Latin Squares with Restricted Transversals

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Abstract: **We prove that for all odd $m \geq 3$ there exists a latin square of order $3m$ that contains an $(m-1)\times m$ latin subrectangle consisting of entries not in any transversal. We prove that for all even $n \geq 10$ there exists a latin square of order n in which there is at least one transversal, but all transversals coincide on a single entry. A corollary is a new proof of the existence of a latin square without an orthogonal mate, for all odd orders $n \geq 11$. Finally, we report on an extensive computational study of transversal-free entries and sets of disjoint transversals in the latin squares of order $n \leq 9$. In particular, we count the number of species of each order that possess an orthogonal mate.** © 2011 Wiley Periodicals, Inc. J Combin Designs 20: 124–141, 2012

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1. INTRODUCTION

A *latin square of order n* is an $n \times n$ matrix L in which n distinct symbols are arranged so that each symbol occurs once in each row and column. We can specify L by a set of n^2 ordered triples $(x, y, z) \in \mathcal{I}(L)^3$, where $\mathcal{I}(L)$ is a set of cardinality n, and no two distinct triples agree in more than one coordinate. The interpretation is that z is the symbol in column y of row x. We say that L is *indexed* by $\mathcal{I}(L)$ and that (x, y, z) is an *entry* of L. A *transversal* in a latin square is a selection of n distinct entries in which each row, column and symbol is represented exactly once. An entry that is not in any transversal will be described as *transversal-free*.

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In this article, we consider several ways in which transversals in latin squares may be restricted. Our two main results are:

Theorem 1.1. *For all even $n \geq 10$, there exists a latin square of order n in which there is at least one transversal, but all transversals coincide on a single entry.*

Theorem 1.2. *For all odd $m \geq 3$, there exists a latin square of order $3m$ that contains an $(m-1)\times m$ latin subrectangle consisting of transversal-free entries.*

Theorems 1.1 and 1.2 will be proven in Sections 2 and 3, respectively. The first theorem resembles the second if we rephrase it as showing the existence of a $1\times(n-1)$ subrectangle of transversal-free entries. Section 4 contains computational evidence which suggests that latin squares typically have a transversal through every entry, so the properties exhibited in both our theorems may be considered rare.

For a given latin square L of order n, we define $\lambda(L)$ to be the maximum m such that L has m disjoint transversals. It is not possible to have $\lambda(L)=n-1$. For all $n\notin\{2,6\}$, the existence of a latin square of order n with $\lambda=n$ follows from a famous result by Bose et al. [1]. A latin square has an orthogonal mate if and only if it can be partitioned into transversals; see [4, p. 55].

A conjecture by Ryser (see [4, p. 486]) says that every latin square of odd order has a transversal. However, it is well known that there are latin squares of even order with $\lambda=0$. For example, as first shown by Euler (e.g. [4, p. 445]), the Cayley table of the cyclic group $\mathbb{Z}_n$ has no transversals whenever n is even. The next lemma, from [7], shows that among the latin squares of even order, λ takes many values other than the extremes of 0 and n.

Lemma 1.3. *For each even $n\geq 6$ and each $j\equiv 0 \bmod 4$ such that $0\leq j\leq n$, there exists a latin square L of order n with $\lambda(L)=j$.*

It is immediate from Theorem 1.1 that $\lambda=1$ is also achieved for all even orders $n\geq 10$. Our proof in Section 2 will show that if $n\equiv 0 \bmod 16$, then a stronger condition holds, as every transversal coincides on two entries. Data in [7, 14] and in Section 4 show that if L is of even order $n\leq 8$, then $\lambda(L)\neq 1$, so Theorem 1.1 cannot be extended to smaller even n. The possibility of an infinite family of latin squares of odd order with constant λ value remains an open question.

For all positive integers n, we define $\mu(n)$ to be the minimum value of λ among the latin squares of order n. Clearly, $\mu(n)=0$ for all even n, so we are concerned with the case when n is odd. If Ryser's Conjecture is true, then $\mu(n)\geq 1$ for all odd n. Following terminology introduced in [13], a latin square with no orthogonal mate is called a *bachelor* latin square. A trivial upper bound, $\mu(n)\leq n-2$, follows immediately from the existence of a bachelor latin square of order n. However a better bound can be derived from two such existence results in the literature. Studying the constructions of Mann [9] for $n\equiv 1$ mod 4 and Evans [8] for $n\equiv 3$ mod 4, we can employ their reasoning to bound the number of disjoint transversals and thereby show:

Theorem 1.4. *If n is odd and $n>3$, then $\mu(n)\leq \frac{1}{2}(n+1)$.*

We know of no general argument that improves on the bound given by Theorem 1.4, although it is not tight in the cases where μ is known. The current authors [7] found by computation that $\mu(9)=3$. For order $n\in\{5,7\}$, there is a latin square whose transversals coincide on one entry, hence $\mu(5)=\mu(7)=1$. For $n\in\{1,3\}$, we have $\mu(n)=n$.

For a given latin square L, we define $\alpha(L)$ to be the minimum cardinality of a maximal set of disjoint transversals. Section 4 includes a classification of the latin squares of order $n \le 9$ according to their values of λ and α. For all positive integers n, we define $\beta(n)$ to be the minimum value of α among the latin squares of order n. By definition, $\alpha(L) \le \lambda(L)$ for all L and $\beta(n) \le \mu(n)$ for all n. Although we know little about the size of $\mu(n)$, we can narrow $\beta(n)$ down to a very small set of possible values. A result in [7] proved that

$$\beta(n) \le \begin{cases} 1 & \text{if } n \equiv 1 \bmod 4, \\ 3 & \text{if } n \equiv 3 \bmod 4. \end{cases} \tag{1}$$

Our final measure of restrictions on transversals is $\tau(L)$, the number of transversal-free entries in a latin square L. Our data in Section 4 suggest that almost all latin squares of large order have $\tau = 0$. A latin square of order n with $\tau > 0$ has $\lambda \le n-2$. In general, the converse does not hold. For example, the constructions that prove Theorem 1.4 often have $\tau = 0$. A latin square with $\tau > 0$ is called a *confirmed* bachelor square [18].

For all odd $n > 3$, the existence of a confirmed bachelor latin square of order n was first proved by Wanless and Webb [18] using a family we shall refer to as $\mathcal{W}_n$. As noted in [18], three transversal-free entries occur in one row so $\lambda(\mathcal{W}_n) \le n-3$. Lemma 2.1, also used in [18], shows that for odd $n \ge 9$ there are at least seven transversal-free entries in $\mathcal{W}_n$. Together with small examples in Section 4 this shows:

Theorem 1.5. *For all $n \ge 4$, there exists a latin square L of order n with $\tau(L) \ge 7$.*

Two new proofs of the existence of confirmed bachelors have been found since [18] but they show fewer transversal-free entries than Theorem 1.5. In [5], a family defined for all odd orders $n \ge 5$ is shown to have at least two transversal-free entries. In Section 2, we show the existence of a latin square of order n with at least one transversal-free entry for all odd $n \ge 11$. This will follow as an easy corollary of Theorem 1.1.

2. TRANSVERSALS COINCIDING ON A SINGLE ENTRY

In this section we prove Theorem 1.1. First, we introduce our notation and a key lemma.

Let G be an Abelian group and let L be a latin square of order $|G|$, where $\mathcal{I}(L) = G$. The function $\Delta : L \to G$ is given by $\Delta(e) = z - x - y$ for each entry $e = (x, y, z)$ of L.

Lemma 2.1. *Let G be an Abelian group with identity ε and let L be a latin square indexed by G. If T is a transversal in L, then*

$$\sum_{e \in T} \Delta(e) = -\sum_{g \in G} g = \begin{cases} \omega & \text{if } G \text{ has a unique involution } \omega, \\ \varepsilon & \text{otherwise.} \end{cases}$$

Lemma 2.1 is shown in [7]. It is a minor variation of the original formulation independently due to Egan and Wanless [6] and Evans [8]. Variations of the lemma have also been used in [2, 3, 5, 12, 18].

When applying Lemma 2.1, we focus on the following subsets of L:

$$\Delta_* = \{e \in L : \Delta(e) \ne \varepsilon\}, \quad \Delta_g = \{e \in L : \Delta(e) = g\},$$

where g is an element of $G \setminus \{\varepsilon\}$.

The members of our families of even order will be indexed by a cyclic group $\mathbb{Z}_n$. All calculations are in $\mathbb{Z}_n$, and we assume our residues to be $\{0, 1, \ldots, n-1\}$. A partition by parity is given by $\mathbb{Z}_n = E \cup F$, where

$$E = \{0, 2, 4, \ldots, n-2\} \quad \text{and} \quad F = \{1, 3, 5, \ldots, n-1\}.$$

We define r_x to be the set of entries in row x of L. That is, $r_x = \{(x, y, z) \in L : y, z \in \mathcal{I}(L)\}$. To specify a transversal T of a large latin square, we simply specify, for each row x, the column $\mathrm{col}(x)$ of the entry in $T \cap r_x$.

We now begin our proof of Theorem 1.1, which we split into three cases. The case for $n \equiv 2 \bmod 4$ will be shown using the family $\mathcal{B}_n$.

Latin square $\mathcal{B}_n$: For $n = 2h$, where h is an odd integer and $h \geq 5$

$$\mathcal{B}_n[x, y] = \begin{cases} x+y+1 & \text{if } x=0 \text{ and } y \in \{n-2, n-1\}, \\ x+y-1 & \text{if } x=1 \text{ and } y \in \{0, n-2, n-1\}, \\ x+y+3 & \text{if } x=1 \text{ and } y=n-3, \\ x+y-3 & \text{if } x=4 \text{ and } y \in \{0, n-3\}, \\ x+y+4 & \text{if } x=0 \text{ and } y \in \{0, 4, 8, \ldots, n-6\}, \\ x+y-4 & \text{if } x=4 \text{ and } y \in \{4, 8, \ldots, n-6\}, \\ x+y+h-7 & \text{if } n>14,\ x=6 \text{ and } y \in E, \\ x+y-h+7 & \text{if } n>14,\ x=h-1 \text{ and } y \in E, \\ x+y & \text{otherwise.} \end{cases}$$

For $\mathcal{B}_n$, there are eight entries $e \in \Delta_*$ for which $\Delta(e)$ is an odd value:

	0	$n-3$	$n-2$	$n-1$
0			1	1
1	-1	3	-1	-1
4	-3	-3		

(2)

For a even, $\Delta_a \subset \mathcal{B}_n$ is contained as follows:

$$\begin{array}{ll} \text{If } n=22, & \Delta_4 \subset (r_0 \cup r_6) \text{ and } \Delta_{-4} \subset (r_4 \cup r_{h-1}). \\ \text{Otherwise,} & \Delta_4 \subset r_0 \text{ and } \Delta_{-4} \subset r_4. \\ \text{If } n>14 \text{ and } n \neq 22, & \Delta_{h-7} \subset r_6 \text{ and } \Delta_{h+7} \subset r_{h-1}. \end{array} \quad (3)$$

Lemma 2.2. *The latin square $\mathcal{B}_n$ has a transversal. Every transversal in $\mathcal{B}_n$ coincides on a single entry e, where $e = (4, 4, 4)$ if $n = 10$ and otherwise $e = (1, n-3, 1)$.*

Proof. Assume that $T \subset \mathcal{B}_n$ is a transversal. Lemma 2.1 requires that $\sum_{e \in T} \Delta(e) = h$, an odd value. The possible choices for odd Δ values are shown in (2). Respecting that T contains only one entry in each row and each column, the odd sums obtained by using odd Δ values alone are ± 1, ± 3. Hence, a sum of $\pm(h-1)$ or $\pm(h-3)$ must be obtained from entries in Δ_a, where a is even. These sets are located as stated in (3).

Since T has one entry from each row, we can choose at most one entry in each of the sets $(\Delta_{-4} \cap r_4) \cup \Delta_{-3}$, $(\Delta_4 \cap r_0) \cup \Delta_1$, $\Delta_{h-7} \cap r_6$ and $\Delta_{h+7} \cap r_{h-1}$. It follows that if $n=10$, then T contains the entry $(4,4,4) \in \Delta_{-4}$ and an entry in Δ_{-1}. If $n>10$, then T must contain the entry $(1, n-3, 1) \in \Delta_3$, some entry in $\Delta_4 \cap r_0$ and, for $n>14$, also an entry in $\Delta_{h-7} \cap r_6$.

Next, we show that a transversal T exists in $\mathcal{B}_n$. If $n=10$, then T is given by

$$\{(0,7,7),\ (1,9,9),\ (2,0,2),\ (3,2,5),\ (4,4,4),$$
$$(5,3,8),\ (6,5,1),\ (7,6,3),\ (8,8,6),\ (9,1,0)\}.$$

If $n=14$, then T is given by

$$\{(0,0,4),\ (1,11,1),\ (2,12,0),\ (3,13,2),\ (4,2,6),\ (5,3,8),\ (6,5,11),$$
$$(7,6,13),\ (8,4,12),\ (9,1,10),\ (10,7,3),\ (11,8,5),\ (12,9,7),\ (13,10,9)\}.$$

For $n>14$, we specify T in two cases.

Case 1: $n \equiv 2 \bmod 8$. Then T is given by

$$\mathrm{col}(x) = \begin{cases} 0 & \text{if } x=0, \\ n-3 & \text{if } x=1, \\ h+3 & \text{if } x=6, \\ h-1 & \text{if } x=h-3 \text{ and } n\equiv 10 \bmod 16, \\ h-5 & \text{if } x=h+1 \text{ and } n \equiv 2 \bmod 16, \\ 1 & \text{if } x=n-1, \\ x+1 & \text{if } x>0 \text{ and } x\equiv 0 \bmod 4, \\ x-2 & \text{if } 1<x<n-1 \text{ and } x\equiv 1 \bmod 4, \\ x-1 & \text{if } x\equiv 3 \bmod 4, \\ x-10 & \text{if } 14\le x<h+5 \text{ and } x\equiv 6 \bmod 8, \\ x+2 & \text{if } x\ge h+5 \text{ and } x\equiv 2 \bmod 4, \\ x+6 & \text{otherwise.} \end{cases}$$

Case 2: $n \equiv 6 \bmod 8$. Then T is given by

$$\mathrm{col}(x) = \begin{cases} h-3 & \text{if } x=6, \\ h-4 & \text{if } x=h+2, \\ x-1 & \text{if } x\in\{h,h-1\}, \\ x-2 & \text{if } x\in\{4,5,7,8\}, \\ x-4 & \text{if } (1\le x\le 3) \text{ or } (11\le x\le h-2 \text{ and } x\in F), \\ x-8 & \text{if } x=9 \text{ or } (12\le x\le h+1 \text{ and } x\equiv 0 \bmod 4), \\ x & \text{if } x=0 \text{ or } (10\le x\le h-5 \text{ and } x\equiv 2 \bmod 4), \\ x-3 & \text{otherwise.} \end{cases}$$

It is routine to verify that T is a transversal. Hence, the lemma is shown. □

Lemma 2.2 proves Theorem 1.1 for $n \equiv 2 \bmod 4$. The next family will be used to prove all cases $n \equiv 0 \bmod 4$, except when n is a power of 2.

Latin square $\mathcal{U}_n$: Suppose that $n = 2qm$, where q is odd, $q \geq 3$ and $m = 2^t$ for some positive integer t. Let $M = \{m, 3m, 5m, \ldots, n-m\}$ and define

$$\mathcal{U}_n[x,y] = \begin{cases} n-m & \text{if } \{x,y\} = \{0, n-\frac{1}{2}m\} \text{ or } \{n-m\}, \\ n-\frac{1}{2}m & \text{if } \{x,y\} = \{0\} \text{ or } \{n-\frac{1}{2}m\}, \\ 0 & \text{if } \{x,y\} = \{0, n-m\}, \\ y+qm-2m & \text{if } q>3,\ x=m \text{ and } y \in M, \\ y-m & \text{if } x = qm-2m \text{ and } y \in M, \\ y+m & \text{if } x = n-m \text{ and } y \in M \setminus \{n-m\}, \\ x+y & \text{otherwise.} \end{cases}$$

For $\mathcal{U}_n$, the value of $\Delta(e)$ for each $e \in \Delta_*$ is as follows:

	0	m	$3m$	$5m$	$\ldots$	$n-3m$	$n-m$	$n-\frac{1}{2}m$
0	$-\frac{1}{2}m$						m	$-\frac{1}{2}m$
m		$(q-3)m$	$(q-3)m$	$(q-3)m$	$\ldots$	$(q-3)m$	$(q-3)m$	
$m(q-2)$		$(1-q)m$	$(1-q)m$	$(1-q)m$	$\ldots$	$(1-q)m$	$(1-q)m$	
$n-m$	m	$2m$	$2m$	$2m$	$\ldots$	$2m$	m	
$n-\frac{1}{2}m$	$-\frac{1}{2}m$							$\frac{1}{2}m$

Lemma 2.3. *The latin square* $\mathcal{U}_n$ *has a transversal. Every transversal in* $\mathcal{U}_n$ *contains the entry* $(0, n-m, 0)$.

Proof. Assume that $T \subset \mathcal{U}_n$ is a transversal. Lemma 2.1 requires that $\sum_{e \in T} \Delta(e) = qm \equiv 0 \bmod m$. Hence, T must contain an even number of entries in $\Delta_{-m/2} \cup \Delta_{m/2}$. As the entries in $\Delta_{-m/2} \cup \Delta_{m/2}$ form a latin subsquare of order two, we conclude that T cannot contain any of them. In fact, the only way to satisfy Lemma 2.1 is for T to include one entry from each of Δ_m and Δ_{2m} and, if $q \neq 3$, also one entry from $\Delta_{(q-3)m}$. So T includes two entries in $\Delta_m \cup \Delta_{2m}$, and one of those must be from outside r_{n-m}. Hence, the entry $(0, n-m, 0)$ is required in T.

Next, we identify a transversal $T \subset \mathcal{U}_n$ in two cases.

Case 1: $n \equiv 4 \bmod 8$. We have $n = 2qm$ with $m = 2$. Then T is given by

$$\text{col}(x) = \begin{cases} n-m & \text{if } x = 0, \\ qm-2m & \text{if } x = m \text{ and } q>3, \\ m & \text{if } x = n-m, \\ 0 & \text{if } x = n-2m, \\ x+2m+1 & \text{if } x \in E \text{ and } qm-2m \leq x < n-2m, \\ x-2m+1 & \text{if } x \in F \text{ and } x > qm, \\ x & \text{otherwise.} \end{cases}$$

Case 2: $n \equiv 0 \bmod 8$. We have $n = 2qm$ with $m \geq 4$. Then T is given by

$$\mathrm{col}(x) = \begin{cases} n-m & \text{if } x=0, \\ qm-2m & \text{if } x=m \text{ and } q>3, \\ m & \text{if } x=n-m, \\ qm+1 & \text{if } x=qm-2m, \\ x+3 & \text{if } x\in E \text{ and } qm\leq x<n-m, \\ x-1 & \text{if } x\in F \text{ and } qm<x<n-m, \\ x+1 & \text{if } n-m<x<n, \\ x & \text{otherwise.} \end{cases}$$

It is routine to verify that T is a transversal. □

The proof of Lemma 2.3 depends on the existence of a proper odd divisor q. It is possible to modify the definition for $\mathcal{U}_n$ to permit $q=1$ but in that case each of the three entries in Δ_m alone satisfy Lemma 2.1, so the most we can say is that $\lambda(\mathcal{U}_n) \leq 3$. Instead, when n is a power of 2, we will use the family $\mathcal{A}_n$.

Latin square $\mathcal{A}_n$: For $n = 16d$, where d is a positive integer, define

$$\mathcal{A}_n[x,y] = \begin{cases} 0 & \text{if } (x,y)\in\{(d,0),\ (0,15d)\}, \\ d & \text{if } (x,y)\in\{(4d,0),\ (d,13d)\}, \\ 4d & \text{if } (x,y)\in\{(4d,10d),\ (10d,0)\}, \\ 10d & \text{if } (x,y)\in\{(0,0),\ (10d,10d)\}, \\ 14d & \text{if } (x,y)\in\{(0,10d),\ (d,14d),\ (4d,13d)\}, \\ 15d & \text{if } (x,y)\in\{(0,14d),\ (d,15d)\}, \\ x+y & \text{otherwise.} \end{cases}$$

For $\mathcal{A}_n$, the value of $\Delta(e)$ for each $e \in \Delta_*$ is given below

	0	$10d$	$13d$	$14d$	$15d$
0	$-6d$	$4d$		d	d
d	$-d$		$3d$	$-d$	$-d$
$4d$	$-3d$	$6d$	$-3d$		
$10d$	$-6d$	$6d$			

(4)

Lemma 2.4. *The latin square $\mathcal{A}_n$ has a transversal. Every transversal in $\mathcal{A}_n$ includes the entries $(10d, 0, 4d)$ and $(4d, 13d, 14d)$.*

Proof. Assume that T is a transversal in $\mathcal{A}_n$. Lemma 2.1 requires that $\sum_{e\in T} \Delta(e) = 8d$. Given (4), the only viable alternative is for T to include one entry from Δ_d together with the entries $(10d, 0, 4d) \in \Delta_{-6d}$ and $(4d, 13d, 14d) \in \Delta_{-3d}$.

A transversal T is given by

$$\mathrm{col}(x)=\begin{cases} 15d & \text{if } x=0,\\ x+10d-1 & \text{if } 1\le x\le 3d,\\ x+10d & \text{if } 3d<x<4d,\\ 13d & \text{if } x=4d,\\ x-4d & \text{if } 4d<x<9d,\\ x & \text{if } 9d\le x<10d \text{ or } 15d<x<n,\\ 0 & \text{if } x=10d,\\ x+4d-1 & \text{if } 10d<x\le 11d,\\ x-6d-1 & \text{if } 11d<x\le 15d. \end{cases}$$

It is routine to verify that T is a transversal. □

Together, Lemmas 2.2, 2.3 and 2.4 cover all cases to prove Theorem 1.1.

We conclude this section by using our latin squares of even order with $\lambda=1$ to build confirmed bachelor latin squares of odd order.

Theorem 2.5. *If L is a latin square of order n and $\alpha(L)=1$, then there exists a latin square L' of order $(n+1)$ such that $\tau(L')>0$.*

Proof. Let $T\subset L$ be a transversal and let $\mathcal{I}(L)=\{0,1,\dots,n-1\}$. We construct L' by prolongation of the transversal T. That is, the latin square L' of order $(n+1)$ is given by

$$L'_{n+1}[x,y]=\begin{cases} n & \text{if } x=y=n \text{ or } (x,y,z)\in T \text{ for some } z\in\mathcal{I}(L),\\ z & \text{if } x=n \text{ and } (x',y,z)\in T \text{ for some } x',z\in\mathcal{I}(L),\\ z & \text{if } y=n \text{ and } (x,y',z)\in T \text{ for some } y',z\in\mathcal{I}(L),\\ L[x,y] & \text{otherwise.} \end{cases}$$

Suppose T' is a transversal of L' including the entry (n,n,n). The entries in $T'\setminus\{(n,n,n)\}$ constitute a transversal of L disjoint from T, which contradicts $\alpha(L)=1$. Hence, (n,n,n) is a transversal-free entry. □

Theorem 2.5 provides a new proof of the existence of latin squares without orthogonal mates for all large enough orders, a result first shown in [8, 18].

3. A LARGE SUBRECTANGLE OF TRANSVERSAL-FREE ENTRIES

In this section we prove Theorem 1.2 using the following family of latin squares.

Latin square $\mathcal{D}_{3m}$: For odd $m \geq 3$, we define the latin square $\mathcal{D}_{3m}$ of order $3m$ and indexed by $\mathbb{Z}_3 \oplus \mathbb{Z}_m$

$$\mathcal{D}_{3m}[(a,b),(c,d)] = \begin{cases} (1,d) & \text{if } (a{=}b{=}c{=}0) \text{ or } (a{=}2, b{=}0 \text{ and } c{=}1), \\ (0,d) & \text{if } (a{=}b{=}0 \text{ and } c{=}1) \text{ or } (a{=}c{=}2 \text{ and } b{=}0), \\ (0,d+1) & \text{if } a=1 \text{ and } b=c=0, \\ (1,d+1) & \text{if } a=1,\ b=0 \text{ and } c=2, \\ (0,d) & \text{if } a=c=0 \text{ and } b=1, \\ (1,b+d+1) & \text{if } a=c=2 \text{ and } b\neq 0, \\ (a+c,b+d) & \text{otherwise.} \end{cases} \tag{5}$$

Let $\ell = m-1$. For each $e \in \Delta_* \subset \mathcal{D}_{3m}$, we display below the abbreviated ordered pairs $\Delta(e)$. The shaded region shows an $(m-1)\times m$ subrectangle. Theorem 1.2 will follow from our proof that this subrectangle consists of transversal-free entries

	00	01	02	…	0ℓ	10	11	12	…	1ℓ	20	21	22	…	2ℓ
00	10	10	10	…	10	20	20	20	…	20					
01	0ℓ	0ℓ	0ℓ	…	0ℓ										
02															
⋮															
0ℓ															
10	21	21	21	…	21						11	11	11	⋯	11
11															
12															
⋮															
1ℓ															
20						10	10	10	…	10	20	20	20	…	20
21											01	01	01	…	01
22											01	01	01	…	01
⋮											⋮	⋮	⋮	⋱	⋮
2ℓ											01	01	01	…	01

Suppose that T is a transversal of $\mathcal{D}_{3m}$. We define x_{ij} to be the number of entries in T of the form $((i,b),(j,d),(e,f))$, where b, d and f are arbitrary and $e = i+j$ in $\mathbb{Z}_3$. We define y_{ij} to be the number of entries in T of the same form, but where $e \neq i+j$ in $\mathbb{Z}_3$. Finally, we let z be the number of entries in T of the form $((0,1),(0,d),(0,d))$, where d is arbitrary. A number of constraints are immediate from the definition of $\mathcal{D}_{3m}$.

We will make repeated implicit use of the bounds $0 \le x_{ij} \le m$, $0 \le y_{ij} \le 1$, $0 \le z \le 1$ and the fact that $y_{02} = y_{11} = y_{20} = 0$. Moreover, the construction of $\mathcal{D}_{3m}$ forces

$$y_{00} + y_{01} \le 1, \quad y_{10} + y_{12} \le 1, \quad y_{21} + y_{22} \le 1, \tag{6}$$

$$0 \le x_{00} - z \le m - 2, \tag{7}$$

$$0 \le x_{22} \le m - 1. \tag{8}$$

Also, the need for T to include one representative from each row, column and symbol of $\mathcal{D}_{3m}$ implies

$$x_{00} + x_{01} + x_{02} + y_{00} + y_{01} = m, \tag{9}$$

$$x_{20} + x_{21} + x_{22} + y_{21} + y_{22} = m, \tag{10}$$

$$x_{00} + x_{10} + x_{20} + y_{00} + y_{10} = m, \tag{11}$$

$$x_{02} + x_{12} + x_{22} + y_{12} + y_{22} = m, \tag{12}$$

$$x_{00} + x_{12} + x_{21} + y_{01} + y_{10} + y_{22} = m, \tag{13}$$

$$x_{01} + x_{10} + x_{22} + y_{00} + y_{12} + y_{21} = m. \tag{14}$$

Adding (9), (11) and (13), then subtracting (10), (12) and (14), gives

$$3x_{00} - 3x_{22} + y_{00} + 2y_{01} + 2y_{10} - 2y_{12} - 2y_{21} - y_{22} = 0. \tag{15}$$

Moreover, Lemma 2.1 necessitates that

$$3 \mid y_{00} + 2y_{01} + 2y_{10} + y_{12} + y_{21} + 2y_{22}, \tag{16}$$

$$m \mid x_{22} + y_{10} + y_{12} - z. \tag{17}$$

The above restrictions are enough to show that certain entries in $\mathcal{D}_{3m}$ are not in any transversal.

Lemma 3.1. *No transversal of* $\mathcal{D}_{3m}$ *includes an entry in row* $(1,0)$ *and column* $(0,d)$, *where* d *is arbitrary.*

Proof. We are required to show $y_{10} = 0$, so assume for the sake of contradiction that $y_{10} = 1$. By (6) it follows that $y_{12} = 0$.

First, suppose that $z = 1$. In this case, (8) and (17) imply that $x_{22} = 0$. To satisfy (16) and (6) the only possibilities are

(i) $y_{01} = y_{22} = 1$, $y_{00} = y_{21} = 0$,
(ii) $y_{00} = 1$, $y_{01} = y_{21} = y_{22} = 0$,
(iii) $y_{21} = 1$, $y_{00} = y_{01} = y_{22} = 0$.

In all the three cases x_{00} may be calculated from (15) but its value violates (7).

We conclude that $z = 0$, and hence $x_{22} = m - 1$ from (17). Again to satisfy (16) and (6) we must have (i), (ii) or (iii). However, (i) and (15) imply $x_{00} = m - 2$, which together with (13) gives the contradiction $x_{12} + x_{21} < 0$.

Similarly, (ii) and (15) imply $x_{00} = m - 2$, which with (11) leads to $x_{20} = 0$. However, this is impossible since having $x_{20} = y_{21} = y_{22} = 0$ prevents the transversal from including any entry in row $(2,0)$.

Finally, (iii) and (15) imply $x_{00} = m-1$ which breaches (7). We have exhausted all possibilities, and are forced to conclude that $y_{10} = 0$ as required. □

Lemma 3.2. *No transversal of* $\mathcal{D}_{3m}$ *includes an entry in row* $(0, b)$ *and column* $(0, d)$, *where* $b \in \{2, 3, \ldots, m-1\}$ *and d is arbitrary.*

Proof. We are required to show that $x_{00} - z = 0$ and may assume, given Lemma 3.1, that $y_{10} = 0$. From (15) we have

$$0 = 3(x_{00} - z) - 3(x_{22} + y_{12} - z) + y_{00} + 2y_{01} + y_{12} - 2y_{21} - y_{22}. \tag{18}$$

As $m > 1$, we see from (17) that the only possible values for $x_{22} + y_{12} - z$ are 0 and m. In the latter case, (18) and (7) yield the immediate contradiction

$$0 \leq 3(m-2) - 3m + y_{00} + 2y_{01} + y_{12} \leq -2.$$

That leaves the case when $x_{22} + y_{12} - z = 0$. Here, (18) and (6) yield

$$0 \geq 3(x_{00} - z) - y_{21} - (y_{21} + y_{22}) \geq 3(x_{00} - z) - 2.$$

Given that $x_{00} - z$ is a non-negative integer, it must be zero and we are done. □

Lemmas 3.1 and 3.2 combine to prove Theorem 1.2. Note that in both lemmas the symbols occurring in the transversal-free entries have the form $(0, f)$ for arbitrary f. Thus the transversal-free entries do form a latin subrectangle as claimed.

Computational results summarized in Section 4 confirm that Lemmas 3.1 and 3.2 identify the only transversal-free entries in $\mathcal{D}_9$. We found that there is no latin square of order 9 whose set of transversal-free entries contains a subrectangle of larger dimensions than 2×3, although three species of order 9 have more than six transversal-free entries. We refer the reader to [4] for a definition of species, also known as main class or paratopy class.

We also computed that $\alpha(\mathcal{D}_9) = 2$ and $\lambda(\mathcal{D}_9) = 5$. It is easy to see that $\lambda(\mathcal{D}_{3m}) \leq 2m$ since there are rows with m transversal-free entries in them. For each $z \in \mathbb{Z}_m$, a transversal of $\mathcal{D}_{3m}$ is given by

$$\text{col}((a, b)) = \begin{cases} (a, z) & \text{if } b = 0, \\ (a-1, b+z) & \text{otherwise.} \end{cases}$$

These transversals are disjoint for different values of z, so $m \leq \lambda(\mathcal{D}_{3m}) \leq 2m$. However, $\mathcal{D}_9$ does not attain either bound.

4. LATIN SQUARES OF SMALL ORDER

The number of transversals in a latin square is well known to be a species invariant. It follows from the same reasoning that so are the values of λ, α and τ. In this section we report the results of a computational study of these parameters for all species of order ≤ 9. To increase reliability, all computations reported in this section were

TABLE I. Species of Order $4 \le n \le 9$ According to Number of Transversal-Free Entries.

	Order n					
τ	4	5	6	7	8	9
0	1	1	2	54	267932	19270833530
1				11	13165	18066
2				26	1427	1853
3				12	253	54
4				12	508	21
5				6	89	7
6			1	8	65	7
7				3	33	1
8				4	48	1
9					25	
10				1	27	1
11				1	9	
12		1	2	6	9	
13				1	2	
14					2	
16	1		1	1	27	
18					1	
20					1	
28					1	
36			6	1		
64					33	

undertaken separately by both authors. Our programs were independent except that they used a common list of species representatives as input. For order 9, we used a program from [11] to generate the species representatives. We refer the reader to [7, 14–16] for a history of earlier studies concerning transversals in latin squares.

In Table I we summarize species of order $4 \le n \le 9$ according to the number of transversal-free entries they contain. The single species of order 1, 2 and 3 have, respectively, 0, 4 and 0 transversal-free entries.

The unique species of order 9 achieving the maximum number of transversal-free entries turns out to be familiar. The representative L_1, shown below, is isotopic to $\mathcal{W}_9$. It is in semisymmetric form (meaning that 3 of its 6 conjugates are equal) and has 415 transversals. Shading shows its ten transversal-free entries which correspond to all of the entries in Δ_* in $\mathcal{W}_9$. By computation $\lambda(L_1)=6$ and $\alpha(L_1)=3$.

$$L_1=\begin{pmatrix} 0 & 3 & 1 & 8 & 7 & 2 & 6 & 5 & 4 \\ 2 & 6 & 4 & 0 & 8 & 7 & 1 & 3 & 5 \\ 5 & 0 & 3 & 2 & 1 & 6 & 4 & 8 & 7 \\ 1 & 7 & 2 & 4 & 3 & 5 & 8 & 6 & 0 \\ 8 & 2 & 6 & 3 & 5 & 4 & 7 & 0 & 1 \\ 7 & 8 & 0 & 5 & 4 & 3 & 2 & 1 & 6 \\ 6 & 1 & 5 & 7 & 2 & 8 & 0 & 4 & 3 \\ 4 & 5 & 8 & 1 & 6 & 0 & 3 & 7 & 2 \\ 3 & 4 & 7 & 6 & 0 & 1 & 5 & 2 & 8 \end{pmatrix}.$$

The next two latin squares, L_2 and L_3 shown below, represent the only other species of order 9 with more than six transversal-free entries. All three squares L_1, L_2 and L_3 have at least one subsquare of order 3. L_2 has 287 transversals, seven transversal-free entries (shaded), $\lambda(L_2)=6$ and $\alpha(L_2)=3$.

$$L_2=\left(\begin{array}{ccc|ccc|ccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 0 & 4 & 5 & 3 & 7 & 8 & 6 \\ 2 & 0 & 1 & 5 & 3 & 4 & 8 & 6 & 7 \\ \hline 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 & 0 \\ 4 & 7 & 3 & 2 & 8 & 6 & 0 & 5 & 1 \\ 7 & 3 & 4 & 8 & 6 & 0 & 5 & 1 & 2 \\ \hline 6 & 5 & 8 & 1 & 2 & 7 & 4 & 0 & 3 \\ 5 & 8 & 6 & 7 & 0 & 1 & 2 & 3 & 4 \\ 8 & 6 & 7 & 0 & 1 & 2 & 3 & 4 & 5 \end{array}\right), \quad L_3=\left(\begin{array}{ccc|ccc|ccc} 0 & 1 & 2 & 4 & 5 & 3 & 6 & 7 & 8 \\ 1 & 2 & 0 & 5 & 3 & 4 & 7 & 8 & 6 \\ 4 & 0 & 1 & 3 & 2 & 5 & 8 & 6 & 7 \\ \hline 2 & 8 & 6 & 0 & 1 & 7 & 4 & 5 & 3 \\ 8 & 6 & 4 & 1 & 7 & 0 & 5 & 3 & 2 \\ 6 & 4 & 8 & 7 & 0 & 1 & 3 & 2 & 5 \\ \hline 3 & 7 & 5 & 2 & 8 & 6 & 0 & 1 & 4 \\ 7 & 5 & 3 & 8 & 6 & 2 & 1 & 4 & 0 \\ 5 & 3 & 7 & 6 & 4 & 8 & 2 & 0 & 1 \end{array}\right)$$

L_3 is isotopic to its transpose by the symbol permutation (38)(57). It has eight transversal-free entries (shaded) and 92 transversals in total. This latin square exhibits another type of restriction on its transversals. It is one of only three species [7] recording the least λ value for order 9, that is $\lambda(L_3)=\mu(9)=3$. Also $\alpha(L_3)=2$.

The other two species that achieve $\lambda=\mu(9)=3$ are illustrated below by L_4 and L_5, neither of which has any transversal-free entries. We found that $\alpha(L_4)=2$ and $\alpha(L_5)=3$. Note that L_3 differs from L_4 only in the six entries shaded in L_4. The latin square L_4 has 84 transversals and is isotopic to its transpose by the symbol permutation (36)(58). Around one page of case analysis using Lemma 2.1 (or a quick computation) can be used to show that any transversal in L_4 must include two entries from the centre subsquare (with symbols $\{0,1,7\}$) and it follows from this that it has at most three disjoint transversals.

$$L_4 = \left(\begin{array}{ccc|ccc|ccc} 0&1&2&4&5&3&6&7&8\\ 1&2&0&5&3&4&7&8&6\\ 2&0&1&3&4&5&8&6&7\\ \hline 4&8&6&0&1&7&2&5&3\\ 8&6&4&1&7&0&5&3&2\\ 6&4&8&7&0&1&3&2&5\\ \hline 3&7&5&2&8&6&0&1&4\\ 7&5&3&8&6&2&1&4&0\\ 5&3&7&6&2&8&4&0&1 \end{array}\right), \quad L_5 = \left(\begin{array}{ccc|cccccc} 2&1&0&4&5&3&7&8&6\\ 1&0&2&7&8&6&4&5&3\\ 0&2&1&8&6&7&5&3&4\\ \hline 5&8&7&3&0&4&6&1&2\\ 3&6&8&5&4&0&2&7&1\\ 4&7&6&0&3&5&1&2&8\\ 8&5&4&6&1&2&3&0&7\\ 6&3&5&2&7&1&8&4&0\\ 7&4&3&1&2&8&0&6&5 \end{array}\right). \quad (19)$$

The latin square L_5 is isotopic to its transpose. It has 168 transversals and one subsquare of order 3. Suppose that we divide L_5 into 3×3 blocks, as follows

$$\begin{pmatrix} A & B & C\\ D & E & F\\ G & H & I \end{pmatrix}.$$

Now consider a transversal T of L_5 that misses block A. To fit in the correct number of occurrences of the symbols 3, 4 and 5, the parity of $|T\cap C|$ and $|T\cap D|$ must agree. But then this makes it impossible for T to include the required occurrences of the symbols 0, 1 and 2. We conclude that any transversal must intersect the order 3 subsquare in block A. As it happens, every transversal in L_5 contains exactly one of the three shaded entries of the subsquare, but an explanation of this fact (which implies that $\lambda(L_5)\le 3$) remains open.

For $n\le 8$, the number of species for each λ was summarized in [14]. Tables II, III, IV and V respectively, classify the species of order 6, 7, 8 and 9 according to their α and λ values. Every latin square L of order $n\le 5$ has $\alpha(L)=\lambda(L)$. Specifically, latin squares of order $n=1, 2, 3$ have $\lambda=\alpha=1,0,3$ respectively. For $n\in\{4,5\}$, there are exactly two species. They have $\lambda=\alpha\in\{0,4\}$ for $n=4$, and $\lambda=\alpha\in\{1,5\}$ for $n=5$.

We next give L_6, a totally symmetric representative (meaning all six of its conjugates are equal) of the unique species of order 7 with $\alpha(L_6)=\lambda(L_6)=1$. The only three

TABLE II. Species of Order 6 According to λ and α.

	λ			
α	0	2	4	Total
0	6			6
2		2	1	3
3			2	2
4			1	1
Total	6	2	4	12

TABLE III. Species of Order 7 According to λ and α.

	λ						
α	1	2	3	4	5	7	Total
1	1	3	5	7	2	1	19
2		2	16	49	22		89
3			3	12	19	4	38
4						1	1
Total	1	5	24	68	43	6	147

TABLE IV. Species of Order 8 According to λ and α.

	λ							
α	0	2	3	4	5	6	8	Total
0	33							33
1			1					1
2		7	39	538	20477	21247	143	42451
3			6	147	50852	188248	1745	240998
4				27	1	10	111	149
5							2	2
6							7	7
8							16	16
Total	33	7	46	712	71330	209505	2024	283657

TABLE V. Species of Order 9 According to λ and α.

	λ						
α	3	4	5	6	7	9	Total
1		7	36000				36007
2	2	4	6765	528	873	5	8177
3	1	12	100150	61085	18786989798	340588766	19127739812
4					135160264	7909243	143069507
5						32	32
6						5	5
7						1	1
Total	3	23	142915	61613	18922150935	348498052	19270853541

transversals of L_6 share the four shaded cells. We also give L_7, a semisymmetric representative of the unique species of order 8 with $\alpha(L_7)=1$. The $\tau(L_7)=9$ transversal-free entries are lightly shaded. There are 24 transversals of L_7 and all of them intersect the darkly shaded transversal.

$$L_6=\begin{pmatrix} 1&0&2&6&5&4&3\\ 0&2&1&4&3&6&5\\ 2&1&0&5&6&3&4\\ 6&4&5&3&1&2&0\\ 5&3&6&1&4&0&2\\ 4&6&3&2&0&5&1\\ 3&5&4&0&2&1&6 \end{pmatrix},\quad L_7=\begin{pmatrix} 0&2&1&4&5&6&7&3\\ 2&3&0&1&4&7&6&5\\ 1&0&7&6&3&5&4&2\\ 7&1&4&5&0&3&2&6\\ 3&4&6&2&1&0&5&7\\ 4&7&5&3&6&2&0&1\\ 5&6&3&7&2&4&1&0\\ 6&5&2&0&7&1&3&4 \end{pmatrix}$$

There are 16 species of order 8 with $\alpha=8$. Four of them have just 8 transversals, and yet have an orthogonal mate (one such example is given in [15, 16]). The others have either 16 or 32 transversals, and have 2, 16, 36 or 144 orthogonal mates.

Several different characterizations of the 36007 species of order 9 with $\alpha=\beta(9)=1$ are given in [7]. One characterization is that these are precisely the species that have a subsquare isotopic to the elementary Abelian group of order 4. A classical argument of Mann [9] (cf. Theorem 1.4) shows that any latin square of order 9 with a subsquare of order 4 has $\lambda\le 5$. In practice, this observation accounts for most of the examples we find with $\lambda\le 5$. There are 106432 species with a subsquare isotopic to the cyclic group of order 4 (all have $\lambda=5$), of which 6628 have $\alpha=2$ and 99804 have $\alpha=3$. Hence, Mann's argument accounts for all but 502 of the 142941 species with $\lambda\le 5$, and helps explain why there are more species with $\lambda=5$ than there are with $\lambda\in\{4,6\}$.

There is a unique species of order 9 with $\alpha=7$. It has 243 transversals and 139968 orthogonal mates. A totally symmetric representative of the species is L_8 below:

$$L_8=\left(\begin{array}{ccc|ccc|ccc} 0&2&1&6&8&7&3&5&4\\ 2&1&0&8&7&6&5&4&3\\ 1&0&2&7&6&8&4&3&5\\ \hline 6&8&7&5&4&3&0&2&1\\ 8&7&6&4&3&5&2&1&0\\ 7&6&8&3&5&4&1&0&2\\ \hline 3&5&4&0&2&1&6&8&7\\ 5&4&3&2&1&0&8&7&6\\ 4&3&5&1&0&2&7&6&8 \end{array}\right)$$

The order 3 subsquare in the centre of L_8 can be traded for a different subsquare to obtain a latin square isotopic to the elementary Abelian group. Both groups of order 9 have $\lambda=5$.

Much of the data generated in this project can be downloaded from [17].

5. CONCLUDING REMARKS

Computational results such as those in Section 4 suggest that a random latin square L of large order can be expected to have $\tau(L)=0$ and $\lambda(L)>1$. In this sense, the restrictions on transversals described by Theorems 1.1 and 1.2 illustrate exotic behavior. It is apparent from our data that only a small proportion of latin squares of orders 7, 8 and 9 possess orthogonal mates. However, most squares of these orders are close to having a mate, in the sense that their value of λ is not much below n. In [11] it is estimated that around 60% of latin squares of order 10 have mates. For higher orders we suspect that almost all latin squares have mates.

Theorem 1.2 shows that the family $\mathcal{D}_{3m}$ has quadratically many transversal-free entries and is thus much stronger than Theorem 1.5. It would be interesting to know if a result similar to Theorem 1.2 applies to orders that are not divisible by 3. In particular:

Problem 5.1. Is $\liminf_{n\to\infty} \max_{L} \frac{1}{n^2}\tau(L)>0$, where L ranges over squares of order n?

Theorem 1.1 illustrates an extreme type of restriction in latin squares of even order that do have transversals. Collectively, Theorems 1.1, Lemma 1.3 and the data in [7, 14] and Section 4 suggest the following.

Conjecture 5.2. *For all even $n\geq 10$ and each $m\in\{0,1,\ldots,n-3,n-2,n\}$, there exists a latin square of order n such that $\lambda(L)=m$.*

For odd n, there appears room to improve on the upper bound for $\mu(n)$ stated by Theorem 1.4. It would be of interest to determine if $\mu(n)<\frac{1}{2}n$ for all odd $n>3$. In particular:

Problem 5.3. Is $\mu(n)$ bounded as $n\to\infty$?

Finally, it may be possible to improve the upper bound on $\beta(n)$ in (1) when $n\equiv 3 \mod 4$. For example, $\beta(7)=1$. The species of order 9 that achieve $\beta(9)$ are characterised by Lemma 4.2 in [7].

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