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STARSHAPED COMPACT HYPERSURFACES WITH PRESCRIBED k-TH MEAN CURVATURE IN HYPERBOLIC SPACE

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Abstract. In this paper we consider the problem of finding a star-shaped compact hypersurface with prescribed k-th mean curvature in hyperbolic space. Under some sufficient conditions, we obtain an existence result by establishing a priori estimates and using degree theory arguments.

1. **Introduction.** Let $\mathbb{S}^n$ be the unit sphere in the Euclidean space $\mathbb{R}^{n+1}$, and let e be the standard metric on $\mathbb{S}^n$ induced from $\mathbb{R}^{n+1}$. Suppose that (u, ρ) are the spherical coordinates in $\mathbb{R}^{n+1}$, where $u \in \mathbb{S}^n$, $\rho \in [0, \infty)$. By choosing the smooth function $\varphi(\rho) := \sinh^2 \rho$ on $[0, \infty)$ we can define a Riemannian metric h on the set $\{(u, \rho) : u \in \mathbb{S}^n, 0 \le \rho < \infty\}$ as follows

$$h = d\rho^2 + \varphi(\rho)e.$$

This gives the space form $\mathcal{R}^{n+1}(-1)$ which is the hyperbolic space $\mathbb{H}^{n+1}$ with sectional curvature -1. For a smooth hypersurface $\mathcal{M}$ in $\mathcal{R}^{n+1}(-1)$, we denote by $\lambda = (\lambda_1, \cdots, \lambda_n)$ its principal curvatures with respect to the metric $g := h|_{\mathcal{M}}$. Then, for each $1 \le k \le n$, the k-th mean curvature of $\mathcal{M}$ is defined as

$$H_k(\lambda) = \begin{pmatrix} n \\ k \end{pmatrix}^{-1} \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

Let $\psi(u, \rho)$, $u \in \mathbb{S}^n$, $\rho \in (0, \infty)$, be a given positive smooth function satisfying suitable conditions. We are interested in the existence of a smooth hypersurface $\mathcal{M}$ embedded in $\mathcal{R}^{n+1}(-1)$ as a graph over $\mathbb{S}^n$ so that its k-th mean curvature is given by ψ. We refer the readers to [8] and [5] for the introductory material and the history of this problem.

It is clear that $\mathcal{M} := \{(u, z(u)) : u \in \mathbb{S}^n\}$ is an embedded hypersurface in $\mathcal{R}^{n+1}(-1)$ for any smooth positive function z on $\mathbb{S}^n$. We call z k-admissible if the principal curvatures $\lambda = (\lambda_1, \cdots, \lambda_n)$ of $\mathcal{M}$ belong to Γ_k, where Γ_k is the connected component of $\{\lambda \in \mathbb{R}^n : H_k(\lambda) > 0\}$ containing the positive cone $\{\lambda \in \mathbb{R}^n : \lambda_1 > 0, \cdots, \lambda_n > 0\}$.

The main result of this paper is the following:

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Theorem 1. *Let $1 \leq k \leq n$, and let ψ be a smooth positive function in the annulus $\overline{\Omega} : u \in \mathbb{S}^n, \rho \in [R_1, R_2], 0 < R_1 < R_2 < \infty$, satisfying the conditions*

$$\psi(u, R_1) \geq \coth^k(R_1) \quad \textit{and} \quad \psi(u, R_2) \leq \coth^k(R_2) \quad \textit{for } u \in \mathbb{S}^n$$

and

$$\frac{\partial}{\partial \rho}\Big(\psi(u,\rho)\sinh^k \rho\Big) \leq 0 \quad \textit{for all } u \in \mathbb{S}^n \textit{ and } \rho \in [R_1, R_2].$$

Then there exists a positive smooth k-admissible function z on $\mathbb{S}^n$ such that the closed hypersurface $\mathcal{M} := \{(u, z(u)) : u \in \mathbb{S}^n\}$ is in $\overline{\Omega} \subset \mathcal{R}^{n+1}(-1)$, and its k-th mean curvature is given by ψ:

$$H_k(\lambda(u, z(u))) = \psi(u, z(u)), \quad \forall\ u \in \mathbb{S}^n.$$

In the Euclidean space $\mathcal{R}^{n+1}(0)$, such results were obtained for the case $k = 1$ by Bakelman and Kantor [3, 4] and by Treibergs and Wei [20], for the case $k = n$ by Oliker [16], and for general k by Caffarelli, Nirenberg and Spruck [8]. In the elliptic space $\mathcal{R}^{n+1}(+1)$, such a result is the combination of the work of Barbosa, Lira and Oliker [5] and that of Li and Oliker [15]. The $k = n$ case in Theorem 1 was established by Oliker [17]. Our proof of Theorem 1 uses the C^0 and C^1 a priori estimates obtained in [5] and the arguments in [15] which is based on the degree theory for fully nonlinear elliptic operators of second order developed in [13]. The main work for us to prove Theorem 1 is to give the C^2 a priori estimates. In establishing the C^2 estimates, we make use of Lemma 2, a quantitative version of a theorem of Davis [10] which, to our knowledge, was given in [1]. The theorem in [10] says that a rotationally invariant function on symmetric matrices is concave if and only if it is concave on the diagonal matrices, while Lemma 2 allows the use of this term in making C^2 a priori estimates. The use of such a concave term in C^2 estimates for solutions of the Monge-Ampère equations has been extensive, see e.g. Calabi [9] and Pogorelov [18]. The use of Lemma 2 in C^2 estimates for solutions of more general equations can be found in [1, 2, 12, 19, 21, 22].

Rcently, Barbosa, Lira and Oliker [6] obtained the uniqueness of solutions under the same hypoethses of Theorem 1.

2. **Some fundamental formulae.** Let us define a function f on Γ_k by

$$f(\lambda) = \left[\begin{pmatrix} n \\ k \end{pmatrix}^{-1} \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}\right]^{1/k},$$

where $\lambda := (\lambda_1, \cdots, \lambda_n) \in \Gamma_k$. It is well known that f is smooth, positive, concave, and strictly increasing with respect to each variable, see e.g. [7]. Now our problem is equivalent to finding a smooth positive k-admissible function z on $\mathbb{S}^n$ so that

$$F(\mathbf{B}) = \overline{\psi} \tag{1}$$

on $\mathcal{M} := \{(u, z(u)) : u \in \mathbb{S}^n\}$, where $\overline{\psi} := \psi^{1/k}$, $\mathbf{B}$ is the second fundamental form of $\mathcal{M}$, and $F(\mathbf{B}) := f(\lambda(\mathbf{B}))$ with $\lambda(\mathbf{B})$ being the eigenvalues of $\mathbf{B}$ with respect to the metric g on $\mathcal{M}$.

Suppose now $\mathcal{M}$ is the graph of a smooth positive k-admissible function z on $\mathbb{S}^n$. Let us recall the formulae given in [5] for the components of $g = (g_{ij})$ and $\mathbf{B} = (b_{ij})$ on $\mathcal{M}$ under a local coordinate. Let $\theta^1, \cdots, \theta^n$ be a smooth local coordinate of $\mathbb{S}^n$, which of course gives a local coordinate of $\mathcal{M}$. If we denote by $\{e_{ij}\}$ the components of e under this local coordinate, and set $z_i = \frac{\partial z}{\partial \theta^i}$ and $z_{ij} = \frac{\partial^2 z}{\partial \theta^i \partial \theta^j}$, then

$$g_{ij} = \varphi e_{ij} + z_i z_j \tag{2}$$

and

$$g^{ij} = \frac{1}{\varphi}\left[e^{ij} - \frac{z^i z^j}{\varphi + |\nabla' z|^2}\right], \qquad z^i = e^{ij} z_j, \tag{3}$$

where $(g^{ij}) = (g_{ij})^{-1}$, $(e^{ij}) = (e_{ij})^{-1}$, and ∇' denotes the Levi-Civita connection on $\mathbb{S}^n$. Moreover, for the second fundamental form we have

$$b_{ij} = \frac{\varphi}{\sqrt{\varphi^2 + \varphi |\nabla' z|^2}}\left\{-\nabla'_{ij} z + \frac{\partial \ln \varphi}{\partial \rho} z_i z_j + \frac{1}{2}\frac{\partial \varphi}{\partial \rho} e_{ij}\right\}. \tag{4}$$

We also need the following well-known fundamental equations for a hypersurface $\mathcal{M}$ in $\mathcal{R}^{n+1}(-1)$:

$$\text{Codazzi equation:} \qquad \nabla_i b_{jk} = \nabla_j b_{ki} = \nabla_k b_{ij} \tag{5}$$

$$\text{Gauss equation:} \qquad R_{ijkl} = (b_{ik} b_{jl} - b_{il} b_{jk}) - (g_{ik} g_{jl} - g_{il} g_{jk}) \tag{6}$$

$$\text{Ricci equation:} \qquad \nabla_l \nabla_k b_{ij} - \nabla_k \nabla_l b_{ij} = b_{ip} g^{pq} R_{qjkl} + b_{jp} g^{pq} R_{qikl} \tag{7}$$

where R_{ijkl} denotes the Riemannian curvature tensor of $\mathcal{M}$, and ∇_i and $\nabla_i \nabla_j$ the covariant differentiations in the metric g with respect to some local coordinates on $\mathcal{M}$.

As the preparation for deriving the C^2-estimates, let us introduce the following two functions on $\mathcal{M}$

$$\tau = \frac{\varphi(z)}{\sqrt{\varphi(z) + |\nabla' z|^2}} \quad \text{and} \quad \eta = -\cosh(z). \tag{8}$$

We have

Lemma 1. *For τ and η the following equations hold*

$$\nabla_i \tau = -\, b_{ip} g^{pq} \nabla_q \eta, \tag{9}$$

$$\nabla_{ij} \tau = -\, \nabla_p b_{ij} g^{pq} \nabla_q \eta - \tau b_{ip} g^{pq} b_{qj} - \eta b_{ij}, \tag{10}$$

$$\nabla_{ij} \eta = \tau b_{ij} + \eta g_{ij}. \tag{11}$$

Proof. These formulae have been derived in [5] by using another model of $\mathcal{R}^{n+1}(-1)$. In fact, we can show them directly. Since (10) is an immediate consequence of (9), (11) and the Codazzi equation (5), it suffices to verify (9) and (11). Let $c(\rho) = \cosh(\rho)$ and $s(\rho) = \sinh(\rho)$. Then

$$\begin{aligned} \nabla_i \tau = & \frac{2sc}{\sqrt{\varphi + |\nabla' z|^2}} z_i - \frac{\varphi}{(\varphi + |\nabla' z|^2)^{3/2}}\left(sc z_i + e^{pq} \nabla'_{ip} z \nabla'_q z\right) \\ = & \frac{1}{(\varphi + |\nabla' z|^2)^{3/2}}\left\{sc\varphi z_i + 2sc z_i |\nabla' z|^2 - \varphi \nabla'_{ip} z z^p\right\} \end{aligned}$$

Noting that $\nabla_q \eta = -s z_q$, we have from (3) and (4) that

$$b_{ip} g^{pq} \nabla_q \eta = -\frac{s}{\varphi + |\nabla' z|^2} b_{ip} z^p = -\nabla_i \tau.$$

Let us now verify (11) for any fixed $\bar{u} \in \mathbb{S}^n$. Noting that the both sides of (11) are tensorial, we may assume that the local coordinates are chosen such that $\frac{\partial g_{jk}}{\partial \theta^i} = 0$

at $\bar{u}$. Then from (2) we have

$$\frac{\partial e_{lj}}{\partial \theta^i} = -\frac{2sc}{\varphi^2}\left(g_{lj} - z_l z_j\right) z_i - \frac{1}{\varphi}\left(z_{il} z_j + z_l z_{ij}\right).$$

Thus the corresponding Christoffel symbols of $\mathbb{S}^n$ are given by

$$\begin{aligned}
\Gamma'^k_{ij} =& \frac{1}{2} e^{kl}\left\{\frac{\partial e_{lj}}{\partial \theta^i} + \frac{\partial e_{li}}{\partial \theta^j} - \frac{\partial e_{ij}}{\partial \theta^l}\right\} \\
=& -\frac{sc}{\varphi^2}\left\{e^{kl} g_{lj} z_i + e^{kl} g_{li} z_j - g_{ij} z^k - z_i z_j z^k\right\} - \frac{1}{\varphi} z_{ij} z^k. \qquad (12)
\end{aligned}$$

This, together with (2), gives

$$\begin{aligned}
\nabla'_{ij} z =& z_{ij} - \Gamma'^k_{ij} z_k \\
=& \frac{\varphi + |\nabla' z|^2}{\varphi} z_{ij} + \frac{sc}{\varphi^2}\left\{g_{lj} z_i z^l + g_{li} z_j z^l - g_{ij} |\nabla' z|^2 - z_i z_j |\nabla' z|^2\right\} \\
=& \frac{\varphi + |\nabla' z|^2}{\varphi} z_{ij} + \frac{sc}{\varphi}\left(2 z_i z_j - |\nabla' z|^2 e_{ij}\right).
\end{aligned}$$

Noting that $\nabla_{ij} z = z_{ij}$ at $\bar{u}$, we thus have

$$\nabla_{ij} z = \frac{1}{\varphi + |\nabla' z|^2}\left(\varphi \nabla'_{ij} z - 2sc z_i z_j + sc |\nabla' z|^2 e_{ij}\right). \qquad (13)$$

Therefore

$$\nabla_{ij} \eta = -c z_i z_j - s \nabla_{ij} z = -c z_i z_j - \frac{s}{\varphi + |\nabla' z|^2}\left(\varphi \nabla'_{ij} z - 2sc z_i z_j + sc |\nabla' z|^2 e_{ij}\right).$$

But from (2) and (4) we can see that the right hand side of the above equation is exactly $\tau b_{ij} + \eta g_{ij}$. □

3. **C^2-estimates.** Now we are in a position to derive the C^2 estimates for any smooth positive k-admissible solutions of (1) in $\mathcal{R}^{n+1}(-1)$. Let us set

$$f_i = \frac{\partial f}{\partial \lambda_i}, \quad F^{ij} = \frac{\partial F}{\partial b_{ij}} \quad \text{and} \quad F^{ij,kl} = \frac{\partial^2 F}{\partial b_{ij} \partial b_{kl}}.$$

We will achieve our aim by choosing suitable test functions and making full use of the terms involving $F^{ij,kl}$. In particular, we need the following

Lemma 2. ([1]) *For any symmetric matrix (η_{ij}) there holds*

$$F^{ij,kl} \eta_{ij} \eta_{kl} = \sum_{i,j} \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} \eta_{ii} \eta_{jj} + \sum_{i \neq j} \frac{f_i - f_j}{\lambda_i - \lambda_j} \eta_{ij}^2,$$

where the second term on the right-hand side must be interpreted as a limit whenever $\lambda_i = \lambda_j$.

This result was, to our knowledge, first stated in [1]; for proofs one may consult [12, 2].

Theorem 2. *Let $1 \le k \le n$ and let ψ be a positive C^2 function in the annulus $\overline{\Omega} : u \in \mathbb{S}^n, \rho \in [R_1, R_2], 0 < R_1 < R_2 < \infty$. Let $z \in C^4(\mathbb{S}^n)$ be a positive k-admissible solution of (1) in $\mathcal{R}^{n+1}(-1)$ satisfying*

$$R_1 \le z \le R_2 \quad \textit{and} \quad |\nabla' z| \le C_0 = \textit{constant} \quad \textit{on } \mathbb{S}^n.$$

Then

$$\|z\|_{C^2(\mathbb{S}^n)} \le C,$$

where the constant C depends only on k, n, R_1, R_2, C_0 and $\|\psi\|_{C^2(\overline{\Omega})}$.

Proof. We will estimate the maximal principal curvature of $\mathcal{M}$. Since z is k-admissible, this estimate, together with the C^0 and C^1 bounds of z and the equation (4), implies an estimate for $\|z\|_{C^2(\mathbb{S}^n)}$. Consider the function

$$\widetilde{W}(u,\xi) = \mathbf{B}(\xi,\xi)\exp\left[\Phi(\tau) - \beta\eta\right],$$

where $u \in \mathbb{S}^n$, ξ is a unit tangent vector of $\mathcal{M}$ at $(u, z(u))$, τ and η are defined as in (8), and the function Φ and the constant $\beta > 0$ will be determined later. Suppose the maximum of $\widetilde{W}$ is attained at some point $\bar{u} \in \mathbb{S}^n$ in the unit tangential direction $\bar{\xi}$ of $\mathcal{M}$ at $(\bar{u}, z(\bar{u}))$. We may choose the local coordinates $\theta^1, \cdots, \theta^n$ around $\bar{u}$ such that

$$g_{ij} = \delta_{ij} \quad \text{and} \quad \frac{\partial g_{ij}}{\partial \theta^k} = 0 \text{ at } \bar{u}.$$

Moreover, since $\bar{\xi}$ is the maximal principal direction of $\mathcal{M}$ at $(\bar{u}, z(\bar{u}))$, such coordinates can be chosen so that $\{b_{ij}\}$ is diagonal at $\bar{u}$ and $b_{11}(\bar{u}) = \mathbf{B}(\bar{\xi},\bar{\xi})$.

Consider the local function $Z = b_{11}/g_{11}$. By direct calculation we have at $\bar{u}$ that

$$\nabla_i Z = \frac{\partial b_{11}}{\partial \theta^i} = \nabla_i b_{11}$$

and

$$\nabla_i\nabla_j Z = \frac{\partial^2 b_{11}}{\partial\theta^i\partial\theta^j} - b_{11}\frac{\partial^2 g_{11}}{\partial\theta^i\partial\theta^j} = \frac{\partial^2 b_{11}}{\partial\theta^i\partial\theta^j} - 2\frac{\partial\Gamma^1_{j1}}{\partial\theta^i} b_{11} = \nabla_i\nabla_j b_{11}.$$

It is clear that the function

$$W(u) = Z(u)\exp\left[\Phi(\tau) - \beta\eta\right].$$

has a local maximum at $\bar{u}$. Thus at $\bar{u}$

$$0 = \nabla_i(\log W) = \frac{\nabla_i Z}{Z} + \Phi'\nabla_i\tau - \beta\nabla_i\eta = \frac{\nabla_i b_{11}}{b_{11}} + \Phi'\nabla_i\tau - \beta\nabla_i\eta \tag{14}$$

and the matrix

$$\begin{aligned}(\nabla_{ij}(\log W)) &= \left(\frac{\nabla_i\nabla_j Z}{Z} - \frac{\nabla_i Z\nabla_j Z}{Z^2} + \Phi'\nabla_{ij}\tau + \Phi''\nabla_i\tau\nabla_j\tau - \beta\nabla_{ij}\eta\right)\\ &= \left(\frac{\nabla_i\nabla_j b_{11}}{b_{11}} - \frac{\nabla_i b_{11}\nabla_j b_{11}}{b_{11}^2} + \Phi'\nabla_{ij}\tau + \Phi''\nabla_i\tau\nabla_j\tau - \beta\nabla_{ij}\eta\right)\end{aligned}$$

is negative semi-definite. Therefore

$$\begin{aligned}0 \geq F^{ij}\nabla_{ij}(\log W) =& \frac{1}{b_{11}}F^{ij}\nabla_i\nabla_j b_{11} - \frac{1}{b_{11}^2}F^{ij}\nabla_i b_{11}\nabla_j b_{11} + \Phi' F^{ij}\nabla_{ij}\tau\\ &+ \Phi'' F^{ij}\nabla_i\tau\nabla_j\tau - \beta F^{ij}\nabla_{ij}\eta. \end{aligned}\tag{15}$$

Since $\{b_{ij}\}$ is diagonal at $\bar{u}$, $\{F^{ij}\}$ is also diagonal there and $F^{ii} = f_i$. For simplicity, we let $\lambda_i = b_{ii}(\bar{u})$ and assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, moreover we may assume $\lambda_1 \geq 1$. Then we have $f_1 \leq f_2 \leq \cdots \leq f_n$, see lemma 2 in [11] or lemma A.2 in [14]. It follows from (15) that

$$\begin{aligned}0 \geq& \frac{1}{\lambda_1}\sum_i f_i\nabla_i\nabla_i b_{11} - \frac{1}{\lambda_1^2}\sum_i f_i|\nabla_i b_{11}|^2 + \Phi'\sum_i f_i\nabla_{ii}\tau\\ &+ \Phi''\sum_i f_i|\nabla_i\tau|^2 - \beta\sum_i f_i\nabla_{ii}\eta. \end{aligned}\tag{16}$$

Now we take the covariant differentiation on (1) to get

$$F^{ij}\nabla_1\nabla_1 b_{ij} + F^{ij,kl}\nabla_1 b_{ij}\nabla_1 b_{kl} = \nabla_{11}\overline{\psi}. \tag{17}$$

From (5), (6) and (7) it follows that

$$\begin{aligned}\nabla_1\nabla_1 b_{ii} =& \nabla_1\nabla_i b_{1i} = \nabla_i\nabla_1 b_{1i} + \sum_k b_{1k}R_{kii1} + \sum_k b_{ik}R_{k1i1}\\ =& \nabla_i\nabla_i b_{11} + b_{11}b_{ii}^2 - b_{11}^2 b_{ii} - (b_{11}\delta_{1i} - b_{11}\delta_{ii} + b_{ii} - b_{ii}\delta_{1i}).\end{aligned}$$

This shows that

$$F^{ij}\nabla_1\nabla_1 b_{ij} = \sum_i f_i\nabla_i\nabla_i b_{11} + \lambda_1\sum_i f_i\lambda_i^2 - \lambda_1^2\sum_i f_i\lambda_i + \lambda_1\mathcal{T} - \sum_i f_i\lambda_i,$$

where $\mathcal{T} := \sum_i f_i$. Since the degree one homogeneity of f implies $\sum_i f_i\lambda_i = \overline{\psi}$, the above equation together with (17) gives

$$\sum_i f_i\nabla_i\nabla_i b_{11} = -F^{ij,kl}\nabla_1 b_{ij}\nabla_1 b_{kl} + \nabla_{11}\overline{\psi} + \lambda_1^2\overline{\psi} + \overline{\psi} - \lambda_1\sum_i f_i\lambda_i^2 - \lambda_1\mathcal{T}.$$

Plugging this into (16), noting that $\overline{\psi} \geq c_0 > 0$ and $\lambda_1 \geq 1$, we therefore obtain

$$\begin{aligned}0 \geq& c_0\lambda_1 - \frac{1}{\lambda_1}F^{ij,kl}\nabla_1 b_{ij}\nabla_1 b_{kl} + \frac{\nabla_{11}\overline{\psi}}{\lambda_1} - \sum_i f_i\lambda_i^2 - \mathcal{T} - \frac{1}{\lambda_1^2}\sum_i f_i|\nabla_i b_{11}|^2\\ &+ \Phi'\sum_i f_i\nabla_{ii}\tau + \Phi''\sum_i f_i|\nabla_i\tau|^2 - \beta\sum_i f_i\nabla_{ii}\eta.\end{aligned} \tag{18}$$

From (10) and (11) we have

$$\beta\sum_i f_i\nabla_{ii}\eta = \beta\tau\sum_i f_i\lambda_i + \beta\eta\mathcal{T} = \beta\tau\overline{\psi} + \beta\eta\mathcal{T}. \tag{19}$$

and

$$\begin{aligned}\Phi'\sum_i f_i\nabla_{ii}\tau =& \Phi'\left\{-\sum_p\nabla_p\eta\left(\sum_i f_i\nabla_p b_{ii}\right) - \tau\sum_i f_i\lambda_i^2 - \eta\sum_i f_i\lambda_i\right\}\\ =& \Phi'\left\{-\sum_p\nabla_p\eta\nabla_p\overline{\psi} - \tau\sum_i f_i\lambda_i^2 - \eta\overline{\psi}\right\}\\ \geq& -C|\Phi'| - \Phi'\tau\sum_i f_i\lambda_i^2.\end{aligned} \tag{20}$$

Here we have used the facts $|\nabla_p\eta| \leq C$ and $|\nabla_p\overline{\psi}| \leq C$ at $\bar{u}$ which can be demonstrated as follows. Since $g_{ij} = \delta_{ij}$ at $\bar{u}$, it follows from (2) that $(z_p)^2 \leq 1$ at $\bar{u}$. Note that $\nabla_p\eta = -\sinh(z)z_p$. Therefore $|\nabla_i\eta| \leq C$ at $\bar{u}$. For $|\nabla_p\overline{\psi}|$, we note that $\nabla_p\overline{\psi} = \overline{\psi}_p + \overline{\psi}_z z_p$. Thus, by using (3), we have at $\bar{u}$ that

$$\begin{aligned}|\nabla_p\overline{\psi}|^2 &\leq C\left(1 + |\overline{\psi}_p|^2\right) \leq C\left(1 + g^{ij}\overline{\psi}_i\overline{\psi}_j\right) \leq C\left(1 + \varphi^{-1}e^{ij}\overline{\psi}_i\overline{\psi}_j\right)\\ &= C\left(1 + \varphi^{-1}|\nabla'\overline{\psi}|^2\right) \leq C.\end{aligned}$$

One can show that

$$\frac{\nabla_{11}\overline{\psi}}{\lambda_1} \geq -C \quad \text{at } \bar{u}. \tag{21}$$

To see this, note that $\frac{\partial g_{ij}}{\partial\theta^k} = 0$ at $\bar{u}$, we have

$$\nabla_{11}\overline{\psi} = \overline{\psi}_{11} + 2\overline{\psi}_{z1}z_1 + \overline{\psi}_{zz}(z_1)^2 + \overline{\psi}_z z_{11}.$$

Similar to the above argument we can show $|\overline{\psi}_{z1}| \leq C$ at $\bar{u}$. Therefore at $\bar{u}$

$$|\nabla_{11}\overline{\psi}| \leq C \left(1 + |\overline{\psi}_{11}| + |z_{11}|\right).$$

Let us estimate $|\overline{\psi}_{11}|$. It follows from (3) that

$$\begin{aligned}
|\nabla'_{11}\overline{\psi}|^2 &\leq g^{ik}g^{jl}\nabla'_{ij}\overline{\psi}\nabla'_{kl}\overline{\psi} \\
&\leq \varphi^{-2}e^{ik}e^{jl}\nabla'_{ij}\overline{\psi}\nabla'_{kl}\overline{\psi} + \varphi^{-2}(\varphi + |\nabla' z|^2)^{-2}\left(z^i z^j \nabla'_{ij}\overline{\psi}\right)^2 \\
&\leq \varphi^{-2}|\nabla'^2\overline{\psi}|^2 + \varphi^{-2}(\varphi^2 + |\nabla' z|^2)^{-2}|\nabla'^2\overline{\psi}|^2|\nabla' z \otimes \nabla' z|^2 \\
&\leq \varphi^{-2}|\nabla'^2\overline{\psi}|^2 + \varphi^{-2}(\varphi^2 + |\nabla' z|^2)^{-2}|\nabla'^2\overline{\psi}|^2|\nabla' z|^4 \\
&\leq C
\end{aligned}$$

By using (2) we obtain at $\bar{u}$ that

$$\sum_i |z^i|^2 = g_{ij}z^i z^j = \varphi|\nabla' z|^2 + |\nabla' z|^4 \leq C.$$

This together with (3) then implies $|e^{ij}| \leq C$. Thus it follows from (12) that $|\Gamma'^k_{11}| \leq C(1 + |z_{11}|)$ at $\bar{u}$. Since $\overline{\psi}_{11} = \nabla'_{11}\overline{\psi} + \Gamma'^k_{11}\overline{\psi}_k$, we therefore have $|\overline{\psi}_{11}| \leq C\,(1 + |z_{11}|)$. Since $z_{11} = \nabla_{11} z$ at $\bar{u}$, from (13) and (4) we finally obtain

$$|\nabla_{11}\overline{\psi}| \leq C\,(1 + |z_{11}|) \leq C\,(1 + \lambda_1)$$

which gives (21).

Combining (18), (19), (20) and (21), we thus obtain

$$\begin{aligned}
0 \geq & c_0\lambda_1 - C(1 + |\Phi'|) - (1 + \beta\eta)\mathcal{T} - \beta\tau\overline{\psi} - (\Phi'\tau + 1)\sum_i f_i\lambda_i^2 \\
& + \Phi''\sum_i f_i|\nabla_i\tau|^2 - \frac{1}{\lambda_1^2}\sum_i f_i|\nabla_i b_{11}|^2 - \frac{1}{\lambda_1}F^{ij,kl}\nabla_1 b_{ij}\nabla_1 b_{kl}.
\end{aligned} \tag{22}$$

Now we will use Lemma 2, similar to the way used in [19].

Case 1. $\lambda_n < -\theta\lambda_1$ for some positive constant θ (to be chosen later).

In this case, using the concavity of F we may discard the last term on the right hand side of (22) since it is nonnegative. Also from (14) we have for any $\varepsilon > 0$

$$\begin{aligned}
\frac{1}{\lambda_1^2}\sum_i f_i|\nabla_i b_{11}|^2 =& \sum_i f_i|\Phi'\nabla_i\tau - \beta\nabla_i\eta|^2 \\
\leq & (1 + \varepsilon^{-1})\beta^2\sum_i f_i|\nabla_i\eta|^2 + (1 + \varepsilon)(\Phi')^2\sum_i f_i|\nabla_i\tau|^2.
\end{aligned}$$

Therefore, from (22) it yields

$$\begin{aligned}
0 \geq & c_0\lambda_1 - C(1 + |\Phi'|) - \left[(1 + \beta\eta) + C(1 + \varepsilon^{-1})\beta^2\right]\mathcal{T} - \beta\tau\overline{\psi} \\
& - (\Phi'\tau + 1)\sum_i f_i\lambda_i^2 + \left[\Phi'' - (1 + \varepsilon)(\Phi')^2\right]\sum_i f_i|\nabla_i\tau|^2.
\end{aligned} \tag{23}$$

Using (9) we have

$$\sum_i f_i|\nabla_i\tau|^2 = \sum_i f_i\lambda_i^2|\nabla_i\eta|^2 \leq C_1\sum_i f_i\lambda_i^2$$

for some constant $C_1 > 0$. If we can choose Φ such that $\Phi'' - (1+\varepsilon)(\Phi')^2 \le 0$, then from (23) we have

$$\begin{aligned} 0 \ge & c_0\lambda_1 - C(1+|\Phi'|) - \left[(1+\beta\eta) + C(1+\varepsilon^{-1})\beta^2\right]\mathcal{T} - \beta\tau\overline{\psi} \\ & + \left[-(\Phi'\tau+1) + C_1\left(\Phi'' - (1+\varepsilon)(\Phi')^2\right)\right]\sum_i f_i\lambda_i^2. \end{aligned} \tag{24}$$

In order to choose Φ, let $a > 0$ be a positive number such that $\tau \ge 2a$ which is guaranteed by our assumption. Then we define

$$\Phi(\tau) = -\log(\tau - a).$$

It is easy to check that $\Phi'' - (1+\varepsilon)(\Phi')^2 < 0$. Moreover, for $\varepsilon = \frac{a^2}{2C_1}$ we have

$$-(\Phi'\tau+1) + C_1\left(\Phi'' - (1+\varepsilon)(\Phi')^2\right) = \frac{a}{\tau-a} - \frac{C_1\varepsilon}{(\tau-a)^2} \ge \frac{a^2}{2(\tau-a)^2} \ge c_2 > 0.$$

Therefore we get from (24) that

$$0 \ge c_0\lambda_1 - C - C\mathcal{T} + c_2\sum_i f_i\lambda_i^2.$$

Since $\lambda_n \le -\theta\lambda_1$ and $f_n \ge \frac{1}{n}\mathcal{T}$, we have $\sum_i f_i\lambda_i^2 \ge f_n\lambda_n^2 \ge \frac{1}{n}\theta^2\mathcal{T}\lambda_1^2$. Hence

$$0 \ge c_0\lambda_1 - C - C\mathcal{T} + \frac{c_2\theta^2}{n}\mathcal{T}\lambda_1^2.$$

This clearly implies λ_1 is bounded from above.

Case 2. $\lambda_n \ge -\theta\lambda_1$.

We now have $\lambda_i \ge -\theta\lambda_1$ for all $i = 1, \cdots, n$. Let us partition $\{1, \cdots, n\}$ into two parts: $I = \{j : f_j \le 4f_1\}$ and $J = \{j : f_j > 4f_1\}$. Using (14) we have for $i \in I$ that

$$\begin{aligned} \frac{1}{\lambda_1^2} f_i|\nabla_i b_{11}|^2 = & f_i|\Phi'\nabla_i\tau - \beta\nabla_i\eta|^2 \\ \le & (1+\varepsilon)(\Phi')^2 f_i|\nabla_i\tau|^2 + (1+\varepsilon^{-1})\beta^2 f_i|\nabla_i\eta|^2 \\ \le & (1+\varepsilon)(\Phi')^2 f_i|\nabla_i\tau|^2 + C(1+\varepsilon^{-1})\beta^2 f_1. \end{aligned}$$

Therefore it follows from (22) that

$$\begin{aligned} 0 \ge & c_0\lambda_1 - C(1+|\Phi'|) - \beta\tau\overline{\psi} - (1+\beta\eta)\mathcal{T} - (\Phi'\tau+1)\sum_i f_i\lambda_i^2 \\ & + \left[\Phi'' - (1+\varepsilon)(\Phi')^2\right]\sum_i f_i|\nabla_i\tau|^2 - C(1+\varepsilon^{-1})\beta^2 f_1 \\ & - \frac{1}{\lambda_1^2}\sum_{j\in J} f_j|\nabla_j b_{11}|^2 - \frac{1}{\lambda_1}F^{ij,kl}\nabla_1 b_{ij}\nabla_1 b_{kl}. \end{aligned}$$

Proceeding exactly as before we have

$$-(\Phi'\tau+1)\sum_i f_i\lambda_i^2 + \left[\Phi'' - (1+\varepsilon)(\Phi')^2\right]\sum_i f_i|\nabla_i\tau|^2 \ge c_2\sum_i f_i\lambda_i^2$$

if we choose $\varepsilon = \frac{a^2}{2C_1}$. So

$$\begin{aligned} 0 \ge & c_0\lambda_1 - C(1+|\Phi'|) - \beta\tau\overline{\psi} - (1+\beta\eta)\mathcal{T} + c_2\sum_i f_i\lambda_i^2 - C(1+\varepsilon^{-1})\beta^2 f_1 \\ & - \frac{1}{\lambda_1^2}\sum_{j\in J} f_j|\nabla_j b_{11}|^2 - \frac{1}{\lambda_1}F^{ij,kl}\nabla_1 b_{ij}\nabla_1 b_{kl}. \end{aligned} \tag{25}$$

By using Lemma 2 and noting $1 \notin J$ we have

$$-\frac{1}{\lambda_1} F^{ij,kl}\nabla_1 b_{ij}\nabla_1 b_{kl} \geq -\frac{2}{\lambda_1}\sum_{j\in J}\frac{f_1-f_j}{\lambda_1-\lambda_j}|\nabla_1 b_{1j}|^2 = -\frac{2}{\lambda_1}\sum_{j\in J}\frac{f_1-f_j}{\lambda_1-\lambda_j}|\nabla_j b_{11}|^2.$$

Therefore

$$\begin{aligned}0 \geq &c_0\lambda_1 - C(1+|\Phi'|) - \beta\tau\overline{\psi} - (1+\beta\eta)\mathcal{T} - C(1+\varepsilon^{-1})\beta^2 f_1 + c_2\sum_i f_i\lambda_i^2\\ &-\frac{2}{\lambda_1}\sum_{j\in J}\frac{f_1-f_j}{\lambda_1-\lambda_j}|\nabla_j b_{11}|^2 - \frac{1}{\lambda_1^2}\sum_{j\in J} f_j|\nabla_j b_{11}|^2.\end{aligned} \tag{26}$$

We claim that

$$-\frac{2}{\lambda_1}\frac{f_1-f_j}{\lambda_1-\lambda_j} \geq \frac{1}{\lambda_1^2} f_j, \qquad \forall j \in J.$$

This is equivalent to showing $2f_1\lambda_1 \leq f_j\lambda_1 + +f_j\lambda_j$. Since $j \in J$, we have $f_j > 4f_1$. If $\lambda_j \geq 0$, this is obviously true. If $\lambda_j < 0$, then $-\theta\lambda_1 \leq \lambda_j < 0$, and hence

$$f_j\lambda_1 + f_j\lambda_j \geq (1-\theta)f_j\lambda_1 \geq 4(1-\theta)f_1\lambda_1 \geq 2f_1\lambda_1$$

if we choose $\theta = \frac{1}{2}$. From this claim and (26) we obtain

$$0 \geq c_0\lambda_1 - C - \beta\tau\overline{\psi} - (1+\beta\eta)\mathcal{T} + c_2\sum_i f_i\lambda_i^2 - C(1+\varepsilon^{-1})\beta^2 f_1.$$

Recall the definition of η, we have $-c_3 \leq \eta \leq -c_4$ for two positive constants c_3 and c_4. Choose β sufficiently large so that $-(1+\beta\eta) \geq 0$. Then we get

$$0 \geq -C + c_0\lambda_1 + c_2 f_1\lambda_1^2 - Cf_1.$$

This clearly implies an upper bound for λ_1. $\square$

4. **Proof of main result.** Since the proof of Theorem 1 essentially follows the lines in [15], only the sketch will be given below.

We may assume that neither $z(u) \equiv R_1$ nor $z(u) \equiv R_2$ is a solution of (1); otherwise we are done. Let us fix some $\overline{R}$ such that $R_1 < \overline{R} < R_2$ and define a family of functions

$$\overline{\psi}^t(u,\rho) := t\overline{\psi}(u,\rho) + (1-t)A^\mu \coth^{1+\mu}(\rho), \qquad t \in [0,1],$$

where $A = 1/\coth(\overline{R})$ and μ is a large number satisfying $\mu \geq \sinh^2(R_2)$. One can check that $\overline{\psi}^t$ satisfy the conditions in Theorem 1. Fix $0 < \alpha < 1$, and denote by $C_a^{4,\alpha}(\mathbb{S}^n)$ the subset of functions from $C^{4,\alpha}(\mathbb{S}^n)$ which is k-admissible. We define a family of operators $\Psi(\cdot,t) : C_a^{4,\alpha}(\mathbb{S}^n) \to C^{2,\alpha}(\mathbb{S}^n)$ by

$$\Psi(z(u),t) \equiv F(\mathbf{B}) - \overline{\psi}^t(u,z(u)), \quad u \in \mathbb{S}^n,$$

where $z \in C_a^{4,\alpha}(\mathbb{S}^n)$ and $\mathbf{B}$ is the second fundamental form of $\mathcal{M} := \{(u,z(u)) : u \in \mathbb{S}^n\}$. Consider the family of equations

$$\Psi(z,t) \equiv 0. \tag{27}$$

Since we assumed that neither $z(u) \equiv R_1$ nor $z(u) \equiv R_2$ is a solution of (1), we can show that neither $z(u) \equiv R_1$ nor $z(u) \equiv R_2$ is a solution of (27) for any $t \in [0,1]$. Therefore, by the strong maximum principle, any solution $z \in C_a^{4,\alpha}(\mathbb{S}^n)$ of (27) satisfying $R_1 \leq z(u) \leq R_2$ for all $u \in \mathbb{S}^n$ must satisfy the strict inequalities

$$R_1 < z(u) < R_2 \quad \text{for all } u \in \mathbb{S}^n. \tag{28}$$

By using the C^1-estimates in [5], Theorem 2, the result of Evans and Krylov, and Schauder theory for second order uniformly elliptic equations one can obtain

$$\|z\|_{C^{4,\alpha}(\mathbb{S}^n)} < C \tag{29}$$

for any solution $z \in C_a^{4,\alpha}(\mathbb{S}^n)$ of (27) satisfying (28), where C is a constant depending only on k, n, R_1, R_2 and $\|\psi\|_{C^{2,\alpha}(\overline{\Omega})}$.

We can choose a constant $\delta > 0$ depending on C such that

$$\delta \le \overline{\psi}^t(u, z(u)) \le \delta^{-1} \quad \text{for } u \in \mathbb{S}^n,$$

where $0 \le t \le 1$ and $z \in C^{4,\alpha}(\mathbb{S}^n)$ satisfying (28) and (29). Consequently we can find an open bounded set V of Γ_k satisfying $\overline{V} \subset \Gamma_k$ such that $\lambda(\mathbf{B}) \in V$ for any $z \in C_a^{4,\alpha}(\mathbb{S}^n)$ satisfying (28), (29) and $\delta \le F(\mathbf{B}) \le \delta^{-1}$. Now we define an open bounded subset O^* of $C^{4,\alpha}(\mathbb{S}^n)$ by

$$O^* := \{z \in C^{4,\alpha}(\mathbb{S}^n) : z \text{ satisfies (28), (29) and } \lambda(\mathbf{B}) \in V\}$$

One can show that

$$\Psi(\cdot, t)^{-1}(0) \cap \partial O^* = \emptyset \quad \text{for } 0 \le t \le 1$$

when $\Psi(\cdot, t)$ are viewed as maps from $\overline{O}^* \subset C^{4,\alpha}(\mathbb{S}^n)$ to $C^{2,\alpha}(\mathbb{S}^n)$. Therefore, the degree $\deg(\Psi(\cdot, t), O^*, 0)$ is defined for all $0 \le t \le 1$ and is independent of t; see [13].

Comparing a solution with spheres $z \equiv$ *constant* and using the maximum principle as usual, we know that $z_0(u) \equiv \overline{R}$ is the unique solution in O^* of the equation $\Psi(z, 0) = 0$. Clearly, the linearized operator $\Psi_z(z_0, 0)$ is of the form

$$\Psi_z(z_0, 0) = -a^{ij}(u)\nabla'_{ij} + b^i(u)\nabla'_i + c(u),$$

where $(a^{ij}(u))$ is positive definite. Since

$$\Psi(sz_0, 0) = \coth(sz_0) - A^\mu \coth^{1+\mu}(sz_0),$$

we have, in view of $A\coth(\overline{R}) = 1$,

$$\begin{aligned} \overline{R}c(u) =& \Psi_z(z_0, 0)(z_0) = \frac{d}{ds}\bigg|_{s=1} \Psi(sz_0, 0) = \frac{d}{d\rho}\bigg|_{\rho=\overline{R}} [\coth(\rho) - A^\mu \coth^{1+\mu}(\rho)] \\ =& -\mu \frac{d}{d\rho}\bigg|_{\rho=\overline{R}} \coth(\rho) > 0. \end{aligned}$$

Thus $\Psi_z(z_0, 0)$ is an invertible operator from $C^{4,\alpha}(\mathbb{S}^n)$ to $C^{2,\alpha}(\mathbb{S}^n)$. It follows, as in [15], that

$$\deg(\Psi(\cdot, 1), O^*, 0) = \deg(\Psi(\cdot, 0), O^*, 0) \ne 0.$$

Therefore, the equation

$$\Psi(z, 1) = 0, \quad z \in O^*$$

has at least one solution. This completes the proof of Theorem 1.

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