3.2.6 there exist a sequence $(\lambda_j)_j$ in ℓ^1 and a sequence $(A_j)_j$ of t^1-atoms such that $\sum_j \lambda_j A_j$ converges to U in t^2 and also $\|(\lambda_j)_j\|_{\ell^1} \lesssim \|U\|_{t^1}$. Then, since $Q^{\mathcal{D}}_{\psi,\phi} f(\mathcal{D}) \mathcal{S}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}$ is bounded on $t^2 \oplus L^2_{\mathscr{Q}}$, we have

$$\mathcal{P}^{2,1}_f U = \textstyle\sum_j \lambda_j \mathcal{P}^{2,1}_f (A_j),$$

where the sum converges in $L^2_{\mathscr{Q}}$. Also, the partial sums $\sum_{j=1}^n \mathcal{P}^{2,1}_f(\lambda_j A_j)$ form a Cauchy sequence in $L^1_{\mathscr{Q}}$ by (3.5.1). Therefore, there exists $v \in L^1_{\mathscr{Q}}$ such that

$$v = \textstyle\sum_j \lambda_j \mathcal{P}^{2,1}_f (A_j),$$

where the sum converges in $L^1_{\mathcal{Q}}$, and $\|v\|_{L^1_{\mathcal{Q}}} \lesssim \|f\|_\infty \|U\|_{t^1}$. Given that both $L^1_{\mathcal{Q}}$ and $L^2_{\mathcal{Q}}$ are continuously embedded in $L^1_{\mathcal{Q}} + L^2_{\mathcal{Q}}$, as in the proof of Theorem 3.3.10, we must have $v = \mathcal{P}^{2,1}_f U$. A similar argument holds for $\mathcal{P}^{1,1}_f U$ to give (3.5.2).

The set $L^1_{\mathcal{Q}} \cap L^2_{\mathcal{Q}}$ is dense in $L^1_{\mathcal{Q}}$ by Proposition 3.3.4. Therefore, to prove that there exist bounded extensions $\mathcal{P}^{1,2}_f : L^1_{\mathcal{Q}} \to t^1$ and $\mathcal{P}^{2,2}_f : L^1_{\mathcal{Q}} \to L^1_{\mathcal{Q}}$, it suffices to show that

$$\|\mathcal{P}^{1,2}_f u\|_{t^1} \lesssim \|f\|_\infty \|u\|_{L^1_{\mathcal{Q}}} \quad \text{and} \quad \|\mathcal{P}^{2,2}_f u\|_{L^1_{\mathcal{Q}}} \lesssim \|f\|_\infty \|u\|_{L^1_{\mathcal{Q}}} \tag{3.5.3}$$

for all $u \in L^1_{\mathcal{Q}} \cap L^2_{\mathcal{Q}}$.

If $u \in L^1_{\mathcal{Q}} \cap L^2_{\mathcal{Q}}$, then by Theorem 3.3.6 there exist a sequence $(\lambda_j)_j$ in ℓ^1 and a sequence $(a_j)_j$ of $L^1_{\mathcal{Q}}$-atoms such that $\sum_j \lambda_j a_j$ converges to u in $L^2_{\mathcal{Q}}$ and also $\|(\lambda_j)_j\|_{\ell^1} \lesssim \|u\|_{L^1_{\mathcal{Q}}}$. Then, since $\mathcal{Q}^{\mathcal{D}}_{\psi,\phi} f(\mathcal{D}) \mathcal{S}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}$ is bounded on $t^2 \oplus L^2_{\mathcal{Q}}$, we have

$$\mathcal{P}^{1,2}_f u = \textstyle\sum_j \lambda_j \mathcal{P}^{1,2}_f (a_j),$$

where the sum converges in t^2. Also, the partial sums $\sum_{j=1}^n \mathcal{P}^{1,2}_f (\lambda_j a_j)$ form a Cauchy sequence in t^1 by (3.5.1). Therefore, there exists $V \in t^1$ such that

$$V = \textstyle\sum_j \lambda_j \mathcal{P}^{1,2}_f (a_j),$$

where the sum converges in t^1, and $\|V\|_{t^1} \lesssim \|f\|_\infty \|u\|_{L^1_{\mathcal{Q}}}$. Given that both t^1 and t^2 are continuously embedded in $t^1 + t^2$, as in the proof of Theorem 3.2.13, we must have $V = \mathcal{P}^{1,2}_f u$. A similar argument holds for $\mathcal{P}^{2,2}_f U$ to give (3.5.3).

The bounds in (3.5.2) and (3.5.3) prove that $\mathcal{Q}^{\mathcal{D}}_{\psi,\phi} f(\mathcal{D}) \mathcal{S}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}$ has a bounded extension satisfying the estimate in (1) on $t^1 \oplus L^1_{\mathcal{Q}}$. Therefore, result (1) follows by the interpolation in Theorems 3.2.13 and 3.3.10.

To prove (2), note that replacing $\mathcal{D}$ with $\mathcal{D}^*$ in the proof of (1) shows that $\mathcal{Q}^{\mathcal{D}^*}_{\psi^*,\phi^*} f^*(\mathcal{D}^*) \mathcal{S}^{\mathcal{D}^*}_{\tilde{\psi}^*,\tilde{\phi}^*}$ has a bounded extension $\mathcal{P}_{f^*}$ satisfying the estimate in (1) on the space $t^1 \oplus L^1_{\mathcal{Q}}$. The duality in Theorems 3.2.12 and 3.3.8 then allows us to define the dual operator $\mathcal{P}'_{f^*}$ satisfying the estimate in (2) on $t^\infty \oplus L^\infty_{\mathcal{Q}}$. We also have $\mathcal{P}'_{f^*} = \mathcal{Q}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}} f(\mathcal{D}) \mathcal{S}^{\mathcal{D}}_{\psi,\phi}$ on $(t^\infty \cap t^2) \oplus (L^\infty_{\mathcal{Q}} \cap L^2_{\mathcal{Q}})$, as $\mathcal{S}^{\mathcal{D}}_{\psi,\phi} = (\mathcal{Q}^{\mathcal{D}^*}_{\psi^*,\phi^*})^*$ on $t^2 \oplus L^2_{\mathcal{Q}}$. Therefore, result (2) follows by the interpolation in Theorems 3.2.13 and 3.3.10. □

The remainder of this section is devoted to proving (3.5.1). The proof is divided into four lemmas. We adopt the notation

$$\mathcal{Q}^{\mathcal{D}}_{\psi,\phi} f(\mathcal{D}) \mathcal{S}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}} = \begin{pmatrix} \mathcal{P}^{1,1}_f & \mathcal{P}^{1,2}_f \\ \mathcal{P}^{2,1}_f & \mathcal{P}^{2,2}_f \end{pmatrix}$$

as in the proof of Theorem 3.5.2.

Lemma 3.5.3. Under the assumptions of Theorem 3.5.2, there exists $c > 0$ such that $\|\mathcal{P}^{1,1}_f A\|_{t^1} \leq c\|f\|_\infty$ for all A that are t^1-atoms.

Proof. Let A be a t^1-atom. There exists a ball B in M with radius $r(B) \leq 2$ such that A is supported in $T^1(B)$ and $\|A\|_{L^2_\bullet} \leq \mu(B)^{-1/2}$. If $r(B) > 1/2$, let $K = 0$. If

$r(B) \leq 1/2$, let K be the positive integer such that $2^K \leq 1/r(B) < 2^{K+1}$. Next, associate B with the characteristic functions $\mathbf{1}_k$ defined by

$$\mathbf{1}_k = \begin{cases} \mathbf{1}_{T^1(4B)} & \text{if} \quad k = 0; \\ \mathbf{1}_{T^1(2^{k+2}B)\setminus T^1(2^{k+1}B)} & \text{if} \quad K \geq 1 \quad \text{and} \quad k \in \{1, \ldots, K\}. \end{cases}$$

Also, define the ball B^* with radius $r(B^*) \in [4, 8]$ by

$$B^* = \begin{cases} 4B & \text{if} \quad K = 0; \\ 2^{K+2}B & \text{if} \quad K \geq 1 \end{cases}$$

and associate it with the characteristic functions $\mathbf{1}_k^*$ defined by

$$\mathbf{1}_k^* = \mathbf{1}_{T^1((k+1)B^*)\setminus T^1(kB^*)}$$

for all $k \in \mathbb{N} = \{1, 2, ...\}$.

Let $\tilde{A}_k = \mathbf{1}_k \mathcal{P}_f^{1,1} A$ and $\tilde{A}_k^* = \mathbf{1}_k^* \mathcal{P}_f^{1,1} A$, so we have $\operatorname{sppt} \tilde{A}_k \subseteq T^1(2^{k+2}B)$, $\operatorname{sppt} \tilde{A}_k^* \subseteq T^1\left((k+1)B^*\right)$ and

$$\mathcal{P}_f^{1,1} A = \sum_{k=0}^{K} \tilde{A}_k + \sum_{k=1}^{\infty} \tilde{A}_k^*.$$

We prove below that there exist $c > 0$ and two sequences $(\lambda_k)_{k\in\{0,\ldots,K\}}$ and $(\lambda_k^*)_{k\in\mathbb{N}}$ in ℓ^1, all of which do not depend on A, such that the following hold:

$$\|\tilde{A}_k\|_{L^2_\bullet} \leq c\|f\|_\infty \lambda_k \, \mu\left(2^{k+2}B\right)^{-\frac{1}{2}} \quad \text{for all} \quad k \in \{0, ..., K\}; \tag{3.5.4}$$

$$\|\tilde{A}_k^*\|_{L^2_\bullet} \leq c\|f\|_\infty \lambda_k^* \, \mu\left((k+1)B^*\right)^{-\frac{1}{2}} \quad \text{for all} \quad k \in \mathbb{N}. \tag{3.5.5}$$

The result then follows by Remark 3.2.5.

To prove (3.5.4) and (3.5.5), choose δ in $(0, \frac{2\beta-\kappa}{3})$ so that $\psi \in \Psi_{2\delta}^{\beta}(S^o_{\theta\cup r})$ and that $\tilde{\psi} \in \Psi_{\beta}^{2\delta}(S^o_{\theta\cup r})$, which is possible because $\beta > \kappa/2$. Also, choose a in $(\frac{\lambda}{2}\frac{C_{\mathcal{D}}C_{\theta\cup r}}{r}, 1)$, which is possible because $r/C_{\mathcal{D}}C_{\theta\cup r} > \lambda/2$. Proposition 3.4.4 applied with $\sigma = \delta$ and $\tau = \beta - \delta$ then shows that

$$\|\mathbf{1}_E(\psi_t f \tilde{\psi}_s)(\mathcal{D})\mathbf{1}_F\| \lesssim \|f\|_\infty e^{-a(r/C_{\mathcal{D}}C_{\theta\cup r})\rho(E,F)} \begin{cases} (\frac{s}{t})^{\beta-\delta}\langle\frac{t}{\rho(E,F)}\rangle^{\beta+\delta} & \text{if} \quad s \leq t; \\ (\frac{t}{s})^{\delta}\langle\frac{s}{\rho(E,F)}\rangle^{\beta+\delta} & \text{if} \quad t \leq s \end{cases} \tag{3.5.6}$$

for all $s, t \in (0, 1]$ and closed subsets E and F of M. Applying the Cauchy–Schwarz inequality and considering the support of A, we also obtain

$$\begin{aligned} |(\mathcal{P}_f^{1,1} A)_t|^2 &= \left| \int_0^{\langle r(B)\rangle} \min\{\tfrac{t}{s}, \tfrac{s}{t}\}^{\frac{\delta}{2}} \left(\min\{\tfrac{t}{s}, \tfrac{s}{t}\}^{-\frac{\delta}{2}} (\psi_t f \tilde{\psi}_s)(\mathcal{D}) A_s\right) \frac{\mathrm{d}s}{s} \right|^2 \\ &\lesssim \int_0^{\langle r(B)\rangle} \min\{\tfrac{t}{s}, \tfrac{s}{t}\}^{-\delta} |(\psi_t f \tilde{\psi}_s)(\mathcal{D}) A_s|^2 \frac{\mathrm{d}s}{s} \end{aligned} \tag{3.5.7}$$

for all $t \in (0,1]$. We now use (3.5.6) and (3.5.7) to prove (3.5.4) and (3.5.5):

Proof of (3.5.4). The operator $\mathcal{Q}^{\mathcal{D}}_{\psi,\phi} f(\mathcal{D}) \mathcal{S}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}$ is bounded on $L^2_\bullet \oplus L^2$, so we have

$$\|\tilde{A}_0\|_{L^2_\bullet} \leq \|\mathcal{P}_f(A,0)\|_{L^2_\bullet} \lesssim \|f\|_\infty \|A\|_{L^2_\bullet} \lesssim \|f\|_\infty \mu(4B)^{-\frac{1}{2}}.$$

Suppose that $K \geq 1$ and that $k \in \{1,\dots,K\}$, which implies that $2^k r(B) \leq 1$. Note that the support of $\tilde{A}_k$ is contained in $T^1(2^{k+2}B) \setminus T^1(2^{k+1}B)$. Also, if (x,t) belongs to $T^1(2^{k+2}B) \setminus T^1(2^{k+1}B)$ and $t \leq 2^k r(B)$, then x belongs to $2^{k+2}B \setminus 2^k B$. Using (3.5.7), we then obtain

$$\begin{aligned}
\|\tilde{A}_k\|^2_{L^2_\bullet} &\lesssim \int_0^{2^k r(B)} \int_0^{r(B)} \min\{\tfrac{t}{s},\tfrac{s}{t}\}^{-\delta} \|\mathbf{1}_{2^{k+2}B\setminus 2^k B}(\psi_t f \tilde{\psi}_s)(\mathcal{D}) A_s\|_2^2 \frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t} \\
&\quad + \int_{2^k r(B)}^{\langle 2^{k+2} r(B)\rangle} \int_0^{r(B)} \min\{\tfrac{t}{s},\tfrac{s}{t}\}^{-\delta} \|(\psi_t f \tilde{\psi}_s)(\mathcal{D}) A_s\|_2^2 \frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t} \\
&= I_1 + I_2.
\end{aligned}$$

To estimate I_1, note that $\rho(2^{k+2}B \setminus 2^k B, B) = (2^k - 1) r(B) \leq 1$, since we are assuming that $2^k r(B) \leq 1$. Using (3.5.6) and $(\mathrm{E}_{\kappa,\lambda})$, we then obtain

$$\begin{aligned}
I_1 &\lesssim \|f\|^2_\infty \int_0^{r(B)} \int_0^s \left(\frac{t}{s}\right)^\delta \left(\frac{s}{2^k r(B)}\right)^{2\beta+2\delta} \frac{\mathrm{d}t}{t} \|A_s\|_2^2 \frac{\mathrm{d}s}{s} \\
&\quad + \|f\|^2_\infty \int_0^{r(B)} \int_s^{2^k r(B)} \left(\frac{s}{t}\right)^{2\beta-3\delta} \left(\frac{t}{2^k r(B)}\right)^{2\beta+2\delta} \frac{\mathrm{d}t}{t} \|A_s\|_2^2 \frac{\mathrm{d}s}{s} \\
&\lesssim \|f\|^2_\infty (2^{-(2\beta+2\delta)k} + 2^{-(2\beta-3\delta)k}) \|A\|^2_{L^2_\bullet} \\
&\lesssim \|f\|^2_\infty 2^{-(2\beta-3\delta)k} \mu(B)^{-1} \\
&\lesssim \|f\|^2_\infty 2^{-(2\beta-\kappa-3\delta)k} e^{\lambda 2^{k+2} r(B)} \mu(2^{k+2}B)^{-1} \\
&\lesssim \|f\|^2_\infty 2^{-(2\beta-\kappa-3\delta)k} \mu(2^{k+2}B)^{-1},
\end{aligned}$$

where $2^k r(B) \leq 1$ was used in the final inequality. We also obtain

$$\begin{aligned}
I_2 &\leq \|f\|^2_\infty \int_{2^k r(B)}^\infty \int_0^{r(B)} \left(\frac{s}{t}\right)^{2\beta-3\delta} \|A_s\|_2^2 \frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t} \\
&\lesssim \|f\|^2_\infty \int_0^{r(B)} \left(\frac{s}{2^k r(B)}\right)^{2\beta-3\delta} \|A_s\|_2^2 \frac{\mathrm{d}s}{s} \\
&\lesssim \|f\|^2_\infty 2^{-(2\beta-3\delta)k} \|A\|^2_{L^2_\bullet} \\
&\lesssim \|f\|^2_\infty 2^{-(2\beta-\kappa-3\delta)k} \mu(2^{k+2}B)^{-1}.
\end{aligned}$$

The bounds for I_1 and I_2 show that

$$\|\tilde{A}_k\|_{L^2_\bullet} \lesssim \|f\|_\infty 2^{-(2\beta-\kappa-3\delta)k/2} \mu(2^{k+2}B)^{-\frac{1}{2}},$$

which proves (3.5.4) with $\lambda_k = 2^{-(2\beta-\kappa-3\delta)k/2}$, since $2\beta - \kappa - 3\delta > 0$.

Proof of (3.5.5). Suppose that $k \in \mathbb{N}$. If (x,t) belongs to $T^1((k+1)B^*) \setminus T^1(kB^*)$, then x belongs to $(k+1)B^* \setminus (k-1/4)B^*$, since the radius $r(B^*) \in [4,8]$. Also, since $r(B) \leq r(B^*)/4$, we have

$$\rho((k+1)B^* \setminus (k-\tfrac{1}{4})B^*, B) \geq (k-\tfrac{1}{4})r(B^*) - r(B) \geq \max\{1, kr(B^*)\}.$$

Using (3.5.6), (3.5.7) and $(\mathrm{E}_{\kappa,\lambda})$, we then obtain

$$\begin{aligned}
\|\tilde{A}_k^*\|_{L^2_\bullet}^2 &\lesssim \|f\|_\infty^2 \int_0^1 \int_0^{\langle r(B)\rangle} \min\{\tfrac{t}{s}, \tfrac{s}{t}\}^{-\delta} \|\mathbf{1}_{(k+1)B^*\setminus(k-1/4)B^*}(\psi_t f \tilde{\psi}_s)(\mathcal{D})A_s\|_2^2 \frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t} \\
&\lesssim \|f\|_\infty^2 e^{-2a(r/C_{\mathcal{D}}C_{\theta\cup r})kr(B^*)} \int_0^{\langle r(B)\rangle} \int_0^s \left(\frac{t}{s}\right)^\delta s^{2\beta+2\delta} \frac{\mathrm{d}t}{t} \|A_s\|_2^2 \frac{\mathrm{d}s}{s} \\
&\quad + \|f\|_\infty^2 e^{-2a(r/C_{\mathcal{D}}C_{\theta\cup r})kr(B^*)} \int_0^{\langle r(B)\rangle} \int_s^1 \left(\frac{s}{t}\right)^{2\beta-3\delta} t^{2\beta+2\delta} \frac{\mathrm{d}t}{t} \|A_s\|_2^2 \frac{\mathrm{d}s}{s} \\
&\lesssim \|f\|_\infty^2 e^{-2a(r/C_{\mathcal{D}}C_{\theta\cup r})kr(B^*)} \langle r(B)\rangle^{2\beta-3\delta} \|A\|_{L^2_\bullet}^2 \\
&\lesssim \|f\|_\infty^2 e^{-2a(r/C_{\mathcal{D}}C_{\theta\cup r})kr(B^*)} r(B)^\kappa \mu(B)^{-1} \\
&\lesssim \|f\|_\infty^2 e^{-(2a(r/C_{\mathcal{D}}C_{\theta\cup r})-\lambda)kr(B^*)} k^\kappa \mu((k+1)B^*)^{-1} \\
&\lesssim \|f\|_\infty^2 e^{-(2a(r/C_{\mathcal{D}}C_{\theta\cup r})-\lambda)2k} \mu((k+1)B^*)^{-1}.
\end{aligned}$$

This shows that

$$\|\tilde{A}_k^*\|_{L^2_\bullet} \lesssim \|f\|_\infty e^{-(2a(r/C_{\mathcal{D}}C_{\theta\cup r})-\lambda)k} \mu((k+1)B^*)^{-\frac{1}{2}},$$

which proves (3.5.5) with $\lambda_k^* = e^{-(2a(r/C_{\mathcal{D}}C_{\theta\cup r})-\lambda)k}$, since $2a(r/C_{\mathcal{D}}C_{\theta\cup r}) - \lambda > 0$. □

Lemma 3.5.4. Under the assumptions of Theorem 3.5.2, there exists $c > 0$ such that $\|\mathcal{P}_f^{2,1}A\|_{L^1_{\mathscr{Q}}} \leq c\|f\|_\infty$ for all A that are t^1-atoms.

Proof. Let A be a t^1-atom. There exists a ball B in M with radius $r(B) \leq 2$ such that A is supported in $T^1(B)$ and $\|A\|_{L^2_\bullet} \leq \mu(B)^{-1/2}$. Define the ball B^* with radius $r(B^*) \in [2,4]$ by

$$B^* = \begin{cases} 2B & \text{if} \quad 1 < r(B) \leq 2; \\ (2/r(B))B & \text{if} \quad r(B) \leq 1 \end{cases}$$

and associate B^* with the characteristic functions $\mathbf{1}_k^*$ defined by

$$\mathbf{1}_k^* = \begin{cases} \mathbf{1}_{2B^*} & \text{if} \quad k = 0; \\ \mathbf{1}_{(k+2)B^*\setminus(k+1)B^*} & \text{if} \quad k = 1, 2, \ldots. \end{cases}$$

Let $\tilde{A}_k^* = \mathbf{1}_k^* \mathcal{P}_f^{2,1}A$, so we have $\operatorname{sppt} \tilde{A}_k^* \subseteq (k+2)B^*$ and $\mathcal{P}_f^{2,1}A = \sum_{k=0}^\infty \tilde{A}_k^*$. We prove below that there exist $c > 0$ and a sequence $(\lambda_k^*)_k$ in ℓ^1, both of which do not depend on A, such that

$$\|\tilde{A}_k^*\|_2 \leq c\|f\|_\infty \lambda_k^* \mu\left((k+2)B^*\right)^{-\frac{1}{2}}. \tag{3.5.8}$$

The result then follows from Theorem 3.3.6.

To prove (3.5.8), choose a as in the proof of Lemma 3.5.3. Proposition 3.4.4 applied with $\tau = \kappa/2$ then shows that

$$\|\mathbf{1}_E(\phi f \tilde{\psi}_s)(\mathcal{D})\mathbf{1}_F\| \lesssim \|f\|_\infty s^{\frac{\kappa}{2}} e^{-a(r/C_{\mathcal{D}}C_{\theta\cup r})\rho(E,F)}$$

for all $s \in (0,1]$ and closed subsets E and F of M. Now note that if $k \geq 0$, then

$$\rho((k+2)B^*\backslash(k+1)B^*, B) = (k+1)r(B^*) - r(B) \geq kr(B^*).$$

Using $(\mathrm{E}_{\kappa,\lambda})$, we then obtain

$$\begin{aligned}
\|\tilde{A}_k^*\|_2^2 &= \int_M \mathbf{1}_k^* \left| \int_0^{\langle r(B)\rangle} s^{\frac{\kappa}{2}} s^{-\frac{\kappa}{2}} (\phi f \tilde{\psi}_s)(\mathcal{D}) A_s \frac{\mathrm{d}s}{s} \right|^2 \mathrm{d}\mu \\
&\lesssim r(B)^\kappa \int_0^{r(B)} s^{-\kappa} \|\mathbf{1}_k^*(\phi f \tilde{\psi}_s)(\mathcal{D}) A_s\|_2^2 \frac{\mathrm{d}s}{s} \\
&\leq \|f\|_\infty^2 e^{-2a(r/C_{\mathcal{D}}C_{\theta\cup r})kr(B^*)} r(B)^\kappa \|A\|_{L^2_\bullet}^2 \\
&\leq \|f\|_\infty^2 e^{-(2a(r/C_{\mathcal{D}}C_{\theta\cup r})-\lambda)kr(B^*)} k^\kappa \mu((k+2)B^*)^{-1} \\
&\lesssim \|f\|_\infty^2 e^{-(2a(r/C_{\mathcal{D}}C_{\theta\cup r})-\lambda)k} \mu((k+2)B^*)^{-1},
\end{aligned}$$

which proves (3.5.8) with $\lambda_k^* = e^{-(2a(r/C_{\mathcal{D}}C_{\theta\cup r})-\lambda)k/2}$. □

Lemma 3.5.5. Under the assumptions of Theorem 3.5.2, there exists $c > 0$ such that $\|\mathcal{P}_f^{1,2} A\|_{t^1} \leq c\|f\|_\infty$ for all A that are $L^1_{\mathscr{Q}}$-atoms.

Proof. Let A be an $L^1_{\mathscr{Q}}$-atom. There exists a ball B in M with radius $r(B) \geq 1$ such that A is supported in B and $\|A\|_2 \leq \mu(B)^{-1/2}$. In view of Remark 3.3.7 and Theorem 3.3.6, however, it suffices to assume that $r(B) = 1$. In that case, associate B with the characteristic functions $\mathbf{1}_k$ defined by

$$\mathbf{1}_k = \begin{cases} \mathbf{1}_{T^1(B)} & \text{if} \quad k = 0; \\ \mathbf{1}_{T^1((k+1)B)\backslash T^1(kB)} & \text{if} \quad k = 1, 2, \ldots. \end{cases}$$

Let $\tilde{A}_k = \mathbf{1}_k \mathcal{P}_f^{1,2} A$, so we have sppt $\tilde{A}_k \subseteq T^1((k+1)B)$ and $\mathcal{P}_f^{1,2} A = \sum_{k=0}^\infty \tilde{A}_k$. We prove below that there exist $c > 0$ and a sequence $(\lambda_k)_k$ in ℓ^1, both of which do not depend on A, such that

$$\|\tilde{A}_k\|_{L^2_\bullet} \leq c\|f\|_\infty \lambda_k \, \mu\left((k+1)B\right)^{-\frac{1}{2}}. \tag{3.5.9}$$

The result then follows by Remark 3.2.5.

To prove (3.5.9), choose δ and a as in the proof of Lemma 3.5.3. Proposition 3.4.4 applied with $\sigma = \delta$ then shows that

$$\|\mathbf{1}_E(\psi_t f \tilde{\phi})(\mathcal{D})\mathbf{1}_F\| \lesssim \|f\|_\infty t^\delta e^{-a(r/C_{\mathcal{D}}C_{\theta\cup r})\rho(E,F)}$$

for all $t \in (0,1]$ and closed subsets E and F of M. Now note that if $k \geq 1$ and (x,t) belongs to $T^1((k+1)B) \setminus T^1(kB)$, then x belongs to $(k+1)B \setminus (k-1)B$, since $t \leq 1$

and $r(B)=1$. Using $(\mathrm{E}_{\kappa,\lambda})$, we then obtain

$$\begin{aligned}\|\tilde{A}_k\|^2_{L^2_\bullet} &= \int_0^1 \|\mathbf{1}_k(\psi_t f\tilde{\phi})(\mathcal{D})A\|_2^2 \frac{\mathrm{d}t}{t} \\ &\lesssim \|f\|_\infty^2 e^{-2a(r/C_\mathcal{D}C_{\theta\cup r})k} \int_0^1 t^{2\delta}\frac{\mathrm{d}t}{t}\|A\|_2^2 \\ &\lesssim \|f\|_\infty^2 e^{-2a(r/C_\mathcal{D}C_{\theta\cup r})k}\mu(B)^{-1} \\ &\lesssim \|f\|_\infty^2 e^{-(2a(r/C_\mathcal{D}C_{\theta\cup r})-\lambda)k}k^\kappa\mu((k+1)B)^{-1} \\ &\lesssim \|f\|_\infty^2 e^{-(2a(r/C_\mathcal{D}C_{\theta\cup r})-\lambda)k/2}\mu((k+1)B)^{-1},\end{aligned}$$

which proves (3.5.9) with $\lambda_k = e^{-(2a(r/C_\mathcal{D}C_{\theta\cup r})-\lambda)k/4}$. □

Lemma 3.5.6. Under the assumptions of Theorem 3.5.2, there exists $c>0$ such that $\sup\|\mathcal{P}_f^{2,2}A\|_{L^1_\mathscr{Q}} \leq c\|f\|_\infty$ for all A that are $L^1_\mathscr{Q}$-atoms.

Proof. Let A be an $L^1_\mathscr{Q}$-atom. As in the proof of Lemma 3.5.5, it is suffices to assume that there exists a ball B in M with radius $r(B)=1$ such that A is supported in B and $\|A\|_2 \leq \mu(B)^{-1/2}$. Associate B with the characteristic functions $\mathbf{1}_k$ defined by

$$\mathbf{1}_k = \begin{cases} \mathbf{1}_{2B} & \text{if} \quad k=0; \\ \mathbf{1}_{(k+2)B\setminus(k+1)B} & \text{if} \quad k=1,2,\dots. \end{cases}$$

Let $\tilde{A}_k = \mathbf{1}_k\mathcal{P}_f^{2,2}A$, so we have sppt $\tilde{A}_k \subseteq (k+2)B$ and $\mathcal{P}_f^{2,2}A = \sum_{k=0}^\infty \tilde{A}_k$. As in the proof of Lemma 3.5.4, it is enough to find $c>0$ and a sequence $(\lambda_k)_k$ in ℓ^1, both of which do not depend on A, such that

$$\|\tilde{A}_k\|_2 \leq c\|f\|_\infty \lambda_k\, \mu\left((k+2)B\right)^{-\frac{1}{2}}. \tag{3.5.10}$$

Choose a as in the proof of Lemma 3.5.3. Using Proposition 3.4.4 and $(\mathrm{E}_{\kappa,\lambda})$, we then obtain

$$\|\tilde{A}_k\|_2 \lesssim \|f\|_\infty e^{-a(r/C_\mathcal{D}C_{\theta\cup r})k}\|A\|_2 \lesssim \|f\|_\infty e^{-(2a(r/C_\mathcal{D}C_{\theta\cup r})-\lambda)k/2}k^{\kappa/2}\mu((k+2)B)^{-\frac{1}{2}},$$

which proves (3.5.10) with $\lambda_k = e^{-(2a(r/C_\mathcal{D}C_{\theta\cup r})-\lambda)k/4}$. □

3.6 Local Hardy Spaces $h^p_\mathcal{D}(\wedge T^*M)$

Throughout this section, let $\kappa,\lambda \geq 0$ and suppose that M is a complete Riemannian manifold satisfying $(\mathrm{E}_{\kappa,\lambda})$. Also, let $\omega \in [0,\pi/2)$ and $R \geq 0$ and suppose that $\mathcal{D}$ is an operator on $L^2(\wedge T^*M)$ of type $S_{\omega\cup R}$ satisfying hypotheses (A1)–(A3) from Section 3.4 with constants $C_{\theta\cup r}>0$ and $C_\mathcal{D}>0$, where $C_{\theta\cup r}$ is defined for each $\theta\in(\omega,\pi/2)$ and $r>R$.

We use the Φ-class of holomorphic functions from Definition 2.2.8 to prove a variant of the Calderón reproducing formula. This allows us to characterise $L^2(\wedge T^*M)$ in terms of square functions involving the operators $\mathcal{Q}^\mathcal{D}_{\psi,\phi}$ and $\mathcal{S}^\mathcal{D}_{\psi,\phi}$ from the previous section, where ϕ is restricted to the Φ-class. We combine this with Theorem 3.5.2 to define local Hardy spaces of differential forms $h^p_{\mathcal{D}}(\wedge T^*M)$ for all $p \in [1,\infty]$ in terms of square functions and a retraction on the space $t^p(\wedge T^*M \times (0,1]) \oplus L^p_{\mathscr{Q}}(\wedge T^*M)$. In what follows, we only consider spaces of differential forms and usually omit writing $\wedge T^*M$ and $\wedge T^*M \times (0,1]$.

Proposition 3.6.1 (Calderón reproducing formula)**.** Let $\theta \in (\omega, \pi/2)$ and $r > R$. Given $\alpha, \beta, \gamma, \sigma, \tau, \upsilon > 0$ and nondegenerate $\psi \in \Psi^\beta_\alpha(S^o_{\theta\cup r})$ and $\phi \in \Phi^\gamma(S^o_{\theta\cup r})$, there exist $\tilde{\psi} \in \Psi^\tau_\sigma(S^o_{\theta\cup r})$ and $\tilde{\phi} \in \Theta^\upsilon(S^o_{\theta\cup r})$ such that

$$\int_0^1 \tilde{\psi}_t(z)\psi_t(z)\frac{\mathrm{d}t}{t} + \tilde{\phi}(z)\phi(z) = 1 \tag{3.6.1}$$

for all $z \in S^o_{\theta\cup r}$. Moreover, we have $\mathcal{S}^{\mathcal{D}}_{\psi,\phi}\mathcal{Q}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}} = \mathcal{S}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}\mathcal{Q}^{\mathcal{D}}_{\psi,\phi} = I$ on L^2.

Proof. Given $f \in H^\infty(S^o_{\theta\cup r})$, let $f_-(z) = f(-z)$ and $f^*(z) = \overline{f(\bar{z})}$ for all $z \in S^o_{\theta\cup r}$. Choose integers M and N so that $4M \geq \max(\frac{\sigma}{\alpha}, \frac{\tau}{\beta}) + 1$ and $4M\beta + (4N-1)\gamma \geq \upsilon$. Let $c = \int_0^\infty |\psi(t)\psi(-t)|^{2M}|\phi(t)\phi(-t)|^{2N}\,\frac{\mathrm{d}t}{t}$ and define the functions

$$\tilde{\psi} = c^{-1}\psi^{M-1}(\psi^*\psi_-\psi^*_-)^M(\phi\phi^*\phi_-\phi^*_-)^N \quad \text{and} \quad \tilde{\phi} = \frac{1}{\phi}\left(1 - \int_0^1 \tilde{\psi}_t\psi_t\frac{\mathrm{d}t}{t}\right),$$

in which case (3.6.1) is immediate and $\tilde{\psi} \in \Psi^{\beta(4M-1)}_{\alpha(4M-1)}(S^o_{\theta\cup r}) \subseteq \Psi^\tau_\sigma(S^o_{\theta\cup r})$. The proof that $\tilde{\phi} \in \Theta^\upsilon(S^o_{\theta\cup r})$ follows as in the proof of Lemma 2.2.9 upon noting that

$$|\tilde{\phi}(z)| \lesssim \sup_{t\geq 1}\frac{|\phi_t(z)|}{|\phi(z)|}\int_1^\infty (t|z|)^{-4M\beta-(4N-1)\gamma}\frac{\mathrm{d}t}{t} \lesssim |z|^{-\upsilon}$$

for all $z \in S^o_\theta$.

The last part of the proposition follows because $\mathcal{D}$ satisfies (A2), since that allows us to apply the McIntosh approximation technique, as in the proof of Lemma 2.2.9, with the function defined by $f(z) = 1$ for all $z \in S^o_{\theta\cup r}$. □

Given $\psi \in \Psi(S^o_{\theta\cup r})$ and $\phi \in \Phi(S^o_{\theta\cup r})$, since $\mathcal{D}$ satisfies (A2), Proposition 2.2.14 and Theorem 2.2.18 show that the local quadratic estimate

$$\|u\|_2 \eqsim \|\mathcal{Q}^{\mathcal{D}}_{\psi,\phi}u\|_{L^2_\bullet \oplus L^2} \tag{3.6.2}$$

holds for all $u \in L^2$. There also exists $\tilde{\psi} \in \Psi(S^o_{\theta\cup r})$ and $\tilde{\phi} \in \Theta(S^o_{\theta\cup r})$ such that $\mathcal{S}^{\mathcal{D}}_{\psi,\phi}\mathcal{Q}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}} = I$ on L^2 by Proposition 3.6.1. This shows that $L^2 = \mathcal{S}^{\mathcal{D}}_{\psi,\phi}(L^2_\bullet \oplus L^2)$ with

$$\|u\|_2 \eqsim \inf\{\|U\|_{L^2_\bullet\oplus L^2} : U \in L^2_\bullet \oplus L^2 \text{ and } u = \mathcal{S}^{\mathcal{D}}_{\psi,\phi}U\} \tag{3.6.3}$$

for all $u \in L^2$, since both $\mathcal{S}^{\mathcal{D}}_{\psi,\phi}$ and $\mathcal{Q}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}$ are bounded operators. These characterisations of L^2 help to motivate our definition of the local Hardy spaces. In particular, we define $h^p_{\mathcal{D}}$ by replacing $L^2_\bullet \oplus L^2$ with $t^p \oplus L^p_{\mathscr{Q}}$ in (3.6.2) and (3.6.3), and suitably extending the operators $\mathcal{Q}^{\mathcal{D}}_{\psi,\phi}$ and $\mathcal{S}^{\mathcal{D}}_{\psi,\phi}$.

There is a fundamental difference here from the Hardy spaces H^p_D in [9]. The reproducing formula used to define H^p_D is based on selecting ψ and $\tilde{\psi}$ in $\Psi(S^o_\theta)$ such that

$\int_0^\infty \tilde{\psi}_t(z)\psi_t(z)\frac{\mathrm{d}t}{t} = 1$ for all $z \in S^o_\theta$. The decay of the $\Psi(S^o_\theta)$-class functions near the origin implies that $\int_0^\infty \tilde{\psi}_t(D)\psi_t(D)\frac{\mathrm{d}t}{t} = \mathsf{P}_{\overline{\mathsf{R}(D)}}$, where $\mathsf{P}_{\overline{\mathsf{R}(D)}}$ denotes the projection onto the closure of $\mathsf{R}(D)$, as given by the Hodge decomposition $L^2 = \overline{\mathsf{R}(D)} \oplus \mathsf{N}(D)$. This leads the authors of [9] to define H^2_D to be $\overline{\mathsf{R}(D)}$. Identity (3.6.1), by contrast, holds on a neighbourhood D^o_r of the origin as well as on the bisector S^o_θ, and since the Φ-class functions are nonzero at the origin, we get $\mathcal{S}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}\mathcal{Q}^{\mathcal{D}}_{\psi,\phi} = I$ on all of L^2. The local Hardy spaces are therefore not subject to the null space considerations that one encounters with the Hardy spaces. In fact, we show that $h^2_{\mathcal{D}}$ can be identified with L^2.

We now define an ambient space $h^0_{\mathcal{D}}$ in order to have $h^p_{\mathcal{D}} \subseteq h^0_{\mathcal{D}}$ for all $p \in [1,\infty]$. This requires that we recall the results concerning the spaces $t^1 + \tilde{t}^\infty$ and $L^1_{\mathcal{Q}} + \tilde{L}^\infty_{\mathcal{Q}}$ in Corollaries 3.2.14 and 3.3.11.

Definition 3.6.2. Let $\theta \in (\omega, \pi/2)$, $r > R$ and $\beta > \kappa/2$ such that $r/C_{\mathcal{D}}C_{\theta\cup r} > \lambda/2$. Fix $\eta \in \Psi^\beta_\beta(S^o_{\theta\cup r})$ and $\varphi \in \Phi^\beta(S^o_{\theta\cup r})$ satisfying

$$\int_0^1 \eta_t^2(z)\frac{\mathrm{d}t}{t} + \varphi^2(z) = 1$$

for all $z \in S^o_{\theta\cup r}$. The ambient space $h^0_{\mathcal{D}}$ is defined to be the abstract completion of L^2 under the norm defined by

$$\|u\|_{h^0_{\mathcal{D}}} = \|\mathcal{Q}^{\mathcal{D}}_{\eta,\varphi}u\|_{(t^1+\tilde{t}^\infty)\oplus(L^1_{\mathcal{Q}}+\tilde{L}^\infty_{\mathcal{Q}})}$$

for all $u \in L^2$. This provides an identification of L^2 with a dense subspace of $h^0_{\mathcal{D}}$. The functions η and φ remain fixed for the remainder of this section.

To check that $\|\cdot\|_{h^0_{\mathcal{D}}}$ is a norm on L^2, suppose that $\|u\|_{h^0_{\mathcal{D}}} = 0$ for some $u \in L^2$. It follows that $\mathcal{Q}^{\mathcal{D}}_{\eta,\varphi}u = 0$, and since $\mathcal{Q}^{\mathcal{D}}_{\eta,\varphi}u \in L^2_\bullet \oplus L^2$, the equivalence in (3.6.2) guarantees that $u = 0$, as required.

The following result allows us to define the local Hardy spaces.

Proposition 3.6.3. The operators $\mathcal{Q}^{\mathcal{D}}_{\eta,\varphi}$ and $\mathcal{S}^{\mathcal{D}}_{\eta,\varphi}$ have bounded extensions

$$\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi} : h^0_{\mathcal{D}} \to (t^1 + \tilde{t}^\infty) \oplus (L^1_{\mathcal{Q}} + \tilde{L}^\infty_{\mathcal{Q}})$$

and

$$\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi} : (t^1 + \tilde{t}^\infty) \oplus (L^1_{\mathcal{Q}} + \tilde{L}^\infty_{\mathcal{Q}}) \to h^0_{\mathcal{D}}$$

such that $\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi} = I$ on $h^0_{\mathcal{D}}$, and the restriction of $\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}$ to $t^p \oplus L^p_{\mathcal{Q}}$ for each $p \in [1,\infty)$ and to $\tilde{t}^\infty \oplus \tilde{L}^\infty_{\mathcal{Q}}$ is bounded.

Proof. We immediately have

$$\|\mathcal{Q}^{\mathcal{D}}_{\eta,\varphi}u\|_{(t^1+\tilde{t}^\infty)\oplus(L^1_{\mathcal{Q}}+\tilde{L}^\infty_{\mathcal{Q}})} = \|u\|_{h^0_{\mathcal{D}}}$$

for all $u \in L^2$, and since L^2 is identified with a dense subspace of $h^0_{\mathcal{D}}$, the bounded extension $\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}$ exists.

It follows from Theorem 3.5.2 that $\mathcal{Q}^{\mathcal{D}}_{\eta,\varphi}\mathcal{S}^{\mathcal{D}}_{\eta,\varphi}$ has a bounded extension to $t^p \oplus L^p_{\mathscr{Q}}$ for each $p \in [1,\infty]$, and hence to $\tilde{t}^\infty \oplus \tilde{L}^\infty_{\mathscr{Q}}$ as well. Moreover, the extensions coincide with a single bounded operator

$$\mathcal{P} : (t^1 + \tilde{t}^\infty) \oplus (L^1_{\mathscr{Q}} + \tilde{L}^\infty_{\mathscr{Q}}) \to (t^1 + \tilde{t}^\infty) \oplus (L^1_{\mathscr{Q}} + \tilde{L}^\infty_{\mathscr{Q}})$$

such that the restriction of $\mathcal{P}$ to $t^p \oplus L^p_{\mathscr{Q}}$ coincides with the extension of $\mathcal{Q}^{\mathcal{D}}_{\eta,\varphi}\mathcal{S}^{\mathcal{D}}_{\eta,\varphi}$ to $t^p \oplus L^p_{\mathscr{Q}}$ for each $p \in [1,\infty)$, and the restriction of $\mathcal{P}$ to $\tilde{t}^\infty \oplus \tilde{L}^\infty_{\mathscr{Q}}$ coincides with the extension of $\mathcal{Q}^{\mathcal{D}}_{\eta,\varphi}\mathcal{S}^{\mathcal{D}}_{\eta,\varphi}$ to $\tilde{t}^\infty \oplus \tilde{L}^\infty_{\mathscr{Q}}$. Therefore, we have

$$\|\mathcal{S}^{\mathcal{D}}_{\eta,\varphi}U\|_{h^0_{\mathcal{D}}} = \|\mathcal{P}U\|_{(t^1+\tilde{t}^\infty)\oplus(L^1_{\mathscr{Q}}+\tilde{L}^\infty_{\mathscr{Q}})} \lesssim \|U\|_{(t^1+\tilde{t}^\infty)\oplus(L^1_{\mathscr{Q}}+\tilde{L}^\infty_{\mathscr{Q}})}$$

for all $U \in t^2 \oplus L^2_{\mathscr{Q}}$, and since $t^2 \oplus L^2_{\mathscr{Q}}$ is dense in $(t^1 + \tilde{t}^\infty) \oplus (L^1_{\mathscr{Q}} + \tilde{L}^\infty_{\mathscr{Q}})$ by Corollaries 3.2.14 and 3.3.11, the bounded extension $\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}$ exists.

It follows that $\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}$ is bounded on $h^0_{\mathcal{D}}$. The formula $\mathcal{S}^{\mathcal{D}}_{\eta,\varphi}\mathcal{Q}^{\mathcal{D}}_{\eta,\varphi} = I$ holds on L^2 by Proposition 3.6.1, so by density $\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi} = I$ on $h^0_{\mathcal{D}}$.

It also follows that $\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}$ is bounded on $(t^1 + \tilde{t}^\infty) \oplus (L^1_{\mathscr{Q}} + \tilde{L}^\infty_{\mathscr{Q}})$, and that $\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi} = \mathcal{P}$ on $(t^p \cap t^2) \oplus (L^p_{\mathscr{Q}} \cap L^2_{\mathscr{Q}})$ for $p \in [1,\infty)$, and on $(\tilde{t}^\infty \cap t^2) \oplus (\tilde{L}^\infty_{\mathscr{Q}} \cap L^2_{\mathscr{Q}})$. Now suppose that $p \in [1,\infty)$ and that $u \in t^p \oplus L^p_{\mathscr{Q}}$. There exists a sequence $(u_n)_n$ in $(t^p \cap t^2) \oplus (L^p_{\mathscr{Q}} \cap L^2_{\mathscr{Q}})$ that converges to u in $t^p \oplus L^p_{\mathscr{Q}}$ by Propositions 3.2.3 and 3.3.4. The continuity of the embedding $t^p \oplus L^p_{\mathscr{Q}} \subseteq (t^1 + \tilde{t}^\infty) \oplus (L^1_{\mathscr{Q}} + \tilde{L}^\infty_{\mathscr{Q}})$, which is a consequence of the interpolation in Corollaries 3.2.14 and 3.3.11, then implies that

$$\begin{aligned}\|\mathcal{P}u - &\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}u\|_{(t^1+\tilde{t}^\infty)\oplus(L^1_{\mathscr{Q}}+\tilde{L}^\infty_{\mathscr{Q}})} \\ &\leq \|\mathcal{P}(u-u_n)\|_{t^p\oplus L^p_{\mathscr{Q}}} + \|\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}(u_n - u)\|_{(t^1+\tilde{t}^\infty)\oplus(L^1_{\mathscr{Q}}+\tilde{L}^\infty_{\mathscr{Q}})}\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, we have $\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi} = \mathcal{P}$ on $t^p \oplus L^p_{\mathscr{Q}}$ for all $p \in [1,\infty)$. We also have $\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi} = \mathcal{P}$ on $\tilde{t}^\infty \oplus \tilde{L}^\infty_{\mathscr{Q}}$ by the density properties in Corollaries 3.2.14 and 3.3.11, so the result follows. □

We now define the local Hardy spaces.

Definition 3.6.4. For each $p \in [1,\infty)$, the *local Hardy space* $h^p_{\mathcal{D}}$ consists of all $u \in h^0_{\mathcal{D}}$ with

$$\|u\|_{h^p_{\mathcal{D}}} = \|\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}u\|_{t^p\oplus L^p_{\mathscr{Q}}} < \infty.$$

For $p = \infty$, the *local Hardy space* space $h^\infty_{\mathcal{D}}$ consists of all $u \in h^0_{\mathcal{D}}$ such that $\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}u \in \tilde{t}^\infty \oplus \tilde{L}^\infty_{\mathscr{Q}}$ with

$$\|u\|_{h^\infty_{\mathcal{D}}} = \|\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}u\|_{\tilde{t}^\infty\oplus\tilde{L}^\infty_{\mathscr{Q}}} = \|\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}u\|_{t^\infty\oplus L^\infty_{\mathscr{Q}}}.$$

The dual of $h^1_{\mathcal{D}}$ should be identified with a bmo type space, as in the classical case due to Goldberg in [36]. To construct the ambient space $h^0_{\mathcal{D}}$, however, we used the closed subspace $\tilde{t}^\infty \oplus \tilde{L}^\infty_{\mathscr{Q}}$ of $t^\infty \oplus L^\infty_{\mathscr{Q}}$. This suggests that $h^\infty_{\mathcal{D}}$ can only be identified with a closed subspace of the dual of $h^1_{\mathcal{D}}$. Therefore, we do not denote $h^\infty_{\mathcal{D}}$ by $\mathrm{bmo}_{\mathcal{D}}$ and we postpone the construction of an appropriate $\mathrm{bmo}_{\mathcal{D}}$ space to the sequel. Note that we do identify the dual of $h^p_{\mathcal{D}}$ for all $p \in (1,\infty)$ in Theorem 3.6.10 below.

The local Hardy spaces are Banach spaces for all $p \in [1, \infty]$. To see this, suppose that $p \in [1, \infty)$ and that $(u_n)_n$ is a Cauchy sequence in $h^p_{\mathcal{D}}$. Then there exists v in $t^p \oplus L^p_{\mathscr{Q}}$ such that $\lim_n \|\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi} u_n - v\|_{t^p \oplus L^p_{\mathscr{Q}}} = 0$. Moreover, the embedding $t^p \oplus L^p_{\mathscr{Q}} \subseteq (t^1 + \tilde{t}^\infty) \oplus (L^1_{\mathscr{Q}} + \tilde{L}^\infty_{\mathscr{Q}})$ implies that $\lim_n \|\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi} u_n - v\|_{(t^1+\tilde{t}^\infty)\oplus(L^1_{\mathscr{Q}}+\tilde{L}^\infty_{\mathscr{Q}})} = 0$, and that there exists u in $h^0_{\mathcal{D}}$ such that $\lim_n \|u_n - u\|_{h^0_{\mathcal{D}}} = 0$. Therefore, we have $\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi} u = v \in t^p \oplus L^p_{\mathscr{Q}}$, which implies that $u \in h^p_{\mathcal{D}}$ and that $\lim_n \|u_n - u\|_{h^p_{\mathcal{D}}} = 0$. The proof for $p = \infty$ is the same but with $\tilde{t}^\infty \oplus \tilde{L}^\infty_{\mathscr{Q}}$ instead of $t^p \oplus L^p_{\mathscr{Q}}$.

The definition of the ambient space allowed us to identify L^2 with a dense subspace of $h^0_{\mathcal{D}}$. It now follows from (3.6.2) that $L^2 \subseteq h^2_{\mathcal{D}}$ under this identification. In fact, we have $L^2 = h^2_{\mathcal{D}}$ under this identification by (3.6.3) and the following proposition, which gives an equivalent definition for $h^p_{\mathcal{D}}$.

Proposition 3.6.5. If $p \in [1, \infty)$, then $h^p_{\mathcal{D}} = \tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}(t^p \oplus L^p_{\mathscr{Q}})$ and

$$\|u\|_{h^p_{\mathcal{D}}} \eqsim \inf\{\|U\|_{t^p \oplus L^p_{\mathscr{Q}}} : U \in t^p \oplus L^p_{\mathscr{Q}} \text{ and } u = \tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi} U\}.$$

If $p = \infty$, then the above holds with $\tilde{t}^\infty \oplus \tilde{L}^\infty_{\mathscr{Q}}$ instead of $t^p \oplus L^p_{\mathscr{Q}}$.

Proof. Suppose that $p \in [1, \infty)$. Proposition 3.6.3 shows that $\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi} \tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi} = I$ on $h^0_{\mathcal{D}}$, and that the restricted operators

$$\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi} : h^p_{\mathcal{D}} \to t^p \oplus L^p_{\mathscr{Q}} \quad \text{and} \quad \tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi} : t^p \oplus L^p_{\mathscr{Q}} \to h^p_{\mathcal{D}}$$

are bounded. Therefore, we have $h^p_{\mathcal{D}} = \tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}(t^p \oplus L^p_{\mathscr{Q}})$ with

$$\inf_{\substack{U \in t^p \oplus L^p_{\mathscr{Q}};\\ u = \tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi} U}} \|U\|_{t^p \oplus L^p_{\mathscr{Q}}} \le \|\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi} u\|_{t^p \oplus L^p_{\mathscr{Q}}} = \|u\|_{h^p_{\mathcal{D}}} = \|\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi} V\|_{h^p_{\mathcal{D}}} \le \|V\|_{t^p \oplus L^p_{\mathscr{Q}}}$$

for all $V \in t^p \oplus L^p_{\mathscr{Q}}$ satisfying $u = \tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi} V$.

The proof for $p = \infty$ is the same but with $\tilde{t}^\infty \oplus \tilde{L}^\infty_{\mathscr{Q}}$ instead of $t^p \oplus L^p_{\mathscr{Q}}$. ☐

This leads us to the following density properties of the local Hardy spaces.

Corollary 3.6.6. For all $p \in [1, \infty]$ and $q \in [1, \infty)$, the set $h^p_{\mathcal{D}} \cap h^q_{\mathcal{D}}$ is dense in $h^p_{\mathcal{D}}$. Moreover, for all $p, q \in [1, \infty)$, we have $h^p_{\mathcal{D}} \cap h^q_{\mathcal{D}} = \tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}((t^p \cap t^q) \oplus (L^p_{\mathscr{Q}} \cap L^q_{\mathscr{Q}}))$. This also holds for $p = \infty$ but with $\tilde{t}^\infty$ and $\tilde{L}^\infty_{\mathscr{Q}}$ instead of t^p and $L^p_{\mathscr{Q}}$.

Proof. If $p, q \in [1, \infty)$, then $h^p_{\mathcal{D}} = \tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}(t^p \oplus L^p_{\mathscr{Q}})$ by Proposition 3.6.5, so the density of $h^p_{\mathcal{D}} \cap h^q_{\mathcal{D}}$ in $h^p_{\mathcal{D}}$ follows from the density properties in Propositions 3.2.3 and 3.3.4. If $p = \infty$, then the result follows from the density properties in Corollaries 3.2.14 and 3.3.11.

If $p, q \in [1, \infty)$ and $u \in h^p_{\mathcal{D}} \cap h^q_{\mathcal{D}}$, then by the reproducing formula in Proposition 3.6.3, we have

$$u = \tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi} \tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi} u \in \tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}((t^p \cap t^q) \oplus (L^p_{\mathscr{Q}} \cap L^q_{\mathscr{Q}})),$$

since $\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi} u \in (t^p \cap t^q) \oplus (L^p_{\mathscr{Q}} \cap L^q_{\mathscr{Q}})$. If $p = \infty$, then this holds with $\tilde{t}^\infty$ and $\tilde{L}^\infty_{\mathscr{Q}}$ instead of t^p and $L^p_{\mathscr{Q}}$, which completes the proof. ☐

The interpolation results for the local tent spaces t^p and the spaces $L^p_{\mathscr{Q}}$ allow us to interpolate the local Hardy spaces.

Theorem 3.6.7. *If $\theta \in (0,1)$ and $1 \leq p_0 < p_1 \leq \infty$, then*

$$[h^{p_0}_{\mathcal{D}}, h^{p_1}_{\mathcal{D}}]_\theta = h^{p_\theta}_{\mathcal{D}},$$

where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$ and $[\cdot,\cdot]_\theta$ denotes complex interpolation.

Proof. The interpolation space $[h^{p_0}_{\mathcal{D}}, h^{p_1}_{\mathcal{D}}]_\theta$ is well-defined because it is an immediate consequence of Definition 3.6.4 that $h^p_{\mathcal{D}} \subseteq h^0_{\mathcal{D}}$ for all $p \in [1,\infty]$.

Suppose that $p_1 \in (1,\infty)$. Theorems 3.2.13 and 3.3.10 show that

$$[t^{p_0} \oplus L^{p_0}_{\mathscr{Q}}, t^{p_1} \oplus L^{p_1}_{\mathscr{Q}}]_\theta = t^{p_\theta} \oplus L^{p_\theta}_{\mathscr{Q}}.$$

Proposition 3.6.3 shows that the reproducing formula $\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi} = I$ holds on $h^0_{\mathcal{D}}$, and that the restricted operators

$$\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi} : h^p_{\mathcal{D}} \to t^p \oplus L^p_{\mathscr{Q}} \quad \text{and} \quad \tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi} : t^p \oplus L^p_{\mathscr{Q}} \to h^p_{\mathcal{D}}$$

are bounded for all $p \in [1,\infty)$. It then follows by Theorem 1.2.4 of [68], which concerns the interpolation of spaces related by a retraction, that $\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}$ is an isomorphism from $[h^{p_0}_{\mathcal{D}}, h^{p_1}_{\mathcal{D}}]_\theta$ onto

$$\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}(t^{p_\theta} \oplus L^{p_\theta}_{\mathscr{Q}}) = \tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}(h^{p_\theta}_{\mathcal{D}})$$

in $t^{p_\theta} \oplus L^{p_\theta}_{\mathscr{Q}}$ for all $p_0 \in [1,p_1)$, where the equality is given by Proposition 3.6.5. The reproducing formula then implies that $[h^{p_0}_{\mathcal{D}}, h^{p_1}_{\mathcal{D}}]_\theta = h^{p_\theta}_{\mathcal{D}}$.

The proof for $p_1 = \infty$ is the same but with $\tilde{t}^\infty \oplus \tilde{L}^\infty_{\mathscr{Q}}$ instead of $t^{p_1} \oplus L^{p_1}_{\mathscr{Q}}$, and it relies on Corollaries 3.2.14 and 3.3.11. □

The next result is an application of the interpolation of the local Hardy spaces.

Lemma 3.6.8. Let $\theta \in (\omega, \frac{\pi}{2})$, $r > R$ and $\beta > \kappa/2$ such that $r/C_{\mathcal{D}}C_{\theta\cup r} > \lambda/2$. For each $\psi \in \Psi^\beta(S^o_{\theta\cup r})$, $\tilde{\psi} \in \Psi_\beta(S^o_{\theta\cup r})$, $\phi \in \Theta^\beta(S^o_{\theta\cup r})$ and $\tilde{\phi} \in \Theta(S^o_{\theta\cup r})$, the following hold:

1. The operators $\mathcal{Q}^{\mathcal{D}}_{\psi,\phi}$ and $\mathcal{S}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}$ have bounded extensions $\tilde{\mathcal{Q}}^{\mathcal{D}}_{\psi,\phi} : h^p_{\mathcal{D}} \to t^p \oplus L^p_{\mathscr{Q}}$ and $\tilde{\mathcal{S}}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}} : t^p \oplus L^p_{\mathscr{Q}} \to h^p_{\mathcal{D}}$ for all $p \in [1,2]$.

2. The operators $\mathcal{Q}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}$ and $\mathcal{S}^{\mathcal{D}}_{\psi,\phi}$ have bounded extensions $\tilde{\mathcal{Q}}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}} : h^p_{\mathcal{D}} \to t^p \oplus L^p_{\mathscr{Q}}$ and $\tilde{\mathcal{S}}^{\mathcal{D}}_{\psi,\phi} : t^p \oplus L^p_{\mathscr{Q}} \to h^p_{\mathcal{D}}$ for all $p \in [2,\infty)$. This also holds for all $p \in [2,\infty]$ but with $\tilde{t}^\infty \oplus \tilde{L}^\infty_{\mathscr{Q}}$ instead of $t^\infty \oplus L^\infty_{\mathscr{Q}}$.

Proof. If $u \in h^1_{\mathcal{D}} \cap L^2$, then $\mathcal{Q}^{\mathcal{D}}_{\eta,\varphi}u \in (t^1 \cap t^2) \oplus (L^1_{\mathscr{Q}} \cap L^2_{\mathscr{Q}})$ and $u = \mathcal{S}^{\mathcal{D}}_{\eta,\varphi}\mathcal{Q}^{\mathcal{D}}_{\eta,\varphi}u$, so by Theorem 3.5.2 we have

$$\|\mathcal{Q}^{\mathcal{D}}_{\psi,\phi}u\|_{t^1\oplus L^1_{\mathscr{Q}}} = \|\mathcal{Q}^{\mathcal{D}}_{\psi,\phi}\mathcal{S}^{\mathcal{D}}_{\eta,\varphi}\mathcal{Q}^{\mathcal{D}}_{\eta,\varphi}u\|_{t^1\oplus L^1_{\mathscr{Q}}} \lesssim \|u\|_{h^1_{\mathcal{D}}}.$$

The set $h^1_{\mathcal{D}} \cap L^2$ is dense in $h^1_{\mathcal{D}}$ by Corollary 3.6.6, so the bounded extension $\tilde{\mathcal{Q}}^{\mathcal{D}}_{\psi,\phi}$ exists for $p = 1$, and hence for all $p \in [1,2]$ by interpolation.

If $U \in (t^1 \cap t^2) \oplus (L^1_{\mathscr{Q}} \cap L^2_{\mathscr{Q}})$, then by Theorem 3.5.2 we have

$$\|\mathcal{S}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}} U\|_{h^1_{\mathcal{D}}} = \|\mathcal{Q}^{\mathcal{D}}_{\eta,\varphi}\mathcal{S}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}} U\|_{t^1 \oplus L^1_{\mathscr{Q}}} \lesssim \|U\|_{t^1\oplus L^1_{\mathscr{Q}}}.$$

The density properties in Propositions 3.2.3 and 3.3.4 then imply that the bounded extension $\tilde{\mathcal{S}}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}$ exists for $p = 1$, and hence for all $p \in [1, 2]$ by interpolation. $\square$

This allows us to construct a family of equivalent norms on the local Hardy spaces.

Proposition 3.6.9. Let $\theta \in (\omega, \frac{\pi}{2})$, $r > R$ and $\beta > \kappa/2$ so that $r/C_{\mathcal{D}}C_{\theta\cup r} > \lambda/2$. For each $\psi \in \Psi^\beta(S^o_{\theta\cup r})$, $\tilde{\psi} \in \Psi_\beta(S^o_{\theta\cup r})$, $\phi \in \Phi^\beta(S^o_{\theta\cup r})$ and $\tilde{\phi} \in \Phi(S^o_{\theta\cup r})$, the following hold:

1. The extension operators from Lemma 3.6.8 satisfy $h^p_{\mathcal{D}} = \tilde{\mathcal{S}}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}(t^p \oplus L^p_{\mathscr{Q}})$ and

$$\|u\|_{h^p_{\mathcal{D}}} \eqsim \|\tilde{\mathcal{Q}}^{\mathcal{D}}_{\psi,\phi} u\|_{t^p\oplus L^p_{\mathscr{Q}}} \eqsim \inf_{u=\tilde{\mathcal{S}}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}U} \|U\|_{t^p\oplus L^p_{\mathscr{Q}}}$$

   for all $u \in h^p_{\mathcal{D}}$ and $p \in [1, 2]$.

2. The extension operators from Lemma 3.6.8 satisfy $h^p_{\mathcal{D}} = \tilde{\mathcal{S}}^{\mathcal{D}}_{\psi,\phi}(t^p \oplus L^p_{\mathscr{Q}})$ and

$$\|u\|_{h^p_{\mathcal{D}}} \eqsim \|\tilde{\mathcal{Q}}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}} u\|_{t^p\oplus L^p_{\mathscr{Q}}} \eqsim \inf_{u=\tilde{\mathcal{S}}^{\mathcal{D}}_{\psi,\phi}U} \|U\|_{t^p\oplus L^p_{\mathscr{Q}}}$$

   for all $u \in h^p_{\mathcal{D}}$ and $p \in [2, \infty)$. This also holds for all $p \in [2, \infty]$ but with $\tilde{t}^\infty \oplus \tilde{L}^\infty_{\mathscr{Q}}$ instead of $t^\infty \oplus L^\infty_{\mathscr{Q}}$.

Proof. Suppose that $p \in [1, 2]$. It follows from Proposition 3.6.1 that there exists $\psi' \in \Psi_\beta(S^o_{\theta\cup r})$ and $\phi' \in \Theta(S^o_{\theta\cup r})$ such that $\mathcal{S}^{\mathcal{D}}_{\psi',\phi'}\mathcal{Q}^{\mathcal{D}}_{\psi,\phi} = I$ on L^2. Lemma 3.6.8 then shows that

$$\|u\|_{h^p_{\mathcal{D}}} = \|\mathcal{Q}^{\mathcal{D}}_{\eta,\varphi}\mathcal{S}^{\mathcal{D}}_{\psi',\phi'}\mathcal{Q}^{\mathcal{D}}_{\psi,\phi} u\|_{t^p\oplus L^p_{\mathscr{Q}}} \lesssim \|\mathcal{Q}^{\mathcal{D}}_{\psi,\phi} u\|_{t^p\oplus L^p_{\mathscr{Q}}} \lesssim \|u\|_{h^p_{\mathcal{D}}}$$

for all $u \in h^p_{\mathcal{D}} \cap L^2$, so by density we have $\|u\|_{h^p_{\mathcal{D}}} \eqsim \|\tilde{\mathcal{Q}}^{\mathcal{D}}_{\psi,\phi} u\|_{t^p\oplus L^p_{\mathscr{Q}}}$ for all $u \in h^p_{\mathcal{D}}$.

There also exists $\tilde{\psi}' \in \Psi^\beta(S^o_{\theta\cup r})$ and $\tilde{\phi}' \in \Theta^\beta(S^o_{\theta\cup r})$ such that $\mathcal{S}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}\mathcal{Q}^{\mathcal{D}}_{\tilde{\psi}',\tilde{\phi}'} = I$ on $h^p_{\mathcal{D}} \cap L^2$, so by density we have $\tilde{\mathcal{S}}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}\tilde{\mathcal{Q}}^{\mathcal{D}}_{\tilde{\psi}',\tilde{\phi}'} = I$ on $h^p_{\mathcal{D}}$. It then follows from Lemma 3.6.8 that $h^p_{\mathcal{D}} = \tilde{\mathcal{S}}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}(t^p \oplus L^p_{\mathscr{Q}})$.

Now suppose that $u \in h^p_{\mathcal{D}}$, in which case $u = \tilde{\mathcal{S}}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}\tilde{\mathcal{Q}}^{\mathcal{D}}_{\tilde{\psi}',\tilde{\phi}'} u$ and there exists V in $t^p \oplus L^p_Q$ such that $u = \tilde{\mathcal{S}}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}} V$ and $\|V\|_{t^p\oplus L^p_Q} \leq 2 \inf_{u=\tilde{\mathcal{S}}^{\mathcal{D}}_{\psi,\phi}U} \|U\|_{t^p\oplus L^p_{\mathscr{Q}}}$. Lemma 3.6.8 then shows that

$$\inf_{u=\tilde{\mathcal{S}}^{\mathcal{D}}_{\psi,\phi}U} \|U\|_{t^p\oplus L^p_{\mathscr{Q}}} \leq \|\tilde{\mathcal{Q}}^{\mathcal{D}}_{\tilde{\psi}',\tilde{\phi}'} u\|_{t^p\oplus L^p_{\mathscr{Q}}} \lesssim \|u\|_{h^p_{\mathcal{D}}} = \|\tilde{\mathcal{S}}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}} V\|_{h^p_{\mathcal{D}}} \leq 2 \inf_{u=\tilde{\mathcal{S}}^{\mathcal{D}}_{\psi,\phi}U} \|U\|_{t^p\oplus L^p_{\mathscr{Q}}},$$

which completes the proof of (1). The proof of (2) is similar. $\square$

All of the equivalent norms on $h^p_{\mathcal{D}}$ are denoted by $\|\cdot\|_{h^p_{\mathcal{D}}}$. As an example, recall the Hodge–Dirac operator D and the Hodge–Laplacian $\Delta = D^2$ from Example 3.4.1. If $\beta > \kappa/2$ and $a > \lambda^2/4$, then by recalling the Φ-class functions listed after Definition 2.2.8, we have

$$\begin{aligned}\|u\|_{h^p_{\mathcal{D}}} &\eqsim \|tDe^{-t\sqrt{\Delta+aI}}u\|_{t^p} + \|e^{-\sqrt{\Delta+aI}}u\|_{L^p_{\mathcal{Q}}}\\ &\eqsim \|t^2\Delta e^{-t^2\Delta}u\|_{t^p} + \|e^{-\Delta}u\|_{L^p_{\mathcal{Q}}}\\ &\eqsim \|tD(t^2\Delta + aI)^{-\beta}u\|_{t^p} + \|(\Delta + aI)^{-\beta}u\|_{L^p_{\mathcal{Q}}}\end{aligned}$$

for all $u \in h^p_{\mathcal{D}}$ and $p \in [1,\infty]$, where the operators are initially defined on L^2 and extended to $h^p_{\mathcal{D}}$.

Finally, the duality results for the local tent spaces t^p and the spaces $L^p_{\mathcal{Q}}$ allow us to derive a duality result for the local Hardy spaces.

Theorem 3.6.10. *If $p \in (1,\infty)$ and $1/p + 1/p' = 1$, then the mapping*

$$v \mapsto \langle u, v\rangle_{h^2_{\mathcal{D}}} = \langle \tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}u, \tilde{\mathcal{Q}}^{\mathcal{D}^*}_{\eta^*,\varphi^*}v\rangle_{L^2_\bullet \oplus L^2}$$

for all $u \in h^p_{\mathcal{D}}$ and $v \in h^{p'}_{\mathcal{D}^*}$, is an isomorphism from $h^{p'}_{\mathcal{D}^*}$ onto the dual $(h^p_{\mathcal{D}})^*$.

Proof. Using Theorems 3.2.12 and 3.3.8, we obtain

$$|\langle \tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}u, \tilde{\mathcal{Q}}^{\mathcal{D}^*}_{\eta^*,\varphi^*}v\rangle_{L^2_\bullet \oplus L^2}| \le \|u\|_{h^p_{\mathcal{D}}}\|v\|_{h^{p'}_{\mathcal{D}^*}}$$

for all $u \in h^p_{\mathcal{D}}$ and $v \in h^{p'}_{\mathcal{D}^*}$, since $\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}u \in t^p \oplus L^p_{\mathcal{Q}}$ by Definition 3.6.4, and $\tilde{\mathcal{Q}}^{\mathcal{D}^*}_{\eta^*,\varphi^*}v$ is in $t^{p'} \oplus L^{p'}_{\mathcal{Q}}$ by Proposition 3.6.9.

Now suppose that $T \in (h^p_{\mathcal{D}})^*$ and define $\tilde{T} \in (t^p \oplus L^p_{\mathcal{Q}})^*$ by

$$\tilde{T}(V) = T(\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}V)$$

for all $V \in t^p \oplus L^p_{\mathcal{Q}}$. It follows from Theorems 3.2.12 and 3.3.8 that there exists $U_T \in t^{p'} \oplus L^{p'}_{\mathcal{Q}}$ such that $\tilde{T}(V) = \langle V, U_T\rangle_{L^2_\bullet \oplus L^2}$ for all $V \in t^p \oplus L^p_{\mathcal{Q}}$. The reproducing formula $\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi} = I$, which is valid on $h^p_{\mathcal{D}}$ by Proposition 3.6.3, then implies that

$$Tu = T(\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}u) = \tilde{T}(\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}u) = \langle \tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}\tilde{\mathcal{S}}^{\mathcal{D}}_{\eta,\varphi}\tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}u, U_T\rangle_{L^2_\bullet \oplus L^2}$$

for all $u \in h^p_{\mathcal{D}}$. If $U_T \in (t^{p'} \cap t^2) \oplus (L^{p'}_{\mathcal{Q}} \cap L^2_{\mathcal{Q}})$, then since $(\mathcal{Q}^{\mathcal{D}}_{\eta,\varphi})^* = \mathcal{S}^{\mathcal{D}^*}_{\eta^*,\varphi^*}$ on $t^2 \oplus L^2_{\mathcal{Q}}$ and $(\mathcal{S}^{\mathcal{D}}_{\eta,\varphi})^* = \mathcal{Q}^{\mathcal{D}^*}_{\eta^*,\varphi^*}$ on L^2, we obtain

$$Tu = \langle \mathcal{Q}^{\mathcal{D}}_{\eta,\varphi}u, \mathcal{Q}^{\mathcal{D}^*}_{\eta^*,\varphi^*}(\mathcal{S}^{\mathcal{D}^*}_{\eta^*,\varphi^*}U_T)\rangle_{L^2_\bullet \oplus L^2}$$

for all $u \in h^p_{\mathcal{D}} \cap L^2$. If $U_T \in t^{p'} \oplus L^{p'}_{\mathcal{Q}}$, then the density properties in Propositions 3.2.3 and 3.3.4 imply that the above result extends to

$$Tu = \langle \tilde{\mathcal{Q}}^{\mathcal{D}}_{\eta,\varphi}u, \tilde{\mathcal{Q}}^{\mathcal{D}^*}_{\eta^*,\varphi^*}(\tilde{\mathcal{S}}^{\mathcal{D}^*}_{\eta^*,\varphi^*}U_T)\rangle_{L^2_\bullet \oplus L^2}$$

for all $u \in h^p_{\mathcal{D}}$, and since $\tilde{\mathcal{S}}^{\mathcal{D}^*}_{\eta^*,\varphi^*}U_T \in h^{p'}_{\mathcal{D}^*}$ by Proposition 3.6.9, the proof is complete. □

3.6.1 Molecular Characterisation

We prove a molecular characterisation of $h^1_{\mathcal{D}}$. The Hardy space H^1_D from [9] is characterised in terms of H^1_D-molecules, which are differential forms a that satisfy $a = D^N b$ for some differential form b and $N \in \mathbb{N}$. In contrast to atoms, molecules are not assumed to be compactly-supported. Instead, the L^2-norms of a and b are concentrated on some ball. The condition $a = D^N b$ is the substitute for the moment condition required of classical atoms. The molecular characterisation of $h^1_{\mathcal{D}}$ proved here involves two different types of molecules, reflecting the atomic characterisation of $h^1(\mathbb{R}^n)$ mentioned in the introduction. The first kind are concentrated on balls of radius less than 1 and are of the type used to characterise H^1_D, whilst the second kind are concentrated on balls of radius larger than 1 and are not required to satisfy a moment condition.

We use the following notation to specify the L^2-norm distribution of molecules.

Notation. Given a ball B in M of radius $r(B) > 0$, let $\mathbf{1}_k(B)$ denote the characteristic function defined by

$$\mathbf{1}_k(B) = \begin{cases} \mathbf{1}_B & \text{if} \quad k = 0; \\ \mathbf{1}_{2^k B \setminus 2^{k-1} B} & \text{if} \quad k = 1, 2, \dots. \end{cases}$$

Definition 3.6.11. Given $N \in \mathbb{N}$ and $q \geq 0$, an *$h^1_{\mathcal{D}}$-molecule of type* (N, q) is a measurable differential form a associated with a ball B in M of radius $r(B) > 0$ such that the following hold:

1. The bound $\|\mathbf{1}_k(B)a\|_2 \leq \exp(-q2^{k-1}r(B))2^{-k}\mu(2^k B)^{-1/2}$ for all $k \geq 0$;
2. If $r(B) < 1$, then there exists a differential form b with $a = \mathcal{D}^N b$ and the bound $\|\mathbf{1}_k(B)b\|_2 \leq r(B)^N \exp(-q2^{k-1}r(B))2^{-k}\mu(2^k B)^{-1/2}$ for all $k \geq 0$.

For all $N \in \mathbb{N}$, $q \geq 0$ and $h^1_{\mathcal{D}}$-molecules a of type (N, q), the uniform bound

$$\|a\|_1 \leq \sum_{k=0}^{\infty} \mu(2^k B)^{\frac{1}{2}} \|\mathbf{1}_k(B)a\|_2 \lesssim 1 \tag{3.6.4}$$

follows from the Cauchy–Schwarz inequality. Moreover, for an $h^1_{\mathcal{D}}$-molecule a of type (N, q) associated with a ball B and a differential form b such that $a = \mathcal{D}^N b$, both a and b are in $L^2 = h^2_{\mathcal{D}} \subseteq h^0_{\mathcal{D}}$ with

$$\|a\|_2 \leq \sum_{k=0}^{\infty} \|\mathbf{1}_k(B)a\|_2 \leq 2e^{-qr(B)/2}\mu(B)^{-\frac{1}{2}} \tag{3.6.5}$$

and

$$\|b\|_2 \leq 2r(B)^N e^{-qr(B)/2}\mu(B)^{-\frac{1}{2}}. \tag{3.6.6}$$

Condition (2) is obviated in Definition 3.6.11 when $r(B) \geq 1$, so we set $N = 0$ in that case. We will see that q is related to the exponential growth parameter λ in $(\mathrm{E}_{\kappa,\lambda})$, and that we can set $q = 0$ when M is doubling, since then $\lambda = 0$. Given $\delta > 1$, note that the results in this section also hold for $h^1_{\mathcal{D}}$-molecules defined by replacing 2^k and 2^{-k} with δ^k and δ^{-k} in Definition 3.6.11.

Definition 3.6.12. Given $N \in \mathbb{N}$ and $q \geq 0$, define $h^1_{\mathcal{D},\mathrm{mol}(N,q)}$ to be the space of all u in $h^0_{\mathcal{D}}$ for which there exist a sequence $(\lambda_j)_j$ in ℓ^1 and a sequence $(a_j)_j$ of $h^1_{\mathcal{D}}$-molecules of type (N,q) such that $\sum_j \lambda_j a_j$ converges to u in $h^0_{\mathcal{D}}$. Moreover, define

$$\|u\|_{h^1_{\mathcal{D},\mathrm{mol}(N,q)}} = \inf\{\|(\lambda_j)_j\|_{\ell^1} : f = \textstyle\sum_j \lambda_j a_j\}$$

for all $u \in h^1_{\mathcal{D},\mathrm{mol}(N,q)}$.

The following is the molecular characterisation of $h^1_{\mathcal{D}}$. Theorem 1.1.3 follows from this result in the case of the Hodge–Dirac operator by Example 3.4.1.

Theorem 3.6.13. *Let $\kappa, \lambda \geq 0$ and suppose that M is a complete Riemannian manifold satisfying* $(\mathrm{E}_{\kappa,\lambda})$. *Suppose that $\mathcal{D}$ is an operator on $L^2(\wedge T^*M)$ satisfying* (A1) – (A3) *from Section 3.4. If $N \in \mathbb{N}$, $N > \kappa/2$ and $q \geq \lambda$, then $h^1_{\mathcal{D}} = h^1_{\mathcal{D},mol(N,q)}$.*

Proof. Fix $N \in \mathbb{N}$ and $q \geq 0$. Let $\tilde{\psi}$ and $\tilde{\phi}$ be the functions from Lemmas 3.6.15 and 3.6.16 below. Suppose that $u \in h^1_{\mathcal{D}} \subseteq h^0_{\mathcal{D}}$. Proposition 3.6.9 then implies that there exists $(V,v) \in t^1 \oplus L^1_{\mathcal{Q}}$ such that $u = \tilde{\mathcal{S}}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}(V,v)$ and $\|(V,v)\|_{t^1\oplus L^1_{\mathcal{Q}}} \lesssim \|u\|_{h^1_{\mathcal{D}}}$. Also, by Theorems 3.2.6 and 3.3.6, there exist a sequence $(A_j)_j$ of t^1-atoms, a sequence $(a_j)_j$ of $L^1_{\mathcal{Q}}$-atoms and two sequences $(\lambda_j)_j$ and $(\tilde{\lambda}_j)_j$ in ℓ^1 such that

$$V = \sum_j \lambda_j A_j \quad \text{and} \quad v = \sum_j \tilde{\lambda}_j a_j, \tag{3.6.7}$$

where these sums converge in t^1 and $L^1_{\mathcal{Q}}$, respectively. Moreover, we can assume that $\|(\lambda_j)_j\|_{\ell^1} \lesssim \|V\|_{t^1}$, $\|(\tilde{\lambda}_j)_j\|_{\ell^1} \lesssim \|v\|_{L^1_{\mathcal{Q}}}$ and, by Remark 3.3.7, that each $L^1_{\mathcal{Q}}$-atom a_j is associated with a ball of radius equal to 1. Therefore, we have

$$u = \sum_j \left(\lambda_j \int_0^1 \tilde{\psi}_t(\mathcal{D}) A_j \frac{\mathrm{d}t}{t} + \tilde{\lambda}_j \tilde{\phi}(\mathcal{D}) a_j \right),$$

where the sum converges in $h^1_{\mathcal{D}}$, and hence also in $h^0_{\mathcal{D}}$, because Proposition 3.6.9 implies that

$$\|u - \textstyle\sum_{j=1}^n \mathcal{S}^{\mathcal{D}}_{\tilde{\psi},\tilde{\phi}}(\lambda_j A_j, \tilde{\lambda}_j a_j)\|_{h^1_{\mathcal{D}}} \lesssim \|V - \sum_{j=1}^n \lambda_j A_j\|_{t^1} + \|v - \sum_{j=1}^n \tilde{\lambda}_j a_j\|_{L^1_{\mathcal{Q}}}$$

for all $n \in \mathbb{N}$. It follows from Lemmas 3.6.15 and 3.6.16 that $u \in h^1_{\mathcal{D},\mathrm{mol}(N,q)}$, and since $\|(\lambda_j)_j\|_{\ell^1} + \|(\tilde{\lambda}_j)_j\|_{\ell^1} \lesssim \|u\|_{h^1_{\mathcal{D}}}$, we have shown that $h^1_{\mathcal{D}} \subseteq h^1_{\mathcal{D},\mathrm{mol}(N,q)}$.

We prove the converse in the case $N \in \mathbb{N}$, $N > \kappa/2$ and $q \geq \lambda$. Let ψ and ϕ be the functions from Lemmas 3.6.17 and 3.6.18 below. Suppose that $u \in h^1_{\mathcal{D},\mathrm{mol}(N,q)} \subseteq h^0_{\mathcal{D}}$. There exist a sequence $(a_j)_j$ of $h^1_{\mathcal{D}}$-molecules of type (N,q) and a sequence $(\lambda_j)_j$ in ℓ^1 such that $\sum_j \lambda_j a_j$ converges to u in $h^0_{\mathcal{D}}$. It follows from Proposition 3.6.9 and Lemmas 3.6.17 and 3.6.18 that $\sum_{j=1}^n \lambda_j a_j$ is in $h^1_{\mathcal{D}}$ with $\|\sum_{j=1}^n \lambda_j a_j\|_{h^1_{\mathcal{D}}} \lesssim \sum_{j=1}^n |\lambda_j|$ for all $n \in \mathbb{N}$. Therefore, there exists v in $h^1_{\mathcal{D}}$ such that $\sum_j \lambda_j a_j$ converges to v in $h^1_{\mathcal{D}}$, and hence also in $h^0_{\mathcal{D}}$. This implies that $u = v \in h^1_{\mathcal{D}}$, so by Proposition 3.6.9 we have

$$\|\tilde{\mathcal{Q}}^{\mathcal{D}}_{\psi,\phi} u - \textstyle\sum_{j=1}^n \lambda_j \mathcal{Q}^{\mathcal{D}}_{\psi,\phi} a_j\|_{t^1\oplus L^1_{\mathcal{Q}}} \lesssim \|u - \sum_{j=1}^n \lambda_j a_j\|_{h^1_{\mathcal{D}}}$$

for all $n \in \mathbb{N}$. It follows from Lemmas 3.6.17 and 3.6.18 that

$$\|u\|_{h^1_{\mathcal{D}}} \eqsim \|\tilde{\mathcal{Q}}^{\mathcal{D}}_{\psi,\phi} u\|_{t^1 \oplus L^1_{\mathscr{Q}}} \leq \sum_j \lambda_j \|(\psi_t(\mathcal{D})a_j, \phi(\mathcal{D})a_j)\|_{t^1 \oplus L^1_{\mathscr{Q}}} \lesssim \|(\lambda_j)_j\|_{\ell^1},$$

which shows that $h^1_{\mathcal{D},\mathrm{mol}(N,q)} \subseteq h^1_{\mathcal{D}}$. □

Remark 3.6.14. The first half of the proof of Theorem 3.6.13 showed that for each $u \in h^1_{\mathcal{D}}$, there exists a sequence $(\lambda_j)_j$ in ℓ^1 and a sequence $(a_j)_j$ of $h^1_{\mathcal{D}}$-molecules of type (N,q) such that $\sum_j \lambda_j a_j$ converges to u in both $h^1_{\mathcal{D}}$ and $h^0_{\mathcal{D}}$. Theorems 3.2.6 and 3.3.6 show that for each $u \in h^1_{\mathcal{D}} \cap h^2_{\mathcal{D}}$, the atomic decompositions in (3.6.7) also converge in t^2 and $L^2_{\mathscr{Q}}$, respectively. In that case, the arguments in the proof show that the series $\sum_j \lambda_j a_j$ converges to u in L^2 as well.

We now prove four lemmas to construct the functions $\tilde{\psi}, \tilde{\phi}, \psi$ and ϕ that were used to prove Theorem 3.6.13.

Lemma 3.6.15. Let $\theta \in (\omega, \frac{\pi}{2})$, $r > R$ and $\beta > \kappa/2$ such that $r/C_{\mathcal{D}}C_{\theta\cup r} > \lambda/2$. For each $N \in \mathbb{N}$ and $q \geq 0$, there exist $c > 0$ and $\tilde{\psi} \in \Psi_\beta(S^o_{\theta\cup r})$ such that

$$c\int_0^1 \tilde{\psi}_t(\mathcal{D})A_t \frac{\mathrm{d}t}{t}$$

is an $h^1_{\mathcal{D}}$-molecule of type (N,q) for all A that are t^1-atoms.

Proof. Let A be a t^1-atom. There exists a ball B in M with radius $r(B) \leq 2$ such that A is supported in $T^1(B)$ and $\|A\|_{L^2_\bullet} \leq \mu(B)^{-1/2}$. Choose $\tilde{r}$ so that $\tilde{r} \geq r$ and $\tilde{r}/C_{\mathcal{D}}C_{\theta\cup\tilde{r}} > \lambda + q$. Also, choose $\tilde{\psi}$ in $\Psi_{\beta+N+1}(S^o_{\theta\cup\tilde{r}})$, in which case $\psi \in \Psi_\beta(S^o_{\theta\cup r})$. Next, define $\tilde{\tilde{\psi}}(z) = z^{-N}\tilde{\psi}(z)$, in which case $\tilde{\tilde{\psi}} \in \Psi_{\beta+1}(S^o_{\theta\cup r})$ and

$$\int_0^1 \tilde{\psi}_t(\mathcal{D})A_t \frac{\mathrm{d}t}{t} = \mathcal{D}^N \left(\int_0^1 t^N \tilde{\tilde{\psi}}_t(\mathcal{D})A_t \frac{\mathrm{d}t}{t}\right).$$

It remains to prove that there exists $c > 0$, which does not depend on A, such that

$$\|\mathbf{1}_k(B)\left(\int_0^1 \tilde{\psi}_t(\mathcal{D})A_t \frac{\mathrm{d}t}{t}\right)\|_2 \leq ce^{-q2^{k-1}r(B)}2^{-k}\mu(2^kB)^{-\frac{1}{2}} \tag{3.6.8}$$

for all $k \geq 0$, and that if $r(B) < 1$, then

$$\|\mathbf{1}_k(B)\left(\int_0^1 t^N \tilde{\tilde{\psi}}_t(\mathcal{D})A_t \frac{\mathrm{d}t}{t}\right)\|_2 \leq cr(B)^N e^{-q2^{k-1}r(B)}2^{-k}\mu(2^kB)^{-\frac{1}{2}} \tag{3.6.9}$$

for all $k \geq 0$.

Now, since $\beta > \kappa/2$ and $\tilde{r}/C_{\mathcal{D}}C_{\theta\cup\tilde{r}} > \lambda + q$, Lemma 3.4.3 implies the following estimates:

$$\|\mathbf{1}_E \tilde{\psi}_t(\mathcal{D})\mathbf{1}_F\| \lesssim \langle t/\rho(E,F)\rangle^{\frac{\kappa}{2}+1} e^{-(\lambda+q)\rho(E,F)}; \tag{3.6.10}$$

$$\|\mathbf{1}_E \tilde{\tilde{\psi}}_t(\mathcal{D})\mathbf{1}_F\| \lesssim \langle t/\rho(E,F)\rangle^{\frac{\kappa}{2}+1} e^{-(\lambda+q)\rho(E,F)} \tag{3.6.11}$$

for all $t \in (0,1]$ and closed subsets E and F of M.

We now prove (3.6.8). If $k = 0$ or $k = 1$, then by (3.6.3) and $r(B) \le 2$, we have

$$\|\mathbf{1}_k(B)\left(\int_0^1 \tilde{\psi}_t(\mathcal{D})A_t \frac{\mathrm{d}t}{t}\right)\|_2 \lesssim \|A\|_{L^2_\bullet} \lesssim \begin{cases} e^{-q2^{-1}r(B)}\mu(B)^{-\frac{1}{2}} & \text{if} \quad k=0; \\ e^{-qr(B)}2^{-1}\mu(2B)^{-\frac{1}{2}} & \text{if} \quad k=1. \end{cases}$$

If $k \ge 2$, then

$$\rho(2^kB \backslash 2^{k-1}B, B) = (2^{k-1}-1)r(B) \gtrsim 2^k r(B)$$

and $\mu(2^kB) \le 2^{k\kappa}e^{\lambda(2^k-1)r(B)}\mu(B)$, so by (3.6.10), and since $r(B) \le 2$, we have

$$\begin{aligned} \|\mathbf{1}_k(B)\left(\int_0^1 \tilde{\psi}_t(\mathcal{D})A_t \frac{\mathrm{d}t}{t}\right)\|_2 &\le \int_0^{r(B)} \|\mathbf{1}_k(B)\tilde{\psi}_t(\mathcal{D})\mathbf{1}_B\|\|A_t\|_2 \frac{\mathrm{d}t}{t} \\ &\lesssim \left(\int_0^{r(B)} \left(\frac{t}{2^k r(B)}\right)^{2(\frac{\kappa}{2}+1)} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}} e^{-(\lambda+q)(2^{k-1}-1)r(B)}\|A\|_{L^2_\bullet} \\ &\le 2^{-k(\frac{\kappa}{2}+1-\frac{\kappa}{2})}e^{-q(2^{k-1}-1)r(B)}e^{\lambda(-2^{k-1}+1+2^{k-1}-\frac{1}{2})r(B)}\mu(2^kB)^{-\frac{1}{2}} \\ &\lesssim e^{-q2^{k-1}r(B)}2^{-k}\mu(2^kB)^{-\frac{1}{2}}. \end{aligned}$$

We prove (3.6.9) similarly. If $k = 0$ or $k = 1$, then we have

$$\begin{aligned} \|\mathbf{1}_k(B)\left(\int_0^1 t^N \tilde{\tilde{\psi}}_t(\mathcal{D})A_t \frac{\mathrm{d}t}{t}\right)\|_2 &\lesssim r(B)^N\|A\|_{L^2_\bullet} \\ &\lesssim r(B)^N \begin{cases} e^{-q2^{-1}r(B)}\mu(B)^{-\frac{1}{2}} & \text{if} \quad k=0; \\ e^{-qr(B)}2^{-1}\mu(2B)^{-\frac{1}{2}} & \text{if} \quad k=1. \end{cases} \end{aligned}$$

If $k \ge 2$, then by (3.6.11) we have

$$\begin{aligned} \|\mathbf{1}_k(B)\left(\int_0^1 t^N \tilde{\tilde{\psi}}_t(\mathcal{D})A_t \frac{\mathrm{d}t}{t}\right)\|_2 &\le r(B)^N \int_0^{r(B)} \|\mathbf{1}_k(B)\tilde{\tilde{\psi}}_t(\mathcal{D})\mathbf{1}_B\|\|A_t\|_2 \frac{\mathrm{d}t}{t} \\ &\lesssim r(B)^N e^{-q2^{k-1}r(B)}2^{-k}\mu(2^kB)^{-\frac{1}{2}}, \end{aligned}$$

which completes the proof. □

Lemma 3.6.16. Let $\theta \in (\omega, \frac{\pi}{2})$, $r > R$ and $\beta > \kappa/2$ such that $r/C_{\mathcal{D}}C_{\theta\cup r} > \lambda/2$. For each $N \in \mathbb{N}$ and $q \ge 0$, there exist $c > 0$ and $\tilde{\phi} \in \Phi(S^o_{\theta\cup r})$ such that $c\tilde{\phi}(\mathcal{D})a$ is an $h^1_{\mathcal{D}}$-molecule of type (N, q) for all a that are $L^1_{\mathscr{Q}}$-atoms supported on balls B of radius $r(B) = 1$ with $\|a\|_2 \le \mu(B)^{-1/2}$.

Proof. Let a and B be as stated in the lemma. Choose $\tilde{r}$ so that $\tilde{r} \ge r$ and $\tilde{r}/C_{\mathcal{D}}C_{\theta\cup\tilde{r}} > \lambda + q$. Also, choose $\tilde{\phi}$ in $\Phi(S^o_{\theta\cup\tilde{r}})$, in which case $\tilde{\phi} \in \Phi(S^o_{\theta\cup r})$. Now, since $r(B) = 1$, it only remains to prove that there exists $c > 0$, which does not depend on a, such that

$$\|\mathbf{1}_k(B)\tilde{\phi}(\mathcal{D})a\|_2 \le ce^{-q2^{k-1}r(B)}2^{-k}\mu(2^kB)^{-\frac{1}{2}}$$

for all $k \ge 0$. To do this, choose δ in $(0, \tilde{r}/C_{\mathcal{D}}C_{\theta\cup\tilde{r}} - (\lambda + q))$. Lemma 3.4.3 then implies that

$$\|\mathbf{1}_E\tilde{\phi}(\mathcal{D})\mathbf{1}_F\| \lesssim e^{-(\lambda+q+\delta)\rho(E,F)} \lesssim \langle 1/\rho(E,F)\rangle^{\frac{\kappa}{2}+1}e^{-(\lambda+q)\rho(E,F)} \tag{3.6.12}$$

for all closed subsets E and F of M.

If $k=0$ or $k=1$, then by (3.6.3), and since $r(B)=1$, we have

$$\|\mathbf{1}_k(B)\tilde{\phi}(\mathcal{D})a\|_2 \lesssim \|a\|_2 \leq \mu(B)^{-\frac{1}{2}} \lesssim \begin{cases} e^{-q2^{-1}r(B)}\mu(B)^{-\frac{1}{2}} & \text{if} \quad k=0; \\ e^{-qr(B)}2^{-1}\mu(2B)^{-\frac{1}{2}} & \text{if} \quad k=1. \end{cases}$$

If $k \geq 2$, then using (3.6.12) and proceeding as in Lemma 3.6.15, we obtain

$$\|\mathbf{1}_k(B)\tilde{\phi}(\mathcal{D})a\|_2 \leq \|\mathbf{1}_k(B)\tilde{\phi}(\mathcal{D})\mathbf{1}_B\|\|a\|_2 \lesssim e^{-q2^{k-1}r(B)}2^{-k}\mu(2^kB)^{-\frac{1}{2}},$$

which completes the proof. □

Lemma 3.6.17. Let $\theta \in (\omega, \frac{\pi}{2})$, $r > R$ and $\beta > \kappa/2$ such that $r/C_{\mathcal{D}}C_{\theta\cup r} > \lambda/2$. For each $N \in \mathbb{N}$, $N > \kappa/2$ and $q \geq \lambda$, there exist $c > 0$ and $\psi \in \Psi^\beta(S^o_{\theta\cup r})$ such that $\|\psi_t(\mathcal{D})a\|_{t^1} \leq c$ for all a that are $h^1_{\mathcal{D}}$-molecules of type (N,q).

Proof. Let a be an $h^1_{\mathcal{D}}$-molecule of type (N,q). There exists a ball B in M of radius $r(B) > 0$ such that the requirements of Definition 3.6.11 are satisfied. Let $C^1_0(B) = C^1(B)$ be the truncated Carleson box over B introduced in Section 3.2, and let $C^1_k(B) = C^1(2^kB) \setminus C^1(2^{k-1}B)$ for each $k \geq 1$. As depicted in Figure 3.1, divide each $C^1_k(B)$ with the following characteristic functions:

$$\eta_k = \mathbf{1}_{C^1_k(B)}\mathbf{1}_{M\times(0,r(B)]};$$
$$\eta'_k = \mathbf{1}_{C^1_k(B)}\mathbf{1}_{M\times(r(B),2^{k-1}r(B)]};$$
$$\eta''_k = \mathbf{1}_{C^1_k(B)}\mathbf{1}_{M\times(2^{k-1}r(B),2^kr(B)]},$$

so we have $\mathbf{1}_{C^1_k(B)} = \eta_k + \eta'_k + \eta''_k$ and $\sum_k \mathbf{1}_{C^1_k(B)} = \mathbf{1}_{M\times(0,1]}$.

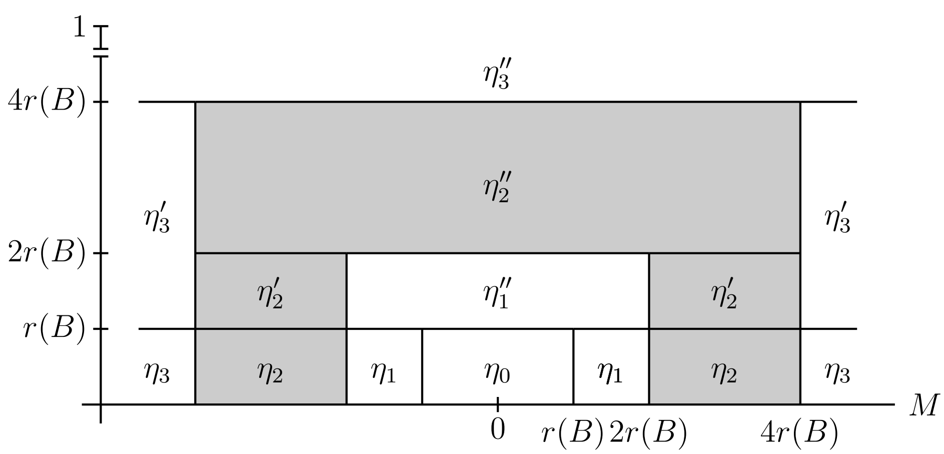

FIGURE 3.1: The division of $C^1_2(B)$ used in Lemma 3.6.17 for a ball B in M of radius $r(B) < 1/4$.

Suppose that there exist $\psi \in \Psi^\beta(S^o_{\theta\cup r})$ and $c, \delta > 0$, all of which do not depend on a, such that the following hold for all $k \geq 0$:

$$\|\eta_k\psi_t(\mathcal{D})a\|_{L^2_\bullet} \leq c2^{-\delta k}\mu(2^kB)^{-\frac{1}{2}}; \tag{3.6.13a}$$
$$\|\eta'_k\psi_t(\mathcal{D})a\|_{L^2_\bullet} \leq c2^{-\delta k}\mu(2^kB)^{-\frac{1}{2}}; \tag{3.6.13b}$$
$$\|\eta''_k\psi_t(\mathcal{D})a\|_{L^2_\bullet} \leq c2^{-\delta k}\mu(2^kB)^{-\frac{1}{2}}. \tag{3.6.13c}$$

In that case, each $(2^{\delta k}/c)\mathbf{1}_{C^1_k(B)}\psi_t(\mathcal{D})a$ is a t^1-Carleson atom, and since

$$\psi_t(\mathcal{D})a = \sum_{k=0}^{\infty} \mathbf{1}_{C^1_k(B)}\psi_t(\mathcal{D})a$$

almost everywhere in $M \times (0,1]$, Proposition 3.2.11 implies that $\psi_t(\mathcal{D})a$ is in t^1 with $\|\psi_t(\mathcal{D})a\|_{t^1} \lesssim c\sum_{k=0}^{\infty} 2^{-\delta k} \lesssim 1$. Therefore, it suffices to prove (3.6.13).

To prove (3.6.13), choose $\tilde{r}$ so that $\tilde{r} \geq r$ and $\tilde{r}/C_{\mathcal{D}}C_{\theta\cup\tilde{r}} > \lambda$. Also, choose δ in $(0, \beta - \kappa/2)$ and choose ψ in $\Psi_\beta^{\beta+N}(S^o_{\theta\cup\tilde{r}})$, in which case $\psi \in \Psi^\beta(S^o_{\theta\cup r})$. Then, since $\beta > \kappa/2$, Lemma 3.4.3 implies that

$$\|\mathbf{1}_E\psi_t(\mathcal{D})\mathbf{1}_F\| \lesssim \langle t/\rho(E,F)\rangle^{\frac{\kappa}{2}+\delta} e^{-\lambda\rho(E,F)} \leq \langle t/\rho(E,F)\rangle^{\delta} \tag{3.6.14}$$

for all closed subsets E and F of M.

We now prove (3.6.13a). If $k = 0$, then by (3.6.2) and (3.6.5) we have

$$\|\eta_0\psi_t(\mathcal{D})a\|_{L^2_\bullet} \leq \|\psi_t(\mathcal{D})a\|_{L^2_\bullet} \lesssim \|a\|_2 \lesssim \mu(B)^{-\frac{1}{2}}.$$

Now consider $k \geq 1$. For each $l \in \mathbb{N}$, define I_l by

$$\|\eta_k\psi_t(\mathcal{D})a\|^2_{L^2_\bullet} \leq \sum_{l=0}^{\infty}\int_0^{\langle r(B)\rangle} \|\mathbf{1}_k(B)\psi_t(\mathcal{D})\mathbf{1}_l(B)a\|_2^2 \,\frac{\mathrm{d}t}{t} = \sum_{l=0}^{\infty} I_l.$$

If $0 \leq l \leq k-2$, then

$$\rho(2^kB\backslash 2^{k-1}B, 2^lB\backslash 2^{l-1}B) = (2^{k-1} - 2^l)r(B) \gtrsim 2^k r(B)$$

and $\mu(2^kB) \leq 2^{(k-l)\kappa}e^{\lambda(2^{k-l}-1)2^l r(B)}\mu(2^lB)$, so by (3.6.14) we have

$$\begin{aligned} I_l &\lesssim \int_0^{r(B)} \left(\frac{t}{2^k r(B)}\right)^{2(\frac{\kappa}{2}+\delta)} \frac{\mathrm{d}t}{t} e^{-2\lambda(2^{k-1}-2^l)r(B)} e^{-q2^l r(B)} 2^{-2l}\mu(2^lB)^{-1} \\ &\lesssim 2^{-2l(\frac{\kappa}{2}+1)}2^{-2k(\frac{\kappa}{2}+\delta-\frac{\kappa}{2})}e^{\lambda(-2^k+2^{l+1}+2^k-2^l-2^l)r(B)}\mu(2^kB)^{-1} \\ &\lesssim 2^{-2l}2^{-2\delta k}\mu(2^kB)^{-1}. \end{aligned}$$

If $k-1 \leq l \leq k+1$, then $\mu(2^kB) \lesssim e^{\lambda 2^l r}\mu(2^lB)$, so we have

$$I_l \leq \|\psi_t(\mathcal{D})\mathbf{1}_l(B)a\|^2_{L^2_\bullet} \lesssim e^{-q2^l r(B)}2^{-2l}\mu(2^lB)^{-1} \lesssim 2^{-2l}\mu(2^kB)^{-1}.$$

If $l \geq k+2$, then

$$\rho(2^kB\backslash 2^{k-1}B, 2^lB\backslash 2^{l-1}B) = (2^{l-1} - 2^k)r(B) \gtrsim 2^l r(B)$$

and $\mu(2^kB) \leq \mu(2^lB)$, so by (3.6.14) we have

$$I_l \lesssim \int_0^{r(B)} \left(\frac{t}{2^l r(B)}\right)^{2\delta} \frac{\mathrm{d}t}{t} 2^{-2l}\mu(2^lB)^{-1} \lesssim 2^{-2l}2^{-2\delta k}\mu(2^kB)^{-1}.$$

Note that we needed $q \geq \lambda$ when $0 \leq l \leq k+1$. This proves (3.6.13a), since now

$$\|\eta_k \psi_t(\mathcal{D})a\|_{L^2_\bullet}^2 \leq \sum_{l=0}^{\infty} I_l \lesssim \sum_{l=0}^{\infty} 2^{-2l}2^{-2\delta k}\mu(2^k B)^{-1} \lesssim 2^{-2\delta k}\mu(2^k B)^{-1}.$$

To prove (3.6.13b) and (3.6.13c) we only need to consider when $r(B) < 1$. In that case, there exists a differential form b such that $a = \mathcal{D}^N b$, as in Definition 3.6.11. Define $\tilde{\psi}(z) = z^N \psi(z)$, in which case $\tilde{\psi} \in \Psi_N(S^o_{\theta\cup\tilde{r}})$, where $\tilde{r} \geq r$ was fixed previously so that $\tilde{r}/C_{\mathcal{D}}C_{\theta\cup\tilde{r}} > \lambda$. Now choose ϵ in $(0, N-\kappa/2)$. Then, since $N > \kappa/2$, Lemma 3.4.3 implies that

$$\|\mathbf{1}_E \tilde{\psi}_t(\mathcal{D})\mathbf{1}_F\| \lesssim \langle t/\rho(E,F)\rangle^{\frac{\kappa}{2}+\epsilon} e^{-\lambda\rho(E,F)} \lesssim \langle t/\rho(E,F)\rangle^{\epsilon} \tag{3.6.15}$$

for all closed subsets E and F of M.

To prove (3.6.13b), we only consider $k \geq 2$, since otherwise $\eta'_k = 0$. For each $l \in \mathbb{N}$, define J_l by

$$\|\eta'_k \psi_t(\mathcal{D})a\|_{L^2_\bullet}^2 \leq \sum_{l=0}^{\infty} \int_{r(B)}^{\langle 2^{k-1}r(B)\rangle} \|\mathbf{1}_k(B)\tilde{\psi}_t(\mathcal{D})\mathbf{1}_l(B)b\|_2^2 \,\frac{\mathrm{d}t}{t^{2N+1}} = \sum_{l=0}^{\infty} J_l.$$

The proof proceeds as for I_l by using (3.6.15) instead of (3.6.14). If $0 \leq l \leq k-2$, then since $N - \kappa/2 - \epsilon > 0$ and $r(B) < 1$, we have

$$\begin{aligned} J_l &\lesssim \int_{r(B)}^{1} \left(\frac{t}{2^k r(B)}\right)^{2(\frac{\kappa}{2}+\epsilon)} \frac{\mathrm{d}t}{t^{2N+1}} e^{-2\lambda(2^{k-1}-2^l)r(B)} r(B)^{2N} e^{-q2^l r(B)} 2^{-2l}\mu(2^l B)^{-1} \\ &\lesssim r(B)^{2(N-\frac{\kappa}{2}-\epsilon)} \int_{r(B)}^{1} t^{-2(N-\frac{\kappa}{2}-\epsilon)}\frac{\mathrm{d}t}{t} 2^{-2l(\frac{\kappa}{2}+1)} 2^{-2k(\frac{\kappa}{2}+\epsilon-\frac{\kappa}{2})}\mu(2^k B)^{-1} \\ &\lesssim 2^{-2l}2^{-2\epsilon k}\mu(2^k B)^{-1}. \end{aligned}$$

If $k-1 \leq l \leq k+1$, then since $r(B) < 1$, we have

$$J_l \leq r(B)^{-2N}\|\tilde{\psi}_t(\mathcal{D})\mathbf{1}_l(B)b\|_{L^2_\bullet}^2 \lesssim e^{-q2^l r(B)} 2^{-2l}\mu(2^l B)^{-1} \lesssim 2^{-2l}\mu(2^k B)^{-1}.$$

If $l \geq k+2$, then since $N - \epsilon > 0$ and $r(B) < 1$, we have

$$\begin{aligned} J_l &\lesssim \int_{r(B)}^{1} \left(\frac{t}{2^l r(B)}\right)^{2\epsilon} \frac{\mathrm{d}t}{t^{2N+1}} r(B)^{2N} 2^{-2l}\mu(2^l B)^{-1} \\ &\leq r(B)^{2(N-\epsilon)} \int_{r(B)}^{1} t^{-2(N-\epsilon)}\frac{\mathrm{d}t}{t} 2^{-2l}2^{-2\epsilon k}\mu(2^k B)^{-1} \\ &\lesssim 2^{-2l}2^{-2\epsilon k}\mu(2^k B)^{-1}. \end{aligned}$$

Note that we needed $q \geq \lambda$ when $0 \leq l \leq k+1$. This proves (3.6.13b), since now $\|\eta'_k \psi_t(\mathcal{D})a\|_{L^2_\bullet}^2 \leq \sum_{l=0}^{\infty} J_l \lesssim 2^{-2\epsilon k}\mu(2^k B)^{-1}$.

To prove (3.6.13c), we only consider $k \geq 1$ for which $2^{k-1}r(B) < 1$, since otherwise $\eta''_k = 0$. For each $l \in \mathbb{N}$, define K_l by

$$\|\eta''_k \psi_t(\mathcal{D})a\|_{L^2_\bullet}^2 \leq \sum_{l=0}^{\infty} \int_{2^{k-1}r(B)}^{\langle 2^k r(B)\rangle} \|\mathbf{1}_{2^k B}\tilde{\psi}_t(\mathcal{D})\mathbf{1}_l(B)b\|_2^2 \,\frac{\mathrm{d}t}{t^{2N+1}} = \sum_{l=0}^{\infty} K_l.$$

The proof proceeds as for J_l. In fact, we only require the weaker estimate obtained by setting $\epsilon = 0$ in (3.6.15). If $0 \leq l \leq k+2$, then $\mu(2^k B) \lesssim 2^{(k-l)\kappa}\mu(2^l B)$, since $2^{k-1}r(B) < 1$, so we have

$$K_l \lesssim (2^k r(B))^{-2N} \|\tilde{\psi}_t(\mathcal{D})\mathbf{1}_l(B)b\|^2_{L^2_\bullet} \leq 2^{-2l(\frac{\kappa}{2}+1)}2^{-2k(N-\frac{\kappa}{2})}\mu(2^k B)^{-1}.$$

If $l \geq k+2$, then we have

$$K_l \lesssim 2^{-2l(\frac{\kappa}{2}+1)} \int_{2^{k-1}r(B)}^{2^k r(B)} \left(\frac{r(B)}{t}\right)^{2(N-\frac{\kappa}{2})} \frac{\mathrm{d}t}{t}\mu(2^l B)^{-1} \leq 2^{-2l}2^{-2k(N-\frac{\kappa}{2})}\mu(2^k B)^{-1}.$$

Note that we did not require $q \geq \lambda$ here. This proves (3.6.13c), since $N > \kappa/2$ and now $\|\eta''_k \psi_t(\mathcal{D})a\|^2_{L^2_\bullet} \leq \sum_{l=0}^\infty K_l \lesssim 2^{-2(N-\frac{\kappa}{2})k}\mu(2^k B)^{-1}$. □

Lemma 3.6.18. Let $\theta \in (\omega, \frac{\pi}{2})$, $r > R$ and $\beta > \kappa/2$ such that $r/C_{\mathcal{D}}C_{\theta\cup r} > \lambda/2$. For each $N \in \mathbb{N}$, $N > \kappa/2$ and $q \geq \lambda$, there exist $c > 0$ and $\phi \in \Phi^\beta(S^o_{\theta\cup r})$ such that $\|\phi(\mathcal{D})a\|_{L^1_{\mathscr{Q}}} \leq c$ for all a that are $h^1_{\mathcal{D}}$-molecules of type (N, q).

Proof. Let a be an $h^1_{\mathcal{D}}$-molecule of type (N, q). There exists a ball B in M of radius $r(B) > 0$ such that the requirements of Definition 3.6.11 are satisfied. Let $B^* = (1/\langle r(B)\rangle)B$, so the radius $r(B^*) \geq 1$.

Suppose that there exist $\phi \in \Phi^\beta(S^o_{\theta\cup r})$ and $c, \delta > 0$, all of which do not depend on a, such that

$$\|\mathbf{1}_k(B^*)\phi(\mathcal{D})a\|_2 \leq c2^{-\delta k}\mu(2^k B^*)^{-\frac{1}{2}} \tag{3.6.16}$$

for all $k \geq 0$. In that case, each $(2^{\delta k}/c)\mathbf{1}_k(B^*)\phi(\mathcal{D})a$ is an $L^1_{\mathscr{Q}}$-atom, and since

$$\phi(\mathcal{D})a = \sum_{k=0}^\infty \mathbf{1}_k(B^*)\phi(\mathcal{D})a$$

almost everywhere on M, Theorem 3.3.6 implies that $\|\phi(\mathcal{D})a\|_{L^1_{\mathscr{Q}}} \lesssim c\sum_{k=0}^\infty 2^{-\delta k}$. Therefore, it suffices to prove (3.6.16).

To prove (3.6.16), choose $\tilde{r}$ so that $\tilde{r} \geq r$ and $\tilde{r}/C_{\mathcal{D}}C_{\theta\cup\tilde{r}} > \lambda$. Also, choose δ in $(0, \tilde{r}/C_{\mathcal{D}}C_{\theta\cup\tilde{r}} - \lambda)$ and choose ϕ in $\Phi^{\beta+N}(S^o_{\theta\cup\tilde{r}})$, in which case $\phi \in \Phi^\beta(S^o_{\theta\cup r})$. Lemma 3.4.3 then implies that

$$\|\mathbf{1}_E\phi(\mathcal{D})\mathbf{1}_F\| \lesssim e^{-(\lambda+\delta)\rho(E,F)} \lesssim \langle 1/\rho(E,F)\rangle^{\frac{\kappa}{2}+1}e^{-\lambda\rho(E,F)} \leq \langle 1/\rho(E,F)\rangle \tag{3.6.17}$$

for all closed subsets E and F of M.

We now prove (3.6.16) when $r(B) \geq 1$, in which case $B^* = B$. If $k = 0$, then by (3.6.2) and (3.6.5) we have

$$\|\mathbf{1}_0(B)\phi(\mathcal{D})a\|_2 \leq \|\phi(\mathcal{D})a\|_2 \lesssim \|a\|_2 \lesssim \mu(B)^{-\frac{1}{2}}.$$

Now consider $k \geq 1$ and for each $l \in \mathbb{N}$, define I'_l by

$$\|\mathbf{1}_k(B)\phi(\mathcal{D})a\|^2_2 \leq \sum_{l=0}^\infty \|\mathbf{1}_k(B)\phi(\mathcal{D})\mathbf{1}_l(B)a\|^2_2 = \sum_{l=0}^\infty I'_l.$$

The proof proceeds as for I_l in Lemma 3.6.17 by using (3.6.17) instead of (3.6.14). If $0 \leq l \leq k-2$, then since $r(B) \geq 1$, we have

$$I_l' \lesssim \left(\frac{1}{2^k r(B)}\right)^{2(\frac{\kappa}{2}+1)} e^{-2\lambda(2^{k-1}-2^l)r(B)} e^{-q2^l r} 2^{-2l} \mu(2^l B)^{-1} \lesssim 2^{-2l} 2^{-2k} \mu(2^k B)^{-1}.$$

If $k-1 \leq l \leq k+1$, then we have

$$I_l' \leq \|\phi(\mathcal{D})\mathbf{1}_l(B)a\|_2^2 \lesssim e^{-q2^l r(B)} 2^{-2l} \mu(2^l B)^{-1} \lesssim 2^{-2l} \mu(2^k B)^{-1}.$$

If $l \geq k+2$, then since $r(B) \geq 1$, we have

$$I_l' \lesssim \left(\frac{1}{2^l r(B)}\right)^2 2^{-2l} \mu(2^l B)^{-1} \lesssim 2^{-2l} 2^{-2k} \mu(2^k B)^{-1}.$$

Note that we needed $q \geq \lambda$ when $0 \leq l \leq k+1$. This proves (3.6.16) when $r(B) \geq 1$, since now $\|\mathbf{1}_k(B)\phi(\mathcal{D})a\|_2^2 \leq \sum_{l=0}^{\infty} I_l' \lesssim 2^{-2k}\mu(2^k B)^{-1}$.

If $r(B) < 1$, then $r(B^*) = 1$ and there exists a differential form b such that $a = \mathcal{D}^N b$, as in Definition 3.6.11. Define $\psi(z) = z^N \phi(z)$, in which case $\psi \in \Psi_N(S^o_{\theta\cup\tilde{r}})$, where $\tilde{r} \geq r$ was fixed previously so that $\tilde{r}/C_{\mathcal{D}}C_{\theta\cup\tilde{r}} > \lambda$. Now choose ϵ in $(0, N-\kappa/2)$. Then, since $N > \kappa/2$, Lemma 3.4.3 implies that

$$\|\mathbf{1}_E \psi(\mathcal{D})\mathbf{1}_F\| \lesssim \langle 1/\rho(E,F)\rangle^{\frac{\kappa}{2}+\epsilon} e^{-\lambda\rho(E,F)} \leq \langle 1/\rho(E,F)\rangle^{\epsilon} \tag{3.6.18}$$

for all closed subsets E and F of M.

We now prove (3.6.16) when $r(B) < 1$. If $k = 0$, then by (3.6.2) and (3.6.6), and since $r(B) < 1$ and $N > \kappa/2$, we have

$$\|\mathbf{1}_0(B^*)\phi(\mathcal{D})a\|_2 \leq \|\psi(\mathcal{D})b\|_2 \lesssim r(B)^N \mu(B)^{-\frac{1}{2}} \lesssim r(B)^{N-\frac{\kappa}{2}} \mu(B^*)^{-\frac{1}{2}} \leq \mu(B^*)^{-\frac{1}{2}}.$$

Now consider $k \geq 1$. For each $l \in \mathbb{N}$, define I_l'' by

$$\|\mathbf{1}_k(B^*)\phi(\mathcal{D})a\|_2^2 \leq \sum_{l=0}^{\infty} \|\mathbf{1}_k(B^*)\psi(\mathcal{D})\mathbf{1}_l(B)b\|_2^2 = \sum_{l=0}^{\infty} I_l''.$$

If $1 \leq 2^l < 2^{k-1}/r(B)$, then

$$\rho(2^k B^* \backslash 2^{k-1}B^*, 2^l B \backslash 2^{l-1}B) = 2^{k-1} - 2^l r(B) \gtrsim 2^k$$

and $\mu(2^k B^*) \leq (2^{k-l}/r(B))^{\kappa} e^{\lambda(2^{k-l}/r(B)-1)2^l r(B)} \mu(2^l B)$, so that by (3.6.18), and since $r(B) < 1$ and $N > \kappa/2$, we have

$$\begin{aligned} I_l'' &\lesssim \left(\frac{1}{2^k}\right)^{2(\frac{\kappa}{2}+\epsilon)} e^{-2\lambda(2^{k-1}-2^l r(B))} r(B)^{2N} e^{-q2^l r(B)} 2^{-2l} \mu(2^l B)^{-1} \\ &\lesssim 2^{-2l(\frac{\kappa}{2}+1)} 2^{-2k(\frac{\kappa}{2}+\epsilon-\frac{\kappa}{2})} r(B)^{2(N-\frac{\kappa}{2})} e^{\lambda(-2^k+2^{l+1}r(B)+2^k-2^l r(B)-2^l r(B))} \mu(2^k B^*)^{-1} \\ &\lesssim 2^{-2l} 2^{-2\epsilon k} \mu(2^k B^*)^{-1}. \end{aligned}$$

If $2^{k-1}/r(B) \leq 2^l \leq 2^{k+1}/r(B)$, then $\mu(2^k B^*) \lesssim e^{\lambda 2^l r}\mu(2^l B)$, and since $r(B) < 1$, we have

$$I''_l \leq \|\psi(\mathcal{D})\mathbf{1}_l(B)b\|_2^2 \lesssim r(B)^{2N} e^{-q2^l r(B)} 2^{-2l}\mu(2^l B)^{-1} \lesssim 2^{-2l}\mu(2^k B^*)^{-1}.$$

If $2^l > 2^{k+1}/r(B)$, then

$$\rho(2^k B^* \backslash 2^{k-1}B^*, 2^l B \backslash 2^{l-1}B) = 2^{l-1}r(B) - 2^k \gtrsim 2^l$$

and $\mu(2^k B^*) \leq \mu(2^l B)$, so that by (3.6.18), and since $r(B) < 1$, we have

$$I''_l \lesssim \left(\frac{1}{2^l}\right)^{2\epsilon} r(B)^{2N} 2^{-2l}\mu(2^l B)^{-1} \lesssim 2^{-2l}2^{-2\epsilon k}\mu(2^k B^*)^{-1}.$$

Note that we needed $q \geq \lambda$ when $1 \leq 2^l < 2^{k+1}/r(B)$. This proves (3.6.16) when $r(B) < 1$, since now $\|\mathbf{1}_k(B^*)\phi(\mathcal{D})a\|_2^2 \leq \sum_{l=0}^{\infty} I''_l \lesssim 2^{-2\epsilon k}\mu(2^k B^*)^{-1}$. □

3.6.2 Local Riesz Transforms and Holomorphic Functional Calculi

We now prove the principal result of this chapter, which is the local analogue of Theorem 5.11 in [9].

Theorem 3.6.19. *Let $\kappa, \lambda \geq 0$ and suppose that M is a complete Riemannian manifold satisfying* (E$_{\kappa,\lambda}$)*. Let $\omega \in [0, \pi/2)$ and $R \geq 0$ and suppose that $\mathcal{D}$ is an operator on $L^2(\wedge T^*M)$ of type $S_{\omega\cup R}$ satisfying hypotheses* (A1) – (A3) *from Section 3.4. Let $\theta \in (\omega, \pi/2)$ and $r > R$ such that $r/C_{\mathcal{D}}C_{\theta\cup r} > \lambda/2$, where $C_{\theta\cup r}$ is from* (A1) *and $C_{\mathcal{D}}$ is from* (A3)*. Then for all $f \in H^\infty(S^o_{\theta\cup r})$, the operator $f(\mathcal{D})$ on $L^2(\wedge T^*M)$ has a bounded extension such that*

$$\|f(\mathcal{D})u\|_{h^p_{\mathcal{D}}} \lesssim \|f\|_\infty \|u\|_{h^p_{\mathcal{D}}}$$

for all $u \in h^p_{\mathcal{D}}$ and $p \in [1, \infty]$.

Proof. If $u \in h^1_{\mathcal{D}} \cap L^2$, then Proposition 3.6.5 gives $U \in t^1 \oplus L^1_{\mathscr{Q}}$ with $\mathcal{S}^{\mathcal{D}}_{\eta,\varphi}U = u$ and $\|U\|_{t^1\oplus L^1_{\mathscr{Q}}} \leq 2\|u\|_{h^1_{\mathcal{D}}}$. Therefore, by Theorem 3.5.2 we have

$$\|f(\mathcal{D})u\|_{h^1_{\mathcal{D}}} = \|\mathcal{Q}^{\mathcal{D}}_{\eta,\varphi}f(\mathcal{D})\mathcal{S}^{\mathcal{D}}_{\eta,\varphi}U\|_{t^1\oplus L^1_{\mathscr{Q}}} \lesssim \|f\|_\infty\|U\|_{t^1\oplus L^1_{\mathscr{Q}}} \lesssim \|f\|_\infty\|u\|_{h^1_{\mathcal{D}}}$$

for all $u \in h^1_{\mathcal{D}} \cap L^2$, and since $h^1_{\mathcal{D}} \cap L^2$ is dense in $h^1_{\mathcal{D}}$ by Corollary 3.6.6, $f(\mathcal{D})$ has a bounded extension to $h^1_{\mathcal{D}}$. The same proof with $\tilde{t}^\infty \oplus \tilde{L}^\infty_{\mathscr{Q}}$ instead of $t^1 \oplus L^1_{\mathscr{Q}}$ shows that $f(\mathcal{D})$ has a bounded extension to $h^\infty_{\mathcal{D}}$. These extensions coincide on $h^1_{\mathcal{D}} \cap h^\infty_{\mathcal{D}}$, since $h^1_{\mathcal{D}} \cap h^\infty_{\mathcal{D}} \subseteq h^2_{\mathcal{D}} = L^2$ is a consequence of the interpolation of the local Hardy spaces in Theorem 3.6.7. Therefore, the required extension exists by interpolation. □

Theorem 1.1.4 follows from this result in the case of the Hodge–Dirac operator by Example 3.4.1, which allows us to prove Corollary 1.1.5.

Proof of Corollary 1.1.5. It was shown in Example 3.4.1 that D satisfies (A1)–(A3) with $\omega = 0$, $R = 0$, $C_{\mathcal{D}} = 1$ and $C_{\theta\cup r} = 1/\sin\theta$ for all $\theta \in (0, \pi/2)$ and $r > 0$. Therefore, Corollary 1.1.5 follows from Theorem 3.6.19 by choosing θ in $(0, \pi/2)$ such that $\lambda/2\sin\theta < \sqrt{a}$, choosing r in $(\lambda/2\sin\theta, \sqrt{a})$ and defining the holomorphic function $f(z) = z(z^2 + a)^{-1/2}$ for all $z \in S^o_{\theta\cup r}$. □

3.7 Embedding $h^p_{\mathcal{D}}(\wedge T^*M)$ in $L^p(\wedge T^*M)$

The local Hardy spaces $h^p_{\mathcal{D}}$ were defined as subspaces of the ambient space $h^0_{\mathcal{D}}$. Amongst other things, this allowed us to interpolate the local Hardy spaces. The ambient space, however, is an *abstract* completion and one would like to identify it with a space of differential forms; a subspace of the space of locally integrable differential forms or the space of measurable differential forms, for instance. In general, however, this appears to be a nontrivial matter that depends on the properties of the operator and the geometry of the manifold. Indeed this may be a limitation of our approach to local Hardy spaces on Riemannian manifolds. The space $h^2_{\mathcal{D}}$, however, was defined so that it could be identified with L^2. This was explained in detail in the paragraph preceding Proposition 3.6.5. In this section, we specify additional properties of the operator $\mathcal{D}$ and the manifold M that allow $h^p_{\mathcal{D}}$ to be identified with a subspace of L^p for all $p \in [1,2]$. In particular, we show that the Hodge–Dirac operator D has the additional properties required of $\mathcal{D}$.

To be precise, let $\imath : L^2 \to h^2_{\mathcal{D}}$ denote the isometric isomorphism that identifies L^2 with the dense subspace $h^2_{\mathcal{D}}$ of $h^0_{\mathcal{D}}$. We use the molecular characterisation to prove the embedding in the case $p = 1$. The result in the case $p \in (1,2)$ then follows by interpolation. Let $\kappa, \lambda \geq 0$ and suppose that M is a complete Riemannian manifold satisfying $(\mathrm{E}_{\kappa,\lambda})$. Let $N \in \mathbb{N}$, $N > \kappa/2$ and $q \geq \lambda$. It follows from Theorem 3.6.13 that there exists $c > 0$ such that the following holds: For each $u \in h^1_{\mathcal{D}}$, there exists a sequence $(\lambda_j)_j$ in ℓ^1 and a sequence $(a_j)_j$ of $h^1_{\mathcal{D}}$-molecules of type (N, q) such that $\sum_j \lambda_j \imath(a_j)$ converges to u in $h^0_{\mathcal{D}}$ and

$$\|(\lambda_j)_j\|_{\ell^1} \leq c\|u\|_{h^1_{\mathcal{D}}}.$$

The molecules a_j are contained in $L^1 \cap L^2$ with $\sup_j \|a_j\|_1 \lesssim 1$ by (3.6.4) and (3.6.5). Therefore, for each $u \in h^1_{\mathcal{D}}$, there exists $\tilde{u} \in L^1$ such that the sequence of partial sums $(u_n)_n$ defined by $u_n = \sum_{j=1}^n \lambda_j a_j$ for each $n \in \mathbb{N}$ converges to $\tilde{u}$ in L^1 with

$$\|\tilde{u}\|_1 \leq \sup_{n\in\mathbb{N}} \sum_{j=1}^{n} |\lambda_j| \|a_j\|_1 \lesssim \|u\|_{h^1_{\mathcal{D}}}.$$

Thus, define the bounded linear operator $\jmath : h^1_{\mathcal{D}} \to L^1$ by $\jmath(u) := \tilde{u}$ for all $u \in h^1_{\mathcal{D}}$. If $u \in h^1_{\mathcal{D}} \cap h^2_{\mathcal{D}}$, then Remark 3.6.14 implies that u_n converges to $\imath^{-1}(u)$ in $L^2 \subseteq L^1_{\mathrm{loc}}$, and since u_n converges to $\jmath(u)$ in $L^1 \subseteq L^1_{\mathrm{loc}}$, we must have

$$\jmath(u) = \imath^{-1}(u) \tag{3.7.1}$$

for all $u \in h^1_{\mathcal{D}} \cap h^2_{\mathcal{D}}$. This shows that $\jmath$ is injective on $h^1_{\mathcal{D}} \cap h^2_{\mathcal{D}}$. In order to identify $h^1_{\mathcal{D}}$ with the subspace $\jmath(h^1_{\mathcal{D}}) \subseteq L^1$, it suffices to show that $\jmath$ is injective on $h^1_{\mathcal{D}}$. We require the following properties of the operator $\mathcal{D}$ to prove this in Theorem 3.7.5.

(B1) The operator $\mathcal{D} : L^2(\wedge T^*M) \to L^2(\wedge T^*M)$ is self-adjoint.

(B2) The operator $\mathcal{D} : C^\infty(\wedge T^*M) \to C^\infty(\wedge T^*M)$ is a first-order differential operator in the following sense: First, if $u, v \in C^\infty(\wedge T^*M)$ satisfy $u = v$ on an open set $\Omega \subseteq M$, then $\mathcal{D}u = \mathcal{D}v$ on Ω; Second, on any coordinate chart $\Omega \subseteq M$

and any local trivialization of the bundle $\wedge T^*M$ over Ω, the operator $\mathcal{D}$ is represented as

$$\mathcal{D}u = \sum_j A_j \frac{\partial u}{\partial x_j} + Bu,$$

where A_j and B are smooth matrix-valued functions on Ω.

Let $\mathrm{End}(\wedge T^*M)$ denote the endomorphism bundle over M. The *principal symbol* of a first-order differential operator $\mathcal{D}$ as defined in (B2) is a vector bundle morphism $\sigma_{\mathcal{D}} : T^*M \to \mathrm{End}(\wedge T^*M)$. If $x \in M$ and $\xi \in T^*_x M$ is given by $\xi = \sum_j \xi_j dx^j$ in a coordinate chart at x, then the principal symbol $\sigma_{\mathcal{D}}(x, \xi)$ is the endomorphism on $\wedge T^*_x M$ given by

$$\sigma_{\mathcal{D}}(x, \xi) = \sum_j A_j \xi_j.$$

This definition is actually independent of the choice of coordinate chart, and for all smooth scalar-valued functions $\eta \in C^\infty(M)$, the principal symbol satisfies

$$\sigma_{\mathcal{D}}(x, d\eta(x))u(x) = [\mathcal{D}, \eta I]u(x) \tag{3.7.2}$$

for all $u \in C^\infty(\wedge T^*M)$ and $x \in M$, where d is the exterior derivative. These facts are standard and can be found in, for instance, Chapter 10 of [40]. The final two properties required of the operator $\mathcal{D}$ are expressed in terms of its principal symbol.

(B3) The *propagation speed*

$$P_{\mathcal{D}} := \sup\{|\sigma_{\mathcal{D}}(x, \xi)| : x \in M,\ \xi \in T^*_x M,\ |\xi|_{T^*_x M} = 1\}$$

is finite.

(B4) The operator $\mathcal{D}$ is *elliptic* in the sense that the principal symbol $\sigma_{\mathcal{D}}(x, \xi)$ is an invertible endomorphism of $\wedge T^*_x M$ for all $\xi \in T^*_x M \setminus \{0\}$.

A self-adjoint operator on $L^2(\wedge T^*M)$ satisfies hypotheses (A1) and (A2) from Section 3.4 with $\omega = 0$, $R = 0$ and $C_{\theta \cup r} \leq 1/\sin\theta$ for all $\theta \in (0, \pi/2)$ and $r > 0$. Moreover, it follows from (3.7.2) that an operator $\mathcal{D}$ satisfying (B3) also satisfies hypothesis (A3) from Section 3.4 with $C_{\mathcal{D}} = P_{\mathcal{D}}$. Therefore, hypotheses (B1)–(B3) imply hypotheses (A1)–(A3) from Section 3.4.

The following example shows that the Hodge–Dirac operator satisfies all of the hypotheses (B1)–(B4).

Example 3.7.1. The Hodge–Dirac operator $D = d + d^*$ is a self-adjoint first-order differential operator so it immediately satisfies (B1)–(B2). The principal symbol of D is given by

$$\sigma_D(x, \xi)u = \xi \wedge u - \xi \lrcorner u$$

for all $x \in M$, $\xi \in T^*_x M$ and $u \in \wedge T^*_x M$, where $\wedge$ and $\lrcorner$ denote the exterior and (left) interior products on $\wedge T^*_x M$, respectively. This shows that

$$|\sigma_D(x, \xi)u|_{\wedge T^*_x M} = |\xi \wedge u - \xi \lrcorner u|_{\wedge T^*_x M} = |\xi|_{T^*_x M} |u|_{\wedge T^*_x M}$$

for all $x \in M$, $\xi \in T^*_x M$ and $u \in \wedge T^*_x M$. Therefore, the Hodge–Dirac operator satisfies (B3)–(B4) with $P_D = 1$.

The constant $P_{\mathcal{D}}$ in (B3) is called the propagation speed because it implies the following finite propagation speed property of $\mathcal{D}$. The proof of this result is standard and can be found in, for instance, Propositions 10.3.1 and 10.5.4 of [40].

Proposition 3.7.2 (Finite propagation speed)**.** Let M be a complete Riemannian manifold and suppose that $\mathcal{D}$ satisfies (B1)–(B3) with propagation speed $P_{\mathcal{D}} > 0$. Let E and F be closed subsets of M. If $|t| < \rho(E,F)/P_{\mathcal{D}}$, then $\mathbf{1}_E e^{it\mathcal{D}} \mathbf{1}_F = 0$.

In what follows, the operator $\mathcal{D}$ is self-adjoint and so it has a functional calculus that is defined for all measurable functions f on $\mathbb{R}$. Note that we continue to use the notation $f_t(x) = f(tx)$ for all $x \in \mathbb{R}$ and $t \in (0,1]$. We use this functional calculus in the following lemma to obtain off-diagonal estimates that are similar to those in Section 3.4 but which hold for more general functions of $\mathcal{D}$.

Lemma 3.7.3. Let M be a complete Riemannian manifold and suppose that $\mathcal{D}$ is an operator on $L^2(\wedge T^*M)$ satisfying (B1)–(B3). Let $f \in \mathcal{S}(\mathbb{R})$ be a real-valued Schwartz function that has a holomorphic extension to the strip of width W about the real axis in the complex plane. For each $a \in (0,1)$ and $b \geq 0$, there exists $c > 0$ such that

$$\|\mathbf{1}_E f_t(\mathcal{D}) \mathbf{1}_F\| \leq c \left\langle \frac{t}{\rho(E,F)} \right\rangle^b \exp\left(-a \frac{W}{P_{\mathcal{D}}} \frac{\rho(E,F)}{t} \right)$$

for all $t \in (0,1]$ and closed subsets E and F of M.

Proof. Let f be as stated in the lemma. The operator $\mathcal{D}$ satisfies (B1), so the functional calculus for self-adjoint operators and the Fourier inversion formula imply that

$$f_t(\mathcal{D})u = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f_t}(s) e^{is\mathcal{D}} u \, \mathrm{d}s = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}\left(\frac{s}{t}\right) e^{is\mathcal{D}} u \, \frac{\mathrm{d}s}{t}$$

for all $u \in L^2$ and $t \in (0,1]$. Given that f has a holomorphic extension to the strip of width W, the properties of the Fourier transform imply that $|\widehat{f}(\sigma)| \lesssim e^{-W\sigma}$ for all $\sigma \in \mathbb{R}$. The operator $\mathcal{D}$ also satisfies the finite propagation speed property from Proposition 3.7.2 because it satisfies (B2) and (B3). Altogether, this shows that

$$\begin{aligned}
\|\mathbf{1}_E f_t(\mathcal{D}) \mathbf{1}_F\| &\lesssim \int_{|s| \geq \frac{\rho(E,F)}{P_{\mathcal{D}}}} |\widehat{f}\left(\frac{s}{t}\right)| \, \|\mathbf{1}_E e^{is\mathcal{D}} \mathbf{1}_F\| \, \frac{\mathrm{d}s}{t} \\
&\leq \int_{|\sigma| \geq \frac{\rho(E,F)}{P_{\mathcal{D}} t}} |\widehat{f}(\sigma)| \, \mathrm{d}\sigma \\
&\lesssim \int_{|\sigma| \geq \frac{\rho(E,F)}{P_{\mathcal{D}} t}} e^{-W\sigma} \, \mathrm{d}\sigma \\
&\lesssim e^{-\frac{W}{P_{\mathcal{D}}} \frac{\rho(E,F)}{t}},
\end{aligned}$$

and the result follows. ∎

The next lemma is also used to prove Theorem 3.7.5.

Lemma 3.7.4. Let M be a complete Riemannian manifold and suppose that $\mathcal{D}$ is an operator on $L^2(\wedge T^*M)$ satisfying (B1)–(B4). Let $f \in \mathcal{S}(\mathbb{R})$ be a real-valued Schwartz function such that $f(0) = 0$ and the Fourier transform $\widehat{f} \in C_0^\infty(\mathbb{R})$. Then for all $U \in t^2$ with compact support, the differential form $\mathcal{S}_f U := \int_0^1 f_t(\mathcal{D})U_t \frac{dt}{t}$ is in $L^\infty(\wedge T^*M)$.

Proof. Let f be as stated in the lemma and suppose that $U \in t^2$ with compact support. There exists $\delta \in (0,1]$ and a ball B in M such that $\operatorname{sppt} U \subseteq B \times [\delta, 1]$. Also, there exists $r > 0$ such that $\operatorname{sppt} \widehat{f} \subseteq [-r, r]$. The operator $\mathcal{D}$ satisfies (B1), so the functional calculus for self-adjoint operators and the Fourier inversion formula imply that

$$f_t(\mathcal{D})U_t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f_t}(s) e^{is\mathcal{D}} U_t \, \mathrm{d}s = \frac{1}{2\pi}\int_{-rt}^{rt} \widehat{f}\left(\frac{s}{t}\right) e^{is\mathcal{D}} U_t \frac{\mathrm{d}s}{t}$$

for all $t \in (0, 1]$. Then, since $\mathcal{D}$ satisfies (B2)–(B3) and $\operatorname{sppt} U_t \subseteq B$, the finite propagation speed property from Proposition 3.7.2 implies that

$$\operatorname{sppt} e^{is\mathcal{D}} U_t \subseteq (1 + 2rtP_{\mathcal{D}})B$$

for all $s \leq rt$. The previous formula then shows that $\operatorname{sppt} f_t(\mathcal{D})U_t \subseteq (1 + 2rP_{\mathcal{D}})B$ for all $t \in (\delta, 1]$, which implies that

$$\operatorname{sppt} \mathcal{S}_f U \subseteq (1 + 2rP_{\mathcal{D}})B.$$

Let $\tilde{B} = (1 + 2rP_{\mathcal{D}})B$, $k \in \mathbb{N}_0$ and $m = \dim M$. We now recall the definition of the Sobolev space $W^{k,2}_{\tilde{B}}(\wedge T^*\tilde{M})$ of differential forms on M that are supported on $\tilde{B}$. Let $(\Omega_j, \varphi_j)_j$ denote a locally finite covering of $\tilde{B}$ by coordinate charts $\varphi_j : \Omega_j \subseteq \tilde{B} \to \mathbb{R}^m$ and let $(\rho_j)_j$ denote a subordinate partition of unity. The Sobolev space $W^{k,2}_{\tilde{B}}(\wedge T^*\tilde{M})$ then consists of all measurable differential forms u on M that are supported on $\tilde{B}$ with

$$\|u\|_{W^{k,2}_{\tilde{B}}(\wedge T^*\tilde{M})} := \left(\sum_j \|(\rho_j |u|_{\wedge T^*M}) \circ \varphi_j^{-1}\|^2_{W^{k,2}(\mathbb{R}^m)}\right)^{\frac{1}{2}} < \infty,$$

where $\|f\|_{W^{k,2}(\mathbb{R}^m)} := \left(\sum_{|\alpha|\leq k} \|D^\alpha f\|_2^2\right)^{1/2}$ for all $f \in C^\infty(\mathbb{R}^m)$ is the Sobolev norm on $\mathbb{R}^m$. This definition is adapted from Section 1.3.3 of [59] and Section 10.4 of [40]. The Sobolev embedding theorem for differential forms on a compact manifold then implies that $W^{k,2}_{\tilde{B}}(\wedge T^*M) \subseteq L^\infty_{\tilde{B}}(\wedge T^*M)$ for all $k > m/2$, where $L^\infty_{\tilde{B}}(\wedge T^*M)$ is the subspace of $u \in L^\infty(\wedge T^*M)$ that are supported on $\tilde{B}$. A precise formulation of the Sobolev embedding theorem is contained in Section I.3.2 of [15] and Proposition 1.1 in Chapter 4 of [70]. In any case, this shows that

$$\|\mathcal{S}_f U\|_{L^\infty(\wedge T^*M)} \lesssim_{\tilde{B}} \|\mathcal{S}_f U\|_{W^{m,2}_{\tilde{B}}(\wedge T^*M)}.$$

The operator $\mathcal{D}$ is elliptic by (B4), which implies that it satisfies Gårding's inequality and the fundamental elliptic estimate. In particular, the later states that for each $k \in \mathbb{N}$, we have

$$\|u\|_{W^{k,2}_{\tilde{B}}(\wedge T^*M)} \lesssim_k \|\mathcal{D}u\|_{W^{k-1,2}_{\tilde{B}}(\wedge T^*M)} + \|u\|_{W^{k-1,2}_{\tilde{B}}(\wedge T^*M)}$$

for all $u \in W^{k-1,2}_{\tilde{B}}(\wedge T^*M)$. This is proved in Section 10.4.4 of [40]. A relatively simple proof in the case of the Hodge–Dirac operator is contained in Theorem 2.46 of [59]; in the case of general Dirac operators see also (3.15) in [58]. Therefore, we have

$$\|\mathcal{S}_f U\|_{W^{m,2}_{\tilde{B}}(\wedge T^*M)} \lesssim_m \sum_{k=0}^{m} \|\mathcal{D}^k \mathcal{S}_f U\|_2.$$

The functional calculus for self-adjoint operators then allows us to conclude that

$$\|\mathcal{S}_f U\|_{L^\infty(\wedge T^*M)} \lesssim_{\tilde{B},m} \sum_{k=0}^{m} \left\| \int_\delta^1 (t\mathcal{D})^k f(t\mathcal{D}) U_t \frac{\mathrm{d}t}{t^{k+1}} \right\|_2 \lesssim \frac{1}{\delta^k} \|U\|_{t^2} < \infty.$$

□

We now prove the injectivity of the operator $\jmath : h^1_{\mathcal{D}} \to L^1$ defined at the beginning of this section.

Theorem 3.7.5. *Let $\kappa, \lambda \geq 0$ and suppose that M is a complete Riemannian manifold satisfying* (E$_{\kappa,\lambda}$) *with* $\inf_{x\in M} V(x,1) > 0$. *Suppose that $\mathcal{D}$ is an operator on $L^2(\wedge T^*M)$ satisfying* (B1) – (B4). *Then the mapping $\jmath : h^1_{\mathcal{D}}(\wedge T^*M) \to L^1(\wedge T^*M)$ is injective.*

Proof. Let $u \in h^1_{\mathcal{D}}$ and suppose the $\jmath(u) = 0$. It suffices to show that $u = 0$. Since $\jmath(u) = 0$, the definition of $\jmath(u)$ implies that there exists a sequence $(u_n)_n$ in $L^1 \cap L^2$ such that $\imath(u_n)$ converges to u in $h^1_{\mathcal{D}}$ and u_n converges to 0 in L^1. The convergence in $h^1_{\mathcal{D}}$, which is stronger than convergence in $h^0_{\mathcal{D}}$, is due to Remark 3.6.14.

Choose $N \in \mathbb{N}$, $\theta \in (0, \pi/2)$ and $r > 0$ such that $N > \kappa/2$ and $r \sin\theta / P_{\mathcal{D}} > \lambda/2$, where $P_{\mathcal{D}}$ is from (B3). Now define $\eta(z) = z^N e^{-z^2}$ and $\varphi(z) = e^{-z^2}$ for all $z \in S^o_{\theta\cup r}$. Next, choose an even real-valued Schwartz function $f \in \mathcal{S}(\mathbb{R})$ with $f(0) = 0$ and compactly supported Fourier transform $\widehat{f} \in C^\infty_0(\mathbb{R})$. The Paley–Wiener–Schwartz Theorem (see, for instance, Theorem 1 in Section 1.2.1 of [69]) guarantees that f has a holomorphic extension to the entire complex plane. Therefore, define the constant $c = (\int_0^\infty f(\pm t)\eta(\pm t)\frac{\mathrm{d}t}{t})^{-1}$ and the functions

$$\tilde{\eta}(z) = c\eta(z) \quad \text{and} \quad g(z) = \frac{1}{\varphi(z)}\left(1 - \int_0^1 f_t(z)\tilde{\eta}_t(z)\frac{\mathrm{d}t}{t}\right)$$

for all $z \in S^o_{\theta\cup r}$. These functions are holomorphic on $S^o_{\theta\cup r}$, and a change of variables shows that

$$g(x) = \frac{1}{\varphi(x)} \int_1^\infty f_t(x)\tilde{\eta}_t(x)\frac{\mathrm{d}t}{t}$$

for all $x \in \mathbb{R} \setminus \{0\}$. A straightforward but tedious calculation verifies that the restriction of g to $\mathbb{R}$ is a real-valued Schwartz function. Applying the functional calculus for self-adjoint operators, we have

$$\int_0^1 f_t(\mathcal{D})\tilde{\eta}_t(\mathcal{D})u\frac{\mathrm{d}t}{t} + g(\mathcal{D})\varphi(\mathcal{D})u = u$$

for all $u \in L^2$.

Let $\mathcal{S}_f U := \int_0^1 f_t(\mathcal{D})U_t \frac{dt}{t}$ for all $U \in t^2$ and $\mathcal{Q}_f u := f_t(\mathcal{D})u$ for all $u \in L^2$. The operators $\mathcal{S}_\eta$ and $\mathcal{Q}_\eta$ are defined in the same way, so we have

$$\mathcal{S}_\eta \mathcal{Q}_f u_n + \varphi(\mathcal{D})g(\mathcal{D})u_n = u_n$$

for all $n \in \mathbb{N}$. The equivalent norms on $h^1_{\mathcal{D}}$ from Propostion 3.6.9 show that

$$\|\imath(u_n - u_m)\|_{h^1_{\mathcal{D}}} \eqsim \|\mathcal{Q}_{\eta,\varphi}(u_n - u_m)\|_{t^1 \oplus L^1_{\mathscr{Q}}}.$$

for all integers $n > m > 0$. Then, since $\imath(u_n)$ is Cauchy in $h^1_{\mathcal{D}}$, the sequence $(\mathcal{Q}_{\eta,\varphi}u_n)_n$ is Cauchy in $t^1 \oplus L^1_{\mathscr{Q}}$. The Calderón reproducing formula in Proposition 3.6.1 shows that there exists $\eta' \in \Psi_N(S^o_{\theta \cup r})$ and $\varphi' \in \Theta(S^o_{\theta \cup r})$ such that $\mathcal{S}_{\eta',\varphi'}\mathcal{Q}_{\eta,\varphi} = I$ on L^2. Using the off-diagonal estimates in Lemma 3.7.3 and following the proof of Theorem 3.5.2, we obtain

$$\|\mathcal{Q}_f(u_n - u_m)\|_{t^1} = \|\mathcal{Q}_f \mathcal{S}_{\eta',\varphi'}\mathcal{Q}_{\eta,\varphi}(u_n - u_m)\|_{t^1} \lesssim \|\mathcal{Q}_{\eta,\varphi}(u_n - u_m)\|_{t^1 \oplus L^1_{\mathscr{Q}}}$$

for all integers $n > m > 0$. Therefore, there exists $V \in t^1$ such that $\mathcal{Q}_f u_n$ converges to V in t^1. Now, for each $F \in t^2$ with compact support, we have

$$\begin{aligned} |\langle V, F\rangle|_{L^2_\bullet} &\leq |\langle V - \mathcal{Q}_f u_n, F\rangle|_{L^2_\bullet} + |\langle u_n, \mathcal{S}_f F\rangle|_{L^2_\bullet} \\ &\leq \|V - \mathcal{Q}_f u_n\|_{t^1}\|F\|_{t^\infty} + \|u_n\|_1 \|\mathcal{S}_f F\|_\infty \end{aligned}$$

for all $n \in \mathbb{N}$. Then, since u_n converges to 0 in L^1, Lemma 3.7.4 implies that $|\langle V, F\rangle|_{L^2_\bullet} = 0$ for all $F \in t^2$ with compact support, which implies that $V = 0$. Lemma 3.6.8 then shows that $\imath(\mathcal{S}_\eta \mathcal{Q}_f u_n)$ converges to 0 in $h^1_{\mathcal{D}}$.

Now let $\delta \in (0,1]$ be the constant associated with the unit cube structure $\mathscr{Q}$ as in Definition 3.3.1. For each $Q_j \in \mathscr{Q}$, there exists a ball B_j in M or radius equal to 1 such that $\delta B_j \subseteq Q_j \subseteq B_j$, so by ($\mathrm{E}_{\kappa,\lambda}$) we have

$$\inf_{x \in M} V(x,1) \lesssim \inf_{Q_j \in \mathscr{Q}} \mu(Q_j).$$

The property that $\inf_{x \in M} V(x,1) > 0$ then implies that

$$\|u\|_2 \leq \sum_{Q_j \in \mathscr{Q}} \|\mathbf{1}_{Q_j}u\|_2 \lesssim \sum_{Q_j \in \mathscr{Q}} \mu(Q_j)^{\frac{1}{2}} \|\mathbf{1}_{Q_j}u\|_2 = \|u\|_{L^1_{\mathscr{Q}}}$$

for all $u \in L^1_{\mathscr{Q}}$. Hence, we have $L^1_{\mathscr{Q}} \subseteq L^2$.

Next, using the off-diagonal estimates in Lemma 3.7.3 and following the proof of Theorem 3.5.2, we obtain

$$\|g(\mathcal{D})u_n\|_{L^1_{\mathscr{Q}}} = \|g(\mathcal{D})\mathcal{S}_{\eta',\varphi'}\mathcal{Q}_{\eta,\varphi}u_n\|_{L^1_{\mathscr{Q}}} \lesssim \|\mathcal{Q}_{\eta,\varphi}u_n\|_{t^1 \oplus L^1_{\mathscr{Q}}}$$

for all $n \in \mathbb{N}$. This implies that there exists $v \in L^1_{\mathscr{Q}} \subseteq L^2$ such that $g(\mathcal{D})u_n$ converges to v in $L^1_{\mathscr{Q}}$. Therefore, Lemma 3.6.8 implies that $\imath(\varphi(\mathcal{D})g(\mathcal{D})u_n)$ converges to $\imath(\varphi(\mathcal{D})v)$ in $h^1_{\mathcal{D}}$, and writing

$$u_n - \varphi(\mathcal{D})v = \mathcal{S}_\eta \mathcal{Q}_f u_n + (\varphi(\mathcal{D})g(\mathcal{D})u_n - \varphi(\mathcal{D})v)$$

shows that $\imath(u_n)$ converges to $\imath(\varphi(\mathcal{D})v)$ in $h^1_{\mathcal{D}}$. This implies that u_n converges to $\varphi(\mathcal{D})v$ in L^1, since by (3.7.1) we have

$$\|u_n - \varphi(\mathcal{D})v\|_1 = \|\jmath\imath(u_n - \varphi(\mathcal{D})v)\|_1 \lesssim \|\imath(u_n) - \imath(\varphi(\mathcal{D})v)\|_{h^1_{\mathcal{D}}}.$$

Given that u_n converges to 0 in L^1 and that $\imath(u_n)$ converges to u in $h^1_{\mathcal{D}}$, we must have $\varphi(\mathcal{D})v = 0$ and $u = \imath(\varphi(\mathcal{D})v) = 0$, as required. □

The embedding $h^p_{\mathcal{D}} \subseteq L^p_{\mathcal{D}}$ for all $p \in (1,2)$ then follows from the interpolation result for the local Hardy spaces in Theorem 3.6.7. The next result is then an immediate corollary of Theorem 3.6.19.

Theorem 3.7.6. *Let $\kappa, \lambda \geq 0$ and suppose that M is a complete Riemannian manifold satisfying* $(\mathrm{E}_{\kappa,\lambda})$ *and that* $\inf_{x\in M} V(x,1) > 0$. *Suppose that $\mathcal{D}$ is an operator on $L^2(\wedge T^*M)$ satisfying hypotheses* (B1) – (B4). *Let $\theta \in (0, \pi/2)$ and $r > 0$ such that $r \sin\theta/P_{\mathcal{D}} > \lambda/2$, where $P_{\mathcal{D}}$ is from* (B3). *Then for all $f \in H^\infty(S^o_{\theta\cup r})$, the operator $f(\mathcal{D})$ on $L^2(\wedge T^*M)$ has a bounded extension such that*

$$\|f(\mathcal{D})u\|_p \lesssim \|f\|_\infty \|u\|_{h^p_{\mathcal{D}}}$$

for all $u \in h^p_{\mathcal{D}}$ and $p \in [1,2]$.

This allows us to state the following $h^p_{\mathcal{D}}(\wedge T^*M)$–$L^p(\wedge T^*M)$ bound for the geometric Riesz transform $\mathcal{D}(\mathcal{D}^2 + aI)^{-\frac{1}{2}}$ associated with $\mathcal{D}$. The proof follows that of Corollary 1.1.5 at the end of Section 3.6.

Corollary 3.7.7. Let $\kappa, \lambda \geq 0$ and suppose that M is a complete Riemannian manifold satisfying $(\mathrm{E}_{\kappa,\lambda})$ and that $\inf_{x\in M} V(x,1) > 0$. Suppose that $\mathcal{D}$ is an operator on $L^2(\wedge T^*M)$ satisfying hypotheses (B1)–(B4) with propagation speed $P_{\mathcal{D}} > 0$ in (B3). If $a > (P_{\mathcal{D}}\lambda/2)^2$, then the operator $\mathcal{D}(\mathcal{D}^2 + aI)^{-\frac{1}{2}}$ has a bounded extension such that

$$\|\mathcal{D}(\mathcal{D}^2 + aI)^{-\frac{1}{2}}u\|_p \lesssim \|u\|_{h^p_{\mathcal{D}}}$$

for all $u \in h^p_{\mathcal{D}}$ and $p \in [1,2]$.

Chapter 4

Local Quadratic Estimates for Dirac Type Operators

In this chapter we develop a general framework for a class of first-order differential operators that act on the trivial bundle over a complete Riemannian manifold. The main result is a local quadratic estimate for certain L^∞ perturbations of these operators on manifolds with at most exponential volume growth and on which a local Poincaré inequality holds. The solution of the Kato square root problem for divergence form operators in Theorem 1.1.7 is shown to be a corollary of this result in Section 4.2. The technical tools required to prove the main result include a local version of the dyadic cube structure developed by Christ in [24] and the local properties of Carleson measures. The relevant details are contained in Section 4.3 and the main local quadratic estimate is proved in Section 4.4. The material in Sections 4.1 and 4.4 follows closely the treatment by Axelsson, Keith and McIntosh in [11, 10] and the reader is advised to have a copy of those papers at hand. Throughout this chapter we adopt the notation from Section 2.1 and the notation for off-diagonal estimates from Section 3.4.

4.1 Dirac Type Operators

We begin with a statement of the operator-theoretic results from [11]. Consider three operators $\{\Gamma, B_1, B_2\}$ acting in a Hilbert space $\mathcal{H}$ that satisfy the following properties:

(H1) The operator $\Gamma : \mathsf{D}(\Gamma) \to \mathcal{H}$ is densely defined, closed and nilpotent with domain $\mathsf{D}(\Gamma) \subseteq \mathcal{H}$. The condition that Γ is nilpotent is defined to mean that $\mathsf{R}(\Gamma) \subseteq \mathsf{N}(\Gamma)$, which implies that $\Gamma^2 = 0$ on $\mathsf{D}(\Gamma)$;

(H2) The operators B_1 and B_2 are bounded and there exist $\kappa_1, \kappa_2 > 0$ such that the following accretivity conditions are satisfied:

$$\begin{aligned} \mathrm{Re}(B_1 u, u) &\geq \kappa_1 \|u\|^2 \quad \text{for all} \quad u \in \mathsf{R}(\Gamma^*); \\ \mathrm{Re}(B_2 u, u) &\geq \kappa_2 \|u\|^2 \quad \text{for all} \quad u \in \mathsf{R}(\Gamma). \end{aligned}$$

The angles of accretivity are then defined as follows:

$$\omega_1 := \sup_{u \in \mathsf{R}(\Gamma^*)\setminus\{0\}} |\arg(B_1 u, u)| < \tfrac{\pi}{2}$$
$$\omega_2 := \sup_{u \in \mathsf{R}(\Gamma)\setminus\{0\}} |\arg(B_2 u, u)| < \tfrac{\pi}{2}.$$

Also, set $\omega := \frac{1}{2}(\omega_1 + \omega_2)$.

(H3) The operators satisfy $\Gamma^* B_2 B_1 \Gamma^* = 0$ on $\mathsf{D}(\Gamma^*)$ and $\Gamma B_1 B_2 \Gamma = 0$ on $\mathsf{D}(\Gamma)$. This implies that $\Gamma B_1^* B_2^* \Gamma = 0$ on $\mathsf{D}(\Gamma)$ and $\Gamma^* B_2^* B_1^* \Gamma^* = 0$ on $\mathsf{D}(\Gamma^*)$.

Now consider the following operators.

Definition 4.1.1. Let $\Pi := \Gamma + \Gamma^*$, $\Gamma_B := B_2^* \Gamma B_1^*$ and $\Pi_B := \Gamma + \Gamma_B^*$.

Lemma 4.1 and Corollary 4.3 in [11] show that the adjoint $\Gamma_B^* = B_1 \Gamma^* B_2$ and $(\Pi_B)^* = \Gamma^* + \Gamma_B$, that each of these operators is closed and densely defined, and that Γ_B and Γ_B^* are nilpotent. The following results are from Lemma 4.2 in [11]:

$$\begin{aligned} \|\Gamma u\| + \|\Gamma_B^* u\| &\eqsim \|\Pi_B u\| \quad \text{for all} \quad u \in \mathsf{D}(\Pi_B); \\ \|\Gamma^* u\| + \|\Gamma_B u\| &\eqsim \|\Pi_B^* u\| \quad \text{for all} \quad u \in \mathsf{D}(\Pi_B^*). \end{aligned} \tag{4.1.1}$$

Proposition 2.2 in [11] establishes the following Hodge decompositions of $\mathcal{H}$ into closed subspaces:

$$\mathcal{H} = \mathsf{N}(\Pi_B) \oplus \overline{\mathsf{R}(\Gamma_B^*)} \oplus \overline{\mathsf{R}(\Gamma)} = \mathsf{N}(\Pi_B^*) \oplus \overline{\mathsf{R}(\Gamma_B)} \oplus \overline{\mathsf{R}(\Gamma^*)},$$

where there is no orthogonality implied by the direct sums (except in the case $B_1 = B_2 = I$) and the decompositions are topological. It is also shown there that $\mathsf{N}(\Pi_B) = \mathsf{N}(\Gamma_B^*) \cap \mathsf{N}(\Gamma)$ and $\overline{\mathsf{R}(\Pi_B)} = \overline{\mathsf{R}(\Gamma_B^*)} \oplus \overline{\mathsf{R}(\Gamma)}$. Furthermore, Proposition 2.5 in [11] establishes that Π_B is of type S_ω, which means that the spectrum of Π_B is contained in the bisector S_ω and that for each $\theta \in (\omega, \pi/2)$ there exists a constant $C_\theta > 0$ such that

$$|z| \|(zI - \Pi_B)^{-1}\| \leq C_\theta \tag{4.1.2}$$

for all $z \in \mathbb{C} \setminus S_\theta$.

We work within this general framework and consider a complete Riemannian manifold M with geodesic distance ρ and Riemannian measure μ. For each $N \in \mathbb{N}$, the space $L^2(M; \mathbb{C}^N)$ consists of all $\mathbb{C}^N$-valued measurable functions $u = (u_j)_{j=1,\dots,N}$ on M with

$$\|u\|_{L^2(M;\mathbb{C}^N)} = \sum_{j=1}^{N} \|u_j\|_{L^2(M)} < \infty.$$

For any scalar-valued smooth function $f \in C^\infty(M)$, let ∇f denote the differential of f, which is the smooth 1-form defined by $\nabla f(X) = X(f)$ for all smooth vector fields X. The space $\mathcal{W}^{1,2}(M; \mathbb{C}^N)$ consists of all $u = (u_j)_{j=1,\dots,N}$ in $C^\infty(M; \mathbb{C}^N)$ with

$$\|u\|^2_{W^{1,2}(M;\mathbb{C}^N)} := \sum_{j=1}^{N} \|u_j\|^2_{L^2(M)} + \sum_{j=1}^{N} \|\nabla u_j\|^2_{L^2(T^*M)} < \infty,$$

where $\| \cdot \|_{L^2(T^*M)}$ is defined by restricting the norm on $L^2(\wedge T^*M)$ from Section 3.4 to sections of the bundle $\wedge^1 T^*M = T^*M$, i.e. differential 1-forms. The Sobolev space $W^{1,2}(M;\mathbb{C}^N)$ is then defined to be the completion of $\mathcal{W}^{1,2}(M;\mathbb{C}^N)$ under the norm $\| \cdot \|_{W^{1,2}(M;\mathbb{C}^N)}$. This completion is identified with the subspace of $L^2(M;\mathbb{C}^N)$ consisting of all $u \in L^2(M;\mathbb{C}^N)$ for which there exists a Cauchy sequence $(u_n)_n$ in $\mathcal{W}^{1,2}(M;\mathbb{C}^N)$ that converges to u in $L^2(M;\mathbb{C}^N)$, and the norm

$$\|u\|_{W^{1,2}(M;\mathbb{C}^N)} := \lim_n \|u_n\|_{W^{1,2}(M;\mathbb{C}^N)}.$$

Further details on this identification are contained in Section 2.2 of [39].

Now consider the following additional hypotheses for the operators $\{\Gamma, B_1, B_2\}$ and the Hilbert space $\mathcal{H}$ analogous to those used by Axelsson, Keith and McIntosh in [10]:

(H4) The Hilbert space $\mathcal{H} = L^2(M;\mathbb{C}^N)$ for some $N \in \mathbb{N}$;

(H5) The operators B_1 and B_2 are matrix-valued pointwise multiplication operators such that the functions defined for all x in M by $x \mapsto B_1(x)$ and $x \mapsto B_2(x)$ belong to $L^\infty(M;\mathcal{L}(\mathbb{C}^N))$.

(H6) The operator Γ is a first-order differential operator in the following sense. There exist $C_\Gamma > 0$ such that for all smooth compactly-supported scalar-valued functions $\eta \in C_0^\infty(M)$, the domain $\mathsf{D}(\Gamma) \subseteq \mathsf{D}(\Gamma \circ \eta I)$ and the commutator $[\Gamma, \eta I]$ is a pointwise multiplication operator such that

$$|[\Gamma, \eta I]u(x)| \leq C_\Gamma \ |\nabla \eta(x)|_{T_x^*M} |u(x)|$$

for all $u \in \mathsf{D}(\Gamma)$ and almost all $x \in M$. This implies that the same hypotheses hold with Γ replaced by Γ^* and Π.

(H7) There exists $c > 0$ such that the following hold for all balls B in M of radius $r \leq 1$:

$$\left| \int_B \Gamma u \ \mathrm{d}\mu \right| \leq c\mu(B)^{\frac{1}{2}} \|u\|_{L^2(M;\mathbb{C}^N)} \text{ for all } u \in \mathsf{D}(\Gamma) \text{ compactly supported in } B;$$
$$\left| \int_B \Gamma^* u \ \mathrm{d}\mu \right| \leq c\mu(B)^{\frac{1}{2}} \|u\|_{L^2(M;\mathbb{C}^N)} \text{ for all } u \in \mathsf{D}(\Gamma^*) \text{ compactly supported in } B.$$

(H8) There exists $c > 0$ such that

$$\|u\|_{W^{1,2}(M;\mathbb{C}^N)} \leq c\|\Pi u\|_{L^2(M;\mathbb{C}^N)}$$

for all $u \in \mathsf{R}(\Gamma) \cup \mathsf{R}(\Gamma^*) \cap \mathsf{D}(\Pi)$.

Remark 4.1.2. Hypothesis (H8) requires the inhomogeneous Sobolev norm as defined above. This is in contrast with the Euclidean setting where the homogeneous norm is often used.

We consider manifolds that have at most exponential volume growth and on which a local Poincaré inequality holds. This is made precise below.

Definition 4.1.3. A complete Riemannian manifold M has *exponential volume growth* if there exist constants $c \geq 1$ and $\kappa, \lambda \geq 0$ such that

$$0 < V(x, \alpha r) \leq c\alpha^{\kappa} e^{\lambda \alpha r} V(x, r) < \infty \tag{$\mathrm{E_{loc}}$}$$

for all $\alpha \geq 1$, $r > 0$ and $x \in M$.

Notation. For all measurable subsets $S \subseteq M$ and functions $u = (u_j)_{j=1,\ldots,N}$ in $L^1_{\mathrm{loc}}(M; \mathbb{C}^N)$, let

$$u_{S_j} = \frac{1}{\mu(S)} \int_S u_j \, \mathrm{d}\mu$$

so that $u_S = (u_{S_j})_{j=1,\ldots,N}$.

Definition 4.1.4. A complete Riemannian manifold M satisfies a *local Poincaré inequality* if there exists a constant $c \geq 1$ such that

$$\|\mathbf{1}_B(u - u_B)\|_{L^2(M;\mathbb{C}^N)} \leq c\, r(B) \|\mathbf{1}_B u\|_{W^{1,2}(M;\mathbb{C}^N)}, \tag{$\mathrm{P_{loc}}$}$$

for all $u \in W^{1,2}(M; \mathbb{C}^N)$ and balls B in M of radius $r(B) \leq 1$.

The following is the main result of this chapter. The proof is in Section 4.4.

Theorem 4.1.5. *Let M be a complete Riemannian manifold satisfying* ($\mathrm{E_{loc}}$) *and* ($\mathrm{P_{loc}}$). *Given operators $\{\Gamma, B_1, B_2\}$ on $L^2(M; \mathbb{C}^N)$ satisfying hypotheses* (H1)–(H8), *the perturbed operator $\Pi_B := \Gamma + B_1 \Gamma^* B_2$ satisfies the quadratic estimate*

$$\int_0^\infty \|t\Pi_B(I + t^2\Pi_B^2)^{-1} u\|_2^2 \, \frac{\mathrm{d}t}{t} \eqsim \|u\|_2^2 \tag{4.1.3}$$

for all u in $\overline{\mathsf{R}(\Pi_B)}$.

It was remarked earlier that the operator Π_B is of type S_ω, where $\omega \in [0, \pi/2)$ is from (H2). Therefore, the quadratic estimate in Theorem 4.1.5 implies that Π_B has a bounded $H^\infty(S^o_\theta)$ functional calculus in $L^2(M; \mathbb{C}^N)$ for all $\theta \in (\omega, \pi/2)$. This is the essential feature of the functional calculus constructed by McIntosh in [53]. Moreover, it implies the following Kato square root estimate, which is proved in the same way as Corollary 2.11 of [11].

Corollary 4.1.6. Assume the hypotheses stated in Theorem 4.1.5. We then have $\mathsf{D}(\sqrt{\Pi_B^2}) = \mathsf{D}(\Pi_B) = \mathsf{D}(\Gamma) \cap \mathsf{D}(\Gamma_B^*)$ with

$$\|\sqrt{\Pi_B^2}\, u\|_2 \eqsim \|\Pi_B u\|_2 \eqsim \|\Gamma u\|_2 + \|\Gamma_B^* u\|_2$$

for all $u \in \mathsf{D}(\sqrt{\Pi_B^2})$.

4.2 Application to Divergence Form Operators

We prove Theorem 1.1.7 as a corollary of Theorem 4.1.5. Let us first fix notation and dispense with some technicalities to handle submanifold geometry. Some references that adopt similar notation to that used here are [39, 61]. For an introduction to submanifold geometry see, for instance, [22, 45, 57].

Let M denote a complete Riemannian manifold and let $m = \dim M$. A coordinate chart (Ω, φ) at $x \in M$ refers to an open neighbourhood $\Omega \subseteq M$ containing x and a diffeomorphism $\varphi : \Omega \to \mathbb{R}^m$. When working in a coordinate chart, we adopt the convention whereby repeated indices are summed over the dimension of M.

Let TM denote the tangent bundle over M and let T^*M denote the cotangent bundle over M. Let $C^\infty(TM)$ denote the space of smooth vector fields and let $C^\infty(T^*M)$ denote the space of smooth covector fields (1-forms). For each $x \in M$, let $(\partial_i)_{i=1,\dots,m}$ denote the standard basis for the tangent space T_xM and let $(dx^i)_{i=1,\dots,m}$ denote the corresponding basis for the cotangent space T^*_xM. More specifically, in a coordinate chart (Ω, φ) at x, the operator $\partial_i : C^\infty(\Omega) \to \mathbb{R}$ is the derivation defined by

$$\partial_i f = [D_i(f \circ \varphi^{-1})](\varphi(x))$$

for all $f \in C^\infty(\Omega)$, where D_i denotes partial differentiation with respect to the i-th Cartesian coordinate. The basis covectors for T^*_xM are then defined by requiring that $dx^i(\partial_j) = \delta^i_j$.

The (k,l)-tensor bundle $T^{k,l}M := \bigotimes^k TM \otimes \bigotimes^l T^*M$. A (k,l)-tensor field T is a section of $T^{k,l}M$; this simply means that T associates each $x \in M$ with a multilinear map

$$T(x) : \overbrace{T_xM \times \cdots T_xM}^{k} \times \overbrace{T^*_xM \times \cdots T^*_xM}^{l} \to \mathbb{R},$$

which is called a (k,l)-tensor at x. A $(0,0)$-tensor field is defined to be a real-valued function on M. Multilinearity implies that the action of a (k,l)-tensor T at x is completely determined by its action on the basis vectors and covectors, which we denote by

$$T^{j_1 \dots j_l}_{i_1 \dots i_k} := T(\partial_{i_1}, \dots, \partial_{i_k}, dx^{j_1}, \dots, dx^{j_k}).$$

Let $\nabla : TM \times C^\infty(TM) \to TM$ denote the Levi-Civita connection on M. The connection is completely determined by its Christoffel symbols, which in a coordinate chart (Ω, φ) are smooth functions $\Gamma^k_{ij} : \Omega \to \mathbb{R}$ defined by

$$\nabla(\partial_i, \partial_j) = \Gamma^k_{ij} \partial_k$$

for all $i, j = 1, \dots, m$. The connection has a natural extension to all smooth tensor fields, which we continue to denote by ∇. For all smooth functions $u \in C^\infty(M)$, the smooth covector field ∇u is given in a coordinate chart by

$$(\nabla u)_i = \partial_i u.$$

This is exactly the differential of u that was introduced in Section 4.1 to define the Sobolev space $W^{1,2}(M)$, so there is no ambiguity in our notation. The smooth

(2,0)-tensor field $\nabla^2 u := \nabla(\nabla u)$, known as the Hessian, is given in a coordinate chart by

$$(\nabla^2 u)_{ij} = \partial_i \partial_j u - \Gamma^k_{ij} \partial_k u. \tag{4.2.1}$$

Let $\langle \cdot, \cdot \rangle_{TM} : TM \times TM \to \mathbb{R}$ denote the Riemannian metric on M. In a coordinate chart at $x \in M$, let

$$g_{ij}(x) = \langle \partial_i, \partial_j \rangle_{T_x M} \quad \text{and} \quad g^{ij}(x) = (g_{ij}(x))^{-1}.$$

Also, let $g(x)$ denote the matrix with entries given by $g_{ij}(x)$. The metric induced on the cotangent bundle $\langle \cdot, \cdot \rangle_{T^*M} : T^*M \times T^*M \to \mathbb{R}$ is then given in a coordinate chart by $\langle dx^i, dx^j \rangle_{T^*_x M} = g^{ij}(x)$.

The curvature R of the connection ∇ is the smooth (3,1)-tensor field on M whose components are given in a coordinate chart by

$$R^l_{ijk} = \partial_j \Gamma^l_{ki} - \partial_k \Gamma^l_{ji} + \Gamma^l_{ja} \Gamma^a_{ki} - \Gamma^l_{ka} \Gamma^a ji$$

and the Ricci curvature Ric is the smooth (2,0)-tensor field on M whose components in a coordinate chart are

$$(\mathrm{Ric})_{ij} = R_{kilj} g^{kl}.$$

The gradient operator $\mathrm{grad} : C^\infty(M) \to C^\infty(TM)$ is defined by requiring that

$$\langle \mathrm{grad}\, f, X \rangle_{TM} = (\nabla f)(X) = X(f) \tag{4.2.2}$$

for all functions $f \in C^\infty(M)$ and vector fields $X \in C^\infty(TM)$. In a coordinate chart, the gradient is given by

$$\mathrm{grad}\, f = g^{ij} (\partial_i f) \partial_j$$

The divergence operator $\mathrm{div} : C^\infty(TM) \to C^\infty(M)$ is defined by requiring that

$$\int_M f \,\mathrm{div}\, X \; \mathrm{d}\mu(x) = -\int_M (\nabla f)(X) \; \mathrm{d}\mu(x) \tag{4.2.3}$$

for all vector fields $X \in C^\infty(TM)$ and compactly supported functions $f \in C^\infty_0(M)$. In a coordinate chart, the divergence of a smooth vector field $X = X^i \partial_i$ is given by

$$\mathrm{div}\, X = \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g}\, X^i).$$

The Laplace–Beltrami operator on functions $\Delta : C^\infty(M) \to C^\infty(M)$ is defined by

$$\Delta f = -\,\mathrm{div}(\mathrm{grad}\, f)$$

for all $f \in C^\infty(M)$. The integration by parts formula

$$\int_M f \Delta g \; \mathrm{d}\mu = \int_M \langle \mathrm{grad}\, f, \mathrm{grad}\, g \rangle_{TM} \; \mathrm{d}\mu = \int_M \langle \nabla f, \nabla g \rangle_{T^*M} \; \mathrm{d}\mu.$$

holds for all $f, g \in C^\infty(M)$.

The gradient has a closed extension $\mathrm{grad} : W^{1,2}(M) \to L^2(TM)$. This follows by the construction of $W^{1,2}(M)$ in Section 4.1 and the definition of the gradient in terms of the differential in (4.2.2). Moreover, we see from (4.2.3) that $-\operatorname{div}$ has a closed extension that is formally adjoint to grad. This requires the density of $C_0^\infty(M)$ in $W^{1,2}(M)$, which was shown to hold on a complete Riemannian manifold by Aubin in [4] (see also Theorem 3.1 in [39]).

Now suppose that M is an *embedded submanifold* of $\mathbb{R}^n$ for some $n > m$. This is defined to mean that there exists a smooth embedding, i.e. an injective immersion, $\iota : M \to \mathbb{R}^n$. In that case, we identify M with its image $\iota(M) \subseteq M$. The differential of ι at $x \in M$ is a linear map from T_xM to $T_{\iota(x)}\mathbb{R}^n$, which we identify with a mapping $(\iota_*)_x : T_xM \to \mathbb{R}^n$. The normal bundle NM is the bundle over M whose fiber at each $x \in M$ is the orthogonal complement of T_xM in $\mathbb{R}^n$, which we denote by N_xM so that $T_xM \oplus N_xM \approx \mathbb{R}^n$. The orthogonal projection from $\mathbb{R}^n$ onto the tangent bundle TM, which we denote by $\pi : L^2(M;\mathbb{C}^n) \to L^2(TM)$, is formally adjoint to the differential of the embedding, i.e. $\pi = (\iota_*)^*$. The second fundamental form is the bundle homomorphism $h : TM \times TM \to NM$ whose components in a coordinate chart are

$$h_{ij} = \nabla_i(\partial_j\iota) = \partial_i\partial_j\iota - \Gamma_{ij}^k\partial_k\iota. \tag{4.2.4}$$

The first fundamental form is simply the metric $g_{ij} = \partial_i\iota\partial_j\iota$, which is induced by the embedding.

Now recall the following from the Introduction. Let I denote the identity operator on $L^2(M)$ and following [10] define the operator

$$S = \begin{bmatrix} I \\ \operatorname{grad} \end{bmatrix} : \mathsf{D}(S) \subseteq L^2(M) \to L^2(M) \oplus L^2(TM)$$

with $\mathsf{D}(S) = W^{1,2}(M)$ and adjoint

$$S^* = \begin{bmatrix} I & -\operatorname{div} \end{bmatrix} : \mathsf{D}(S^*) \subseteq L^2(M) \oplus L^2(TM) \to L^2(M).$$

These operators are closed and densely defined.

Given $A_{00} \in L^\infty(M)$ as well as $A_{01} \in \mathscr{L}^\infty(TM,\mathbb{C})$, $A_{10} \in \mathscr{L}^\infty(\mathbb{C},TM)$ and $A_{11} \in \mathscr{L}^\infty(TM)$, define the operator $A : L^2(M) \oplus L^2(TM) \to L^2(M) \oplus L^2(TM)$ by

$$(Au)_x = \begin{bmatrix} (A_{00})_x & (A_{01})_x \\ (A_{10})_x & (A_{11})_x \end{bmatrix} \begin{bmatrix} (u_0)_x \\ (u_1)_x \end{bmatrix}$$

for all $u = (u_0, u_1) \in L^2(M) \oplus L^2(TM)$ and $x \in M$, where $(\cdot)_x$ denotes the value of a function or section at x. Furthermore, given $a \in L^\infty(M)$, suppose that there exists κ_1, $\kappa_2 > 0$ such that the following accretivity conditions are satisfied:

$$\begin{aligned} \operatorname{Re}\langle a\,u, u\rangle_{L^2(TM)} &\geq \kappa_1\|u\|_{L^2(M)}^2 \quad \text{for all} \quad u \in L^2(M); \\ \operatorname{Re}\langle ASu, Su\rangle_{L^2(M)\oplus L^2(TM)} &\geq \kappa_2\|u\|_{W^{1,2}(M)}^2 \quad \text{for all} \quad u \in W^{1,2}(M). \end{aligned} \tag{4.2.5}$$

The divergence form operator $L_A : \mathsf{D}(L_A) \to L^2(M)$ is then defined by

$$L_Au = aS^*ASu = -a\operatorname{div}(A_{11}\operatorname{grad} u) - a\operatorname{div}(A_{10}u) + aA_{01}\operatorname{grad} u + aA_{00}u \tag{4.2.6}$$

for all $u \in \mathsf{D}(L_A) = \{u \in W^{1,2}(M) : ASu \in \mathsf{D}(S^*)\}$.

To prove Theorem 1.1.7, we define operators $\{\Gamma, B_1, B_2\}$ acting in a Hilbert space $\mathcal{H}$ such that L_A is a component of the first-order system $\Pi_B^2 := \Gamma + B_1\Gamma^* B_2$. First, define the operator

$$\tilde{S} = \begin{bmatrix} I \\ \mathrm{grad}_M \end{bmatrix} := \begin{bmatrix} I & 0 \\ 0 & \iota_* \end{bmatrix} S : \mathsf{D}(\tilde{S}) \subseteq L^2(M) \to L^2(M; \mathbb{C}^{1+n})$$

with $\mathsf{D}(\tilde{S}) = W^{1,2}(M)$ and adjoint

$$\tilde{S}^* = \begin{bmatrix} I & -\mathrm{div}_M \end{bmatrix} := S^* \begin{bmatrix} I & 0 \\ 0 & \pi \end{bmatrix} : \mathsf{D}(\tilde{S}^*) \subseteq L^2(M; \mathbb{C}^{1+n}) \to L^2(M).$$

Next, define the pointwise multiplication operator $\tilde{A} \in L^\infty(M; \mathcal{L}(\mathbb{C}^{1+n}))$ by

$$\tilde{A}(x) = \begin{bmatrix} 1 & 0 \\ 0 & (\iota_*)_x \end{bmatrix} \begin{bmatrix} (A_{00})_x & (A_{01})_x \\ (A_{10})_x & (A_{11})_x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \pi_x \end{bmatrix}$$

for almost all $x \in M$. This consists of the four mappings $\tilde{A}_{11} \in L^\infty(M; \mathcal{L}(\mathbb{C}^n))$, $\tilde{A}_{10} \in L^\infty(M; \mathcal{L}(\mathbb{C}; \mathbb{C}^n))$, $\tilde{A}_{01} \in L^\infty(M; \mathcal{L}(\mathbb{C}^n; \mathbb{C}))$ and $\tilde{A}_{00} \in L^\infty(M; \mathcal{L}(\mathbb{C}))$ with

$$\tilde{A}(x) = \begin{bmatrix} \tilde{A}_{00}(x) & \tilde{A}_{01}(x) \\ \tilde{A}_{10}(x) & \tilde{A}_{11}(x) \end{bmatrix} := \begin{bmatrix} (A_{00})_x & (A_{01})_x \pi_x \\ (\iota_*)_x (A_{10})_x & (\iota_*)_x (A_{11})_x \pi_x \end{bmatrix}.$$

The operators $\{\Gamma, B_1, B_2\}$ acting in the Hilbert space $\mathcal{H} = L^2(M) \oplus L^2(M; \mathbb{C}^{1+n})$ are defined below:

$$\Gamma = \begin{bmatrix} 0 & 0 \\ \tilde{S} & 0 \end{bmatrix}; \quad \Gamma^* = \begin{bmatrix} 0 & \tilde{S}^* \\ 0 & 0 \end{bmatrix}; \quad B_1 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}; \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{A} \end{bmatrix}. \tag{4.2.7}$$

In that case, the operators from Definition 4.1.1 are as follows:

$$\Gamma_B^* = B_1 \Gamma^* B_2 = \begin{bmatrix} 0 & a\tilde{S}^*\tilde{A} \\ 0 & 0 \end{bmatrix}; \quad \Pi_B = \Gamma + \Gamma_B^* = \begin{bmatrix} 0 & a\tilde{S}^*\tilde{A} \\ \tilde{S} & 0 \end{bmatrix};$$
$$\Pi_B^2 = \begin{bmatrix} a\tilde{S}^*\tilde{A}\tilde{S} & 0 \\ 0 & \tilde{S}a\tilde{S}^*\tilde{A} \end{bmatrix} = \begin{bmatrix} L_A & 0 \\ 0 & \tilde{S}a\tilde{S}^*\tilde{A} \end{bmatrix}.$$

We require additional geometric assumptions in order to verify that these operators satisfy the requirements of Theorem 4.1.5. This is the content of the following proposition.

Proposition 4.2.1. If the Ricci curvature Ric of M is uniformly bounded below and the second fundamental form h of the embedding $\iota : M \to \mathbb{R}^n$ is uniformly bounded, then the operators $\{\Gamma, B_1, B_2\}$ on the Hilbert space $\mathcal{H} = L^2(M; \mathbb{C}^{2+n})$ satisfy hypotheses (H1)–(H8) from Section 4.1.

Proof. Let $\|\cdot\|$ denote the norm on $L^2(M; \mathbb{C}^{2+n})$. Hypotheses (H1) and (H3)–(H6) are immediate and do not require the geometric assumptions in the proposition.

(H2). If $u \in \mathsf{R}(\Gamma^*)$, then $u = (\tilde{S}^*\tilde{u}, 0)$ for some $\tilde{u} \in \mathsf{D}(\tilde{S}^*)$ such that $\tilde{S}^*\tilde{u} \in L^2(M)$. The accretivity assumption on a in (4.2.5) then implies that

$$\mathrm{Re}(B_1 u, u) = \mathrm{Re}(a\tilde{S}^*\tilde{u}, \tilde{S}^*\tilde{u}) \geq \kappa_1 \|\tilde{S}^*\tilde{u}\|^2 = \kappa_1 \|u\|^2.$$

If $u \in \mathsf{R}(\Gamma)$, then $u = (0, \tilde{S}u_0)$ for some $u_0 \in \mathsf{D}(\tilde{S}) = W^{1,2}(M)$. The accretivity assumption on A in (4.2.5) then implies that

$$\begin{aligned}\operatorname{Re}(B_2 u, u) &= \operatorname{Re}(\begin{bmatrix} I & 0 \\ 0 & \iota_* \end{bmatrix} A \begin{bmatrix} I & 0 \\ 0 & \pi \end{bmatrix} \tilde{S}u_0, \tilde{S}u_0) \\ &= \operatorname{Re}\langle ASu_0, Su_0\rangle_{L^2(M)\oplus L^2(TM)} \\ &\geq \kappa_2 \|u_0\|^2_{W^{1,2}(M)} \\ &= \kappa_2 \|Su_0\|^2_{L^2(M)\oplus L^2(TM)} \\ &= \kappa_2 \|\tilde{S}u_0\|^2 \\ &= \kappa_2 \|u\|^2.\end{aligned}$$

(H7). For the first part, it suffices to show that there exists $c > 0$ such that for all balls B in M the following hold:

$$\left| \int_B u \; \mathrm{d}\mu \right| \leq \mu(B)^{\frac{1}{2}} \|u\|_{L^2(M)}; \qquad \left| \int_B \operatorname{grad}_M u \; \mathrm{d}\mu \right| \leq c\, \mu(B)^{\frac{1}{2}} \|u\|_{L^2(M)}$$

for all $u \in W^{1,2}(M)$ with compact support in B. The first of these is given by the Cauchy–Schwarz inequality. Now recall that ι_* is the differential of the embedding map ι. Integrating by parts, we then obtain

$$\begin{aligned}\left| \int_B \operatorname{grad}_M u \; \mathrm{d}\mu \right| &= \left| \int_B \iota_* \operatorname{grad} u \; \mathrm{d}\mu \right| \\ &= \left| \int_B (\operatorname{grad} u)(\iota) \; \mathrm{d}\mu(x) \right| \\ &= \left| \int_B \langle \nabla u, \nabla \iota \rangle_{T^*M} \; \mathrm{d}\mu \right| \\ &= \left| \int_B u(\Delta \iota) \; \mathrm{d}\mu \right| \\ &\leq \sup_{x\in M} |h_x| \left| \int_B u \; \mathrm{d}\mu \right| \\ &\lesssim \mu(B)^{1/2} \|u\|_2,\end{aligned}$$

where the penultimate inequality follows from the definition of the second fundamental form h in (4.2.4). This proves the second inequality above.

To verify the second part of (H7), suppose that $u = (u_0, u_1, \tilde{u}) \in L^2(M; \mathbb{C}^{1+1+n})$ has compact support in a ball B in M and that $(u_1, \tilde{u}) \in \mathsf{D}(S^*)$. This implies that $\pi\tilde{u} \in \mathsf{D}(\operatorname{div})$, and since π is defined pointwise on M, the vector field $\pi\tilde{u}$ is compactly supported in B. Therefore, using the Cauchy–Schwarz inequality and the Riemannian divergence theorem (see, for instance, Section III.7.5 in [22]), we obtain

$$\begin{aligned}\left|\int_B \Gamma^* u \, \mathrm{d}\mu\right| &= \left|\int_B u_1 - \operatorname{div}_M \tilde{u} \, \mathrm{d}\mu\right| \\ &\leq \mu(B)^{\frac{1}{2}} \|u_1\|_{L^2(M)} + \left|\int_B \operatorname{div}(\pi\tilde{u}) \, \mathrm{d}\mu\right| \\ &= \mu(B)^{\frac{1}{2}} \|u_1\|_{L^2(M)} \\ &= \mu(B)^{\frac{1}{2}} \|u\|.\end{aligned}$$

Note that we did not require the condition that the radius $r(B) \leq 1$ to verify (H7).

(H8). Consider two cases:
(i) Let $u \in \mathsf{R}(\Gamma^*) \cap \mathsf{D}(\Pi)$. This implies that $u = (u_0, 0)$ for some $u_0 \in L^2(M)$ and

$$\|\Pi u\| = \|\Gamma u\| = \|\tilde{S} u_0\|_{L^2(M;\mathbb{C}^{1+n})} = \|S u_0\|_{L^2(M)\oplus L^2(TM)} = \|u\|_{W^{1,2}(M;\mathbb{C}^{2+n})}.$$

(ii) Let $u \in \mathsf{R}(\Gamma) \cap \mathsf{D}(\Pi)$. This implies that $u = (0, \tilde{S} u_0)$ for some $u_0 \in W^{1,2}(M)$ and

$$\|\Pi u\| = \|\Gamma^* u\| = \|\tilde{S}^* \tilde{S} u_0\|_{L^2(M)} = \|u_0 - \operatorname{div} \operatorname{grad} u_0\|_{L^2(M)} = \|(I + \Delta) u_0\|_{L^2(M)}.$$

Also, we have

$$\begin{aligned}\|u\|^2_{W^{1,2}(M;\mathbb{C}^{2+n})} &= \|\tilde{S} u_0\|^2_{W^{1,2}(M;\mathbb{C}^{1+n})} \\ &= \|u_0\|^2_{W^{1,2}(M)} + \|\iota_* \operatorname{grad} u_0\|^2_{W^{1,2}(M;\mathbb{C}^n)} \\ &= \|u_0\|^2_{L^2(M)} + \|\nabla u_0\|^2_{L^2(T^*M)} + \|\iota_* \operatorname{grad} u_0\|^2_{L^2(M;\mathbb{C}^n)} + \|\nabla(\iota_* \operatorname{grad} u_0)\|^2_{L^2(T^*M)} \\ &= \|u_0\|^2_{L^2(M)} + 2\|\nabla u_0\|^2_{L^2(T^*M)} + \|\nabla(\iota_* \operatorname{grad} u_0)\|^2_{L^2(T^*M)}.\end{aligned}$$

In a coordinate chart, the integrand in the last term is given by

$$\nabla_k[g^{ij}(\partial_i u_0)(\partial_j \iota)] = \nabla_k(g^{ij})(\partial_i u_0)(\partial_j \iota) + g^{ij} \nabla_k(\partial_i u_0)(\partial_j \iota) + g^{ij}(\partial_i u_0) \nabla_k(\partial_j \iota).$$

The compatibility of the Levi-Civita connection ∇ with the metric on M implies that $\nabla_k(g^{ij}) = 0$. The second term in the equation above is related to the first fundamental form, which is simply the metric g induced by the embedding, and the hessian ∇^2 from (4.2.1). The last term is related to the second fundamental form h of the embedding from (4.2.4). Moreover, we have

$$\|\nabla(\iota_* \operatorname{grad} u_0)\|^2_{L^2(T^*M)} \leq \|\nabla^2 u_0\|^2_{L^2(T^*M)} + \sup_{x\in M} |h_x| \|\nabla u_0\|^2_{L^2(T^*M)}.$$

The Bochner–Lichnerowicz–Weitzenböck formula, as applied in Proposition 3.3 of [39], shows that

$$\begin{aligned}\|\nabla^2 u_0\|^2_{L^2(T^*M)} &= \|\Delta u_0\|^2_{L^2(M)} - \int_M \operatorname{Ric}\langle \nabla u_0, \nabla u_0 \rangle_{T^*M} \, \mathrm{d}\mu \\ &\leq \|\Delta u_0\|^2_{L^2(M)} - \inf_{x\in M} \operatorname{Ric}_x \|\nabla u_0\|^2_{L^2(T^*M)}.\end{aligned}$$

Altogether, the bounds on the Ricci curvature and the second fundamental form then imply that

$$\|u\|^2_{W^{1,2}(M;\mathbb{C}^{2+n})} \lesssim \|u_0\|^2_{L^2(M)} + \|\nabla u_0\|^2_{L^2(T^*M)} + \|\Delta u_0\|^2_{L^2(M)}.$$

Finally, we integrate by parts and use the functional calculus for the self-adjoint operator Δ to obtain

$$\|u\|^2_{W^{1,2}(M;\mathbb{C}^{2+n})} \lesssim \|u_0\|^2_{L^2(M)} + \|\Delta u_0\|^2_{L^2(M)} \lesssim \|(I+\Delta)u_0\|^2_{L^2(M)},$$

which allows us to conclude that

$$\|u\|_{W^{1,2}(M;\mathbb{C}^{2+n})} \lesssim \|\Pi u\|_{L^2(M;\mathbb{C}^{2+n})}.$$

□

We can now solve the Kato square root problem for the divergence form operator L_A defined by (4.2.6). This is the content of Theorem 1.1.7 from the Introduction, which is proved below as a corollary of the quadratic estimate in Theorem 4.1.5.

Proof of Theorem 1.1.7. The lower bound on the Ricci curvature implies that both ($\mathrm{E_{loc}}$) and ($\mathrm{P_{loc}}$) are satisfied on M. The volume growth condition ($\mathrm{E_{loc}}$) is a consequence of the Bishop–Gromov volume comparison theorem, which can be found in, for instance, [22]. The local Poincaré inequality is a result of Buser in [17]. A concise summary of these and other properties of manifolds with Ricci curvature bounded below can be found in Section 5.6.3 of [61].

Now let $\{\Gamma, B_1, B_2\}$ denote the operators defined in (4.2.7). It follows from Proposition 4.2.1 that the requirements of Theorem 4.1.5 are satisfied. Therefore, Corollary 4.1.6 implies that $\mathsf{D}(\sqrt{\Pi_B^2}) = \mathsf{D}(\Pi_B) = \mathsf{D}(\Gamma) \cap \mathsf{D}(\Gamma_B^*)$ with

$$\|\sqrt{\Pi_B^2}\, u\|_2 \eqsim \|\Pi_B u\|_2 \eqsim \|\Gamma u\|_2 + \|\Gamma_B^* u\|_2$$

for all $u \in \mathsf{D}(\sqrt{\Pi_B^2})$. When we restrict this result to $u \in L^2(M)$, we obtain $\mathsf{D}(\sqrt{L_A}) = \mathsf{D}(\tilde{S}) = W^{1,2}(M)$ with

$$\|\sqrt{L_A}\, u\|_2 \eqsim \|\tilde{S}u\|_2 = \|Su\|_2 = \|u\|_{W^{1,2}(M)}$$

for all $u \in W^{1,2}(M)$. □

4.3 Christ's Dyadic Cubes and Carleson Measures

The results in this section do not require a differentiable structure. Therefore, as in Section 3.1, let X denote a metric measure space with metric ρ and Borel measure μ. Also, recall property ($\mathrm{D_{loc}}$) from Definition 3.1.1, which was used to define a locally doubling metric measure space.

The proof of Theorem 4.1.5 in the case $M = \mathbb{R}^n$ in [11] relies on the dyadic cube structure of $\mathbb{R}^n$. In [24], Christ constructs an analogue of the dyadic cube structure for a doubling metric measure space. This construction can also be applied on a locally doubling metric measure space to provide a truncated dyadic cube structure. This is the content of the following proposition. The proof follows as in [24].

Proposition 4.3.1. Let X be a locally doubling metric measure space. There exists a countable collection $\Delta_{(0,1]} = (Q^k_\alpha)_{\alpha\in I_k, k\in\mathbb{N}_0}$ of open subsets of X, indexed by some set I_k for each integer $k \geq 0$, and a sequence $(x^k_\alpha)_{\alpha\in I_k, k\in\mathbb{N}_0}$ of points in X, together with constants $\delta, \eta \in (0,1)$ and $a_0, a_1, c > 0$, such that the following hold:

1. $\mu\left(X \setminus \bigcup_{\alpha\in I_k} Q^k_\alpha\right) = 0$;
2. $Q^k_\alpha \cap Q^k_\beta = \emptyset$ for all $\alpha, \beta \in I_k$;
3. For each integer $l > k$, $\alpha \in I_k$ and $\beta \in I_l$, either $Q^l_\beta \subset Q^k_\alpha$ or $Q^k_\alpha \cap Q^l_\beta = \emptyset$;
4. For each integer $l \in [0, k)$ and $\alpha \in I_k$ there is a unique $\beta \in I_l$ such that $Q^k_\alpha \subset Q^l_\beta$;
5. $B(x^k_\alpha, a_0\delta^k) \subseteq Q^k_\alpha \subseteq B(x^k_\alpha, a_1\delta^k)$ for all $\alpha \in I_k$;
6. $\mu(\{x \in Q^k_\alpha : \rho(x, X \setminus Q^k_\alpha) \leq s\delta^k\}) \leq cs^\eta \mu(Q^k_\alpha)$ for all $\alpha \in I_k$ and $s > 0$.

Any collection of sets $\Delta_{(0,1]} = (Q^k_\alpha)_{\alpha\in I_k, k\in\mathbb{N}_0}$ with the properties in Proposition 4.3.1 is called a *truncated dyadic cube structure on* X; the sets in $\Delta_{(0,1]}$ are called *dyadic cubes*. Given $t \in (0,1]$, define the collection of dyadic cubes $\Delta_t := (Q^k_\alpha)_{\alpha\in I_k}$ by requiring that $k \in \mathbb{N}_0$ satisfy $\delta^{k+1} < t \leq \delta^k$. For all $Q \in \Delta_t$, define the *side length of* Q by $l(Q) := \delta^k$ and the *Carleson box over* Q by $C(Q) := Q \times (0, l(Q)]$. Note that $t \leq l(Q) < t/\delta$ and so $l(Q) \eqsim t$. The *dyadic averaging operator* A_t is then defined for all $u \in L^1_{\text{loc}}(X)$ by

$$A_t u(x) = \fint_Q u(y)\, \mathrm{d}\mu(y) := \frac{1}{\mu(Q)} \int_Q u(y)\, \mathrm{d}\mu(y)$$

for all $t \in (0,1]$ and almost all $x \in X$, where Q is the unique dyadic cube in Δ_t containing x.

Notation. Given a truncated dyadic cube structure on X with the constants specified in Proposition 4.3.1, the constant $a := \max\{1, a_1/\delta\}$.

It is useful to record the following inequalities, which will be used frequently. Given $t \in (0,1]$, dyadic cubes $Q, R \in \Delta_t$ and points $x_Q, x_R \in X$ such that

$$B(x_Q, a_0 l(Q)) \subseteq Q \subseteq B(x_Q, a_1 l(Q)) \quad \text{and} \quad B(x_R, a_0 l(R)) \subseteq R \subseteq B(x_R, a_1 l(R)),$$

the following are easily verified:

$$\begin{aligned} \rho(Q,R) \leq \rho(x_Q, x_R) &\leq a\,(2t + \rho(Q,R))\,; \\ \rho(Q,R) \leq \rho(Q,x) &\leq a\,(2t + \rho(Q,R)) \quad \text{for all} \quad x \in R; \\ \rho(Q,x) \leq \rho(x_Q,x) &\leq a\,(t + \rho(Q,x)) \quad \text{for all} \quad x \in X. \end{aligned} \tag{4.3.1}$$

In the next section, we reduce the proof of Theorem 4.1.5 to verifying a local analogue of Carleson's condition. The relevant properties of Carleson measures are recorded in the following two results. In these proofs, we use the notation introduced for tent spaces at the beginning of Section 3.2.

Lemma 4.3.2. Let X be a locally doubling metric measure space and suppose that $t_0 \in (0,1]$. If ν is a (positive) measure on $X \times (0,t_0]$ satisfying the following local analogue of Carleson's condition

$$\|\nu\|_{\mathcal{C}} := \sup_{t\in(0,t_0]} \sup_{Q\in\Delta_t} \frac{1}{\mu(Q)} \iint_{C(Q)} d\nu(x,t) < \infty,$$

then

$$\sup_{B\in\mathscr{B}_0} \frac{1}{\mu(B)} \iint_{T(B)} d\nu(x,t) \lesssim \|\nu\|_{\mathcal{C}},$$

where $\mathscr{B}_0$ denotes the collection of all balls in X of radius $r \in (0,t_0]$.

Proof. Let $B = B(x,r)$ denote a ball in X of radius $r \in (0,t_0]$ and let

$$\Delta_r(B) = \{Q \in \Delta_r \mid Q \cap B \neq \emptyset\}.$$

Let $N(B) = \sharp\Delta_r(B)$ and let $\{Q_\alpha\}_{\alpha=1,\ldots,N(B)}$ denote an enumeration of $\Delta_r(B)$. The local Carleson condition and the inclusion $T(B) \subseteq \bigcup_{\alpha=1}^{N(B)} C(Q_\alpha)$ then imply that

$$\nu(T(B)) \leq N(B)\|\nu\|_{\mathcal{C}} \max_{\alpha=1,\ldots,N(B)} \mu(Q_\alpha).$$

Proposition 4.3.1 shows that there exists a set of disjoint balls $\{B_\alpha\}_{\alpha=1,\ldots,N(B)}$ of radius $a_0 l(Q)$ such that $B_\alpha \subseteq Q_\alpha$ for all $\alpha \in \{1,\ldots,N(B)\}$. Each $B_\alpha \subseteq Q_\alpha \in \Delta_r(B)$ is contained in $B(x, 2a_1 l(Q)+r) \subseteq B(x,(2a_1+1)l(Q))$, since $r \leq l(Q)$ for all $Q \in \Delta_r$. Moreover, the centres of the balls B_α are separated by a distance of at least $2a_0 l(Q)$. Then, since $l(Q) < r/\delta \leq t_0/\delta$, the local property of homogeneity from Remark 3.1.6 implies that there exists $N \in \mathbb{N}$ such that $N(B) \leq N$ for all $B \in \mathscr{B}_0$. The estimate from the previous paragraph, the inclusion $Q_\alpha \subseteq (2a_1/\delta+1)B$ and the local doubling property ($\mathrm{D}_{\mathrm{loc}}$) then imply that

$$\nu(T(B)) \leq N\|\nu\|_{\mathcal{C}}\, \mu((2a_1/\delta+1)B) \lesssim \|\nu\|_{\mathcal{C}}\, \mu(B)$$

for all $B \in \mathscr{B}_0$, as required. □

The following proof is a localized version of results by Stein in Section II.2.3 of [64].

Theorem 4.3.3. *Let X be a locally doubling metric measure space and suppose that $t_0 \in (0,1]$. If ν is a (positive) measure on $X \times (0,t_0]$ satisfying the local Carleson condition in Lemma 4.3.2, then*

$$\iint_{X\times(0,t_0]} |A_t u(x)|^2 \,\mathrm{d}\nu(x,t) \lesssim \|\nu\|_{\mathcal{C}} \|u\|^2$$

for all $u \in L^2(X)$.

Proof. Let $t_1 = t_0/2$ and let $\mathcal{A}^\infty_{\mathrm{loc}} f(x) = \sup_{(y,t)\in\Gamma^{t_1}(x)} |f(y,t)|$ for all measurable functions on f on $X \times (0,t_1]$. We will reduce the proof to verifying the following two facts:

1. For all $\alpha \geq 0$, we have
$$\{(x,t) \in X \times (0,t_1] : |f(x,t)| > \alpha\} \subseteq T^{t_1}(\{x \in X : \mathcal{A}^\infty_{\text{loc}} f(x) > \alpha\});$$
2. There exists $c > 0$ such that for all open sets $O \subseteq X$, we have
$$\nu(T^{t_1}(O)) \leq c\|\nu\|_{\mathcal{C}}\mu(O).$$

To see that 1 and 2 imply the result, set $O = \{x \in X \mid \mathcal{A}^\infty_{\text{loc}} f(x) > \alpha\}$ in 2, which is an open set by the lower semicontinuity of $\mathcal{A}^\infty_{\text{loc}} f$, then by 1 we obtain
$$\nu(\{(x,t) \in X \times (0,t_1] : |f(x,t)| > \alpha\}) \leq c\|\nu\|_{\mathcal{C}}\mu(O).$$
Integrating over all positive α and applying Fubini's Theorem shows that
$$\iint_{X\times(0,t_1]} |f(x,t)| \, \mathrm{d}\nu(x,t) \lesssim \|\nu\|_{\mathcal{C}} \int_X \mathcal{A}^\infty_{\text{loc}} f(x) \, \mathrm{d}\mu(x).$$
Now set $f(x,t) = |A_t u(x)|^2$. The local doubling property (D_{loc}) and the properties of the local maximal operator $\mathcal{M}_{\text{loc}}$ from Proposition 3.1.10 then imply that
$$\begin{aligned}\iint_{X\times(0,t_1]} |A_t u(x)|^2 \, \mathrm{d}\nu(x,t) &\lesssim \|\nu\|_{\mathcal{C}} \int_X \sup_{(y,t)\in\Gamma^{t_1}(x)} |A_t u(y)|^2 \, \mathrm{d}\mu(x) \\ &\lesssim \|\nu\|_{\mathcal{C}}\|\mathcal{M}_{\text{loc}} u\|^2 \\ &\lesssim \|\nu\|_{\mathcal{C}}\|u\|^2.\end{aligned}$$
It is a straightforward matter to verify that
$$\iint_{X\times(t_1,t_0]} |A_t u(x)|^2 \, \mathrm{d}\nu(x,t) \lesssim \|\nu\|_{\mathcal{C}}\|u\|^2$$
so the result follows.

To prove 1, let $(x,t) \in X \times (0,t_1]$ with $|f(x,t)| > \alpha \geq 0$. If $y \in B(x,t)$, then $(x,t) \in \Gamma^{t_1}(y)$ and so $\mathcal{A}^\infty_{\text{loc}} f(y) > \alpha$. This shows that
$$B(x,t) \subseteq \{x \in X : \mathcal{A}^\infty_{\text{loc}} f(x) > \alpha\}$$
and hence
$$(x,t) \in T^{t_1}(B(x,t)) \subseteq T^{t_1}(\{x \in X : \mathcal{A}^\infty_{\text{loc}} f(x) > \alpha\}).$$

To prove 2, we apply the local Whitney type covering lemma from Proposition 3.1.9 to decompose O into a sequence of disjoint balls $(B_j)_j$ with centre $x_j \in X$ and radius $r_j = \frac{1}{8}\min(\rho(x_j, {}^cO), t_1)$ such that $O = \bigcup_j 4B_j$. We claim that
$$T^{t_1}(O) \subseteq \bigcup_j T^{t_1}(13B_j). \tag{4.3.2}$$
To see this, let $(x,t) \in T^{t_1}(O)$, in which case $t \leq \min(\rho(x, {}^cO), t_1)$. Choose $k \in \mathbb{N}$ so that $x \in 4B_k$ and let x_k be the centre of B_k. If $y \in B(x,t)$, then
$$\rho(y,x_k) \leq \rho(y,x)+\rho(x,x_k) \leq \min(\rho(x,{}^cO),t_1)+4r_k \leq \min(\rho(x_k,{}^cO),t_1)+5r_k = 13r_k.$$

This shows that $B(x,t) \subseteq 13B_k$, which implies that $(x,t) \in T^{t_1}(B(x,t)) \subseteq T^{t_1}(13B_k)$ and that (4.3.2) holds. The radius $r(13B_j) \leq 13t_1/8 < t_0$ so by Lemma 4.3.2 and ($\mathrm{D}_{\mathrm{loc}}$), we now have

$$\nu(T^{t_1}(O)) \leq \sum_j \nu(T(13B_j)) \lesssim \|\nu\|_{\mathcal{C}} \sum_j \mu(13B_j) \lesssim \|\nu\|_{\mathcal{C}}\mu(O)$$

and the proof is complete. □

We conclude this section by recording two technical results for use later on. Recall that for all $x \geq 0$, there is the notation $\langle x \rangle = \min\{1, x\}$.

Lemma 4.3.4. Let X be a metric measure space satisfying ($\mathrm{E}_{\mathrm{loc}}$). Let $\Delta_{(0,1]}$ denote a truncated unit cube structure on X with the constant $a = \max\{1, a_1/\delta\}$. Then

$$\left\langle \frac{t}{\rho(Q,R)} \right\rangle^{\kappa} e^{-a\lambda\rho(Q,R)} \lesssim \frac{\mu(Q)}{\mu(R)} \lesssim \left\langle \frac{t}{\rho(Q,R)} \right\rangle^{-\kappa} e^{a\lambda\rho(Q,R)}$$

for all $Q, R \in \Delta_t$ and $t \in (0,1]$.

Proof. It suffices to show the second inequality, since the estimate is symmetric in R and Q. It follows from ($\mathrm{E}_{\mathrm{loc}}$) that

$$V(x,r) \leq A\left(1 + \frac{\rho(x,y)}{r}\right)^{\kappa} e^{\lambda(r+\rho(x,y))} V(y,r),$$

for all $x, y \in X$ and $r > 0$, since $B(x,r) \subseteq B(y, (1 + \frac{\rho(x,y)}{r})r)$. Given $t \in (0,1]$ and $Q, R \in \Delta_t$, it then follows from Proposition 4.3.1 and ($\mathrm{E}_{\mathrm{loc}}$) that there exists $x_Q, x_R \in X$ such that

$$\begin{aligned}
\frac{\mu(Q)}{\mu(R)} &\leq \frac{V(x_Q, a_1 l(Q))}{V(x_R, a_0 l(R))} \\
&\leq A(\tfrac{a_1}{a_0})^{\kappa} e^{\lambda a_1 l(Q)} \frac{V(x_Q, a_0 l(Q))}{V(x_R, a_0 l(R))} \\
&\lesssim A\left(1 + \frac{\rho(x_Q, x_R)}{a_0 l(Q)}\right)^{\kappa} e^{\lambda(a_0 l(Q) + \rho(x_Q, x_R))} \\
&\lesssim \left(1 + \frac{\rho(x_Q, x_R)}{t}\right)^{\kappa} e^{\lambda\rho(x_Q, x_R)}.
\end{aligned}$$

For all $x > 0$, we have $1 + x \leq 2\max\{1, x\} = 2\langle 1/x \rangle^{-1}$. Using this and the above estimate with (4.3.1), we conclude that

$$\frac{\mu(Q)}{\mu(R)} \lesssim \left\langle \frac{t}{\rho(x_Q, x_R)} \right\rangle^{-\kappa} e^{\lambda\rho(x_Q,x_R)} \lesssim \left\langle \frac{t}{\rho(Q,R)} \right\rangle^{-\kappa} e^{a\lambda\rho(Q,R)}$$

for all $Q, R \in \Delta_t$. □

Lemma 4.3.5. Let X be a metric measure space satisfying ($\mathrm{E}_{\mathrm{loc}}$). Let $\Delta_{(0,1]}$ denote a truncated unit cube structure on X. If $t \in (0,1]$, $M > \kappa$ and $m > \lambda t$, then

$$\sup_{R \in \Delta_t} \sum_{Q \in \Delta_t} \frac{\mu(Q)}{\mu(R)} \left\langle \frac{t}{\rho(Q,R)} \right\rangle^{M} e^{-m\frac{\rho(Q,R)}{t}} \lesssim 1.$$

Proof. Suppose that $t \in (0,1]$, $M > \kappa$ and $m > \lambda t$. Let $\sigma = m/\lambda t > 1$ and for each $R \in \Delta_t$, let

$$\Delta_t^j(R) = \begin{cases} \{Q \in \Delta_t : \rho(Q,R)/t \leq 1\} & \text{if} \quad j = 0; \\ \{Q \in \Delta_t : \sigma^{j-1} < \rho(Q,R)/t \leq \sigma^j\} & \text{if} \quad j \in \mathbb{N}. \end{cases}$$

For each $R \in \Delta_t$, Proposition 4.3.1 implies that there exists $x_R \in X$ such that

$$B(x_R, a_0 l(R)) \subseteq R \subseteq B(x_R, a_1 l(R)).$$

A simple calculation then shows that

$$\bigcup \Delta_t^j(R) \subseteq B(x_R, 3a_1 l(R) + \sigma^j t)$$

for all $j \in \mathbb{N}_0$, and it follows from ($\mathrm{E_{loc}}$) that

$$\mu\big(\bigcup \Delta_t^j(R)\big) \lesssim \sigma^{j\kappa} e^{\lambda \sigma^j t} \mu(R)$$

for all $j \in \mathbb{N}_0$. Therefore, we have

$$\begin{aligned}
&\sup_{R\in\Delta_t} \sum_{Q\in\Delta_t} \frac{\mu(Q)}{\mu(R)} \left\langle \frac{t}{\rho(Q,R)} \right\rangle^M e^{-m\frac{\rho(Q,R)}{t}} \\
&\qquad = \sup_{R\in\Delta_t} \sum_{j=0}^{\infty} \sum_{Q\in\Delta_t^j(R)} \frac{\mu(Q)}{\mu(R)} \left\langle \frac{t}{\rho(Q,R)} \right\rangle^M e^{-m\frac{\rho(Q,R)}{t}} \\
&\qquad \leq \sup_{R\in\Delta_t} \frac{1}{\mu(R)} \Big[\mu\big(\bigcup \Delta_t^0(R)\big) + \sum_{j=1}^{\infty} \sigma^{-(j-1)M} e^{-m\sigma^{j-1}} \mu\big(\bigcup \Delta_t^j(R)\big)\Big] \\
&\qquad \lesssim \sum_{j=0}^{\infty} \sigma^{-j(M-\kappa)} e^{-(m-\sigma\lambda t)2^{j-1}} \\
&\qquad \lesssim 1,
\end{aligned}$$

as required. ∎

4.4 The Main Local Quadratic Estimate

This section contains the proof of Theorem 4.1.5. We consider a complete Riemannian manifold M satisfying ($\mathrm{E_{loc}}$) and ($\mathrm{P_{loc}}$) with constants $\kappa, \lambda \geq 0$, and suppose that $\{\Gamma, B_1, B_2\}$ are operators on $L^2(M;\mathbb{C}^N)$ satisfying hypotheses (H1)–(H8). We use the symbol $\|\cdot\|$ to denote the norm on $L^2(M;\mathbb{C}^N)$. Now fix a truncated dyadic cube structure $\Delta_{(0,1]}$ with constants $\delta, \eta \in (0,1)$ and $a_1 > a_0 > 0$ as in Proposition 4.3.1, and let $a = \max\{1, a_1/\delta\}$. We follow [11, 10] and introduce the following operators.

Definition 4.4.1. Given $t \in \mathbb{R} \setminus \{0\}$, define the following bounded operators:

$$\begin{aligned} R_t^B &:= (I + it\Pi_B)^{-1}; \\ P_t^B &:= (I + t^2\Pi_B^2)^{-1} = \tfrac{1}{2}(R_t^B + R_{-t}^B); \\ Q_t^B &:= t\Pi_B(I + t^2\Pi_B^2)^{-1} = \tfrac{1}{2i}(-R_t^B + R_{-t}^B); \\ \Theta_t^B &:= t\Gamma_B^*(I + t^2\Pi_B^2)^{-1}. \end{aligned}$$

The operators R_t, P_t and Q_t are defined as above by replacing Π_B with Π.

The uniform estimate

$$\sup_{t\in\mathbb{R}\setminus\{0\}} \|U_t\| \lesssim 1 \tag{4.4.1}$$

holds when $U_t = R_t^B$, P_t^B, Q_t^B and Θ_t^B. This follows immediately from (4.1.1) and the resolvent bounds in (4.1.2), since $R_t^B = (i/t)[(i/t)I - \Pi_B]^{-1}$ for all $t \in \mathbb{R} \setminus \{0\}$.

The operator Π is self-adjoint, so by the functional calculus for self-adjoint operators, we have the quadratic estimate

$$\int_0^\infty \|Q_t u\|^2 \, \frac{\mathrm{d}t}{t} \approx \|u\|^2 \tag{4.4.2}$$

for all $u \in \overline{\mathsf{R}(\Pi)}$.

The following result, which is an immediate consequence of Proposition 4.8 in [11] and the inhomogeneity assumed in hypothesis (H8), shows that Theorem 4.1.5 can be reduced to finding $t_0 > 0$ small enough such that a certain local quadratic estimate holds.

Proposition 4.4.2. If there exists $t_0 \in (0, 1]$ such that

$$\int_0^{t_0} \|\Theta_t^B P_t u\|^2 \, \frac{\mathrm{d}t}{t} \lesssim \|u\|^2 \tag{4.4.3}$$

for all $u \in \mathsf{R}(\Gamma)$, as well as the three similar estimates obtained upon replacing $\{\Gamma, B_1, B_2\}$ by $\{\Gamma^*, B_2, B_1\}$, $\{\Gamma^*, B_2^*, B_1^*\}$ and $\{\Gamma, B_1^*, B_2^*\}$, then (4.1.3) holds for all u in $\overline{\mathsf{R}(\Pi_B)}$.

Proof. Suppose that there exists $t_0 \in (0, 1]$ such that (4.4.3) holds for all $u \in \mathsf{R}(\Gamma)$, as well as the three similar estimates mentioned in the proposition. If $u \in \mathsf{R}(\Gamma)$ and $t > 0$, then $P_t u = u - t\Pi Q_t u \in \mathsf{R}(\Pi)$, since the Hodge decomposition guarantees that $\mathsf{R}(\Gamma) \subseteq \mathsf{R}(\Pi)$. Therefore, hypothesis (H8) implies that $\|P_t u\| \lesssim \|\Pi P_t u\|$ for all $u \in \mathsf{R}(\Gamma)$ and $t > 0$. The uniform bound in (4.4.1) then implies that

$$\int_{t_0}^\infty \|\Theta_t^B P_t u\|^2 \, \frac{\mathrm{d}t}{t} \lesssim \int_{t_0}^\infty \|Q_t u\|^2 \, \frac{\mathrm{d}t}{t^3} \lesssim \|u\|^2 \int_{t_0}^\infty \frac{\mathrm{d}t}{t^3} \lesssim \|u\|^2$$

for all $u \in \mathsf{R}(\Gamma)$, which shows that

$$\int_0^\infty \|\Theta_t^B P_t u\|^2 \, \frac{\mathrm{d}t}{t} \lesssim \|u\|^2$$

for all $u \in \mathsf{R}(\Gamma)$, as well as the three similar estimates obtained upon replacing $\{\Gamma, B_1, B_2\}$ by $\{\Gamma^*, B_2, B_1\}$, $\{\Gamma^*, B_2^*, B_1^*\}$ and $\{\Gamma, B_1^*, B_2^*\}$. It then follows from Proposition 4.8 in [11] that (4.1.3) holds for all u in $\overline{\mathsf{R}(\Pi_B)}$. □

The above result allows us to work locally, in the sense that we only need to consider $t \in (0, 1]$, which means that we are not restricted to considering manifolds that are doubling. The metric-measure interaction is instead restricted by the strength of the following off-diagonal estimates. The proof below is based on that of Proposition 5.2 in [11]. Recall that for all $x \geq 0$, there is the notation $\langle x \rangle = \min\{1, x\}$.

Proposition 4.4.3. Let U_t denote either R_t^B, P_t^B, Q_t^B or Θ_t^B for all $t \in \mathbb{R} \setminus \{0\}$. There exists a constant $C_\Theta > 0$, which depends only on the constants in (H1)–(H8), such that the following holds: For each $M \geq 0$, there exists $c > 0$ such that

$$\|\mathbf{1}_E U_t \mathbf{1}_F\| \leq c \left\langle \frac{|t|}{\rho(E,F)} \right\rangle^M \exp\left(-C_\Theta \frac{\rho(E,F)}{|t|} \right)$$

for all closed subsets E and F of M.

Proof. In the case $U_t = R_t^B = (i/t)[(i/t)I - \Pi_B]^{-1}$, the result follows exactly as in the proof of Lemma 3.4.2, since Π_B is of type $S_{\omega\cup 0}$ and (H5)–(H6) imply that

$$\begin{aligned} |[\Pi_B, \eta I]u(x)| &= |[\Gamma, \eta I]u(x) + B_1[\Gamma^*, \eta I]B_2 u(x)| \\ &\leq C_\Gamma(1 + \|B_1\|_\infty \|B_2\|_\infty)|\nabla\eta(x)|_{T_xM}|u(x)| \end{aligned}$$

for all $\eta \in C_0^\infty(M)$, $u \in \mathsf{D}(\Pi_B)$ and almost all $x \in M$. The results for P_t^B and Q_t^B then follow by linearity. In the case $U_t = Q_t^B$, the result is also given by the proof of Lemma 3.4.3.

Now consider $U_t = \Theta_t^B = t\Gamma_B^* P_t^B$. Let E and F be closed subsets of M with $\rho(E,F) > 0$. Let $\tilde{E} = \{x \in M : \rho(x, E) \leq \rho(E,F)/2\}$ and choose $\eta : M \to [0,1]$ in $C_0^\infty(M)$ satisfying

$$\eta(x) = \begin{cases} 1, & \text{if} \quad x \in E; \\ 0, & \text{if} \quad x \in M \setminus \tilde{E} \end{cases}$$

and $\|\nabla\eta\|_\infty \leq 3/\rho(E,F)$, as in the proof of Lemma 3.4.2. Using both (4.1.1) and (H5)–(H6), we obtain

$$\begin{aligned} \|\mathbf{1}_E \Theta_t^B \mathbf{1}_F\| &\leq \|(\eta I)t\Gamma_B^* P_t^B \mathbf{1}_F\| \\ &\leq \|t[\eta I, \Gamma_B^*]P_t^B \mathbf{1}_F\| + \|t\Gamma_B^*(\eta I)P_t^B \mathbf{1}_F\| \\ &\lesssim |t|\|\nabla\eta\|_\infty \|\mathbf{1}_{\tilde{E}} P_t^B \mathbf{1}_F\| + \|t\Pi_B(\eta I)P_t^B \mathbf{1}_F\| \\ &\leq |t|\|\nabla\eta\|_\infty \|\mathbf{1}_{\tilde{E}} P_t^B \mathbf{1}_F\| + |t|\|[\Pi_B, (\eta I)]P_t^B \mathbf{1}_F\| + \|(\eta I)Q_t^B \mathbf{1}_F\| \\ &\leq |t|\|\nabla\eta\|_\infty \|\mathbf{1}_{\tilde{E}} P_t^B \mathbf{1}_F\| + |t|\|\nabla\eta\|_\infty \|\mathbf{1}_{\tilde{E}} P_t^B \mathbf{1}_F\| + \|\mathbf{1}_{\tilde{E}} Q_t^B \mathbf{1}_F\| \end{aligned}$$

for all $t \in \mathbb{R} \setminus \{0\}$. The result then follows from the corresponding estimates for P_t^B and Q_t^B, since $\rho(\tilde{E}, F) = 2\rho(E,F)$. ∎

The off-diagonal estimates imply the following result.

Lemma 4.4.4. The operator Θ_t^B on $L^2(M; \mathbb{C}^N)$ has a bounded extension

$$\Theta_t^B : L^\infty(M; \mathbb{C}^N) \to L^2_{\mathrm{loc}}(M; \mathbb{C}^N)$$

for all $t \in (0, \langle C_\Theta/2a\lambda \rangle]$. Moreover, there exists $c > 0$ such that

$$\|\Theta_t^B u\|^2_{L^2(Q)} \leq c\mu(Q)\|u\|^2_\infty$$

for all $u \in L^\infty(M; \mathbb{C}^N)$, $Q \in \Delta_t$ and $t \in (0, \langle C_\Theta/2a\lambda \rangle]$.

Proof. Let $t \in (0, \langle C_\Theta/2a\lambda \rangle]$ and $Q \in \Delta_t$. There exists $x_Q \in M$ such that

$$B(x_Q, a_0 t) \subseteq Q \subseteq B(x_Q, (a_1/\delta)t).$$

Let $\Delta_t^{m,n}(Q) = \{R \in \Delta_t : m < \rho(Q,R) \leq n\}$ for all integers $n > m \geq 0$. Let $u \in L^\infty(M;\mathbb{C}^N)$ and define $u_n = \mathbf{1}_{\Delta_t^{0,n}(Q)} u$ for all $n \in \mathbb{N}$. If $n > m$, then

$$\begin{aligned}
&\|\Theta_t^B(u_n - u_m)\|^2_{L^2(Q)} \\
&\leq \Bigg[\sum_{R \in \Delta_t^{m,n}(Q)} \left(\frac{\mu(R)}{\mu(Q)} \frac{\mu(Q)}{\mu(R)} \right)^{\frac{1}{2}} \|\mathbf{1}_Q \Theta_t^B \mathbf{1}_R\| \|\mathbf{1}_R u\| \Bigg]^2 \\
&\leq \Bigg(\sum_{R \in \Delta_t^{m,n}(Q)} \frac{\mu(R)}{\mu(Q)} \|\mathbf{1}_Q \Theta_t^B \mathbf{1}_R\| \Bigg) \sum_{R \in \Delta_t^{m,n}(Q)} \frac{\mu(Q)}{\mu(R)} \|\mathbf{1}_Q \Theta_t^B \mathbf{1}_R\| \|\mathbf{1}_R u\|^2 \\
&\leq \Bigg(\sum_{R \in \Delta_t^{m,n}(Q)} \frac{\mu(R)}{\mu(Q)} \|\mathbf{1}_Q \Theta_t^B \mathbf{1}_R\| \Bigg) \sum_{R \in \Delta_t^{m,n}(Q)} \frac{\mu(Q)}{\mu(R)} \|\mathbf{1}_Q \Theta_t^B \mathbf{1}_R\| \|u\|^2_\infty \mu(R) \\
&\leq \Bigg(\sum_{R \in \Delta_t^{m,n}(Q)} \frac{\mu(R)}{\mu(Q)} \|\mathbf{1}_Q \Theta_t^B \mathbf{1}_R\| \Bigg) \Bigg(\sum_{R \in \Delta_t^{m,n}(Q)} \frac{\mu(R)}{\mu(Q)} \frac{\mu(Q)}{\mu(R)} \|\mathbf{1}_Q \Theta_t^B \mathbf{1}_R\| \Bigg) \mu(Q) \|u\|^2_\infty,
\end{aligned}$$

Now choose $M > 2\kappa$. The off-diagonal estimates from Proposition 4.4.3 then show that

$$\|\mathbf{1}_Q \Theta_t^B \mathbf{1}_R\| \lesssim \left\langle \frac{t}{\rho(Q,R)} \right\rangle^M \exp\left(-C_\Theta \frac{\rho(Q,R)}{t} \right)$$

for all $Q, R \in \Delta_t$ and $t \in (0,1]$. Moreover, Lemma 4.3.4 shows that

$$\frac{\mu(Q)}{\mu(R)} \lesssim \left\langle \frac{t}{\rho(Q,R)} \right\rangle^{-\kappa} e^{a\lambda\rho(Q,R)}$$

for $Q, R \in \Delta_t$ and $t \in (0,1]$. Then, since $M - \kappa > \kappa$ and $C_\Theta - a\lambda t > \lambda t$, Lemma 4.3.5 guarantees that both of the partial sums in the estimate above converge. Therefore, the sequence $(\Theta_t^B u_n)_n$ is Cauchy in $L^2(Q)$ and

$$\sup_{n \in \mathbb{N}} \|\Theta_t^B u_n\|^2_{L^2(Q)} \lesssim \mu(Q) \|u\|^2_\infty$$

for all $Q \in \Delta_t$ and $t \in (0, \langle C_\Theta/2a\lambda \rangle]$, which implies the result. ☐

As in [11, 10], we now introduce the following operator to prove (4.4.3).

Definition 4.4.5. For each $w \in \mathbb{C}^N$, let $\tilde{w} \in L^\infty(M;\mathbb{C}^N)$ denote the constant function that is equal to w on M. For each $x \in M$ and $t \in (0, \langle C_\Theta/\lambda \rangle]$, the multiplication operator $\gamma_t(x) \in \mathcal{L}(\mathbb{C}^N)$ is defined by

$$[\gamma_t(x)]w := (\Theta_t^B \tilde{w})(x)$$

for all $w \in \mathbb{C}^N$, where Θ_t^B is defined on $L^\infty(M;\mathbb{C}^N)$ by Lemma 4.4.4.

Corollary 4.4.6. The functions $\gamma_t := (x \mapsto \gamma_t(x)\ \forall\ x \in M)$ are in $L^2_{\mathrm{loc}}(M; \mathcal{L}(\mathbb{C}^N))$ and there exists $c > 0$ such that

$$\fint_Q |\gamma_t(x)|^2\ \mathrm{d}\mu(x) \leq c \tag{4.4.4}$$

for all $Q \in \Delta_t$ and $t \in (0, \langle C_\Theta/2a\lambda\rangle]$. Moreover, $\sup_{t\in(0,\langle C_\Theta/2a\lambda\rangle]} \|\gamma_t A_t\| \lesssim 1$.

Proof. The first property follows from Proposition 4.4.4 and the definition of γ_t. It then follows that

$$\begin{aligned}
\|\gamma_t A_t u\|^2 &= \sum_{Q\in\Delta_t} \int_Q |\gamma_t(y) A_t u(y)|^2\ \mathrm{d}\mu(y) \\
&= \sum_{Q\in\Delta_t} \int_Q \left|\gamma_t(y) \fint_Q u(x)\ \mathrm{d}\mu(x)\right|^2\ \mathrm{d}\mu(y) \\
&= \sum_{Q\in\Delta_t} \left|\fint_Q u(x)\ \mathrm{d}\mu(x)\right|^2 \int_Q |\gamma_t(y)|^2\ \mathrm{d}\mu(y) \\
&\lesssim \sum_{Q\in\Delta_t} \|u\|^2_{L^2(Q)} \\
&= \|u\|^2
\end{aligned}$$

for all $t \in (0, \langle C_\Theta/2a\lambda\rangle]$ and $u \in L^2(M; \mathbb{C}^N)$, which completes the proof. □

To prove (4.4.3), we follow [11, 10] and estimate each of the following terms separately:

$$\begin{aligned}
\int_0^{t_0} \|\Theta_t^B P_t u\|^2\ \frac{\mathrm{d}t}{t} \lesssim \int_0^{t_0} \|\Theta_t^B P_t u - \gamma_t A_t P_t u\|^2\ \frac{\mathrm{d}t}{t} + \int_0^{t_0} \|\gamma_t A_t (P_t - I) u\|^2\ \frac{\mathrm{d}t}{t} \\
+ \int_0^{t_0} \int_M |A_t u(x)|^2 |\gamma_t(x)|^2\ \frac{\mathrm{d}\mu(x)\mathrm{d}t}{t}.
\end{aligned} \tag{4.4.5}$$

The following weighted Poincaré inequality is used to estimate the first term above. The proof is based on techniques contained in Lemma 5.4 of [11] that have been adapted to suit off-diagonal estimates of exponential type.

Lemma 4.4.7. Given $M > \kappa + 3$ and $m \geq a\lambda$, we have

$$\begin{aligned}
&\int_M |u(x) - u_Q|^2 \left\langle \frac{t}{\rho(x,Q)} \right\rangle^M e^{-m\rho(x,Q)/t}\ \mathrm{d}\mu(x) \\
&\qquad \lesssim t^2 \int_M (|\nabla u(x)|^2 + |u(x)|^2) \left\langle \frac{t}{\rho(x,Q)} \right\rangle^{M-(\kappa+3)} e^{-(\frac{m}{a} - \lambda t)\rho(x,Q)/t}\ \mathrm{d}\mu(x)
\end{aligned}$$

for all $u \in W^{1,2}(M; \mathbb{C}^N)$, $Q \in \Delta_t$ and $t \in (0, 1]$.

Proof. Let $t \in (0,1]$ and $Q \in \Delta_t$. There exists $x_Q \in M$ such that

$$B(x_Q, a_0 t) \subseteq Q \subseteq B(x_Q, (a_1/\delta)t).$$

Let $r \geq a$ and $u \in W^{1,2}(M;\mathbb{C}^N)$. We have

$$\|\mathbf{1}_{B(x_Q,rt)}(u - u_Q)\|_2^2 \leq \|\mathbf{1}_{B(x_Q,rt)}(u - u_{B(x_Q,rt)})\|_2^2 + \|\mathbf{1}_{B(x_Q,rt)}(u_{B(x_Q,rt)} - u_Q)\|_2^2.$$

The Cauchy–Schwarz inequality and ($\mathrm{E}_{\mathrm{loc}}$) imply that

$$\begin{aligned}
\|\mathbf{1}_{B(x_Q,rt)}(u_{B(x_Q,rt)} - u_Q)\|_2^2 &= V(x_Q, rt)|u_Q - u_{B(x_Q,rt)}|^2 \\
&= V(x_Q, rt)\left|\fint_Q (u - u_{B(x_Q,rt)})\right|^2 \\
&\leq \frac{V(x_Q, rt)}{\mu(Q)} \int_Q |u - u_{B(x_Q,rt)}|^2 \\
&\lesssim r^\kappa e^{\lambda rt} \|\mathbf{1}_{B(x_Q,rt)}(u - u_{B(x_Q,rt)})\|_2^2,
\end{aligned}$$

where $r \geq a_1/\delta$ ensured that $Q \subseteq B(x_Q, rt)$. It then follows from ($\mathrm{P}_{\mathrm{loc}}$) that

$$\|\mathbf{1}_{B(x_Q,rt)}(u - u_Q)\|_2^2 \lesssim (1 + r^\kappa e^{\lambda rt})(rt)^2 \|\mathbf{1}_{B(x_Q,rt)} u\|_{W^{1,2}}^2.$$

Now let $\nu(r) := -r^{-M} e^{-(m/a)r}$ for all $r \geq a$, in which case

$$\mathrm{d}\nu(r) = (Mr^{-M-1} + (m/a)r^{-M})e^{-(m/a)r}\mathrm{d}r$$

is a positive measure on (a, ∞). Integrating the above estimate with respect to ν, we obtain

$$\begin{aligned}
&\int_a^\infty \int_M \mathbf{1}_{B(x_Q,rt)}|u(x) - u_Q|^2 \, \mathrm{d}\mu(x)\mathrm{d}\nu(r) \\
&\qquad \lesssim t^2 \int_a^\infty r^{\kappa+2} e^{\lambda rt} \int_M \mathbf{1}_{B(x_Q,rt)}(|\nabla u(x)|^2 + |u(x)|^2) \, \mathrm{d}\mu(x)\mathrm{d}\nu(r).
\end{aligned}$$

It then follows from (4.3.1) and Fubini's theorem that

$$\begin{aligned}
&\int_M |u(x) - u_Q|^2 \left\langle \frac{t}{\rho(x,Q)} \right\rangle^M e^{-m\rho(x,Q)/t} \, \mathrm{d}\mu(x) \\
&\quad \lesssim \int_M |u(x) - u_Q|^2 \left\langle \frac{t}{\rho(x,x_Q)} \right\rangle^M e^{-\frac{m}{a}\rho(x,x_Q)/t} \, \mathrm{d}\mu(x) \\
&\quad \lesssim \int_M |u(x) - u_Q|^2 (\max\{\rho(x,x_Q)/t, a\})^{-M} e^{-\frac{m}{a}\max\{\rho(x,x_Q)/t, a\}} \, \mathrm{d}\mu(x) \\
&\quad = \int_M |u(x) - u_Q|^2 \int_{\max\{\rho(x,x_Q)/t, a\}}^\infty \mathrm{d}\nu(r)\mathrm{d}\mu(x) \\
&\quad = \int_a^\infty \int_M \mathbf{1}_{B(x_Q,rt)}|u(x) - u_Q|^2 \, \mathrm{d}\mu(x)\mathrm{d}\nu(r) \\
&\quad \lesssim t^2 \int_a^\infty r^{\kappa+2} e^{\lambda rt} \int_M \mathbf{1}_{B(x_Q,rt)}(|\nabla u(x)|^2 + |u(x)|^2) \, \mathrm{d}\mu(x)\mathrm{d}\nu(r) \\
&\quad = t^2 \int_M (|\nabla u(x)|^2 + |u(x)|^2)\left(\int_{\max\{\rho(x,x_Q)/t, a\}}^\infty r^{\kappa+2} e^{\lambda rt}\mathrm{d}\nu(r)\right)\mathrm{d}\mu(x). \\
&\quad = t^2 \int_M (|\nabla u(x)|^2 + |u(x)|^2)\left(\int_{\max\{\rho(x,x_Q)/t, a\}}^\infty r^{\kappa+2} e^{\lambda rt}(Mr^{-M-1} + \tfrac{m}{a}r^{-M})e^{-\frac{m}{a}r} \, \mathrm{d}r\right)\mathrm{d}\mu(x).
\end{aligned}$$

The term in brackets is bounded by

$$e^{-(\frac{m}{a}-\lambda t)\rho(x,Q)/t}\int_{\rho(x,Q)/t}^{\infty} r^{-(M-(\kappa+2))}\,\mathrm{d}r \lesssim e^{-(\frac{m}{a}-\lambda t)\rho(x,Q)/t}\left\langle\frac{t}{\rho(x,Q)}\right\rangle^{M-(\kappa+3)},$$

which completes the proof. □

The first term in (4.4.5) is now estimated in a manner similar to that of Proposition 5.5 in [11]. The idea to replace the cube counting techniques used in [11] with the measure based approach below was suggested by Pascal Auscher.

Proposition 4.4.8. Let $C_\Theta > 0$ be the constant from Proposition 4.4.3. We have

$$\int_0^{\langle C_\Theta/4a^3\lambda\rangle} \|\Theta_t^B P_t u - \gamma_t A_t P_t u\|^2\,\frac{\mathrm{d}t}{t} \lesssim \|u\|^2$$

for all $u \in \mathsf{R}(\Pi)$.

Proof. Choose $M > 4\kappa+3$ and let $t_0 = \langle C_\Theta/4a^3\lambda\rangle$. Let $t \in (0,t_0]$, $u \in \mathsf{R}(\Pi)$ and set $v = P_t u$. The Cauchy–Schwarz inequality shows that

$$\begin{aligned}
\|\Theta_t^B P_t u - \gamma_t A_t P_t u\|^2 &= \sum_{Q\in\Delta_t}\|\Theta_t^B\sum_{R\in\Delta_t}\mathbf{1}_R(v-v_Q)\|^2_{L^2(Q)}\\
&\le \sum_{Q\in\Delta_t}\left(\sum_{R\in\Delta_t}\left(\frac{\mu(R)}{\mu(Q)}\frac{\mu(Q)}{\mu(R)}\right)^{\frac12}\|\mathbf{1}_Q\Theta_t^B\mathbf{1}_R(v-v_Q)\|\right)^2\\
&\le \sup_{Q\in\Delta_t}\left(\sum_{R\in\Delta_t}\frac{\mu(R)}{\mu(Q)}\|\mathbf{1}_Q\Theta_t^B\mathbf{1}_R\|\right)\sum_{Q\in\Delta_t}\sum_{R\in\Delta_t}\frac{\mu(Q)}{\mu(R)}\|\mathbf{1}_Q\Theta_t^B\mathbf{1}_R\|\|\mathbf{1}_R(v-v_Q)\|^2.
\end{aligned}$$

Then, since $C_\Theta > \lambda t$, Lemma 4.3.5 and the off-diagonal estimates from Proposition 4.4.3 show that the supremum term is uniformly bounded. Lemma 4.3.4 and (4.3.1) show that the remaining term is bounded by

$$\begin{aligned}
&\sum_{Q\in\Delta_t}\sum_{R\in\Delta_t}\left\langle\frac{t}{\rho(Q,R)}\right\rangle^{-\kappa} e^{a\lambda\rho(Q,R)}\left\langle\frac{t}{\rho(Q,R)}\right\rangle^{M} e^{-C_\Theta\frac{\rho(Q,R)}{t}}\|\mathbf{1}_R(v-v_Q)\|^2\\
&\lesssim \sum_{Q\in\Delta_t}\sum_{R\in\Delta_t}\int_R\left\langle\frac{t}{\rho(Q,R)}\right\rangle^{M-\kappa} e^{-(C_\Theta-a\lambda t_0)\frac{\rho(Q,R)}{t}}|v(x)-v_Q|^2\,\mathrm{d}\mu(x)\\
&\lesssim \sum_{Q\in\Delta_t}\int_M\left\langle\frac{t}{\rho(Q,x)}\right\rangle^{M-\kappa} e^{-(\frac{C_\Theta}{a}-\lambda t_0)\frac{\rho(Q,x)}{t}}|v(x)-v_Q|^2\,\mathrm{d}\mu(x).
\end{aligned}$$

The weighted Poincaré inequality from Lemma 4.4.7, Lemma 4.3.4 and (H8) show that this is bounded by

$$
\begin{aligned}
&t^2 \sum_{Q\in\Delta_t} \int_M \left\langle \frac{t}{\rho(Q,x)} \right\rangle^{M-(2\kappa+3)} e^{-(\frac{C_\Theta}{a^2}-(\frac{\lambda}{a}+\lambda)t_0)\frac{\rho(Q,x)}{t}} (|\nabla v(x)|^2 + |v(x)|^2)\, \mathrm{d}\mu(x) \\
&\leq t^2 \sum_{R\in\Delta_t} \int_R (|\nabla v(x)|^2 + |v(x)|^2)\, \mathrm{d}\mu(x) \sum_{Q\in\Delta_t} \left\langle \frac{t}{\rho(Q,R)} \right\rangle^{M-(2\kappa+3)} e^{-(\frac{C_\Theta}{a^2}-2\lambda t_0)\frac{\rho(Q,R)}{t}} \\
&\lesssim t^2 \|v\|_{W^{1,2}}^2 \sup_{R\in\Delta_t} \sum_{Q\in\Delta_t} \frac{\mu(Q)}{\mu(R)} \left\langle \frac{t}{\rho(Q,R)} \right\rangle^{M-(3\kappa+3)} e^{-(\frac{C_\Theta}{a^2}-3a\lambda t_0)\frac{\rho(Q,R)}{t}} \\
&\lesssim t^2 \|v\|_{W^{1,2}}^2 \\
&\lesssim t^2 \|\Pi v\|^2,
\end{aligned}
$$

where the penultimate inequality is implied by Lemma 4.3.5 because $M-(3\kappa+3) > \kappa$ and $\frac{C_\Theta}{a^2} - 3a\lambda t_0 > \lambda t$. Therefore, we have

$$\|\Theta_t^B P_t u - \gamma_t A_t P_t u\|^2 \lesssim \|Q_t u\|^2$$

for all $u \in \mathsf{R}(\Pi)$ and $t \in (0, t_0]$. The result then follows from the quadratic estimate for the unperturbed operator in (4.4.2). □

The following interpolation inequality is used to estimate the remaining terms in (4.4.5). It is an extension of Lemma 6 in [10]. The result relies on having a certain control of the volume of dyadic cubes near their boundary. This control is given by property 6 in Proposition 4.3.1.

Lemma 4.4.9. Let Υ denote either Π, Γ or Γ^*, then

$$\left| \fint_Q \Upsilon u \right|^2 \lesssim \frac{1}{l(Q)^\eta} \left(\fint_Q |u|^2 \right)^{\frac{\eta}{2}} \left(\fint_Q |\Upsilon u|^2 \right)^{1-\frac{\eta}{2}} + \fint_Q |u|^2$$

for all $u \in \mathsf{D}(\Upsilon)$, $Q \in \Delta_t$ and $t \in (0,1]$, where $\eta > 0$ is from Proposition 4.3.1.

Proof. Let $s = \|\mathbf{1}_Q u\| / \|\mathbf{1}_Q \Upsilon u\|$. If $s \geq a_0 l(Q)/2$, then the Cauchy–Schwarz inequality implies that

$$
\begin{aligned}
\left| \fint_Q \Upsilon u \right|^2 &\leq \fint_Q |\Upsilon u|^2 \\
&= \frac{s^{-\eta}}{\mu(Q)} \left(\int_Q |u|^2 \right)^{\frac{\eta}{2}} \left(\int_Q |\Upsilon u|^2 \right)^{1-\frac{\eta}{2}} \\
&\lesssim \frac{1}{l(Q)^\eta} \left(\fint_Q |u|^2 \right)^{\frac{\eta}{2}} \left(\fint_Q |\Upsilon u|^2 \right)^{1-\frac{\eta}{2}}.
\end{aligned}
$$

Now suppose that $0 < s \leq a_0 l(Q)/2$. Let $Q_s = \{x \in Q : \rho(x, M \setminus Q) > s\} \subset Q$. It follows from Proposition 4.3.1 that there exists $c > 0$ such that

$$\mu(M \setminus Q_s) \leq c(s/l(Q))^\eta \mu(Q)$$

Choose $\eta : M \to [0,1]$ in $C_0^\infty(M)$ satisfying $\operatorname{sppt}\eta \subseteq Q$ as well as

$$\eta(x) = \begin{cases} 1, & \text{if} \quad x \in Q_s; \\ 0, & \text{if} \quad x \in M \setminus Q \end{cases}$$

and $\|\nabla\eta\|_\infty \lesssim 1/s$. The existence of such functions follows as in the proof of Lemma 3.4.2. Using (H6)–(H7), we then obtain

$$\begin{aligned}
\left|\int_Q \Upsilon u\right| &= \left|\int_Q [\eta,\Upsilon]u + \int_Q (1-\eta)\Upsilon u + \int_Q \Upsilon(\eta u)\right| \\
&\lesssim \|\nabla\eta\|_\infty \int_{\operatorname{sppt}(\nabla\eta)} |u| + \int_{Q\cap\operatorname{sppt}(1-\eta)} |\Upsilon u| + \mu(Q)^{\frac{1}{2}} \left(\int_Q |u|^2\right)^{\frac{1}{2}} \\
&\lesssim \mu(M\setminus Q_s)^{\frac{1}{2}}\big(\|\mathbf{1}_Q u\|/s + \|\mathbf{1}_Q \Upsilon u\|\big) + \mu(Q)^{\frac{1}{2}}\|\mathbf{1}_Q u\| \\
&\lesssim (s/l(Q))^{\frac{\eta}{2}}\mu(Q)^{\frac{1}{2}}\|\mathbf{1}_Q\Upsilon u\| + \mu(Q)^{\frac{1}{2}}\|\mathbf{1}_Q u\|.
\end{aligned}$$

This shows that

$$\begin{aligned}
\left|\fint_Q \Upsilon u\right|^2 &\lesssim \frac{1}{l(Q)^\eta}\frac{s^\eta}{\mu(Q)}\int_Q |\Upsilon u|^2 + \fint_Q |u|^2 \\
&= \frac{1}{l(Q)^\eta}\left(\fint_Q |u|^2\right)^{\eta/2}\left(\fint_Q |\Upsilon u|^2\right)^{1-\eta/2} + \fint_Q |u|^2,
\end{aligned}$$

as required. ☐

The second term in (4.4.5) is now estimated by following the proof of Proposition 5 in [10].

Proposition 4.4.10. We have

$$\int_0^1 \|\gamma_t A_t(P_t - I)u\|^2 \,\frac{\mathrm{d}t}{t} \lesssim \|u\|^2$$

for all $u \in L^2(M;\mathbb{C}^N)$.

Proof. Lemma 4.4.9 and Hölder's inequality imply that

$$\begin{aligned}
\|A_t Q_s u\|^2 &= s^2 \sum_{Q\in\Delta_t} \mu(Q)\left|\fint_Q \Pi P_s u\right|^2 \\
&\lesssim s^2 \sum_{Q\in\Delta_t} \frac{\mu(Q)}{l(Q)^\eta}\left(\fint_Q |P_s u|^2\right)^{\frac{\eta}{2}}\left(\fint_Q |\Pi P_s u|^2\right)^{1-\frac{\eta}{2}} + s^2\|P_s u\|^2 \\
&\lesssim \left(\frac{s}{t}\right)^\eta \sum_{Q\in\Delta_t}\left(\int_Q |P_s u|^2\right)^{\frac{\eta}{2}}\left(\int_Q |Q_s u|^2\right)^{1-\frac{\eta}{2}} + s^2\|P_s u\|^2 \\
&\le \left(\frac{s}{t}\right)^\eta \|P_s u\|^\eta \|Q_s u\|^{2-\eta} + t^2\left(\frac{s}{t}\right)^2\|u\|^2 \\
&\lesssim (\tfrac{s}{t})^\eta \|u\|^2
\end{aligned}$$

for all $u \in L^2(M;\mathbb{C}^N)$ and $0 < s < t \le 1$. The result then follows by the arguments in the proof of Proposition 5 in [10]. ☐

To estimate the third and final term in (4.4.5), it follows from Theorem 4.3.3 that it suffices to show that there exists $t_0 \in (0,1]$ such that

$$\iint_{C(Q)} |\gamma_t(x)|^2 \, \mathrm{d}\mu(x) \frac{\mathrm{d}t}{t} \lesssim \mu(Q) \tag{4.4.6}$$

for all dyadic cubes $Q \in \bigcup_{t \in (0,t_0]} \Delta_t$.

Following [11], we let $\sigma > 0$ to be fixed later. Given $v \in \mathcal{L}(\mathbb{C}^N)$ with $|v| = 1$, define the cone of aperture σ by

$$K_{v,\sigma} := \{v' \in \mathcal{L}(\mathbb{C}^N) \setminus \{0\} : \left| \frac{v'}{|v'|} - v \right| \leq \sigma\}.$$

Let $\mathcal{V}_\sigma$ be a finite set of $v \in \mathcal{L}(\mathbb{C}^N)$ with $|v| = 1$ such that $\bigcup_{v \in \mathcal{V}_\sigma} K_{v,\sigma} = \mathcal{L}(\mathbb{C}^N) \setminus \{0\}$. To prove (4.4.6), it suffices to prove that there exists $t_0 > 0$ and $\sigma > 0$ such that

$$\iint_{\substack{(x,t) \in C(Q) \\ \gamma_t(x) \in K_{v,\sigma}}} |\gamma_t(x)|^2 \, \mathrm{d}\mu(x) \frac{\mathrm{d}t}{t} \lesssim \mu(Q) \tag{4.4.7}$$

for each $v \in \mathcal{V}_\sigma$ and for all $Q \in \bigcup_{t \in (0,t_0]} \Delta_t$. This in turn reduces to proving the following proposition.

Proposition 4.4.11. Let $t_0 = \langle C_\Theta / 4a^3 \lambda \rangle$, where $C_\Theta > 0$ is the constant from Proposition 4.4.3. There exist $\sigma, \tau, c > 0$ such that for all $Q \in \bigcup_{t \in (0,t_0]} \Delta_t$ and $v \in \mathcal{L}(\mathbb{C}^N)$ with $|v| = 1$, there exists a collection $\{Q_k\}_k \subseteq \Delta_{(0,1]}$ of disjoint subsets of Q such that the set $E_Q := Q \setminus \bigcup_k Q_k$ satisfies $\mu(E_Q) > \tau \mu(Q)$ and the set $E_Q^* := C(Q) \setminus \bigcup_k C(Q_k)$ satisfies

$$\iint_{\substack{(x,t) \in E_Q^* \\ \gamma_t(x) \in K_{v,\sigma}}} |\gamma_t(x)|^2 \, \mathrm{d}\mu(x) \frac{\mathrm{d}t}{t} \leq c\mu(Q).$$

To see that Proposition 4.4.11 implies (4.4.7), write

$$\begin{aligned} \{(x,t) \in C(Q) : \gamma_t(x) \in K_{v,\sigma}\} &= E_Q^* \cup \left(\bigcup_{k_1} \{(x,t) \in C(Q_{k_1}) : \gamma_t(x) \in K_{v,\sigma}\} \right) \\ &= E_Q^* \cup E_{Q_{k_1}}^* \cup \left(\bigcup_{k_2} \{(x,t) \in C(Q_{k_2}) : \gamma_t(x) \in K_{v,\sigma}\} \right) \\ &= \bigcup_{j=0}^{\infty} \bigcup_{k_j=0}^{\infty} E_{Q_{k_j}}^*. \end{aligned}$$

Monotone convergence then implies that

$$\begin{aligned}
\iint_{\substack{(x,t)\in C(Q)\\ \gamma_t(x)\in K_{v,\sigma}}} |\gamma_t(x)|^2 \, \mathrm{d}\mu(x)\frac{\mathrm{d}t}{t} &= \iint \sum_{j=0}^{\infty}\sum_{k_j=0}^{\infty} \mathbf{1}_{E^*_{Q_{k_j}}}(x,t)|\gamma_t(x)|^2 \, \mathrm{d}\mu(x)\frac{\mathrm{d}t}{t} \\
&= \sum_{j=0}^{\infty}\sum_{k_j=0}^{\infty} \iint_{E^*_{Q_{k_j}}} |\gamma_t(x)|^2 \, \mathrm{d}\mu(x)\frac{\mathrm{d}t}{t} \\
&\lesssim \sum_{j=0}^{\infty}\sum_{k_j=0}^{\infty} \mu(Q_{k_j}) \\
&= \sum_{j=0}^{\infty} \mu(\bigcup_{k_j=0}^{\infty} Q_{k_j}) \\
&< \sum_{j=0}^{\infty} (1-\tau)^j \mu(Q) \\
&= \frac{1}{\tau}\mu(Q).
\end{aligned}$$

The proof of Proposition 4.4.11 is a matter of constructing suitable test functions and applying a stopping-time argument. The test functions are constructed as in [11], with some minor modifications. Fix $v \in \mathcal{L}(\mathbb{C}^N)$ with $|v| = 1$ and choose $\hat{w}, w \in \mathbb{C}^N$ such that $|\hat{w}| = |w| = 1$ and $v^*(\hat{w}) = w$. For each $Q \in \bigcup_{t\in(0,1]} \Delta_t$, let B_Q denote a ball of radius $a_1 l(Q)$ such that $(a_0/a_1)B_Q \subseteq Q \subseteq B_Q$. Then let $\eta_Q : M \to [0,1]$ be a smooth function supported on $3B_Q$ and equal to 1 on $2B_Q$. The existence of such functions follows as in the proof of Lemma 3.4.2. Define $w_Q := \eta_Q w$, and for each $\epsilon > 0$, define the test function

$$f^w_{Q,\epsilon} := w_Q - i\epsilon l(Q)\Gamma R^B_{\epsilon l(Q)} w_Q = (I + i\epsilon l(Q)\Gamma^*_B) R^B_{\epsilon l(Q)} w_Q.$$

These functions have the following properties. The proof is almost identical to that of Lemma 7 in [10] but we include it for completeness.

Lemma 4.4.12. There exists $c > 0$ such that the following hold for all $Q \in \Delta_{(0,1]}$ and $\epsilon > 0$:

1. $\|f^w_{Q,\epsilon}\| \leq c\mu(Q)^{\frac{1}{2}}$;
2. $\displaystyle\iint_{C(Q)} |\Theta^B_t f^w_{Q,\epsilon}|^2 \, \mathrm{d}\mu(x)\frac{\mathrm{d}t}{t} \leq c\epsilon^{-2}\mu(Q)$;
3. $\displaystyle\left|\fint_Q f^w_{Q,\epsilon} - w\right| < c\epsilon^{\frac{\eta}{2}}$,

where $\eta > 0$ is the constant Proposition 4.3.1.

Proof. 1. Let $Q \in \bigcup_{t\in(0,1]} \Delta_t$. Using (4.1.1), Proposition 4.4.3 and ($\mathrm{E}_{\mathrm{loc}}$), we obtain

$$\|f^w_{Q,\epsilon}\| \lesssim \|\eta_Q\| + \|i\epsilon l(Q)\Pi_B R^B_{\epsilon l(Q)}\eta_Q\| \lesssim \|\eta_Q\| \leq \mu(2B_Q)^{1/2} \lesssim \mu(Q)^{1/2},$$

where the constant in the last inequality is uniform for all $Q \in \bigcup_{t\in(0,1]} \Delta_t$.

2. Next, by the nilpotency of Γ_B^* and $[\Gamma_B^*, P_t^B] = 0$ on $\mathsf{D}(\Gamma_B^*)$, we have

$$\Theta_t^B f_{Q,\epsilon}^w = tP_t^B\Gamma_B^*(I + i\epsilon l(Q)\Gamma_B^*)R_{i\epsilon l(Q)}^B w_Q = tP_t^B\Gamma_B^* R_{i\epsilon l(Q)}^B w_Q.$$

Therefore, using (4.1.1), Proposition 4.4.3 and ($\mathrm{E}_{\mathrm{loc}}$) again, we obtain

$$\begin{aligned}\iint_{C(Q)} |\Theta_t^B f_{Q,\epsilon}^w|^2 \, \mathrm{d}\mu(x)\frac{\mathrm{d}t}{t} &\le \int_0^{l(Q)} \|tP_t^B\Gamma_B^* R_{i\epsilon l(Q)}^B w_Q\|^2 \, \frac{\mathrm{d}t}{t} \\ &\lesssim \int_0^{l(Q)} \frac{t}{(\epsilon l(Q))^2}\|i\epsilon l(Q)\Pi_B R_{i\epsilon l(Q)}^B \eta_Q\|^2 \, \mathrm{d}t \\ &\lesssim \frac{1}{\epsilon^2}\mu(Q).\end{aligned}$$

3. Finally, since $\eta_Q = 1$ on Q, by Lemma 4.4.9 with $\Upsilon = \Gamma$ and $u = R_{\epsilon l(Q)}^B w_Q$, and using (4.1.1), Proposition 4.4.3 and ($\mathrm{E}_{\mathrm{loc}}$) again, we obtain

$$\begin{aligned}&\left|\fint_Q f_{Q,\epsilon}^w - w\right| = \epsilon l(Q)\left|\fint_Q \Gamma R_{\epsilon l(Q)}^B w_Q\right| \\ &\lesssim \epsilon l(Q)^{1-\frac{\eta}{2}}\left(\fint_Q |R_{\epsilon l(Q)}^B w_Q|^2\right)^{\frac{\eta}{4}}\left(\fint_Q |\Gamma R_{\epsilon l(Q)}^B w_Q|^2\right)^{\frac{1}{2}-\frac{\eta}{4}} + \epsilon l(Q)\left(\fint_Q |R_{\epsilon l(Q)}^B w_Q|^2\right)^{\frac{1}{2}} \\ &\lesssim \mu(Q)^{-\frac{1}{2}}\|R_{\epsilon l(Q)}^B w_Q\|^{\frac{\eta}{2}}\left(\int_Q |i\epsilon l(Q)\Pi_B R_{\epsilon l(Q)}^B w_Q|^2\right)^{\frac{1}{2}-\frac{\eta}{4}} + \epsilon\mu(Q)^{-\frac{1}{2}}\|R_{\epsilon l(Q)}^B w_Q\| \\ &\lesssim \epsilon^{\frac{\eta}{2}}\mu(Q)^{-\frac{1}{2}}\|R_{\epsilon l(Q)}^B w_Q\|^{\frac{\eta}{2}}\|(I - R_{\epsilon l(Q)}^B)w_Q\|^{1-\frac{\eta}{2}} + \epsilon\mu(Q)^{-\frac{1}{2}}\|\eta_Q\| \\ &\lesssim \epsilon^{\frac{\eta}{2}}\mu(Q)^{-\frac{1}{2}}\|\eta_Q\| \\ &\lesssim \epsilon^{\frac{\eta}{2}},\end{aligned}$$

as required. □

We now fix $\epsilon = (\frac{1}{2c})^{2/\eta}$ and the test functions $f_Q^w := f_Q^{w,\epsilon}$, where c is the constant from Lemma 4.4.12. The preceding result then implies that

$$\mathrm{Re}\left(w, \fint_Q f_Q^w\right) \ge \frac{1}{2}.$$

The stopping-time argument from Lemma 5.11 in [11] can then be applied to obtain the following result. The properties of the dyadic cube structure in Proposition 4.3.1 suffice for this purpose.

Lemma 4.4.13. Let $t_0 = \langle C_\Theta/4a^3\lambda\rangle$. There exist $\alpha, \beta > 0$ such that for all dyadic cubes $Q \in \bigcup_{t\in(0,t_0]}\Delta_t$ there exists a collection $\{Q_k\}_k \subseteq \Delta_{(0,1]}$ of disjoint subsets of Q such that the set $E_Q := Q \setminus \bigcup_k Q_k$ satisfies $\mu(E_Q) > \beta\mu(Q)$ and the set $E_Q^* := C(Q) \setminus \bigcup_k C(Q_k)$ has the following property:

$$\mathrm{Re}\left(w, \fint_{Q'} f_Q^w\right) \ge \alpha \quad \text{and} \quad \fint_{Q'} |f_Q^w| \le \frac{1}{\alpha}$$

for all $Q' \in \Delta_{(0,1]}$ that are contained in Q and satisfy $C(Q') \cap E_Q^* \neq \emptyset$.

We can now prove Proposition 4.4.11 by following closely the ideas at the end of Section 5 in [11].

Proof of Proposition 4.4.11. Choose $\sigma \in (0, \alpha^2)$ and let $\tau = \beta$, where $\alpha, \beta > 0$ are the constants from Lemma 4.4.13.

Let $Q \in \bigcup_{t\in(0,t_0]} \Delta_t$ and $v \in \mathcal{L}(\mathbb{C}^N)$ with $|v| = 1$. Let $\{Q_k\}_k \subseteq \Delta_{(0,1]}$ denote the collection of disjoint subsets of Q given by Lemma 4.4.13 and suppose that $(x,t) \in E^*_Q$. This implies that $(x,t) \in C(Q)$ and that $t \leq l(Q) \leq t_0/\delta$. Now let Q' be the unique dyadic cube in Δ_t that contains x. Then, since $l(Q') \geq t$, we must have $(x,t) \in C(Q')$ and so $C(Q') \cap E^*_Q \neq \emptyset$. Lemma 4.4.13 and the Cauchy–Schwarz inequality then imply that

$$|v(A_t f^w_Q(x))| \geq \operatorname{Re}(\hat{w}, v(A_t f^w_Q(x))) = \operatorname{Re}\left(w, \fint_{Q'} f^w_Q(x)\right) \geq \alpha$$

and that

$$|A_t f^w_Q(x)| = \left|\fint_{Q'} f^w_Q(x)\right| \leq \frac{1}{\alpha}.$$

The choice of σ then implies that

$$\left|\frac{\gamma_t(x)}{|\gamma_t(x)|} A_t f^w_Q(x)\right| \geq |v(A_t f^w_Q(x))| - \left|\frac{\gamma_t(x)}{|\gamma_t(x)|} - v\right| \left|A_t f^w_Q(x)\right| \geq \alpha - \frac{\sigma}{\alpha} \gtrsim 1.$$

Therefore, we have

$$\begin{aligned}\iint_{\substack{(x,t)\in E^*_Q \\ \gamma_t(x)\in K_{v,\sigma}}} |\gamma_t(x)|^2 \, \mathrm{d}\mu(x)\frac{\mathrm{d}t}{t} &\lesssim \iint_{C(Q)} |\gamma_t(x) A_t f^w_Q(x)|^2 \, \mathrm{d}\mu(x)\frac{\mathrm{d}t}{t} \\ &\lesssim \iint_{C(Q)} |\Theta^B_t f^w_Q - \gamma_t A_t f^w_Q|^2 \, \mathrm{d}\mu\frac{\mathrm{d}t}{t} + \iint_{C(Q)} |\Theta^B_t f^w_Q|^2 \, \mathrm{d}\mu\frac{\mathrm{d}t}{t}.\end{aligned}$$

Lemma 4.4.12 shows that the last term above is bounded by $c(2c)^{4/\eta}\mu(Q)$. It remains to show that

$$\iint_{C(Q)} |\Theta^B_t f^w_Q - \gamma_t A_t f^w_Q|^2 \, \mathrm{d}\mu\frac{\mathrm{d}t}{t} \lesssim \mu(Q).$$

Now let $v = i\epsilon l(Q)\Gamma R^B_{\epsilon l(Q)} w_Q$ and write

$$\Theta^B_t f^w_Q - \gamma_t A_t f^w_Q = -(\Theta^B_t - \gamma_t A_t)v + (\Theta^B_t - \gamma_t A_t)w_Q. \tag{4.4.8}$$

Then, since $v \in \mathsf{R}(\Gamma)$, by (i) in Proposition 4.8 of [11], Proposition 4.4.8 and Proposition 4.4.10, we have

$$\begin{aligned}\iint_{C(Q)} |(\Theta^B_t - \gamma_t A_t)v|^2 \, \mathrm{d}\mu\frac{\mathrm{d}t}{t} \lesssim & \int_0^{t_0} \|\Theta^B_t (I - P_t)v\|^2 \frac{\mathrm{d}t}{t} \\ &+ \int_0^{t_0} \|(\Theta^B_t P_t - \gamma_t A_t P_t)v\|^2 \frac{\mathrm{d}t}{t} \\ &+ \int_0^{t_0} \|\gamma_t A_t (P_t - I)v\|^2 \frac{\mathrm{d}t}{t} \\ \lesssim & \ \mu(Q).\end{aligned}$$

To handle the remaining term in (4.4.8), recall that $(a_0/a_1)B_Q \subseteq Q \subseteq B_Q$ and that $\eta_Q = 1$ on $2B_Q$. This implies that if $x \in Q$ and $t \in (0, l(Q)]$, then

$$(\Theta_t^B - \gamma_t A_t)w_Q(x) = \Theta_t^B((\eta_Q - 1)w)(x).$$

Now choose $M > \kappa/2$ and consider the characteristic functions $\mathbf{1}_j(B_Q)$ defined by

$$\mathbf{1}_j(B_Q) = \begin{cases} \mathbf{1}_{2B_Q} & \text{if} \quad j = 0; \\ \mathbf{1}_{2^{j+1}B_Q \setminus 2^j B_Q} & \text{if} \quad j = 1, 2, \ldots. \end{cases}$$

Then, since $\eta_Q - 1 = 0$ on $2B_Q$, the off-diagonal estimates from Proposition 4.4.3 and ($\mathrm{E}_{\mathrm{loc}}$) imply that

$$\begin{aligned} \|\Theta_t^B(\eta_Q - 1)w\|_{L^2(Q)}^2 &\leq \sum_{j=1}^{\infty} \|\mathbf{1}_{B_Q}\Theta_t^B \mathbf{1}_j(B_Q)\|^2 \|\mathbf{1}_j(B_Q)(\eta_Q - 1)\|^2 \\ &\lesssim \sum_{j=1}^{\infty} \left(\frac{t}{(2^j - 1)a_1 l(Q)}\right)^{2M} e^{-2C_\Theta(2^j-1)a_1 l(Q)/t} \mu(2^{j+1}B_Q) \\ &\lesssim \frac{t}{l(Q)}\mu(B_Q) \sum_{j=1}^{\infty} 2^{-j(2M-\kappa)} e^{-(C_\Theta - \lambda t_0)2^{j+1}a_1 l(Q)/t} \\ &\leq \frac{t}{l(Q)}\mu(Q) \end{aligned}$$

for all $t \in (0, l(Q)]$. This shows that

$$\iint_{C(Q)} |(\Theta_t^B - \gamma_t A_t)w_Q|^2 \,\mathrm{d}\mu \frac{\mathrm{d}t}{t} \lesssim \mu(Q),$$

so the proof is complete. □

As shown previously, Proposition 4.4.11 implies (4.4.7), which in turn implies (4.4.6) and allows us to estimate the final term in (4.4.5). In summary, as a consequence of Propositions 4.4.8, 4.4.10 and 4.4.11, we have proved the local quadratic estimate

$$\int_0^{\langle C_\Theta/4a^3\lambda\rangle} \|\Theta_t^B P_t u\|^2 \,\frac{\mathrm{d}t}{t} \lesssim \|u\|^2$$

for all $u \in \mathsf{R}(\Gamma)$. The hypothesis (H1)–(H8) are invariant upon replacing $\{\Gamma, B_1, B_2\}$ with $\{\Gamma^*, B_2, B_1\}$, $\{\Gamma^*, B_2^*, B_1^*\}$ and $\{\Gamma, B_1^*, B_2^*\}$. This completes the proof of the main result in this chapter, since Proposition 4.4.2 then allows us to conclude that the quadratic estimate in Theorem 4.1.5 holds.

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