Universal Shot-Noise Limit for Quantum Metrology with Local Hamiltonians

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Quantum many-body interactions can induce quantum entanglement among particles, rendering them valuable resources for quantum-enhanced sensing. In this work, we establish a link between the bound on the growth of the quantum Fisher information and the Lieb-Robinson bound, which characterizes the operator growth in locally interacting quantum many-body systems. We show that for initial separable states, despite the use of local many-body interactions, the precision cannot surpass the shot noise limit at all times. This conclusion also holds for an initial state that is the nondegenerate ground state of a local and gapped Hamiltonian. These findings strongly hint that when one can only prepare separable initial states, nonlocal and long-range interactions are essential resources for surpassing the shot noise limit. This observation is confirmed through numerical analysis on the long-range Ising model. Our results bridge the field of many-body quantum sensing and operator growth in many-body quantum systems and open the possibility to investigate the interplay between quantum sensing and control, many-body physics and information scrambling.

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Introduction.—Quantum entanglement is a valuable resource in quantum information processing. In quantum metrology, quantum Fisher information (QFI) [1–5], quantifying the precision of the sensing parameter, scales linearly with the number of uncorrelated probes, known as the shot noise limit (SNL), which also appears in sensing with classical resources. Quantum entanglement can achieve the Heisenberg limit (HL), featuring quadratic scaling, or even surpass it to reach the super-HL. Entanglement manifests its efficacy in two primary ways: during state preparation [6–11] or during signal sensing via the many-body interactions among individual sensors [12–16], which is the main essence of many-body quantum metrology. Recently, the subject matter has gained renewed interest. However, the existing protocols of dynamic sensing require to prepare the initial state in the highly entangled Greenberger-Horne-Zeilinger (GHZ)-like states, whose preparation is very challenging and time consuming. An effective strategy to address this issue involves merging the protocols of quantum state preparation and quantum metrology, see, e.g., Refs. [17–19], where an entangled initial state is prepared before the sensing process. Nevertheless, evaluating the time required to prepare a highly entangled state from separable ones, while considering restrictions imposed by accessible Hamiltonians, proves to be extremely challenging [20–22]. On the other hand, the protocols of quantum critical sensing either necessitate an initial state that near the vicinity of quantum criticality [23–33] or involve critical quantum dynamics [34,35], which is time consuming due to critical slow down. The time required for initial state preparation in quantum critical sensing is also largely ignored [36].

To circumvent the overhead of quantum state preparation, in this work, we propose to prepare the probes or sensors initially in a separable state, which can be prepared with the current experimentally feasible technology [37–39]. In our protocol depicted in Fig. 1(a), entanglement emerges during the signal sensing process due to the interactions in the many-body sensing Hamiltonian. This contrasts sharply with the protocol in Ref. [17] illustrated in Fig. 1(b), where the entangled initial state is explicitly prepared through the time evolution driven by a locally interacting preparation Hamiltonian, while the sensing Hamiltonian is noninteracting.

It is well known in the literature that for separable initial states and a noninteracting sensing Hamiltonian, the precision is limited by the SNL [6–8]. In our protocol, due to the many-body interactions, the state can become

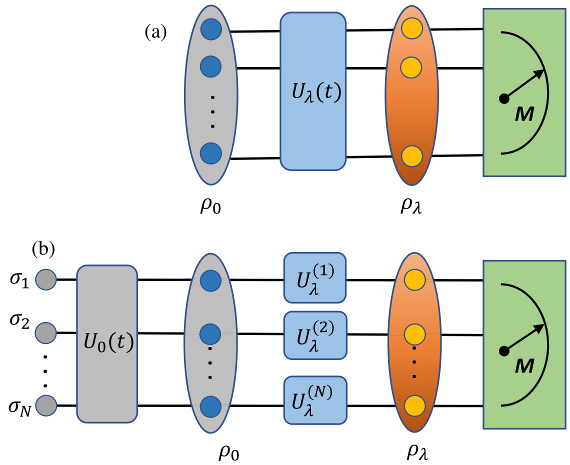

FIG. 1. Comparison between our protocol (a) with the protocol in Ref. [17] (b). In our protocol (a), the information of the estimation parameter is encoded into the many-body quantum states through the many-body dynamics $U_\lambda(t) = e^{-\mathrm{i}(\lambda \sum_i h_{X_i} + H_1)t}$ while in Ref. [17], the encoding dynamics given by $U_\lambda = e^{-\mathrm{i}\lambda \sum_i h_{X_i}}$ with $X_j = \{j\}$. In our protocol, the initial state is chosen to be either a separable state or the nondegenerate ground states of a gapped and local Hamiltonian while in Ref. [17] the initial state is prepared through the many-body dynamics $U_0(t)$.

entangled after the sensing process. This prompts the central question: whether many-body interactions can break the SNL. This question is also intimately related to recent studies on operator growth and quantum chaos in quantum many-body systems [40–44].

To answer this question, we derive a universal bound governing the growth of QFI over time, which can characterize the role of quantum entanglement in information scrambling, operator growth, and quantum chaos. We apply our bound to dynamic quantum sensing protocols with time-independent many-body Hamiltonians as shown in Fig. 1(a) and estimate the bound using the celebrated Lieb-Robinson bound [45–48] for quantum many-body systems with local interactions. We find that it is impossible to surpass the SNL with local interactions. This observation holds not only for separable initial states but also extends to cases where the initial state is the nondegenerate ground state of a locally gapped Hamiltonian—a state feasible for experimental preparation through cooling processes. Therefore, if only separable states are accessible in experiments, nonlocal or long-range interactions are essential to beat the SNL and bring real quantum advantage in many-body quantum metrology. We exemplify our findings in magnetometry with the short-range transverse-field Ising (TFI) model, the chaotic Ising (CI) model, and the long-range Ising (LRI) model.

Universal bound on the growth of the QFI.—We consider the following sensing Hamiltonian:

$$H_\lambda(t) = H_{0\lambda}(t) + H_1(t), \tag{1}$$

where $H_{0\lambda}(t)$ is a simple Hamiltonian encoding the estimation parameter λ, and $H_1(t)$ involves interactions among sensors induced by either intrinsic interactions or external coherent controls. In the formal case, H_1 is usually time independent, while in the later case, $H_1(t)$ becomes time dependent. The generator for sensing λ [12,49] is given by

$$G(t) = \int_0^t [\partial_\lambda H_\lambda(\tau)]^{(\mathrm{H})} d\tau, \tag{2}$$

where an operator in the Heisenberg picture is defined as $\mathcal{O}^{(\mathrm{H})}(t) = U^\dagger(t)\mathcal{O}^{(\mathrm{S})}(t)U(t)$. The QFI is determined by the variance of $G(t)$ over the initial state $|\psi_0\rangle$, i.e.,

$$I(t) = 4\mathrm{Var}[G(t)]_{|\psi_0\rangle}. \tag{3}$$

Optimal control theory has been proposed to simultaneously optimize the initial state $|\psi_0\rangle$ and $H_1(t)$, resulting in a bound $I(t) \le 4\left(\int_0^t \|\partial_\lambda H_\lambda^{(\mathrm{S})}(\tau)\| d\tau\right)^2$ [14,49–51]. Here, the seminorm $\|\cdot\|$ denotes the spectrum width of an operator, i.e., the difference between its maximum eigenvalue and minimum eigenvalue.

By taking the derivative of Eq. (3) and applying the Cauchy-Schwarz inequality, we derive a universal bound [52,53] that characterizes the growth of QFI:

$$\frac{d\sqrt{I(t)}}{dt} \le \Gamma(t) \equiv 2\sqrt{\mathrm{Var}\left([\partial_\lambda H_\lambda(t)]^{(\mathrm{H})}\right)_{|\psi_0\rangle}}. \tag{4}$$

The saturation condition is provided in the Supplemental Material (SM) [52]. Alternatively, one can rewrite

$$\Gamma(t) = 2\sqrt{\mathrm{Var}\left(\partial_\lambda H_\lambda^{(\mathrm{S})}(t)\right)_{|\psi(t)\rangle}}, \tag{5}$$

where $|\psi(t)\rangle = U(t)|\psi_0\rangle$.

It is worth noting that Eq. (4) universally holds for all initial states, including time-independent and driven quantum systems. $\Gamma(t)$ depends on the control Hamiltonian $H_1(t)$ and the initial state $|\psi_0\rangle$. Optimizing $\Gamma(t)$ over all possible unitary dynamics and initial states yields $\Gamma(t) \le 2\|\partial_\lambda H_\lambda^{(\mathrm{S})}(t)\|$. By combining this bound with $I(t) \le \left(\int_0^t \Gamma(\tau) d\tau\right)^2$, which can be obtained by integrating both sides of Eq. (4), one immediately reproduces the bound given in previous works [14,49–51]. Compared to these studies, our bound (4) provides a feasible approach to study the scaling behavior of the QFI when the initial state $|\psi_0\rangle$ is limited to a specific set of states.

SNL for short-range local interactions.—We will show the close connection between our bound (4), depicting QFI growth, and the Lieb-Robinson bound, which characterizes operator complexity in quantum many-body with short-range local interactions. We consider time-independent Hamiltonians as follows:

$$H_\lambda = \lambda \sum_{i=1}^{N} h_{X_i} + H_1, \tag{6}$$

where h_{X_i} is supported on the set X_i with cardinality $|X_i| = R$ and diameter $\mathrm{diam}(X_i) = \max_{k,l\in X_j}|k-l|$. H_1 denotes the interactions. We require H_λ to contain only local and short-range interactions, imposing that $\mathrm{diam}(X_j)$ is independent of N [46,48,54] and h_{X_j} is a local operator. Equation (6) represents the model used in magnetometry, where λ represents the magnetic field [55].

According to Eq. (6), the bound (4) can be reformulated in relation to dynamic correlation matrices of local operators as

$$\Gamma(t) = 2\sqrt{\sum_{jk} \mathrm{Cov}[h_{X_j}^{(\mathrm{H})}(t) h_{X_k}^{(\mathrm{H})}(t)]_{|\psi_0\rangle}}, \tag{7}$$

where $\mathrm{Cov}[AB]\big|_{|\psi_0\rangle} \equiv \langle\{A,B\}\rangle/2 - \langle A\rangle\langle B\rangle$ and

$$h_{X_i}^{(\mathrm{H})}(t) \equiv e^{\mathrm{i}H_\lambda t} h_{X_i} e^{-\mathrm{i}H_\lambda t}. \tag{8}$$

In this case, we observe that $\Gamma(t) \le 2N$, implying $I(t) \le 4N^2t^2$ [49,51]. If H_1 commutes with $\sum_{i=1}^N h_{X_i}$, then such an HL can be saturated only when using GHZ-like entangled initial states [12,14,15]. However, preparing such states experimentally is challenging. Conversely, if H_1 does not commute with $\sum_{i=1}^N h_{X_i}$, then entanglement maybe generated by signal sensing from separable initial states. So, for separable initial states, what precisely is the tight bound that limits the precision? Is it possible to surpass the SNL using many-body interactions?

We emphasize that the Lieb-Robinson bound [45–48] imposes a strong restriction on the scaling of QFI for local Hamiltonians. Specifically, if the sensing Hamiltonian (6) only contains local or short-range interactions, the static correlation $\mathrm{Cov}[h_{X_j}h_{X_k}]\big|_{|\psi_0\rangle}$ between two disjoint local operators h_{X_j} and h_{X_k} decays exponentially, provided the initial state $|\psi_0\rangle$ is separable or the nondegenerate ground state of some local and gapped Hamiltonians. In this case, the dynamic correlation function also decays exponentially,

$$\left|\mathrm{Cov}[h_{X_j}^{(\mathrm{H})}(t) h_{X_k}^{(\mathrm{H})}(t)]_{|\psi_0\rangle}\right| \le \mathcal{C}\exp(-[d(X_j,X_k) - v_{\mathrm{LR}}t]/\xi), \tag{9}$$

where $\mathcal{C}$ and ξ are constants that solely depend on the topology of the sites, $d(X_j,X_k)$ is the distance between X_j and X_k, and v_{LR} is the celebrated Lieb-Robinson velocity. Substituting Eq. (9) into Eq. (7), we rigorously show that the scaling of $\Gamma(t)$ is lower bounded by $\sqrt{N}$. The crucial observation here is that upon factoring out the time-dependent term $\exp(v_{\mathrm{LR}}t/\xi)$, thanks to the exponential decay of dynamic correlation functions, only initially overlapping local operators will contribute to the scaling of $\Gamma(t)$. This results in [52]

$$\Gamma(t) \le 2\gamma(t)\sqrt{N}, \tag{10}$$

where $\gamma(t)$ is only a function of time and independent of N. It remains finite as long as t is finite and behaves as $e^{v_{\mathrm{LR}}t/\xi}$ as $t \to \infty$.

Equation (10) is the main result of this work. Clearly, for finite but fixed times, the QFI is limited by the SNL. On the other hand, at sufficiently long times, for time-independent systems, one can show that $I(t)/t^2$ is independent of time [52,56] and is only a function of N. Moreover, the timescale to reach this regime corresponds to the case where t is much larger than the inverse of the minimum energy gap for the system. In this regime, when N is large, $I(t) \sim t^2N^\alpha$. Since Eq. (10) is valid for all times and all N, combined with Eq. (4) we conclude $\alpha \le 1$. Therefore, in local short-range models where operator growth is constrained by the Lieb-Robinson bound, the SNL cannot be surpassed.

Nevertheless, for the same initial state, the many-body interaction H_1 may increase the prefactor of the QFI compared to the noninteracting case, though it is not always the case. For example, if the initial state is prepared in a state slightly deviating from the ground state of $\sum_{i=1}^N h_{X_i}$, then the QFI for the noninteracting case, being the fluctuations of $G(t)$ over the initial state, grows very slowly as time evolves. Meanwhile, if a many-body interaction that does not commute with the noninteracting Hamiltonian is added, the noncommutativity introduces significant fluctuations of $G(t)$, which can lead to a QFI significantly larger than the noninteracting case and thus enhance the prefactor of QFI.

The spread of the generator of the metrological bound.— The manifestation of the SNL in locally interacting systems and for separable initial states can also be understood from the perspective of operator growth. Despite the spread of $h_{X_i}^{(\mathrm{H})}(t)$ over the lattice, the metrological generator $[\partial_\lambda H_\lambda(t)]^{(\mathrm{H})}$, being a sum of these nonlocal operators, may still be reformulated as a sum of local operators, thus keeping the precision limited to the SNL. A trivial example is when H_1 commutes with $\sum_i h_{X_i}$ while H_λ does not commute with each individual h_{X_i}, in which case $\Gamma(t)$ remains at the SNL.

Generally, we assume $[\partial_\lambda H_\lambda(t)]^{(\mathrm{H})}$ can be expanded in terms of two-body basis operators

$$[\partial_\lambda H_\lambda(t)]^{(\mathrm{H})} = \sum_{i=1}^{N}\sum_{j\ge i}^{N}\sum_{\alpha}\eta_{ij}^{\alpha}\mathcal{O}_{ij}^{\alpha} \tag{11}$$

where we have suppressed the time dependence for simplicity and for spin systems $\mathcal{O}_{ij}^{\alpha}$ is a basis spin operator, such as $\sigma_i^x\sigma_j^y$ while for fermionic systems $\mathcal{O}_{ij}^{\alpha}$

is a Hermitian basis fermionic operator, such as $c_i^\dagger c_j + \mathrm{H.c.}$ or $\mathrm{i}(c_i^\dagger c_j - \mathrm{H.c.})$. We emphasize that the number of different types of operators indexed by α is finite and does not scale with N. If the initial state is separable and $[\partial_\lambda H_\lambda(t)]^{(\mathrm{H})}$ is a sum of fast-decaying long-range two-body interactions, i.e.,

$$\lim_{k\to\infty}\lim_{N\to\infty}\sum_{i\le k}\sum_{j\ge k}|\eta_{ij}^\alpha| = \lim_{k\to\infty}\int_1^k dx\int_k^\infty dy|\eta_{xy}^\alpha| < \infty, \quad (12)$$

then the SNL cannot be surpassed by using the bound (4) [52]. It follows from Eq. (11) that $[\partial_\lambda H_\lambda(t)]^{(\mathrm{H})} = \sum_{i=1}^N \tilde{\mathscr{O}}_i$, where $\tilde{\mathscr{O}}_i \equiv \frac{1}{2}\sum_\alpha\left(\sum_{j\ge i}^N \eta_{ij}^\alpha \mathcal{O}_{ij}^\alpha + \sum_{j\le i}^N \eta_{ji}^\alpha \mathcal{O}_{ji}^\alpha\right)$. The condition (12) ensures that $\tilde{\mathscr{O}}_i$ behaves effectively as a local operator and can be different from $h_{X_i}^{(\mathrm{H})}(t)$, which is generically nonlocal. Essentially, the locality of components of the metrological generator leads to the SNL for separable initial states. We will further elaborate this observation using the TFI model.

SNL in the TFI model.—We consider the integrable TFI chain

$$H_\lambda^{\mathrm{TFI}} = -\left(J\sum_{i=1}^N \sigma_i^x\cdot\sigma_{i+1}^x + \lambda\sum_{i=1}^N \sigma_i^z\right), \quad (13)$$

with the periodic boundary condition $\sigma_1^z = \sigma_{N+1}^z$, and $J, \lambda > 0$. In the thermodynamic limit $N\to\infty$, when $J \gg \lambda$ the ground state is ferromagnetic and degenerate, represented by $|++\cdots+\rangle$ or $|--\cdots-\rangle$, while for $J \ll \lambda$ the ground state is paramagnetic $|\uparrow\uparrow\cdots\uparrow\rangle$.

For any initial separable state, Eq. (4) predicts that the QFI cannot surpass the SNL. On the other hand, this model can be exactly solved by mapping it to a free fermion model [57–59] and therefore one can compute $[\partial_\lambda H_\lambda(t)]^{(\mathrm{H})}$ explicitly. In our SM [52], we demonstrate that the metrological generator $[\partial_\lambda H_\lambda(t)]^{(\mathrm{H})}$ in this case explicitly follows the structure of Eq. (11) with four types of fermionic operators: $\mathcal{O}_{ij}^{(1)} = (c_i^\dagger c_j + \mathrm{H.c.})$, $\mathcal{O}_{ij}^{(2)} = (c_i c_j^\dagger + \mathrm{H.c.})$, $\mathcal{O}_{ij}^{(3)} = (c_i^\dagger c_j^\dagger + \mathrm{H.c.})$, and $\mathcal{O}_{ij}^{(4)} = \mathrm{i}(c_i^\dagger c_j^\dagger - \mathrm{H.c.})$. The expression for the η functions characterizing the weights of these operators spreading from the ith site to the jth site can be found in the SM [52]. In the thermodynamic limit, η_{ij}^α behaves like p^{j-i} for $j\ge i$, where $p = J/\lambda$ for $J<\lambda$, $p = \lambda/J$ for $\lambda < J$, and $p = 0$ for $J = \lambda$ [52]. The power-law decay of η functions indicates that the evolved operator remains extremely local, as shown in Fig. 2(b), ensuring the condition (12), i.e.,

$$\int_1^k dx\int_k^\infty dy|\eta_{xy}^\alpha| \sim \frac{1-p^{k-1}}{(\ln p)} < \infty, \quad (14)$$

as $k\to\infty$. Therefore, the locality of the evolved operator suggests that QFI beyond the SNL cannot be achieved by

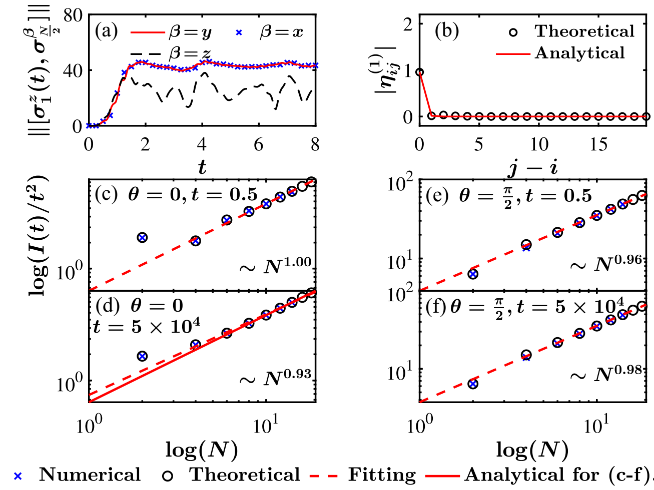

FIG. 2. (a) Numerical calculation of the operator diffusion in the TFI chain with $N = 10$. (b) Coefficient $|\eta_{ij}^{(1)}|$ characterizing the decay of the two-body interactions. (c)–(f) Scaling of the QFI with respect to the number of spins at different times for differential initial separable spin coherent states $|\psi_0\rangle = \otimes_{i=1}^N [\cos(\theta/2)|\uparrow\rangle_i + \sin(\theta/2)e^{\mathrm{i}\phi}|\downarrow\rangle_i]$. Here numerical data are obtained by directly diagonalizing the Hamiltonian of the TFI model, while theoretical data are derived using results by mapping the TFI model to the free fermion model. The analytical result refers to Eq. (16). Other parameters used for the calculations are $J = 2$, $\lambda = 5$, and $\phi = 0$.

initial separable probe states in this integrable TFL model. Figure 2(a) characterizes the diffusion of the correlators, suggesting that the numerical choices of $t = 0.5$ and $t = 5\times 10^4$ can be considered as the timescales for the part and full spread of local operators, respectively. Figures 2(c) and 2(d) numerically verify that only the SNL can be achieved for the different initial separable spin coherent states parameterized by $|\psi_0\rangle = \otimes_{i=1}^N [\cos(\theta/2)|\uparrow\rangle_i + \sin(\theta/2)e^{\mathrm{i}\phi}|\downarrow\rangle_i]$.

Furthermore, if we consider the initial state as the ground state of the TFI model with known values of parameters λ_* and J, achievable through cooling processes, then the asymptotic behavior of the QFI with respect to the unknown parameter λ under the Hamiltonian (13) is

$$\lim_{t,N\to\infty}\frac{I(t)}{Nt^2} = f(\lambda, J, \lambda_*) \sim O(1), \quad (15)$$

where the function f is N independent [60], confirming the claim that only the SNL can be achieved even with the ground state of local and gapped Hamiltonians. Taking $\lambda_*\to+\infty$, where the ground state becomes the spin coherent state with $\theta = \phi = 0$, we find

$$\lim_{t,N\to\infty}\frac{I(t)}{Nt^2} = \frac{J^2(4\lambda^2 - 3J^2)}{\lambda^4} \quad (16)$$

for $J < \lambda$, which is also verified in Fig. 2(d), and $\lim_{t,N\to\infty} I(t)/(Nt^2) = 1$ for $J \geq \lambda$. We observe that the prefactor of the QFI, both for the ground state and for other separable states [as depicted in Figs. 2(c)–2(f)], does not exhibit a significant difference in order of magnitude, i.e., < 10, when compared to the optimal noninteraction scheme involving separable states, where $I(t)/(Nt^2) = \|\sigma_i^z\|^2 = 4$.

SNL in the chaotic Ising model.—Different from integrable models, the operator complexity in chaotic models grows very rapidly [40–44]. Nevertheless, according to Eqs. (4) and (10), even in locally chaotic models, the SNL cannot be surpassed by using separable states. For instance, we consider the Ising model with both transverse and longitudinal fields described by the following Hamiltonian:

$$H_\lambda^{\mathrm{CI}} = -\sum_{i=1}^{N} (J\sigma_i^x\sigma_{i+1}^x + h\sigma_i^x + \lambda\sigma_i^z), \tag{17}$$

where open boundary conditions are adopted. Energy-level spacing statistics indicate that this model is quantum chaotic for $J = h = \lambda$ [42,61]. Figures 3(c)–3(f) verify the prediction by Eq. (10) that separable states cannot surpass the SNL even in such chaotic short-range systems. To surpass the SNL, we are thus motivated to explore long-range models. The effect of quantum chaos in quantum metrology has been studied in Ref. [62] within the context of a kicked top which involves long-range interactions. Here, we show that quantum chaos plays no enhancement in the scaling of QFI in locally chaotic many-spin models.

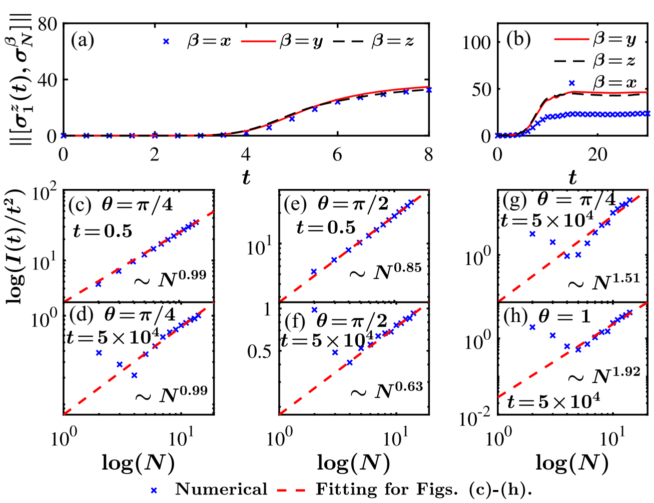

FIG. 3. Numerical calculation of the operator diffusion in (a) the CI model (b) the LRI model with $N = 10$. The scaling of the QFI with respect to the number of spins at different times for differential initial separable spin coherent states $|\psi_0\rangle = \otimes_{i=1}^N [\cos(\theta/2)|\uparrow\rangle_i + \sin(\theta/2)e^{i\phi}|\downarrow\rangle_i]$ in (c)–(f) the CI model and (g) and (h) the LRI model. Other parameters used for the calculations are $J = \lambda = h = 1$, $\phi = 0$ in the CI model, and $J = 1$, $\lambda = 0.5$, $\alpha = 3$ in the LRI model.

Beyond the SNL with the LRI model.—As demonstrated before, breaking the SNL is solely feasible within long-range and nonlocal systems, which violates the Lieb-Robinson inspired bound (10). Thus, we consider the long-range Ising model with power-law decay,

$$H_\lambda^{\mathrm{LRI}} = -\left(J\sum_{i<j}\frac{\sigma_i^x\sigma_j^x}{|i-j|^\alpha} + \lambda\sum_i \sigma_i^z\right), \tag{18}$$

which reduces to the TFI model as $\alpha \to \infty$. For $\alpha = 0$, this model corresponds to the Lipkin-Meshkov-Glick model [63]. In this long-range model, the breakdown of exponential decay in connected correlation function Eq. (9) will result in the failure of the bound presented in (10). Consequently, we expect that for small α where the long-range interactions decay sufficiently slowly, it is possible to surpass the SNL with separable initial states. As depicted in Figs. 3(g) and 3(h), we have identified specific instances of this scenario.

Conclusion and outlook.—In conclusion, we have derived a universal bound on the growth of the QFI under arbitrary dynamics and initial states. We apply our bound to the case of separable initial states or the nondegenerate ground state of a gapped and local sensing Hamiltonian. We prove that with these particular initial states, the QFI cannot surpass the SNL, as we have explicitly demonstrated with TFI and CI models. Our results give an important guideline for many-body sensing: either initial entanglement or long-range interactions are essential resources to achieve quantum advantage in many-body quantum sensing, as demonstrated in the LRI model. Our results shed light on various aspects of the interplay between many-body physics, quantum control theory, quantum chaos, operator growth, and information scrambling. We leave these studies for future exploration.

Note added.—Recently, we noted that a bound similar to Eq. (4) also appears in Ref. [53] with the focus on non-Hermitian sensing.

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