New Results in Negative Imaginary Systems Theory

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Declaration

I hereby declare that this submission is my own work and to the best of my knowledge it contains no materials previously published or written by another person, or substantial proportions of material which have been accepted for the award of any other degree or diploma at the ANU or any other educational institution, except where due acknowledgement is made in the thesis. Any contribution made to the research by others, with whom I have worked at the ANU or elsewhere, is explicitly acknowledged in the thesis. I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in the project's design and conception or in style, presentation and linguistic expression is acknowledged.

Abstract

Negative imaginary (NI) systems property naturally arises in flexible structures with colocated force actuators and position sensors. NI systems theory provides a robust control approach to these systems and has attracted attention among control theorists since it was introduced. NI systems theory has found its application in many areas, such as nanopositioning in atomic force microscopy. This thesis provides some new results in NI systems theory.

We provide a definition of nonlinear NI systems to allow for systems with free body motion. Roughly speaking, a system is said to be nonlinear NI if there exists a positive semidefinite storage function, whose time derivative is less than or equal to the inner product of the system input and the time derivative of the system output. Under some assumptions, we show that the interconnection of a nonlinear NI system and a nonlinear output strictly negative imaginary (OSNI) system is asymptotically stable. We also provide a separate stability result for nonlinear NI plants with positive definite storage functions, which corresponds to the case that the systems do not have free body motion.

The problem of state feedback equivalence to NI systems is investigated. The necessary and sufficient conditions under which a linear system can be made NI using state feedback control is that there exists an output transformation such that the resulting system is of relative degree less than or equal to two and is weakly minimum phase. Similar results are also provided for the state feedback equivalence to OSNI systems and strongly strictly negative imaginary systems. For an input-affine nonlinear system, we provide sufficient conditions under which it can be made NI using state feedback. We also provide necessary and sufficient conditions for a nonlinear system in a particular normal form to be state feedback equivalent to an NI system. These results lead to alternative approaches to address the stabilization problems for systems with NI uncertainty.

We provide control frameworks for the output feedback consensus of three types of networked nonlinear NI systems. They are networked identical nonlinear NI systems, networked heterogeneous nonlinear NI systems, and networked heterogeneous

nonlinear NI systems with free body motion. In these three cases, we use OSNI systems as controllers, whose outputs are distributed to the plants according to the network topology.

We extend the NI systems theory to switched systems. A system is said to be switched NI if each subsystem is NI and the storage functions of the subsystems in the switching sequence are nonincreasing. Under some assumptions, the interconnection of a switched NI system and a switched OSNI system is asymptotically stable.

For the digital control of NI systems, we provide a new discrete-time NI systems definition for general nonlinear systems. This definition is automatically satisfied for a ZOH sampled continuous-time NI system. We provide linear matrix inequality conditions and frequency-domain conditions for a linear discrete-time system to be NI. Under certain assumptions, asymptotic stability is achieved for the interconnection of a discrete-time NI system and a discrete-time step advanced OSNI system, and also the interconnection of a discrete-time step advanced NI (SANI) system and a discrete-time OSNI system.

Hybrid integrator-gain systems (HIGS), are investigated as controllers for NI systems due to their own NI property. We show that a single HIGS, a multi-HIGS, and the cascade of two HIGS are all nonlinear NI systems. They can be applied to achieve feedback stability for linear NI plants. This motivates the application of a multi-HIGS on a microelectromechanical nanopositioner, which improves the system performance. Also, a discrete-time HIGS is shown to be an SANI system, which can be applied to stabilize a discrete-time NI system.

This thesis includes several examples to illustrate the results.

List of Publications

Journal Papers:

1. K. Shi, I. R. Petersen, and I. G. Vladimirov, "Output feedback consensus for networked heterogeneous nonlinear negative-imaginary systems with free body motion," IEEE Transactions on Automatic Control, vol. 68, no. 9, pp. 5536–5543, 2023.
2. K. Shi, I. R. Petersen, and I. G. Vladimirov, "Making nonlinear systems negative imaginary via state feedback," Automatica, vol. 155, p. 111127, 2023.
3. K. Shi, N. Nikooienejad, I. R. Petersen, and S. O. R. Moheimani, "Negative imaginary control using hybrid integrator-gain systems: Application to MEMS nanopositioner," IEEE Transactions on Control Systems Technology, vol. 32, no. 3, pp. 718–730, 2024.
4. K. Shi, I. R. Petersen, and I. G. Vladimirov,"Necessary and sufficient conditions for state feedback equivalence to negative imaginary systems," IEEE Transactions on Automatic Control, vol. 69, no. 7, pp. 4657–4672, 2024.

Conference Papers:

1. K. Shi, I. G. Vladimirov, and I. R. Petersen, "Robust output feedback consensus for networked heterogeneous nonlinear negative-imaginary systems," in 2020 Australian and New Zealand Control Conference (ANZCC). IEEE, 2020, pp. 214–219.
2. K. Shi, I. G. Vladimirov, and I. R. Petersen, "Robust output feedback consensus for networked identical nonlinear negative-imaginary systems," in 24th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2020), Cambridge, United Kingdom, IFAC-PapersOnLine, vol. 54, no. 9, pp. 239–244, 2021.
3. K. Shi, I. R. Petersen, and I. G. Vladimirov, "Negative imaginary state feedback equivalence for systems of relative degree one and relative degree two,"

in 2021 60th IEEE Conference on Decision and Control (CDC). IEEE, 2021, pp. 3948–3953.

4. K. Shi, N. Nikooienejad, I. R. Petersen, and S.O. R. Moheimani, "A negative imaginary approach to hybrid integrator-gain system control," in 2022 IEEE 61st Conference on Decision and Control (CDC). IEEE, 2022, pp. 1968–1973.
5. K. Shi, I. R. Petersen, and I. G. Vladimirov, "Making nonlinear systems negative imaginary via state feedback," in 2022 IEEE 61st Conference on Decision and Control (CDC). IEEE, 2022, pp. 3827–3832.
6. K. Shi, I. R. Petersen, and I. G. Vladimirov, "Negative imaginary state feedback equivalence for a class of nonlinear systems," in 12th IFAC Symposium on Nonlinear Control Systems (NOLCOS 2022), Canberra, Australia, IFAC-PapersOnLine, vol. 56, no. 1, pp. 288–293, 2023.
7. K. Shi, I. R. Petersen, and I. G. Vladimirov, "Nonlinear negative imaginary systems with switching," in 22nd IFAC World Congress Yokohama, Japan, IFAC-PapersOnLine, vol. 56, no. 2, pp. 3936–3941, 2023.
8. K. Shi, I. R. Petersen, and I. G. Vladimirov, "Discrete-time negative imaginary systems from ZOH sampling," To appear in the 26th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2024), arXiv preprint:2312.05419, 2024..
9. K. Shi and I. R. Petersen, "Digital control of negative imaginary systems: a discrete-time hybrid integrator-gain system approach," in 2024 European Control Conference (ECC). IEEE, 2024, pp. 2611–2616.

Contents

List of Figures

Notation

$\equiv$	identically equal
$\approx$	approximately equal
$:=$	defined as
$<$ $(>)$	less (greater) than
$\leq$ $(\geq)$	less (greater) than or equal to
$\mathbb{R}$	the set of real numbers
$j\mathbb{R}$	the set of imaginary numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{N}$	the set of nonnegative integers
$\mathbb{R}^{m\times n}$	the space of real matrices of dimension $m \times n$
A^T (x^T)	the transpose of a matrix A (a vector x)
A^{-1}	the inverse of a square matrix A
A^{-T}	the transpose of the inverse of a square matrix A
$\ker(A)$	the kernel of a matrix A
$Re[z]$	the real part of a complex variable z
$Im[z]$	the imaginary part of a complex variable z
$\forall$	for all
$\exists$	there exists
$\in$	belongs to
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$\rightarrow$	tend to
$\Longrightarrow$	implies
$\Longleftrightarrow$	if and only if
$\sum$	summation
C^r	functions with r-continuous derivatives
$spec(A)$	the spectrum of a matrix A
$\lambda_{max}(A)$	the largest eigenvalue of a matrix A with real spectrum

$\|x\|$	the standard Euclidean norm of a vector x
max	maximum
min	minimum
sup	supremum, the least upper bound
inf	infimum, the greatest lower bound
$f : S_1 \to S_2$	a function f mapping a set S_1 into a set S_2
$f_2 \circ f_1$	the composition of two functions
$f^{-1}(\cdot)$	the inverse of a function f
$\frac{\partial f}{\partial x}$	the Jacobian matrix
$\dot{y}$	the first derivative of y with respect to time
$\ddot{y}$	the second derivative of y with respect to time
$y^{(i)}$	the ith derivative of y with respect to time
$L_f h$	the Lie derivative of h with respect to the vector field f
$[f, g]$	the Lie bracket of the vector fields f and g
$ad_f^k g$	$[f, adj_f^{k-1} g]$
$diag\{a_1, \cdots, a_n\}$	a diagonal matrix with diagonal elements a_1 to a_n
$P > (<) 0$	a symmetric positive (negative) definite matrix P
$P \geq (\leq) 0$	a symmetric positive (negative) semidefinite matrix P
$P^{\frac{1}{2}}$	the unique positive definite square root of a positive definite matrix P
$\mathbf{1}$ $(\mathbf{1}_n)$	the vector of ones (of dimension n)
I (I_n)	the identity matrix (of dimension $n \times n$)

List of Acronyms

SISO	single-input single-output
MIMO	multi-input multi-output
TITO	two-input two-output
NI	negative imaginary
SNI	strictly negative imaginary
OSNI	output strictly negative imaginary
SSNI	strongly strictly negative imaginary
SANI	step advanced negative imaginary
SAOSNI	step advanced output strictly negative imaginary
PR	positive real
LMI	linear matrix inequality
AS	asymptotic stability
GAS	globally asymptotic stability
ISS	input-to-state stability
LISS	locally input-to-state stability
LIRD	leading incomplete relative degree
CLHP	closed left-half plane
OLHP	open left-half plane
LHS	left-hand side
RHS	right-hand side
HIGS	hybrid integrator-gain system
FORE	first-order reset element
MEMS	microelectromechanical system
ZOH	zero-order hold

Chapter 1

Introduction

In this chapter, we provide the background to negative imaginary systems theory and also provide an outline for the rest of the chapters.

1.1 Background

Negative imaginary (NI) systems theory was initially introduced by Lanzon and Petersen in [2] and [3] to address robust control problems for flexible structures [4–6]. Examples of flexible structures include bridges [7], buildings [8], large space structures [9], robots [10], atomic force microscopy [11, 12], etc. Generally speaking, flexible structures are systems that experience significant deformations or displacements when subjected to loads or external forces. As a consequence of small internal damping, flexible structures are sensitive to external disturbances. This may lead to severe vibrations especially near their natural frequencies, which can significantly degrade system performance or even cause structural failure [4]. Hence, it is often necessary to design control systems that introduce additional energy dissipation in order to suppress vibrations.

Negative-velocity feedback controllers, which have been intensively studied in the literature [13], provide a useful vibration control approach. Such a controller measures the velocity of a mechanical system and provides a force control that is colocated with the sensor. Stability can be guaranteed using results in the passivity theory (see e.g., [13–15]). To be specific, a system with colocated force actuators and velocity sensors are usually passive in nature if it only consumes energy. For such a system, its energy rate of change cannot exceed the power supplied by the force input, which equals to the inner product of input and output. In the case of single-input single-output (SISO) linear systems, this property corresponds to the fact that, roughly speaking, the frequency response has positive real parts for all frequencies. Hence, linear passive systems are also referred to as positive real (PR) systems.

However, for some systems, we prefer to measure position rather than velocity, which means that passivity theory is no longer a suitable tool for controller synthesis. A lightly damped flexible structure with force actuators and position sensors has a transfer function matrix of the form

$$G_\Delta(s) = \sum_{i=1}^{\infty} \frac{1}{s^2 + \kappa_i s + \omega_i^2} \Psi_i \Psi_i^T, \tag{1.1}$$

where $\kappa_i > 0$, $\omega_i > 0$ for all i, and $\Psi_i \in \mathbb{R}^p$ with p being the input and output dimension [16]. Such an infinite-dimensional system is usually truncated into a finite-dimensional system so that only the first few modes are retained, as shown in the following

$$G(s) = \sum_{i=1}^{h} \frac{1}{s^2 + \kappa_i s + \omega_i^2} \Psi_i \Psi_i^T, \tag{1.2}$$

where $h \geq 1$ is an integer. The spill-over dynamics

$$\Delta(s) = \sum_{i=h+1}^{\infty} \frac{1}{s^2 + \kappa_i s + \omega_i^2} \Psi_i \Psi_i^T \tag{1.3}$$

are unmodelled and hence introduce uncertainty into the system. When a controller is designed according to the truncated plant $G(s)$ to achieve stability, it is difficult yet important to ensure that the controller can stabilize the actual plant $G_\Delta(s)$. That is, the controller needs to be robust against the uncertain spill-over effects from $\Delta(s)$.

One property of the transfer function matrix $G_\Delta(s)$, as is pointed out in [2, 3], is that the Hermitian imaginary part of the frequency response $G_\Delta(j\omega)$, which equals to

$$\Im_H[G_\Delta(s)] = -\frac{1}{2}j[G_\Delta(j\omega) - G_\Delta(j\omega)^*] = -\omega \sum_{i=1}^{\infty} \frac{\kappa_i}{(\omega_i^2 - \omega^2)^2 + \omega^2\kappa_i^2} \Psi_i \Psi_i^T,$$

is negative semidefinite for all frequencies $\omega \geq 0$. If a transfer function matrix has its poles contained in the open-left half of the complex plane and its frequency response has a negative semidefinite Hermitian imaginary part for all nonnegative frequencies, then it is said to be negative imaginary (NI) [2, 3]. For a SISO system, the NI property implies that its Nyquist plot is contained in the lower half of the complex plane for all positive frequencies. The definition of NI systems in [2, 3] was extended in [17] to allow for poles on the imaginary axis except the origin, which corresponds to the undamped case that $\kappa_i = 0$ for some i in (1.1). The definition was extended again in [18] to allow for systems with free body motion. That is, a pole is allowed to be at the origin; i.e., $\omega_i = 0$ for some i in (1.1). In [2,17], necessary and sufficient linear matrix inequality (LMI) conditions were provided under which a minimal system is NI without poles at the origin. This result is usually referred to as the NI lemma. The minimality assumption in the NI lemma is lifted in [19].

A positive-position feedback control approach was provided in [2] for systems with the NI property. To be specific, under some assumptions, the positive feedback interconnection of an NI system and a strictly negative imaginary (SNI) system is internally stable if and only if the DC gain of their open-loop interconnection has all its eigenvalues less than unity. This stability result provides a solution to the robust control problem for flexible structures with colocated force actuators and position sensors. The transfer function matrices $G_\Delta(s)$, $G(s)$ and $\Delta(s)$ given in (1.1), (1.2) and (1.3) respectively are guaranteed by the physical properties of such systems to be all NI. Consider a controller that is designed for the nominal plant $G(s)$ in the presence of the uncertainty $\Delta(s)$ as shown in Fig. 1.1. The system in Fig. 1.1 can be rearranged as the closed-loop interconnection of $\Delta(s)$ and the transfer function from the plant output disturbance w to the plant input u, denoted by $M(s)$, as is shown in Fig. 1.2. Here, $M(s) = (I + C(s)G(s))^{-1}C(s)$. Since the uncertainty is naturally NI, stability of the closed-loop system can be achieved using the results in [2] by choosing a suitable controller $C(s)$ such that $M(s)$ is SNI and the required DC gain condition is satisfied.

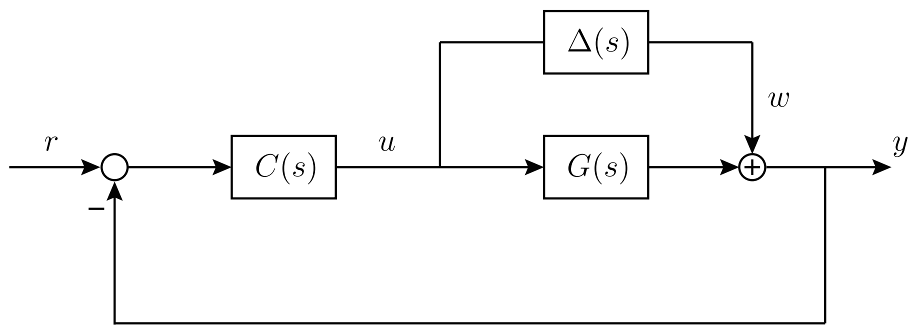


Figure 1.1: A feedback controlled flexible structure where the controller is designed for the truncated plant $G(s)$ in the presence of the spill-over dynamics $\Delta(s)$.

Although NI systems theory was motivated by the robust control problem of second-order mechanical systems, the NI property also arises in other physical systems. For example, [20] provides physical interpretations of NI systems theory in electrical circuits. An advantage of NI systems theory is that it can deal with systems of relative degree zero, one and two. Therefore, NI systems theory can be regarded as a complement to the PR systems theory, which can only deal with systems of

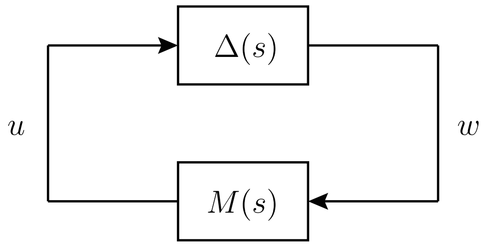


Figure 1.2: Rearranged block diagram of the feedback controlled system.

relative degree zero and one [13].

NI systems have found applications in a wide variety of control systems including robotic manipulators [18], hard disk drives [21], quadrotors [22], nanopositioning in atomic force microscopy [11, 23–25], and micrograds [26], just to list a few. Besides control applications, NI systems theory has also been extended and developed theoretically since it was introduced. In the following, we review some important results, which motivated, or are related to the contents in this thesis.

Output strictly negative imaginary (OSNI) systems were introduced in [27,28]. Compared with SNI systems, OSNI systems have different type of strictness and can be used as alternative controllers for NI plants. That is, the interconnection of an NI system and an OSNI system is asymptotically stable under some assumptions. The intersection of SNI systems and OSNI systems contains another class of systems called strongly strictly negative imaginary (SSNI) systems, which was introduced in [29].

A version of NI systems theory is applied to nonlinear systems in [30] through the notion of counterclockwise input-output dynamics [31]. In [30], a SISO nonlinear system is said to be NI if there is a positive definite storage function such that its time derivative is less than or equal to the product of the system input and the derivative of the system output. In other words, a nonlinear NI system is dissipative respect to the supply rate $u\dot{y}$, where u and y are the system input and output, respectively. It is shown in [30] that the interconnection of a nonlinear NI system and a nonlinear weakly strictly negative imaginary system is asymptotically stable under some assumptions.

An output feedback consensus problem was investigated for networked NI systems with identical system models in [32]. A similar problem was considered in [33] for networked heterogeneous NI systems. Control frameworks were established in both cases to force the outputs of the plants in the network to converge to the same limit trajectory. To be specific, the information exchange between the networked plants is modelled as an undirected graph, where the plants correspond to nodes and the controllers correspond to the edges. Suppose that the graph is connected, SNI controllers are applied to the plants via the Laplacian matrix and incidence matrix in [32] and [33], respectively, to achieve consensus. These consensus results have motivated both theoretical development and control applications since they were established; see e.g., [34–38].

NI systems theory was also extended to discrete-time systems in [39]. Some fundamental results in continuous-time NI systems theory was extended to discrete-time systems via the use of the bilinear transform [39–43]

$$s = \frac{z-1}{z+1}.$$

LMI conditions and frequency domain conditions are provided under which a discrete-time system is NI [39,40]. Also, a discrete-time stability result was provided by applying a bilinear transform to the continuous-time stability results.

Some other interesting results on NI systems theory are listed in the following. NI systems theory is extended to linear time-varying systems in [44] using the dissipativity property of NI systems. Non-proper systems with poles at $j\infty$ are included in NI systems theory in [45]. Necessary and sufficient LMI conditions are given for a descriptor system to be NI in [46]. NI systems theory is investigated for non-square systems with redundant actuators and sensors in [47]. Input-output negative imaginary systems are introduced in [48,49]. Two subclasses of SNI systems called input strictly negative imaginary systems and very strictly negative imaginary systems are also introduced in [49].

1.2 Thesis Objectives and Outline

In this thesis, we establish some new results on NI systems. Considering the nonlinear nature of most real-world control systems, we investigate several control problems directly in a nonlinear systems framework. However, for some nonlinear problems that we are interested in, if the corresponding linear problems remain unsolved in the literature, we also provide solutions in this thesis using linear systems theory.

To be specific, we provide some fundamental nonlinear NI systems results based on the results in [30] in Chapter 3. As [30] only considers SISO systems, we provide a new definition of nonlinear NI systems for MIMO systems. The new definition also allows systems with free body motion, which includes linear NI systems with poles at the origin. Also, we propose nonlinear OSNI systems as controllers, which is different to [30], where nonlinear weakly strictly negative imaginary systems are used as controllers. For a plant with NI uncertainty, we can apply an output feedback controller as in Fig. 1.1 to make the nominal closed-loop system SNI so that stability can be achieved. An alternative approach is to apply a state feedback controller to the plant so that controlled plant becomes NI. However, for a general system, it is unclear whether it can be made NI using state feedback control. We provide in Chapters 4 and 5 the required conditions for linear and nonlinear systems, respectively, to be made NI via state feedback control. Since output feedback consensus problems have only been investigated for networked linear NI systems, we provide in Chapter 6 control frameworks for the output feedback consensus of networked nonlinear NI systems. In Chapter 7, we investigate the nonlinear NI property and stability results for systems with switching. In Chapter 8, we establish a new discrete-time NI systems theory, where ZOH sampling is applied for the conversion between continuous time and discrete time. In comparison with the results in [39], our results provides a more straightforward method for the digital control of NI systems. A specific controller called a hybrid integrator-gain system (HIGS) controller is investigated in Chapter 9. We show that a HIGS will have the NI property and can be applied as a controller for an NI system. This motivates a hardware experiment where a microelectromechanical system (MEMS) nanopositioner is controlled by a variant of the HIGS controller.

A more detailed outline of the rest of the thesis is provided in the following.

1.2.1 Chapter 2. Preliminaries

We provide some preliminary results that will be used throughout the thesis. To be specific, we first provide results on NI systems theory. We also provide the definition of nonlinear NI systems given in [30]. Then we provide some results on linear and nonlinear systems theory. We also provide some preliminaries on discrete-time systems and graph theory.

1.2.2 Chapter 3. Nonlinear Negative Imaginary Systems Theory

We provide a new definition for MIMO nonlinear NI systems with free body motion. We also provide a definition for nonlinear OSNI systems using the dissipativity property of linear OSNI systems. We establish stability results for nonlinear NI systems with and without free body motion. Under some conditions, the interconnection of a nonlinear NI system and a nonlinear OSNI system is asymptotically stable. Different conditions are required and different types of nonlinear OSNI controllers are applied for nonlinear NI systems with and without free body motion. We provide an example for each case.

1.2.3 Chapter 4. State Feedback Equivalence to Negative Imaginary Systems

We investigate necessary and sufficient conditions for a linear system to be state feedback equivalent to an NI system, an OSNI system or an SSNI system. The conditions are given as a relative degree condition and a minimum phase condition. We apply the NI state feedback equivalence results to stabilize a system with SNI uncertainty. An illustrative example is provided.

1.2.4 Chapter 5. State Feedback Equivalence to Nonlinear Negative Imaginary Systems

We consider the problem addressed in Chapter 4 for nonlinear systems. For an input-affine nonlinear system, we provide sufficient conditions under which it can be made nonlinear NI or OSNI using state feedback control. We apply these results to the stabilization of a system with or without a nonlinear NI uncertainty. An illustrative example is provided. Also, we consider the same problem for a special normal form. For this normal form, we provide necessary and sufficient conditions under which it is state feedback equivalent to a nonlinear NI system or a nonlinear OSNI system with a positive definite storage function. This result can also be applied in the stabilization of this special normal form in the presence of a nonlinear OSNI uncertainty, which is illustrated using an example.

1.2.5 Chapter 6. Output Feedback Consensus for Networked Nonlinear Negative Imaginary Systems

We provide control frameworks for the output feedback consensus of networked identical nonlinear NI systems, networked heterogeneous nonlinear NI systems, and networked heterogeneous nonlinear NI systems with free body motion. In each case, the information exchange between the networked systems are represented by a graph where the plants are modelled as nodes and controllers correspond to the edges. OSNI systems as applied as controllers. In each case, we show that the controlled networked system can be regarded as the interconnection of an augmented nonlinear NI system and an augmented nonlinear OSNI system, which is then used to prove output consensus. An illustrative example is provided to show the effect of the proposed control framework for output feedback consensus of networked heterogeneous nonlinear NI systems.

1.2.6 Chapter 7. Switched Nonlinear Negative Imaginary Systems

In this chapter, we provide definitions of switched nonlinear NI systems and also switched nonlinear OSNI systems. We show that the positive feedback interconnection of two nonlinear NI (OSNI) systems is also nonlinear NI (OSNI). Under some assumptions, the interconnection of a switched nonlinear NI system and a switched nonlinear OSNI system is asymptotically stable.

1.2.7 Chapter 8. Discrete-Time Negative Imaginary Systems from Zero-Order Hold Sampling

We provide a new definition of discrete-time NI systems for general nonlinear systems, which is automatically satisfied for ZOH sampled continuous-time NI systems. We provide LMI conditions and frequency-domain conditions for a linear system to satisfy this definition. We show that the interconnection of a discrete-time NI system and a discrete-time step-advanced OSNI system is asymptotically stable. The interconnection of a discrete-time step-advance NI system and a discrete-time OSNI system is also shown to be asymptotically stable. These stability results are illustrated using an example.

1.2.8 Chapter 9. Negative Imaginary Control Using Hybrid Integrator-Gain Systems

We show that a single HIGS, as well as its two variants, the multi-HIGS and the cascade of two HIGS are all nonlinear NI systems. We prove that these three types of HIGS can be applied as feedback controllers to NI systems to achieve stability. These stability results motivates a hardware experiment where a multi-HIGS controller is applied to a MEMS nanopositioner in order to achieve better system performance. Based on the discrete-time NI systems theory established in Chapter 8, we show that a discrete-time HIGS is a step-advance NI system and can be applied in the digital control of a NI plant. This is also illustrated on a mass-spring system example.

1.2.9 Chapter 10. Conclusions and Future Work

We conclude the thesis. Also, we provide some potential directions for future work.

Chapter 2

Preliminaries

In this chapter, we provide some fundamental results on linear and nonlinear NI systems theory. Also, we give some useful results on linear systems, nonlinear systems, discrete-time systems and graph theory.

2.1 Facts on Matrices

Definition 2.1. *[50] A square matrix A is said to be a Hurwitz matrix if each eigenvalue of A has strictly negative real part; that is*

$$Re(\lambda_i) < 0$$

for each eigenvalue λ_i. A matrix A with such a property is also called a stable matrix.

Definition 2.2. *[51] Let λ be an eigenvalue of a matrix A. The dimension of the eigenspace of A associated with λ is the geometric multiplicity of λ. The multiplicity of λ as a zero of the characteristic polynomial of A is the algebraic multiplicity of λ. We say that λ is simple if its algebraic multiplicity is 1; it is semisimple if its algebraic and geometric multiplicities are equal.*

Definition 2.3. *[52] A symmetric matrix $P \in \mathbb{R}^{n\times n}$ is said to be positive definite if $x^T Px > 0$, for all $x \in \mathbb{R}^n \backslash \{0\}$. P is said to be positive semidefinite if $x^T Px \geq 0$ for all $x \in \mathbb{R}^n$.*

Definition 2.4. *[52] A symmetric matrix $P \in \mathbb{R}^{n\times n}$ is said to be negative definite if $x^T Px < 0$, for all $x \in \mathbb{R}^n \backslash \{0\}$. P is said to be negative semidefinite if $x^T Px \leq 0$ for all $x \in \mathbb{R}^n$.*

Lemma 2.1. *[52] The following statements are equivalent for a symmetric matrix $P \in \mathbb{R}^{n\times n}$:*

1. *P is positive definite.*
2. *All eigenvalues of P are strictly positive.*
3. *There exists an $n \times n$ nonsingular real matrix H such that*

$$P = H^T H.$$

We provide some results on the Schur complement of a 2×2 block matrix in the following.

Definition 2.5. *[53] Consider the block matrix*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times q}$, $C \in \mathbb{R}^{q\times p}$, $D \in \mathbb{R}^{q\times q}$, and $p, q > 0$. If the block A is invertible, then the Schur complement of the block A of the matrix M is defined by

$$M/A := D - CA^{-1}B.$$

If the block D is invertible, then the Schur complement of the block D of the matrix M is defined by

$$M/D := A - BD^{-1}C.$$

Lemma 2.2. *[53] Let X be a symmetric matrix of real numbers given by $X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$. We have the following statements:*

1. *X is positive definite if and only if A and its Schur complement X/A are both positive definite; that is*

$$X > 0 \iff A > 0 \text{ and } X/A = C - B^T A^{-1} B > 0.$$

2. *X is positive definite if and only if C and its Schur complement X/C are both positive definite; that is*

$$X > 0 \iff C > 0 \text{ and } X/C = A - BC^{-1}B^T > 0.$$

2.2 Linear Negative Imaginary Systems

2.2.1 NI Systems

Definition 2.6. *[2, 3, 17] A square, real, rational, proper transfer function matrix $G(s)$ is said to be NI if the following conditions are satisfied:*

1. *$G(s)$ has no poles at the origin and in $Re[s] > 0$.*
2. *$j(G(j\omega) - G(j\omega)^*) \geq 0$ for all $\omega \in (0, \infty)$ except the values of ω where $j\omega$ is a pole of $G(s)$.*
3. *If $s = j\omega_0$ with $\omega_0 > 0$ is a pole of $G(s)$, then it is at most a simple pole and the residue matrix $K = \lim_{s \to j\omega_0}(s - j\omega_0)jG(s)$ is Hermitian and positive semidefinite.*

The following lemma provides LMI conditions for a system to be NI as defined in Definition 2.6.

Lemma 2.3. *(NI Lemma) [2, 17] Let (A, B, C, D) be a minimal state-space realisation of a $p \times p$ real-rational proper transfer function matrix $G(s)$ where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times p}$. Then $G(s)$ is NI if and only if:*

1. *$\det(A) \neq 0$, $D = D^T$.*
2. *There exists a matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n \times n}$ such that*

$$AY + YA^T \leq 0, \quad \text{and} \quad B + AYC^T = 0.$$

2.2.2 Generalized NI Systems

Definition 2.7. *[18] A square, real, rational, proper transfer function matrix $G(s)$ is said to be generalized NI if the following conditions are satisfied:*

1. *$G(s)$ has no poles in $Re[s] > 0$.*
2. *$j(G(j\omega) - G(j\omega)^*) \geq 0$ for all $\omega \in (0, \infty)$ except the values of ω where $j\omega$ is a pole of $G(s)$.*
3. *If $s = j\omega_0$ with $\omega_0 > 0$ is a pole of $G(s)$, then it is at most a simple pole and the residue matrix $K = \lim_{s \to j\omega_0}(s - j\omega_0)jG(s)$ is Hermitian and positive semidefinite.*
4. *If $s = 0$ is a pole of $G(s)$, then $\lim_{s \to 0} s^k G(s) = 0$ for all $k \geq 3$ and $\lim_{s \to 0} s^2 G(s)$ is Hermitian and positive semidefinite.*

Both Definitions 2.6 and 2.7 have been used as definitions for NI systems in the literature. Definition 2.7 generalizes Definition 2.6 to allow for systems with poles at the origin. We use both of the definitions in this thesis. To avoid confusion, we call a system satisfying Definition 2.7 a *generalized NI system*.

2.2.3 Strictly Negative Imaginary Systems

Definition 2.8. *[17] A square real-rational proper transfer function matrix $G(s)$ is said to be strictly negative imaginary if the following conditions are satisfied:*

1. *$G(s)$ has no poles in $Re[s] \geq 0$;*
2. *$j[G(j\omega) - G^*(j\omega)] > 0$ for all $\omega \in (0, \infty)$.*

2.2.4 Output Strictly Negative Imaginary Systems

Definition 2.9. *(e.g., see [28]) Let $G(s) \in \mathbb{R}^{m \times m}$ be a real-rational, proper, stable transfer function matrix. Then, $G(s)$ is said to be output strictly negative-imaginary (OSNI) if there exists a scalar $\delta > 0$ such that*

$$j\omega[G(j\omega) - G(j\omega)^*] - \delta\omega^2 \widetilde{G}(j\omega)^* \widetilde{G}(j\omega) \geq 0,$$

$\forall \omega \in \mathbb{R} \cup \{\infty\}$, where $\widetilde{G}(j\omega) = G(j\omega) - G(\infty)$. The quantity $\delta > 0$ is the level of output strictness of the OSNI system.

Theorem 2.1. *(e.g., see [28]) Let $G(s) \in \mathbb{R}^{m \times m}$ be a real-rational, proper, stable transfer function matrix with the minimal realization $\dot{x} = Ax + Bu$ and $y = Cx + Du$, where A is Hurwitz, $D = D^T$. Let $\widetilde{y} = y - Du$. Let $\delta > 0$ be a given scalar. Then, the following statements are equivalent:*

1. *$G(s)$ is OSNI with output strictness δ;*
2. *There exists a real matrix $Y = Y^T > 0$ such that*

$$AY + YA^T + \delta(CAY)^T(CAY) \leq 0, \quad \text{and} \quad B = -AYC^T;$$

3. *The realization (A, B, C, D) is dissipative (see e.g. [13] for dissipativity) with respect to the supply rate $w(u, \dot{\widetilde{y}}) = u^T \dot{\widetilde{y}} - \frac{\delta}{2} \left\| \dot{\widetilde{y}} \right\|^2$.*

2.2.5 Strongly Strictly Negative Imaginary Systems

Definition 2.10. *[29] A square real-rational proper transfer function matrix $G(s)$ is said to be strongly strictly negative imaginary if the following conditions are satisfied:*

1. *$G(s)$ is SNI.*
2. *$\lim_{\omega \to \infty} j\omega[G(j\omega) - G^*(j\omega)] > 0$ and $\lim_{\omega \to 0} j\frac{1}{\omega}[G(j\omega) - G^*(j\omega)] > 0$.*

Lemma 2.4. *(SSNI Lemma) [29] Given a square transfer function matrix $G(s) \in \mathbb{R}^{p\times p}$ with a state-space realisation (A, B, C, D), where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times p}$. Suppose $G(s) + G(-s)^T$ has normal rank p and (A, B, C, D) has no observable uncontrollable modes. Then A is Hurwitz and $G(s)$ is SSNI if and only if $D = D^T$ and there exists a matrix $Y = Y^T > 0$ that satisfies the conditions*

$$AY + YA^T < 0, \quad \text{and} \quad B + AYC^T = 0.$$

2.2.6 Stability Results

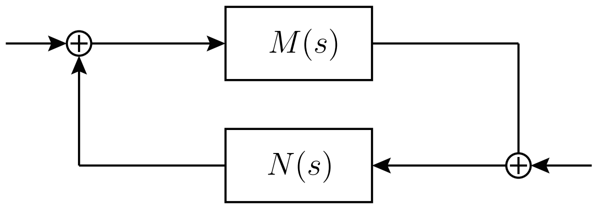


Figure 2.1: Positive feedback interconnection of the transfer functions $M(s)$ and $N(s)$.

Theorem 2.2. *[54] Let $M(s)$ be an NI system and $N(s)$ be an SNI system. Then, $[M(s), N(s)]$ is internally stable if and only if*

$$\begin{aligned}
&I - M(\infty)N(\infty) \text{ is nonsingular,}\\
&\lambda_{\max}[[I - M(\infty)N(\infty)]^{-1}(M(\infty)N(0) - I)] < 0,\\
&\text{and} \quad \lambda_{\max}[[I - N(0)M(\infty)]^{-1}(N(0)M(0) - I)] < 0.
\end{aligned}$$

Corollary 2.1. *[54] Consider an NI transfer function matrix $M(s)$ satisfying $M(\infty) = 0$ and an SNI system $N(s)$. Then the positive feedback interconnection $[M(s), N(s)]$ shown in Fig. 2.1 is internally stable if and only if $\lambda_{max}(M(0)N(0)) < 1$.*

2.3 Nonlinear Negative Imaginary Systems

The following definition of nonlinear NI systems was introduced in [30] for SISO systems. Consider a nonlinear system of the following form

$$\begin{aligned}
\dot{x} &= f(x(t), u(t)),\\
y &= h(x(t)),
\end{aligned} \tag{2.1}$$

where $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a Lipschitz continuous function and $h : \mathbb{R}^n \to \mathbb{R}$ is a class C^1 function.

Definition 2.11. *[30] The system (2.1) is said to be a nonlinear negative-imaginary system if there exists a positive definite storage function $V : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that*

$$\dot{V}(x(t)) \leq u(t)\dot{y}(t)$$

for all $t \geq 0$.

2.4 Linear Systems Theory

2.4.1 Controllability and Observability

Lemma 2.5. *(Eigenvector Test for Controllability) [52] The pair (A, B) is controllable if and only if there is no eigenvector of A^T in the kernel of B^T.*

Lemma 2.6. *(Eigenvector Test for Observability) [52] The pair (A, C) is observable if and only if no eigenvector of A is in the kernel of C.*

2.4.2 Stability

Definition 2.12. *(Lyapunov Stability) [55] A square matrix A is said to be Lyapunov stable if spec$(A) \subset CLHP$ and every purely imaginary eigenvalue of A is semisimple.*

Lemma 2.7. *(Lyapunov Stability Theorem - Asymptotic Stablity) [52] Consider a continuous-time homogeneous linear time-invariant (LTI) system*

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n, \tag{2.2}$$

the following statements are equivalent:

1. *The system (2.2) is asymptotically stable.*
2. *All the eigenvalues of A have strictly negative real parts.*
3. *For every symmetric positive definite matrix Q, there exists a unique solution P to the following Lyapunov equation*

$$A^T P + PA = -Q \tag{2.3}$$

Moreover, P is symmetric and positive definite.

4. *There exists a symmetric positive definite matrix P for which the following Lyapunov matrix inequality holds:*

$$A^T P + PA < 0.$$

Lemma 2.8. *(Lyapunov Stability Theorem - Lyapunov Stable) [55] Let $A \in \mathbb{R}^{n\times n}$ and assume there exists a positive semidefinite matrix $Q \in \mathbb{R}^{n\times n}$ and a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that (2.3) is satisfied, then A is Lyapunov stable.*

2.5 Nonlinear Systems Theory

2.5.1 Lyapunov Stability Theorem

Consider the autonomous system

$$\dot{x} = f(x), \tag{2.4}$$

where $f : \mathcal{D} \to \mathbb{R}^n$ is a locally Lipschitz map from a domain $\mathcal{D} \subseteq \mathbb{R}^n$ into $\mathbb{R}^n$. Without loss of generality, assume $f(0) = 0$.

Definition 2.13. *[15] The equilibrium point $x = 0$ of (2.4) is*

- *Lyapunov stable, if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that*

$$\|x(0)\| < \delta \implies \|x(t)\| < \epsilon, \forall t \geq 0.$$

- *unstable if it is not Lyapunov stable.*
- *asymptotically stable if it is Lyapunov stable and δ can be chosen such that*

$$\|x(0)\| < \delta \implies \lim_{t\to\infty} x(t) = 0.$$

Theorem 2.3. *[15] Let $x = 0$ be an equilibrium point for (2.4) and $\mathcal{D} \subseteq \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that*

$$V(0) = 0, \quad \text{and} \quad V(x) > 0 \text{ for all } x \in \mathcal{D}\backslash\{0\}; \tag{2.5}$$

$$\dot{V}(x) \leq 0 \text{ for all } x \in \mathcal{D}. \tag{2.6}$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \text{ for all } x \in \mathcal{D}\backslash\{0\},$$

then $x = 0$ is asymptotically stable.

A function $V(x)$ satisfying (2.5) and (2.6) is referred to as a *Lyapunov function* for the system (2.4). The property (2.5) is called positive definiteness.

Definition 2.14. *A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be positive definite on a neighbourhood $\mathcal{D}$ of the origin if $V(0) = 0$ and $V(x) > 0$ for all $x \in \mathcal{D}\backslash\{0\}$.*

2.5.2 LaSalle's Invariance Principle

According to Theorem 2.3, if the system (2.4) has a Lyapunov function such that $\dot{V}(x) \leq 0$, then only Lyapunov stability is implied. However, if the Lyapunov derivative $\dot{V}(x)$ cannot stay identically at zero unless $x \equiv 0$, then we can still show asymptotically stability using LaSalle's invariance principle.

Let $x(t)$ be a solution of (2.4). A point p is said to be a *positive limit point* of $x(t)$ if there is a sequence $\{t_n\}$, with $t_n \to \infty$ as $n \to \infty$, such that $x(t_n) \to p$ as $n \to \infty$. The set of all positive limit points of $x(t)$ is called the positive limit set of $x(t)$. A set M is said to be an *invariant set* with respect to (2.4) if

$$x(0) \in M \implies x(t) \in M, \ \forall t \in \mathbb{R}.$$

A set M is said to be a *positively invariant set* if

$$x(0) \in M \implies x(t) \in M, \ \forall t \geq 0.$$

The following theorem is known as LaSalle's invariance principle.

Theorem 2.4. *Let $\Omega \in \mathcal{D}$ be a compact set that is positively invariant with respect to (2.4). Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V} \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.*

We also provide in the following a corollary that extends Theorem 2.3 using Theorem 2.4.

Corollary 2.2. *[15] Let $x = 0$ be an equilibrium point for (2.4). Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable positive definite on a domain $\mathcal{D}$ containing the origin $x = 0$, such that $\dot{V} \leq 0$ in $\mathcal{D}$. Let $S = \{x \in \mathcal{D} | \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$. Then the origin is asymptotically stable.*

Corollary 2.2 is used throughout the thesis in the proofs of asymptotic stability.

2.6 Discrete-Time Systems

Consider a discrete-time nonlinear dynamical system

$$x(k+1) = f(x(k)), \qquad k \in \mathbb{N}, \tag{2.7}$$

where $x(k) \in \mathcal{D} \subseteq \mathbb{R}^n$ is the system state. Here, $\mathcal{D}$ is an open set, $0 \in \mathcal{D}$, $f : \mathcal{D} \to \mathcal{D}$ and $f(0) = 0$. We assume that f is continuous on $\mathcal{D}$.

The *equilibrium point* of the system (2.7) is a point $x \in \mathcal{D}$ satisfying $f(x) = x$ for all $k \in \mathbb{N}$.

Definition 2.15. *[56] The zero solution $x \equiv 0$ is*

- *Lyapunov stable, if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that*

$$\|x(0)\| < \delta \implies \|x(k)\| < \epsilon, \forall k \in \mathbb{N}.$$

- *unstable if it is not Lyapunov stable.*
- *asymptotically stable if it is Lyapunov stable and δ can be chosen such that*

$$\|x(0)\| < \delta \implies \lim_{k\to\infty} x(k) = 0.$$

Theorem 2.5. *[56] Consider the discrete-time nonlinear dynamical system (2.7) and assume there exists a continuous function $V : \mathcal{D} \to \mathbb{R}$ such that*

$$V(0) = 0, \quad \text{and} \quad V(x) > 0 \text{ for all } x \in \mathcal{D}\backslash\{0\}$$

$$V(f(x)) - V(x) \leq 0 \text{ for all } x \in \mathcal{D}.$$

Then the zero solution $x \equiv 0$ to (2.7) is Lyapunov stable. If, in addition,

$$V(f(x)) - V(x) \leq 0 \text{ for all } x \in \mathcal{D}\backslash\{0\},$$

then the zero solution $x \equiv 0$ to (2.7) is asymptotically stable.

Theorem 2.6. *[56] Consider the nonlinear dynamical system (2.7), assume that $\mathcal{D}_c$ is a compact invariant set with respect to (2.7) such that $0 \in \mathcal{D}_c$, and assume that there exists a continuous function $V : \mathcal{D}_c \to \mathbb{R}$ such that $V(0) = 0$, $V(x) > 0$ for all $x \neq 0$, and $V(f(x)) - V(x) \leq 0$ for all $x \in \mathcal{D}_c$. Furthermore, assume that the set $\mathcal{R} := \{x \in \mathcal{D}_c | V(f(x)) = V(x)\}$ contains no invariant set other than the set $\{0\}$. Then the zero solution $x(k) \equiv 0$ to (2.7) is asymptotically stable.*

2.7 Graph Theory

Information exchange between agents in a networked system can be modelled by undirected or directed graphs [57]. $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ and $\mathcal{E} = \{e_1, e_2, \cdots, e_l\} \subseteq \mathcal{V} \times \mathcal{V}$ describes a graph with N nodes (or vertices) and l edges. For an undirected graph, the pairs of nodes are unordered, where the edge (v_i, v_j) denotes that agents corresponding to nodes v_i and v_j can obtain information from each other. A sequence of unrepeated edges in $\mathcal{E}$ that joins a sequence of nodes in $\mathcal{V}$ defines a path. An undirected graph is connected if there is a path between every pair of distinct nodes. For an undirected graph, the adjacency matrix denoted by $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is defined so that $a_{ii} = 0$, and for all $i \neq j$, $a_{ij} = 1$ if $(v_i, v_j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. $\mathcal{D} = diag\{d_1, d_2, \cdots, d_n\} \in \mathbb{R}^{N \times N}$ denotes the degree matrix where $d_i = \sum_{j=1}^{N} a_{ij}$ denotes the degree of node v_i. The Laplacian matrix of a graph $\mathcal{G}$ is given by $\mathcal{L}_N = \mathcal{D} - \mathcal{A}$.

For a directed graph, the ordered pair of nodes (v_i, v_j) denotes an edge from node v_i to v_j. In this thesis, for an edge $(v_i, v_j) \in \mathcal{E}$ in a directed graph, we also allow agents corresponding to nodes v_i and v_j to obtain information from each other. We consider a directed graph to be weakly connected if the underlying undirected graph obtained by replacing all directed edges of the graph with undirected edges is a connected graph. The incidence matrix $\mathcal{Q} = [q_{ev}] \in \mathbb{R}^{l \times N}$ of a directed graph is defined so that the elements in $\mathcal{Q}$ are given by

$$q_{ev} := \begin{cases} 1 & \text{if } v \text{ is the initial vertex of edge } e, \\ -1 & \text{if } v \text{ is the terminal vertex of edge } e, \\ 0 & \text{if } v \text{ does not belong to edge } e. \end{cases} \tag{2.8}$$

Chapter 3

Nonlinear Negative Imaginary Systems Theory

3.1 Introduction

In this chapter, we provide some fundamental results on nonlinear NI systems. Based on Definition 2.11, which considers SISO systems only, we provide a definition for MIMO nonlinear NI systems. According to the new definition, a system is said to be nonlinear NI if there exists a positive semidefinite storage function such that the time derivative of the storage function is less than or equal to the inner product of the system input and the derivative of the system output. The new definition also allows for systems with free body motion such as single and double integrators. NI systems with free body motion arise in many applications [18] such as rotary cranes [58], flexible link manipulators [59], rotary flexible spacecraft [60], and robotics, which is often of a nonlinear nature. Therefore, we provide a stability result for nonlinear NI systems with free body motion in this chapter, which can be regarded as a nonlinear generalization of the stability results in [18]. Also, we provide a separate stability result for nonlinear NI systems without free body motion where the controller can take a relatively simpler form.

The controllers we apply to achieve stability for nonlinear NI systems are called nonlinear OSNI systems. We define nonlinear OSNI systems, according to the dissipativity property (see Theorem 2.1) of linear OSNI systems described in Definition 2.9. We provide stability analysis for two closed-loop systems: (i) the interconnection of a nonlinear NI system with positive semidefinite storage function and a nonlinear OSNI system with a direct feedthrough; (ii) the interconnection of a nonlinear NI system with a positive definite storage function and a nonlinear OSNI system without a direct feedthrough. Case (i) deals with NI systems with free body motion while Case (ii) deals with NI systems without free body motion. Different assumptions are required for the stability of these two closed-loop systems. Under the given assumptions, the closed-loop interconnection is asymptotically stable if a certain candidate Lyapunov function is positive definite. We provide examples to illustrate both stability results. The results in this chapter have been published in the papers [61, 62].

3.2 Definitions of Nonlinear NI and OSNI systems

In this section, definitions of nonlinear NI systems and nonlinear OSNI systems are provided. Consider a nonlinear system H_1:

$$H_1: \quad \dot{x}_1(t) = f_1(x_1(t), u_1(t)), \tag{3.1a}$$
$$y_1(t) = h_1(x_1(t)), \tag{3.1b}$$

where $x_1(t) \in \mathbb{R}^n$ is the state, $u_1(t) \in \mathbb{R}^p$ is a locally integrable input, and $y_1(t) \in \mathbb{R}^p$ is the output, $f_1 : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a Lipschitz continuous function and $h_1 : \mathbb{R}^n \to \mathbb{R}^p$ is a class C^1 function.

Definition 3.1. *The system (3.1) is said to be a nonlinear negative imaginary (NI) system if there exists a positive semidefinite storage function $V_1 : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that for any input $u_1(t)$ and solution $x_1(t)$ to (3.1a),*

$$\dot{V}_1(x_1(t)) \leq u_1(t)^T \dot{y}_1(t), \tag{3.2}$$

for all $t \geq 0$.

In contrast to Definition 2.11, which considers SISO systems, Definition 3.1 now considers MIMO systems. Also, Definition 3.1 allows for systems with free body motion by weakening the required positive definiteness condition for the storage function in Definition 2.11 to a positive semidefiniteness condition. In the case of linear systems, Definition 3.1 allows for systems with poles at the origin. We illustrate this by the following example.

Example 3.1. *Consider a single integrator system*

$$\dot{x} = u,$$
$$y = x,$$

where $x, u, y \in \mathbb{R}$. This system has the transfer function $G(s) = 1/s$, which is generalized NI according to Definition 2.7. To show that it is nonlinear NI as per Definition 2.11, we need to find a positive definite $V(x)$ such that $\dot{V}(x) \leq u\dot{y} = u^2$. Considering that $\dot{V}(x) = \frac{\partial V(x)}{\partial x}\dot{x} = \frac{\partial V(x)}{\partial x}u$, then $\frac{\partial V(x)}{\partial x} = \lambda u$ should hold with $\lambda < 1$. This is impossible for a positive definite function $V(x)$. However, if $V(x)$ is allowed to be positive semidefinite, then $V(x) = 0$ is a function that satisfies the NI condition (3.2).

Indeed, Definition 3.1 can be regarded as a nonlinear generalization of Definition 2.7. However, for some problems investigated in the thesis, the system of interest does not have free body motion, and we require Definition 3.1 to be satisfied with a positive definite storage function. Therefore, we also use the term *nonlinear NI with a positive definite storage function*, which can be regarded as a nonlinear generalization of the NI property defined in Definition 2.6.

We provide a definition of nonlinear OSNI systems in the following. Throughout the thesis, we frequently use nonlinear OSNI systems as feedback controllers to stabilize nonlinear NI systems. Consider a nonlinear system H_2 with the following state-space model:

$$H_2: \quad \dot{x}_2(t) = f_2(x_2(t), u_2(t)), \tag{3.3a}$$

$$y_2(t) = h_2(x_2(t)) + D_2u_2(t), \tag{3.3b}$$

where $x_2(t) \in \mathbb{R}^m$ is the state, $u_2(t) \in \mathbb{R}^p$ is a locally integrable input, and $y_2(t) \in \mathbb{R}^p$ is the output, $f_2 : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$ is a Lipschitz continuous function, $h_2 : \mathbb{R}^m \to \mathbb{R}^p$ is a class C^1 function and $D_2 \in \mathbb{R}^{p\times p}$ is a symmetric matrix; i.e., $D_2 = D_2^T$.

Definition 3.2. *The system (3.3) is said to be a nonlinear output strictly negative-imaginary (OSNI) system if there exists a positive semidefinite storage function $V_2 : \mathbb{R}^m \to \mathbb{R}$ of class C^1 and a scalar $\epsilon > 0$ such that*

$$\dot{V}_2(x_2(t)) \leq u_2(t)^T\dot{\widetilde{y}}_2(t) - \epsilon \left\|\dot{\widetilde{y}}_2(t)\right\|^2, \tag{3.4}$$

for all $t \geq 0$, where $\widetilde{y}_2(t) = h_2(x_2(t))$. In this case, we also say that system (3.3) is nonlinear OSNI with degree of output strictness ϵ.

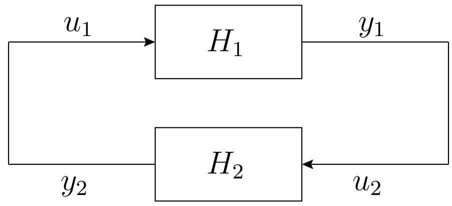


Figure 3.1: Closed-loop interconnection of a nonlinear NI system H_1 and a nonlinear OSNI system H_2.

3.3 Stability for the inteconnection of a nonlinear NI system and a nonlinear OSNI system

In this section, we consider the stability of the closed-loop system shown in Fig. 3.1. Before presenting these stability results, we need to make some assumptions.

For the system H_1 with the state-space model (3.1), we suppose the following assumption is satisfied.

Assumption 3.1. *Consider a constant input* $u_1(t) \equiv \overline{u}_1$ *applied to the system* H_1 *such that* H_1 *is in a steady state; i.e.,* $x_1(t) \equiv \overline{x}_1$ *and* $y_1(t) = h_1(\overline{x}_1) \equiv \overline{y}_1$ *are also constants. For any constant input* $\overline{u}_1$ *and corresponding constant output* $\overline{y}_1$*, we suppose that* $\overline{u}_1^T \overline{y}_1 \geq 0$.

Assumption 3.1 corresponds to a property of all linear NI systems as stated in Lemma 2 in [2]. Although this property holds for all linear NI systems, it does not hold for some special nonlinear systems. Hence, we make this assumption in order to exclude these cases.

For the system H_2 with the state-space model (3.3), we suppose that the following assumption is satisfied.

Assumption 3.2. *Consider a constant input* $u_2(t) \equiv \overline{u}_2$ *applied to the system* H_2 *such that* H_2 *is in a steady state; i.e.,* $x_2(t) \equiv \overline{x}_2$ *and* $y_2(t) = h_2(\overline{x}_2) \equiv \overline{y}_2$ *are also constants. For any constant input* $\overline{u}_2$ *and corresponding constant output* $\overline{y}_2$*, we suppose that* $\overline{u}_2^T \overline{y}_2 \leq -\gamma \|\overline{u}_2\|^2$ *with* $\gamma > 0$.

It might be observed that as nonlinear OSNI systems is also a nonlinear NI system with an additional direct feedthrough, Assumption 3.2 seems to be in conflict with Assumption 3.2 if the latter was applied to H_2. In fact, Assumption 3.2 can be satisfied because of the term $D_2 u_2(t)$ in the output equation (3.3b). Note that when stability is considered for feedback controlled nonlinear NI systems, Assumption 3.2 only poses a restriction on the choice of controllers. Hence, we can suppose Assumption 3.2, and Assumptions 3.3 and 3.4 below hold for the nonlinear OSNI controllers without loss of generality in the nonlinear NI plants to which our results are applicable.

In addition to the above assumptions, both of the systems H_1 and H_2 are assumed to satisfy the following Assumptions 3.3 and 3.4. For the system H_1 with input $u_1(t)$, state $x_1(t)$ and output $y_1(t) = h_1(x_1(t))$ described by the state-space model (3.1) and the system H_2 with input $u_2(t)$, state $x_2(t)$ and the auxiliary output $\widetilde{y}_2(t) = h_2(x_2(t))$ described by the state-space model (3.3), we suppose for $i = 1$ and 2, the following conditions are satisfied.

Assumption 3.3. *Over any time interval $[t_a, t_b]$ where $t_b > t_a$, $h_i(x_i(t))$ remains constant if and only if $x_i(t)$ remains constant; i.e., $\dot{h}_i(x_i(t)) \equiv 0 \iff \dot{x}_i(t) \equiv 0$. Moreover, $h_i(x_i(t)) \equiv 0 \iff x_i(t) \equiv 0$.*

Assumption 3.4. *Over any time interval $[t_a, t_b]$ where $t_b > t_a$, $x_i(t)$ remains constant only if $u_i(t)$ remains constant; i.e., $x_i(t) \equiv \overline{x}_i \implies u_i(t) \equiv \overline{u}_i$. Moreover, $x_i(t) \equiv 0 \implies u_i(t) \equiv 0$.*

Assumptions 3.3 and 3.4 are nonlinear generalizations of standard properties for linear systems. Assumption 3.3 is an observability assumption. Assumption 3.4 requires all inputs to affect the system dynamics, which can be supposed without loss of generality. In the case of linear systems, Assumption 3.4 corresponds to the matrix B in the realization (A, B, C, D) of the linear system having full column rank.

Theorem 3.1. *Consider the positive feedback interconnection of the system H_1 with state-space model (3.1) and the system H_2 with state-space model (3.3), as shown in Fig. 3.1. Suppose that Assumptions 3.1—3.4 are satisfied, and the function, defined as*

$$W(x_1, x_2) := V_1(x_1) + V_2(x_2) - h_1(x_1)^T h_2(x_2) - \frac{1}{2} h_1(x_1)^T D_2 h_1(x_1), \tag{3.5}$$

is positive definite, where $V_1(x_1)$ and $V_2(x_2)$ are positive semidefinite storage functions that satisfy (3.2) for the system H_1 and (3.4) for the system H_2, respectively. Then, the closed-loop interconnection of the systems H_1 and H_2 is asymptotically stable.

Proof. According to the nonlinear NI property (3.2) for the system H_1, the nonlinear OSNI property (3.4) for the system H_2 and the system setting $u_1(t) \equiv y_2(t)$ and $u_2(t) \equiv y_1(t)$ in Fig. 3.1, we have

$$\begin{aligned} \dot{V}_1(x_1) &\le u_1^T \dot{y}_1 = y_2^T \dot{y}_1 = [h_2(x_2) + D_2 u_2]^T \dot{h}_1(x_1) = [h_2(x_2) + D_2 y_1]^T \dot{h}_1(x_1) \\ &= [h_2(x_2) + D_2 h_1(x_1)]^T \dot{h}_1(x_1); \end{aligned} \tag{3.6}$$

$$\dot{V}_2(x_2) \leq u_2^T \dot{\tilde{y}}_2 - \epsilon \|\dot{\tilde{y}}_2\|^2 = y_1^T \dot{\tilde{y}}_2 - \epsilon \|\dot{\tilde{y}}_2\|^2 = \; h_1(x_1)^T \dot{h}_2(x_2) - \epsilon \|\dot{h}_2(x_2)\|^2, \tag{3.7}$$

where the above equalities also use (3.1b) and (3.3b). We now apply the Lyapunov stability theorem (Theorem 2.3) to prove stability. We obtain the time derivative of the function $W(x_1, x_2)$ in (3.5) using (3.6) and (3.7):

$$\begin{aligned}
\dot{W}(x_1, x_2) =& \dot{V}_1(x_1) + \dot{V}_2(x_2) - \dot{h}_1(x_1)^T h_2(x_2) - h_1(x_1)^T \dot{h}_2(x_2) - h_1(x_1)^T D_2 \dot{h}_1(x_1) \\
\leq& \left[h_2(x_2) + D_2 h_1(x_1)\right]^T \dot{h}_1(x_1) + h_1(x_1)^T \dot{h}_2(x_2) - \epsilon \|\dot{h}_2(x_2)\|^2 \\
&- \dot{h}_1(x_1)^T h_2(x_2) - h_1(x_1)^T \dot{h}_2(x_2) - h_1(x_1)^T D_2 \dot{h}_1(x_1) \\
=& - \epsilon \|\dot{h}_2(x_2)\|^2 \\
\leq& \, 0.
\end{aligned} \tag{3.8}$$

From this it follows that $\dot{W}(x_1, x_2) = 0$ is only possible when $\dot{h}_2(x_2) = 0$. Hence, $\dot{W}(x_1, x_2)$ can remain zero only if $\dot{h}_2(x_2)$ remains zero; i.e., $\dot{W}(x_1, x_2) \equiv 0$ implies $\dot{h}_2(x_2(t)) \equiv 0$. According to Assumptions 3.3 and 3.4, $\dot{h}_2(x_2(t)) \equiv 0$ implies $\dot{x}_2(t) \equiv 0$, which implies that $u_2(t) \equiv \overline{u}_2$ remains constant. Hence, the system H_2 is in a steady state. According to the system setting in Fig. 3.1, $y_1(t) \equiv u_2(t)$. Hence, using Assumptions 3.3 and 3.4, $\dot{y}_1(t) \equiv 0$ implies $\dot{x}_1(t) \equiv 0$, which implies $u_1(t) \equiv \overline{u}_1$ remains constant. Thus, the system H_1 is also in a steady state. Then, according to Assumption 3.2, we have

$$\overline{u}_2^T \overline{y}_2 \leq -\gamma \|\overline{u}_2\|^2 \tag{3.9}$$

for some $\gamma > 0$. If $\overline{u}_2 = 0$, then $\overline{u}_2^T \overline{y}_2 = 0$. According to Assumptions 3.3 and 3.4, and the system setting in Fig. 3.1, $\overline{u}_2 = 0$ sequentially implies $\overline{y}_1 = 0$, $\overline{x}_1 = 0$, $\overline{u}_1 = 0$, $\overline{y}_2 = 0$, $\overline{x}_2 = 0$. Hence, in this case, the system is in equilibrium. Otherwise, if $\overline{u}_2 \neq 0$, then from (3.9), we have

$$\overline{u}_2^T \overline{y}_2 < 0. \tag{3.10}$$

Also, according to Assumption 3.1, we have

$$\overline{u}_1^T \overline{y}_1 \geq 0. \tag{3.11}$$

According to the system setting in Fig. 3.1, we have $\overline{u}_1 = \overline{y}_2$ and $\overline{y}_1 = \overline{u}_2$. Hence, (3.11) can be rewritten as $\overline{u}_2^T \overline{y}_2 \geq 0$, which contradicts (3.10). Thus, we can conclude that $\dot{W}(x_1, x_2)$ cannot remain zero unless $(x_1, x_2) = (0, 0)$. Thus, according to LaSalle's invariance principle (see Corollary 2.2), $W(x_1, x_2)$ will keep decreasing until $W(x_1, x_2) = 0$. Hence, the equilibrium at $(x_1, x_2) = (0, 0)$ of the closed-loop interconnection is asymptotically stable. □

Remark 3.1. *In Theorem 3.1, the positive definiteness of the candidate Lyapunov function (3.5) is one of the conditions of the theorem. In the linear case, the DC gain condition implies the positive definiteness of the candidate Lyapunov function of the interconnection of an NI system and an SNI system (see, for example, [2]). However, in the nonlinear case, the positive definiteness of (3.5) does not automatically follow from Assumption 3.2, which is a nonlinear version of the DC gain condition. This is due to the nonlinear nature of the problem and the non-quadratic nature of the storage function (3.5). However, since the purpose of controller synthesis is to find suitable controllers for given plants, this positive definiteness condition and the assumptions on the controllers pose no restriction on the nonlinear plants to which this theorem can be applied. The theorem is applicable to all nonlinear NI plants given in (3.1) satisfying Assumptions 3.1, 3.3 and 3.4, which, as has been explained, are all mild assumptions.*

We consider feedback stability for the system H_1 as given in (3.1) in the case that it is nonlinear NI with a positive definite storage function. This excludes systems with free body motion. However, such a system can be stabilized using a nonlinear OSNI controller without a direct feedthrough. Also, instead of requiring Assumptions 3.1 and 3.2 to be satisfied for H_1 and H_2 respectively, we suppose the open-loop interconnection of H_1 and H_2 satisfies Assumption 3.5 in the following.

Assumption 3.5. *Given any nonzero constant input $\overline{u}_1$ for the system H_1, we obtain a corresponding output $y_1(t)$. Set the input of the system H_2 to be $u_2(t) \equiv y_1(t)$ in the open-loop interconnection shown in Fig. 3.2. If the corresponding output of the system H_2 is constant; i.e., $y_2(t) \equiv \overline{y}_2$, we suppose that*

$$\overline{u}_1 \neq \overline{y}_2. \tag{3.12}$$

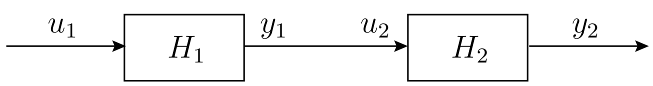


Figure 3.2: Open-loop interconnection of a nonlinear NI system and a nonlinear OSNI system.

Theorem 3.2. *Suppose the system H_1 given in (3.1) is nonlinear NI with a positive definite storage function $V_1(x_1)$, and the system H_2 given in (3.3) has $D_2 = 0$ and is nonlinear OSNI with a positive definite storage function $V_2(x_2)$. Suppose Assumptions 3.3—3.5 are satisfied and the function*

$$\widetilde{W}(x_1, x_2) := V_1(x_1) + V_2(x_2) - h_1(x_1)^T h_2(x_2), \tag{3.13}$$

is positive definite, then the closed-loop interconnection of the systems H_1 and H_2 as shown in Fig. 3.1 is asymptotically stable.

Proof. We apply Lyapunov stability theorem and use $\widetilde{W}(x_1, x_2)$ given in (3.13) as a candidate Lyapunov function. Taking the time derivative of the function $W(x_1, x_2)$ with (3.2) and (3.4) also used, we obtain:

$$\dot{\widetilde{W}}(x_1, x_2) = \dot{V}_1(x_1) + \dot{V}_2(x_2) - u_1^T \dot{y}_1 - u_2^T \dot{y}_2 \leq -\epsilon \|\dot{y}_2\|^2 \leq 0. \tag{3.14}$$

This inequality shows that the system is Lyapunov stable. We will prove in the following that the equilibrium point at $(x_1, x_2) = (0, 0)$ is asymptotically stable.

According to (3.14), $\dot{\widetilde{W}}(x_1, x_2) = 0$ can only hold when $\dot{y}_2 = 0$. In other words, $\dot{\widetilde{W}}(x_1, x_2)$ cannot remain zero unless $\dot{y}_2$ remains zero. According to Assumptions 3.3 and 3.4, $\dot{y}_2(t) \equiv 0$ implies $\dot{x}_2(t) \equiv 0$, which implies that $u_2(t)$ remains constant. According to the system setting $u_2(t) \equiv y_1(t)$, we have that $\dot{y}_1(t) \equiv 0$, which implies $\dot{x}_1(t) \equiv 0$ and in the sequel that $\dot{u}_1(t)$ remains constant. Since $u_1(t)$ and $y_2(t)$ both remain constant; i.e., $u_1(t) \equiv \overline{u}_1$ and $y_2(t) \equiv \overline{y}_2$. If $\overline{u}_1 \neq 0$, then according to Assumption 3.5, $\overline{y}_2 \neq \overline{u}_1$, which contradicts the system setting that $u_1(t) \equiv y_2(t)$. Therefore, we have $\overline{u}_1 = \overline{y}_2 = 0$. Then according to Assumptions 3.3 and 3.4, $y_2(t) \equiv 0$ implies that $x_2(t) \equiv 0$, which implies $u_2(t) \equiv 0$. Also, $y_1(t) \equiv u_2(t) \equiv 0$ implies $x_1(t) \equiv 0$. In this case, the system is already at the equilibrium. This implies that $\dot{W}(x_1, x_2)$ cannot remain zero unless $(x_1, x_2) = (0, 0)$. Hence, $\widetilde{W}(x_1, x_2)$ will keep decreasing until $x_1 = x_2 = 0$. According to LaSalle's invariance principle, it follows that the equilibrium point of the closed-loop interconnection of H_1 and H_2 at $(x_1, x_2) = (0, 0)$ is asymptotically stable. □

3.4 Illustrative Examples

We provide two illustrative examples for the stability results presented in Theorems 3.1 and 3.2.

3.4.1 Example: stabilizing a nonlinear NI system with free body motion

Consider a nonlinear single integrator with the state-space model:

$$\dot{x} = u^3, \tag{3.15a}$$

$$y = x, \tag{3.15b}$$

where $x, u, y \in \mathbb{R}$. This system is nonlinear NI with the storage function

$$V(x) = 0,$$

which is positive semidefinite and satisfies

$$\dot{V}(x) = 0 \leq u^4 = u\dot{y}.$$

We aim to find a nonlinear OSNI controller of the form (3.3) to stabilize the nonlinear NI system (3.15). We consider the following system

$$\dot{x}_c = -x_c + u_c, \tag{3.16a}$$

$$y_c = x_c - 10u_c, \tag{3.16b}$$

where $x_c, u_c, y_c \in \mathbb{R}$. This system is nonlinear OSNI with the storage function

$$V_c(x_c) = \frac{1}{2}x_c^2,$$

which satisfies

$$\dot{V}_c(x_c) \leq u_c\dot{x}_c - \epsilon\dot{x}_c^2,$$

for $\epsilon \in (0, 1]$. Now we show that Assumption 3.1 is satisfied by the system (3.15). Given a constant input $\overline{u}$, if the the output is also a constant $\overline{y}$, then we have that $0 = \dot{y} = \dot{x} = \overline{u}^3$. Hence, $\overline{u} = 0$ and $\overline{u}^T\overline{y} \geq 0$. Now we show that Assumption 3.2 is satisfied by the system (3.16). If both $\overline{u}_c$ and $\overline{y}_c$ are constants, then x_c is also a constant, which we denote by $\overline{x}_c$. Considering $0 = \dot{x}_c = -\overline{x}_c + \overline{u}_c$, we have that $\overline{u}_c = \overline{x}_c$. Therefore, $\overline{y}_c = \overline{x}_c - 10\overline{x}_c = -9\overline{x}_c$. We have that $\overline{u}_c^T\overline{y}_c = -9\overline{x}_c^2 \leq -\gamma\overline{x}_c^2$ for some $\gamma \in (0, 9]$. Assumptions 3.3 and 3.4 are also satisfied. We now construct the candidate Lyapunov function W using the formula (3.5). We have

$$W(x, x_c) = \frac{1}{2}x_c^2 - xx_c + 5x^2,$$

which is positive definite and hence the interconnection of the systems (3.15) and (3.16) is asymptotically stable according to Theorem 3.1. This can be verified using a simulation. Let the initial state of the plant be $x = 2$ and the initial state of the controller be $x_c = 0$. It can be observed from Fig. 3.3 that the state trajectories of the closed-loop system converge to the origin. Because of the cubic nonlinearity in the plant, its state have different rates of convergence in the domains where the state are far and close to zero. Therefore, a log scale is used for the time axis in Fig. 3.3.

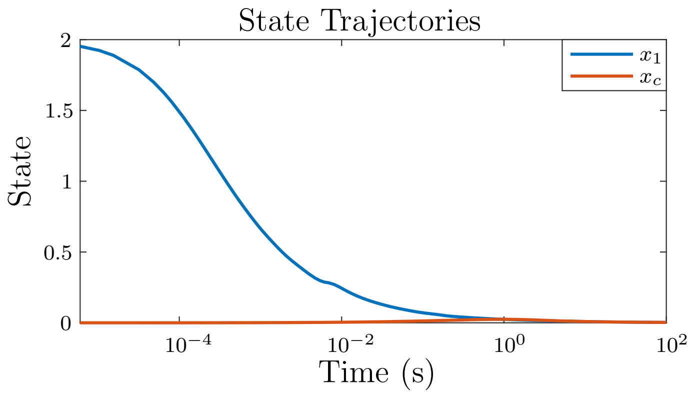


Figure 3.3: The state trajectories of the closed-loop interconnection of the systems (3.15) and (3.16) converge to the origin.

3.4.2 Example: stabilizing a nonlinear NI system without free body motion

Consider a mass-spring system with an additional nonlinear spring, which has the following state-space model:

$$\dot{x}_1 = x_2, \tag{3.17a}$$

$$\dot{x}_2 = -x_1^3 - x_1 + u, \tag{3.17b}$$

$$y = x_1, \tag{3.17c}$$

where $x_1, x_2, u, y \in \mathbb{R}$. This system is nonlinear NI with the positive definite storage function

$$V(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2,$$

which satisfies

$$\dot{V}(x_1, x_2) = x_1^3 x_2 + x_1 x_2 + x_2(-x_1^3 - x_1 + u) = ux_2 = u\dot{y}.$$

We aim to find a nonlinear OSNI controller to achieve closed-loop asymptotic stablility. A suitable such system is

$$\dot{x}_c = -\frac{3}{2}x_c + u_c, \tag{3.18a}$$

$$y_c = x_c, \tag{3.18b}$$

where $x_c, u_c, y_c \in \mathbb{R}$. This system is nonlinear OSNI with the storage function

$$V_c(x_c) = \frac{3}{4}x_c^2,$$

which is positive definite and satisfies

$$\dot{V}_c(x_c) \leq u_c\dot{y}_c - \epsilon\dot{y}_c^2,$$

for $\epsilon \in (0, 1]$. Assumptions 3.3 and 3.4 are satisfied by the systems (3.17) and (3.18). We now show that Assumption 3.5 is satisfied. Suppose that the systems (3.17) and (3.18) are interconnected as shown in Fig. 3.2; i.e., $u_c \equiv y$. In this case, if y_c is a constant, then so is x_c. Then $\dot{x}_c = 0$ and hence u_c is a constant and $u_c \equiv \frac{3}{2}x_c \equiv \frac{3}{2}y_c$. Therefore, according to the open-loop setting, we have $y \equiv u_c \equiv \frac{3}{2}y_c$. This implies that $x_1 \equiv y$ is also a constant, which implies $x_2 \equiv 0$. According to (3.17b), we have that u is also a constant and $u \equiv x_1^3 + x_1 \equiv \frac{27}{8}y_c^3 + \frac{3}{2}y_c$. If $u = y_c$, then $\frac{27}{8}y_c^3 + \frac{1}{2}y_c = 0$, which implies $y_c \equiv 0$ and, in the sequel, $u \equiv 0$. Therefore, Assumption 3.5 is satisfied. We now construct the function W as given in (3.13).

$$W(x_1, x_2, x_c) = \frac{1}{4}x_1^4 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{3}{4}x_c^2 - x_1x_c,$$

which is positive definite and hence is a Lyapunov function which establishes the stability of the closed-loop interconnection of the systems (3.17) and (3.18). This can be verified using a simulation. Let the initial states of the plant be $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and the initial state of the controller be $x_c = 0$. It can be observed from Fig. 3.4 that the state trajectories of the closed-loop system converge to the origin.

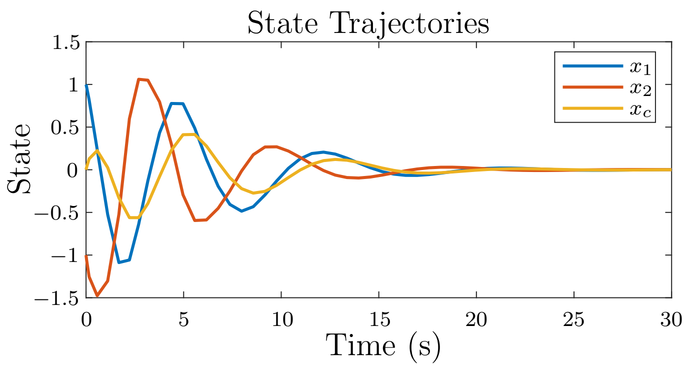


Figure 3.4: The state trajectories of the closed-loop interconnection of the systems (3.17) and (3.18) converge to the origin.

3.5 Conclusion

In this chapter, we provided a new definition of nonlinear NI systems. We also provided the definition of nonlinear OSNI systems, which are applied as feedback controllers to stabilize NI plants. We provided two stability results for the stability of nonlinear NI systems with and without free body motion. The stability results are then illustrated using two examples.

Chapter 4

State Feedback Equivalence to Negative Imaginary Systems

4.1 Introduction

In Chapters 1–3, we have shown that NI systems theory is useful because (i) the NI property arises in many control applications; (ii) the NI property has physical interpretations both in mechanical systems and electrical circuits; (iii) feedback stability results have been established for NI systems. Considering the importance of NI systems theory, we wonder whether it can be applied to a broader class of systems. In other words, we would like to make a system NI so that system analysis and controller synthesis can follow from the framework established for NI systems. In this chapter, we aim to use state feedback control to make a linear system NI.

A similar problem, which considers making a system PR using state feedback control, has been investigated in many papers (e.g., see [63, 64]) in order to achieve stabilization. For example, [64] renders a linear system PR and this result is then generalized to nonlinear systems in [65] using passivity theory. Further nonlinear generalizations of these ideas are presented in the papers [66–69]. In these papers, such PR or passivity state feedback equivalence results are then applied to stabilize systems with specific nonlinearities.

One of the necessary and sufficient conditions for state feedback equivalence to a passive or PR system is that the original system must have relative degree one. This restriction stems from the nature of passivity and PR systems and, as a result, rules out a wide variety of control systems with relative degree two, such as mechanical systems with force actuators and position sensors. Since NI systems can have relative degree less than or equal to two, the results in the present chapter can overcome this limitation.

We investigate necessary and sufficient conditions under which a linear system with the minimal realization $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is state feedback equivalent to an NI system. Suppose the system has no zeros at the origin. We show that such a system can be made NI via the use of state feedback control if and only if (a) it can be output transformed into a system with relative degree less than or equal to two; and (b) the transformed system is weakly minimum phase. The idea of applying an output transformation comes from the fact that a system in general does not always have a relative degree vector and hence does not always have a normal form (see e.g. [14]). However, we show that the property of NI state feedback equivalence is invariant under a nonsingular output transformation because its effect can be compensated for by an additional input transformation. Moreover, we show that a system can be made OSNI if and only if it can be made NI. In particular, we show that a system is state feedback equivalent to a strongly strictly negative imaginary (SSNI) system if and only if it has a relative degree vector $\{1, \cdots, 1\}$ and is minimum phase. The proposed NI state feedback equivalence results are then applied to asymptotically stabilize an uncertain system with SNI uncertainty.

This work is important because simple necessary and sufficient conditions for NI state feedback equivalence are provided for a general linear system. The conditions are stated as a relative degree and minimum phase condition, which is easy to verify. We also provide results for the special cases in which a system has relative degree of either one or two. This work also contributes in providing a method to stabilize systems with SNI uncertainty by making the plant NI. Indeed, the problem of making a system NI using state feedback was also considered in [3] and [70]. However, the NI state feedback control results in [3] and [70] are given in terms of the existence of solutions to LMIs or Riccati equations, which may be difficult to solve. Also, we provide necessary and sufficient conditions while [3] and [70] only provide sufficient conditions. In addition, we provide explicit formulas for the construction of the required state feedback control input. The contents of this chapter are included in the papers [71] and [72].

4.2 NI State Feedback Equivalence

In this section, we provide the main results of this chapter by deriving necessary and sufficient conditions under which a system is state feedback equivalent to an NI system and an OSNI system, respectively. Consider a system with the state-space model:

$$\dot{x} = \mathcal{A}x + \mathcal{B}u, \tag{4.1a}$$
$$y = \mathcal{C}x, \tag{4.1b}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input and $y \in \mathbb{R}^p$ is the output. Here, $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B} \in \mathbb{R}^{n\times p}$ and $\mathcal{C} \in \mathbb{R}^{p\times n}$. We assume that $rank(\mathcal{B}) = rank(\mathcal{C}) = p$.

It will be shown later in this section that one essential condition for NI state feedback equivalence is on a system's relative degree. We provide the following definition of the relative degree vector for the system (4.1).

Definition 4.1. *(see also [14, 73]) A vector $r = \{r_1, \cdots, r_p\} \in \mathbb{N}^p$ is called the relative degree vector of system (4.1) if the following conditions are satisfied:*

1. *For all $i = 1, \cdots, p$,*

$$\begin{aligned} &\mathcal{C}_i\mathcal{A}^j\mathcal{B} = 0 \quad \text{for} \quad j = 0, \cdots, r_i - 2; \\ &\text{and} \quad H(r)_i := \mathcal{C}_i\mathcal{A}^{r_i-1}\mathcal{B} \neq 0. \end{aligned} \tag{4.2}$$

2. $\det(H(r)) \neq 0$.

Here $\mathcal{C}_i$ denotes the i-th row of the matrix $\mathcal{C} \in \mathbb{R}^{p\times n}$ and

$$H(r) = \begin{bmatrix} \mathcal{C}_1\mathcal{A}^{r_1-1}\mathcal{B} \\ \vdots \\ \mathcal{C}_p\mathcal{A}^{r_p-1}\mathcal{B} \end{bmatrix}. \tag{4.3}$$

Condition 1 in Definition 4.1 indicates that the i-th output has its r_i-th time derivative depending explicitly on the inputs.

As is explained in [73], in the case that (4.1) is a MIMO system; i.e., $p \geq 2$, Condition 2 in Definition 4.1 is not always satisfied. The components in the relative degree vector r are invariant under a nonsingular state transformation. However, a nonsingular output transformation can change the components in the vector r and in some cases transform the realization $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ to $(\mathcal{A}, \mathcal{B}, \widetilde{\mathcal{C}})$, where $\widetilde{\mathcal{C}} = T_y\mathcal{C}$, $T_y \in \mathbb{R}^{p\times p}$ and $\det(T_y) \neq 0$, which satisfies Condition 2 in Definition 4.1.

Note that in general there does not always exist an output transformation for a system of the form (4.1) such that the resulting system has a relative degree vector. In what follows, we need the following notion of a leading incomplete relative degree vector [73].

Definition 4.2. *[73] A vector $r = \{r_1, ..., r_p\} \in \mathbb{N}^p$ is called a leading incomplete relative degree (LIRD) vector of the system (4.1) if the following conditions are satisfied:*

1. *For all $i = 1, \cdots, p$,*

$$\begin{aligned} &\mathcal{C}_i \mathcal{A}^j \mathcal{B} = 0 \quad \text{for} \quad j = 0, \cdots, r_i - 2; \\ &\text{and} \quad \mathcal{C}_i \mathcal{A}^{r_i - 1} \mathcal{B} \neq 0. \end{aligned}$$

2. *$r_i \leq r_{i+1}$ for $i = 1, \cdots, p-1$.*
3. *For any set of pairwise distinct indices $i_1, \cdots, i_q \in \{1, 2, \cdots, p\}$ such that $r_{i_1} = r_{i_2} = \cdots = r_{i_q}$, the rows $H(r)_{i_1}, \cdots, H(r)_{i_q}$ are linearly independent, where $H(r)$ is defined in (4.3) and $H(r)_i$ is defined in (4.2).*

As is explained in [73] and [74], if a LIRD vector is such that all rows in $H(r)$ are linearly independent, then this LIRD vector is a relative degree vector as defined in Definition 4.1. This relationship can also be observed by comparing Definitions 4.1 and 4.2.

Lemma 4.1. *[74] For any controllable system with the realization $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, there exists a nonsingular output transformation such that the resulting system has an LIRD vector.*

Proof. This follows directly from Remark 4 and Lemma 4 in [74]. □

We consider systems whose relative degree vector only consists of components that are less than or equal to two. We show later that this is one of the necessary conditions for state feedback equivalence to NI systems.

Definition 4.3. *The system (4.1) is said to have relative degree less than or equal to two if it has a relative degree vector $r = \{r_1, \cdots, r_p\}$, where $1 \leq r_i \leq 2$ for all $i = 1, \cdots, p$.*

4.2.1 State feedback equivalence to NI systems

In this section, we derive conditions for the NI state feedback equivalence of the system (4.1) by investigating an auxiliary system. The auxiliary system is obtained by transforming the system (4.1) into a form which has a relative degree vector via the use of an output transformation. The existence of such an output transformation will be shown later to be a necessary condition for NI state feedback equivalence. First, we show in the following that the property of NI state feedback equivalence is invariant under a nonsingular output transformation. State feedback equivalence to an NI system is defined as follows.

Definition 4.4. *A system in the form of (4.1) is said to be state feedback equivalent to an NI system if there exists a state feedback control law*

$$u = K_x x + K_v v,$$

with $K_x \in \mathbb{R}^{p\times n}$ and $K_v \in \mathbb{R}^{p\times p}$, such that the resulting closed-loop system with the new input $v \in \mathbb{R}^p$ is minimal and NI.

Note that state feedback equivalence problems do not allow for a change of output. However, they allow for a free change of inputs. We show in the following two lemmas that for system (4.1), its NI state feedback equivalence property is invariant under a nonsingular output transformation.

Lemma 4.2. *Suppose $T \in \mathbb{R}^{p\times p}$ is nonsingular. Then the transfer function matrix $TG(s)T^T$ is NI if and only if $G(s)$ is NI.*

Proof. The proof is based on Definition 2.6. $G(s)$ is NI if and only if Conditions 1, 2 and 3 in Definition 2.6 are satisfied. However, the positive definiteness (semi-definiteness) of the matrices in Conditions 1, 2 and 3 in Definition 2.6 is invariant under the transformation $G(s) \mapsto TG(s)T^T$. □

The following lemma shows that state feedback equivalence to an NI system is invariant under state, input and output transformations, which will be used in the process of making a system NI.

Lemma 4.3. *Consider the system (4.1) and the state, input and output transformations $\widetilde{x} = T_x x$, $\widetilde{u} = T_u u$ and $\widetilde{y} = T_y y$, where $T_x \in \mathbb{R}^{n\times n}$, $T_u, T_y \in \mathbb{R}^{p\times p}$ are nonsingular. Then the system (4.1) is state feedback equivalent to an NI system if and only if the transformed system is also state feedback equivalent to an NI system.*

Proof. If the transformed system with state $\widetilde{x}$, input $\widetilde{u}$ and output $\widetilde{y}$ is state feedback equivalent to an NI system, then there exists a control law

$$\widetilde{u} = K_x \widetilde{x} + K_v \widetilde{v},$$

under which the system with input $\widetilde{v}$ and output $\widetilde{y}$ is minimal and NI. According to Lemma 4.2, now the system with output $y = T_y^{-1}\widetilde{y}$ and input $v = T_y^T \widetilde{v}$ is also minimal and NI. This means that the original system with state x, input u and output y is also state feedback equivalent to an NI system. The corresponding feedback control law can be derived as shown in the following:

$$u = T_u^{-1}\widetilde{u} = T_u^{-1}(K_x\widetilde{x} + K_v\widetilde{v}) = T_u^{-1}(K_xT_xx + K_vT_y^{-T}v). \tag{4.4}$$

This completes the sufficiency part of the proof. Since the state, input and output transformation matrices T_x, T_u and T_y are all nonsingular, the necessity part of the proof follows in the same manner as the sufficiency part with the inverses of the transformations considered. □

Consider the case that there exists an output transformation $\widetilde{y} = T_yy$, where $T_y \in \mathbb{R}^{p\times p}$ and $\det T_y \neq 0$, that transforms the system (4.1) into a form with a relative degree vector $r = \{r_1, \cdots, r_p\}$. Let $\widetilde{\mathcal{C}} = T_y\mathcal{C} \in \mathbb{R}^{p\times n}$, then we can write the transformed system in the form:

$$\dot{x} = \mathcal{A}x + \mathcal{B}u, \tag{4.5a}$$
$$\widetilde{y} = \widetilde{\mathcal{C}}x, \tag{4.5b}$$

where $rank(\mathcal{B}) = rank(\widetilde{\mathcal{C}}) = p$. For ease of presentation of our main NI state feedback equivalence results, we aim to transform the system (4.5) into a normal form. We show in the following lemma that this is always achievable via input and state transformations in the case that the system (4.5) has relative degree less than or equal to two.

Lemma 4.4. *Suppose the system (4.5) has relative degree less than or equal to two. Then there exist input and state transformations that transform (4.5) into the following normal form:*

$$\dot{z} = A_{00}z + A_{01}x_1 + A_{02}x_2 + A_{03}x_3, \tag{4.6a}$$
$$\dot{x}_1 = A_{10}z + A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + u_1, \tag{4.6b}$$
$$\dot{x}_2 = x_3, \tag{4.6c}$$
$$\dot{x}_3 = A_{30}z + A_{31}x_1 + A_{32}x_2 + A_{33}x_3 + u_2, \tag{4.6d}$$
$$\widetilde{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{4.6e}$$

where $\widetilde{x} = \begin{bmatrix} z^T & x_1^T & x_2^T & x_3^T \end{bmatrix}^T \in \mathbb{R}^n$ *is the state,* $\widetilde{u} = \begin{bmatrix} u_1^T & u_2^T \end{bmatrix}^T \in \mathbb{R}^p$ *is the input and* $\widetilde{y} \in \mathbb{R}^p$ *is the output of the transformed system. Here,* $x_1, u_1 \in \mathbb{R}^{p_1}$ *and* $x_2, x_3, u_2 \in \mathbb{R}^{p_2}$*, where* $0 \leq p_1 \leq p$ *and* $p_2 := p - p_1$*. Also,* $z \in \mathbb{R}^m$*, where* $m := n - p - p_2$*.*

Proof. Without loss of generality, suppose the components in the relative degree vector r of the system (4.5) are sorted in nondecreasing order, i.e., $r = \{1,\cdots,1,2\cdots,2\}$. Let p_1 $(0 \leq p_1 \leq p)$ be the number of ones in r and $p_2 = p - p_1$ be the number of twos in r. Also, define the following matrices:

$$\widetilde{\mathcal{C}}_O = \begin{bmatrix} \widetilde{\mathcal{C}}_1 \\ \vdots \\ \widetilde{\mathcal{C}}_{p_1} \end{bmatrix} \in \mathbb{R}^{p_1 \times n}, \quad \text{and} \quad \widetilde{\mathcal{C}}_T = \begin{bmatrix} \widetilde{\mathcal{C}}_{p_1+1} \\ \vdots \\ \widetilde{\mathcal{C}}_p \end{bmatrix} \in \mathbb{R}^{p_2 \times n},$$

where $\widetilde{\mathcal{C}}_i$ is the i-th row in the matrix $\widetilde{\mathcal{C}}$. Hence, $\widetilde{\mathcal{C}}_O$ is the block matrix in $\widetilde{\mathcal{C}}$ which determines the output entries corresponding to the ones in r. $\widetilde{\mathcal{C}}_T$ is the block matrix in $\widetilde{\mathcal{C}}$ which determines the output entries corresponding to the twos in r. According to Definition 4.1, we have that $rank(\widetilde{\mathcal{C}}_O\mathcal{B}) = p_1$, $\widetilde{\mathcal{C}}_T\mathcal{B} = 0$ and $rank(\widetilde{\mathcal{C}}_T\mathcal{A}\mathcal{B}) = p_2$. Also, Condition 2 in Definition 4.1 implies that

$$\det \begin{bmatrix} \widetilde{\mathcal{C}}_O\mathcal{B} \\ \widetilde{\mathcal{C}}_T\mathcal{A}\mathcal{B} \end{bmatrix} \neq 0. \tag{4.7}$$

Therefore, the rows of the matrix $\begin{bmatrix} \widetilde{\mathcal{C}}_O \\ \widetilde{\mathcal{C}}_T\mathcal{A} \end{bmatrix}$ are linearly independent. Since $rank(\mathcal{C}) = p$ and $\det T_y \neq 0$, then $rank(\widetilde{\mathcal{C}}) = p$. Hence, $\widetilde{\mathcal{C}}_T$ has full row rank. Also, according to Condition 1 in Definition 4.1, we have that $\widetilde{\mathcal{C}}_T\mathcal{B} = 0$. Then we can prove by contradiction that the rows of $\widetilde{\mathcal{C}}_T$ are linearly independent of the rows of $\begin{bmatrix} \widetilde{\mathcal{C}}_O \\ \widetilde{\mathcal{C}}_T\mathcal{A} \end{bmatrix}$. Indeed, suppose there exists a row $(\widetilde{\mathcal{C}}_T)_\kappa$ of $\widetilde{\mathcal{C}}_T$, which is a linear combination of the rows of $\begin{bmatrix} \widetilde{\mathcal{C}}_O \\ \widetilde{\mathcal{C}}_T\mathcal{A} \end{bmatrix}$. Then $(\widetilde{\mathcal{C}}_T)_\kappa\mathcal{B} \neq 0$ according to (4.7), which contradicts the equation $\widetilde{\mathcal{C}}_T\mathcal{B} = 0$. Therefore, the matrix $\begin{bmatrix} \widetilde{\mathcal{C}}_O \\ \widetilde{\mathcal{C}}_T \\ \widetilde{\mathcal{C}}_T\mathcal{A} \end{bmatrix}$ has full row rank. Define the new state as

$$\begin{aligned} x_1 &= \widetilde{\mathcal{C}}_O x, \\ x_2 &= \widetilde{\mathcal{C}}_T x, \\ x_3 &= \dot{x}_2 = \widetilde{\mathcal{C}}_T\mathcal{A}x. \end{aligned}$$

We also need a complementary state $z \in \mathbb{R}^m$ where $m := n - p - p_2 \geq 0$. Let $z = \widetilde{\mathcal{C}}_z x$, where $\widetilde{\mathcal{C}}_z$ is such that $\widetilde{\mathcal{C}}_z\mathcal{B} = 0$ and the matrix $T_x = \begin{bmatrix} \widetilde{\mathcal{C}}_z \\ \widetilde{\mathcal{C}}_O \\ \widetilde{\mathcal{C}}_T \\ \widetilde{\mathcal{C}}_T\mathcal{A} \end{bmatrix}$ is nonsingular.

Let $\widetilde{x} = T_x x$ be the new state. Also, let $\widetilde{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \widetilde{\mathcal{C}}_O \\ \widetilde{\mathcal{C}}_T \mathcal{A} \end{bmatrix} \mathcal{B} u$. According to (4.7), the input transformation matrix $T_u = \begin{bmatrix} \widetilde{\mathcal{C}}_O \\ \widetilde{\mathcal{C}}_T \mathcal{A} \end{bmatrix} \mathcal{B}$ is nonsingular. The new system has a state-space model

$$\frac{d}{dt}\begin{bmatrix} z \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = T_x \mathcal{A} T_x^{-1} \begin{bmatrix} z \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ u_1 \\ 0 \\ u_2 \end{bmatrix}, \tag{4.8a}$$

$$\widetilde{y} = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} z \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{4.8b}$$

By considering the blocks of $T_x \mathcal{A} T_x^{-1}$ including the relation $\dot{x}_2 = x_3$, we can write (4.8) in the form (4.6). □

We now consider necessary and sufficient conditions under which the system (4.6) is state feedback equivalent to an NI system. For the system (4.6), choose the control inputs u_1 and u_2 to be

$$u_1 = v_1 + (K_{10} - A_{10})z + (K_{11} - A_{11})x_1 + (K_{12} - A_{12})x_2 + (K_{13} - A_{13})x_3; \tag{4.9}$$

$$u_2 = v_2 + (K_{20} - A_{30})z + (K_{21} - A_{31})x_1 + (K_{22} - A_{32})x_2 + (K_{23} - A_{33})x_3, \tag{4.10}$$

which allows the system (4.6) to be represented by the form

$$\dot{\widetilde{x}} = A\widetilde{x} + B\widetilde{v}, \tag{4.11a}$$

$$\widetilde{y} = C\widetilde{x}, \tag{4.11b}$$

where $\widetilde{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is the new input and

$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ K_{10} & K_{11} & K_{12} & K_{13} \\ 0 & 0 & 0 & I \\ K_{20} & K_{21} & K_{22} & K_{23} \end{bmatrix}, \tag{4.12}$$

$$B = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}, \tag{4.13}$$

$$C = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}. \tag{4.14}$$

We need to find the state feedback matrices

$$K_{10} \in \mathbb{R}^{p_1 \times m}, K_{11} \in \mathbb{R}^{p_1 \times p_1}, K_{12} \in \mathbb{R}^{p_1 \times p_2}, K_{13} \in \mathbb{R}^{p_1 \times p_2}, K_{20} \in \mathbb{R}^{p_2 \times m}, K_{21} \in \mathbb{R}^{p_2 \times p_1},$$
$$K_{22} \in \mathbb{R}^{p_2 \times p_2}, \text{ and } K_{23} \in \mathbb{R}^{p_2 \times p_2} \tag{4.15}$$

such that the system (4.11) is minimal and NI. The following lemma provides necessary and sufficient conditions for such state feedback matrices to exist.

Lemma 4.5. *Suppose the system (4.6) satisfies* $\det A_{00} \neq 0$. *Then it is state feedback equivalent to an NI system if and only if it is controllable and* A_{00} *is Lyapunov stable.*

Proof. The system (4.6) is state feedback equivalent to an NI system if and only if there exist state feedback matrices (4.15) such that the system (4.11) is NI and the realization (A, B, C) in (4.12)–(4.14) is minimal.

First, we prove that the controllability of the system (4.6) is equivalent to the controllability of the system (4.11). Let

$$\breve{A} = \begin{bmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{10} & A_{11} & A_{12} & A_{13} \\ 0 & 0 & 0 & I \\ A_{30} & A_{31} & A_{32} & A_{33} \end{bmatrix}.$$

Then we need to prove that the controllability of $(\breve{A}, B)$ is equivalent to that of (A, B). Applying the eigenvector test for controllability (see Lemma 2.5), $(\breve{A}, B)$ is controllable if and only if any non-zero vector in the kernal of B^T is not an eigenvector of $\breve{A}^T$. Considering the structure of B in (4.13), a non-zero vector $\eta \in \ker(B^T)$ must take the form $\eta = \begin{bmatrix} \eta_1^T & 0 & \eta_3^T & 0 \end{bmatrix}^T$, where $\eta_1 \neq 0$ or $\eta_3 \neq 0$.

Therefore, for any scalar λ_c, we have that $\breve{A}^T\eta \neq \lambda_c\eta$. Substituting for $\breve{A}$, we obtain

$$\begin{bmatrix} A_{00}^T\eta_1 \\ A_{01}^T\eta_1 \\ A_{02}^T\eta_1 \\ A_{03}^T\eta_1 + \eta_3 \end{bmatrix} \neq \lambda_c \begin{bmatrix} \eta_1 \\ 0 \\ \eta_3 \\ 0 \end{bmatrix} \tag{4.16}$$

for any scalar λ_c. This condition depends only on the matrices A_{00}, A_{01}, A_{02} and A_{03}, which form the common first block row of the matrices $\breve{A}$ and A. Hence, the controllability of $(\breve{A}, B)$ is equivalent to that of (A, B).

Sufficiency. The condition (4.16) is satisfied if and only if for any eigenvector η_1 of A_{00}^T with eigenvalue λ_c, we have that $A_{01}^T\eta_1 \neq 0$ or $\begin{bmatrix} A_{02}^T\eta_1 \\ A_{03}^T\eta_1 + \eta_3 \end{bmatrix} \neq \lambda_c \begin{bmatrix} \eta_3 \\ 0 \end{bmatrix}$. The condition $A_{01}^T\eta_1 \neq 0$ holds if and only if (A_{00}, A_{01}) is controllable. The condition $\begin{bmatrix} A_{02}^T\eta_1 \\ A_{03}^T\eta_1 + \eta_3 \end{bmatrix} \neq \lambda_c \begin{bmatrix} \eta_3 \\ 0 \end{bmatrix}$ holds if and only if for any $\eta_3 = -A_{03}^T\eta_1$, we have that $A_{02}^T\eta_1 \neq \lambda_c\eta_3 = -\lambda_c A_{03}^T\eta_1 = -A_{03}^T A_{00}^T\eta_1$. That is $(A_{03}^T A_{00}^T + A_{02}^T)\eta_1 \neq 0$, which holds if and only if $(A_{00}, A_{00}A_{03} + A_{02})$ is controllable. Therefore, we conclude that (A, B) is controllable if and only if (A_{00}, A_{01}) or $(A_{00}, A_{00}A_{03} + A_{02})$ is controllable.

We now derive necessary and sufficient conditions under which (A, C) is observable. Given the structure of C in (4.14), any non-zero vector $\sigma \in \ker(C)$ must take the form $\sigma = \begin{bmatrix} \sigma_1^T & 0 & 0 & \sigma_4^T \end{bmatrix}^T$, where $\sigma_1 \neq 0$ or $\sigma_4 \neq 0$. According to the eigenvector test for observability (see Lemma 2.6), (A, C) is observable if and only if $A\sigma \neq \lambda_o\sigma$ for any scalar λ_o. Substituting A from (4.12), we obtain

$$\begin{bmatrix} A_{00}\sigma_1 + A_{03}\sigma_4 \\ K_{10}\sigma_1 + K_{13}\sigma_4 \\ \sigma_4 \\ K_{20}\sigma_1 + K_{23}\sigma_4 \end{bmatrix} \neq \lambda_o \begin{bmatrix} \sigma_1 \\ 0 \\ 0 \\ \sigma_4 \end{bmatrix}. \tag{4.17}$$

When $\sigma_4 \neq 0$, (4.17) is always true. Now we consider the case that $\sigma_1 \neq 0$ and $\sigma_4 = 0$. In this case, (4.17) becomes

$$\begin{bmatrix} A_{00}\sigma_1 \\ K_{10}\sigma_1 \\ 0 \\ K_{20}\sigma_1 \end{bmatrix} \neq \lambda_o \begin{bmatrix} \sigma_1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which holds if and only if $K_{10}\sigma_1 \neq 0$ or $K_{20}\sigma_1 \neq 0$ for any eigenvector σ_1 of A_{00}. Therefore, we conclude that (A, C) is observable if and only if (A_{00}, K_{10})

or (A_{00}, K_{20}) is observable.

The nonsingular matrix A_{00} is Lyapunov stable if and only if there exists a state transformation $A_{00} \mapsto SA_{00}S^{-1}$ which allows A_{00} to be represented, without loss of generality, as $A_{00} = diag(A_{00}^a, A_{00}^b)$, where

$$spec(A_{00}^a) \subset j\mathbb{R}\backslash\{0\}, \quad spec(A_{00}^b) \subset OLHP, \quad \text{and} \quad A_{00}^a + (A_{00}^a)^T = 0. \tag{4.18}$$

Here $A_{00}^a \in \mathbb{R}^{m_a \times m_a}$ and $A_{00}^b \in \mathbb{R}^{m_b \times m_b}$, where $0 \leq m_a \leq m$ and $m_b := m - m_a$. The conditions in (4.18) are achievable according to the proof of Proposition 11.9.6 in [55]. Decomposing A_{01}, A_{02}, A_{03}, K_{10} and K_{20} accordingly using the same state-space transformation, we can write (4.11) as

$$\dot{z}_1 = A_{00}^a z_1 + A_{01}^a x_1 + A_{02}^a x_2 + A_{03}^a x_3, \tag{4.19a}$$
$$\dot{z}_2 = A_{00}^b z_2 + A_{01}^b x_1 + A_{02}^b x_2 + A_{03}^b x_3, \tag{4.19b}$$
$$\dot{x}_1 = K_{10}^a z_1 + K_{10}^b z_2 + K_{11} x_1 + K_{12} x_2 + K_{13} x_3 + v_1, \tag{4.19c}$$
$$\dot{x}_2 = x_3, \tag{4.19d}$$
$$\dot{x}_3 = K_{20}^a z_1 + K_{20}^b z_2 + K_{21} x_1 + K_{22} x_2 + K_{23} x_3 + v_2, \tag{4.19e}$$
$$y = C \begin{bmatrix} z_1 \\ z_2 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix}. \tag{4.19f}$$

Since A_{00}^b is Hurwitz, there exist $\mathcal{Y}_1^b = (\mathcal{Y}_1^b)^T > 0$ and $Q_b = Q_b^T > 0$ such that

$$A_{00}^b \mathcal{Y}_1^b + \mathcal{Y}_1^b (A_{00}^b)^T = -Q_b.$$

Let K_{20} be defined as

$$K_{20} = \begin{bmatrix} K_{20}^a & K_{20}^b \end{bmatrix}, \tag{4.20}$$

where

$$\begin{aligned} K_{20}^a &= (A_{02}^a)^T (A_{00}^a)^{-1} - (A_{03}^a)^T; \\ K_{20}^b &= \left(-(A_{02}^b)^T (A_{00}^b)^{-T} - (A_{03}^b)^T + \mathcal{H}\right) \left(\mathcal{Y}_1^b\right)^{-1}. \end{aligned} \tag{4.21}$$

Here, $\mathcal{H}$ is a matrix that belongs to the set

$$S_{\mathcal{H}} = \{\mathcal{H} \in \mathbb{R}^{p_2 \times m_b} \mid \mathcal{H}^T \mathcal{H} \leq Q_b\}. \tag{4.22}$$

If $(A_{00}, A_{00}A_{03} + A_{02})$ is controllable, we can always find $\mathcal{H}$ such that (A_{00}, K_{20}) is observable. This is proved in the following. According to Lemma 2.5, the controllability of $(A_{00}, A_{00}A_{03} + A_{02})$ implies that no eigenvector of $diag\left((A_{00}^a)^T, (A_{00}^b)^T\right)$ is in the kernel of $\begin{bmatrix}(A_{03}^a)^T(A_{00}^a)^T + (A_{02}^a)^T & (A_{03}^b)^T(A_{00}^b)^T + (A_{02}^b)^T\end{bmatrix}$. This implies that both $(A_{00}^a, A_{00}^aA_{03}^a + A_{02}^a)$ and $(A_{00}^b, A_{00}^bA_{03}^b + A_{02}^b)$ are controllable, which can be proved by applying the eigenvector test for controllability to the vectors $\begin{bmatrix}\eta_a \\ 0\end{bmatrix}$ and $\begin{bmatrix}0 \\ \eta_b\end{bmatrix}$, where η_a and η_b are eigenvectors of $(A_{00}^a)^T$ and $(A_{00}^b)^T$, respectively. The pair (A_{00}, K_{20}) is observable if and only if for any non-zero vector $\delta_K = \begin{bmatrix}\delta_a \\ \delta_b\end{bmatrix}$, which is an eigenvector of A_{00}, we have $K_{20}\delta_K \neq 0$. Since A_{00}^a and A_{00}^b have no common eigenvalues, then δ_K is an eigenvector of A_{00} only if $\delta_a = 0$ or $\delta_b = 0$. We consider two cases:

Case 1. $\delta_a \neq 0$ and $\delta_b = 0$. In this case, δ_a is an eigenvector of A_{00}^a; i.e., $A_{00}^a\delta_a = \lambda_a\delta_a$ for some scalar λ_a. Considering (4.18), δ_a is also an eigenvector of $(A_{00}^a)^T$. Because $(A_{00}^a, A_{00}^aA_{03}^a + A_{02}^a)$ is controllable, $\left((A_{03}^a)^T(A_{00}^a)^T + (A_{02}^a)^T\right)\delta_a \neq 0$. Therefore,

$$\begin{aligned}
K_{20}\delta_K &= K_{20}^a\delta_a \\
&= \left((A_{02}^a)^T(A_{00}^a)^{-1} - (A_{03}^a)^T\right)\delta_a \\
&= \left((A_{02}^a)^T - (A_{03}^a)^TA_{00}^a\right)(A_{00}^a)^{-1}\delta_a \\
&= \frac{1}{\lambda_a}\left((A_{02}^a)^T + (A_{03}^a)^T(A_{00}^a)^T\right)\delta_a \\
&\neq 0,
\end{aligned}$$

where (4.18) is also used.

Case 2. $\delta_a = 0$ and $\delta_b \neq 0$. In this case, δ_b is an eigenvector of A_{00}^b. Because $-(A_{02}^b)^T(A_{00}^b)^{-T} - (A_{03}^b)^T$ in (4.21) is fixed, and $S_{\mathcal{H}}$ has nonempty interior due to the positive definiteness of Q_b, then we can always find $\mathcal{H}$ such that

$$K_{20}\delta_K = K_{20}^b\delta_b = \left(-(A_{02}^b)^T(A_{00}^b)^{-T} - (A_{03}^b)^T + \mathcal{H}\right)(\mathcal{Y}_1^b)^{-1}\delta_b \neq 0,$$

for all δ_b that are eigenvalues of A_{00}^b. We conclude that there exists $\mathcal{H}$ such that (A_{00}, K_{20}) is observable. We choose such a matrix $\mathcal{H}$ in the following proof.

Let K_{10} be defined as

$$K_{10} = \begin{bmatrix}K_{10}^a & K_{10}^b\end{bmatrix}, \tag{4.23}$$

where

$$K_{10}^a = (A_{01}^a)^T (A_{00}^a)^{-1};$$
$$K_{10}^b = -\left((A_{01}^b)^T (A_{00}^b)^{-T} + K_{13}\mathcal{H}\right)(\mathcal{Y}_1^b)^{-1}.$$

Here K_{13} is contained in the set

$$S_K = \{K_{13} \in \mathbb{R}^{p_1 \times p_2} \mid K_{13}K_{13}^T \leq 2I\}. \tag{4.24}$$

If (A_{00}, A_{01}) is controllable, we can always find K_{13} such that (A_{00}, K_{10}) is observable. This is proved in the following. The controllability of (A_{00}, A_{01}) implies that no eigenvector of $diag\left((A_{00}^a)^T, (A_{00}^b)^T\right)$ is in $ker\left(\begin{bmatrix}(A_{01}^a)^T & (A_{01}^b)^T\end{bmatrix}\right)$. This implies that both (A_{00}^a, A_{01}^a) and (A_{00}^b, A_{01}^b) are controllable, which can be proved by applying the eigenvector test for controllability to the vectors $\begin{bmatrix}\eta_a \\ 0\end{bmatrix}$ and $\begin{bmatrix}0 \\ \eta_b\end{bmatrix}$, where η_a and η_b are eigenvectors of $(A_{00}^a)^T$ and $(A_{00}^b)^T$, respectively. The pair (A_{00}, K_{10}) is observable if and only if for any non-zero vector $\phi_K = \begin{bmatrix}\phi_a \\ \phi_b\end{bmatrix}$, which is an eigenvector of A_{00}, we have that $K_{10}\phi_K \neq 0$. Since A_{00}^a and A_{00}^b have no common eigenvalues, then ϕ_K is an eigenvector of A_{00} only if $\phi_a = 0$ or $\phi_b = 0$. We consider two cases:

Case 1. $\phi_a \neq 0$ and $\phi_b = 0$. In this case, ϕ_a is an eigenvector of A_{00}^a; i.e., $A_{00}^a\phi_a = \mu_a\phi_a$ for some scalar μ_a. Considering (4.18), ϕ_a is also an eigenvector of $(A_{00}^a)^T$. Because (A_{00}^a, A_{01}^a) is controllable, $(A_{01}^a)^T\phi_a \neq 0$. Therefore,

$$K_{10}\phi_K = K_{10}^a\phi_a = (A_{01}^a)^T(A_{00}^a)^{-1}\phi_a = \frac{1}{\mu_a}(A_{01}^a)^T\phi_a \neq 0,$$

where (4.18) is also used.

Case 2. $\phi_a = 0$ and $\phi_b \neq 0$. In this case, ϕ_b is an eigenvector of A_{00}^b. Because $(A_{01}^b)^T(A_{00}^b)^{-T}$ is fixed and the set S_K has a nonempty interior, then we can always find K_{13}, together with an $\mathcal{H}$ that makes (A_{00}, K_{20}) observable, such that

$$K_{10}\phi_K = K_{10}^b\phi_b = -\left((A_{01}^b)^T\left(A_{00}^b\right)^{-T} + K_{13}\mathcal{H}\right)(\mathcal{Y}_1^b)^{-1}\phi_b \neq 0,$$

for all ϕ_b that are eigenvectors of A_{00}^b. Therefore, with this particular choice of K_{13}, we have that (A_{00}, K_{10}) is observable.

Now recall that (A, B) is controllable if and only if (A_{00}, A_{01}) or $(A_{00}, A_{00}A_{03} + A_{02})$ is controllable. Also, (A, C) is observable if and only if (A_{00}, K_{10}) or (A_{00}, K_{20}) is observable. Since the controllability of (A_{00}, A_{01}) implies the observability of (A_{00}, K_{10}) and the controllability of $(A_{00}, A_{00}A_{03} + A_{02})$ implies the observability of (A_{00}, K_{20}), then the controllability of (A, B) implies the observability of (A, C), when suitable $\mathcal{H}$ and K_{13} are chosen. Therefore, with those choices of $\mathcal{H}$ and K_{13}, the controllability of the system (4.6) implies that the realization (A, B, C) in (4.12), (4.13) and (4.14) is minimal. Choose the other state feedback matrices as follows:

$$K_{11} = K_{10}A_{00}^{-1}A_{01} - \mathcal{Y}_2^{-1}, \tag{4.25}$$

$$K_{12} = K_{10}A_{00}^{-1}A_{02}, \tag{4.26}$$

$$K_{21} = K_{20}A_{00}^{-1}A_{01}, \tag{4.27}$$

$$K_{22} = K_{20}A_{00}^{-1}A_{02} - \mathcal{Y}_3^{-1}, \tag{4.28}$$

$$K_{23} = -\frac{1}{2}I, \tag{4.29}$$

where $\mathcal{Y}_2 \in \mathbb{R}^{p_1 \times p_1}$ and $\mathcal{Y}_3 \in \mathbb{R}^{p_2 \times p_2}$ can be any symmetric positive definite matrices; i.e., $\mathcal{Y}_2 = \mathcal{Y}_2^T > 0$ and $\mathcal{Y}_3 = \mathcal{Y}_3^T > 0$. We will apply Lemma 2.3 in the following in order to prove that the system (4.11) is an NI system. We construct the matrix Y as follows:

$$Y = \begin{bmatrix} Y_{11} & -A_{00}^{-1}A_{01}\mathcal{Y}_2 & -A_{00}^{-1}A_{02}\mathcal{Y}_3 & 0 \\ -\mathcal{Y}_2 A_{01}^T A_{00}^{-T} & \mathcal{Y}_2 & 0 & 0 \\ -\mathcal{Y}_3 A_{02}^T A_{00}^{-T} & 0 & \mathcal{Y}_3 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \tag{4.30}$$

where $Y_{11} = \mathcal{Y}_1 + A_{00}^{-1}A_{01}\mathcal{Y}_2 A_{01}^T A_{00}^{-T} + A_{00}^{-1}A_{02}\mathcal{Y}_3 A_{02}^T A_{00}^{-T}$. Here, $\mathcal{Y}_1 = diag(y_1^a I, \mathcal{Y}_1^b)$ with $y_1^a > 0$ being a scalar. It can be verified that $Y > 0$ using the Schur complement theorem.

In order to verify Condition 1 in Lemma 2.3, we note that for the determinant of the matrix A in (4.12) we have

$$\begin{aligned}
(-1)^{p_2} \det A &= \det \begin{bmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ K_{10} & K_{11} & K_{12} & K_{13} \\ K_{20} & K_{21} & K_{22} & K_{23} \\ 0 & 0 & 0 & I \end{bmatrix} \\
&= \det \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ K_{10} & K_{11} & K_{12} \\ K_{20} & K_{21} & K_{22} \end{bmatrix} \\
&= \det A_{00} \det \left(\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} K_{10} \\ K_{20} \end{bmatrix} A_{00}^{-1} \begin{bmatrix} A_{01} & A_{02} \end{bmatrix} \right) \\
&= \det A_{00} \det \begin{bmatrix} -\mathcal{Y}_2^{-1} & 0 \\ 0 & -\mathcal{Y}_3^{-1} \end{bmatrix}
\end{aligned}$$

$$= \det A_{00} \det(-\mathcal{Y}_2^{-1}) \det(-\mathcal{Y}_3^{-1}) \neq 0,$$

where the equalities also use (4.25)–(4.28). Also, the input feedthrough matrix in the system (4.11) is zero, and hence symmetric. Hence, Condition 1 in Lemma 2.3 is satisfied. For Condition 2 in Lemma 2.3, with Y defined in (4.30), we have

$$AY = \begin{bmatrix} y_1^a A_{00}^a & 0 & 0 & 0 & A_{03}^a \\ 0 & A_{00}^b \mathcal{Y}_1^b & 0 & 0 & A_{03}^b \\ 0 & -K_{13}\mathcal{H} & -I & 0 & K_{13} \\ 0 & 0 & 0 & 0 & I \\ -(A_{03}^a)^T & \mathcal{H} - (A_{03}^b)^T & 0 & -I & -\frac{1}{2}I \end{bmatrix}. \tag{4.31}$$

Therefore, we have that

$$AYC^T = -B; \tag{4.32}$$

$$AY + YA^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathcal{Q}_b & -\mathcal{H}^T K_{13}^T & 0 & \mathcal{H}^T \\ 0 & -K_{13}\mathcal{H} & -2I & 0 & K_{13} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{H} & K_{13}^T & 0 & -I \end{bmatrix}.$$

Let $M = \begin{bmatrix} -\mathcal{Q}_b & -\mathcal{H}^T K_{13}^T & \mathcal{H}^T \\ -K_{13}\mathcal{H} & -2I & K_{13} \\ \mathcal{H} & K_{13}^T & -I \end{bmatrix}$, where (4.29) is used. For the matrix $-M$, we have that $I > 0$ and the Schur complement of the block I is

$$\begin{aligned} (-M)/I &= \begin{bmatrix} \mathcal{Q}_b & \mathcal{H}^T K_{13}^T \\ K_{13}\mathcal{H} & 2I \end{bmatrix} - \begin{bmatrix} \mathcal{H}^T \\ K_{13} \end{bmatrix} \begin{bmatrix} \mathcal{H} & K_{13}^T \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{Q}_b - \mathcal{H}^T\mathcal{H} & 0 \\ 0 & 2I - K_{13}K_{13}^T \end{bmatrix} \geq 0, \end{aligned}$$

where (4.22) and (4.24) are also used. Therefore, we have that $M \leq 0$. Hence $AY + YA^T \leq 0$, and Condition 2 in Lemma 2.3 is satisfied. Hence, the system (4.11) is an NI system with a minimal realization.

Necessity. If the realization (A, B, C) in (4.12), (4.13) and (4.14) is minimal and NI, then according to the proof of Lemma 2.3 (see Lemma 7 in [17]), there exists an $X = X^T > 0$ such that

$$\begin{bmatrix} XA + A^T X & XB - A^T C^T \\ B^T X - CA & -\left(CB + B^T C^T\right) \end{bmatrix} \leq 0.$$

Therefore, for any z, x_1, x_2, x_3 and v, we have

$$\begin{bmatrix}\begin{pmatrix} z \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ v \end{bmatrix}^T \begin{bmatrix} XA + A^TX & XB - A^TC^T \\ B^TX - CA & -\left(CB + B^TC^T\right) \end{bmatrix} \begin{bmatrix}\begin{pmatrix} z \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ v \end{bmatrix} \leq 0. \tag{4.33}$$

Let $X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{12}^T & X_{22} & X_{23} & X_{24} \\ X_{13}^T & X_{23}^T & X_{33} & X_{34} \\ X_{14}^T & X_{24}^T & X_{34}^T & X_{44} \end{bmatrix}$ and substitute (4.12)–(4.14) into (4.33). Also, let $x_1 = 0$, $x_2 = x_3 = 0$ and $\widetilde{v} = \begin{bmatrix} -K_{10} \\ -K_{20} \end{bmatrix} z$. We get

$$z^T(X_{11}A_{00} + A_{00}^TX_{11})z \leq 0$$

for any z, which implies that $X_{11}A_{00} + A_{00}^TX_{11} \leq 0$. Considering $X = X^T > 0$, we have $X_{11} > 0$. Also, since $\det A_{00} \neq 0$, then A_{00} is Lyapunov stable. $\square$

To facilitate the description of the conditions for NI state feedback equivalence to a system in the general form (4.1), we recall the following terminology (see [14, 15]). In the case when the system (4.5) has relative degree less than or equal to two, the system (4.6) is said to be the *normal form* of (4.5). The dynamics (4.6a), which are not controlled by the input u directly or through chains of integrators, are called the *internal dynamics*. The other part of the state, described by (4.6b)–(4.6d), are called the *external dynamics*. Setting the states described by the external dynamics to be zero in the internal dynamics, we obtain the *zero dynamics*:

$$\dot{z} = A_{00}z. \tag{4.34}$$

We now provide the definition of the weakly minimum phase property.

Definition 4.5. *(Weakly Minimum Phase) [64, 65] The system (4.5) of relative degree less than or equal to two is said to be weakly minimum phase if its zero dynamics (4.34) are Lyapunov stable.*

We present the main NI state feedback equivalence result of this chapter in the following.

Theorem 4.1. *Suppose the system (4.1) satisfying rank($\mathcal{B}$) = rank($\mathcal{C}$) = p is minimal with no zero at the origin. Then it is state feedback equivalent to an NI* *system if and only if there exists an output transformation* $\widetilde{y} = T_y y$*, where* $T_y \in \mathbb{R}^{p\times p}$ *and* $\det T_y \neq 0$*, such that the transformed system has relative degree less than or equal to two, and the transformed system is weakly minimum phase.*

Proof. ***Sufficiency.*** The sufficiency part of the proof directly follows from Lemmas 4.3, 4.4 and 4.5. According to Lemma 4.4, the system (4.1) can always be transformed into the form (4.6) using nonsingular input, output and state transformations. Since the system (4.1) has no zero at the origin, then $\det A_{00} \neq 0$ because nonsingular input, output and state transformations do not change the zeros of a system. Also, since the transformed system is weakly minimum phase, then A_{00} is Lyapunov stable. Since the input, output and state transformations are all nonsingular, the minimality of the system (4.1) is preserved in (4.6). According to Lemma 4.5, the output transformed system (4.6) is state feedback equivalent to an NI system. According to Lemma 4.3, the original system (4.1) is also state feedback equivalent to an NI system. This completes the sufficiency part of the proof.

Necessity. We first prove that if the system (4.1) is state feedback equivalent to an NI system, then there exists an output transformation $\widetilde{y} = T_y y$ that transforms the system (4.1) into a system with relative degree less than or equal to two.

If the system (4.1) is state feedback equivalent to an NI system, then according to Lemma 4.2, it is still feedback equivalent NI after a nonsingular output transformation. We apply an output transformation to the system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in order that the transformed system has a leading incomplete relative degree vector. Since the output transformed system is feedback equivalent to an NI system, then under a state feedback control law, we can make it NI with a minimal realization $(\widehat{A}, \widehat{B}, \widehat{C})$.

Since the system with the realization $(\widehat{A}, \widehat{B}, \widehat{C})$ has a leading incomplete relative degree vector r, we denote by $p_1 \geq 0$ the number of components in r that equal to one; i.e., $r_1, \cdots, r_{p_1} = 1$, and $r_{p_1+1}, \cdots, r_p \geq 2$. We decompose the matrix $\widehat{C}$ as $\widehat{C} = \begin{bmatrix} \widehat{C}_O \\ \widehat{C}_G \end{bmatrix}$, where $\widehat{C}_O \in \mathbb{R}^{p_1\times n}$ and $\widehat{C}_G \in \mathbb{R}^{(p-p_1)\times n}$. Here, $\widehat{C}_O$ determines the output entries corresponding to the ones in r, and $\widehat{C}_G$ determines the output entries corresponding to the components greater than one in r. According to Definition 4.2, $rank(\widehat{C}_O\widehat{B}) = p_1$ and $\widehat{C}_G\widehat{B} = 0$.

According to the proof of Lemma 2.3 (see [17]), the fact that $(\widehat{A}, \widehat{B}, \widehat{C})$ is NI implies that there exists $X = X^T > 0$ such that

$$\begin{bmatrix} X\widehat{A} + \widehat{A}^T X & X\widehat{B} - \widehat{A}^T\widehat{C}^T \\ \widehat{B}^T X - \widehat{C}\widehat{A} & -(\widehat{C}\widehat{B} + \widehat{B}^T\widehat{C}^T) \end{bmatrix} \leq 0. \tag{4.35}$$

Decomposing $\widehat{B}$ accordingly as $\widehat{B} = \begin{bmatrix}\widehat{B}_O & \widehat{B}_G\end{bmatrix}$ where $\widehat{B}_O \in \mathbb{R}^{n\times p_1}$ and $\widehat{B}_G \in \mathbb{R}^{n\times(p-p_1)}$, the inequality (4.35) can be expanded to be

$$\begin{bmatrix} X\widehat{A}+\widehat{A}^TX & X\widehat{B}_O-\widehat{A}^T\widehat{C}_O^T & X\widehat{B}_G-\widehat{A}^T\widehat{C}_G^T \\ \widehat{B}_O^TX-\widehat{C}_O\widehat{A} & -(\widehat{C}_O\widehat{B}_O+\widehat{B}_O^T\widehat{C}_O^T) & -\widehat{C}_O\widehat{B}_G \\ \widehat{B}_G^TX-\widehat{C}_G\widehat{A} & -\widehat{B}_G^T\widehat{C}_O^T & 0 \end{bmatrix} \leq 0, \tag{4.36}$$

where the condition $\widehat{C}_G\widehat{B} = 0$ is also used. The condition (4.36) implies that $\widehat{C}_O\widehat{B}_G = 0$ and $\widehat{B}_G^TX - \widehat{C}_G\widehat{A} = 0$. We have that $rank(\widehat{B}_G) = p - p_1$ because $rank(\widehat{B}) = rank(\mathcal{B}) = p$. Then, $\widehat{B}_G^TX - \widehat{C}_G\widehat{A} = 0$ implies that $\widehat{C}_G\widehat{A}\widehat{B}_G = \widehat{B}_G^TX\widehat{B}_G > 0$. The positive definiteness of $\widehat{C}_G\widehat{A}\widehat{B}_G$ implies that the largest component in the leading incomplete relative degree vector r of the system is two. Moreover, we have that

$$\begin{bmatrix}\widehat{C}_O\widehat{B} \\ \widehat{C}_G\widehat{A}\widehat{B}\end{bmatrix} = \begin{bmatrix}\widehat{C}_O\widehat{B}_O & 0 \\ \widehat{C}_G\widehat{A}\widehat{B}_O & \widehat{C}_G\widehat{A}\widehat{B}_G\end{bmatrix}. \tag{4.37}$$

Since $\widehat{C}_O\widehat{B}_G = 0$ and $rank(\widehat{C}_O\widehat{B}) = p_1$, we have that $\det(\widehat{C}_O\widehat{B}_O) \neq 0$. Considering that $\widehat{C}_O\widehat{B}_O$ and $\widehat{C}_G\widehat{A}\widehat{B}_G$ in (4.37) are both nonsingular, we have that $\det\begin{bmatrix}\widehat{C}_O\widehat{B} \\ \widehat{C}_G\widehat{A}\widehat{B}\end{bmatrix} \neq 0$. This implies that the leading incomplete relative degree vector r of the realization $(\widehat{A}, \widehat{B}, \widehat{C})$ is indeed a relative degree vector, whose components are either one or two. Therefore, we conclude that the system (4.1) can be output transformed into a system with a relative degree vector $r = \{r_1, \cdots, r_p\}$ with $1 \leq r_i \leq 2$ for all $i = 1, \cdots, p$. Therefore, the system (4.1) can be transformed into the form (4.6) using input, output and state transformations. The necessity part of Lemma 4.5 implies that the weakly minimum phase property of the output transformed system is another necessary condition. This completes the necessity part of the proof. □

We provide in the following the necessary and sufficient conditions for systems with relative degree of either one or two to be state feedback equivalent to NI systems.

Corollary 4.1. *Suppose the system (4.1) is minimal and of relative degree one; i.e., $r = \{1, \cdots, 1\}$, or relative degree two; i.e., $r = \{2, \cdots, 2\}$ and has no zero at the origin. Then it is state feedback equivalent to an NI system if and only if it is weakly minimum phase.*

Proof. This corollary considers two special cases of Theorem 4.1 and hence the proof directly follows from Theorem 4.1. □

4.2.2 State feedback equivalence to OSNI systems

We also derive necessary and sufficient conditions under which the system (4.1) can be rendered OSNI.

Definition 4.6. *A system in the form of (4.1) is said to be state feedback equivalent to an OSNI system if there exists a state feedback control law*

$$u = K_x x + K_v v,$$

where $K_x \in \mathbb{R}^{p\times n}$ and $K_v \in \mathbb{R}^{p\times p}$, such that the closed-loop system with the new input $v \in \mathbb{R}^p$ is minimal and OSNI.

Similar to Lemma 4.2, we show in the following that the OSNI state feedback equivalence property is invariant under a nonsingular output transformation.

Lemma 4.6. *If the transfer matrix $G(s)$ is OSNI, then $TG(s)T^T$ is also OSNI, where $T \in \mathbb{R}^{p\times p}$ and $\det T \neq 0$.*

Proof. The proof follows from Definition 2.9. If $G(s)$ is OSNI, then we have that

$$j\omega[G(j\omega) - G(j\omega)^*] - \epsilon\omega^2\overline{G}(j\omega)^*\overline{G}(j\omega) \geq 0,$$

$\forall\omega \in \mathbb{R}\cup\{\infty\}$ where $\overline{G}(j\omega) = G(j\omega) - G(\infty)$. We have that $T^TT \leq \lambda_{max}(T^TT)I$. Therefore,

$$\begin{aligned}
&j\omega[G(j\omega) - (G(j\omega))^*] - \frac{\epsilon}{\lambda_{max}(T^TT)}\omega^2\overline{G}(j\omega)^*T^TT\overline{G}(j\omega)\\
&\geq j\omega[G(j\omega) - (G(j\omega))^*] - \frac{\epsilon}{\lambda_{max}(T^TT)}\omega^2\overline{G}(j\omega)^*T^TT\overline{G}(j\omega)\\
&+\frac{\epsilon}{\lambda_{max}(T^TT)}\omega^2\overline{G}(j\omega)^*(T^TT - \lambda_{max}(T^TT)I)\overline{G}(j\omega)\\
&= j\omega[G(j\omega) - G(j\omega)^*] - \epsilon\omega^2\overline{G}(j\omega)^*\overline{G}(j\omega) \geq 0. \qquad (4.38)
\end{aligned}$$

The transfer matrix $TG(s)T^T$ satisfies Definition 2.9 via (4.38). Therefore, the transformed system $TG(s)T^T$ is OSNI with the output strictness $\frac{\epsilon}{\lambda_{max}(T^TT)}$. $\square$

We show in the following that the same conditions in Theorem 4.1 are also necessary and sufficient for the state feedback equivalence of system (4.1) to an OSNI system.

Lemma 4.7. *Suppose the system (4.6) has $\det A_{00} \neq 0$. Then it is state feedback equivalent to an OSNI system if and only if it is controllable and A_{00} is Lyapunov stable.*

Proof. The necessity part of this lemma follows from the necessity part of Lemma 4.5 because OSNI systems belong to the class of NI systems.

For the sufficiency part, we need to show that the condition

$$AY + YA^T + \epsilon(CAY)^T CAY \leq 0$$

in Theorem 2.1 is satisfied for some scalar $\epsilon > 0$ in addition to what is shown in the sufficiency proof of Lemma 4.5. Following from the sufficiency proof of Lemma 4.5, we add restrictions on the choices of K_{13} and $\mathcal{H}$ such that $K_{13}^T K_{13} = I$ and $\mathcal{H} \in \widetilde{S}_{\mathcal{H}}$, where

$$\widetilde{S}_{\mathcal{H}} = \left\{ \mathcal{H} \in \mathbb{R}^{p_2 \times m_b} \,\middle|\, \mathcal{H}^T \mathcal{H} \leq \frac{\epsilon^2 - 3\epsilon + 1}{1 - 2\epsilon} Q_b \right\}.$$

Here, we have that $\frac{\epsilon^2-3\epsilon+1}{1-2\epsilon} \in (0,1)$ for $\epsilon \in \left(0, \frac{1}{2}\left(3 - \sqrt{5}\right)\right)$. Note that this additional restriction does not change the results in Lemma 4.5. Using C and AY in (4.19f) and (4.31), we have that

$$CAY = \begin{bmatrix} 0 & -K_{13}\mathcal{H} & -I & 0 & K_{13} \\ 0 & 0 & 0 & 0 & I \end{bmatrix}.$$

Therefore,

$$(CAY)^T CAY = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{H}^T\mathcal{H} & \mathcal{H}^T K_{13}^T & 0 & -\mathcal{H}^T \\ 0 & K_{13}\mathcal{H} & I & 0 & -K_{13} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathcal{H} & -K_{13}^T & 0 & 2I \end{bmatrix}.$$

Hence,

$$\begin{aligned} & AY + YA^T + \epsilon(CAY)^T CAY \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathcal{Q}_b + \epsilon\mathcal{H}^T\mathcal{H} & -(1-\epsilon)\mathcal{H}^T K_{13}^T & 0 & (1-\epsilon)\mathcal{H}^T \\ 0 & -(1-\epsilon)K_{13}\mathcal{H} & -(2-\epsilon)I & 0 & (1-\epsilon)K_{13} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & (1-\epsilon)\mathcal{H} & (1-\epsilon)K_{13}^T & 0 & -(1-2\epsilon)I \end{bmatrix}. \end{aligned}$$

Let

$$\widetilde{M} = \begin{bmatrix} -\mathcal{Q}_b + \epsilon\mathcal{H}^T\mathcal{H} & -(1-\epsilon)\mathcal{H}^T K_{13}^T & (1-\epsilon)\mathcal{H}^T \\ -(1-\epsilon)K_{13}\mathcal{H} & -(2-\epsilon)I & (1-\epsilon)K_{13} \\ (1-\epsilon)\mathcal{H} & (1-\epsilon)K_{13}^T & -(1-2\epsilon)I \end{bmatrix}.$$

We apply the Schur complement theorem in the following to find the range of ϵ. We choose $\epsilon \in (0, \frac{1}{2})$ and therefore $-(1-2\epsilon)I < 0$. The Schur complement of the block $(1-2\epsilon)I$ of $-\widetilde{M}$ is

$$
\begin{aligned}
(-\widetilde{M})/[(1-2\epsilon)I] &= \begin{bmatrix} \mathcal{Q}_b - \epsilon\mathcal{H}^T\mathcal{H} & (1-\epsilon)\mathcal{H}^T K_{13}^T \\ (1-\epsilon)K_{13}\mathcal{H} & (2-\epsilon)I \end{bmatrix} - \frac{(1-\epsilon)^2}{1-2\epsilon}\begin{bmatrix} \mathcal{H}^T \\ K_{13} \end{bmatrix}\begin{bmatrix} \mathcal{H} & K_{13}^T \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{Q}_b - \left(\epsilon + \frac{(1-\epsilon)^2}{1-2\epsilon}\right)\mathcal{H}^T\mathcal{H} & \left(1-\epsilon-\frac{(1-\epsilon)^2}{1-2\epsilon}\right)\mathcal{H}^T K_{13}^T \\ \left(1-\epsilon-\frac{(1-\epsilon)^2}{1-2\epsilon}\right)K_{13}\mathcal{H} & \left(2-\epsilon-\frac{(1-\epsilon)^2}{1-2\epsilon}\right)I \end{bmatrix},
\end{aligned}
$$

which is positive semi-definite when $\epsilon \in \left(0, \frac{1}{2}\left(3-\sqrt{5}\right)\right)$. In this case, $AY + YA^T + \epsilon(CAY)^T CAY \leq 0$. Therefore, the system with the realization (A, B, C) in (4.12), (4.13) and (4.14) is OSNI. □

Theorem 4.2. *Suppose the system (4.1) is minimal with no zero at the origin. Then it is state feedback equivalent to an OSNI system if and only if there exists an output transformation* $\widetilde{y} = T_y y$, *where* $T_y \in \mathbb{R}^{p\times p}$ *and* $\det T_y \neq 0$, *such that the transformed system has relative degree less than or equal to two, and the transformed system is weakly minimum phase.*

Proof. This proof is similar to the proof of Theorem 4.1 except that Lemmas 4.6 and 4.7 are used instead of Lemmas 4.2 and 4.5. □

Considering the results in Theorem 4.1 and 4.2, we have the following corollary which shows that the same necessary and sufficient conditions are shared for NI and OSNI state feedback equivalence.

Corollary 4.2. *Suppose the system (4.1) is minimal with no zero at the origin. Then the following statements are equivalent:*

1. *The system (4.1) is state feedback equivalent to an NI system;*
2. *The system (4.1) is state feedback equivalent to an OSNI system;*
3. *There exists an output transformation* $\widetilde{y} = T_y y$, *where* $T_y \in \mathbb{R}^{p\times p}$ *and* $\det T_y \neq 0$, *such that the transformed system has relative degree less than or equal to two, and the transformed system is weakly minimum phase.*

Remark 4.1. *Although the necessary and sufficient conditions for state feedback equivalence to NI and OSNI systems are the same, as shown in Corollary 4.2, different state feedback matrices are required. That is, additional restrictions on state feedback matrices should be satisfied in order to make a system OSNI. Considering that under certain assumptions, the interconnection of an NI system and an OSNI system is asymptotically stable [27, 28], we can decide to make a system NI or OSNI depending on the other system in the interconnection.*

4.3 State Feedback Equivalence to an SSNI System

In this section, we derive necessary and sufficient conditions under which a system in the form of (4.1) is state feedback equivalent to an SSNI system. First, we define state feedback equivalence to an SSNI system as follows.

Definition 4.7. *A system in the form of (4.1) is said to be state feedback equivalent to an SSNI system if there exists a state feedback control law*

$$u = K_x x + K_v v,$$

where $K_x \in \mathbb{R}^{p\times n}$ and $K_v \in \mathbb{R}^{p\times p}$, such that the closed-loop system with the new input $v \in \mathbb{R}^p$ is SSNI.

It will be shown later in this section that having a relative degree vector $r = \{1, \cdots, 1\}$ is one of the necessary conditions for this system to be state feedback equivalent to an SSNI system. Therefore, we start with the derivation of the normal form for the system (4.1) with a relative degree vector $r = \{1, \cdots, 1\}$.

Lemma 4.8. *Suppose the system (4.1) satisfying $rank(\mathcal{B}) = rank(\mathcal{C}) = p$ has a relative degree vector $r = \{1, \cdots, 1\}$. Then there exists input and state transformations such that the resulting transformed system is of the form*

$$\dot{z} = A_{00}z + A_{01}y, \tag{4.39a}$$

$$\dot{x}_1 = A_{10}z + A_{11}x_1 + \widetilde{u}, \tag{4.39b}$$

$$y = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} z \\ x_1 \end{bmatrix}. \tag{4.39c}$$

Proof. If (4.1) has a relative degree vector $r = \{1, \cdots, 1\}$, then $\det(\mathcal{CB}) \neq 0$. The rest of the proof follows from Lemma 4.4 with $p_1 = p$ and $p_2 = 0$. □

Choose the input u to be

$$\widetilde{u} = v + (K_1 - A_{10})z + (K_2 - A_{11})y,$$

and the system (4.39) takes the form

$$\dot{z} = A_{00}z + A_{01}y, \tag{4.40a}$$

$$\dot{y} = K_1 z + K_2 y + v, \tag{4.40b}$$

$$y = [0 \quad I] \begin{bmatrix} z \\ y \end{bmatrix}. \tag{4.40c}$$

We need to find the state feedback matrices $K_1 \in \mathbb{R}^{p\times m}$ and $K_2 \in \mathbb{R}^{p\times p}$ such that the system (4.40) is SSNI.

Lemma 4.9. *Suppose the system (4.40) has (A_{00}, A_{01}) controllable. Then the following statements are equivalent:*

1. *A_{00} is Hurwitz;*
2. *There exist K_1 and K_2 such that the system (4.40) is an SSNI system with realization (A, B, C), where A is Hurwitz, and the transfer function $G(s) := C(sI - A)^{-1}B$ is such that $G(s) + G(-s)^T$ has full normal rank.*

Proof. Let us define the following:

$$A = \begin{bmatrix} A_{00} & A_{01} \\ K_1 & K_2 \end{bmatrix}, \tag{4.41}$$

$$B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{4.42}$$

$$C = \begin{bmatrix} 0 & I \end{bmatrix}. \tag{4.43}$$

From the proof of Lemma 4.5, (A, B) is controllable if and only if (A_{00}, A_{01}) is controllable. Therefore, there are no observable uncontrollable modes in this system.

Sufficiency. Let A_{00} be Hurwitz. Then we can always find a matrix $\mathcal{Y}_1 > 0$ such that

$$A_{00}\mathcal{Y}_1 + \mathcal{Y}_1 A_{00}^T + \frac{1}{2} A_{00}^{-1} A_{01} A_{01}^T A_{00}^{-T} < 0$$

is satisfied. In the sequel, we will find a matrix K_1 such that

$$A_{00}\mathcal{Y}_1 + \mathcal{Y}_1 A_{00}^T + \frac{1}{2}(\mathcal{Y}_1 K_1^T + A_{00}^{-1} A_{01})(K_1 \mathcal{Y}_1 + A_{01}^T A_{00}^{-T}) < 0 \tag{4.44}$$

is satisfied. One possible choice is $K_1 = -A_{01}^T A_{00}^{-T} \mathcal{Y}_1^{-1}$, which simplifies (4.44) to be $A_{00}\mathcal{Y}_1 + \mathcal{Y}_1 A_{00}^T < 0$. Let $K_2 = K_1 A_{00}^{-1} A_{01} - \mathcal{Y}_2^{-1}$, where $\mathcal{Y}_2 \in \mathbb{R}^{p\times p}$ can be any symmetric positive definite matrix; i.e., $\mathcal{Y}_2 = \mathcal{Y}_2^T > 0$. We apply Lemma 2.4 in the following to prove that the system (4.40) is an SSNI system. We construct the matrix Y as follows:

$$Y = \begin{bmatrix} \mathcal{Y}_1 + A_{00}^{-1} A_{01} \mathcal{Y}_2 A_{01}^T A_{00}^{-T} & -A_{00}^{-1} A_{01} \mathcal{Y}_2 \\ -\mathcal{Y}_2 A_{01}^T A_{00}^{-T} & \mathcal{Y}_2 \end{bmatrix}.$$

We have $Y > 0$ because $\mathcal{Y}_2 > 0$ and the Schur complement of the block $\mathcal{Y}_2$ is $\mathcal{Y}_1$, which is positive definite. Now, we have $B + AYC^T = 0$ and

$$AY + YA^T = \begin{bmatrix} A_{00}\mathcal{Y}_1 + \mathcal{Y}_1 A_{00}^T & \mathcal{Y}_1 K_1^T + A_{00}^{-1} A_{01} \\ K_1 \mathcal{Y}_1 + A_{01}^T A_{00}^{-T} & -2I \end{bmatrix}.$$

We have $-2I < 0$ and the Schur complement of the block $2I$ in the matrix $-(AY + YA^T)$ is

$$\begin{aligned}&(-(AY + YA^T))/(2I) \\ &\qquad = -A_{00}\mathcal{Y}_1 - \mathcal{Y}_1 A_{00}^T - \frac{1}{2}(\mathcal{Y}_1 K_1^T + A_{00}^{-1} A_{01})(K_1 \mathcal{Y}_1 + A_{01}^T A_{00}^{-T}) > 0,\end{aligned}$$

according to (4.44). Hence $AY + YA^T < 0$ and A^T is Hurwitz. Therefore A is Hurwitz. Now we prove that $G(s) + G(-s)^T$ has full normal rank. For A, B and C given by (4.41)–(4.43), we have

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} sI - A_{00} & -A_{01} \\ -K_1 & sI - K_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \\ &= \left(sI - K_1(sI - A_{00})^{-1} A_{01} - K_2\right)^{-1}. \end{aligned} \tag{4.45}$$

Substituting $s = 0$ in (4.45), we have

$$G(0) = (K_1 A_{00}^{-1} A_{01} - K_2)^{-1} = \mathcal{Y}_2 > 0.$$

Hence $G(s) + G(-s)^T$ must have full normal rank. Therefore, according to Lemma 2.4, the system (4.40) is SSNI.

Necessity. If A is Hurwitz, $G(s) + G(-s)^T$ has full normal rank and the system (4.40) is SSNI, then according to Lemma 2.4, there exists a matrix $Y = Y^T > 0$ such that $B = -AYC^T$ and $AY + YA^T < 0$.

Letting $X = Y^{-1}$, then $X = X^T > 0$. Also letting $Q = -(AY + YA^T)$, then we have $Q = Q^T > 0$ and $XA + A^TX = -XQX < 0$. Since $B = -AYC^T$, we have $CB + B^TC^T = -CAYC^T - CYA^TC^T = CQC^T$. Also, $XB - A^TC^T = -XAX^{-1}C^T - A^TC^T = -(XA + A^TX)X^{-1}C^T = XQXX^{-1}C^T = XQC^T$. Since $Q = Q^T > 0$, let $H := Q^{\frac{1}{2}}$. Hence $H = H^T > 0$. We have

$$\begin{bmatrix} XA + A^TX & XB - A^TC^T \\ B^TX - CA & -(CB + B^TC^T) \end{bmatrix} = -\begin{bmatrix} L^T \\ W^T \end{bmatrix} \begin{bmatrix} L & W \end{bmatrix} \leq 0, \tag{4.46}$$

where $L = HX$ and $W = -HC^T$. (4.46) implies that for any $z \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $v \in \mathbb{R}^p$, we have

$$\begin{aligned}
&\begin{bmatrix} z^T & y^T & v^T \end{bmatrix} \begin{bmatrix} XA + A^T X & XB - A^T C^T \\ B^T X - CA & -(CB + B^T C^T) \end{bmatrix} \begin{bmatrix} z \\ y \\ v \end{bmatrix} \\
&= - \begin{bmatrix} z^T & y^T & v^T \end{bmatrix} \begin{bmatrix} L^T \\ W^T \end{bmatrix} \begin{bmatrix} L & W \end{bmatrix} \begin{bmatrix} z \\ y \\ v \end{bmatrix} \leq 0, \qquad (4.47)
\end{aligned}$$

where equality holds if and only if $\begin{bmatrix} L & W \end{bmatrix} \begin{bmatrix} z \\ y \\ v \end{bmatrix} = 0$. That is $L \begin{bmatrix} z \\ y \end{bmatrix} + Wv = 0$, which is equivalent to $H \left(X \begin{bmatrix} z \\ y \end{bmatrix} - C^T v \right) = 0$. Because $H > 0$, this equation holds if and only if

$$X \begin{bmatrix} z \\ y \end{bmatrix} - C^T v = 0. \qquad (4.48)$$

Let $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}$ and choose $y = 0$ and $v = -K_1 z$. With C given by (4.43), (4.48) becomes

$$\begin{bmatrix} X_{11} \\ X_{12}^T + K_1 \end{bmatrix} z = 0,$$

which holds only if $X_{11} z = 0$. Since $X = X^T > 0$, then $X_{11} = X_{11}^T > 0$. Hence $X_{11} z = 0$ if and only if $z = 0$. This implies that with the choice $y = 0$ and $v = -K_1 z$, strict inequality holds in (4.47) for all $z \neq 0$. Substituting (4.41)–(4.43) together with $y = 0$ and $v = -K_1 z$ into (4.47), we obtain

$$z^T (X_{11} A_{00} + A_{00}^T X_{11}) z < 0$$

for all $z \neq 0$. This implies that $X_{11} A_{00} + A_{00}^T X_{11} < 0$. Therefore, A_{00} is Hurwitz. □

We provide in the following the definition of the minimum phase property, which is needed in the presentation of the SSNI state feedback equivalence result.

Definition 4.8. *(Minimum Phase) [15, 65] A system (4.1) satisfying rank($\mathcal{B}$) = rank($\mathcal{C}$) = p with relative degree vector $\{1, \cdots, 1\}$ is said to be minimum phase if its zero dynamics $\dot{z} = A_{00} z$ are asymptotically stable.*

Theorem 4.3. *Suppose the system (4.1) satisfying rank($\mathcal{B}$) = rank($\mathcal{C}$) = p is* *minimal. Then the following statements are equivalent:*

1. *The system has a relative degree vector* $r = \{1, \cdots, 1\}$ *and is minimum phase;*
2. *The system is state feedback equivalent to an SSNI system with realization* (A, B, C)*, where* A *is Hurwitz, and the transfer function* $G(s) := C(sI - A)^{-1}B$ *is such that* $G(s) + G(-s)^T$ *has full normal rank.*

Proof. The proof from Statement 1 to Statement 2 follows directly from Lemmas 4.8 and 4.9. Note that the minimum phase condition is equivalent to the condition that A_{00} is Hurwitz in Lemma 4.9. Now we prove that Condition 2 implies that the system has a relative degree vector $r = \{1, \cdots, 1\}$. SSNI systems form a subclass of all NI systems according to Definition 2.10. The analysis in the necessity proof of Theorem 4.1 also holds for SSNI systems except that strict inequalities hold for both (4.35) and (4.36), where this additional restriction comes from the strict inequality in Lemma 2.4. Strict inequality for (4.36) holds only if the zero block matrix has zero dimension, which is true only if $p_2 = 0$. This implies that statement 2 is true only if the original system (4.1) with realization $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ can be output transformed by a nonsingular matrix $T_y \in \mathbb{R}^{p \times p}$ into a system with a relative degree vector $r = \{1, \cdots, 1\}$. According to Definition 4.1, that is, the output transformed system satisfies $\det(\widetilde{\mathcal{C}}\mathcal{B}) \neq 0$, where $\widetilde{\mathcal{C}} = T_y\mathcal{C}$. Since $\det T_y \neq 0$, we have that $\det(\mathcal{C}\mathcal{B}) \neq 0$. This means that the original system (4.1) itself is already in a form with a relative degree vector $r = \{1, \cdots, 1\}$. Therefore, according to Lemma 4.8, (4.39) is the normal form of the system (4.1). The rest of the proof follows directly from Lemma 4.9. □

Remark 4.2. *Note that SSNI systems belong to the class of OSNI systems and also the class of SNI systems (see [29]). Therefore, based on the control objective and conditions on the systems of interest, we can decide on whether to achieve the SSNI property or the OSNI property via the use of state feedback control.*

4.4 Control of Systems with SNI Uncertainty

One useful application of state feedback equivalence to NI systems is to robustly stabilize systems for a class of uncertainties. More precisely, for a system having SNI uncertainty, we can render the nominal closed-loop system NI with the DC gain condition satisfied when full state measurement is available. A similar controller synthesis problem is investigated in [3], where the existence of such a state feedback controller depends on the solvability of a series of LMIs. However, in this chapter,

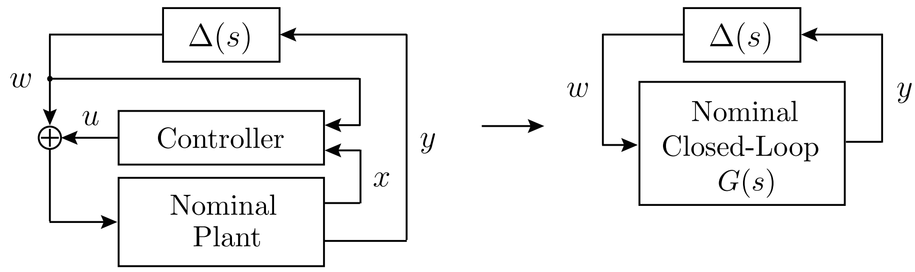


Figure 4.1: A feedback control system. The plant uncertainty $\Delta(s)$ is SNI and satisfies $\lambda_{max}(\Delta(0)) \leq \gamma$. Under some assumptions, we can find a controller such that the closed-loop transfer function $G(s)$ is NI with $G(\infty) = 0$ and $\lambda_{max}(G(0)) < 1/\gamma$. Then the closed-loop system is aymptotically stable.

the LMI conditions in [3] are replaced by some simpler conditions with respect to the relative degree vector and the weakly minimum phase property.

Consider the uncertain feedback control system in Fig. 5.1 and suppose that full state feedback is available. Then Theorem 4.1 can be used in order to synthesize a state feedback controller such that the nominal closed-loop system is NI. Suppose the state-space model of the uncertain system in Fig. 5.1 is

$$\dot{x} = \mathcal{A}x + \mathcal{B}(u + w), \tag{4.49a}$$

$$y = \mathcal{C}x, \tag{4.49b}$$

$$w = \Delta(s)y, \tag{4.49c}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ and $y \in \mathbb{R}^p$ are the state, input and output of the nominal plant. Here, (4.49c) models the uncertainty, and the uncertainty transfer function $\Delta(s)$ is assumed to be SNI satisfying $\lambda_{max}(\Delta(0)) \leq \gamma$ for some constant $\gamma > 0$. Here, since $\Delta(s)$ is SNI, $\Delta(0)$ must be positive semidefinite.

The general idea used to stabilize the system (4.49) is to choose a control law u such that the system described by (4.49a) and (4.49b) is NI with input w and output y. Therefore, since $\Delta(s)$ is SNI, the system (4.49) forms a positive feedback interconnection of an NI system and an SNI system, whose equilibrium is asymptotically stable if the DC gain condition in Corollary 2.1 is satisfied.

Theorem 4.4. *Consider the uncertain system (4.49). Suppose the realization $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is minimal with no zero at the origin. If there exists an output transformation $\widetilde{y} = T_y y$, where $T_y \in \mathbb{R}^{p\times p}$ and $\det T_y \neq 0$, such that the realization $(\mathcal{A}, \mathcal{B}, T_y\mathcal{C})$ has relative degree less than or equal to two and is weakly minimum phase, then there exist $K_x \in \mathbb{R}^{p\times n}$ and $K_w \in \mathbb{R}^{p\times p}$ such that the control law*

$$u = K_x x + K_w w$$

stabilizes the system (4.49).

Proof. According to Theorem 4.1 and its proof, the conditions here imply that the nominal plant in (4.49), described by

$$\dot{x} = \mathcal{A}x + \mathcal{B}u,$$
$$y = \mathcal{C}x,$$

is state feedback equivalent to an NI system. Suppose the corresponding state feedback control law is

$$u = K_x x + K_v v. \tag{4.50}$$

Therefore, the nominal plant, described by (4.49a) and (4.49b), is NI with input w and output y under the control law

$$u = K_x x + K_w w, \tag{4.51}$$

where

$$K_w = K_v - I. \tag{4.52}$$

Now the system (4.49) is an interconnection of the nominal closed-loop NI system and the SNI uncertainty. To stabilize this interconnection, we investigate the DC gain conditions of Corollary 2.1. As is shown in the proof of Theorem 4.1, the output transformed system $(\mathcal{A}, \mathcal{B}, T_y\mathcal{C})$ is rendered NI with a transfer function $\widehat{G}(s)$ where $\widehat{G}(s) = C(sI - A)^{-1}B$ with A, B and C given by (4.12), (4.13) and (4.14). We have that $\widehat{G}(\infty) = 0$. With the state feedback matrices given in the proof of Theorem 4.1, we also have that

$$\widehat{G}(0) = -CA^{-1}B = CA^{-1}AYC^T = CYC^T = \begin{bmatrix} \mathcal{Y}_2 & 0 \\ 0 & \mathcal{Y}_3 \end{bmatrix},$$

where we also use (4.30) and (4.32). Since the NI state feedback equivalence of the realization $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ follows from the NI state feedback equivalence of the output transformed system $(\mathcal{A}, \mathcal{B}, T_y\mathcal{C})$, using Lemma 4.2, then the nominal closed-loop system can be rendered to be an NI system whose transfer function is $G(s) = T_y^{-1}\widehat{G}(s)T_y^{-T}$. Since $\mathcal{Y}_2$ and $\mathcal{Y}_3$ can be any positive definite matrices, we choose them to be such that

$$\lambda_{max}\left(T_y^{-1}\begin{bmatrix} \mathcal{Y}_2 & 0 \\ 0 & \mathcal{Y}_3 \end{bmatrix}T_y^{-T}\right) < \frac{1}{\gamma}.$$

Therefore, $\lambda_{max}(G(0)) < \frac{1}{\gamma}$. Hence, $\lambda_{max}(G(0)\Delta(0)) < 1$. According to Corollary 2.1, it now follows that the system (4.49) is asymptotically stable. This completes the proof. □

Remark 4.3. *In the case that the uncertainty (4.49c) in the system (4.49) is NI, we can make the nominal closed-loop system (4.49a) and (4.49b) OSNI using the results of Theorem 4.2 in order to achieve asymptotic stability (see [27, 28] for the corresponding OSNI stability results).*

4.5 Illustrative Example

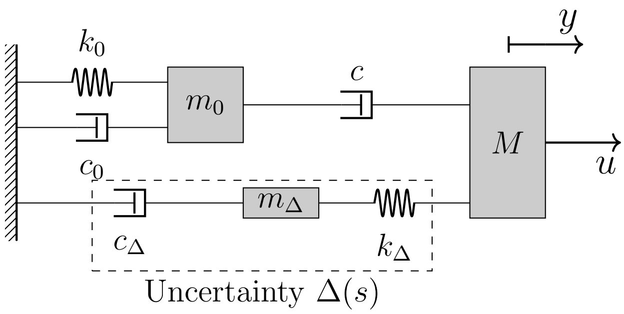


Figure 4.2: The top view of a mass-spring-damper system containing uncertain parameters. The system consists of three masses, which move rectilinearly on a frictionless floor. The known parameters are $m_0 = M = 1kg$, $k_0 = 1N/m$ and $c_0 = c = 1Ns/m$ while the parameters $m\Delta$, k_Δ and c_Δ are unknown. A force input u is applied to the mass M and the displacement of M is measured as y.

In this section, we illustrate our results provided in Section 4.4 using a real-world control example. As shown in Fig. 4.2, we consider three masses that slide on a frictionless floor. Mass m_0 is connected to a wall on its left by a spring k_0 and a damper c_0, and is connected to another mass M on its right by a damper c. The parameters m_0, k_0, c_0, M and c are known; i.e., $m_0 = M = 1kg$, $k_0 = 1N/m$ and $c_0 = c = 1Ns/m$. We apply a force u to the mass M and measure its displacement y. A mass m_Δ is connected to M via a spring k_Δ and m_Δ is connected to the wall via a damper c_Δ. The parameters m_Δ, k_Δ and c_Δ are unknown. This system can be regarded as arising in a vehicle suspension system (e.g., see [75]) or a vibration isolation system.

In Fig. 4.2, the part contained in the dashed rectangle can be regarded as the system uncertainty $\Delta(s)$. The uncertainty $\Delta(s)$ takes the displacement of M as its input u_Δ and produces a force output w, which is the force applied to M by the spring k_Δ. Denoting the displacement of m_Δ by d, we obtain the equation of motion for the uncertainty $\Delta(s)$ using Newton's law as follows

$$m_\Delta \ddot{d} = k_\Delta (y - d) - c_\Delta \dot{d}.$$

Letting $x_{\Delta 1} = d$ and $x_{\Delta 2} = \dot{x}_{\Delta 1}$, we obtain a state-space model for the uncertainty $\Delta(s)$ as:

$$\dot{x}_{\Delta 1} = x_{\Delta 2}, \tag{4.53a}$$

$$\dot{x}_{\Delta 2} = -\frac{k_\Delta}{m_\Delta} x_{\Delta 1} - \frac{c_\Delta}{m_\Delta} x_{\Delta 2} + \frac{k_\Delta}{m_\Delta} u_\Delta, \tag{4.53b}$$

$$w = k_\Delta x_{\Delta 1} - k_\Delta u_\Delta. \tag{4.53c}$$

Therefore, the transfer function of the uncertainty $\Delta(s)$ given in (4.53) from input $u_\Delta = y$ to output w is

$$\Delta(s) = \frac{k_\Delta^2}{m_\Delta s^2 + c_\Delta s + k_\Delta} - k_\Delta. \tag{4.54}$$

Let z_0 denote the displacement of m_0. Using Newton's law, we obtain the following dynamic equations for the masses m_0 and M, respectively.

$$m_0 \ddot{z}_0 = c(\dot{y} - \dot{z}_0) - c_0 \dot{z}_0 - k_0 z_0; \tag{4.55}$$

$$M \ddot{y} = u + w + c(\dot{z}_0 - \dot{y}). \tag{4.56}$$

Using these two equations of motion, we can derive a state-space model for the system in Fig. 4.2 with input u and output y. Letting $z_1 = \dot{z}_0$, $x_1 = y$ and $x_2 = \dot{x}_1$, and substituting the values of m_0, M, k_0, c_0, c into (4.55) and (4.56), we have

$$\dot{z}_0 = z_1, \tag{4.57a}$$

$$\dot{z}_1 = -z_0 - 2z_1 + x_2, \tag{4.57b}$$

$$\dot{x}_1 = x_2, \tag{4.57c}$$

$$\dot{x}_2 = z_1 - x_2 + (u + w), \tag{4.57d}$$

$$y = x_1. \tag{4.57e}$$

This system is a system of the form (4.49a), (4.49b). Hence, the system can be described by the left diagram in Fig. 5.1, where (4.57) is the nominal plant and (4.53) is the uncertainty $\Delta(s)$. We aim to apply the NI state feedback equivalence stabilization method in Theorem 4.4 to asymptotically stabilize this uncertain system. The system (4.57) is of relative degree two and is minimum phase since its internal dynamics described by (4.57a), (4.57b) is asymptotically stable when $x_2 = 0$. We first make the system (4.57) NI by following the procedure given after Theorem 4.1. The system (4.57) is already in the form of (4.6). Indeed, the equations (4.57a) and (4.57b) correspond to (4.6a). The equations (4.57c) and (4.57d) correspond to (4.6c) and (4.6d), respectively. Note that in this example, the equation (4.6b) is irrelevant. For the system (4.57), the corresponding matrices in the normal form (4.6) are $A_{00} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$, $A_{02} = 0$, $A_{03} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A_{30} = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $A_{32} = 0$ and $A_{33} = -1$. Since the matrix A_{00} is Hurwitz, then according to (4.18) and (4.20), we construct the matrix K_{20} using (4.21). Since A_{00} is Hurwitz, we can always find $\mathcal{Y}_1$ such that $A_{00}\mathcal{Y}_1 + \mathcal{Y}_1 A_{00}^T < 0$. For example, we choose $\mathcal{Y}_1 = \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$. Then we have that $A_{00}\mathcal{Y}_1 + \mathcal{Y}_1 A_{00}^T = -\mathcal{Q} = \begin{bmatrix} -4 & 3 \\ 3 & -4 \end{bmatrix}$. Therefore, $\mathcal{Q} = \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}$. According to (4.22), choose $\mathcal{H} = \begin{bmatrix} 0 & 0 \end{bmatrix}$. Therefore, according to (4.21), we have

$$K_{20} = -A_{03}\mathcal{Y}_1^{-1} = \begin{bmatrix} -1 & -1.5 \end{bmatrix}.$$

The pair (A_{00}, K_{20}) is observable. Now, we construct the matrices K_{22} and K_{23} using (4.28) and (4.29), respectively. Since $A_{02} = 0$, we have $K_{22} = -\mathcal{Y}_3^{-1}$ and $K_{23} = -\frac{1}{2}I$, where $\mathcal{Y}_3 < 0$. Choosing $\mathcal{Y}_3 = 1$, then we have $K_{22} = -1$ and $K_{23} = -0.5$. According to (4.10) and (4.50), we construct the state feedback control input as

$$\begin{aligned} u &= v + (K_{20} - A_{30})z + (K_{22} - A_{32})x_1 + (K_{23} - A_{33})x_2 \\ &= v + \begin{bmatrix} -1 & -2.5 \end{bmatrix} z - x_1 - 0.5x_2, \end{aligned} \tag{4.58}$$

where $z = \begin{bmatrix} z_0 & z_1 \end{bmatrix}^T$. According to Theorem 4.1, the system (4.57) is made NI by the control input (4.58). Under the control input (4.58), the system (4.57) now becomes

$$\dot{z}_0 = z_1, \tag{4.59a}$$

$$\dot{z}_1 = -z_0 - 2z_1 + x_2, \tag{4.59b}$$

$$\dot{x}_1 = x_2, \tag{4.59c}$$

$$\dot{x}_2 = -z_0 - 1.5z_1 - x_1 - 0.5x_2 + v + w, \tag{4.59d}$$

$$y = x_1. \tag{4.59e}$$

Let $G(s)$ denote the transfer function of the system (4.59) from v to y. We have that

$$G(s) = \frac{(s+1)^2}{s^4 + 2.5s^3 + 4.5s^2 + 3.5s + 1}.$$

It can be verified that $G(s)$ satisfies Definition 2.6. We aim to let the system (4.57) have the form of (4.59) but with input w instead of v. Then we implement a controller of the form (4.51) using (4.52) as described in the proof of Theorem 4.4. Based on (4.58), we obtain the corresponding control input

$$u = -z_0 - 2.5z_1 - x_1 - 0.5x_2, \tag{4.60}$$

since in this example $K_w = K_v - 1 = 0$. Now the entire closed-loop system becomes the interconnection of $G(s)$ and $\Delta(s)$; i.e., $y = G(s)w$ and $w = \Delta(s)y$. We have that $G(\infty) = 0$, and $\Delta(0) = 0$ for any m_Δ, k_Δ and c_Δ. Therefore, $\lambda(G(0)\Delta(0)) = 0 < 1$. The closed-loop interconnection of $G(s)$ and $\Delta(s)$ is asymptotically stable. Hence, the state feedback control input (4.60) asymptotically stabilizes the system shown in Fig. 4.2, which is described by (4.57) with $w = \Delta(s)y$, where $\Delta(s)$ is given in (4.54). This can also be verified via simulation. Since all of the parameters need to be known in the simulation, we let $m_\Delta = 0.5kg$, $k_\Delta = 0.5N/m$ and $c_\Delta = 0.1Ns/m$. Let the initial state variables of the system (4.57) and (4.53) be $\begin{bmatrix} z_0(0) & z_1(0) & x_1(0) & x_2(0) \end{bmatrix}^T = \begin{bmatrix} -0.5 & 1 & 2 & -1 \end{bmatrix}^T$ and $\begin{bmatrix} x_{\Delta 1}(0) & x_{\Delta 2}(0) \end{bmatrix}^T = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$, respectively. The state trajectories of the uncertain system shown in Fig. 4.2 are shown in Fig. 5.2. It can be observed that all of the state variables converge to zero.

4.6 Conclusion

In this chapter, we provide necessary and sufficient conditions under which a system can be made NI using state feedback control. To be specific, a minimal system without zeros at the origin is state feedback equivalent to an NI system if and only if there exists a nonsingular output transformation such that the resulting system has relative degree less than or equal to two and is weakly minimum phase. We provide a formula for the construction of such a state feedback controller. The OSNI and SSNI state feedback equivalence problems are also investigated. Using the proposed results, we provide an approach for the stabilization of a system with an SNI uncertainty. The proposed results are then illustrated using a physical system example.

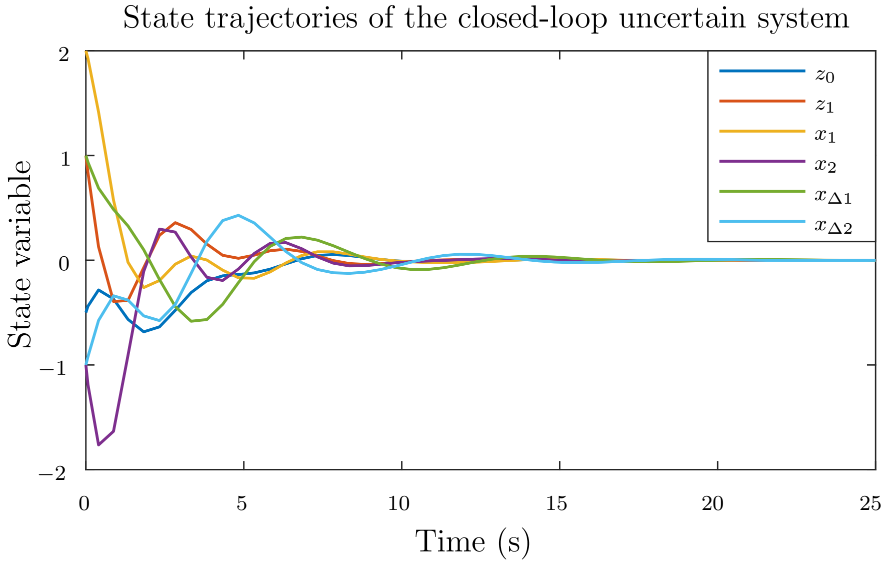


Figure 4.3: The state trajectory of the closed-loop system corresponding to the system shown in Fig. 4.2, which is also described by (4.57) and (4.53). Using the results in Theorems 4.1 and 4.4, we construct the state feedback control input (4.60). Under the input (4.60), all of the state variables converge to zero. This verifies that the system in Fig. 4.2 is asymptotically stabilized by the controller defined in (4.60).

Chapter 5

State Feedback Equivalence to Nonlinear Negative Imaginary Systems

5.1 Introduction

In the previous chapter, the NI state feedback equivalence problem has been investigated for linear systems. Considering the nonlinear nature of most control systems, this chapter investigates the problem of making nonlinear systems nonlinear NI using state feedback, in order to provide a method of stabilization for nonlinear systems of relative degree less than or equal to two. The main contribution of this chapter is in providing sufficient conditions under which an input-affine system can be rendered nonlinear NI or OSNI. Formulas for the corresponding control laws are also provided. Roughly speaking, if an input-affine nonlinear system of relative degree less than or equal to two can be transformed into a normal form (see [66, 76] for a description of the normal form), then there exists state feedback control such that the resulting system is NI or OSNI. If, in addition, the internal dynamics of the normal form are input-to-state stable (ISS), then there exists a state feedback controller that stabilizes the system. Furthermore, such a system with a nonlinear NI plant uncertainty can also be stabilized if in addition there exists a storage function for this system, which is positive definite with respect to a specific subset of the state variables. These results are presented in Sections 5.2, 5.3, 5.5 and are illustrated using an example in Section 5.5.

In Section 5.6, we consider the state feedback equivalence problem for a nonlinear system that is already in a particular normal form. Although the normal form is not as general as the system considered in Section 5.2, the NI state feedback equivalence results in Section 5.6 is still significant for the following reasons. (i). Many control applications can still fit into the special normal form in spite of its specialty. (ii). Necessary and sufficient conditions are provided for the system in Section 5.6 to be state feedback equivalent to a nonlinear NI system while only sufficient conditions are provided for the general input-affine system in Section 5.2.

From the technical point of view, the contribution of this work is in providing an alternative approach to the passivity-based state feedback stabilization results (e.g., [65]) to overcome their limitations by allowing systems with output entries of relative degree two. This broadens the class of systems to which nonlinear NI systems theory is applicable. Although there might be other control approaches that can achieve closed-loop robust stability for the systems of interest, our approach is straightforward to follow and easy to understand due to the physical interpretations of NI systems theory [20]. The contents of this chapter are included in the papers [77, 78].

5.2 State Feedback Equivalence to Nonlinear NI Systems

Consider the following general input-affine nonlinear system:

$$\Sigma: \quad \dot{x} = f(x) + g(x)u, \tag{5.1a}$$

$$y = h(x), \tag{5.1b}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is a locally integrable input and $y \in \mathbb{R}^p$ is the output. The function $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz, $g : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ and $h : \mathbb{R}^n \to \mathbb{R}^p$. Here, f, h and the columns $g^1, \cdots, g^p$ are of class C^∞. We suppose that the vector field f has at least one equilibrium. Then without loss of generality, we can assume $f(0) = 0$ and $h(0) = 0$ after a coordinate shift.

Definition 5.1. *[76] A multivariable nonlinear system of the form (5.1) has vector relative degree $\{r_1, \cdots, r_m\}$ at a point x° if*

1. *$L_g L_f^k h_i(x) = 0$ for all $k < r_i - 1$, for all $1 \le i \le p$ and for all x in a neighbourhood of x°.*
2. *The $p \times p$ matrix*

$$A(x) = \begin{bmatrix} L_g L_f^{r_1-1} h_1(x) \\ \vdots \\ L_g L_f^{r_p-1} h_p(x) \end{bmatrix} \tag{5.2}$$

is nonsingular at $x = x^\circ$*.*

Here $h_i(x)$ *denotes the* i*-th entry of the vector* $h(x) \in \mathbb{R}^p$*.*

Definition 5.2. *[66] The system (5.1) is said to have uniform relative degree* $\{r_1, \cdots, r_p\}$ *if it has vector relative degree* $\{r_1, \cdots, r_p\}$ *at all* $x \in \mathbb{R}^n$*.*

Definition 5.3. *[15] A continuous function* $\alpha : [0, a) \to [0, \infty)$ *is said to belong to class* $\mathcal{K}$ *if it is strictly increasing and* $\alpha(0) = 0$*. It is said to belong to class* $\mathcal{K}_\infty$ *if* $a = \infty$ *and* $\alpha(r) \to \infty$ *as* $r \to \infty$*.*

Definition 5.4. *[15] A continuous function* $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ *is said to belong to class* $\mathcal{KL}$ *if for each fixed* s*, the mapping* $\beta(r, s)$ *belongs to class* $\mathcal{K}$ *with respect to* r *and for each fixed* r*, the mapping* $\beta(r, s)$ *is decreasing with respect to* s *and* $\beta(r, s) \to 0$ *as* $s \to \infty$*.*

Let us consider a nonlinear system of the form (5.1). We aim to use state feedback control to make it a nonlinear NI system or even a nonlinear OSNI system. One condition required for the system (5.1) to be nonlinear NI is that it has relative degree less than or equal to two, as also shown in Chapter 4. Hence, we provide the following definition of relative degree less than or equal to two for nonlinear systems.

Definition 5.5. *A system of the form (5.1) is said to have relative degree less than or equal to two if it has a vector relative degree* $r = \{r_1, \cdots, r_p\}$*, where* $r_i \in \{1, 2\}$ *for all* $i = 1, \cdots, p$*. Without loss of generality, assume the components of the output vector are sorted such that the components in the vector relative degree are in nondecreasing order; i.e.,* $r_i = 1$ *for* $i = 1, 2, \cdots, p_1$ *and* $r_i = 2$ *for* $i = p_1 + 1, p_1 + 2, \cdots, p$*, where* p_1 *is the number of ones in the vector relative degree* r*.*

The paper [66] provides conditions for a system with a vector relative degree to have local and global normal forms. Here, we focus on the specific case that the system has relative degree less than or equal to two.

Lemma 5.1. *(see also [66]) Suppose the system (5.1) has relative degree less than or equal to two at* $x = 0$*. If the distribution*

$$G = span\{g^1, g^2, \cdots, g^p\} \tag{5.3}$$

is involutive, then the system (5.1) can be described locally around $x = 0$ *by the following normal form:*

$$\Sigma: \quad \dot{z} = f^*(z, \xi), \tag{5.4a}$$

$$\dot{\xi}_1 = a_1(z, \xi) + b_1(z, \xi)u, \tag{5.4b}$$

$$\dot{\xi}_2 = \xi_3, \tag{5.4c}$$

$$\dot{\xi}_3 = a_2(z,\xi) + b_2(z,\xi)u, \tag{5.4d}$$

$$y = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{5.4e}$$

where u and y are still the input and output of the system. The vector $[z^T\ \xi^T]^T$ is the new state of the system, where $z \in \mathbb{R}^m$ $(m \geq 0)$ and $\xi = [\xi_1^T\ \xi_2^T\ \xi_3^T]^T$. The vector $\xi_1 \in \mathbb{R}^{p_1}$ contains the state vector entries corresponding to the ones in the vector relative degree. The vector $\xi_2 \in \mathbb{R}^{p_2}$ $(p_2 := p - p_1)$ contains the state vector entries corresponding to the twos in the vector relative degree and $\xi_3 \in \mathbb{R}^{p_2}$ is defined as the derivative of ξ_2. Here, f^*, a_1, b_1, a_2, b_2 are functions of suitable dimensions. Also,

$$a_1(z,\xi) = \begin{bmatrix} L_f h_1(x) \\ \vdots \\ L_f h_{p_1}(x) \end{bmatrix}, \quad a_2(z,\xi) = \begin{bmatrix} L_f^2 h_{p_1+1}(x) \\ \vdots \\ L_f^2 h_p(x) \end{bmatrix},$$

and

$$b_1(z,\xi) = \begin{bmatrix} L_g h_1(x) \\ \vdots \\ L_g h_{p_1}(x) \end{bmatrix}, \quad b_2(z,\xi) = \begin{bmatrix} L_g L_f h_{p_1+1}(x) \\ \vdots \\ L_g L_f h_p(x) \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} b_1(z,\xi) \\ b_2(z,\xi) \end{bmatrix} = A(x)$$

as in (5.2) and is nonsingular for (z,ξ) at $(0,0)$.

Proof. The proof directly follows from Propositions 3.2a and 3.2b in [66] as this lemma is a special case in which the vector relative degree only contains components with values one and two. Note that there are differences between the notation used here and in [66]. □

In the normal form (5.4), the dynamics described by (5.4a) are called the *internal dynamics* of the system. When the output is identically zero, the internal dynamics are called the *zero dynamics* [15, 66, 76]. In the case of system (5.4), y being identically zero implies $\xi = 0$. Therefore, the zero dynamics are described by

$$\dot{z} = f^*(z, 0).$$

Lemma 5.2. *(see [66]) The system (5.1) is globally diffeomorphic to a system having the normal form (5.4) if:*

H1: *the system has uniform relative degree less than or equal to two;*
H2: *the vector fields*

$$X_i^k = ad_{\widetilde{f}}^{k-1}\widetilde{g}_i, \quad 1 \le i \le p, \quad 1 \le k \le r_i$$

are complete;
H3: $[X_i^1, X_j^1] = 0$ *for all* $1 \le i, j \le p$.
Here,

$$\widetilde{f} = f - gA^{-1}(x)\begin{bmatrix} L_f^{r_1}h_1(x) \\ \vdots \\ L_f^{r_p}h_p(x) \end{bmatrix}, \quad \widetilde{g} = gA^{-1}(x).$$

Proof. See Corollary 5.6 in [66]. Note that here the non-singularity of $A(x)$ follows from condition ***H1***, according to Definitions 5.1 and 5.2. □

Lemma 5.3. *Consider the system (5.4) where* $\begin{bmatrix} b_1(z,\xi) \\ b_2(z,\xi) \end{bmatrix}$ *is nonsingular. Then it can be rendered a nonlinear NI system as in Definition 3.1 using the state feedback control law*

$$u = \begin{bmatrix} b_1(z,\xi) \\ b_2(z,\xi) \end{bmatrix}^{-1} \left(v - \begin{bmatrix} a_1(z,\xi) + \left(\frac{\partial V_1(\xi_1)}{\partial \xi_1}\right)^T \\ a_2(z,\xi) + \left(\frac{\partial V_2(\xi_2)}{\partial \xi_2}\right)^T + \lambda\xi_3 \end{bmatrix} \right), \tag{5.5}$$

where $v \in \mathbb{R}^p$ *is the new input,* $V_1(\xi_1)$ *and* $V_2(\xi_2)$ *can be any positive semidefinite functions, and* $\lambda \ge 0$ *is a scalar. Moreover, if* $\lambda > 0$*, then the resulting system is a nonlinear OSNI system as per Definition 3.2 with degree of output strictness* $\epsilon = \min\{1, \lambda\}$*. The storage function of the nonlinear NI (OSNI) system is*

$$V(z,\xi) = \widetilde{V}(\xi) = V_1(\xi_1) + V_2(\xi_2) + \frac{1}{2}\xi_3^T\xi_3. \tag{5.6}$$

Proof. Let $v = [v_1^T \; v_2^T]^T$ in (5.5), where $v_1 \in \mathbb{R}^{p_1}$ and $v_2 \in \mathbb{R}^{p_2}$. With the state feedback control (5.5), the system (5.4) now becomes

$$\dot{z} = f^*(z,\xi), \tag{5.7a}$$

$$\dot{\xi}_1 = v_1 - \left(\frac{\partial V_1(\xi_1)}{\partial \xi_1}\right)^T, \tag{5.7b}$$

$$\dot{\xi}_2 = \xi_3, \tag{5.7c}$$

$$\dot{\xi}_3 = v_2 - \left(\frac{\partial V_2(\xi_2)}{\partial \xi_2}\right)^T - \lambda\xi_3, \tag{5.7d}$$

$$y = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}. \tag{5.7e}$$

The nonlinear NI inequality is satisfied for the resulting system with the positive semidefinite storage function (5.6), which is shown in the following:

$$\begin{aligned}
\dot{V}(z,\xi) - v^T \dot{y} =& \frac{\partial V(z,\xi)}{\partial z}\dot{z} + \frac{\partial V(z,\xi)}{\partial \xi_1}\dot{\xi}_1 + \frac{\partial V(z,\xi)}{\partial \xi_2}\dot{\xi}_2 + \frac{\partial V(z,\xi)}{\partial \xi_3}\dot{\xi}_3 - v^T\dot{y} \\
=& 0 + \frac{\partial V_1(\xi_1)}{\partial \xi_1}\left(v_1 - \left(\frac{\partial V_1(\xi_1)}{\partial \xi_1}\right)^T\right) + \frac{\partial V_2(\xi_2)}{\partial \xi_2}\xi_3 \\
&+ \xi_3^T\left(v_2 - \left(\frac{\partial V_2(\xi_2)}{\partial \xi_2}\right)^T - \lambda\xi_3\right) - v_1^T\left(v_1 - \left(\frac{\partial V_1(\xi_1)}{\partial \xi_1}\right)^T\right) - v_2^T\xi_3 \\
=& -\left(v_1^T - \frac{\partial V_1(\xi_1)}{\partial \xi_1}\right)\left(v_1 - \left(\frac{\partial V_1(\xi_1)}{\partial \xi_1}\right)^T\right) - \lambda\xi_3^T\xi_3 \\
=& -\|\dot{\xi}_1\|^2 - \lambda\|\dot{\xi}_2\|^2 \\
\leq& -\epsilon\|\dot{y}\|^2,
\end{aligned} \tag{5.8}$$

where $\epsilon = \min\{1,\lambda\}$. The system is a nonlinear NI system as per Definition 3.1. Moreover, if $\lambda > 0$, then $\epsilon > 0$. In this case, the system is a nonlinear OSNI system and ϵ is the degree of output strictness of the system. □

Theorem 5.1. *Suppose the system (5.1) has relative degree less than or equal to two at $x = 0$ and the distribution (5.3) is involutive. Then the system (5.1) can be rendered a nonlinear NI (OSNI) system locally around $x = 0$ using the state feedback control*

$$u = A(x)^{-1}\left(v - \begin{bmatrix} L_f^{r_1}h_1(x) \\ \vdots \\ L_f^{r_p}h_p(x) \end{bmatrix} - \begin{bmatrix} \left(\frac{\partial V_1(\xi_1)}{\partial \xi_1}\right)^T \\ \left(\frac{\partial V_2(\xi_2)}{\partial \xi_2}\right)^T + \lambda\xi_3 \end{bmatrix}\right), \tag{5.9}$$

where $A(x)$ is defined in (5.2), $v \in \mathbb{R}^p$ is the new input, $V_1(\xi_1)$ and $V_2(\xi_2)$ can be any positive semidefinite functions, and $\lambda \geq 0$ ($\lambda > 0$) is a scalar. Also, the function (5.6) is a storage function for the resulting nonlinear NI (OSNI) system.

Proof. This result directly follows from Lemmas 5.1 and 5.3. □

Theorem 5.2. *Suppose the system (5.1) satisfies **H1**, **H2** and **H3**. Then the system (5.1) can be globally rendered a nonlinear NI (OSNI) system using the state feedback control (5.9). Also, the function $V(z,\xi)$ defined in (5.6) is a storage function for the resulting nonlinear NI (OSNI) system.*

Proof. See the proof of Theorem 5.1, using Lemma 5.2 instead of Lemma 5.1. □

Considering Condition 2 in Definition 5.1, a system does not have a vector relative degree in general. However, for a system without a vector relative degree, sometimes there exists an output transformation that transforms it into a system with a vector relative degree. This has also been discussed for linear systems in Chapter 4. We generalize Theorem 5.1 by showing that the result is invariant to a nonsingular output transformation. The following lemma can be regarded as a nonlinear version of Lemma 4.2.

Lemma 5.4. *A system with output y and input u is a nonlinear NI (OSNI) system if and only if the system with output $\widetilde{y} = T_y y$ and input $\widetilde{u} = T_y^{-T} u$ is a nonlinear NI (OSNI) system. Here T_y is a nonsingular constant matrix.*

Proof. Consider the following equation:

$$\widetilde{u}^T \dot{\widetilde{y}} = u^T T_y^{-1} T_y \dot{y} = u^T \dot{y}.$$

This implies that the nonlinear NI inequality (3.2) is satisfied for one of these two systems if and only if it is satisfied for the other. Also, we have that

$$\|\dot{\widetilde{y}}\|^2 = \dot{y}^T T_y^T T_y \dot{y} \leq \lambda_{max}(T_y^T T_y) \|\dot{y}\|^2.$$

Therefore, if the system with input u and output y is a nonlinear OSNI system, then there exists a positive semidefinite storage function $V(x)$ such that

$$\dot{V}(x) \leq u^T \dot{y} - \epsilon \|\dot{y}\|^2 \leq \widetilde{u}^T \dot{\widetilde{y}} - \frac{\epsilon}{\lambda_{max}(T_y^T T_y)} \|\dot{\widetilde{y}}\|^2.$$

This implies that the system with input $\widetilde{u}$ and output $\widetilde{y}$ is also a nonlinear OSNI system. The sufficiency part can be proved similarly by considering the inverses of the transformations. □

Lemma 5.5. *Consider a system of the form (5.1) and an output transformation $\widetilde{y} = T_y y$ where $T_y \in \mathbb{R}^{p\times p}$ is a nonsingular constant matrix. If there exists a state feedback control law*

$$u = k_x(x)x + k_u(x)v,$$

under which the system with the new input $v \in \mathbb{R}^p$ is a nonlinear NI (OSNI) system, then the output transformed system; i.e., the system with input u and output $\widetilde{y} = T_y y$, can also be rendered a nonlinear NI (OSNI) system using the state feedback control law

$$u = k_x(x)x + k_u(x) T_y^T \widetilde{v},$$

where $\widetilde{v} \in \mathbb{R}^p$ is the new input.

Proof. According to Lemma 5.5, the system with input v and output y is nonlinear NI (OSNI) if and only if the system with input $\widetilde{v} = T_y^{-T} v$ and output $\widetilde{y} = T_y y$ is nonlinear NI (OSNI). This completes the proof. $\square$

Corollary 5.1. *Suppose the system (5.1) can be output transformed into a system with relative degree less than or equal to two at $x = 0$ using the output transformation*

$$\widetilde{y} = T_y y, \tag{5.10}$$

where $T_y \in \mathbb{R}^{p\times p}$ is a nonsingular constant matrix. Also, suppose the distribution (5.3) is involutive. Then the system (5.1) can be rendered a nonlinear NI (OSNI) system locally around $x = 0$ using state feedback control.

Proof. The proof follows directly from Theorem 5.1 and Lemma 5.5. $\square$

Corollary 5.2. *Suppose the system (5.1) can be output transformed into a system satisfying **H1**, **H2** and **H3** using the output transformation (5.10). Then the system (5.1) can be globally rendered a nonlinear NI (OSNI) system using state feedback control.*

Proof. The proof follows directly from Theorem 5.2 and Lemma 5.5. $\square$

Remark 5.1. *In the state feedback control law (5.9) applied in Theorems 5.1 and 5.2 and Corollaries 5.1 and 5.2, the only choice of λ that makes the resulting system nonlinear NI but not nonlinear OSNI is $\lambda = 0$. Since full state feedback is available, we can simply choose $\lambda > 0$ and render the system a nonlinear OSNI system in order to obtain more strictness. This strictness is helpful in the stabilization of the system.*

5.3 Stabilization of systems having relative degree less than or equal to two

The motivation and objective for addressing the problem of state feedback equivalence to nonlinear NI systems is to stabilize systems that have relative degree less than or equal to two. Based on the dissipativity property of nonlinear NI systems, in the case that a system can be made NI, we can attempt to stabilize it if some additional conditions are satisfied. To be specific, consider the system (5.1). If it is rendered a nonlinear OSNI system with a storage function which is positive definite with respect to ξ, then according to the dissipation inequality (3.4), giving zero input to the system will result the boundedness of the state ξ. Indeed, as will be shown later, the state ξ will converge to zero under zero input. Given the stability of the state ξ, we consider the question of what additional conditions are needed in order to make the state z also stable. First, we need to provide several definitions regarding to the internal dynamics in the normal form of a nonlinear system.

Definition 5.6. *[65, 76] A nonlinear system which has a normal form is said to be (globally) minimum phase if its zero dynamics have a (globally) asymptotically stable equilibrium at the origin.*

In the case of system (5.4), the minimum phase property guarantees that when $\xi = 0$, if z is finite, it will also converge to zero. In other words, if we view the state ξ as the input to the internal dynamics $\dot{z} = f^*(z, \xi)$, then (globally) minimum phase property implies that the internal dynamics are (globally) asymptotically stable with zero input, namely 0-AS (0-GAS) for short. However, examples in [15, 79, 80] show that for systems that are 0-AS (0-GAS), its state may diverge under a bounded input that converges to zero. This phenomena motivated the concept of input-to-state stability (ISS) [79, 80]. As is discussed in [81], the asymptotic stability of the zero dynamics is sometimes insufficient for control design purposes until it is combined with the ISS property of the internal dynamics. This is a common requirement (see for example [82]). Let us now recall the definitions of ISS and locally ISS (LISS) systems.

To avoid introducing new system models, let us consider the system of the form (5.4a). Let us rewrite it in the following as a seperate system:

$$\dot{z} = f^*(z, \xi), \tag{5.11}$$

where ξ acts as the input to this system.

Definition 5.7. *[15, 76, 79, 80, 83] The system (5.11) is said to be input-to-state stable (ISS) if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state $z(t_0)$ and any bounded input $\xi(t)$, the solution $z(t)$ exists for all $t \geq t_0$ and satisfies*

$$\|z(t)\| \leq \beta(\|z(t_0)\|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|\xi(\tau)\|\right). \tag{5.12}$$

Definition 5.8 (Locally Input-to-State Stability)**.** *[84] The system (5.11) is said to be locally input-to-state stable (LISS) if there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ and constants $\rho_z, \rho_\xi > 0$ such that for any initial state $z(t_0)$ with $\|z(t_0)\| \leq \rho_z$ and any bounded input $\xi(t)$ with $\sup_{t_0 \leq \tau \leq t} \|\xi(\tau)\| \leq \rho_\xi$ for all $t \geq t_0$, the solution $z(t)$ exists and satisfies (5.12) for all $t \geq t_0$.*

Lemma 5.6. *[15, 83] Suppose the system (5.11) is ISS. If $\xi(t) \to 0$ as $t \to \infty$, so does $z(t)$.*

Proof. (See also Exercise 4.58 in [15] and its solution manual). We show that for any $\epsilon > 0$, there exists a $T > 0$ such that $\|z(t)\| \leq \epsilon$, $\forall t \geq T$. Since γ is a class $\mathcal{K}$ function, then given $\epsilon > 0$, there exists $\epsilon_1 > 0$ such that $\gamma(\epsilon_1) \leq \frac{\epsilon}{2}$. Since $\lim_{t\to\infty} \xi(t) = 0$, given ϵ_1 there is a $T_1 > 0$ such that $\|\xi(t)\| \leq \epsilon_1$ for $t \geq T_1$. Take $t_1 \geq T_1$. Then for $t \geq t_1$, we have

$$\|z(t)\| \leq \beta(\|z(t_1)\|, t - t_1) + \gamma(\epsilon_1) \leq \beta(c, t - t_1) + \frac{\epsilon}{2},$$

where $c = \|z(t_1)\|$ is a constant. Since β is a class $\mathcal{KL}$ function and $\|z(t_1)\|$ is bounded, then $\|\beta(c, t - t_1)\| \to 0$ as $t \to 0$. There exists a $T_2 > 0$ such that $\|\beta(c, t - t_1)\| \leq \frac{\epsilon}{2}$, for all $t > T_2$. Thus,

$$\|z(t)\| \leq \epsilon, \forall t \geq T = \max\{T_1, T_2\},$$

which shows that $\lim_{t\to\infty} z(t) \to 0$. □

Lemma 5.7. *[15,83] If the system (5.11) is LISS, then there exist constants $\widetilde{\rho}_z, \widetilde{\rho}_\xi > 0$ such that for any initial state $z(t_0)$ with $\|z(t_0)\| \leq \widetilde{\rho}_z$ and any bounded input $\xi(t)$ with $\sup_{t_0 \leq \tau < \infty} \|\xi(\tau)\| \leq \widetilde{\rho}_\xi$ and $\xi(t) \to 0$ as $t \to \infty$, we have $z(t) \to 0$ as $t \to \infty$.*

Proof. Suppose the system is LISS, then there exist ρ_z and ρ_ξ such that for any initial state $z(t_0)$ with $\|z(t_0)\| \leq \rho_z$ and any bounded input $u(t)$ with $\sup_{t_0 \leq \tau \leq t} \|\xi(\tau)\| \leq \rho_\xi$ for all $t \geq t_0$, we have that $z(t)$ exists and satisfies (5.12). Considering that γ is a class $\mathcal{K}$ function, then there exists $\widetilde{\rho}_\xi \leq \rho_\xi$ such that

$$\gamma(\widetilde{\rho}_\xi) \leq \frac{\rho_z}{2}.$$

Also, considering β is a class $\mathcal{KL}$ function, there exists $\widetilde{\rho}_z \leq \rho_z$ such that

$$\beta(\widetilde{\rho}_z, 0) \leq \frac{\rho_z}{2}.$$

Let $\|z(t_0)\| \leq \widetilde{\rho}_z$ and $\sup_{t_0 \leq \tau < \infty} \|\xi(\tau)\| \leq \widetilde{\rho}_\xi$. According to the LISS property, for all $t \geq t_0$, $z(t)$ exists and satisfies

$$\begin{aligned} \|z(t)\| &\leq \beta(\|z(t_0)\|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|\xi(\tau)\|\right) \\ &\leq \beta(\widetilde{\rho}_z, 0) + \gamma(\widetilde{\rho}_\xi) \leq \rho_z. \end{aligned}$$

Since $\|z(t)\| \leq \rho_z$, the property in (5.12) still holds if t_0 is substituted by any $t_1 \geq t_0$. The rest of the proof follows as in the proof of Lemma 5.6, using Definition 5.8 instead of Definition 5.7. $\square$

As is proved in [84], ISS implies 0-GAS and LISS implies 0-AS. We now introduce a new version of the (global) minimum phase property in the following, in which the 0-AS (0-GAS) requirement is replaced by an LISS (ISS) requirement. Hence, the new definition is stricter than Definition 5.6.

Definition 5.9. *A system is said to be (globally) strictly minimum phase if its internal dynamics of the form*

$$\dot{z} = f^*(z, \xi)$$

are LISS (ISS) with respect to ξ.

Theorem 5.3. *After possible output transformation (5.10), suppose the system (5.1) satisfies the following:*

1. *It has relative degree less than or equal to two at $x = 0$.*
2. *The distribution (5.3) is involutive.*
3. *The system (5.1) is strictly minimum phase around $x = 0$.*

Then the system (5.1) can be locally asymptotically stabilized using the state feedback control law

$$u = -A(x)^{-1} \left(\begin{bmatrix} L_f^{r_1} h_1(x) \\ \vdots \\ L_f^{r_p} h_p(x) \end{bmatrix} + \begin{bmatrix} \left(\frac{\partial V_1(\xi_1)}{\partial \xi_1}\right)^T \\ \left(\frac{\partial V_2(\xi_2)}{\partial \xi_2}\right)^T + \lambda \xi_3 \end{bmatrix} \right), \tag{5.13}$$

where $A(x)$ is defined in (5.2), $V_1(\xi_1)$ and $V_2(\xi_2)$ can be any positive definite functions, and $\lambda > 0$ is a scalar.

Proof. Note that the functions $V_1(\xi_1)$ and $V_2(\xi_2)$ are now positive definite. Under the state feedback (5.13), the system becomes the system (5.7) but with $v = 0$. We define the storage function $\widetilde{V}(\xi)$ of this system to be the same as that in (5.6) but with $V_1(\xi_1)$ and $V_2(\xi_2)$ positive definite. Therefore, $\widetilde{V}(\xi)$ is positive definite. The inequality (5.8) implies that

$$\dot{\widetilde{V}}(\xi) \leq -\epsilon \|\dot{y}\|^2,$$

where $\epsilon > 0$. This means that $\dot{\widetilde{V}}(\xi) \equiv 0$ is only possible if $\dot{y} \equiv 0$. This implies that $\dot{\xi}_1 \equiv 0$, $\dot{\xi}_2 \equiv 0$ and therefore $\xi_3 \equiv 0$. According to (5.7b) and (5.7d) and considering that $v = 0$, this is only possible if $\frac{\partial V_1(\xi_1)}{\partial \xi_1} = 0$ and $\frac{\partial V_2(\xi_2)}{\partial \xi_2} = 0$. Since $V_1(\xi_1)$ and $V_2(\xi_2)$ can be any positive definite functions, then we can choose $V_1(\xi_1)$ and $V_2(\xi_2)$

such that $\frac{\partial V_1(\xi_1)}{\partial \xi_1} = 0$ only at $\xi_1 = 0$ and $\frac{\partial V_2(\xi_2)}{\partial \xi_2} = 0$ only at $\xi_2 = 0$. For example, a suitable choice is $V_1(\xi_1) = \frac{1}{2}\xi_1^T \xi_1$ and $V_2(\xi_2) = \frac{1}{2}\xi_2^T \xi_2$. Therefore, with these $V_1(\xi_1)$ and $V_2(\xi_2)$ used in the state feedback, $\dot{y} \equiv 0$ only at $\xi = 0$. Otherwise, $\dot{\widetilde{V}}(\xi)$ cannot remain zero and $\widetilde{V}(\xi)$ will keep decreasing until $\xi = 0$. In both cases, $\xi \to 0$ as $t \to \infty$. Since the system is strictly minimum phase, then the internal dynamics (5.7a) are LISS. According to Lemma 5.7, z will also converge to 0. Therefore, the system is asymptotically stabilized locally around $x = 0$. □

Remark 5.2. *In Theorem 5.3, if the strictly minimum phase requirement in Condition (iii) is replaced by the standard global minimum phase requirement as defined in Definition 5.6, then Theorem 5.3 still holds. This is because 0-GAS implies LISS as is proved in [84], and hence global minimum phase implies strictly minimum phase. Reference [84] includes many equivalences and implications between different stability properties. Here, we only use one of these equivalent properties as a requirement in our stability result.*

Theorem 5.4. *After possible output transformation (5.10), suppose the system (5.1) satisfies **H1**, **H2** and **H3**. Also, suppose the system (5.1) is globally strictly minimum phase. Then the system (5.1) can be globally asymptotically stabilized using the state feedback (5.13).*

Proof. See the proof of Theorem 5.3, using Lemmas 5.2 and 5.6 instead of Lemmas 5.1 and 5.7 for a global result. □

5.4 Controller Synthesis for a system with nonlinear NI Uncertainty

Suppose a system of the form (5.1) has uncertainty that can be modelled as a nonlinear NI system. Denote the uncertainty by H_c. The system model of H_c is

$$H_c : \quad \dot{x}_c = f_c(x_c, u_c), \tag{5.14a}$$

$$y_c = h_c(x_c), \tag{5.14b}$$

where $x_c \in \mathbb{R}^{n_c}$ is the state, $u_c \in \mathbb{R}^p$ is a locally integrable input, and $y_c \in \mathbb{R}^p$ is the output, $f_c : \mathbb{R}^{n_c} \times \mathbb{R}^p \to \mathbb{R}^{n_c}$ is a Lipschitz continuous function and $h_c : \mathbb{R}^{n_c} \to \mathbb{R}^p$ is a class C^1 function. Suppose the system has at least one equilibrium. Then without loss of generality, we can assume $f_c(0,0) = 0$ and $h_c(0) = 0$ after a possible coordinate shift.

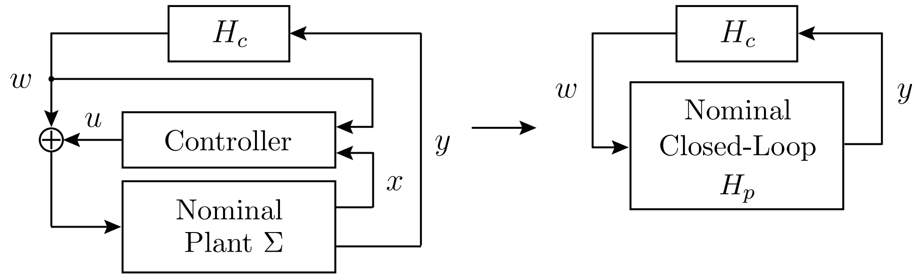


Figure 5.1: A feedback control system. The nominal plant Σ has a plant uncertainty H_c, which can be described as a nonlinear NI system. Under suitable assumptions, we can find a state feedback control input such that the resulting closed-loop system is guaranteed to be asymptotically stable.

When full state information is available, we aim to stabilize the uncertain system using a state feedback controller as shown in the left-hand side (LHS) of Fig. 5.1. The interconnection can be described by the following equations:

$$\dot{x} = f(x) + g(x)(u + w), \tag{5.15a}$$

$$y = h(x), \tag{5.15b}$$

$$\dot{x}_c = f_c(x_c, u_c), \tag{5.15c}$$

$$y_c = h_c(x_c), \tag{5.15d}$$

$$w = y_c, \tag{5.15e}$$

$$u_c = y. \tag{5.15f}$$

Theorem 5.5. *Suppose the nominal plant Σ of the form (5.1) is strictly minimum phase and has relative degree less than or equal to two around $x = 0$ and the distribution (5.3) is involutive. Let $\xi_1 = [y_1^T, \cdots y_{p_1}^T]^T$ and $\xi_2 = [y_{p_1+1}^T, \cdots, y_p^T]^T$ denote the vectors containing the output entries corresponding to the ones and twos in the vector relative degree, respectively. Let $\xi_3 = \dot{\xi}_2$. Suppose that the systems (5.14) is nonlinear NI with storage function $V_c(x_c)$. If there exist positive definite functions $V_1(\xi_1)$ and $V_2(\xi_2)$ such that the function defined as*

$$W(\xi, x_c) = V_1(\xi_1) + V_2(\xi_2) + \frac{1}{2}\xi_3^T\xi_3 + V_c(x_c) - h_c(x_c)^T \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{5.16}$$

is positive definite, then the system (5.15) is locally asymptotically stabilized by the *state feedback control law*

$$u = \left(A(x)^{-1} - I\right) w - A(x)^{-1} \left(\begin{bmatrix} L_f^{r_1} h_1(x) \\ \vdots \\ L_f^{r_p} h_p(x) \end{bmatrix} + \begin{bmatrix} \left(\frac{\partial V_1(\xi_1)}{\partial \xi_1}\right)^T \\ \left(\frac{\partial V_2(\xi_2)}{\partial \xi_2}\right)^T + \lambda \xi_3 \end{bmatrix} \right), \tag{5.17}$$

where $A(x)$ is defined in (5.2), $w \in \mathbb{R}^p$ is the output of the uncertainty H_c, and λ is a positive scalar.

Proof. With the state feedback control (5.17), the nominal plant (5.15a), (5.15b) now becomes the nominal closed-loop system H_p, as shown on the right-hand side (RHS) of Fig. 5.1. The system H_p has a normal form similar to (5.7), where v_1 and v_2 are replaced by w_1 and w_2, respectively. $w = [w_1^T \; w_2^T]^T$. According to Theorem 5.1, the system H_p is a nonlinear OSNI system with the storage function

$$V(z, \xi) = \widetilde{V}(\xi) = V_1(\xi_1) + V_2(\xi_2) + \frac{1}{2}\xi_3^T \xi_3,$$

which is positive semidefinite because $V(z, \xi) = W(\xi, 0)$. This storage function satisfies the nonlinear OSNI inequality:

$$\dot{V}(z, \xi) \leq w^T \dot{y} - \epsilon \|\dot{y}\|^2,$$

where $\epsilon > 0$ quantifies the output strictness of the system. For the interconnection of the nonlinear OSNI system H_p and the nonlinear NI system H_c that is shown on the RHS of Fig. 5.1, we use the function (5.16) as a Lyapunov function. Since H_c is a nonlinear NI uncertainty with storage function $V_c(x_c)$, we have that

$$\dot{V}_c(x_c) \leq u_c^T \dot{y}_c$$

according to Definition 3.1. We show in the following that the stability of this interconnection is guaranteed according to Lyapunov's stability theorem. We have that

$$\begin{aligned} \dot{W}(\xi, x_c) &= \dot{V}(z, \xi) + \dot{V}_c(x_c) - \dot{h}_c(x_c)^T \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} - h_c(x_c)^T \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} \\ &\leq w^T \dot{y} - \epsilon \|\dot{y}\|^2 + u_c^T \dot{h}_c(x_c) - \dot{h}_c(x_c)^T \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} - h_c(x_c)^T \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} \\ &= -\epsilon \|\dot{y}\|^2 \leq 0, \end{aligned}$$

where the equality also uses (5.15b), (5.15d), (5.15e) and (5.15f). Therefore,

$\dot{W}(y, x_2) \equiv 0$ is only possible if $\dot{y} \equiv 0$. In this case, $\dot{\xi}_1 \equiv 0$ and $\dot{\xi}_2 \equiv 0$. This implies that ξ_1, ξ_2 remain constant and $\xi_3 \equiv 0$. Also, according to (5.7b) and (5.7d), we have that $w_1 = \left(\frac{\partial V_1(\xi_1)}{\partial \xi_1}\right)^T$ and $w_2 = \left(\frac{\partial V_2(\xi_2)}{\partial \xi_2}\right)^T$, which now both remain constant. Considering the setting (5.15e) of the interconnection, the output y_c of the nonlinear NI uncertainty H_c now remains constant. For the nonlinear NI uncertainty H_c, its input $u_c = y$ and output $y_c = w$ both remain constant. Moreover, given constant input $[\xi_1^T \ \xi_2^T]^T$ to the system H_c, we get constant output $[\frac{\partial V_1(\xi_1)}{\partial \xi_1} \ \frac{\partial V_2(\xi_2)}{\partial \xi_2}]^T$. We prove in the following that this situation can be avoided. Suppose steady-state input-output relationship of the uncertainty H_c is described by some function $\overline{y} = \kappa(\overline{u})$, where $\overline{y}$ and $\overline{u}$ are the constant output and input respectively in steady state. Then we can always add additional positive definite functions to $V_1(\xi_1)$ and $V_2(\xi_2)$ such that the curve of $[\frac{\partial V_1(\xi_1)}{\partial \xi_1} \ \frac{\partial V_2(\xi_2)}{\partial \xi_2}]^T$ intersects with the curve of $\kappa(\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix})$ only at the origin. That is, the entire closed-loop system cannot remain in a steady state unless $y \equiv 0$. Note that this will not affect the positive definiteness of $W(\xi, x_c)$ because we are adding positive definite functions to $V_1(\xi_1)$ and $V_2(\xi_2)$. Otherwise if y does not identically remain zero, $\dot{W}(\xi, x_c)$ cannot remain zero. It will keep decreasing until $y = 0$ and $x_c = 0$. This means that eventually $\xi \to 0$. Since Σ is strictly minimum phase, z will also converge to zero. Therefore, the closed-loop system as described in (5.15) is asymptotically stabilized locally around $x = 0$. □

Theorem 5.6. *Suppose the nominal plant Σ of the form (5.1) is globally strictly minimum phase and satisfies **H1**, **H2** and **H3**. Let $\xi_1 = [y_1^T, \cdots y_{p_1}^T]^T$ and $\xi_2 = [y_{p_1+1}^T, \cdots, y_p^T]^T$ denote the vectors containing the output entries corresponding to the ones and twos in the vector relative degree, respectively. Let $\xi_3 = \dot{\xi}_2$. Suppose that the system (5.14) is nonlinear NI with storage function $V_c(x_c)$. If there exist positive definite functions $V_1(\xi_1)$ and $V_2(\xi_2)$ such that the function (5.16) is positive definite, then the system (5.15) is globally robustly stabilized by the state feedback control (5.17).*

Proof. See the proof of Theorem 5.5, using Lemmas 5.2 and 5.6 instead of Lemmas 5.1 and 5.7. □

5.5 Illustrative Example

In this section, we illustrate the stabilization process for a system with nonlinear NI uncertainty. We show that if the conditions in Theorems 5.5 and 5.6 are satisfied by choosing a suitable state feedback, the uncertain system can be asymptotically stabilized.

As the conditions for the existence of normal forms provided in Lemma 5.1 and 5.2 are not the main focus of this work, we investigate an uncertain system whose nominal plant is already in its normal form. Consider an uncertain system as shown in the LHS of Fig. 5.1. Suppose the nominal plant Σ has the state-space model:

$$\Sigma: \quad \dot{z} = -z - z^3 + \xi_1^2, \tag{5.18a}$$

$$\dot{\xi}_1 = \sin z + (u_1 + w_1), \tag{5.18b}$$

$$\dot{\xi}_2 = \xi_3, \tag{5.18c}$$

$$\dot{\xi}_3 = \xi_1 + \xi_2^2 + \xi_3 + (u_2 + w_2), \tag{5.18d}$$

$$y = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \tag{5.18e}$$

where $x = [z\ \xi_1\ \xi_2\ \xi_3]^T$, $u = [u_1\ u_2]^T$ and $y = [\xi_1\ \xi_2]^T$ are the state, nominal input and output of the system, respectively. Also, $w = [w_1\ w_2]^T$ is the output of the plant uncertainty. The sum of the nominal input and the uncertainty output; i.e., $u+w$, acts as the actual input of the system (5.18). Here $z, \xi_1, \xi_2, \xi_3, u_1, u_2, w_1, w_2 \in \mathbb{R}$. This system is globally strictly minimum phase because the internal dynamics (5.18a) are ISS with respect to the state ξ (see for example [15, Theorem 4.19], using the Lyapunov function $V(z) = z^2$).

Suppose the plant uncertainty is a nonlinear NI system and has the model:

$$\dot{x}_{c1} = -x_{c1} + u_{c1}, \tag{5.19a}$$

$$\dot{x}_{c2} = -x_{c2}^3 + u_{c2}, \tag{5.19b}$$

$$y_c = \begin{bmatrix} x_{c1} \\ x_{c2} \end{bmatrix}, \tag{5.19c}$$

where $x_c = [x_{c1}\ x_{c2}]^T$, $u_c = [u_{c1}\ u_{c2}]^T$ and $y_c = [x_{c1}\ x_{c2}]^T$ are the state, input and output of the system, respectively. Here, $x_{c1}, x_{c2}, u_{c1}, u_{c2} \in \mathbb{R}$. The system (5.19) is nonlinear NI with the positive definite storage function

$$V_c(x_c) = \frac{1}{2}x_{c1}^2 + \frac{1}{4}x_{c2}^4,$$

which satisfies the nonlinear NI property

$$\dot{V}(x_c) \leq u_c^T \dot{y}_c.$$

The interconnection between the nominal plant (5.18) and the plant uncertainty, as shown in Fig. 5.1, is

$$u_c = y; \quad \text{and} \quad w = y_c. \tag{5.20}$$

We choose positive definite functions $V_1(\xi_1)$ and $V_2(\xi_2)$ to be

$$V_1(\xi_1) = \xi_1^2 \quad \text{and} \quad V_2(\xi_2) = \xi_2^{\frac{4}{3}},$$

which makes the storage function of the entire system, constructed using the formula (5.16), positive definite. The storage function is

$$W(\xi, x_c) = \xi_1^2 + \xi_2^{\frac{4}{3}} + \frac{1}{2}\xi_3^2 + \frac{1}{2}x_{c1}^2 + \frac{1}{4}x_{c2}^4 - \xi_1 x_{c1} - \xi_2 x_{c2}. \tag{5.21}$$

The corresponding state feedback control input, according to (5.17), is

$$u = -\begin{bmatrix} \sin z + 2\xi_1 \\ \xi_1 + \xi_2^2 + \frac{4}{3}\xi_2^{\frac{1}{3}} + 2\xi_3 \end{bmatrix}. \tag{5.22}$$

We show in the following that this state feedback control law stabilizes the system. Under the state feedback (5.22), the nominal plant (5.18) now becomes the nominal closed-loop system H_p, as shown in the RHS of Fig. 5.1. It has the following system model

$$H_p: \quad \dot{z} = -z - z^3 + \xi_1^2, \tag{5.23a}$$

$$\dot{\xi}_1 = -2\xi_1 + w_1, \tag{5.23b}$$

$$\dot{\xi}_2 = \xi_3, \tag{5.23c}$$

$$\dot{\xi}_3 = -\frac{4}{3}\xi_2^{\frac{1}{3}} - \xi_3 + w_2, \tag{5.23d}$$

$$y = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}. \tag{5.23e}$$

Using Lyapunov's direct method, the time derivative of the storage function (5.21) is

$$\begin{aligned} \dot{W}(\xi, x_c) &= -5\xi_1^2 + 6\xi_1 x_{c1} - 2x_{c1}^2 - \xi_3^2 - x_{c2}^6 + 2x_{c2}^3\xi_2 - \xi_2^2 \\ &= -\|\dot{y}\|^2 - (\xi_1 - x_{c1})^2 - (\xi_2 - x_{c2}^3)^2 \\ &\le 0. \end{aligned} \tag{5.24}$$

It can observed from (5.24) that $\dot{W}(\xi, x_c) \equiv 0$ only if $\dot{y} = 0$, $\xi_1 = x_{c1}$ and $\xi_2 = x_{c2}^3$. This implies that $2\xi_1 = w_1 = x_{c1} = \xi_1 = 0$ and $\frac{4}{3}\xi_2^{\frac{1}{3}} = w_2 = x_{c2} = \xi_2^{\frac{1}{3}} = 0$, where (5.20) and (5.23b)-(5.23e) are also used. This implies that $W(\xi, x_c)$ will keep decreasing until $\xi = 0$ and $x_c = 0$. According to Lemma 5.6, z will also converge to zero. We also simulate this uncertain system under the state feedback (5.22). Let the initial state of the nominal plant (5.18) be $x(0) = [z(0)\ \xi_1(0)\ \xi_2(0)\ \xi_3(0)]^T = [10,\ 3,\ -5,\ 7]^T$ and the initial state of the plant uncertainty (5.19) be $x_c(0) = [x_{c1}(0)\ x_{c2}(0)]^T = [-8,\ 2]^T$. Fig. 5.2 shows the state trajectories of the nominal closed-loop system H_p; i.e., the nominal plant (5.18) under the state feedback (5.22). Despite the presence of the nonlinear NI uncertainty H_c as described by (5.19), the plant states still converge to zero.

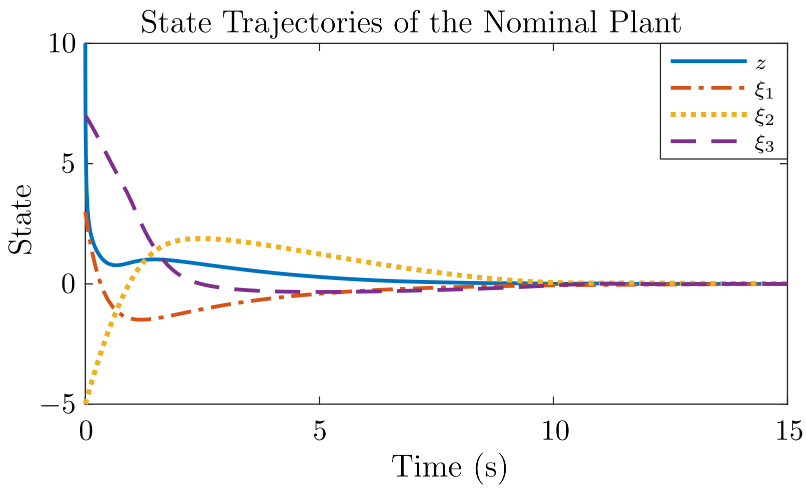


Figure 5.2: State trajectories of the uncertain system (5.18) under the state feedback control (5.22) constructed according to Theorem 5.6. Starting from nonzero initial values, the states of the nominal closed-loop system converge to zero, despite the presence of a nonlinear NI plant uncertainty (5.19).

5.6 NI State Feedback Equivalence for a Class of Nonlinear Systems

In this section, we provide the necessary and sufficient conditions for the following special normal form to be state feedback equivalent to a nonlinear NI system with a positive definite storage function.

$$\dot{z} = A_{11}z + p(y), \tag{5.25a}$$

$$\dot{\xi}_1 = u_1, \tag{5.25b}$$

$$\dot{\xi}_2 = \xi_3, \tag{5.25c}$$

$$\dot{\xi}_3 = u_2, \tag{5.25d}$$

$$y = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \tag{5.25e}$$

where $u = \begin{bmatrix} u_1^T & u_2^T \end{bmatrix}^T \in \mathbb{R}^q$ is the system input, $y \in \mathbb{R}^q$ is the system output and $\begin{bmatrix} z^T & \xi \end{bmatrix}^T$ is the system state. Here, $z \in \mathbb{R}^m$ is a complementary state variable. $\xi = \begin{bmatrix} \xi_1^T & \xi_2^T & \xi_3^T \end{bmatrix}^T$, where $\xi_1 \in \mathbb{R}^{q_1}$ and $\xi_2, \xi_3 \in \mathbb{R}^{q_2}$ $(q_2 := q - q_1)$. $p(0) = 0$. Due to the complexity of nonlinear systems and the fact that NI systems can have relative degree two, only sufficient conditions are derived for the input-affine system (5.1) to be made nonlinear NI using state feedback control. However, for the special normal form (5.25), we can provide necessary and sufficient conditions. Also, we aim to not only make a system nonlinear NI but also let the resulting system have a positive definite storage function. The motivation for guaranteeing the positive definiteness of the storage function of the resulting system is to make it easy to construct a positive definite closed-loop Lyapunov function for feedback controller synthesis (see for example [15] for Lyapunov's direct method). Many passivity feedback equivalence results also require the positive definiteness of the storage function of the resulting systems in order to achieve stability (see for example [65]). With the help of a positive definite storage function for the resulting closed-loop system, the ISS condition required in Section 5.3 is no longer required. The trade-off, on the other hand, is that internal dynamics (see [15, 66, 76] for internal dynamics) of the original system in question should have a constrained nonlinearity. From a different perspective, the results presented in this section can be regarded as a generalization for the NI state feedback equivalence results provided in Chapter 4, in the case that the system (4.1) is already in its normal form.

Besides the necessary and sufficient conditions under which the system (5.25) is state feedback equivalent to a nonlinear NI system or a nonlinear OSNI system, formulas for the corresponding state feedback controllers are also provided. In the case that the system (5.25) has a nonlinear OSNI uncertainty, the NI state feedback equivalence results can be applied to achieve stablization. The process of making a system nonlinear NI and stabilizing it against nonlinear OSNI uncertainty is illustrated in a numerical example.

5.6.1 State Feedback Equivalence

First, let us provide the definition for state feedback equivalence to a nonlinear NI system with positive definition storage function.

Definition 5.10. *A system of the form (5.25) is said to be state feedback equivalent to a nonlinear NI (OSNI) system with a positive definite storage function if there exists a state feedback control law*

$$u = k(x) + v, \tag{5.26}$$

where $v \in \mathbb{R}^q$ is the new input, such that the system with the input v and the output y is a nonlinear NI (OSNI) system with a positive definite storage function.

In the case of system (5.25), it is said to be *weakly minimum phase* if A_{11} is Lyapunov stable (see [15,66,76]). We provide necessary and sufficient conditions for the nonlinear NI state feedback equivalence of the system (5.25).

Theorem 5.7. *Suppose the system (5.25) satisfies* $\det A_{11} \neq 0$. *Then the following statements are equivalent:*

1. *A_{11} is Lyapunov stable.*
2. *The system (5.25) is state feedback equivalent to a nonlinear NI system with a positive definite storage function.*
3. *The system (5.25) is state feedback equivalent to a nonlinear OSNI system with a positive definite storage function.*

Proof. **Statement 1 implies Statements 2 & 3**. Given that $\dot{z} = A_{11}z$ is Lyapunov stable, there exists a storage function $V_1(z) > 0$ under which for any z we have that (see [15])

$$\frac{\partial V_1(z)}{\partial z} A_{11} z \leq 0. \tag{5.27}$$

Let us define a function $V(z, \xi)$ as

$$V(z, \xi) = V_1(\alpha) + V_2(y) + \frac{1}{2}\xi_3^T \xi_3, \tag{5.28}$$

where $V_1(\cdot)$ satisfies (5.27) and $V_2(y)$ can be any class C^1 positive definite function. Here, a new vector α is defined as

$$\alpha = z + A_{11}^{-1} p(y). \tag{5.29}$$

In this case, $V(z, \xi)$ is a positive definite function. Consider the following inequality, which will be used later. According to (5.27), we have that

$$\frac{\partial V(z,\xi)}{\partial z}[A_{11}z + p(y)] = \frac{\partial V_1(\alpha)}{\partial z} A_{11}\alpha = \frac{\partial V_1(\alpha)}{\partial \alpha} A_{11}\alpha \leq 0. \tag{5.30}$$

Let us consider the following change of input. We change the input entries u_1 to be

$$u_1 = v_1 - \left(\frac{\partial V(z,\xi)}{\partial \xi_1}\right)^T, \tag{5.31}$$

and we change the input entries u_2 to be

$$u_2 = v_2 - \left(\frac{\partial V(z,\xi)}{\partial \xi_2}\right)^T - \lambda \xi_3, \tag{5.32}$$

where $\lambda \geq 0$ is an arbitrary constant. Here, $v_1 \in \mathbb{R}^{q_1}$ and $v_2 \in \mathbb{R}^{q_2}$, and $v = \begin{bmatrix} v_1^T & v_2^T \end{bmatrix}^T$ is the new system input. The resulting new system has the model:

$$\dot{z} = A_{11}z + p(y), \tag{5.33a}$$

$$\dot{\xi}_1 = v_1 - \left(\frac{\partial V(z,\xi)}{\partial \xi_1}\right)^T, \tag{5.33b}$$

$$\dot{\xi}_2 = \xi_3, \tag{5.33c}$$

$$\dot{\xi}_3 = v_2 - \left(\frac{\partial V(z,\xi)}{\partial \xi_2}\right)^T - \lambda \xi_3, \tag{5.33d}$$

$$y = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}. \tag{5.33e}$$

Now, let us verify the nonlinear NI property of the new system (5.33). We have that

$$\begin{aligned}
&\dot{V}(z,\xi) - v^T \dot{y} \\
=&\frac{\partial V(z,\xi)}{\partial z}\dot{z} + \frac{\partial V(z,\xi)}{\partial \xi}\dot{\xi} - v^T\dot{y} \\
=&\frac{\partial V(z,\xi)}{\partial z}[A_{11}z + p(y)] + \frac{\partial V(z,\xi)}{\partial \xi_1}\dot{\xi}_1 + \frac{\partial V(z,\xi)}{\partial \xi_2}\dot{\xi}_2 + \frac{\partial V(z,\xi)}{\partial \xi_3}\dot{\xi}_3 - v_1^T\dot{\xi}_1 - v_2^T\dot{\xi}_2 \\
\leq& -\left(v_1^T - \frac{\partial V(z,\xi)}{\partial \xi_1}\right)\dot{\xi}_1 - \left(v_2^T - \frac{\partial V(z,\xi)}{\partial \xi_2}\right)\dot{\xi}_2 + \frac{\partial V(z,\xi)}{\partial \xi_3}\dot{\xi}_3
\end{aligned}$$

$$
\begin{aligned}
&\leq -\left(v_1^T - \frac{\partial V(z,\xi)}{\partial \xi_1}\right)\left(v_1 - \left(\frac{\partial V(z,\xi)}{\partial \xi_1}\right)^T\right) - \left(v_2^T - \frac{\partial V(z,\xi)}{\partial \xi_2}\right)\xi_3 \\
&\quad + \xi_3^T\left(v_2 - \left(\frac{\partial V(z,\xi)}{\partial \xi_2}\right)^T - \lambda \xi_3\right) \\
&= -\left(v_1^T - \frac{\partial V(z,\xi)}{\partial \xi_1}\right)\left(v_1 - \left(\frac{\partial V(z,\xi)}{\partial \xi_1}\right)^T\right) - \lambda \xi_3^T \xi_3 \\
&= -\|\dot{\xi}_1\|^2 - \lambda\|\dot{\xi}_2\|^2 \\
&\leq -\epsilon\|\dot{y}\|^2 \\
&\leq 0,
\end{aligned}
$$

where $\epsilon = \min\{1, \lambda\}$. For all $\lambda \geq 0$, the new system is nonlinear NI. For $\lambda > 0$, we have that $\epsilon > 0$. Hence, the new system is nonlinear OSNI. Note that **Statement 2 implies Statement 3** since nonlinear OSNI systems are also nonlinear NI systems.

Statement 3 implies Statement 1. Note that nonlinear OSNI systems belong to the class of nonlinear NI systems. Therefore, suppose the system is state feedback equivalent to an NI system with a positive definite storage function. This implies that there is a state feedback control law

$$
u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} k_1(z,\xi) + v_1 \\ k_2(z,\xi) + v_2 \end{bmatrix},
$$

such that the resulting system

$$
\begin{aligned}
\dot{z} &= A_{11}z + p(y), \\
\dot{\xi}_1 &= v_1 + k_1(z,\xi), \\
\dot{\xi}_2 &= \xi_3, \\
\dot{\xi}_3 &= v_2 + k_2(z,\xi), \\
y &= \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}
\end{aligned}
$$

with the new input $v = \begin{bmatrix} v_1^T & v_2^T \end{bmatrix}^T$ is a nonlinear NI system with a positive definite storage function $V(z,\xi)$ that satisfies

$$
\frac{\partial V(z,\xi)}{\partial z}\dot{z} + \frac{\partial V(z,\xi)}{\partial \xi}\dot{\xi} \leq v^T \dot{y}. \tag{5.34}
$$

Choose v such that y stays identically at zero. In this case, $\xi = 0$, $\dot{\xi} = 0$ and $p(y) = p(0) = 0$. Then $V(z,0)$, which is a positive definite storage function of z, shows the Lyapunov stability of the system $\dot{z} = A_{11}z$ according to (5.34). $\square$

Remark 5.3. *The system considered in Theorem 5.7 is in normal form where a change of input is already applied. A more general version of the normal form is*

$$\dot{z} = A_{11}z + p(y), \tag{5.35a}$$
$$\dot{\xi}_1 = j_1(z,\xi) + l_1(z,\xi)\widetilde{u}_1, \tag{5.35b}$$
$$\dot{\xi}_2 = \xi_3, \tag{5.35c}$$
$$\dot{\xi}_3 = j_2(z,\xi) + l_2(z,\xi)\widetilde{u}_2, \tag{5.35d}$$
$$y = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \tag{5.35e}$$

where $\begin{bmatrix} l_1(z,\xi) \\ l_2(z,\xi) \end{bmatrix}$ *is nonsingular for all* (z,ξ) *in the region of interest, e.g., near the equilibrium point* $(0,0)$ *or globally. Then, the system (5.35) can be transformed into the form (5.25) using the following change of input:*

$$\begin{bmatrix} \widetilde{u}_1 \\ \widetilde{u}_2 \end{bmatrix} = \begin{bmatrix} l_1(z,\xi) \\ l_2(z,\xi) \end{bmatrix}^{-1} \begin{bmatrix} u_1 - j_1(z,\xi) \\ u_2 - j_2(z,\xi) \end{bmatrix},$$

where $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ *is the new input.*

Remark 5.4. *Although the form (5.25) is not as general as the input-affine system (5.1) in Section 5.2, it still includes many control applications. If we consider the potential state, input and output transformations operated before achieving the normal form (5.25), then a larger set of systems are included. The constraints in the normal form (5.25) exist in the internal dynamics only. Indeed, the constraints are automatically satisfied if the system is linear. Also, the constraints do not rule out any system with zero dimensional internal dynamics; i.e.,* $m = 0$.

5.6.2 Stabilization of Uncertain Systems with nonlinear OSNI uncertainties

Consider a system Σ of the form (5.25) having a system uncertainty $\mathcal{H}_\sigma$ that can be described as a nonlinear OSNI system. We show in the following that such an uncertain system can be asymptotically stabilized using a state feedback controller, as shown on the LHS of Fig. 5.3, that makes the nominal closed-loop system nonlinear OSNI. Suppose the uncertainty $\mathcal{H}_\sigma$ can be described by the following

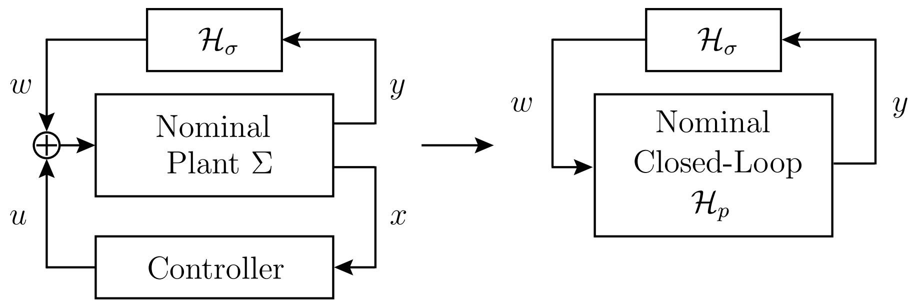


Figure 5.3: A feedback control system. The plant uncertainty $\mathcal{H}_\sigma$ is a nonlinear OSNI system. Under some assumptions, we can find a state feedback controller such that the closed-loop system is asymptotically stable.

equations:

$$H_\sigma: \quad \dot{x}_\sigma = f_\sigma(x_\sigma, u_\sigma),$$
$$y_\sigma = h_\sigma(x_\sigma),$$

where $x_\sigma \in \mathbb{R}^{n_\sigma}$ is the state, $u_\sigma \in \mathbb{R}^q$ is the input, and $y_\sigma \in \mathbb{R}^q$ is the output, $f_\sigma : \mathbb{R}^{n_\sigma} \times \mathbb{R}^q \to \mathbb{R}^{n_\sigma}$ is a Lipschitz continuous function and $h_\sigma : \mathbb{R}^{n_\sigma} \to \mathbb{R}^q$ is a class C^1 function. Suppose the system has at least one equilibrium. Then without loss of generality, we can assume $f_\sigma(0,0) = 0$ and $h_\sigma(0) = 0$ after a possible coordinate shift.

The closed-loop interconnection of the nominal plant Σ and the uncertainty $\mathcal{H}_\sigma$ can be described as follows:

$$\dot{z} = A_{11}z + p(y), \tag{5.36a}$$
$$\dot{\xi}_1 = (u_1 + w_1), \tag{5.36b}$$
$$\dot{\xi}_2 = \xi_3, \tag{5.36c}$$
$$\dot{\xi}_3 = (u_2 + w_2), \tag{5.36d}$$
$$y = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \tag{5.36e}$$
$$\dot{x}_\sigma = f_\sigma(x_\sigma, u_\sigma), \tag{5.36f}$$
$$y_\sigma = h_\sigma(x_\sigma), \tag{5.36g}$$
$$w = y_\sigma, \tag{5.36h}$$
$$u_\sigma = y, \tag{5.36i}$$

where w_1 contains the first q_1 entries of the output w of the uncertainty $\mathcal{H}_\sigma$ and w_2 contains the rest of the output entries; i.e., $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$.

Theorem 5.8. *Consider the interconnection of the uncertain system Σ and the system uncertainty $\mathcal{H}_\sigma$ as described by (5.36) and shown in Fig. 5.3. Suppose* $\det A_{11} \neq 0$ *and A_{11} is Lyapunov stable. Also, suppose the system $\mathcal{H}_\sigma$ is OSNI with the storage function $V_\sigma(x_\sigma)$. If the function, given by*

$$W(z,\xi,x_\sigma) = V(z,\xi) + V_\sigma(x_\sigma) - h_\sigma(x_\sigma)^T \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{5.37}$$

with $V(z,\xi)$ defined in (5.28), is positive definite, then the system (5.36) is asymptotically stabilized by the state feedback control laws

$$u_1 = -\left(\frac{\partial V(z,\xi)}{\partial \xi_1}\right)^T, \tag{5.38}$$

and

$$u_2 = -\left(\frac{\partial V(z,\xi)}{\partial \xi_2}\right)^T - \lambda \xi_3, \tag{5.39}$$

where $\lambda > 0$ is a scalar.

Proof. After applying the state feedback control laws (5.38) and (5.39), the nominal plant Σ becomes the nominal closed-loop system $\mathcal{H}_p$, which is shown on the RHS of Fig. 5.3. The input of the system $\mathcal{H}_p$ is the output w of the uncertainty $\mathcal{H}_\sigma$. As is shown in Section 5.7, under the control laws (5.38) and (5.39), the resulting nominal closed-loop system $\mathcal{H}_p$ is a nonlinear OSNI system with input w. The corresponding positive definite storage function $V(z,\xi)$ satisfies

$$\dot{V}(z,\xi) \leq w^T \dot{y} - \epsilon \|\dot{y}\|^2,$$

where $\epsilon = \min\{1,\lambda\}$, as is shown in the proof of Theorem 5.7. Also, because $\mathcal{H}_\sigma$ is also a nonlinear OSNI system, its storage function $V_\sigma(x_\sigma)$ satisfies

$$\begin{aligned} \dot{V}_\sigma(x_\sigma) &\leq u_\sigma^T \dot{y}_\sigma - \epsilon_\sigma \|\dot{y}_\sigma\|^2 \\ &= y^T \dot{w} - \epsilon_\sigma \|\dot{w}\|^2, \end{aligned}$$

where the equality also uses the equations (5.36h) and (5.36i). Here, the scalar $\epsilon_\sigma > 0$ quantifies the level of output strictness in $\mathcal{H}_\sigma$. Now, we use Lyapunov's direct method to prove the stability of the interconnection shown on the RHS of Fig. 5.3. Let us consider $W(z,\xi,x_\sigma)$ given in (5.37) as the storage function of the interconnection. Taking the time derivative of $W(z,\xi,x_\sigma)$, we have

$$\begin{aligned}
\dot{W}(z,\xi,x_\sigma) &= \dot{V}(z,\xi) + \dot{V}(x_\sigma) - \dot{h}_\sigma(x_\sigma)^T \begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} - h_\sigma(x_\sigma)^T \begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\end{bmatrix}\\
&\le w^T\dot{y} - \epsilon\|\dot{y}\|^2 + u_\sigma^T\dot{y}_\sigma - \epsilon_\sigma\|\dot{y}_\sigma\|^2 - \dot{h}_\sigma(x_\sigma)^T \begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} - h_\sigma(x_\sigma)^T \begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\end{bmatrix}\\
&= w^T\dot{y} - \epsilon\|\dot{y}\|^2 + y^T\dot{w} - \epsilon_\sigma\|\dot{w}\|^2 - \dot{w}^T y - w^T\dot{y}\\
&= -\epsilon\|\dot{y}\|^2 - \epsilon_\sigma\|\dot{w}\|^2\\
&\le 0.
\end{aligned} \tag{5.40}$$

We have that $\dot{W}(z,\xi,x_\sigma) \le 0$ and $\dot{W}(z,\xi,x_\sigma) = 0$ only if $\dot{y} = 0$ and $\dot{w} = 0$. We apply LaSalle's invariance principal in the following. Observing (5.40), it is possible for $\dot{W}(z,\xi,x_\sigma)$ to remain at zero only if $\dot{y} = 0$ and $\dot{w} = 0$ hold over a time interval $[t_a, t_b]$ with $t_a < t_b$; i.e., $\dot{y} \equiv 0$ and $\dot{w} \equiv 0$. And $\dot{y} \equiv 0$ implies that $\dot{\xi}_1 \equiv 0$ and $\dot{\xi}_2 = \xi_3 \equiv 0$. Consider (5.33), with v_1 and v_2 replaced by w_1 and w_2, respectively. We have that

$$\dot{\xi}_1 \equiv 0 \implies w_1 \equiv \left(\frac{\partial V(z,\xi)}{\partial \xi_1}\right)^T;$$

$$\dot{\xi}_2 \equiv 0 \implies w_2 \equiv \left(\frac{\partial V(z,\xi)}{\partial \xi_2}\right)^T.$$

Also, since $\dot{w} \equiv 0$, the system $\mathcal{H}_\sigma$ is in steady state. That is, given constant input $u = \begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}$, the system $\mathcal{H}_\sigma$ also has constant output $y_\sigma = \begin{bmatrix}\left(\frac{\partial V(z,\xi)}{\partial \xi_1}\right)^T\\ \left(\frac{\partial V(z,\xi)}{\partial \xi_2}\right)^T\end{bmatrix}$. Consider that

$$\begin{bmatrix}\left(\frac{\partial V(z,\xi)}{\partial \xi_1}\right)^T\\ \left(\frac{\partial V(z,\xi)}{\partial \xi_2}\right)^T\end{bmatrix} = \left(\frac{\partial V(z,\xi)}{\partial y}\right)^T = \left(\frac{\partial V_1(\alpha)}{\partial y}\right)^T + \left(\frac{\partial V_2(y)}{\partial y}\right)^T,$$

which is a function of z and y. Also, since $\dot{y} \equiv 0$, we have that $\left(\frac{\partial V_2(y)}{\partial y}\right)^T$ is constant. Therefore, the term

$$\frac{\partial V_1(\alpha)}{\partial y} = \frac{\partial V_1(\alpha)}{\partial \alpha}\frac{\partial \alpha}{\partial y} = \frac{\partial V_1(\alpha)}{\partial \alpha}\left(\frac{\partial z}{\partial y} + A_{11}^{-1}\frac{\partial p(y)}{\partial y}\right)$$

must be constant. Since y is constant, then z must also be constant; i.e., $\dot{z} = 0$. We prove in the following that under the situation that $\dot{y} = 0$ and $\dot{z} = 0$, we also have $z = 0$ and $\xi = 0$ if a suitable $V_2(y)$ is chosen. Suppose the steady state input-output relation of the system $\mathcal{H}_\sigma$ can be described by some function $\overline{y} = \kappa(\overline{u})$, where $\overline{y}$

and $\overline{u}$ are the constant output and input in steady state, respectively. Then we can always add additional positive definite function $\widetilde{V}_2(y)$ to $W(z,\xi,x_\sigma)$ such that the curve of $\begin{bmatrix}\left(\frac{\partial V(z,\xi)}{\partial \xi_1}\right)^T \\ \left(\frac{\partial V(z,\xi)}{\partial \xi_2}\right)^T\end{bmatrix}$ intersects with $\kappa(\begin{bmatrix}\xi_1 \\ \xi_2\end{bmatrix})$ only at the origin. In this case, the interconnection shown on the RHS of Fig. 5.3 cannot remain in steady state unless $\xi_1 \equiv 0$ and $\xi_2 \equiv 0$. Therefore, $\xi_3 = \dot{\xi}_2 = 0$ and $y = 0$. This implies that $p(y) = p(0) = 0$ and $\dot{z} = A_{11}z$. Since $\dot{z} = 0$, we have that $z = A_{11}^{-1}\dot{z} = 0$. Therefore, in the case that $\dot{y}$ and $\dot{w}$ are zero, the system state is already at the origin. Otherwise, $W(z,\xi,x_\sigma)$ will keep decreasing until z,ξ,x_σ all become zero. This completes the proof. □

5.6.3 Example

We illustrate the process of making a system of the form (5.25) nonlinear NI with a positive definite storage function using state feedback control. Consider the following system:

$$\dot{z} = -z + \xi_1^2\xi_2, \tag{5.41a}$$
$$\dot{\xi}_1 = u_1, \tag{5.41b}$$
$$\dot{\xi}_2 = \xi_3, \tag{5.41c}$$
$$\dot{\xi}_3 = u_2, \tag{5.41d}$$
$$y = \begin{bmatrix}\xi_1 \\ \xi_2\end{bmatrix}. \tag{5.41e}$$

where $u = \begin{bmatrix}u_1 & u_2\end{bmatrix}^T \in \mathbb{R}^2$ is the system input, $y \in \mathbb{R}^2$ is the system output and $\begin{bmatrix}z & \xi_1 & \xi_2 & \xi_3\end{bmatrix}^T$ is the system state. Here, $z,\xi_1,\xi_2,\xi_3,u_1,u_2 \in \mathbb{R}$. The corresponding A_{11} matrix is -1. According to Theorem 5.7, the system is state feedback equivalent to a nonlinear NI (OSNI) system with a positive definite storage function. We show in the following the corresponding state feedback control and we prove that the resulting system is nonlinear NI. We construct

$$V_1(\alpha) = \alpha^2,$$

where α is defined as

$$\alpha = z + A_{11}^{-1}\xi_1^2\xi_2 = z - \xi_1^2\xi_2,$$

according to (5.29). We also construct

$$V_2(y) = \xi_1^{\frac{4}{3}} + \xi_2^2.$$

Therefore, the function $V(z, \xi)$ as given in (5.28) is

$$V(z,\xi) = (z - \xi_1^2\xi_2)^2 + \xi_1^{\frac{4}{3}} + \xi_2^2 + \frac{1}{2}\xi_3^2 = z^2 - 2z\xi_1^2\xi_2 + \xi_1^4\xi_2^2 + \xi_1^{\frac{4}{3}} + \xi_2^2 + \frac{1}{2}\xi_3^2. \quad (5.42)$$

We apply the control laws

$$u_1 = v_1 - \left(\frac{\partial V(z,\xi)}{\partial \xi_1}\right)^T = v_1 + 4z\xi_1\xi_2 - 4\xi_1^3\xi_2^2 - \frac{4}{3}\xi_1^{\frac{1}{3}}, \quad (5.43)$$

and

$$u_2 = v_2 - \left(\frac{\partial V(z,\xi)}{\partial \xi_2}\right)^T - \lambda\xi_3 = v_2 + 2z\xi_1^2 - 2\xi_1^4\xi_2 - 2\xi_2 - \xi_3, \quad (5.44)$$

where $v = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$ is the new input. We choose $\lambda = 1$ in (5.44). The resulting system now becomes

$$\dot{z} = -z + \xi_1^2\xi_2, \quad (5.45a)$$
$$\dot{\xi}_1 = v_1 + 4z\xi_1\xi_2 - 4\xi_1^3\xi_2^2 - \frac{4}{3}\xi_1^{\frac{1}{3}}, \quad (5.45b)$$
$$\dot{\xi}_2 = \xi_3, \quad (5.45c)$$
$$\dot{\xi}_3 = v_2 + 2z\xi_1^2 - 2\xi_1^4\xi_2 - 2\xi_2 - \xi_3, \quad (5.45d)$$
$$y = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}. \quad (5.45e)$$

We prove that this system is nonlinear NI (OSNI) with the positive definite storage function (5.42). Taking the derivative of $V(z, \xi)$, we get

$$\begin{aligned}
&\dot{V}(z,\xi) \\
&= 2z\dot{z} - 2\dot{z}\xi_1^2\xi_2 - 4z\xi_1\xi_2\dot{\xi}_1 - 2z\xi_1^2\dot{\xi}_2 + 4\xi_1^3\xi_2^2\dot{\xi}_1 + 2\xi_1^4\xi_2\dot{\xi}_2 + \frac{4}{3}\xi_1^{\frac{1}{3}}\dot{\xi}_1 + 2\xi_2\dot{\xi}_2 + \xi_3\dot{\xi}_3 \\
&= (2z - 2\xi_1^2\xi_2)\dot{z} + (4\xi_1^3\xi_2^2 - 4z\xi_1\xi_2 + \frac{4}{3}\xi_1^{\frac{1}{3}})\dot{\xi}_1 + (2\xi_1^4\xi_2 + 2\xi_2 - 2z\xi_1^2)\dot{\xi}_2 + \xi_3\dot{\xi}_3 \\
&= -2\dot{z}^2 + (v_1 - \dot{\xi}_1)\dot{\xi}_1 + \xi_3(v_2 - \xi_3 - \dot{\xi}_3) + \xi_3\dot{\xi}_3 \\
&= -2\dot{z}^2 + v_1\dot{\xi}_1 - \dot{\xi}_1^2 + v_2\xi_3 - \xi_3^2 \\
&= -2\dot{z}^2 + v_1\dot{\xi}_1 + v_2\dot{\xi}_2 - \dot{\xi}_1^2 - \dot{\xi}_2^2 \\
&\leq v^T\dot{y} - \|\dot{y}\|^2.
\end{aligned}$$

Therefore, the system (5.45) is nonlinear OSNI and hence also nonlinear NI.

Now we consider an uncertainty for the system (5.41). Suppose the uncertainty $\mathcal{H}_\sigma$ has the following model:

$$\dot{x}_{\sigma 1} = -x_{\sigma 1}^3 + u_{\sigma 1}, \tag{5.46a}$$

$$\dot{x}_{\sigma 2} = -x_{\sigma 2} + u_{\sigma 2}, \tag{5.46b}$$

$$y_u = \begin{bmatrix} x_{\sigma 1} \\ x_{\sigma 2} \end{bmatrix}, \tag{5.46c}$$

where $x_\sigma = [x_{\sigma 1} \; x_{\sigma 2}]^T$, $u_\sigma = [u_{\sigma 1} \; u_{\sigma 2}]^T$ and $y_\sigma = [x_{\sigma 1} \; x_{\sigma 2}]^T$ are the state, input and output of the system, respectively. Here, $x_{\sigma 1}, x_{\sigma 2}, u_{\sigma 1}, u_{\sigma 2} \in \mathbb{R}$. The system (5.46) is nonlinear OSNI with the positive definite storage function

$$V_\sigma(x_\sigma) = \frac{1}{4}x_{\sigma 1}^4 + \frac{1}{2}x_{\sigma 2}^2,$$

which satisfies the nonlinear OSNI property since

$$\begin{aligned}
\dot{V}(x_\sigma) &= x_{\sigma 1}^3 \dot{x}_{\sigma 1} + x_{\sigma 2}\dot{x}_{\sigma 2} \\
&= (u_{\sigma 1} - \dot{x}_{\sigma 1})\dot{x}_{\sigma 1} + (u_{\sigma 2} - \dot{x}_{\sigma 2})\dot{x}_{\sigma 2} \\
&= u_{\sigma 1}\dot{x}_{\sigma 1} + u_{\sigma 2}\dot{x}_{\sigma 2} - \dot{x}_{\sigma 1}^2 - \dot{x}_{\sigma 2}^2 \\
&= u_\sigma^T \dot{y}_\sigma - \|\dot{y}_\sigma\|^2.
\end{aligned}$$

Suppose the uncertain system has a structure shown on the LHS of Fig. 5.3, we apply the state feedback control laws (5.43) and (5.44) to the system, with w_1 and w_2 replacing v_1 and v_2, respectively. Then the entire system becomes the system shown on the RHS of Fig. 5.3. We show in the following that this interconnection is asymptotically stable. We construct the storage function of the interconnection using the formula given in (5.37):

$$W(z, \xi, x_\sigma) = V(z, \xi) + V_\sigma(x_\sigma) - \xi_1 x_{\sigma 1} - \xi_2 x_{\sigma 2}.$$

It can be verified that this storage function is positive definite. Then

$$\begin{aligned}
\dot{W}(z, \xi, x_\sigma) &= \dot{V}(z, \xi) + \dot{V}_\sigma(x_\sigma) - \dot{\xi}_1 x_{\sigma 1} - \dot{\xi}_2 x_{\sigma 2} - \xi_1 \dot{x}_{\sigma 1} - \xi_2 \dot{x}_{\sigma 2} \\
&= -2\dot{z}^2 + w_1\dot{\xi}_1 + w_2\dot{\xi}_2 - \dot{\xi}_1^2 - \dot{\xi}_2^2 + u_{\sigma 1}\dot{x}_{\sigma 1} + u_{\sigma 2}\dot{x}_{\sigma 2} - \dot{x}_{\sigma 1}^2 - \dot{x}_{\sigma 2}^2 - \dot{\xi}_1 x_{\sigma 1} \\
&\quad - \dot{\xi}_2 x_{\sigma 2} - \xi_1 \dot{x}_{\sigma 1} - \xi_2 \dot{x}_{\sigma 2} \\
&= -2\dot{z}^2 + x_{\sigma 1}\dot{\xi}_1 + x_{\sigma 2}\dot{\xi}_2 - \dot{\xi}_1^2 - \dot{\xi}_2^2 + \xi_1\dot{x}_{\sigma 1} + \xi_2\dot{x}_{\sigma 2} - \dot{x}_{\sigma 1}^2 - \dot{x}_{\sigma 2}^2 - \dot{\xi}_1 x_{\sigma 1} \\
&\quad - \dot{\xi}_2 x_{\sigma 2} - \xi_1 \dot{x}_{\sigma 1} - \xi_2 \dot{x}_{\sigma 2}
\end{aligned}$$

$$
\begin{aligned}
&= -2\dot{z}^2 - \dot{\xi}_1^2 - \dot{\xi}_2^2 - \dot{x}_{\sigma 1}^2 - \dot{x}_{\sigma 2}^2 \\
&\leq 0.
\end{aligned}
$$

Using LaSalle's invariance principal, $\dot{W}(z, \xi, x_\sigma)$ remains at zero only if $\dot{z}, \dot{\xi}_1, \dot{\xi}_2, \dot{x}_{\sigma 1}, \dot{x}_{\sigma 2} \equiv 0$. In this case, we have that

$$
\begin{aligned}
\dot{z} &= -z + \xi_1^2 \xi_2 \equiv 0, \\
\dot{\xi}_1 &= v_1 + 4z\xi_1\xi_2 - 4\xi_1^3\xi_2^2 - \frac{4}{3}\xi_1^{\frac{1}{3}} \equiv 0, \\
\dot{\xi}_2 &= \xi_3 \equiv 0 \implies \dot{\xi}_3 = v_2 + 2z\xi_1^2 - 2\xi_1^4\xi_2 - 2\xi_2 - \xi_3 \equiv 0, \\
\dot{x}_{\sigma 1} &= -x_{\sigma 1}^3 + u_{\sigma 1} \equiv 0, \\
\dot{x}_{\sigma 2} &= -x_{\sigma 2} + u_{\sigma 2} \equiv 0.
\end{aligned}
$$

Solving the equations, we have $z, \xi_1, \xi_2, \xi_3, x_{\sigma 1}, x_{\sigma 2} = 0$. Otherwise, $W(z, \xi, x_\sigma)$ will keep decreasing until z, ξ, x_σ all become zero. Therefore, the interconnection is asymptotically stable.

We also illustrate that the uncertain system is asymptotically stabilized via simulation. Let the initial state of the nominal plant be $x(0) = \begin{bmatrix} z(0) & \xi_1(0) & \xi_2(0) & \xi_3(0) \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 & 2 \end{bmatrix}$ and the initial state of the uncertainty be zero. It is shown in Fig. 5.4 that despite the presence of the uncertainty, the system (5.41) is asymptotically stabilized by the state feedback control (5.43) and (5.44).

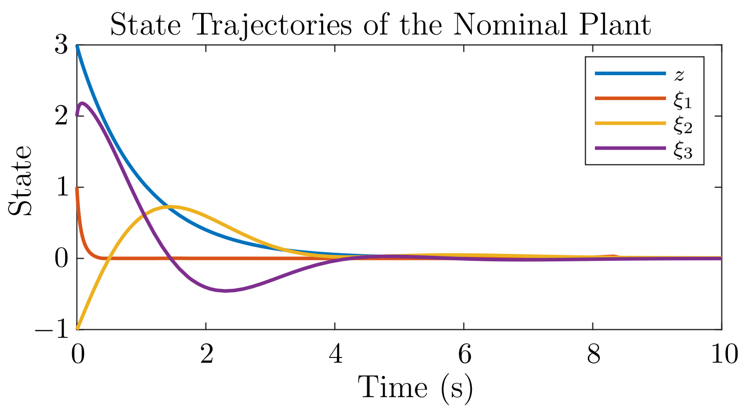


Figure 5.4: State trajectories of the uncertain system (5.41). The uncertainty of the system is described by (5.46) and stability is achieved using the state feedback control (5.43) and (5.44) constructed according to Theorem 5.8. Starting from nonzero initial values, the state variables of the nominal closed-loop system converge to zero, despite the presence of nonlinear OSNI plant uncertainty (5.46).

5.7 Conclusion

In this chapter, we investigated the problem of state feedback equivalence to a nonlinear NI (OSNI) system in two cases. First, for an input-affine nonlinear system, we provided sufficient conditions under which it can be made NI. Using the dissipativity of NI systems, we proposed state feedback controllers that can stabilize nonlinear systems by making them NI (OSNI). For a system with a nonlinear NI uncertainty, we use the NI state feedback equivalence results to asymptotically stabilize it.

We also provided the necessary and sufficient conditions for a special normal form to be state feedback equivalent to a nonlinear NI system with a positive definite storage function. This result is also used in addressing a stabilization problem for a system with a nonlinear OSNI uncertainty. Examples are provided to illustrate the results for both cases.

Chapter 6

Output Feedback Consensus for Networked Nonlinear Negative Imaginary Systems

6.1 Introduction

In this chapter, we investigate the output feedback consensus problem for networked nonlinear NI systems. Cooperative control for multi-agent systems has been a highly active research area over the past two decades [85]. This control paradigm enables multiple systems to perform team missions and has been applied to autonomous vehicles including mobile robots, unmanned air vehicles (UAVs), autonomous underwater vehicles (AUVs) and other applications [86]. Consensus, that is convergence to an agreement through information sharing among agents, is one of the most important problems in the area of cooperative control [87]. Consensus algorithms were first studied for first-order dynamics (see [88, 89], etc) and then extended to second-order systems (see [90–92], etc).

As NI properties arise naturally in autonomous vehicles and a wide variety of other applications [93], consensus problems were investigated in [32, 33] for networked NI systems. Using NI systems theory, the consensus algorithms in [33] only require outputs to be shared among agents, and consensus is guaranteed if a simple DC gain condition is satisfied. The theoretical results in [33] have already been applied to real-world cooperative control problems (see [35, 94, 95]).

Considering that many control systems have a nonlinear nature, we aim to find a control framework in order to achieve output feedback consensus for networked nonlinear NI systems. The output consensus problem is investigated for three types of networked systems: (i) We first consider a network consisting of multiple identical nonlinear NI systems. (ii) Then we consider a network of multiple heterogeneous NI systems. (iii) Finally, we consider a network of multiple heterogeneous NI systems with free body motion. The systems of interest in the cases (i) and (ii) are nonlinear NI with positive definite storage functions, which means, they do not have free body motion. Case (iii) allows for systems in a network to be nonlinear NI with positive semidefinite storage functions. This allows them to have free body motion. Indeed, it can be observed that Case (i) can be regarded as a subset of Case (ii) and Case (ii) can be regarded as a subset of Case (iii). However, due to the relevance of the cases (i) and (ii), we still present them as different control frameworks, which can be used as alternative approaches to achieve output consensus. The results in this chapter have been published in the papers [61, 62, 96].

6.2 Problem statement

Consider N heterogeneous nonlinear plants H_{pi} $(i = 1, 2, \cdots, N)$ with the system model

$$H_{pi}: \quad \begin{aligned} \dot{x}_{pi}(t) &= f_{pi}(x_{pi}(t), u_{pi}(t)), \\ y_{pi}(t) &= h_{pi}(x_{pi}(t)), \end{aligned} \tag{6.1}$$

where $x_{pi}(t) \in \mathbb{R}^{n_{pi}}$ is the state, $u_{pi}(t) \in \mathbb{R}^m$ is the input, and $y_{pi}(t) \in \mathbb{R}^m$ is the output, $f_{pi} : \mathbb{R}^{n_{pi}} \times \mathbb{R}^m \to \mathbb{R}^{n_{pi}}$ are Lipschitz continuous functions and $h_{pi} : \mathbb{R}^{n_{pi}} \to \mathbb{R}^m$ are class C^1 functions. These systems operate independently in parallel, as shown in Fig. 6.1. The subscript "p" indicates that this system will play the role of a plant in what follows.

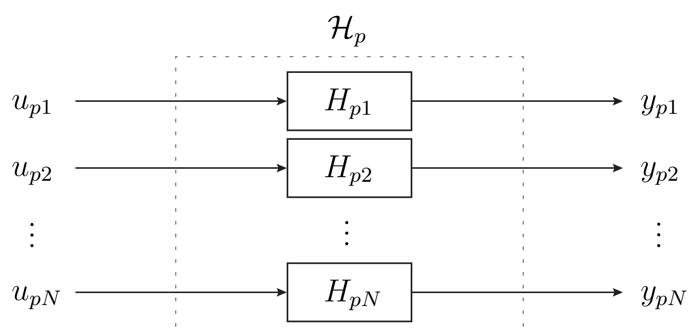


Figure 6.1: A plant $\mathcal{H}_p$ consisting of N independent systems H_{pi} $(i = 1, 2, \cdots, N)$ of the form (6.1), with independent inputs and outputs combined as the input and output of the networked system $\mathcal{H}_p$.

To describe the networked plants using a single system model, we combine their states, inputs and outputs respectively as vectors $X_p = [x_{p1}^T, \cdots, x_{pN}^T]^T \in \mathbb{R}^{n_p}$, where $n_p = \sum_{i=1}^{N} n_{pi}$, $U_p = [u_{p1}^T, \cdots, u_{pN}^T]^T \in \mathbb{R}^{Nm}$ and $Y_p = [y_{p1}^T, \cdots, y_{pN}^T]^T \in \mathbb{R}^{Nm}$. We denote by $\mathcal{H}_p$ the network of nonlinear NI plants consisting of systems H_{pi} ($i = 1, 2, \cdots, N$). The system $\mathcal{H}_p$ is formally defined by the following equations:

$$\mathcal{H}_p: \; \dot{X}_p = \begin{bmatrix} f_{p1}(x_{p1}, u_{p1}) \\ \vdots \\ f_{pN}(x_{pN}, u_{pN}) \end{bmatrix}, \; Y_p = \begin{bmatrix} h_{p1}(x_{p1}) \\ \vdots \\ h_{pN}(x_{pN}) \end{bmatrix}. \tag{6.2}$$

We would like to find an output feedback control framework so that the outputs of the systems H_{pi} are synchronized to the same limit trajectory. Now we give a definition of output feedback consensus for a network of systems H_{pi} ($i = 1, 2, \cdots, N$) of the form (6.1).

Definition 6.1. *A distributed output feedback control law achieves output feedback consensus for a network of systems if $\|y_{pi}(t) - y_{pj}(t)\| \to 0$ as $t \to +\infty$, $\forall i, j \in \{1, 2, \cdots, N\}$.*

We aim to find control frameworks for the output feedback consensus of three types of networked system: networked identical nonlinear NI systems with positive definite storage functions, networked heterogeneous nonlinear NI systems with positive definite storage functions and networked heterogeneous nonlinear NI systems with positive semidefinite storage functions.

6.3 Output feedback consensus for networked identical nonlinear NI systems with positive definite storage functions

In the case that all the plants in the network are identical, they share the same system model. That is, the system model (6.1) is the same for all the plants; i.e., $f_{pi} = f_{pj}$ and $h_{pi} = h_{pj}$ for all $i, j \leq N$. Suppose the information exchange between agents in the network can be modelled as an undirected connected graph $\mathcal{G}$.

To achieve output feedback consensus, we apply N identical linear OSNI systems as controllers, which have the transfer function matrix $R(s)$ and the following minimal realization.

$$\begin{aligned} \dot{x}_{ci} &= Ax_{ci} + Bu_{ci}, \\ y_{ci} &= Cx_{ci}, \end{aligned} \tag{6.3}$$

where $x_{ci} \in \mathbb{R}^q$ is the state, $u_{ci} \in \mathbb{R}^m$ is the input and $y_{ci} \in \mathbb{R}^m$ is the output. Here, the subscript "c" indicates that the system will play the role of a controller in what follows. We choose the controller such that $\det A \neq 0$. We consider the control framework shown in Fig. 6.2 to achieve consensus, where $\mathcal{L}_N$ denotes the Laplacian matrix of the graph $\mathcal{G}$.

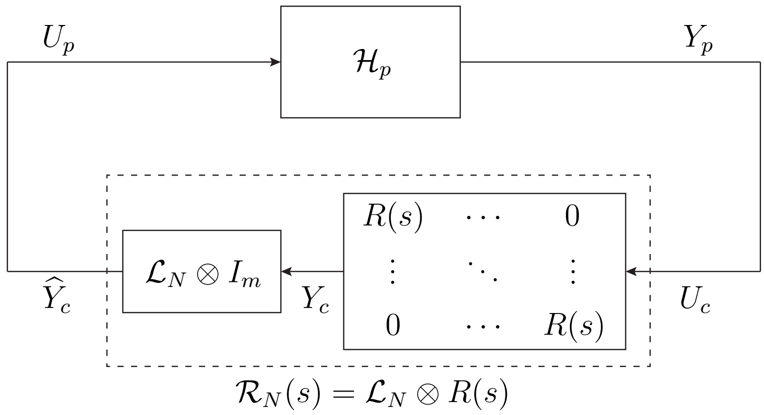


Figure 6.2: Networked identical nonlinear NI systems with identical OSNI controllers connected according to a specified graph topology.

We show in the following that the system $\mathcal{H}_p$ in Fig. 6.2 is a nonlinear NI system if each individual plant of the form (6.1) is nonlinear NI.

Lemma 6.1. *Consider the parallel combination of multiple nonlinear NI plants as shown in Fig. 6.1. If for all $i \leq N$, the plant H_{pi} is nonlinear NI with a positive definite storage function, then the system $\mathcal{H}_p$ is also a nonlinear NI system with a positive definite storage function.*

Proof. It follows from the assumption of the lemma that for all $i \leq N$, there exists a positive definite storage function $V_{pi}(x_{pi})$ such that $\dot{V}_{pi}(x_{pi}) \leq u_{pi}^T \dot{y}_{pi}$. For the system $\mathcal{H}_p$, we define its storage function as $V_p(X_p) := \sum_{i=1}^N V_{pi}(x_{pi}) > 0$. Then

$$\dot{V}_p(X_p) = \sum_{i=1}^{N} \dot{V}_{pi}(x_{pi}) \leq \sum_{i=1}^{N} u_{pi}^T \dot{y}_{pi} = U_P^T \dot{Y}_p, \tag{6.4}$$

which satisfies Definition 3.1. Therefore, $\mathcal{H}_p$ is a nonlinear NI system with a positive definite storage function. $\square$

Then we show that the controller $\mathcal{L}_N \otimes R(s)$ has an OSNI-like property.

Lemma 6.2. *Suppose $R(s)$ with the minimal realization (6.3) is OSNI. Then the system $\mathcal{L}_n \otimes R(s)$ shown in Fig. 6.2 has an OSNI-like property in the sense that there exists a storage function V_c such that*

$$\dot{V}_c \leq U_c \dot{\widehat{Y}}_c - \frac{1}{2}\epsilon \sum_{(v_i,v_j)\in\mathcal{E}} \|\dot{y}_{ci} - \dot{y}_{cj}\|^2, \tag{6.5}$$

where $\epsilon > 0$ is the level of output strictness of $R(s)$.

Proof. According to Definition 3.2, there exists a storage function $V_{ci}(x_{ci})$ for the system (6.3) such that $\dot{V}_{ci}(x_{ci}) \leq u_{ci}^T \dot{y}_{ci} - \epsilon \|\dot{y}_{ci}\|^2$. For any two identical controllers with the state-space model (6.3), denote their states, inputs and outputs by x_{ci}, x_{cj}, u_{ci}, u_{cj}, y_{ci}, y_{cj}, respectively. Then the system described by the model

$$\begin{aligned}\frac{d}{dt}(x_{ci} - x_{cj}) &= A(x_{ci} - x_{cj}) + B(u_{ci} - u_{cj}),\\ y_{ci} - y_{cj} &= C(x_{ci} - x_{cj}),\end{aligned}$$

also has the transfer function matrix $R(s)$ and hence is also OSNI. Therefore, we have that

$$\dot{V}_{ci}(x_{ci} - x_{cj}) \leq (u_{ci} - u_{cj})^T (\dot{y}_{ci} - \dot{y}_{cj}) - \epsilon \|\dot{y}_{ci} - \dot{y}_{cj}\|^2.$$

Note that V_{ci} is identical for all $i \leq N$. Now for the networked controllers $\mathcal{L}_N \otimes R(s)$, we construct its storage function V_c as follows:

$$V_c := \frac{1}{2} \sum_{(v_i, v_j) \in \mathcal{E}} V_{ci}(x_{ci} - x_{cj}).$$

We introduce the coefficient $\frac{1}{2}$ because the each edge is counted twice in the summation. Now taking the time derivative of V_c, we have that

$$\begin{aligned}\dot{V}_c &= \frac{1}{2} \sum_{(v_i, v_j) \in \mathcal{E}} \dot{V}_{ci}(x_{ci} - x_{cj})\\ &\leq \frac{1}{2} \sum_{(v_i, v_j) \in \mathcal{E}} (u_{ci} - u_{cj})^T (\dot{y}_{ci} - \dot{y}_{cj}) - \frac{1}{2}\epsilon \sum_{(v_i, v_j) \in \mathcal{E}} \|\dot{y}_{ci} - \dot{y}_{cj}\|^2\\ &= \sum_{i=1}^{N} u_{ci} \sum_{j=1}^{N} a_{ij} (\dot{y}_{ci} - \dot{y}_{cj}) - \frac{1}{2}\epsilon \sum_{(v_i, v_j) \in \mathcal{E}} \|\dot{y}_{ci} - \dot{y}_{cj}\|^2\\ &= U_c^T \dot{\hat{Y}}_c - \frac{1}{2}\epsilon \sum_{(v_i, v_j) \in \mathcal{E}} \|\dot{y}_{ci} - \dot{y}_{cj}\|^2,\end{aligned}$$

where $\mathcal{A} = [a_{ij}]$ is the adjacency matrix of the graph $\mathcal{G}$ (see Section 2.7). □

Before we present the main result, we suppose that the networked system satisfy the following assumption.

Assumption 6.1. *Consider the parallel combination $\mathcal{H}_p$ of the networked identical plants. Given an input U_p to the system $\mathcal{H}_p$, if the corresponding output of $\mathcal{H}_p$ is such that*

$$Y_p = \mathbf{1}_N \otimes (\rho I_m) + \overline{Y}_D,$$

where $\rho \in \mathbb{R}$ *is a constant scalar and* $\overline{Y}_D \in \mathbb{R}^{Nm}$ *is a nonzero constant vector, then* $U_p \neq (\mathcal{L}_N \otimes R(0))\overline{Y}_D$.

Assumption 6.1 considers the case in which the outputs of the networked plants have constant differences. In this case, the control inputs u_{pi} generated by the controller $\mathcal{L}_N \otimes R(s)$ will also have constant differences between each other. Assumption 6.1 rules out the case that given the corresponding control input U_p to the open-loop interconnection of $\mathcal{H}_p$ and $\mathcal{L}_N \otimes R(s)$, the system still produces the same signal; i.e., $\widehat{Y}_c = U_p$.

Theorem 6.1. *Consider an undirected and connected graph* $\mathcal{G}$ *that models the communication links for a network of identical systems with the model* H_{pi} *given in (6.1) where* $f_{pi} = f_{pj}$ *and* $h_{pi} = h_{pj}$ *for all* $i, j \leq N$. *Suppose the system* H_{pi} *is a nonlinear NI system with a positive definite storage function. Consider an OSNI transfer function matrix* $R(s)$ *with the minimal realization (6.3) where* $\det A \neq 0$. *Suppose* H_{pi} *satisfies Assumptions 3.3 and 3.4. Also, suppose Assumption 6.1 is satisfied, and the function*

$$\widehat{W} = V_p + V_c - Y_p^T \widehat{Y}_c \tag{6.6}$$

is positive definite, where $V_p(X_p) = \sum_{i=1}^N V_{pi}(x_{pi}) > 0$, $V_c = \frac{1}{2}\sum_{(v_i,v_j)\in\mathcal{E}} V_{ci}(x_{ci} - x_{cj})$ *and* $\widehat{Y}_c = \mathcal{L}_N \otimes R(s) Y_p$. *Then output feedback consensus is achieved via the control law*

$$U_p = (\mathcal{L}_N \otimes R(s))Y_p$$

as shown in Fig. 6.2.

Proof. Taking the time derivative of $\widehat{W}$ in (6.6), we have that

$$\begin{aligned}
\dot{\widehat{W}} &= \dot{V}_p + \dot{V}_c - \dot{Y}_p^T \widehat{Y}_c - Y_p^T \dot{\widehat{Y}}_c \\
&\leq U_P^T \dot{Y}_p + U_c \dot{\widehat{Y}}_c - \frac{1}{2}\epsilon \sum_{(v_i,v_j)\in\mathcal{E}} \|\dot{y}_{ci} - \dot{y}_{cj}\|^2 - \dot{Y}_p^T \widehat{Y}_c - Y_p^T \dot{\widehat{Y}}_c \\
&= -\frac{1}{2}\epsilon \sum_{(v_i,v_j)\in\mathcal{E}} \|\dot{y}_{ci} - \dot{y}_{cj}\|^2 \\
&\leq 0,
\end{aligned}$$

where the system settings $Y_p = U_c$ and $\widehat{Y}_c = U_p$ are also used. Hence, the system is stable in the sense of Lyapunov. We apply LaSalle's invariance principle in the following. In the case that $\dot{\widehat{W}}$ stays at zero, we have that $\dot{y}_{ci} - \dot{y}_{cj}$ stays at zero for all $(v_i, v_j) \in \mathcal{E}$. Since the system (6.3) is minimal, we have that $\dot{x}_{ci} - \dot{x}_{cj} \equiv 0$ due to observability and then $\dot{u}_{ci} - \dot{u}_{cj} \equiv 0$ due to controllability. Therefore, we have

$$0 = \dot{x}_{ci} - \dot{x}_{cj} = A(x_{ci} - x_{cj}) + B(u_{ci} - u_{cj}),$$

which implies $x_{ci} - x_{cj} = -A^{-1}B(u_{ci} - u_{cj})$ given that A is nonsingular. In the sequel, we have that $y_{ci} - y_{cj} = -CA^{-1}B(u_{ci} - u_{cj}) = R(0)(u_{ci} - u_{cj})$. The constant differences between u_{ci} and u_{cj} implies that $U_c = \mathbf{1}_N \otimes (\rho I_m) + \overline{Y}_D$. Therefore, we have $U_p = \widehat{Y}_c = (\mathcal{L}_N \otimes R(0))U_c = (\mathcal{L}_N \otimes R(0))\overline{Y}_D$. If $\dot{y}_{ci} - \dot{y}_{cj}$ stays at zero, then $\widehat{Y}_c$ will remain a constant matrix $(\mathcal{L}_N \otimes R(0))\overline{Y}_D$. This violates Assumption 6.1 unless $\overline{Y}_D = 0$. In the case $\overline{Y}_D = 0$, we have $Y_p = U_c = \mathbf{1}_N \otimes (\rho I_m)$, which implies that output consensus is already achieved. Otherwise, $\dot{y}_{ci} - \dot{y}_{cj}$ cannot remain zero and $\dot{\widehat{W}}$ cannot remain zero. Hence, it will keep decreasing until $\widehat{W} = 0$, which will also guarantee output consensus. □

6.4 Output feedback consensus for networked heterogeneous nonlinear NI systems

In this section, we provide a control framework for the output feedback consensus for networked heterogeneous nonlinear NI systems with positive definite storage functions. Consider a series of heterogeneous nonlinear OSNI controllers H_{ck} ($k = 1, 2, \cdots, l$). The controllers have the following state-space models:

$$\begin{aligned} H_{ck}: \quad \dot{x}_{ck}(t) &= f_{ck}(x_{ck}(t), u_{ck}(t)), \\ y_{ck}(t) &= h_{ck}(x_{ck}(t)), \end{aligned} \tag{6.7}$$

where $x_{ck}(t) \in \mathbb{R}^{n_{ck}}$ is the state, $u_{ck}(t) \in \mathbb{R}^m$ is the input, and $y_{ck}(t) \in \mathbb{R}^m$ is the output, $f_{ck} : \mathbb{R}^{n_{ck}} \times \mathbb{R}^m \to \mathbb{R}^{n_{ck}}$ are Lipschitz continuous functions, $h_{ck} : \mathbb{R}^{n_{ck}} \to \mathbb{R}^m$ are class C^1 functions. These systems operate independently in parallel. The subscript "c" indicates that this system will play the role of a controller in what follows. We combine the states, inputs and outputs respectively as vectors $X_c = [x_{c1}^T, \cdots, x_{cl}^T]^T \in \mathbb{R}^{n_c}$, where $n_c = \sum_{k=1}^{l} n_{ck}$, $U_c = [u_{c1}^T, \cdots, u_{cl}^T]^T \in \mathbb{R}^{lm}$ and $Y_c = [y_{c1}^T, \cdots, y_{cl}^T]^T \in \mathbb{R}^{lm}$. We denote by $\mathcal{H}_c$ the network of nonlinear OSNI controllers consisting of systems H_{ck} ($k = 1, 2, \cdots, l$), as is shown in Fig. 6.3. The system $\mathcal{H}_c$ is formally defined by the following equations:

$$\mathcal{H}_c: \ \dot{X}_c = \begin{bmatrix} f_{c1}(x_{c1}, u_{c1}) \\ \vdots \\ f_{cl}(x_{cl}, u_{cl}) \end{bmatrix}, \ Y_c = \begin{bmatrix} h_{c1}(x_{c1}) \\ \vdots \\ h_{cl}(x_{cl}) \end{bmatrix}. \tag{6.8}$$

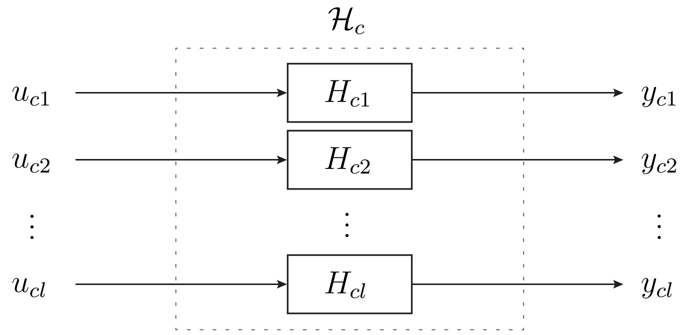


Figure 6.3: System $\mathcal{H}_c$: a nonlinear system consisting of l independent and heterogeneous nonlinear systems H_{ck} ($k = 1, 2, \cdots, l$), with independent inputs and outputs combined as the input and output of the networked system $\mathcal{H}_c$.

Lemma 6.3. *If the controllers H_{ck} are nonlinear OSNI systems for all $k = 1, 2, \cdots, l$, then the networked controller $\mathcal{H}_c$ is also a nonlinear OSNI system.*

Proof. For every nonlinear OSNI system H_{ck}, we have a positive semidefinite storage function $V_{ck}(x_{ck})$ and a constant $\epsilon_k > 0$ such that

$$\dot{V}_{ck}(x_{ck}) \leq u_{ck}^T \dot{y}_{ck} - \epsilon_k \|\dot{y}_{ck}\|^2, \tag{6.9}$$

where ϵ_k is the level of output strictness of the system H_{ck}. For the system $\mathcal{H}_c$, we define its storage function V_c as the sum of the storage functions of all the networked controllers; i.e., $V_c := \sum_{k=1}^{l} V_{ck}(x_{ck})$, which is positive semidefinite. The time derivative of V_c is:

$$\begin{aligned} \dot{V}_c &= \sum_{k=1}^{l} \dot{V}_{ck}(x_{ck}) \\ &\leq \sum_{k=1}^{l} u_{ck}^T \dot{y}_{ck} - \sum_{k=1}^{l} \epsilon_k \|\dot{y}_{ck}\|^2 \\ &\leq \sum_{k=1}^{l} u_{ck}^T \dot{y}_{ck} - \epsilon_{\min} \sum_{k=1}^{l} \|\dot{y}_{ck}\|^2 \\ &= U_c^T \dot{Y}_c - \epsilon_{\min} \|\dot{Y}_c\|^2, \end{aligned} \tag{6.10}$$

where $\epsilon_{\min} = \min\{\epsilon_1, \epsilon_2, \cdots, \epsilon_l\} > 0$. Hence, the system $\mathcal{H}_c$ satisfies the definition of a nonlinear OSNI system and $\epsilon_{\min}$ quantifies the level of output strictness of this system. □

We model the communication links between the networked nonlinear NI plants H_{pi} ($i = 1, 2, \cdots, N$) using a weakly connected directed graph $\mathcal{G}$, which has the corresponding incidence matrix $\mathcal{Q}$ defined by (2.8). Note that the we allow information exchange in both direction for each edge in the directed graph. Indeed, this directed graph can be obtained by arbitrarily assigning directions to the edges of an undirected graph that models the network topology. In this section and the next section, output consensus of heterogeneous nonlinear NI plants is achieved by constructing a control system with the block diagram shown in Fig. 6.4 and choosing suitable controllers that satisfy certain conditions. The plants and controllers correspond to the nodes and edges of the directed graph, respectively. Each controller takes the difference between the outputs of the two plants connected to it as input. Each plant takes the sum of the outputs of the controllers connected to it with suitable signs as input. As is shown in Fig. 6.4, the input to the system $\mathcal{H}_c$ in (6.8) is given

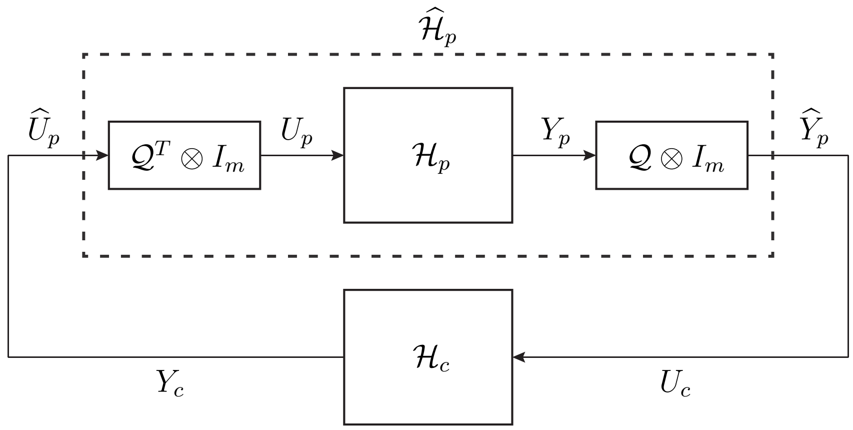


Figure 6.4: Positive feedback interconnection of heterogeneous nonlinear NI plants and nonlinear OSNI controllers according to the directed graph network topology.

by

$$U_c = (\mathcal{Q} \otimes I_m) Y_p. \tag{6.11}$$

Then U_p is defined as

$$U_p = \left(\mathcal{Q}^T \otimes I_m\right) Y_c, \tag{6.12}$$

where Y_c is the corresponding output of the system $\mathcal{H}_c$ defined in (6.8). Equivalently, the distributed control protocol for the plant H_{pi} $(i = 1, 2, \cdots, N)$ is defined by the equations

$$\dot{x}_{ck}(t) = f_{ck}\left(x_{ck}(t), u_{ck}\right), \tag{6.13}$$

$$y_{ck}(t) = h_{ck}(x_{ck}(t)) + D_{ck} u_{ck}, \tag{6.14}$$

$$u_{pi} = \sum_{k=1}^{l} q_{ki} y_{ck}, \tag{6.15}$$

where q_{kj} and q_{ki} are the j-th and i-th elements in the k-th row of the incidence matrix $\mathcal{Q}$, respectively, as defined in (2.8). Here, $u_{ck} = \sum_{j=1}^{N} q_{kj} y_{pj}$ represents the difference between the outputs of the two plants connected by the edge e_k.

We illustrate the connections between the plants and controllers using an example in Fig. 6.5.

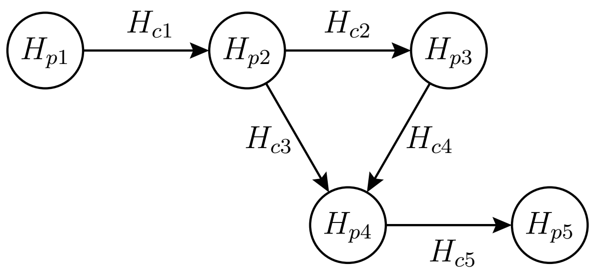


Figure 6.5: An example of a weakly connected directed graph where the nodes correspond to the plants and the edges correspond to the nodes.

In Fig. 6.5, the nodes H_{pi} ($i = 1, \cdots, 5$ in this example) represent the heterogeneous nonlinear NI plants, while the heterogeneous nonlinear OSNI controllers H_{ck} ($k = 1, \cdots, 5$ in this example) correspond to the edges. We illustrate in Fig. 6.6 how the controller H_{c1} influences the dynamics of the two plants that correspond to the nodes linked to it. The controller H_{c1} takes the difference between the outputs of H_{p1} and H_{p2} as its input; i.e., $u_{c1} = y_{p1} - y_{p2}$. Then the corresponding output of H_{c1} given by its dynamics (6.7) is fed into H_{p1} and H_{p2} with opposite signs. Since H_{p1} is only connected to the edge correspond to H_{c1} and it is the initial vertex, then $u_{p1} = y_{c1}$. The plant H_{p2} is connected to three nodes via three edges, which correspond to controllers H_{c1}, H_{c2} and H_{c3}. Therefore, the outputs of these three controllers all contribute in the input of H_{p2}. Since H_{p2} is the terminal vertex of the edge corresponding to H_{c1} and the initial vertices of the edges corresponding to H_{c2} and H_{c3}, then $u_{p2} = -y_{c1} + y_{c2} + y_{c3}$.

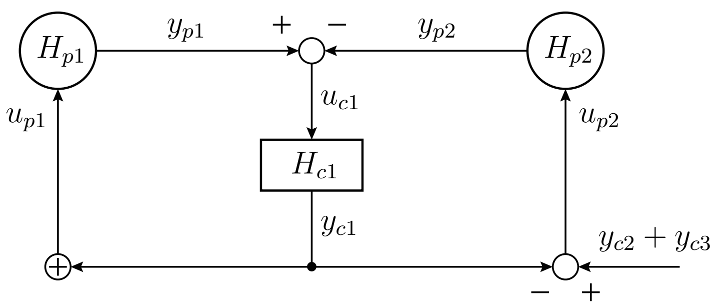


Figure 6.6: Detailed block diagram that shows the interconnection between the plants H_{p1}, H_{p2} and the controller H_{c1} in Fig. 6.5.

We have the following lemma for the system $\widehat{\mathcal{H}}_p$:

Lemma 6.4. *If the plants H_{pi} are nonlinear NI systems with a positive definite storage function for all $i = 1, 2, \cdots, N$, then the system $\widehat{\mathcal{H}}_p$ as shown in Fig. 6.4 is also a nonlinear NI system with a positive definite storage function.*

Proof. According to Definition 3.1, the storage function $V_{pi}(x_{pi})$ of each nonlinear NI system H_{pi} $(i = 1, 2, \cdots, N)$ must satisfy

$$\dot{V}_{pi}(x_{pi}) \leq u_{pi}^T \dot{y}_{pi}. \tag{6.16}$$

We define the storage function for the system $\widehat{\mathcal{H}}_p$ as $\widehat{V}_p = \sum_{i=1}^{N} V_{pi}(x_{pi})$, which is positive definite. Then

$$\dot{\widehat{V}}_p = \sum_{i=1}^{N} \dot{V}_{pi}(x_{pi}) \leq \sum_{i=1}^{N} u_{pi}^T \dot{y}_{pi} = U_p^T \dot{Y}_p. \tag{6.17}$$

According to the system setting in Fig. 6.4,

$$U_p = (\mathcal{Q}^T \otimes I_m)\widehat{U}_p, \quad \text{and} \quad \widehat{Y}_p = (\mathcal{Q} \otimes I_m)Y_p. \tag{6.18}$$

Therefore, we have

$$U_p^T Y_p = [(\mathcal{Q}^T \otimes I_m)\widehat{U}_p]^T Y_p = \widehat{U}_p^T (\mathcal{Q} \otimes I_m) Y_p = \widehat{U}_p^T \widehat{Y}_p. \tag{6.19}$$

According to (6.17) and (6.19), we obtain the nonlinear NI inequality for the system $\widehat{\mathcal{H}}_p$:

$$\dot{\widehat{V}}_p \leq \widehat{U}_p^T \dot{\widehat{Y}}_p. \tag{6.20}$$

Therefore, $\widehat{\mathcal{H}}_p$ is a nonlinear NI system with a positive definite storage function. $\square$

Before presenting the main result, we require the following assumption to be satisfied. For the open-loop interconnection of the systems $\widehat{\mathcal{H}}_p$ and $\mathcal{H}_c$ shown in Fig. 6.7, suppose:

Assumption 6.2. *Given any nonzero constant input $\widehat{U}_p(t) \equiv \overline{\widehat{U}}_p$ for the system $\widehat{\mathcal{H}}_p$, we obtain a corresponding output $\widehat{Y}_p(t)$. Set the input of the system $\mathcal{H}_c$ to be $U_c(t) \equiv \widehat{Y}_p(t)$ in the cascade interconnection shown in Fig. 6.7. If the corresponding output of the system $\mathcal{H}_c$ is a constant; that is $Y_c(t) \equiv \overline{Y}_c$, then we have $\overline{\widehat{U}}_p \neq \overline{Y}_c$.*

Theorem 6.2. *Consider a weakly connected directed graph $\mathcal{G}$ with the incidence matrix $\mathcal{Q}$ that models the communication links for a network of heterogeneous nonlinear NI systems H_{pi} $(i = 1, 2, \cdots, N)$ which have positive definite storage functions and are of the form (6.1). Also, consider the heterogeneous nonlinear OSNI control laws H_{ck} $(k = 1, 2, \cdots, l)$ of the form (6.7) for all of the edges. Suppose the plants H_{pi}*

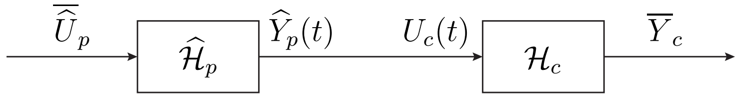


Figure 6.7: Open-loop interconnection of the networked nonlinear NI plants $\widehat{\mathcal{H}}_p$ and the networked nonlinear OSNI controllers $\mathcal{H}_c$.

and controllers H_{ck} satisfy Assumptions 3.3 and 3.4, and Assumption 6.2 is satisfied for the open-loop interconnection of the networked plants and controllers shown in Fig. 6.7. Also, suppose the storage function, defined as

$$W := \sum_{i=1}^{N} V_{pi}(x_{pi}) + \sum_{k=1}^{l} V_{ck}(x_{ck}) - \widehat{Y}_p^T Y_c, \tag{6.21}$$

is positive definite, where V_{pi} and V_{ck} are the corresponding storage functions that satisfy (6.16) for the system $\mathcal{H}_{pi}$ and (6.9) for the system $\mathcal{H}_{ck}$, respectively. Here, $\widehat{Y}_p$ and Y_c are the outputs of the system $\widehat{\mathcal{H}}_p$ and $\mathcal{H}_c$, respectively. Then output feedback consensus can be achieved via the controller defined by equations (6.11), (6.8), (6.12), or equivalently (6.13)-(6.15) in a distributed manner for each plant H_{pi}, as shown in Fig. 6.4.

Proof. We apply the Lyapunov's direct method and take the time derivative of the storage function W. Define $\widehat{V}_p = \sum_{i=1}^{N} V_{pi}(x_{pi})$ and $V_c := \sum_{k=1}^{l} V_{ck}(x_{ck})$ as in the proofs of Lemmas 6.4 and 6.3, respectively. According to (6.17) and (6.10), we have

$$\begin{aligned}
\dot{W} &= \dot{\widehat{V}}_p + \dot{V}_c - \dot{\widehat{Y}}_p^T Y_c - \widehat{Y}_p^T \dot{Y}_c \\
&\leq \widehat{U}_p^T \dot{\widehat{Y}}_p + U_c^T \dot{Y}_c - \epsilon_{\min} \left\| \dot{Y}_c \right\|^2 - \dot{\widehat{Y}}_p^T Y_c - \widehat{Y}_p^T \dot{Y}_c \\
&= - \epsilon_{\min} \left\| \dot{Y}_c \right\|^2 \\
&\leq 0,
\end{aligned} \tag{6.22}$$

where the system setting $\widehat{U}_p = Y_c$ and $U_c = \widehat{Y}_p$ is also used. Hence, the closed-loop system is at least Lyapunov stable. Now we apply LaSalle's invariance principle. According to (6.22), $\dot{W}$ can remain zero only if $\epsilon_{\min} \left\| \dot{Y}_c \right\|^2$ remains zero, which means that $\dot{y}_{ck}(t)$ remain zero for all $k = 1, 2, \cdots, l$. According to Assumptions 3.3 and 3.4, for the system H_{ck}, $\dot{y}_{ck}(t) \equiv 0$ implies $\dot{x}_{ck}(t) \equiv 0$, which holds only if $u_{ck}(t) \equiv \overline{u}_{ck}$. In other words, for all $k = 1, 2, \cdots, l$, the controllers H_{ck} are in steady-state; i.e., $u_{ck}(t) \equiv \overline{u}_{ck}$, $x_{ck}(t) \equiv \overline{x}_{ck}$ and $y_{ck}(t) \equiv \overline{y}_{ck}$. This implies that $U_c(t)$ and $Y_c(t)$ in Fig. 6.4 are both constant vectors; i.e., $U_c(t) \equiv \overline{U}_c$ and $Y_c(t) \equiv \overline{Y}_c$. According to the closed-loop setting that $\widehat{U}_p(t) \equiv Y_c(t)$, we have that $\widehat{U}_p(t) \equiv \overline{\widehat{U}}_p$ is also a constant vector. According to Assumption 6.2, this can only hold if $\overline{\widehat{U}}_p = 0$, which implies $\overline{Y}_c = 0$. Hence, $\overline{y}_{ck} = 0$ for all $k = 1, 2, \cdots, l$. According to Assumptions 3.3 and 3.4, $\overline{y}_{ck} = 0$ implies $\overline{x}_{ck} = 0$ and then $\overline{u}_{ck} = 0$ for all $k = 1, 2, \cdots, l$. This implies $y_{pi}(t) \equiv y_{pj}(t)$ for all $(v_i, v_j) \in \mathcal{E}$, which means output consensus is achieved for all of the heterogeneous nonlinear NI plants. Otherwise, $\dot{W}$ cannot remain at zero and W will keep decreasing until output consensus is achieved or $W = 0$, which also implies output consensus. This completes the proof. □

6.5 Output feedback consensus for networked heterogeneous nonlinear NI systems with free body motion

In this section, we allow the plants (6.1) in the network to have free body motion. That is, we allow the plants H_{pi} to be nonlinear NI with a positive semidefinite storage function, as in Definition 3.1. In this section, output feedback consensus is also achieved using the control framework in Fig. 6.4 as described in Section 6.4, except that the controllers H_{ck} used in the framework have slightly different system models.

Consider a series of heterogeneous nonlinear OSNI controllers H_{ck} $(k = 1, 2, \cdots, l)$. The controllers have the following state-space models:

$$H_{ck}: \quad \begin{aligned} \dot{x}_{ck}(t) &= f_{ck}(x_{ck}(t), u_{ck}(t)), \\ y_{ck}(t) &= h_{ck}(x_{ck}(t)) + D_{ck}u_{ck}(t), \end{aligned} \tag{6.23}$$

where $x_{ck}(t) \in \mathbb{R}^{n_{ck}}$ is the state, $u_{ck}(t) \in \mathbb{R}^m$ is the input, and $y_{ck}(t) \in \mathbb{R}^m$ is the output, $f_{ck} : \mathbb{R}^{n_{ck}} \times \mathbb{R}^m \to \mathbb{R}^{n_{ck}}$ are Lipschitz continuous functions, $h_{ck} : \mathbb{R}^{n_{ck}} \to \mathbb{R}^m$ are class C^1 functions and $D_{ck} \in \mathbb{R}^{m\times m}$ are symmetric matrices. These systems operate independently in parallel as shown in Fig. 6.3. We combine the states, inputs and outputs respectively as vectors $X_c = [x_{c1}^T, \cdots, x_{cl}^T]^T \in \mathbb{R}^{n_c}$, where $n_c = \sum_{k=1}^{l} n_{ck}$, $U_c = [u_{c1}^T, \cdots, u_{cl}^T]^T \in \mathbb{R}^{lm}$ and $Y_c = [y_{c1}^T, \cdots, y_{cl}^T]^T = \Pi_c + D_cU_c \in \mathbb{R}^{lm}$, where

$$\Pi_c = \left[h_{c1}(x_{c1})^T, \cdots, h_{cl}(x_{cl})^T\right]^T \in \mathbb{R}^{lm}; \tag{6.24}$$

$$D_c = diag\{D_{c1}, \cdots, D_{cl}\} \in \mathbb{R}^{lm\times lm}. \tag{6.25}$$

We denote by $\mathcal{H}_c$ the network of nonlinear OSNI controllers consisting of systems H_{ck}

$(k = 1, 2, \cdots, l)$. The system $\mathcal{H}_c$ is formally defined by the following equations:

$$\mathcal{H}_c: \ \dot{X}_c = \begin{bmatrix} f_{c1}(x_{c1}, u_{c1}) \\ \vdots \\ f_{cl}(x_{cl}, u_{cl}) \end{bmatrix}, \ Y_c = \begin{bmatrix} h_{c1}(x_{c1}) + D_{c1}u_{c1} \\ \vdots \\ h_{cl}(x_{cl}) + D_{cl}u_{cl} \end{bmatrix}. \tag{6.26}$$

The following lemma is similar to Lemma 6.3.

Lemma 6.5. *If the controllers H_{ck} are nonlinear OSNI systems for all $k = 1, 2, \cdots, l$, then the networked controller $\mathcal{H}_c$ is also a nonlinear OSNI system.*

Proof. For every nonlinear OSNI system H_{ck}, we have a positive semidefinite storage function $V_{ck}(x_{ck})$ and a constant $\epsilon_k > 0$ such that

$$\dot{V}_{ck}(x_{ck}) \leq u_{ck}^T \dot{\widetilde{y}}_{ck} - \epsilon_k \|\dot{\widetilde{y}}_{ck}\|^2, \tag{6.27}$$

where $\widetilde{y}_{ck} = h_{ck}(x_{ck})$ and ϵ_k is the level of output strictness of the system H_{ck}. For the system $\mathcal{H}_c$, we define its storage function V_c as the sum of the storage functions of all the networked controllers; i.e., $V_c := \sum_{k=1}^{l} V_{ck}(x_{ck})$, which is positive semidefinite. The time derivative of V_c is:

$$\begin{aligned} \dot{V}_c &= \sum_{k=1}^{l} \dot{V}_{ck}(x_{ck}) \\ &\leq \sum_{k=1}^{l} u_{ck}^T \dot{\widetilde{y}}_{ck} - \sum_{k=1}^{l} \epsilon_k \|\dot{\widetilde{y}}_{ck}\|^2 \\ &\leq \sum_{k=1}^{l} u_{ck}^T \dot{\widetilde{y}}_{ck} - \epsilon_{\min} \sum_{k=1}^{l} \|\dot{\widetilde{y}}_{ck}\|^2 \\ &= U_c^T \dot{\Pi}_c - \epsilon_{\min} \|\dot{\Pi}_c\|^2, \end{aligned} \tag{6.28}$$

where $\epsilon_{\min} = \min\{\epsilon_1, \epsilon_2, \cdots, \epsilon_l\} > 0$. Hence, the system $\mathcal{H}_c$ satisfies the definition of a nonlinear OSNI system and $\epsilon_{\min}$ quantifies the level of output strictness of this system. □

Lemma 6.6. *If the plants H_{pi} are nonlinear NI systems for all $i = 1, 2, \cdots, N$, then the system $\widehat{\mathcal{H}}_p$ as shown in Fig. 6.4 is also a nonlinear NI system.*

Proof. See the proof of Lemma 6.4. Since the storage function of each plants is positive semidefinite, then the storage function of the networked plant $\mathcal{H}_p$, which is the sum of the storage functions of all the plants H_{pi}, is also positive semidefinite. □

We need the following assumption for the networked plant $\mathcal{H}_p$. This assumption can be regarded as a networked version of Assumption 3.1.

Assumption 6.3. *Given a constant input $\widehat{U}_p(t) \equiv \overline{\widehat{U}}_p$ to the system $\widehat{\mathcal{H}}_p$, if its output is also constant; i.e., $\widehat{Y}_p(t) \equiv \overline{\widehat{Y}}_p$, then we suppose $\overline{\widehat{U}}_p$ and $\overline{\widehat{Y}}_p$ satisfy $\overline{\widehat{U}}_p^T \overline{\widehat{Y}}_p \geq 0$.*

In fact, Assumption 3.1 applied to all the plants H_{pi} implies Assumption 6.3 in the case that H_{pi} are in a steady state for all $i = 1, 2, \cdots, N$. This is because when all the plants satisfy Assumption 3.1 and are in a steady state, we have $\overline{U}_p^T \overline{Y}_p \geq 0$ and according to the system setting in Fig. 6.4 we have $\overline{U}_p^T \overline{Y}_p = [(\mathcal{Q}^T \otimes I_m)^T \overline{\widehat{U}}_p]^T \overline{Y}_p = \overline{\widehat{U}}_p^T (\mathcal{Q} \otimes I_m) \overline{Y}_p = \overline{\widehat{U}}_p^T \overline{\widehat{Y}}_p$ similarly to (6.19). Hence $\overline{\widehat{U}}_p^T \overline{\widehat{Y}}_p \geq 0$. However, Assumption 6.3 is made for the networked plants $\widehat{\mathcal{H}}_p$ in the following theorem instead of Assumption 3.1 for each individual plant because Assumption 6.3 also allows for the situation in which the input and output of the system $\widehat{\mathcal{H}}_p$ are constant, but the individual plants H_{pi} are not all in a steady state. This situation is possible because the matrix $\mathcal{Q} \otimes I_m$ takes the difference between the outputs of the plants. Under constant inputs, if the plants oscillate with a constant difference between their outputs, then this situation is allowed under Assumption 6.3.

Theorem 6.3. *Consider a weakly connected directed graph $\mathcal{G}$ with the incidence matrix $\mathcal{Q}$ that models the communication links for a network of heterogeneous nonlinear NI systems H_{pi} ($i = 1, 2, \cdots, N$) of the form (6.1). Also, consider the heterogeneous nonlinear OSNI control laws H_{ck} ($k = 1, 2, \cdots, l$) of the form (6.23) for all of the edges. Suppose Assumptions 3.3 and 3.4 are satisfied for the plants H_{pi}, Assumptions 3.2, 3.3 and 3.4 are satisfied for the controllers H_{ck} (with $\gamma = \gamma_k$ for the controller H_{ck} in Assumption 3.2) and Assumption 6.3 is satisfied by the system $\widehat{\mathcal{H}}_p$. Also, suppose the storage function, defined as*

$$\widehat{W} := \sum_{i=1}^{N} V_{pi}(x_{pi}) + \sum_{k=1}^{l} V_{ck}(x_{ck}) - \widehat{Y}_p^T \Pi_c - \frac{1}{2} \widehat{Y}_p^T D_c \widehat{Y}_p, \tag{6.29}$$

is positive definite, where V_{pi} and V_{ck} are the corresponding positive semidefinite storage functions that satisfy (6.16) for the system $\mathcal{H}_{pi}$ and (6.27) for the system $\mathcal{H}_{ck}$, respectively. Here, $\widehat{Y}_p$ is the output of the system $\widehat{\mathcal{H}}_p$. Π_c and D_c are terms in the output Y_c of the system $\mathcal{H}_c$ and are defined in (6.24) and (6.25). Then output feedback consensus can be achieved via the control defined by equations (6.11), (6.26), (6.12), or equivalently (6.13)–(6.15) in a distributed manner for each plant H_{pi}, as shown in Fig. 6.4.

Proof. Define $\widehat{V}_p = \sum_{i=1}^{N} V_{pi}(x_{pi})$ and $V_c := \sum_{k=1}^{l} V_{ck}(x_{ck})$ as in the proofs of Lemmas 6.6 and 6.5, respectively. According to (6.20), (6.28) and the system setting

$\widehat{U}_p \equiv Y_c$ and $U_c \equiv \widehat{Y}_p$ shown in Fig. 6.4, we have

$$\dot{\widehat{V}}_p \leq \widehat{U}_p^T \dot{\widehat{Y}}_p = Y_c^T \dot{\widehat{Y}}_p = \dot{\widehat{Y}}_p^T [\Pi_c + D_c U_c] = \dot{\widehat{Y}}_p^T [\Pi_c + D_c \widehat{Y}_p], \tag{6.30}$$

and

$$\dot{V}_c \leq U_c^T \dot{\Pi}_c - \epsilon_{min} \|\dot{\Pi}_c\|^2 = \widehat{Y}_p^T \dot{\Pi}_c - \epsilon_{min} \|\dot{\Pi}_c\|^2. \tag{6.31}$$

According to (6.30), (6.31) and the symmetry of D_c in (6.25), the time derivative of the storage function $\widehat{W}$ satisfies the following inequality:

$$\begin{aligned}
\dot{\widehat{W}} &= \dot{\widehat{V}}_p + \dot{V}_c - \dot{\widehat{Y}}_p^T \Pi_c - \widehat{Y}_p^T \dot{\Pi}_c - \frac{1}{2} \dot{\widehat{Y}}_p^T (D_c + D_c^T) \widehat{Y}_p \\
&\leq \dot{\widehat{Y}}_p^T [\Pi_c + D_c \widehat{Y}_p] + \widehat{Y}_p^T \dot{\Pi}_c - \epsilon_{\min} \|\dot{\Pi}_c\|^2 - \dot{\widehat{Y}}_p^T \Pi_c - \widehat{Y}_p^T \dot{\Pi}_c - \dot{\widehat{Y}}_p^T D_c \widehat{Y}_p \\
&\leq \epsilon_{\min} \|\dot{\Pi}_c\|^2 \\
&\leq 0.
\end{aligned}$$

Hence, the closed-loop system is at least Lyapunov stable. Moreover, $\dot{\widehat{W}} = 0$ can hold only if $\dot{\Pi}_c = 0$. In other words, $\dot{\widehat{W}}$ can remain zero only if $\dot{h}_{ck}(x_{ck}(t))$ remains zero for all $k = 1, 2, \cdots, l$. According to Assumptions 3.3 and 3.4, $\dot{h}_{ck}(x_{ck}(t)) \equiv 0 \implies \dot{x}_{ck}(t) \equiv 0 \implies u_{ck}(t) \equiv \overline{u}_{ck}$. Hence, H_{ck} is in a steady state for all $k = 1, 2, \cdots, l$. We have $U_c(t) \equiv \overline{U}_c$ and $Y_c(t) \equiv \overline{Y}_c$. According to the system setting in Fig. 6.4 that $\widehat{U}_p(t) \equiv Y_c(t)$ and $U_c(t) \equiv \widehat{Y}_p(t)$, we also have $\widehat{U}_p(t) \equiv \overline{\widehat{U}}_p$ and $\widehat{Y}_p(t) \equiv \overline{\widehat{Y}}_p$. According to Assumption 6.3,

$$\overline{\widehat{U}}_p^T \overline{\widehat{Y}}_p \geq 0. \tag{6.32}$$

According to Assumption 3.2, we have

$$\overline{U}_c^T \overline{Y}_c = \sum_{k=1}^{l} \overline{u}_{ck}^T \overline{y}_{ck} \leq -\sum_{k=1}^{l} \gamma_k \|\overline{u}_{ck}\|^2 \leq -\gamma_{\min} \|\overline{U}_c\|^2,$$

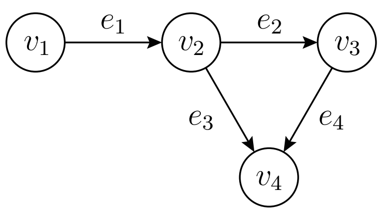
where $\gamma_{\min} = \min\{\gamma_1, \gamma_2 \cdots, \gamma_l\}$. In the case that $\overline{U}_c \neq 0$,

$$\overline{U}_c^T \overline{Y}_c < 0,$$

which contradicts (6.32) because $\overline{U}_c^T \overline{Y}_c = \overline{\widehat{U}}_p^T \overline{\widehat{Y}}_p$. In the case that $\overline{U}_c = 0$, all connected plants have the difference between their system outputs being zero. Hence, output consensus has already been achieved. Otherwise, $\dot{\widehat{W}}$ cannot remain zero. According to LaSalle's invariance principle, $\widehat{W}$ will keep decreasing until either $\overline{U}_c = 0$ or $\widehat{W} = 0$. Thus, output consensus is achieved in both cases. □

Remark 6.1. *The controllers proposed in Theorem 6.1, 6.2 and 6.3 are robust against parameter perturbation for nonlinear NI plants connected in a network. Indeed, for physical systems, the NI property is invariant to parameter variations so that consensus can be always guaranteed as long as the required conditions are satisfied.*

6.6 Illustrative example



$$\mathcal{Q} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Figure 6.8: A weakly connected directed graph $\mathcal{G}$ consisting of four nodes, and its incidence matrix $\mathcal{Q}$.

The example given below illustrates the consensus result presented in this chapter. Since the case considered in Section 6.5 covers those considered in Sections 6.3 and 6.4, we use the results in Theorem 6.3 to synchronize the outputs of networked heterogeneous nonlinear NI systems with free body motion.

Consider four nonlinear NI plants H_{pi} at the vertices v_i of the graph in Fig. 6.8, where the directions of the edges are $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, $e_3 = (v_2, v_4)$ and $e_4 = (v_3, v_4)$. Then $\mathcal{Q}$ in Fig. 6.8 is the incidence matrix of the directed graph $\mathcal{G}$. The plants are nonlinear single integrators which have the following state-space models:

$$\begin{aligned} H_{pi}: \quad & \dot{x}_{pi}(t) = \mu_i u_{pi}(t)^3, \\ & y_{pi}(t) = x_{pi}(t), \quad i = 1, 2, \cdots, 4, \end{aligned} \tag{6.33}$$

where $\mu_1 = 10$, $\mu_2 = 30$, $\mu_3 = 25$ and $\mu_4 = 5$ are constant coefficients. Such plant models involve free body motion. The storage functions for these four plants are all $V_{pi}(x_{pi}) = 0$. The states x_{p1}, x_{p2}, x_{p3} and x_{p4} of these four plants have initial values 30, 15, -5 and -10, respectively. We aim to synchronise the outputs of these four plants to the same limit trajectory by using nonlinear OSNI controllers H_{ck} at the edges e_k of the graph in Fig. 6.8.

$$\begin{aligned} H_{ck}: \quad & \dot{x}_{ck}(t) = -\alpha_k x_{ck}(t) - \beta_k x_{ck}(t)^3 + u_{ck}(t), \\ & y_{ck}(t) = x_{ck}(t) - u_{ck}(t), \quad k = 1, 2, \cdots, 4, \end{aligned} \tag{6.34}$$

where α_k and β_k are constant coefficients. $\alpha_1 = 5$, $\alpha_2 = 8$, $\alpha_3 = 7$, $\alpha_4 = 3$; $\beta_1 = 3$, $\beta_2 = 2$, $\beta_3 = 5$ and $\beta_4 = 2$, respectively. Controllers 1, 2, 3 and 4 are nonlinear OSNI systems with storage functions $V_{c1} = \frac{5}{2}x_{c1}^2 + \frac{3}{4}x_{c1}^4$, $V_{c2} = 4x_{c2}^2 + \frac{1}{2}x_{c2}^4$, $V_{c3} = \frac{7}{2}x_{c3}^2 + \frac{5}{4}x_{c3}^4$ and $V_{c4} = \frac{3}{2}x_{c4}^2 + \frac{1}{2}x_{c4}^4$, respectively. We now construct the networked plant system $\widehat{\mathcal{H}}_p$ for the plants (6.33) using (6.18) with $m = 1$. Substituting the equations

$$\begin{bmatrix} u_{p1} \\ u_{p2} \\ u_{p3} \\ u_{p4} \end{bmatrix} = \mathcal{Q}^T \begin{bmatrix} \widehat{u}_{p1} \\ \widehat{u}_{p2} \\ \widehat{u}_{p3} \\ \widehat{u}_{p4} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \widehat{y}_{p1} \\ \widehat{y}_{p2} \\ \widehat{y}_{p3} \\ \widehat{y}_{p4} \end{bmatrix} = \mathcal{Q} \begin{bmatrix} y_{p1} \\ y_{p2} \\ y_{p3} \\ y_{p4} \end{bmatrix}$$

into the system model (6.33) and letting $\widehat{x}_{pi} := \widehat{y}_{pi}$, we can obtain the state-space model for the system $\widehat{\mathcal{H}}_p$ as follows.

$$\begin{aligned} \dot{\widehat{x}}_{p1} &= 10\widehat{u}_{p1}^3 + 30(\widehat{u}_{p1} - \widehat{u}_{p2} - \widehat{u}_{p3})^3, \\ \dot{\widehat{x}}_{p2} &= 30(-\widehat{u}_{p1} + \widehat{u}_{p2} + \widehat{u}_{p3})^3 + 25(\widehat{u}_{p2} - \widehat{u}_{p4})^3, \\ \dot{\widehat{x}}_{p3} &= 30(-\widehat{u}_{p1} + \widehat{u}_{p2} + \widehat{u}_{p3})^3 + 5(\widehat{u}_{p3} + \widehat{u}_{p4})^3, \\ \dot{\widehat{x}}_{p4} &= 25(-\widehat{u}_{p2} + \widehat{u}_{p4})^3 + 5(\widehat{u}_{p3} + \widehat{u}_{p4})^3, \\ \widehat{y}_{p1} &= \widehat{x}_{p1}, \quad \widehat{y}_{p2} = \widehat{x}_{p2}, \quad \widehat{y}_{p3} = \widehat{x}_{p3}, \quad \widehat{y}_{p4} = \widehat{x}_{p4}. \end{aligned} \tag{6.35}$$

Note that here we also have

$$\widehat{y}_{p4} = \widehat{y}_{p3} - \widehat{y}_{p2}, \tag{6.36}$$

which expresses the output difference between H_{p3} and H_{p4} in terms of the output difference between H_{p2} and H_{p3} and the output difference between H_{p2} and H_{p4}. However, for ease of calculation for the candidate Lyapunov function, we keep the redundant state $\widehat{x}_{p4}$ in (6.35).

The candidate Lyapunov function of the closed-loop system provided by Theorem 6.3 can be constructed using the formula (6.29) and also the equations (6.24) and (6.25):

$$\begin{aligned} \widehat{W} &= \sum_{i=1}^{4} V_{pi} + \sum_{k=1}^{4} V_{ck} + \begin{bmatrix} \widehat{x}_{p1} \\ \widehat{x}_{p2} \\ \widehat{x}_{p3} \\ \widehat{x}_{p4} \end{bmatrix}^T \begin{bmatrix} x_{c1} \\ x_{c2} \\ x_{c3} \\ x_{c4} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \widehat{x}_{p1} \\ \widehat{x}_{p2} \\ \widehat{x}_{p3} \\ \widehat{x}_{p4} \end{bmatrix}^T (-I_4) \begin{bmatrix} \widehat{x}_{p1} \\ \widehat{x}_{p2} \\ \widehat{x}_{p3} \\ \widehat{x}_{p4} \end{bmatrix} \\ &= \frac{5}{2}x_{c1}^2 + \frac{3}{4}x_{c1}^4 + 4x_{c2}^2 + \frac{1}{2}x_{c2}^4 + \frac{7}{2}x_{c3}^2 + \frac{5}{4}x_{c3}^4 + \frac{3}{2}x_{c4}^2 + \frac{1}{2}x_{c4}^4 - \widehat{x}_{p1}x_{c1} - \widehat{x}_{p2}x_{c2} \\ &\quad - \widehat{x}_{p3}x_{c3} - \widehat{x}_{p4}x_{c4} + \frac{1}{2}\widehat{x}_{p1}^2 + \frac{1}{2}\widehat{x}_{p2}^2 + \frac{1}{2}\widehat{x}_{p3}^2 + \frac{1}{2}\widehat{x}_{p4}^2, \\ &= \begin{bmatrix} \widehat{x}_{p1} & x_{c1} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} \widehat{x}_{p1} \\ x_{c1} \end{bmatrix} + \frac{3}{4}x_{c1}^4 + \begin{bmatrix} \widehat{x}_{p2} & x_{c2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 4 \end{bmatrix} \begin{bmatrix} \widehat{x}_{p2} \\ x_{c2} \end{bmatrix} + \frac{1}{2}x_{c2}^4 \end{aligned}$$

$$+\begin{bmatrix}\widehat{x}_{p3} & x_{c3}\end{bmatrix}\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{7}{2}\end{bmatrix}\begin{bmatrix}\widehat{x}_{p3}\\ x_{c3}\end{bmatrix}+\frac{5}{4}x_{c3}^4+\begin{bmatrix}\widehat{x}_{p4} & x_{c4}\end{bmatrix}\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{3}{2}\end{bmatrix}\begin{bmatrix}\widehat{x}_{p4}\\ x_{c4}\end{bmatrix}+\frac{1}{2}x_{c4}^4,$$

which is positive definite. We show in the following that the assumptions required in Theorem 6.3 are satisfied. For the plants (6.33), $\dot{y}_{pi}\equiv 0$ if and only if $\dot{x}_{pi}\equiv 0$, which implies $u_{pi}\equiv 0$. Moreover, if $y_{pi}\equiv 0$, then $x_{pi}\equiv 0$. Therefore, the plants (6.33) satisfy Assumptions 3.3 and 3.4. For the controllers (6.34), the auxiliary output with the feedthrough term removed is $h_{ck}(x_{ck})=x_{ck}$. Therefore, $h_{ck}(x_{ck})\equiv 0$ if and only if $x_{ck}\equiv 0$. When $x_{ck}\equiv 0$, we have that $\dot{x}_{ck}\equiv 0$, which implies that $u_{ck}\equiv \alpha_k x_{ck}+\beta_k x_{ck}^3$ is constant. Moreover, in the case that $h_{ck}(x_{ck})\equiv 0$, we have $x_{ck}\equiv 0$ and $u_{ck}\equiv 0$. Hence, the controllers (6.34) also satisfy Assumptions 3.3 and 3.4. We show in the following that the controllers (6.34) also satisfy Assumption 3.2. In the case that the inputs u_{ck} and outputs y_{ck} are both constant, the state $x_{ck}=y_{ck}+u_{ck}$ is also a constant. Denote the constant state, input and output by $\overline{x}_{ck}$, $\overline{u}_{ck}$ and $\overline{y}_{ck}$, respectively. We have

$$\overline{u}_{ck}=\alpha_k\overline{x}_{ck}+\beta_k\overline{x}_{ck}^3 \tag{6.37}$$

because $\dot{x}_{ck}=0$. We also have that

$$\overline{y}_{ck}=\overline{x}_{ck}-\overline{u}_{ck}=(1-\alpha_k)\overline{x}_{ck}-\beta_k\overline{x}_{ck}^3. \tag{6.38}$$

We want the inequality

$$\overline{u}_{ck}\overline{y}_{ck}\le\gamma_k\|\overline{u}_{ck}\|^2 \tag{6.39}$$

to be satisfied for some $\gamma_k>0$. That is

$$\left((\alpha_k\overline{x}_{ck}+\beta_k\overline{x}_{ck}^3)\right)\left((1-\alpha_k)\overline{x}_{ck}-\beta_k\overline{x}_{ck}^3\right)\le-\gamma_k\left((\alpha_k\overline{x}_{ck}+\beta_k\overline{x}_{ck}^3\right)^2. \tag{6.40}$$

Substituting (6.37) and (6.38) in (6.40), we have that

$$(1-\gamma_k)\beta_k^2\overline{x}_{ck}^6+((2-\gamma_k)\alpha_k\beta_k-\beta_k)\overline{x}_{ck}^4+(\alpha_k^2-\alpha_k-\gamma_k\alpha_k^2)\overline{x}_{ck}^2\ge 0. \tag{6.41}$$

We require the existence of a constant $\gamma_k>0$ such that (6.41) is satisfied for all $\overline{x}_{ck}\in\mathbb{R}$. Considering that $\overline{x}_{ck}^2\ge 0$, let $\xi_k:=\overline{x}_{ck}^2$, then (6.41) is satisfied for all $\overline{x}_{ck}\in\mathbb{R}$ if and only if

$$(1-\gamma_k)\beta_k^2\xi_k^2+((2-\gamma_k)\alpha_k\beta_k-\beta_k)\xi_k+(\alpha_k^2-\alpha_k-\gamma_k\alpha_k^2)\ge 0 \tag{6.42}$$

is satisfied for all $\xi_k\ge 0$. The graph of the LHS of (6.42) as a function of ξ_k is a parabola for $\gamma_k\ne 0$. To let (6.42) to be satisfied, we want the parabola to open to the top. Therefore, we let $1-\gamma_k>0$, which implies $0<\gamma_k<1$. Also, in the case that $\xi_k=0$, we want $\alpha_k^2-\alpha_k-\gamma_k\alpha_k^2\geq 0$. That is, $\gamma_k\leq 1-\frac{1}{\alpha_k}$ considering that $\alpha_k>0$. To let γ_k exist, we need $\alpha_k>1$, which is the case in the controllers (6.34). Hence, $0<\gamma_k\leq 1-\frac{1}{\alpha_k}$. Considering that $\xi_k\geq 0$, then (6.42) is satisfied if the axis of symmetry of the function of the parabola is nonpositive. That is

$$-\frac{(2-\gamma_k)\alpha_k\beta_k-\beta_k}{(1-\gamma_k)\beta_k^2}\leq 0,$$

which is satisfied if $(2-\gamma_k)\alpha_k\beta_k-\beta_k\geq 0$. Also, considering that $\beta_k>0$, then we need $(2-\gamma_k)\alpha_k-1\geq 0$. That is $\gamma_k\leq 2-\frac{1}{\alpha_k}$, which is always satisfied for $0<\gamma_k\leq 1-\frac{1}{\alpha_k}$. Therefore, Assumption 3.2 is satisfied for all the controllers (6.34) in this example.

We show in the following that Assumption 6.3 is satisfied for the networked plant $\widehat{H}_p$ given in (6.35). In the case that the system $\widehat{H}_p$ has a constant input $\overline{\widehat{U}}_p = \begin{bmatrix}\overline{\widehat{u}}_{p1} & \overline{\widehat{u}}_{p2} & \overline{\widehat{u}}_{p3} & \overline{\widehat{u}}_{p4}\end{bmatrix}^T$ and also a constant output $\overline{\widehat{Y}}_p = \begin{bmatrix}\overline{\widehat{y}}_{p1} & \overline{\widehat{y}}_{p2} & \overline{\widehat{y}}_{p3} & \overline{\widehat{y}}_{p4}\end{bmatrix}^T = \begin{bmatrix}\overline{\widehat{x}}_{p1} & \overline{\widehat{x}}_{p2} & \overline{\widehat{x}}_{p3} & \overline{\widehat{x}}_{p4}\end{bmatrix}^T$, we have that $\dot{\widehat{x}}_{p1}=\dot{\widehat{x}}_{p2}=\dot{\widehat{x}}_{p3}=\dot{\widehat{x}}_{p4}=0$. According to the state equations in (6.35), we have that

$$\overline{\widehat{U}}_p = \begin{bmatrix}0\\1\\-1\\1\end{bmatrix}\overline{\widehat{u}}_{p4}.$$

Then using (6.36), we have

$$\overline{\widehat{U}}_p^T\overline{\widehat{Y}}_p = \overline{\widehat{u}}_{p4}\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}^T\begin{bmatrix}\overline{\widehat{y}}_{p1}\\\overline{\widehat{y}}_{p2}\\\overline{\widehat{y}}_{p3}\\\overline{\widehat{y}}_{p4}\end{bmatrix} = \overline{\widehat{u}}_{p4}\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}^T\begin{bmatrix}\overline{\widehat{y}}_{p1}\\\overline{\widehat{y}}_{p2}\\\overline{\widehat{y}}_{p3}\\\overline{\widehat{y}}_{p3}-\overline{\widehat{y}}_{p2}\end{bmatrix} = 0. \tag{6.43}$$

Assumption 6.3 is satisfied. Since all the required conditions in Theorem 6.3 are satisfied, we expect to achieve output feedback consensus.

This was illustrated using a simulation. As is shown in Fig. 6.9, the output trajectories of all the plants converge to the same limit trajectory. Because of the cubic nonlinearity in the plants and the controllers, their outputs have different rates of convergence in the domains where the states are far and close to their limit values. Therefore, a log scale is used for the time axis in the plot shown in Fig. 6.9.

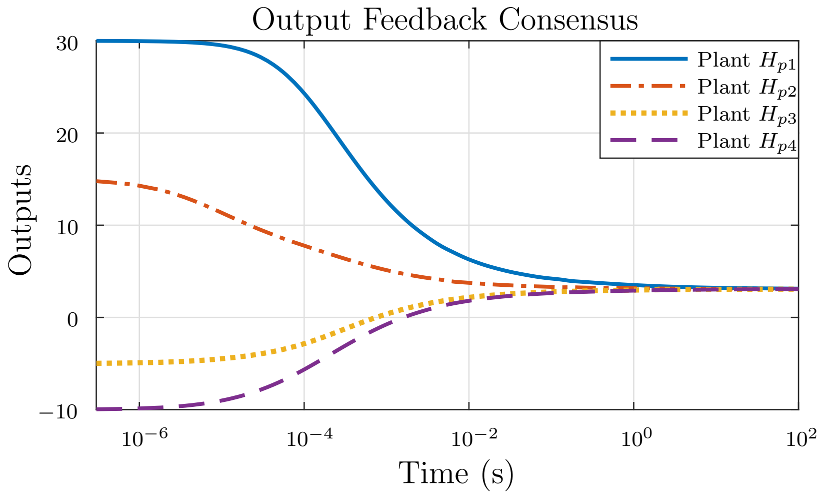


Figure 6.9: Outputs of four nonlinear single integrator plants (6.33), which are synchronized by controllers (6.34). The figure shows the trajectories of the outputs of the plants from the time that the proposed control law is applied to the time that the outputs converge to the same limit trajectory. Starting from different initial conditions, the outputs of the plants converge to the same limit trajectory under the effect of the proposed control framework in Section 6.5.

6.7 Conclusion

We have provided control frameworks to achieve output feedback consensus for a network of identical nonlinear NI plants, a network of heterogeneous nonlinear NI plants and a network of heterogeneous nonlinear NI plants with free body motion. In the case of identical plants, we model the networked system using an undirected connected graph. We apply identical linear OSNI controllers to each plant and the controller outputs are distributed to the plants via the Laplacian matrix of the graph. In the case of heterogeneous plants, we model the networked system using a weakly connected directed graph. The controllers take the output differences of the plants and distribute the corresponding controller outputs to the plants according to the graph topology. For these three different networks, we require different assumptions to be satisfied. In addition, we require certain candidate Lyapunov functions to be positive definite. Then output feedback consensus is achieved. As the result for networked heterogeneous nonlinear NI plants with free body motion is also applicable in the other two cases, it is illustrated using an example.

Chapter 7

Switched Nonlinear Negative Imaginary Systems

7.1 Introduction

In this chapter, we consider extending NI system theory to switched systems. Switched systems, which form an important class of hybrid systems, have drawn much attention because many real-world dynamical systems are of hybrid nature; e.g., see [83, 97]. Such systems are organized as continuous-time systems with discrete-time switching events and arise in many control applications [98–100]. Besides this, the study of switched systems is also motivated by the improvement of control performance that can be achieved using switched controllers [101–104].

Stability problems for switched systems have been investigated in many books and papers for general systems (e.g., see [83,105,106] and also specifically for dissipative systems including passive systems [107–109]. However, control problems for switched nonlinear NI systems have never been investigated. As nonlinear NI systems theory complements passivity theory by allowing systems to have relative degree two, and such systems arise in many control applications (see [93] and references therein), it is worth investigating the class of switched nonlinear NI systems and the corresponding stability problems.

In this chapter, we provide definitions of switched nonlinear NI and OSNI systems. A switched nonlinear system is said to be NI if all the subsystems being switched to are NI, and in addition, the storage functions for the subsystems in the switching sequence are nonincreasing. Switched nonlinear OSNI systems are also defined accordingly. It is shown that with external inputs and outputs considered, the closed-loop interconnection of two switched nonlinear NI (OSNI) systems is also switched NI (OSNI). Under suitable assumptions and zero input, the interconnection of two switched nonlinear NI systems are stable in the sense of Lyapunov. The main result of this chapter is that the closed-loop interconnection of a switched nonlinear NI system and a switched nonlinear OSNI system is asymptotically stable under some assumptions. This stability result is illustrated by an example where a nonlinear mass-spring-damper system is asymptotically stabilized by a certain switched nonlinear NI controller. The results in this chapter have been published in the paper [110].

7.2 Switched nonlinear NI systems

In this section, we provide definitions of nonlinear NI and OSNI properties for switched systems. We show that the positive feedback interconnection of two switched nonlinear NI systems, with external inputs and outputs considered, is also a switched nonlinear NI system. A similar result holds for switched nonlinear OSNI systems.

Consider a continuous time switched system of the following form (e.g., see [83, 106, 111]):

$$\dot{x} = f_\sigma(x, u), \tag{7.1a}$$

$$y = h(x), \tag{7.1b}$$

where $x(t) \in \mathbb{R}^n$ is the absolutely continuous state, $u(t) \in \mathbb{R}^p$ is a locally integrable input, and $y(t) \in \mathbb{R}^p$ is the output. Here, $h : \mathbb{R}^n \to \mathbb{R}^p$ is a class C^1 function and $\{f_i : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n, i \in \mathcal{I}\}$ is a family of Lipschitz continuous functions parametrized by an index set $\mathcal{I} = \{1, 2, \cdots, N\}$. The *switching signal* $\sigma : [0, \infty) \to \mathcal{I}$ is a piecewise constant function of time. The index $i = \sigma(t)$ is called the active mode at time t. We assume that σ is right continuous everywhere; i.e., $\sigma(t) = \lim_{\tau \to t^+} \sigma(\tau)$ for each $t \geq 0$. In general, the value of σ at time t can depend on t, $x(t)$ or both, or be generated using hybrid feedback with memory in the loop. Since the state $x(t)$ of (7.1) is an absolutely continuous function of time, it does not jump at switching instants. We assume that the system switches only a finite number of times on any finite time interval; i.e., there is no chattering. Then, we can denote by $\{(i_0, t_0), (i_1, t_1), \cdots, (i_k, t_k), \cdots\}$, $(k \in \mathbb{N})$ the switching index sequence. Here, t_0 is the initial time and $\mathbb{N}$ denotes the set of nonnegative integers. This means that at time t_k, the state equation of the system becomes $\dot{x} = f_{i_k}(x, u)$. Note that the switched system has a common state, a common input and a common output, while the dynamic equation of the system switches to another map f at every switching instant.

Now, we define switched nonlinear NI systems as follows.

Definition 7.1. *The switched nonlinear system (7.1) is said to be a switched nonlinear NI system if the following two conditions hold:*

1. *For all $i \in \mathcal{I}$, there exist positive semidefinite functions $V_i(x)$ such that for any input u and any solution x to (7.1) with $\sigma = i$,*

$$\frac{\partial V_i(x)}{\partial x} f_i(x,u) \le u^T \frac{\partial h(x)}{\partial x} f_i(x,u). \tag{7.2}$$

2. *For all $k \in \mathbb{N}$, $V_{i_{k+1}}(x) \le V_{i_k}(x)$ for all $x \in \mathbb{R}^n$.*

Similarly, we define switched nonlinear OSNI systems as follows.

Definition 7.2. *The switched nonlinear system (7.1) is said to be a switched nonlinear OSNI system if the following two conditions hold:*

1. *For all $i \in \mathcal{I}$, there exist positive semidefinite functions $V_i(x)$ such that for any input $u \in \mathbb{R}^p$ and any solution $x \in \mathbb{R}^n$ to (7.1) with $\sigma = i$,*

$$\frac{\partial V_i(x)}{\partial x} f_i(x,u) \le u^T \frac{\partial h(x)}{\partial x} f_i(x,u) - \epsilon \| \frac{\partial h(x)}{\partial x} f_i(x,u) \|^2. \tag{7.3}$$

2. *For all $k \in \mathbb{N}$, $V_{i_{k+1}}(x) \le V_{i_k}(x)$ for all $x \in \mathbb{R}^n$.*

Remark 7.1. *Definitions 7.1 and 7.2 state that a system is switched nonlinear NI (OSNI) if every subsystem satisfies the regular nonlinear NI (OSNI) system definition as given in Definitions 3.1 and 3.2 and in addition, the system switches in a pattern such that the storage functions of the subsystems in the switching index sequence are nonincreasing. That is, the function $V_{i_k}(x) - V_{i_{k+1}}(x)$ is positive semidefinite. Note that inequalities (7.2) and (7.3) correspond to inequalities (3.2) and (3.4) in Definitions 3.1 and 3.2 for non-switching systems, respectively. The time derivatives are expressed using the directional derivatives in order to specify the NI and OSNI inequalities for the subsystem that corresponds to the current switching index i_k.*

We investigate in the following the switched nonlinear NI (OSNI) property for the interconnection of two switched nonlinear NI (OSNI) systems with external inputs and outputs, as shown in Fig. 7.1. In what follows, the symbol "tilde", that is $\tilde{\cdot}$, indicates the variables associated with the system H_2.

Suppose the switched systems H_1 and H_2 in the closed-loop interconnection shown in Fig. 7.1 have, without loss of generality, synchronized switching instants. This also applies to the interconnection shown below in Fig. 7.2.

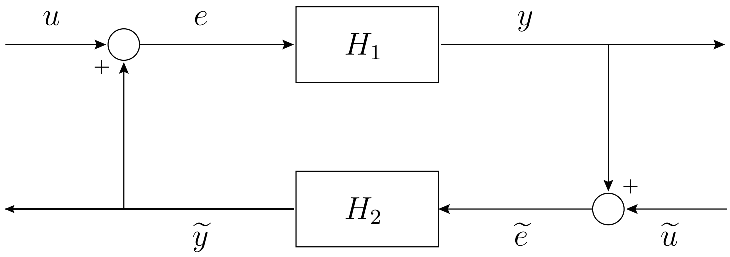


Figure 7.1: Closed-loop interconnection of switched nonlinear systems H_1 and H_2 with external inputs u, $\widetilde{u}$ and external outputs y, $\widetilde{y}$.

Theorem 7.1. *Suppose the systems H_1 and H_2 in the positive feedback interconnection in Fig. 7.1 are switched nonlinear NI (OSNI) systems of the form (7.1) with states $x \in \mathbb{R}^n, \widetilde{x} \in \mathbb{R}^{\widetilde{n}}$, inputs $e, \widetilde{e} \in \mathbb{R}^p$ and outputs $y, \widetilde{y} \in \mathbb{R}^p$, respectively. Also, suppose that for all $i_k \in \mathcal{I}$,*

$$V_{i_k}(x) + \widetilde{V}_{i_k}(\widetilde{x}) - h(x)^T \widetilde{h}(\widetilde{x}) \geq 0, \tag{7.4}$$

for all $x \in \mathbb{R}^n, \widetilde{x} \in \mathbb{R}^{\widetilde{n}}$, where $V_{i_k}(x)$ and $\widetilde{V}_{i_k}(\widetilde{x})$ are positive semidefinite storage functions that satisfy Definition 7.1 (Definition 7.2) for the systems H_1 and H_2, respectively. Then the closed-loop system with input $\widehat{u} = \begin{bmatrix} u^T & \widetilde{u}^T \end{bmatrix}^T$ and output $\widehat{y} = \begin{bmatrix} y^T & \widetilde{y}^T \end{bmatrix}^T$ is also a switched nonlinear NI (OSNI) system.

Proof. We first prove the OSNI part of the theorem. Suppose H_1 and H_2 are switched nonlinear OSNI systems. Then according to Definition 7.2, there exists positive semidefinite functions $V_{i_k}(x)$ and $\widetilde{V}_{i_k}(\widetilde{x})$ for every switching index $i_k \in \mathcal{I}$ such that

$$\begin{aligned}
\frac{\partial V_{i_k}(x)}{\partial x} f_{i_k}(x, e) &\leq e^T \frac{\partial h(x)}{\partial x} f_{i_k}(x, e) - \epsilon \| \frac{\partial h(x)}{\partial x} f_{i_k}(x, e) \|^2; \\
\frac{\partial \widetilde{V}_{i_k}(\widetilde{x})}{\partial \widetilde{x}} \widetilde{f}_{i_k}(\widetilde{x}, \widetilde{e}) &\leq \widetilde{e}^T \frac{\partial \widetilde{h}(\widetilde{x})}{\partial \widetilde{x}} \widetilde{f}_{i_k}(\widetilde{x}, \widetilde{e}) - \widetilde{\epsilon} \| \frac{\partial \widetilde{h}(\widetilde{x})}{\partial \widetilde{x}} \widetilde{f}_{i_k}(\widetilde{x}, \widetilde{e}) \|^2,
\end{aligned}$$

for all e, $\widetilde{e}$ and x, $\widetilde{x}$. In order to satisfy the conditions in Definition 7.2, we construct the storage function for the closed-loop system to be

$$W_{i_k}(x, \widetilde{x}) := V_{i_k}(x) + \widetilde{V}_{i_k}(\widetilde{x}) - h(x)^T \widetilde{h}(\widetilde{x}) \tag{7.5}$$

for the switching index $i_k \in \mathcal{I}$. According to (7.4), the function $W_{i_k}(x, \widetilde{x})$ is positive semidefinite. For Condition 1 in Definition 7.2, we have that

$$\dot{W}_{i_k}(x, \widetilde{x}) = \dot{V}_{i_k} + \dot{\widetilde{V}}_{i_k} - \dot{h}(x)^T \widetilde{h}(\widetilde{x}) - h(x)^T \dot{\widetilde{h}}(\widetilde{x})$$

$$
\begin{aligned}
&\leq e^T\dot{y} - \epsilon\|\dot{y}\|^2 + \widetilde{e}^T\dot{\widetilde{y}} - \widetilde{\epsilon}\|\dot{\widetilde{y}}\|^2 - \dot{y}^T\widetilde{y} - y^T\dot{\widetilde{y}} \\
&= (u+\widetilde{y})^T\dot{y} - \epsilon\|\dot{y}\|^2 + (\widetilde{u}+y)^T\dot{\widetilde{y}} - \widetilde{\epsilon}\|\dot{\widetilde{y}}\|^2 - \dot{y}^T\widetilde{y} - y^T\dot{\widetilde{y}} \\
&= u^T\dot{y} + \widetilde{u}^T\dot{\widetilde{y}} - \epsilon\|\dot{y}\|^2 - \widetilde{\epsilon}\|\dot{\widetilde{y}}\|^2 \\
&= \widehat{u}^T\dot{\widehat{y}} - \epsilon_{\min}\|\dot{\widehat{y}}\|^2, \qquad (7.6)
\end{aligned}
$$

where the system settings $e = u+\widetilde{y}$ and $\widetilde{e} = \widetilde{u}+y$ as shown in Fig. 7.1 are also used. Here, $\epsilon_{\min} = \min\{\epsilon, \widetilde{\epsilon}\} > 0$ is the level of output strictness for the interconnection of H_1 and H_2. For Condition 2 in Definition 7.2, we have that

$$W_{i_{k+1}}(x,\widetilde{x}) \leq W_{i_k}(x,\widetilde{x}) \qquad (7.7)$$

for all $k \in \mathbb{N}$ because $V_{i_{k+1}}(x) \leq V_{i_k}(x)$, $\widetilde{V}_{i_{k+1}}(\widetilde{x}) \leq \widetilde{V}_{i_k}(\widetilde{x})$, and $h(x)$ and $\widetilde{h}(\widetilde{x})$ are continuous. Therefore, since Conditions 1 and 2 in Definition 7.2 are both satisfied, then the system shown in Fig. 7.1 is a switched nonlinear OSNI system. In the case when H_1 and H_2 are both switched nonlinear NI, the proof of the switched nonlinear NI property for their interconnection follows from a similar analysis in the limiting case of $\epsilon = \widetilde{\epsilon} = 0$. □

7.3 Stability for the interconnection of switched nonlinear NI systems

In this section, we show that the interconnection of two switched nonlinear NI systems without external inputs is Lyapunov stable. Also, under some assumptions, the interconnection of a switched nonlinear NI system and a switched nonlinear OSNI system is asymptotically stable.

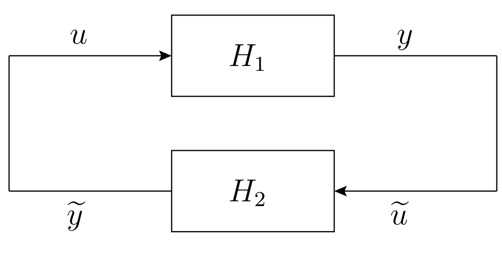


Figure 7.2: Closed-loop interconnection of switched nonlinear systems H_1 and H_2.

Theorem 7.2. *Suppose the systems H_1 and H_2 in the interconnection in Fig. 7.2 are switched nonlinear NI systems of the form (7.1) with states $x \in \mathbb{R}^n, \widetilde{x} \in \mathbb{R}^{\widetilde{n}}$, inputs $u, \widetilde{u} \in \mathbb{R}^p$ and outputs $y, \widetilde{y} \in \mathbb{R}^p$, respectively. Also, suppose for all $i_k \in \mathcal{I}$,* *the function*

$$V_{i_k}(x) + \widetilde{V}_{i_k}(\widetilde{x}) - h(x)^T \widetilde{h}(\widetilde{x}) \tag{7.8}$$

is positive definite for all $x \in \mathbb{R}^n, \widetilde{x} \in \mathbb{R}^{\widetilde{n}}$*, where* $V_{i_k}(x)$ *and* $\widetilde{V}_{i_k}(\widetilde{x})$ *are storage functions satisfying Definition 7.1 for systems* H_1 *and* H_2*, respectively. Then the interconnection of* H_1 *and* H_2 *shown in Fig. 7.2 is Lyapunov stable.*

Proof. According to Theorem 7.1, the interconnection of two switched nonlinear NI systems is also a switched nonlinear NI system. The function in (7.5) satisfies $\dot{W}_{i_k}(x, \widetilde{x}) \leq 0$ when there is no external input, according to (7.6). Assuming the function given by (7.8) to be positive definite ensures that $W_{i_k}(x, \widetilde{x})$ is a Lyapunov function for every switching index i_k. Also, since $W_{i_k}(x, \widetilde{x})$ satisfies (7.7), then we conclude that $W_{\sigma(t_2)}(x, \widetilde{x}) \leq W_{\sigma(t_1)}(x, \widetilde{x})$ for any $t_2 > t_1 \geq 0$. Therefore, the interconnection of H_1 and H_2 is Lyapunov stable. □

It is natural to ask whether the interconnection of H_1 and H_2 shown in Fig. 7.2 is asymptotically stable if H_1 and H_2 are both switched nonlinear OSNI systems. Indeed, the interconnection in Fig. 7.2 is asymptotically stable if at least one of the systems H_1 and H_2 is OSNI. This is shown in Theorem 7.3, where we require Assumptions 3.3–3.5 given in Chapter 3.

Theorem 7.3. *Consider a switched nonlinear NI system* H_1 *and a switched nonlinear OSNI system* H_2*, both of the form (7.1) with states* $x \in \mathbb{R}^n, \widetilde{x} \in \mathbb{R}^{\widetilde{n}}$*, inputs* $u, \widetilde{u} \in \mathbb{R}^p$ *and outputs* $y, \widetilde{y} \in \mathbb{R}^p$*, respectively. Suppose for all* $i_k \in \mathcal{I}$*,* H_1 *and* H_2 *satisfy Assumptions 3.3 and 3.4 and their cascade interconnection as shown in Fig. 3.2 satisfies Assumption 3.5. Also suppose the function*

$$W_{i_k}(x, \widetilde{x}) := V_{i_k}(x) + \widetilde{V}_{i_k}(\widetilde{x}) - h(x)^T \widetilde{h}(\widetilde{x}), \tag{7.9}$$

is positive definite for all $x \in \mathbb{R}^n, \widetilde{x} \in \mathbb{R}^{\widetilde{n}}$*, where* $V_{i_k}(x)$ *and* $\widetilde{V}_{i_k}(\widetilde{x})$ *are positive semidefinite storage functions satisfying Definition 7.1 for the system* H_1 *and Definition 7.2 for the system* H_2*, respectively. Then the closed-loop interconnection of the systems* H_1 *and* H_2 *in Fig. 7.2 is asymptotically stable.*

Proof. We apply Lyapunov's direct method. The candidate Lyapunov function defined in (7.9) is positive definite for all switching indices $i_k \in \mathcal{I}$. Therefore, the candidate Lyapunov function for the closed-loop system $W_\sigma(x, \widetilde{x})$ is positive definite. We need to prove that $W_\sigma(x, \widetilde{x})$ keeps decreasing until it reaches zero. We break the proof into two parts. First we show that for $t \in (t_k, t_{k+1})$ where $k \in \mathbb{N}$, the function $W_{i_k}(x, \widetilde{x})$ keeps decreasing unless $x = 0$. Taking the time derivative of

$W_{i_k}(x, \widetilde{x})$, we have

$$\begin{aligned}
\dot{W}_{i_k}(x, \widetilde{x}) &= \dot{V}_{i_k}(x) + \dot{\widetilde{V}}_{i_k}(\widetilde{x}) - \dot{h}(x)^T \widetilde{h}(\widetilde{x}) - h(x)^T \dot{\widetilde{h}}(\widetilde{x}) \\
&= \frac{\partial V_{i_k}(x)}{\partial x} f_{i_k}(x, u) + \frac{\partial \widetilde{V}_{i_k}(\widetilde{x})}{\partial \widetilde{x}} f_{i_k}(\widetilde{x}, \widetilde{u}) - \left(\frac{\partial h(x)}{\partial x} f_{i_k}(x, u) \right)^T \widetilde{h}(\widetilde{x}) \\
&\quad - h(x)^T \frac{\partial \widetilde{h}(\widetilde{x})}{\partial \widetilde{x}} \widetilde{f}(\widetilde{x}, \widetilde{u}) \\
&\leq u^T \frac{\partial h(x)}{\partial x} f_{i_k}(x, u) + \widetilde{u}^T \frac{\partial \widetilde{h}(\widetilde{x})}{\partial \widetilde{x}} \widetilde{f}_{i_k}(\widetilde{x}, \widetilde{u}) - \widetilde{\epsilon} \left\| \frac{\partial \widetilde{h}(\widetilde{x})}{\partial \widetilde{x}} \widetilde{f}_{i_k}(\widetilde{x}, \widetilde{u}) \right\|^2 \\
&\quad - \left(\frac{\partial h(x)}{\partial x} f_{i_k}(x, u) \right)^T \widetilde{h}(\widetilde{x}) - h(x)^T \frac{\partial \widetilde{h}(\widetilde{x})}{\partial \widetilde{x}} \widetilde{f}(\widetilde{x}, \widetilde{u}) \\
&= -\widetilde{\epsilon} \left\| \frac{\partial \widetilde{h}(\widetilde{x})}{\partial \widetilde{x}} \widetilde{f}_{i_k}(\widetilde{x}, \widetilde{u}) \right\|^2 \\
&\leq 0,
\end{aligned} \tag{7.10}$$

where the equalities also use the closed-loop setting that $u = \widetilde{y} = \widetilde{h}(\widetilde{x})$ and $\widetilde{u} = y = h(x)$. The inequality (7.10) implies that $\dot{W}_{i_k}(x, \widetilde{x})$ is non-increasing. Now we apply LaSalle's invariance principle. According to (7.10), $\dot{W}_{i_k}(x, \widetilde{x})$ remains zero only if $\frac{\partial \widetilde{h}(\widetilde{x})}{\partial \widetilde{x}} \widetilde{f}_{i_k}(\widetilde{x}, \widetilde{u})$ remains zero. This implies that $\widetilde{x}$ remains constant according to Assumption 3.3. According to Assumption 3.4, $\widetilde{x}$ remaining constant implies that $\widetilde{u}$ remains constant. Considering the setting $\widetilde{u} = y$ as shown in Fig. 7.2, we have that y also remains constant. This implies that both x and u remain constant according to Assumptions 3.3 and 3.4. Therefore, the system is in a steady state. In this case, the inputs $u(t)$, $\widetilde{u}(t)$, states $x(t)$, $\widetilde{x}(t)$ and outputs $y(t)$, $\widetilde{y}(t)$ are all constant vectors, which we denote by $\overline{u}$, $\overline{\widetilde{u}}$, $\overline{x}$, $\overline{\widetilde{x}}$, $\overline{y}$, $\overline{\widetilde{y}}$, respectively. According to Assumption 3.5, if $\overline{u} \neq 0$, then $\overline{u} \neq \overline{\widetilde{y}}$, which contradicts $\overline{u} = \overline{\widetilde{y}}$. This implies that (7.10) can hold only if $\overline{u} = \overline{\widetilde{y}} = 0$. According to Assumptions 3.3 and 3.4, we have that $\overline{\widetilde{y}} = 0 \implies \overline{\widetilde{x}} = 0 \implies \overline{\widetilde{u}} = \overline{y} = 0 \implies \overline{x} = 0$. This means that the system is already in equilibrium. Otherwise, $W_{i_k}(x, \widetilde{x})$ keeps decreasing until it reaches zero. This completes the first part of the proof.

We also need to prove that $W_\sigma(x, \widetilde{x})$ does not increase at all switching instants t_k. At a switching instant t_{k+1}, the switching index changes from i_k to i_{k+1}, which leads to a jump of $W_\sigma(x, \widetilde{x})$ from $W_{i_k}(x(t_{k+1}), \widetilde{x}(t_{k+1}))$ to $W_{i_{k+1}}(x(t_{k+1}), \widetilde{x}(t_{k+1}))$. Note that there is no jump in the states x and $\widetilde{x}$. Therefore, using (7.9), we have that

$$\begin{aligned}
&W_{i_{k+1}}(x(t_{k+1}), \widetilde{x}(t_{k+1})) - W_{i_k}(x(t_{k+1}), \widetilde{x}(t_{k+1})) \\
&\quad = V_{i_{k+1}}(x(t_{k+1})) + \widetilde{V}_{i_{k+1}}(\widetilde{x}(t_{k+1})) - h(x(t_{k+1}))^T \widetilde{h}(\widetilde{x}(t_{k+1})) - V_{i_k}(x(t_{k+1}))
\end{aligned}$$

$$
\begin{aligned}
&- \widetilde{V}_{i_k}(\widetilde{x}(t_{k+1})) + h(x(t_{k+1}))^T \widetilde{h}(\widetilde{x}(t_{k+1})) \\
&= V_{i_{k+1}}(x(t_{k+1})) - V_{i_k}(x(t_{k+1})) + \widetilde{V}_{i_{k+1}}(\widetilde{x}(t_{k+1})) - \widetilde{V}_{i_k}(\widetilde{x}(t_{k+1})) \\
&\leq 0,
\end{aligned}
$$

according to Condition 2 in Definitions 7.1 and 7.2. Therefore, we conclude that the storage function $W_\sigma(x, \widetilde{x})$ keeps decreasing monotonically except when $(x, \widetilde{x}) = (0, 0)$. Since $W_\sigma(x, \widetilde{x}) \geq 0$, then $W_\sigma(x, \widetilde{x})$ is bounded from below and hence $W_\sigma(x, \widetilde{x})$ has a limit c as $t \to \infty$. We prove by contradiction that $c = 0$. Suppose $c > 0$, then it implies that $\begin{bmatrix} x(t)^T & \widetilde{x}(t)^T \end{bmatrix}^T \neq 0$ for all $t \geq 0$. In this case, $(x, \widetilde{x})$ converges to a limit set $\mathcal{S}$ such that $(x, \widetilde{x}) \in \mathcal{S}$ implies $\dot{W}_\sigma(x, \widetilde{x}) = 0$. However, as is proved above, $\dot{W}_\sigma(x, \widetilde{x})$ remains zero only at $(x, \widetilde{x}) = (0, 0)$. This contradicts $\begin{bmatrix} x(t)^T & \widetilde{x}(t)^T \end{bmatrix}^T \neq 0$. Therefore, $c = 0$ and the system is asymptotically stable. ☐

Remark 7.2. *A special case of Theorem 7.3 is when there exists a common storage function for all switching indices for the system H_1, and respectively the system H_2; that is $V_i(\cdot) = V_j(\cdot)$ for all $i, j \in \mathcal{I}$ and $\widetilde{V}_i(\cdot) = \widetilde{V}_j(\cdot)$ for all $i, j \in \widetilde{\mathcal{I}}$. Although this special case is included in Theorem 7.3, it is important for switched systems because it allows a switched system with a finite index set $\mathcal{I}$ to have infinite and arbitrary switching sequence. Hence, we provide a stability result for this particular case.*

Corollary 7.1. *For the system setting in Theorem 7.3, suppose $V_i(\cdot) = V_j(\cdot)$ for all $i, j \in \mathcal{I}$ and $\widetilde{V}_i(\cdot) = \widetilde{V}_j(\cdot)$ for all $i, j \in \widetilde{\mathcal{I}}$. Also, suppose Assumptions I-III are satisfied, and the storage function given by*

$$
W(x, \widetilde{x}) := \; V(x) + \widetilde{V}(\widetilde{x}) - h(x)^T \widetilde{h}(\widetilde{x}),
$$

is positive definite for all $x \in \mathbb{R}^n, \widetilde{x} \in \mathbb{R}^{\widetilde{n}}$, where $V(x)$ and $\widetilde{V}(\widetilde{x})$ are the common positive semidefinite storage functions satisfying Definition 7.1 for the system H_1 and Definition 7.2 for the system H_2, respectively. Then the closed-loop interconnection for the systems H_1 and H_2 in Fig. 7.2 is asymptotically stable.

Proof. Since Corollary 7.1 considers a special yet important case of Theorem 7.3, the proof follows directly from that of Theorem 7.3. ☐

7.4 Illustrative example

In this section, we provide a numerical example demonstrating the stability of the interconnection of a switched nonlinear NI system and a switched nonlinear OSNI system using the results given in Section 7.3. Here, we apply a state dependent switched nonlinear NI system called hybrid integrator-gain system (HIGS) as a controller to a mass-spring-damper system with a nonlinear spring. We will investigate HIGS as an NI controller in particular later in Chapter 9. The HIGS was developed in [103] to improve low-frequency disturbance rejection behaviour and to reduce high-frequency amplification and overshoot [104]. This is realized by a switching mechanism between an integrator mode and a gain mode. The system model of a HIGS is given in the following (see [103]):

$$\mathcal{H}:\begin{cases}\dot{x}_h(t)=\omega_h e(t), & \text{if}\,(e(t),u(t),\dot{e}(t))\in\mathcal{F}_1\\ x_h(t)=k_h e(t), & \text{if}\,(e(t),u(t),\dot{e}(t))\in\mathcal{F}_2\\ u(t)=x_h(t)\end{cases}\tag{7.11}$$

where $x_h(t), e(t), u(t) \in \mathbb{R}$ denote the HIGS state, input and output, respectively. For convenience, we omit the time arguments in what follows. Here, $\dot{e}$ is the time derivative of the input e which is assumed to be continuous and piecewise differentiable. The parameters $\omega_h \in [0,\infty)$ and $k_h \in (0,\infty)$ are the integrator frequency and gain value, respectively. The sets $\mathcal{F}_1, \mathcal{F}_2 \in \mathbb{R}^3$ determine the HIGS modes of operation; that is, the integrator and gain modes, respectively. By construction, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ represents the sector $[0, k_h]$ as (see [103])

$$\mathcal{F}=\{(e,u,\dot{e})\in\mathbb{R}^3|\,eu\geq\frac{u^2}{k_h}\},$$

and $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined as

$$\begin{aligned}&\mathcal{F}_1:=\mathcal{F}\setminus\mathcal{F}_2;\\ &\mathcal{F}_2:=\{(e,u,\dot{e})\in\mathbb{R}^3|u=k_h e\quad\text{and}\quad\omega_h e^2>k_h e\dot{e}\}.\end{aligned}$$

The HIGS primarily operates in the integrator mode. It will be switched to the gain mode once the system dynamics reach the boundary of $\mathcal{F}$ and tend to leave $\mathcal{F}$ if it stays in the integrator mode.

The two modes of the HIGS (7.11) satisfy the inequality (7.2) with a common Lyapunov function

$$V(x)=\frac{1}{2k_h}x_h^2.$$

We consider a nonlinear OSNI mass-spring-damper system of the following form to be the plant

$$\begin{aligned} H_1: \quad \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1^3 - x_1 - x_2 + u, \\ y &= x_1. \end{aligned} \tag{7.12}$$

The system is a nonlinear OSNI system with the storage function

$$V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2,$$

which satisfies inequality (3.4). According to Theorem 7.3, if the required conditions are satisfied, then the system H_1 of the form (7.12) can be asymptotically stabilized by applying the HIGS $\mathcal{H}$ of the form (7.11) in positive feedback, as shown in Fig. 7.2.

It can be verified that Assumptions 3.3 and 3.4 are satisfied. For Assumption 3.5, given any nonzero input $\overline{u}$ to the system H_1, suppose the HIGS output remains constant $x_h = \overline{x}_h$, then in the case of integrator mode, we have $e = 0$. This implies that $x_1 \equiv e \equiv 0$. Hence, $x_2 = \dot{x}_1 \equiv 0$ and $\dot{x}_2 = 0$. Therefore, $\overline{u} = x_1^3 + x_1 = 0$. In this case, the system must be in the gain mode. We have $e = \overline{e} = \frac{1}{k_h}\overline{x}_h$. This implies that $x_1 = \overline{x}_1 = \overline{e} = \frac{1}{k_h}\overline{x}_h$. Similarly, we have $x_2 \equiv 0$ and $\dot{x}_2 = 0$, which implies that $\overline{u} = \overline{x}_1^3 + \overline{x}_1$. Hence, we need to show that

$$\overline{x}_1^3 + \overline{x}_1 \neq k_h \overline{x}_1.$$

By choosing $k_h \in (0, 1]$, this inequality is satisfied except for $\overline{x}_1 = 0$. This implies that Assumption 3.5 is satisfied. Since $k_h \leq 1$, then the candidate Lyapunov function of the interconnection given by

$$W = \frac{1}{4}x_1^4 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2k_h}x_h^2 - x_h x_1$$

is positive definite. According to Theorem 7.3, the interconnection of the system H_1 and the HIGS of the form (7.11) with $k_h \leq 1$ is asymptotically stable.

By choosing $k_h = 0.8$ and $\omega_h = 20$ and letting the system H_1 have an initial displacement $x_1 = 5$, the state trajectories of the closed-loop interconnection of the system H_1 and the HIGS $\mathcal{H}$ are obtained via simulation and are shown in Fig. 7.3.

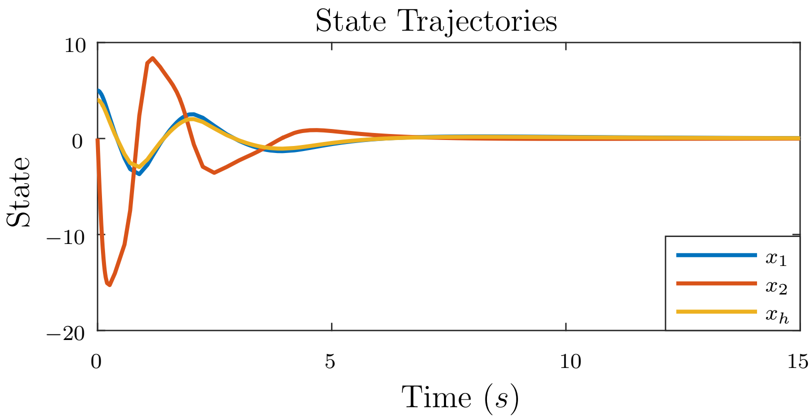


Figure 7.3: State trajectories of the state variables of the closed-loop interconnection of the system H_1 given by (7.12) and the HIGS $\mathcal{H}$ given by (7.11).

7.5 Conclusion

In this chapter, the nonlinear NI systems theory has been extended to switched systems. We have provided definitions for switched nonlinear NI and OSNI systems. It has been shown that the closed-loop interconnection of two switched nonlinear NI systems with external inputs and outputs is also switched nonlinear NI, and the closed-loop interconnection is Lyapunov stable under zero input. Under some assumptions, the closed-loop interconnection of a switched nonlinear NI system and a switched nonlinear OSNI system has been shown to be asymptotically stable. This stability result has been illustrated using an example where a nonlinear mass-spring-damper system is asymptotically stabilized by a HIGS controller, which is a switched nonlinear NI system.

Chapter 8

Discrete-Time Negative Imaginary Systems from Zero-Order Hold Sampling

8.1 Introduction

In Chapters 3–7, the systems we studied are all continuous-time systems. However, in practice, digital computers are almost always used in the control of physical systems [112]. Therefore, it is important to establish a discrete-time NI systems theory. In this chapter, we provide a definition of discrete-time NI systems and propose positive feedback control approaches to achieve stability for such systems.

A different definition of discrete-time NI systems for linear systems has already been introduced in [39, 40]. Inspired by the discrete-time PR systems theory [113], discrete-time NI systems were defined by analogy to continuous-time NI systems. It is shown in [39, 40] that the continuous-time NI systems definition as well as the stability results can be mapped to the discrete-time case using a bilinear transform. However, as pointed out by many papers (see e.g. [114–117]), the definition of discrete-time passive systems [118] and PR systems [113] lead to several issues as compared to their original continuous-time analogues. A commonly observed issue is that discretizating a continuous-time passive system using a zero-order hold (ZOH) device does not in general yield a discrete-time passive system [13]. This issue is also present in the discrete-time NI systems definition introduced in [39, 40] due to the use of bilinear transform.

In the digital control of a continuous-time physical system, a computer reads the output signal of the plant at discrete instants of time and obtains a discrete signal [112]. Then the computer generates a discrete control signal according to a certain control law. The discrete control signal is fed into the plant as a constant continuous-time signal for an entire sampling period until the next control signal is generated. This process is termed ZOH sampling. Therefore, in order to develop digital controllers for physical systems with NI properties, it is useful to find a definition such that the property of NI systems is preserved under ZOH sampling. This would provide a practical approach to digital controller synthesis for NI systems. Also, as mentioned above, the results in [39,40] were obtained from continuous-time results using a bilinear transform, which is a frequency-domain method and hence is only applicable to linear time-invariant systems. Therefore, it is useful to develop an alternative time-domain formulation of discrete-time NI properties for nonlinear systems with a clearer link to work-energy balance relations.

In this chapter, we provide a new discrete-time NI systems definition that describes the energy property of a ZOH sampled continuous-time NI system. Nonlinear systems are allowed in this definition. We propose a feedback control framework to stablize a system that satisfies this definition. To be specific, we introduce discrete-time output strictly negative imaginary (OSNI) systems, which is similar and yet different to continuous-time OSNI systems defined in Definition 3.2. A modification of an OSNI system, which takes one step advance (yet is physically realizable), is called a step advanced output strictly negative imaginary (SAOSNI) system. We prove that the closed-loop interconnection of an NI system and an SAOSNI system is asymptotically stable, if certain assumptions are satisfied. A similar stability result also holds for the interconnection of a step advanced negative imaginary system and an OSNI system. In the case of linear systems, we provide linear matrix inequality (LMI) conditions and frequency-domain conditions under which a system is NI, which are different from, yet related to, the conditions provided in [39]. For linear systems, the assumptions used in the stability results can be replaced by a simple DC gain condition. We also provide an example, where a ZOH sampled mass-spring system, that is shown to be NI, is stabilized using an SAOSNI controller according to the proposed results. The results in this chapter are included in the paper [119].

8.2 Discrete-time NI systems

Consider the following discrete-time system

$$H_1\colon \quad x_{k+1} = f(x_k, u_k), \tag{8.1a}$$

$$y_k = h(x_k), \tag{8.1b}$$

where $f\colon \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ and $h\colon \mathbb{R}^n \to \mathbb{R}^p$. Here $u_k, y_k \in \mathbb{R}^p$ and $x_k \in \mathbb{R}^n$ are the input, output and state of the system, respectively, at time step $k \in \mathbb{N}$.

Definition 8.1. *The system (8.1) is said to be a discrete-time negative imaginary (NI) system if there exists a continuous positive semidefinite storage function $V\colon \mathbb{R}^n \to \mathbb{R}$ such that for arbitrary states x_k and inputs u_k,*

$$V(x_{k+1}) - V(x_k) \leq u_k^T \left(y_{k+1} - y_k\right), \tag{8.2}$$

for all k.

We show in the following that discretizing a continuous-time NI system yields a discrete-time NI system. Consider a continuous-time system

$$\begin{aligned} \dot{x}(t) &= \mathfrak{f}(x(t), u(t)), \\ y(t) &= \mathfrak{h}(x(t)) \end{aligned} \tag{8.3}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^p$ and output $y \in \mathbb{R}^p$. Here, $\mathfrak{f}\colon \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a Lipschitz function and $\mathfrak{h}\colon \mathbb{R}^n \to \mathbb{R}^p$ is a continuously differentiable function.

Proposition 8.1. *Suppose the system (8.3) is NI, then discretizing the system (8.3) using ZOH sampling yields a discrete-time NI system.*

Proof. According to Definition 3.1, the system (8.3) is said to be NI if there exists a continuously differentiable positive semidefinite function $V\colon \mathbb{R}^n \to \mathbb{R}$ such that for any locally integrable input u and any solution x to (8.3),

$$\dot{V}(x(t)) \leq u(t)^T \dot{y}(t), \tag{8.4}$$

for almost all $t \geq 0$. Using ZOH sampling, we keep the input of the system constant over the time interval $[t_k, t_{k+1})$; i.e., $u(t) = u_k$ for $t \in [t_k, t_{k+1})$. Then, integrating the NI inequality (8.4) over the time interval $[t_k, t_{k+1})$ yields

$$\int_{t_k}^{t_{k+1}} \dot{V}(x(\tau)) d\tau \leq \int_{t_k}^{t_{k+1}} u_k^T \dot{y}(\tau) d\tau.$$

That is

$$V(x_{k+1}) - V(x_k) \leq u_k^T (y_{k+1} - y_k),$$

which is the discrete-time NI inequality (8.2) in Definition 8.1, with $x_k = x(t_k)$, $y_k = y(t_k)$ and similarly defined x_{k+1} and y_{k+1}. $\square$

Inspired by the above discrete-time negative imaginary systems definition, we provide a different version of the definition of continuous-time NI systems, which is equivalent to Definition 3.1.

Definition 8.2 (continuous-time NI system)**.** *The system (8.3) is said to be negative imaginary if there exists a continuously differentiable positive semidefinite storage function* $V : \mathbb{R}^n \to \mathbb{R}$ *such that for any time* $T > 0$*, initial condition* $x(0)$ *and any constant input vector* $u(t) = v$*,*

$$V(x(T)) - V(x(0)) \leq v^T(y(T) - y(0)),$$

for $t \in [0, T]$*.*

An advantage of this definition is that it does not involve any derivative or integral terms (though it is obtained by integration over $[0, T]$), in comparison to Definition 3.1 and the definition of passive systems [15].

Remark 8.1. *Definition 8.1 does not reduce to the definition of discrete-time NI systems in [39] in the linear time-invariant case. This is because [39] considers a bilinear transform for conversion of a continuous-time system to a discrete-time system, while this chapter considers the ZOH sampling. Throughout the remainder of the chapter, the term "NI" refers to the discrete-time NI property as defined in Definition 8.1, unless stated otherwise.*

8.3 The NI property for linear systems

In this section, we investigate the conditions for a linear system to satisfy Definition 8.1. Consider the case where the system (8.1) is linear and has a state-space realization as follows

$$x_{k+1} = Ax_k + Bu_k, \tag{8.5a}$$
$$y_k = Cx_k. \tag{8.5b}$$

The following theorem provides necessary and sufficient conditions under which the system (8.5) is NI, with a quadratic storage function.

Theorem 8.1. *The linear system (8.5) is NI with a quadratic storage function of the form*

$$V(x_k) = \frac{1}{2}x_k^T P x_k, \tag{8.6}$$

satisfying the dissipation inequality (8.2), if and only if the following matrix is pos-

itive semidefinite:

$$M = \begin{bmatrix} P - A^T PA & (A^T - I)C^T - A^T PB \\ C(A - I) - B^T PA & CB + B^T C^T - B^T PB \end{bmatrix}. \tag{8.7}$$

Furthermore, in the case when

$$\det(I - A) \neq 0, \tag{8.8}$$

the NI property (8.2) with the storage function (8.6) for the system (8.5) is equivalent to the fulfillment of the following two conditions:

$$C = B^T (I - A)^{-T} P, \tag{8.9}$$

$$P - A^T PA \geq 0. \tag{8.10}$$

Proof. In view of the system dynamics (8.5), the NI property (8.2) can be represented by the inequality

$$u^T C(Ax + Bu - x) - (V(Ax + Bu) - V(x)) \geq 0 \tag{8.11}$$

for any state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^p$. This inequality is obtained by subtracting the left-hand side of (8.2) from its right-hand side. In application to the quadratic storage function (8.6), the condition (8.11) takes the form

$$\begin{aligned} 0 &\leq u^T C((A - I)x + Bu) + \frac{1}{2} x^T Px - \frac{1}{2}(Ax + Bu)^T P(Ax + Bu) \\ &= \frac{1}{2}(x^T (P - A^T PA)x + u^T (CB + B^T C^T - B^T PB)u) + u^T (C(A - I) - B^T PA)x \\ &= \frac{1}{2}\xi^T M\xi, \end{aligned} \tag{8.12}$$

where M is defined as in (8.7) and the right-hand side is a quadratic function of the augmented state-input vector

$$\xi := \begin{bmatrix} x \\ u \end{bmatrix}, \tag{8.13}$$

which can take any value in $\mathbb{R}^{n+p}$. The inequality (8.12) holds for arbitrary ξ if and only if the matrix M is positive semi-definite:

$$M \geq 0. \tag{8.14}$$

Therefore, the NI property (8.2) with the quadratic storage function (8.6) is indeed equivalent to (8.14). We will now prove that, under the condition (8.8), the inequality (8.14) is equivalent to (8.9) and (8.10). Note that (8.14) implies (8.10) regardless of (8.8) because $P - A^T PA$ is a diagonal block of M. Now since (8.14) implies the NI property (8.11), the latter holds, in particular, when the system state x is related to an arbitrary input $u \in \mathbb{R}^p$ by

$$x = (I - A)^{-1} Bu, \tag{8.15}$$

where, this time, we have used (8.8). Since the relation (8.15) is equivalent to $Ax + Bu = x$, any such pair (x, u) makes the left-hand side of (8.11) and the right-hand side of (8.12) vanish. Therefore, the corresponding vector ξ in (8.13) takes the form

$$\xi = \begin{bmatrix} (I - A)^{-1} Bu \\ u \end{bmatrix} = Fu, \text{ where } F := \begin{bmatrix} (I - A)^{-1} B \\ I \end{bmatrix} \in \mathbb{R}^{(n+p)\times p}, \tag{8.16}$$

which satisfies $\xi^T M \xi = 0$ (recall that (8.14) holds), and hence, belongs to the null space of the positive semi-definite matrix M:

$$0 = M\xi = MFu. \tag{8.17}$$

Since $u \in \mathbb{R}^p$ was arbitrary, the relation (8.17) implies

$$MF = 0. \tag{8.18}$$

In particular, (8.18) implies the product of the first block-row $M_{1\bullet}$ of the matrix M in (8.7) and the matrix F from (8.16) vanish:

$$\begin{aligned} 0 &= M_{1\bullet} F \\ &= \begin{bmatrix} P - A^T PA & (A^T - I)C^T - A^T PB \end{bmatrix} \begin{bmatrix} (I - A)^{-1} B \\ I \end{bmatrix} \\ &= (P - A^T PA)(I - A)^{-1} B + (A^T - I)C^T - A^T PB \\ &= P(I - A)^{-1} B - A^T P((I - A)^{-1} - I)B + (A^T - I)C^T - A^T PB \\ &= P(I - A)^{-1} B - A^T P(I - A)^{-1} B + (A^T - I)C^T \\ &= (A^T - I)(C^T - P(I - A)^{-1} B), \end{aligned}$$

which leads to (8.9) due to the nonsingularity of $A^T - I$ which follows from (8.8). This completes the proof that (8.8), (8.14) imply (8.9), (8.10). It now remains to prove that (8.8), (8.9), (8.10) imply (8.14). To this end, we note that (8.8) ensures (8.9) is well-defined, and the substitution of (8.9) into the appropriate blocks of (8.7) yields

$$\begin{aligned}
M_{22} &= CB + B^T C^T - B^T PB \\
&= B^T(I-A)^{-T}PB + B^T P(I-A)^{-1}B - B^T PB \\
&= B^T(I-A)^{-T}(P(I-A) + (I-A^T)P - (I-A^T)P(I-A))(I-A)^{-1}B \\
&= B^T(I-A)^{-T}(P - A^T PA)(I-A)^{-1}B, && (8.19) \\
M_{12} &= (A^T - I)C^T - A^T PB \\
&= (A^T - I)P(I-A)^{-1}B - A^T PB \\
&= \left((A^T - I)P - A^T P(I-A)\right)(I-A)^{-1}B \\
&= -(P - A^T PA)(I-A)^{-1}B. && (8.20)
\end{aligned}$$

By substituting (8.19) and (8.20) along with $M_{21} = M_{12}^T$ into (8.7), it follows that the matrix M can be factorized as

$$M = G^T(P - A^T PA)G, \tag{8.21}$$

where $G := \begin{bmatrix} -I & (I-A)^{-1}B \end{bmatrix} \in \mathbb{R}^{n\times(n+p)}$. The factorization (8.21) shows that (8.10) implies (8.14). Thus (8.8), (8.9), (8.10) imply (8.14). This completes the proof of the theorem. □

Theorem 8.2 below gives conditions under which a linear system will have the NI property. This theorem uses the following preliminary result.

Lemma 8.1. *Suppose the system (8.5) is minimal and satisfies* $\det(I + A) \neq 0$ *and* $\det(I - A) \neq 0$. *Then there exists* $X = X^T > 0$ *satisfying*

$$X - A^T XA \geq 0, \text{ and } C = B^T(I-A)^{-T}X(A+I)$$

if and only if the transfer matrix $\check{G}(z) = C(zI - A)^{-1}B$ *satisfies the following conditions:*

1. $\check{G}(z)$ *has no poles in* $|z| > 1$.
2. $i\left[\check{G}(e^{i\theta}) - \check{G}(e^{i\theta})^*\right] \geq 0$ *for all* $\theta \in (0, \pi)$ *except for the values of* θ *for which* $z = e^{i\theta}$ *is a pole of* $\check{G}(z)$.
3. *If* $z_0 = e^{i\theta_0}$, *with* $\theta_0 \in (0, \pi)$, *is a pole of* $\check{G}(z)$, *then it is a simple pole and the normalized residue matrix*

$$K_0 := \frac{1}{z_0}\lim_{z\to z_0}(z - z_0)i\check{G}(z)$$

is Hermitian and positive semidefinite.

Proof. See Lemma 3.2 and Theorem 3.2 of [39]. Note that the cases considered in Conditions (iv) and (v) in Lemma 3.2 in [39] where $z_0 = -1$ and $z_0 = 1$ are excelueded here. This is because of the assumptions $\det(I + A) \neq 0$ and $\det(I - A) \neq 0$ in Theorem 3.2 in [39]. □

The following theorem provides frequency-domain conditions for a linear system to be NI.

Theorem 8.2. *Suppose the minimal system (8.5) satisfies* $\det(I + A) \neq 0$ *and* $\det(I - A) \neq 0$*. Then it is NI if and only if its transfer matrix* $G(z) = C(zI - A)^{-1}B$ *satisfies the following conditions:*

1. $G(z)$ *has no poles in* $|z| > 1$.
2. $i\left[(e^{i\theta} + 1)G(e^{i\theta}) - (e^{-i\theta} + 1)G(e^{i\theta})^* - L + L^T\right] \geq 0$ *for all* $\theta \in (0, \pi)$ *except for the values of* θ *for which* $z = e^{i\theta}$ *is a pole of* $G(z)$*. Here,*

$$L = \lim_{z \to \infty} zG(z). \tag{8.22}$$

3. *If* $z_0 = e^{i\theta_0}$*, with* $\theta_0 \in (0, \pi)$*, is a pole of* $G(z)$*, then it is a simple pole and the normalized residue matrix*

$$K_0 := \left(1 + \frac{1}{z_0}\right) \lim_{z \to z_0} (z - z_0) iG(z) \tag{8.23}$$

is Hermitian and positive semidefinite.

Proof. By comparing the LMI conditions in Theorem 8.1 and Lemma 8.1, we have that the system (8.5) is NI if and only if the transfer matrix $\check{G}(z)$ of the realization $(A, B, C(A+I))$ satisfies the frequency-domain conditions in Lemma 8.1. By letting $\check{y}_k = C(A + I)x_k$, we have

$$\check{y}_k = CAx_k + Cx_k = y_{k+1} + y_k - CBu_k.$$

This implies that

$$\check{G}(z) = (z + 1)G(z) - CB. \tag{8.24}$$

The matrix CB can be represented as $CB = \lim_{z\to\infty} zG(z)$. We show in the following that $\check{G}(z)$ satisfies Conditions 1–3 in Lemma 8.1 if and only if $G(z)$ satisfies Conditions 1–3 in the present Theorem. We substitute (8.24) into the corresponding conditions in Lemma 8.1. More precisely, the transfer matrix $\check{G}(z)$ satisfies Condition 1 in Lemma 8.1 if and only if Condition 1 in the present theorem is satisfied. Condition 2 is established by substituting (8.24) into $i\left[\check{G}(e^{i\theta}) - \check{G}(e^{i\theta})^*\right] \geq 0$.

Condition 3 is established via the following derivation

$$\begin{aligned} K_0 &= \frac{1}{z_0} \lim_{z \to z_0} (z - z_0) i \breve{G}(z) \\ &= \frac{1}{z_0} \lim_{z \to z_0} (z - z_0) i \left((z+1)G(z) - CB\right) \\ &= \frac{z_0 + 1}{z_0} \lim_{z \to z_0} (z - z_0) i G(z). \end{aligned}$$

□

8.4 Stability for the interconnection of NI systems

In this section, we investigate the stability of the feedback interconnection of NI systems with positive definite storage functions. First, we introduce several variations of NI systems. Consider a system with the model

$$H_2\colon \quad \widehat{x}_{k+1} = \widehat{f}(\widehat{x}_k, \widehat{u}_k), \tag{8.25a}$$

$$\widehat{y}_k = \widehat{h}(\widehat{x}_k), \tag{8.25b}$$

where $\widehat{f}\colon \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$ and $\widehat{h}\colon \mathbb{R}^m \to \mathbb{R}^p$. Here $\widehat{u}_k, \widehat{y}_k \in \mathbb{R}^p$ and $\widehat{x}_k \in \mathbb{R}^m$ are the input, output and state of the system, respectively, at time step $k \in \mathbb{N}$.

Definition 8.3. *The system (8.25) is said to be a discrete-time output strictly negative imaginary (OSNI) system if there exists a continuous positive semidefinite storage function $\widehat{V}\colon \mathbb{R}^m \to \mathbb{R}$ and a scalar $\epsilon > 0$ such that for arbitrary states $\widehat{x}_k$ and inputs $\widehat{u}_k$,*

$$\widehat{V}(\widehat{x}_{k+1}) - \widehat{V}(\widehat{x}_k) \le \widehat{u}_k^T (\widehat{y}_{k+1} - \widehat{y}_k) - \epsilon \|\widehat{y}_{k+1} - \widehat{y}_k\|^2,$$

for all k. In this case, we also say that system (8.25) is OSNI with degree of output strictness ϵ.

Remark 8.2. *Note that unlike the relationship between continuous-time and discrete-time NI systems definition, discrete-time OSNI systems are not defined in a way such that ZOH sampling a continuous-time OSNI system yields a discrete-time OSNI system. The purpose of defining OSNI systems is to use the output strictness to achieve stability. Such discrete-time OSNI systems exist and can be applied in the control of discrete-time NI systems, which is shown in the example in Section 8.5.*

We show in the following that the NI system H_1 can be asymptotically stabilized by an auxiliary system associated with the OSNI system H_2 in positive feedback. The auxiliary system, denoted by $\widetilde{H}_2$, is described by (8.25a) and

$$\widetilde{y}_k = \widehat{h}(\widehat{f}(\widehat{x}_k, \widehat{u}_k)) = \widehat{y}_{k+1}. \tag{8.26}$$

For an OSNI system (8.25), we call the auxiliary system given by (8.25a), (8.26) a step advanced output strictly negative imaginary (SAOSNI) system. A formal definition of SAOSNI systems is as follows.

Consider the system

$$\widetilde{H}_2\colon \quad \widehat{x}_{k+1} = \widehat{f}(\widehat{x}_k, \widehat{u}_k), \tag{8.27a}$$
$$\widetilde{y}_k = \widetilde{h}(\widehat{x}_k, \widehat{u}_k), \tag{8.27b}$$

where $\widehat{f}\colon \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$ and $\widetilde{h}\colon \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^p$. Here $\widehat{u}_k, \widetilde{y}_k \in \mathbb{R}^p$ and $\widehat{x}_k \in \mathbb{R}^m$ are the input, output and state of the system, respectively, at time step $k \in \mathbb{N}$.

Definition 8.4. *The system* (8.27) *is said to be a discrete-time step advanced output strictly negative imaginary (SAOSNI) system if there exists a continuous function* $\widehat{h}(\widehat{x}_k)$ *such that:*

1. $\widetilde{h}(\widehat{x}_k, \widehat{u}_k) = \widehat{h}(\widehat{f}(\widehat{x}_k, \widehat{u}_k))$.
2. *There exists a continuous positive semidefinite storage function* $\widehat{V}\colon \mathbb{R}^m \to \mathbb{R}$ *and a scalar* $\epsilon > 0$ *such that for arbitrary states* $\widehat{x}_k$ *and inputs* $\widehat{u}_k$*,*

$$\widehat{V}(\widehat{x}_{k+1}) - \widehat{V}(\widehat{x}_k) \le \widehat{u}_k^T \left(\widehat{h}(\widehat{x}_{k+1}) - \widehat{h}(\widehat{x}_k)\right) - \epsilon\|\widehat{h}(\widehat{x}_{k+1}) - \widehat{h}(\widehat{x}_k)\|^2,$$

for all k.

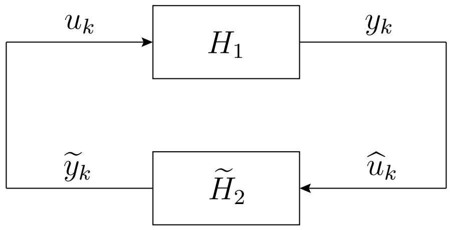


Figure 8.1: Closed-loop interconnection of the NI system H_1 and the SAOSNI system $\widetilde{H}_2$.

Consider the feedback interconnection in Fig. 8.1, which is described by the following equations:

$$\widehat{u}_k = y_k;$$

$$u_k = \widetilde{y}_k.$$

In order to present the stability result, we need to make the following assumption for the open-loop interconnection of the systems H_1 and $\widetilde{H}_2$ as shown in Fig. 8.2.

Assumption 8.1. *Giving an input $u_k = \overline{u}$ for the system H_1 which is constant for all time steps $k \geq k_c$, we obtain a corresponding output y_k. Also, setting the input of the system $\widetilde{H}_2$ to be $\widehat{u}_k = y_k$ in the open-loop interconnection shown in Fig. 8.2 with corresponding output $\widetilde{y}_k$. If $\widetilde{y}_k = \overline{\overline{y}}$ remains constant for all $k \geq k_c$, and $\overline{\overline{y}} = \overline{u}$, then $x_k, \widehat{x}_k = 0$ for all $k \geq k^*$, where k^* is some integer such that $k^* \geq k_c$.*

Figure 8.2: Open-loop interconnection of the NI plant H_1 and the SAOSNI controller $\widetilde{H}_2$.

We have the following stability result.

Theorem 8.3. *Consider the NI system H_1 given by (8.1) and the SAOSNI system $\widetilde{H}_2$ given by (8.27). Also, suppose Assumption 8.1 is satisfied and the function*

$$W(x_k, \widehat{x}_k) = V(x_k) + \widehat{V}(\widehat{x}_k) - h(x_k)^T \widehat{h}(\widehat{x}_k) \tag{8.28}$$

is continuous and positive definite, where $V(x_k)$ and $\widehat{V}(\widehat{x}_k)$ are the storage functions of the systems H_1 and $\widetilde{H}_2$, respectively. Then the closed-loop interconnection of the system H_1 and the system $\widetilde{H}_2$ shown in Fig. 8.1 is asymptotically stable.

Proof. Using Lyapunov's direct method [120], consider the increment of the function $W(x_k, \widehat{x}_k)$ in (8.28):

$$\begin{aligned}
W(x_{k+1}, \widehat{x}_{k+1}) - W(x_k, \widehat{x}_k) &= V(x_{k+1}) + \widehat{V}(\widehat{x}_{k+1}) - V(x_k) - \widehat{V}(\widehat{x}_k) - y_{k+1}^T \widehat{y}_{k+1} \\
&\quad + y_k^T \widehat{y}_k \\
&\leq u_k^T (y_{k+1} - y_k) + \widehat{u}_k^T (\widehat{y}_{k+1} - \widehat{y}_k) - \epsilon \|\widehat{y}_{k+1} - \widehat{y}_k\|^2 \\
&\quad - y_{k+1}^T \widehat{y}_{k+1} + y_k^T \widehat{y}_k \\
&= \widehat{y}_{k+1}^T (y_{k+1} - y_k) + y_k^T (\widehat{y}_{k+1} - \widehat{y}_k) - \epsilon \|\widehat{y}_{k+1} - \widehat{y}_k\|^2 \\
&\quad - y_{k+1}^T \widehat{y}_{k+1} + y_k^T \widehat{y}_k \\
&= -\epsilon \|\widehat{y}_{k+1} - \widehat{y}_k\|^2 \\
&\leq 0.
\end{aligned}$$

The closed-loop system is stable in the sense of Lyapunov. Moreover, $W(x_k, \widehat{x}_k) -$

$W(x_k, \widehat{x}_k) = 0$ implies that $\widehat{y}_{k+1} - \widehat{y}_k = 0$. We use LaSalle's invariance principle in the following; e.g., see Theorem 1 of Reference [121]. In the case that $W(x_k, \widehat{x}_k) - W(x_k, \widehat{x}_k)$ stays at zero for all future time steps k, we have $\widehat{y}_{k+1} - \widehat{y}_k \equiv 0$. Since $\widehat{y}_k$ is the output of the OSNI system associated with the SAOSNI system $\widetilde{H}_2$; i.e., $\widetilde{y}_k = \widehat{y}_{k+1}$, we also have that $\widetilde{y}_k = \widetilde{y}_{k-1}$ for all future time steps k. That is $\widetilde{y}_{k+1} = \widetilde{y}_k$ for all future time steps. In this case, according to the system setting $u_k = \widetilde{y}_k$, the input u_k also remains constant and it equals to $\widetilde{y}_k$. According to Assumption 8.1, this implies that x_k and $\widehat{x}_k$ equal to zero after some time step k^*. Otherwise, the function $W(x_k, \widehat{x}_k)$ will keep decreasing until $W(x_k, \widehat{x}_k) = 0$. That is, $x_k = 0$ and $\widehat{x}_k = 0$, which is the equilibrium of the closed-loop interconnection. □

We consider the linear case of Theorem 8.3 in the following corollary. The positive definiteness condition of the closed-loop Lyapunov function (8.28) is reduced to a DC gain condition.

Consider a minimal system

$$x_{k+1} = A_1 x_k + B_1 u_k; \quad y_k = C_1 x_k \tag{8.29}$$

where $x_k \in \mathbb{R}^n$, $u_k, y_k \in \mathbb{R}^p$ is the state, input and output, respectively. Consider another minimal system

$$\widetilde{x}_{k+1} = A_2 \widetilde{x}_k + B_2 \widetilde{u}_k; \quad \widetilde{y}_k = C_2 \widetilde{x}_k + D_2 \widetilde{u}_k \tag{8.30}$$

where $\widetilde{x}_k \in \mathbb{R}^m$, $\widetilde{u}_k, \widetilde{y}_k \in \mathbb{R}^p$ is the state, input and output, respectively.

Corollary 8.1. *Consider the minimal systems (8.29) and (8.30) with transfer function matrices $G(z)$ and $H(z)$, respectively. Suppose $G(z)$ is NI and $H(z)$ is SAOSNI, both with positive definite quadratic storage functions. If $\lambda_{max}(G(1)H(1)) < 1$, then the closed-loop interconnection of the systems (8.29) and (8.30) is asymptotically stable.*

Proof. Since the system (8.30) is SAOSNI, then according to Definition 8.4 there exists an OSNI system of the form

$$\widetilde{x}_{k+1} = A_2 \widetilde{x}_k + B_2 \widetilde{u}_k; \quad \widehat{y}_k = \widetilde{C}_2 \widetilde{x}_k \tag{8.31}$$

where $C_2 = \widetilde{C}_2 A_2$ and $D_2 = \widetilde{C}_2 B_2$. In other words, $\widetilde{y}_k = \widehat{y}_{k+1}$. Due to the step advance, the system (8.31) has transfer matrix $\widehat{H}(z)$ such that $H(z) = z\widehat{H}(z)$. Therefore, $\widehat{H}(1) = H(1)$. According to Theorem 8.3, a Lyapunov function for the closed-loop interconnection of the systems (8.29) and (8.30) is given by

$$W(x,\widetilde{x}) = \begin{bmatrix} x^T & \widetilde{x}^T \end{bmatrix} \begin{bmatrix} P & -C_1^T\widehat{C}_2 \\ -\widehat{C}_2^T C_1 & \widetilde{P} \end{bmatrix} \begin{bmatrix} x \\ \widetilde{x} \end{bmatrix} \tag{8.32}$$

where x and $\widetilde{x}$ are the states of the state-space realizations of $G(z)$ and $\widehat{H}(z)$, respectively. Also, P and $\widetilde{P}$ are symmetric positive definite matrices satisfying condition (8.9) for the state-space realizations of $G(z)$ and $\widehat{H}(z)$, respectively. Note that the existence of such P and $\widetilde{P}$ follows from the NI properties of $G(z)$ and $H(z)$ according to Theorem 8.1. We prove that if $\lambda_{max}(G(1)H(1)) < 1$, then the Lyapunov function given by (8.32) is positive definite. We have that

$$\begin{aligned} G(1)H(1) &= G(1)\widehat{H}(1) \\ &= C_1(I-A_1)^{-1}B_1\widehat{C}_2(I-A_2)^{-1}B_2 \\ &= C_1P^{-1}C_1^T\widehat{C}_2\widetilde{P}^{-1}\widehat{C}_2^T, \end{aligned}$$

where the last equality follows from condition (8.9). Therefore, $\lambda_{max}(G(1)H(1)) < 1$ implies

$$\lambda_{max}(P^{-1}C_1^T\widehat{C}_2\widetilde{P}^{-1}\widehat{C}_2^T C_1) < 1.$$

This implies that

$$P > C_1^T\widehat{C}_2\widetilde{P}^{-1}\widehat{C}_2^T C_1. \tag{8.33}$$

By considering the Schur's complement, inequality (8.33) and the fact $\widetilde{P} > 0$ imply that $\begin{bmatrix} P & -C_1^T\widehat{C}_2 \\ -\widehat{C}_2^T C_1 & \widetilde{P} \end{bmatrix} > 0$. Hence, $W(x,\widetilde{x})$ given in (8.32) is positive definite. Note that in this case, Assumption 8.1 is satisfied because the DC loop gain is less than unity. The rest of the proof follows from Theorem 8.3. □

In a similar manner to the above, an OSNI system can be stabilized using an auxiliary system for an NI system, which we call a step advanced negative imaginary (SANI) system. We provide the definition for SANI systems as follows. Consider the system

$$\overline{H}_1: \quad x_{k+1} = f(x_k, u_k), \tag{8.34a}$$

$$\overline{y}_k = \overline{h}(x_k, u_k), \tag{8.34b}$$

where $f\colon \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ and $\overline{h}\colon \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$. Here $u_k, \overline{y}_k \in \mathbb{R}^p$ and $x_k \in \mathbb{R}^n$ are the input, output and state of the system, respectively, at time step $k \in \mathbb{N}$.

Definition 8.5. *The system* (8.34) *is said to be a discrete-time step advanced nega-*

tive imaginary (SANI) system if there exists a continuous function $h(x_k)$ such that:

1. $\overline{h}(x_k, u_k) = h(f(x_k, u_k))$.
2. *There exists a continuous positive semidefinite storage function $V\colon \mathbb{R}^n \to \mathbb{R}$ such that for arbitrary states x_k and inputs u_k,*

$$V(x_{k+1}) - V(x_k) \leq u_k^T \left(h(x_{k+1}) - h(x_k)\right)$$

for all k.

Remark 8.3. *Definition 8.5 can be regarded as a variant of Definition 8.1 in a way such that the system output y_k takes the next step output value; i.e., $h(x_{k+1})$. To be specific, suppose the system (8.1) is NI as per Definition 8.1. Then a system with the same state equation (8.1a) and an output equation $\overline{y}(k) = h(x(k+1)) = h(f(x_k, u_k))$ is an SANI system. Note that this does not affect the causality of the system because $h(f(x_k, u_k))$ is a function of the state x_k and input u_k at the current step k.*

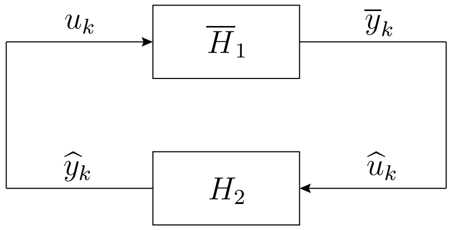


Figure 8.3: Closed-loop interconnection of the SANI system $\overline{H}_1$ and the OSNI system H_2.

The feedback interconnection setting to be considered is shown in Fig. 8.1 and is described by the following equations:

$$\widehat{u}_k = \overline{y}_k;$$
$$u_k = \widehat{y}_k.$$

We make the following assumption for the open-loop interconnection of the systems $\overline{H}_1$ and H_2 as shown in Fig. 8.4.

Assumption 8.2. *Giving an input $u_k = \overline{u}$ for the system $\overline{H}_1$ which is constant for all time steps $k \geq k_c$, we obtain a corresponding output $\overline{y}_k$. Also, setting the input of the system H_2 to be $\widehat{u}_k = \overline{y}_k$ in the open-loop interconnection shown in Fig. 8.4 with corresponding output $\widehat{y}_k$. If $\widehat{y}_k = \overline{\widehat{y}}$ remains constant for all $k \geq k_c$, and $\overline{\widehat{y}} = \overline{u}$, then $x_k, \widehat{x}_k = 0$ for all $k \geq k^*$, where k^* is some integer such that $k^* \geq k_c$.*

Figure 8.4: Open-loop interconnection of the SANI plant $\overline{H}_1$ and the OSNI controller H_2.

Theorem 8.4. *Consider the SANI system $\overline{H}_1$ given by (8.34) and the OSNI system H_2 given by (8.25). Also, suppose Assumption 8.2 is satisfied and the function*

$$W(x_k, \widehat{x}_k) = V(x_k) + \widehat{V}(\widehat{x}_k) - h(x_k)^T \widehat{h}(\widehat{x}_k)$$

is continuous and positive definite, where $V(x_k)$ and $\widehat{V}(\widehat{x}_k)$ are the storage functions of the systems $\overline{H}_1$ and H_2, respectively. Then the closed-loop interconnection of the system $\overline{H}_1$ and the system H_2 shown in Fig. 8.3 is asymptotically stable.

Proof. The proof of Theorem 8.4 follows a similar derivation to the proof of Theorem 8.3. □

Corollary 8.2. *Consider the minimal systems (8.29) and (8.30) with transfer function matrices $G(z)$ and $H(z)$, respectively. Suppose $G(z)$ is SANI and $H(z)$ is OSNI, both with positive definite quadratic storage functions. If $\lambda_{max}(G(1)H(1)) < 1$, then the closed-loop interconnection of the systems (8.29) and (8.30) is asymptotically stable.*

Proof. The proof is similar to that of Corollary 8.1, with Theorem 8.4 used instead of Theorem 8.3. □

Remark 8.4. *For the case of a linear SANI system with transfer matrix $H(z)$, the conditions in Definition 8.5 reduce to the condition that $H(z) = zH_1(z)$ where $H_1(z)$ is an NI system. Under the assumptions of Lemma 8.1, $H(z)$ is SANI if and only if $(1 + \frac{1}{z})H(z) - \lim_{z\to\infty} H(z)$ satisfies Conditions 1–3 in Lemma 8.1.*

8.5 Illustrative example

In this section, we provide an example to illustrate the proposed results. To be specific, we apply an SAOSNI controller to asymptotically stabilize an NI plant.

Consider a mass-spring system shown in Fig. 8.5. The system has mass $m = 0.04kg$ and spring constant $k = 1N/m$. A discrete-time model of the system is obtained by sampling the continuous-time system with period $h = 0.02s$ using the ZOH method

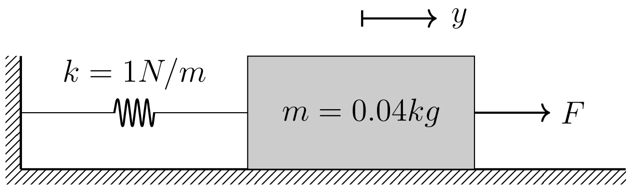


Figure 8.5: A mass-spring system with mass $m = 0.04kg$ and spring constant $k = 1N/m$.

(see also [112]).

$$x_{k+1} = Ax_k + Bu_k, \tag{8.35a}$$
$$y_k = Cx_k, \tag{8.35b}$$

where

$$\begin{aligned}
A &= \begin{bmatrix} \cos(\omega h) & \sin(\omega h) \\ -\sin(\omega h) & \cos(\omega h) \end{bmatrix} = \begin{bmatrix} \cos 0.1 & \sin 0.1 \\ -\sin 0.1 & \cos 0.1 \end{bmatrix}, \\
B &= \begin{bmatrix} \frac{1}{k}\left(1 - \cos(\omega h)\right) \\ \frac{1}{k}\sin(\omega h) \end{bmatrix} = \begin{bmatrix} 1 - \cos 0.1 \\ \sin 0.1 \end{bmatrix}, \\
C &= \begin{bmatrix} 1 & 0 \end{bmatrix},
\end{aligned}$$

with $\omega = \sqrt{k/m} = 0.5rad/sec$. Here, $x_k = \begin{bmatrix} x_{ak} & x_{bk} \end{bmatrix}^T \in \mathbb{R}^2$, $u_k, y_k \in \mathbb{R}$ are the state, input and output of the system, respectively. This system is NI according to Definition 8.1 with the storage function

$$V(x_k) = \frac{1}{2}\left(x_{ak}^2 + x_{bk}^2\right).$$

Note that here x_a corresponds to the position of the mass and x_b corresponds to the displacement of the mass in $\sqrt{m/k} = 0.2sec$. This scaling of the state variables is used in simplifying the system model. The NI property of the plant can also be verified using Theorem 8.2. The transfer function of the system (8.35) is

$$G(z) = \frac{(1 - \cos 0.1)(z + 1)}{z^2 - 2\cos 0.1\ z + 1}.$$

This transfer function has a pair of complex conjugate poles at $z = \cos 0.1 \pm i \sin 0.1$. Therefore, Condition 1 is satisfied. We have that

$$i\left[(e^{i\theta} + 1)G(e^{i\theta}) - (e^{-i\theta} + 1)G(e^{i\theta})^* - L + L^T\right] = 0,$$

for all $\theta \in (0, \pi)$ such that $e^{i\theta}$ is not a pole of $G(z)$. Here, $L = CB$ according to (8.22). Hence, Condition 2 is satisfied. As for Condition 3, the pole $z = \cos 0.1 + i \sin 0.1$ is of the form $e^{i0.1}$ with $0.1 \in (0, \pi)$. This is a simple pole and the residue matrix is $K_0 = \sin 0.1$ according to (8.23). Therefore, the system (8.35) satisfies all the conditions in Theorem 8.2 and hence is NI. This can also be seen in the Bode plot of the transfer function $(z+1)G(z)$ shown in Fig. 8.6.

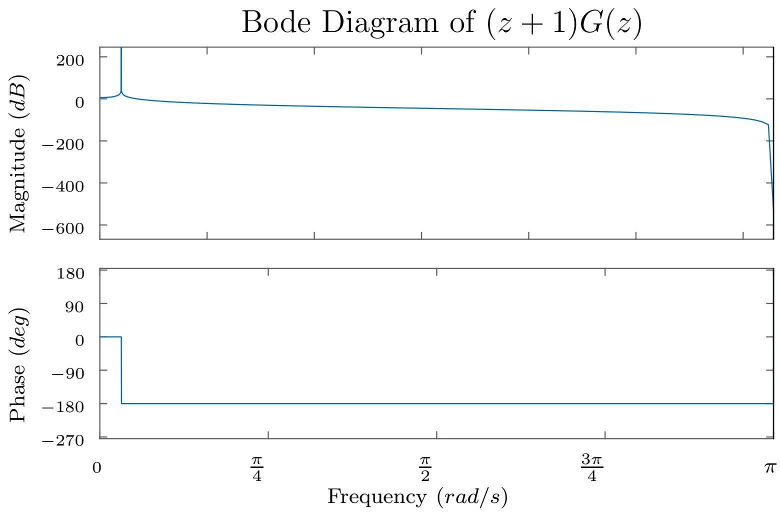


Figure 8.6: Bode plot of the system with transfer function $(z+1)G(z)$. From the phase plot, we can see that its phase is in the range $[-\pi, 0]$ for frequencies in $(0, \pi)$.

According to Theorem 8.3, we can use the following controller, which is SAOSNI according to Definition 8.4:

$$\widehat{x}_{k+1} = \frac{1}{4}\widehat{x}_k + \frac{1}{2}\widehat{u}_k, \tag{8.36a}$$

$$\widetilde{y}_k = \frac{1}{4}\widehat{x}_k + \frac{1}{2}\widehat{u}_k \tag{8.36b}$$

where $\widehat{x}_k, \widehat{u}_k, \widetilde{y}_k \in \mathbb{R}$ are the state, input and output of the system, respectively. This system is SAOSNI with the storage function

$$\widehat{V}(\widehat{x}_k) = \frac{3}{4}\widehat{x}_k^2,$$

and $\epsilon \in (0, \frac{5}{4}]$. The system (8.36) has the transfer matrix

$$H(z) = \frac{2z}{4z-1}$$

and is SAOSNI, which implies that it is also SANI. Hence, according to Remark 8.4, $(1 + \frac{1}{z})H(z)$ should satisfy the Conditions 1–3 in Lemma 8.1. The Bode plot of

$(1+\frac{1}{z})H(z)$ is shown in Fig. 8.7.

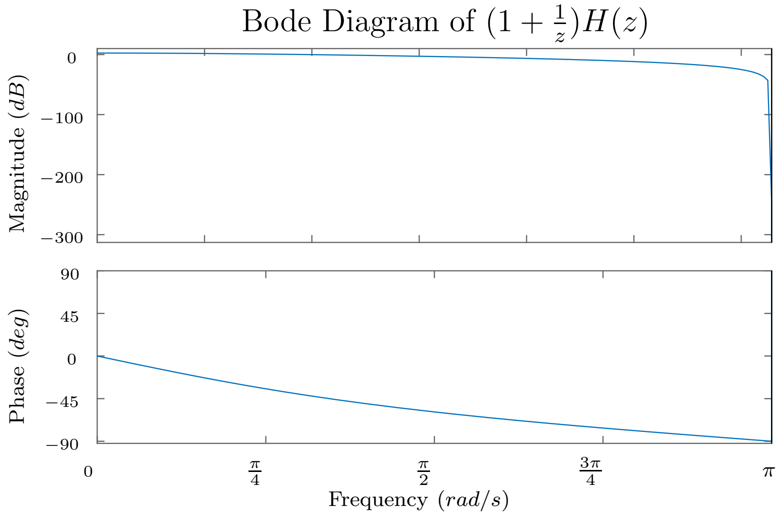


Figure 8.7: Bode plot of the system with transfer function $(1+\frac{1}{z})H(z)$. From the phase plot, we can see that its phase is in the range $(-\pi, 0)$ for frequencies in $(0, \pi)$.

Consider the positive feedback interconnection of the plant (8.35) and the controller (8.36). It can be verified that Assumption 8.1 is satisfied. We construct the Lyapunov function of the interconnection as follows according to (8.28),

$$W(x_k, \widehat{x}_k) = \frac{1}{2}x_{ak}^2 + \frac{1}{2}x_{bk}^2 + \frac{3}{4}\widehat{x}_k^2 - x_{ak}\widehat{x}_k,$$

which is positive definite. We set the initial states of the systems to be $x_0 = \begin{bmatrix} 5 & -3 \end{bmatrix}^T$ and $\widehat{x}_0 = 2$. We can see that the states x_k and $\widehat{x}_k$ converge to zero as shown in Fig. 8.8. Starting from nonzero initial values, the states converge to zero, which implies that the system is asymptotically stable. This is consistent with our expectations according to Theorem 8.3. The stability of the closed-loop interconnection can also be concluded from Corollary 8.1 because $G(1)H(1) = \frac{2}{3} < 1$ in this case.

Remark 8.5. *Closed-loop stability in this example can also be analyzed in the frequency-domain. Let the transfer functions of the systems (8.35) and (8.36) be denoted by $G(z)$ and $H(z)$, respectively. Since $G(z)$ is NI, then $G_0(z) = (z+1)G(z) + \lim_{z\to\infty} G(z)$ satisfies Conditions 1–3 of Lemma 8.2. Since $H(z)$ is SAOSNI, then it is also SANI. Hence, $H_0(z) = (1+\frac{1}{z})H(z) - \lim_{z\to\infty} H(z)$ satisfies Conditions 1–3 of Lemma 8.1. In the case of SISO systems, we need $G_1(z) = (z+1)G(z)$ and $H_1(z) = (1+\frac{1}{z})H(z)$ to satisfy those conditions. Now we consider the loop gain $G(z)H(z)$ of the interconnection. We have that*

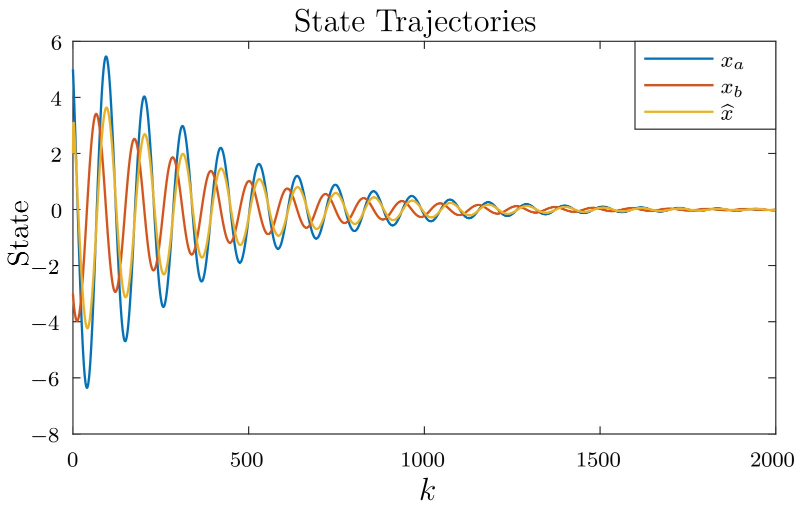


Figure 8.8: State trajectories for the feedback interconnection of the systems (8.35) and (8.36).

$L(z) = G(z)H(z) = \frac{z}{z^2+2z+1}G_1(z)H_1(z)$. Let $z = e^{i\theta}$. Hence,

$$\begin{aligned} L(e^{i\theta}) =& G(e^{i\theta})H(e^{i\theta}) = \frac{1}{e^{i\theta}+2+e^{-i\theta}}G_1(e^{i\theta})H_1(e^{i\theta}) \\ =& \frac{1}{2\cos\theta+2}G_1(e^{i\theta})H_1(e^{i\theta}). \end{aligned} \tag{8.37}$$

Since $G_1(z)$ and $H_1(z)$ satisfy Condition 2 in Lemma 8.2, then for $\theta \in (0,\pi)$, we have $\arg(G_1(e^{i\theta})) \in [-\pi, 0]$ and $\arg(H_1(e^{i\theta})) \in [-\pi, 0]$. If $H_1(z)$ is such that $\arg(H_1(e^{i\theta})) \in (-\pi, 0)$ for $\theta \in (0,\pi)$, then $\arg(G_1(e^{i\theta})H_1(e^{i\theta})) \in (-2\pi, 0)$. Since $2\cos\theta + 2 > 0$ for all $\theta \in (0,\pi)$, then according to (8.37), we also have $\arg(G_1(e^{i\theta})H_1(e^{i\theta})) \in (-2\pi, 0)$ for $\theta \in (0,\pi)$. Hence, the closed-loop interconnection must be asymptotically stable according to the discrete-time Nyquist criterion. The phase of $G(z)H(z)$ is shown in the Bode plot in Fig. 8.9.

8.6 Conclusion

In this chapter, we provided a new definition of the NI property for discrete-time systems. According to this definition, ZOH sampling of a continuous-time NI system yields a discrete-time NI system. We also provide LMI conditions and frequency-domain conditions for this definition to be satisfied in the case of linear systems. We prove that the closed-loop interconnection of an NI system and an SAOSNI system is asymptotically stable, under certain assumptions. Under similar assumptions, asymptotic stability is also proved for the closed-loop interconnection of an SANI system and an OSNI system. These assumptions reduce to a simple DC gain condition in the linear case. We show in an example that a discrete-time mass-

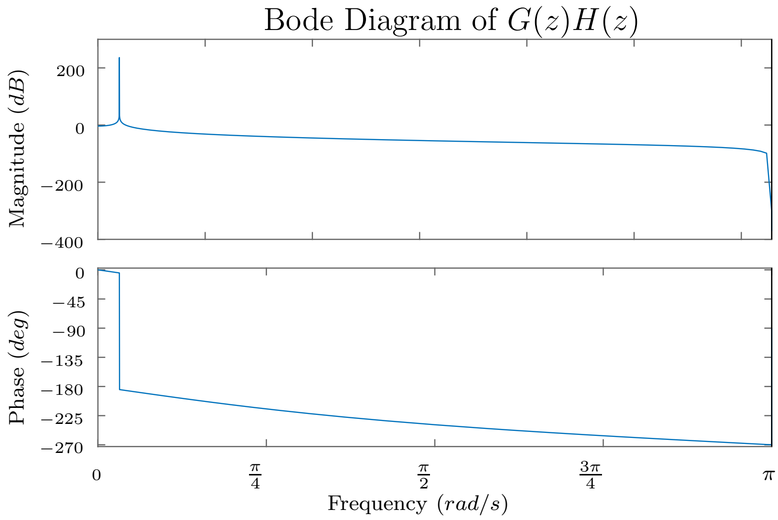


Figure 8.9: Bode plot of the open-loop interconnection of the NI system (8.35) and the SAOSNI system (8.36). The interconnection has the transfer function $G(z)H(z)$ whose phase is in the range $(-2\pi, 0)$ for frequencies in $(0, \pi)$ according to the above explanation and also the Bode plot. Hence the positive feedback interconnection is asymptotically stable.

spring system, which is NI, can be asymptotically stabilized using the proposed results.

Chapter 9

Negative Imaginary Systems Control Using Hybrid Integrator-Gain Systems

9.1 Introduction

In Chapter 7, we have investigated the NI property for switched systems and proposed a feedback stability result. In the example given in Section 7.4, we applied a switched controller called a hybrid integrator-gain system (HIGS) to a nonlinear mass-spring damper system to achieve stability. In this chapter, we study the application of HIGS controllers in the control of linear NI systems. Besides the HIGS of its original form, we also investigate a multi-HIGS, the cascade of two HIGS and a discrete-time HIGS.

The motivation of using this specific controller in the control of NI systems is that the HIGS may overcome some of the limitations of linear controllers. Linear feedback control has inherent limitations which have been discussed in [122, 123]. Bode's phase-gain relationship shows a trade-off between control system performance and robustness [124, 125]. That is, the desired large low-frequency gain and small high-frequency gain can only be achieved at the cost of a lower stability margin [126]. To be specific, an integrator, which is commonly used to eliminate steady-state error [127] introduces a 90° phase lag. The time delay caused by such a phase lag will unavoidably lead to an overshoot and even instability [128, 129]. Clegg introduced a nonlinear integrator in [129], known as a Clegg integrator, to overcome these limitations. The output of the Clegg integrator is reset to zero whenever its input crosses zero. With such a resetting mechanism, the input and output of a Clegg integrator always have the same sign. The describing function of a Clegg integrator has a magnitude slope identical to a linear integrator but a phase lag of only 38.1°. In comparison to a linear integrator, the reduction of 51.9° of phase lag will lead to a significant reduction in time delay and, as a consequence, overshoot. The concept of output resetting was generalized to a first-order filter with the transfer function $1/(s+b)$ in [130]. This is known as the first-order reset element (FORE); see [131,132] for a discussion of reset control systems. A concrete example provided in [133] shows that a reset control system can meet control objectives that are unachievable for any linear controller: under FORE control, the plant tracks a unit step reference with no overshoot even for a large rise time; i.e., when the bandwidth is low (see [122]).

A drawback of conventional reset control systems such as Clegg integrators and FOREs is that they generate discontinuous control signals when the reset happens. Discontinuous control signals can cause chattering which may excite high-frequency dynamics and lead to poor performance or even instability [15, 134]. To overcome this drawback of reset control systems, the HIGS was introduced in [103]. Instead of resetting the state to zero, the HIGS alternates between integrator and gain modes resulting in a continuous (but non-smooth) control signal. This prevents the excitation of high-frequency harmonics induced by conventional reset control systems. The input-output relation of the HIGS is also restricted to a sector in which the HIGS operates as an integrator. The tendency to violate the sector constraint in the integrator mode enforces switching to the gain mode, ensuring that the input and output of the HIGS would have identical signs [135]. The frequency response of the describing function of a HIGS has the same 38.1° of phase lag as the reset control system [103]. Hence, the HIGS has similar advantages as the reset control systems described above in terms of overcoming the limitations of linear controllers. The paper [136] illustrated the advantages of HIGS using a concrete example where overshoot is completely eliminated by HIGS control, which is unachievable by any linear controller.

The demand for high-precision, high-speed, and reliable mechatronics systems has increased tremendously, and to address this need, the HIGS element has been exploited successfully in the semiconductor industry for motion tracking [103,137,138], vibration isolation and damping [135,139,140]. In [103], a HIGS-based proportional double integral derivative (PI^{2}D) controller is designed and applied to a wafer stage system of an industrial wafer scanner. Owing to the enhanced phase behaviour of HIGS-based filters compared to the linear counterparts, HIGS-based second-order low-pass filters [137] and HIGS-based notch filters [141] are proposed to achieve substantial low-frequency disturbance rejection and increase the bandwidth of a wafer scanner. A HIGS-based bandpass filter is also constructed in [135], featuring a series connection of two HIGS elements applied for vibration isolation. By replacing a standard integrator with the HIGS element, [136] demonstrates a novel application of the HIGS in reducing overshoot in linear time-invariant plants with a real unstable pole. The HIGS is also studied in a multivariable configuration [139] applied to a multivariable active vibration isolation system in the form of a HIGS-based bandpass filter.

Stability and performance analysis of closed-loop systems featuring HIGS is challenging due to the nonlinear nature of this hybrid system. Based on the stability analysis of reset control systems in general [142], a frequency-domain approach is proposed in [103] to graphically verify the stability of a controlled system with the HIGS using measured frequency response data of the linear part of the system and the Kalman-Yakubovich-Popov lemma. In this approach, the closed-loop system is rearranged in Lur'e form by isolating the nonlinearity from the linear counterpart of the system. Then the input-to-state stability (ISS) of the closed-loop system is guaranteed based on the detectability of the HIGS element and the circle criterion. This results in frequency-domain conditions, less stringent than the strictly positive criterion [142]. Using a modified version of the circle criterion, the stability of the closed-loop system in a MIMO configuration is also investigated in [139].

Since the switching strategy in HIGS is not taken into account in the proposed stability analysis of the controlled systems featuring HIGS, frequency-domain conditions give a conservative estimate of stability. This has been addressed in [143] by proposing novel conditions that guarantee the existence of the Lyapunov functions in subregions of the state-space where the HIGS is active. The stability of nonlinear closed-loop systems with the HIGS element is also explored through a time-domain approach where an ISS condition is proposed in terms of LMIs that guarantee the existence of a piecewise quadratic Lyapunov function [144]. This approach is less conservative compared to the frequency-domain approach of [144].

In this chapter, we investigate the application of HIGS on the control of linear NI systems. We propose different types of variations of the original HIGS which was introduced in [103]. To be specific, we investigate a single HIGS, a multi-HIGS, and the cascade of two HIGS elements. We show that all these three types of HIGS are nonlinear NI systems, and they can be applied in the control of linear NI plants. We provide an intuitive description of the multi-HIGS, which was originally introduced in [139]. A multi-HIGS, which is a parallel combination of multiple HIGS, can be used in the control of a MIMO NI plant. The cascade connection of two HIGS can be used as an alternative to the single HIGS. We do not consider a cascade of more than two HIGS because NI systems can only have relative degree less than or equal to two and each HIGS can have a relative degree of one.

The NI property of HIGS elements motivates a methodology of using HIGS in the control of flexible structures with colocated force actuators and position sensors. This methodology is applied in this chapter to control a microelectromechanical system (MEMS) nanopositioner. A real-world experiment was implemented, where a 2-DOF MEMS nanopositioner was controlled by a multi-HIGS controller.

In addition, we investigate a discrete-time HIGS using the results proposed in Chapter 8 for the purpose of digital control. The discrete-time HIGS, which was introduced in [145], is shown to be a discrete-time SANI system as defined in Definition 8.5. We prove that discrete-time HIGS controllers can be used to control linear discrete-time NI systems. This result is then illustrated using an example of a mass-spring system. The results in this chapter are included in the papers [146–148].

9.2 A single HIGS

In this section, we provide the system model of the original form of HIGS. We prove that a HIGS is a nonlinear NI system. Also, we show that for any minimal linear NI system, there exists a HIGS that can stabilize it.

9.2.1 System description

A SISO hybrid integrator-gain system (HIGS) $\mathcal{H}$ is represented by the following differential algebraic equations [103]:

$$\mathcal{H}: \begin{cases} \dot{x}_h = \omega_h e, & \text{if } (e, u, \dot{e}) \in \mathcal{F}_1 \\ x_h = k_h e, & \text{if } (e, u, \dot{e}) \in \mathcal{F}_2 \\ u = x_h, \end{cases} \tag{9.1}$$

where $x_h, e, u \in \mathbb{R}$ denote the state, input, and output of the HIGS, respectively. Here, $\dot{e}$ is the time derivative of the input e, which is assumed to be continuous and piecewise differentiable. Also, $\omega_h \in [0, \infty)$ and $k_h \in (0, \infty)$ represents the integrator frequency and gain value, respectively. These tunable parameters allow for desired control performance. The sets $\mathcal{F}_1$ and $\mathcal{F}_2 \in \mathbb{R}^3$ determine the HIGS modes of operation; i.e. the integrator and gain modes, respectively. The HIGS is designed to operate under the sector constraint $(e, u, \dot{e}) \in \mathcal{F}$ (see [103, 139]) where

$$\mathcal{F} = \{(e, u, \dot{e}) \in \mathbb{R}^3 | \, eu \geq \frac{1}{k_h} u^2\}, \tag{9.2}$$

and $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined as

$$\begin{aligned} &\mathcal{F}_1 := \mathcal{F} \setminus \mathcal{F}_2; \\ &\mathcal{F}_2 := \{(e, u, \dot{e}) \in \mathbb{R}^3 | u = k_h e \quad \text{and} \quad \omega_h e^2 > k_h e \dot{e}\}. \end{aligned} \tag{9.3}$$

The HIGS (9.1) operates in the integrator mode unless the HIGS output u is on the boundary of the sector $\mathcal{F}$, and tends to exit the sector; i.e. $(e, u, \dot{e}) \in \mathcal{F}_2$. In this case, the HIGS is enforced to operate in the gain mode. At the time instants when switching happens, the state x_h still remains continuous, as can be seen from (9.1).

In this section, we show that a single HIGS, represented by (9.1), is nonlinear NI. Then, we show that a single HIGS can be used as a controller for a SISO NI system by proving that the positive feedback interconnection of an NI system and a HIGS is asymptotically stable.

9.2.2 Nonlinear NI property of SISO HIGS

We first present a property of the HIGS (9.1) in Lemma 9.1, which is implied by the sector constraint (9.2). This property will be used later in the proofs of the main results.

Lemma 9.1. *Consider a HIGS element with the system model (9.1). This system satisfies*

$$e x_h - k_h e^2 \leq 0,$$

where the equality only holds when $x_h = k_h e$.

Proof. Consider the following inequality

$$\left(\sqrt{\frac{1}{k_h}} x_h - \sqrt{k_h} e\right)^2 \geq 0,$$

where equality only holds when $x_h = k_h e$. Therefore,

$$\frac{1}{k_h} x_h^2 - 2 e x_h + k_h e^2 \geq 0.$$

Considering the condition in $\mathcal{F}$ as given in (9.2), this implies that

$$e x_h - k_h e^2 \leq \frac{1}{k_h} x_h^2 - e x_h \leq 0,$$

where equality only holds when $x_h = k_h e$. □

The nonlinear NI property of a SISO HIGS is shown in the following theorem.

Theorem 9.1. *Consider a SISO HIGS as in (9.1), then the HIGS is a nonlinear NI system from input e to the output u with the positive definite storage function formulated as*

$$V(x_h) = \frac{1}{2k_h} x_h^2$$

satisfying

$$\dot{V}(x_h) \leq e\dot{u}. \tag{9.4}$$

Proof. The storage function $V(x_h)$ is positive definite since $k_h > 0$. Here, we prove that (9.4) holds in both integrator and gain modes. Taking the time derivative of $V(x_h)$, we have that

$$\dot{V}(x_h) = \frac{1}{k_h} x_h \dot{x}_h.$$

Case 1. The HIGS operates in the integrator mode; i.e., $(e, u, \dot{e}) \in \mathcal{F}_1$. In this case, we have that $\dot{x}_h = \omega_h e$. Therefore, $\dot{V}$ is obtained as

$$\dot{V}(x_h) = \frac{1}{k_h} \omega_h e x_h \leq \omega_h e^2 = e\dot{u}. \tag{9.5}$$

where the inequality follows from Lemma 9.1.

Case 2. The HIGS operates in the gain mode; i.e., $(e, u, \dot{e}) \in \mathcal{F}_2$. In this case, we have that $u = x_h = k_h e$. Therefore,

$$\dot{V}(x_h) = \frac{1}{k_h} k_h e \dot{x}_h = e\dot{u}. \tag{9.6}$$

According to (9.5) and (9.6), and Definition 3.1, the HIGS is a nonlinear NI system. □

9.2.3 Stability for the interconnection of a SISO NI system and a HIGS

Consider the interconnection of a SISO linear NI plant $G(s)$ and a HIGS controller $\mathcal{H}$ as shown in Fig. 9.1. We analyze the stability of the closed-loop system in the following. Note that here and also in Sections 9.3, 9.4 and 9.5, HIGS controllers are applied in positive feedback, according to the control framework used in NI systems theory [2, 62].

Consider a SISO NI system with the transfer function matrix $G(s)$ and the minimal realization:

$$\dot{x} = Ax + Bu, \tag{9.7a}$$

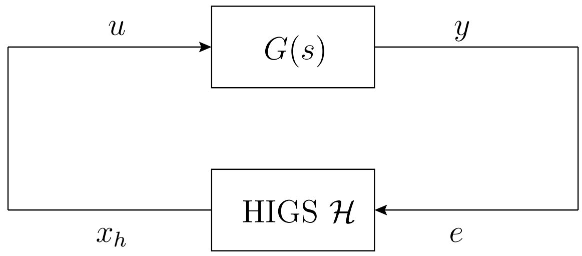


Figure 9.1: Closed-loop interconnection of a linear NI system and a HIGS.

$$y = Cx, \tag{9.7b}$$

where $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}$ are the state, input and output of the system, respectively. Here, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$.

Theorem 9.2. *Consider the SISO minimal linear NI system (9.7). There exists a HIGS element $\mathcal{H}$ of the form (9.1) such that the closed-loop interconnection of the system (9.7) and the HIGS $\mathcal{H}$ as shown in Fig. 9.1 is asymptotically stable.*

Proof. This result is a special case of Theorem 9.4. Since (9.7) is a minimal realization of the linear NI system $G(s)$, then according to Lemma 2.3, we have that $\det A \neq 0$ and there exists $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$ such that

$$AY + YA^T \leq 0, \qquad \text{and} \qquad B + AYC^T = 0. \tag{9.8}$$

Using Lyapunov's direct method, let the storage function of the closed-loop interconnection be

$$W(x, x_h) = \frac{1}{2}x^TY^{-1}x + \frac{1}{2k_h}x_h^2 - Cxx_h = \frac{1}{2}\begin{bmatrix} x^T & x_h \end{bmatrix}\begin{bmatrix} Y^{-1} & -C^T \\ -C & \frac{1}{k_h} \end{bmatrix}\begin{bmatrix} x \\ x_h \end{bmatrix}. \tag{9.9}$$

Using Schur Complement theorem, $W(x, x_h) > 0$ for all $(x, x_h) \neq (0, 0)$ if

$$\frac{1}{k_h} - CYC^T > 0. \tag{9.10}$$

Using (9.8), we have that $CYC^T = -CA^{-1}B = G(0)$. Then, (9.10) becomes

$$k_hG(0) < 1.$$

Taking the time derivative of $W(x, x_h)$ defined in (9.9), we have

$$\dot{W}(x, x_h) = x^TY^{-1}\dot{x} + \frac{1}{k_h}x_h\dot{x}_h - C\dot{x}x_h - Cx\dot{x}_h$$

$$
\begin{aligned}
&= \left(x^T Y^{-1} - x_h C\right)\dot{x} + \left(\frac{1}{k_h}x_h - Cx\right)\dot{x}_h \\
&= \left(x^T Y^{-1} - uC\right)\dot{x} + \left(\frac{1}{k_h}x_h - e\right)\dot{x}_h \\
&= \left(x^T Y^{-1} + uB^T A^{-T} Y^{-1}\right)\dot{x} + \left(\frac{1}{k_h}x_h - e\right)\dot{x}_h \\
&= \left(x^T A^T + uB^T\right)(A^{-T}Y^{-1})\dot{x} + \left(\frac{1}{k_h}x_h - e\right)\dot{x}_h \\
&= \frac{1}{2}\dot{x}^T(A^{-T}Y^{-1} + Y^{-1}A^{-1})\dot{x} + \left(\frac{1}{k_h}x_h - e\right)\dot{x}_h,
\end{aligned}
$$

where $u = x_h$ and $e = y = Cx$ are also used. We have that

$$
\begin{aligned}
\left(\frac{1}{k_h}x_h - e\right)\dot{x}_h &= \begin{cases} \left(\frac{1}{k_h}x_h - e\right)\omega_h e, & \text{if } (e, y_h, \dot{e}) \in \mathcal{F}_1 \\ \left(\frac{1}{k_h}x_h - e\right)k_h \dot{e}, & \text{if } (e, y_h, \dot{e}) \in \mathcal{F}_2 \end{cases} \\
&= \begin{cases} \frac{\omega_h}{k_h}\left(e x_h - k_h e^2\right), & \text{if } (e, y_h, \dot{e}) \in \mathcal{F}_1 \\ \dot{e}\left(x_h - k_h e\right), & \text{if } (e, y_h, \dot{e}) \in \mathcal{F}_2. \end{cases}
\end{aligned} \tag{9.11}
$$

In $\mathcal{F}_2$, according to (9.3), we have that $x_h = u = k_h e$. Hence, $\left(\frac{1}{k_h}x_h - e\right)\dot{x}_h = \dot{e}\left(x_h - k_h e\right) = 0$. In $\mathcal{F}_1$, according to Lemma 9.1, we have that

$$
e x_h - k_h e^2 \le 0,
$$

where equality holds only if $x_h = k_h e$. Therefore, following from (9.11), we have that

$$
\left(\frac{1}{k_h}x_h - e\right)\dot{x}_h \le 0,
$$

and $\left(\frac{1}{k_h}x_h - e\right)\dot{x}_h = 0$ only if $x_h = k_h e$. We also have that $\frac{1}{2}\dot{x}^T(A^{-T}Y^{-1} + Y^{-1}A^{-1})\dot{x} \le 0$ because $A^{-T}Y^{-1} + Y^{-1}A^{-1} \le 0$ according to (9.8). Therefore, $\dot{W}(x, x_h) \le 0$ and $\dot{W}(x, x_h) = 0$ only if $x_h = k_h e$ and $\dot{x}^T(A^{-T}Y^{-1} + Y^{-1}A^{-1})\dot{x} = 0$. Using LaSalle's invariance principle, $\dot{W}(x, x_h)$ stays at zero only if $x_h \equiv k_h e$ and $\dot{x}^T(A^{-T}Y^{-1} + Y^{-1}A^{-1})\dot{x} \equiv 0$. The condition $x_h \equiv k_h e$ implies that

$$
u \equiv k_h y \equiv k_h C x,
$$

according to the setting of the interconnection $u = y_h = x_h$ and $e = y = Cx$. The condition $\dot{x}^T(A^{-T}Y^{-1} + Y^{-1}A^{-1})\dot{x} \equiv 0$ holds in the following two cases:

Case 1. $\dot{x} \equiv 0$. In this case, $x(t) \equiv \overline{x}$ is constant and so is $u(t) \equiv \overline{u}$. Since $A\overline{x} + B\overline{u} \equiv 0$, we have that $\overline{x} = -A^{-1}B\overline{u}$. Recall that $\overline{u} = k_h C\overline{x}$, then we have $\overline{u} =$

$k_hC\overline{x} = -k_hCA^{-1}B\overline{u} = k_hG(0)\overline{u} \neq \overline{u}$ according to the assumption that $k_hG(0) < 1$. This means that $\dot{W}(x, x_h)$ cannot stay at zero.

Case 2. $\dot{x} \neq 0$ and $\dot{x}$ is an eigenvector of $A^{-T}Y^{-1} + Y^{-1}A^{-1}$ that corresponds to a possible zero eigenvalue of $A^{-T}Y^{-1} + Y^{-1}A^{-1}$ and $\dot{x}$ changes along this eigenvector. In this case, we have that

$$\left(A^{-T}Y^{-1} + Y^{-1}A^{-1}\right)\dot{x} = 0\dot{x} = 0,$$

where $\dot{x} = Ax + Bu = (A + k_hBC)x$. And we have two subcases:

Subcase 2a. $\dot{x}$ is constant. That is

$$\dot{x} = (A + k_hBC)x = const.$$

This only holds if x is an eigenvector of the matrix $A + k_hBC$ which corresponds to the zero eigenvalue. However, in this case, $\dot{x} = 0$, which contradicts the condition $\dot{x} \neq 0$ and falls into Case 1.

Subcase 2b. $\dot{x}$ is not constant. Then, $(A^{-T}Y^{-1} + Y^{-1}A^{-1})\dot{x}$ stays at zero only if $\dot{x}$ changes along a line. That is $\ddot{x} = \alpha\dot{x}$ for some scalar $\alpha \neq 0$. Since we have

$$\ddot{x} = (A + k_hBC)\dot{x},$$

we may conclude that $A + k_hBC = \alpha I$ must hold. (i). In the case that $A = \gamma I$ for some constant γ, we have that $\gamma < 0$ according to (9.8). In this case, we have that $A^{-T}Y^{-1} + Y^{-1}A^{-1} = \frac{2}{\gamma}Y^{-1} < 0$. This contradicts with the assumption that $A^{-T}Y^{-1} + Y^{-1}A^{-1}$ has a zero eigenvalue. (ii). Otherwise, if $A \neq \gamma I$, since the set $\{k_h \in \mathbb{R} | k_hG(0) < 1 \text{ and } k_h > 0\}$ has nonempty interior, then there always exists k_h such that $A + k_hBC \neq \alpha I$ for all $\alpha \neq 0$. In this case, $\dot{W}(x, x_h)$ cannot stay at zero.

Therefore, with $k_h > 0$ satisfying $k_hG(0) < 1$ and $(A + k_hBC) \neq \alpha I$ for all $\alpha \neq 0$, we have that $\dot{W}(x, x_h)$ cannot stay at zero. Thus, $W(x, x_h)$ will keep decreasing until $x = 0$ and $x_h = 0$. Hence, the interconnection is asymptotically stable. □

9.3 Multi-HIGS

In this section, we provide a description of a multi-HIGS and show that a multi-HIGS is a nonlinear NI system. Also, we prove that for any MIMO linear NI system, there exists a multi-HIGS controller such that their closed-loop interconnection is asymptotically stable.

9.3.1 System Description

Consider N HIGS of the form (9.1) with different integrator frequencies $\omega_{h,1},\cdots,\omega_{h,N}$ and gain values $k_{h,1},\cdots,k_{h,N}$, connected in parallel as shown in Fig. 9.2. The HIGS are denoted by $\mathcal{H}_1,\mathcal{H}_2,\cdots,\mathcal{H}_N$ while their inputs, outputs and states are denoted by $e_1,e_2,\cdots,e_N$, $u_1,u_2,\cdots,u_N$ and $x_{h,1},x_{h,2},\cdots,x_{h,N}$, respectively. The entire system denoted by $\widehat{\mathcal{H}}$ is called a multi-HIGS (see also [139]). The input, output and state of the system $\widehat{\mathcal{H}}$ are

$$E_h = \begin{bmatrix} e_1 & e_2 & \cdots & e_N \end{bmatrix}^T, \tag{9.12}$$

$$U_h = \begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix}^T, \tag{9.13}$$

and

$$X_h = \begin{bmatrix} x_{h,1} & x_{h,2} & \cdots & x_{h,N} \end{bmatrix}^T, \tag{9.14}$$

respectively.

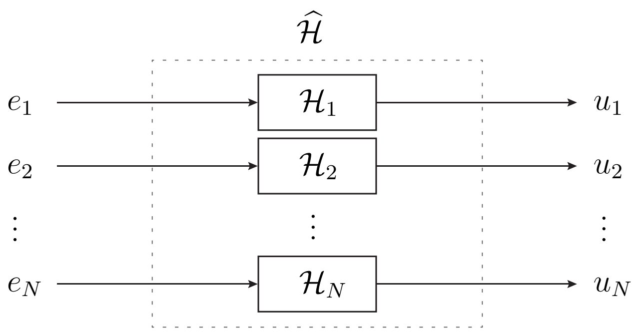


Figure 9.2: A multi-HIGS $\widehat{\mathcal{H}}$, which is constructed by connecting N HIGS of the form (9.1) in parallel.

The system can also be described by the following equations:

$$\widehat{\mathcal{H}}: \begin{cases} e_i = \theta_i^T E_h, & \\ \dot{x}_{h,i} = \omega_{h,i} e_i, & \text{if } (e_i, x_{h,i}, \dot{e}_i) \in \mathcal{F}_{1,i} \\ x_{h,i} = k_{h,i} e_i, & \text{if } (e_i, x_{h,i}, \dot{e}_i) \in \mathcal{F}_{2,i} \\ X_h = \begin{bmatrix} x_{h,1}, x_{h,2}, \cdots, x_{h,N} \end{bmatrix}^T, & \\ U_h = X_h, & \end{cases} \tag{9.15}$$

where $\theta_i \in \mathbb{R}^N$ is a standard unit vector; i.e., the i-th element of θ_i is one and all other elements are zeros. Also, $\mathcal{F}_{1,i}$ and $\mathcal{F}_{2,i}$ are given by:

$$\mathcal{F}_{1,i} := \mathcal{F}_i \setminus \mathcal{F}_{2,i};$$

$$\mathcal{F}_{2,i} := \{(e_i, x_{h,i}, \dot{e}_i) \in \mathbb{R}^3 | x_{h,i} = k_{h,i} e_i \text{ and } \omega_{h,i} e_i^2 > k_{h,i} e_i \dot{e}_i\},$$

where

$$\mathcal{F}_i := \{(e_i, x_{h,i}, \dot{e}_i) \in \mathbb{R}^3 |\, e_i x_{h,i} \geq \frac{1}{k_{h,i}} x_{h,i}^2\}. \tag{9.16}$$

9.3.2 Nonlinear NI property of multi-HIGS

Consider the system $\widehat{\mathcal{H}}$ in Fig. 9.2. It is shown in the following that if for all $i = 1, 2, \cdots, N$, the system $\mathcal{H}_i$ is nonlinear NI, then the system $\widehat{\mathcal{H}}$ is also nonlinear NI.

Lemma 9.2. *Consider N nonlinear NI systems connected in parallel with inputs $u_1, u_2, \cdots, u_N$, outputs $y_1, y_2, \cdots, y_N$ and states $x_1, x_2, \cdots, x_N$. They have storage functions $V_1(x_1), V_2(x_2), \cdots, V_N(x_N)$ that satisfy Definition 3.1; i.e., $\dot{V}_i(x_i) \leq u_i(t)^T \dot{y}_i(t)$, $\forall t \geq 0$, $\forall i = 1, 2, \cdots, N$. Then the nonlinear system having input $U = \begin{bmatrix} u_1^T & u_2^T & \cdots & u_N^T \end{bmatrix}^T$, output $U = \begin{bmatrix} y_1^T & y_2^T & \cdots & y_N^T \end{bmatrix}^T$ and state $X = \begin{bmatrix} x_1^T & x_2^T & \cdots & x_N^T \end{bmatrix}^T$ is also nonlinear NI with the storage function*

$$\widehat{V}(X) = \Sigma_{i=1}^N V_i(x_i). \tag{9.17}$$

Proof. The storage function $\widehat{V}(X)$ defined in (9.17) satisfies

$$\dot{\widehat{V}}(X) = \Sigma_{i=1}^N \dot{V}_i(x_i) \leq \Sigma_{i=1}^N u_i^T \dot{y}_i = U^T \dot{Y}.$$

Therefore, the system with input U and output Y also satisfies Definition 3.1. □

Theorem 9.3. *Consider the multi-HIGS $\widehat{\mathcal{H}}$ represented by (9.15), which is also shown in Fig. 9.2. The system $\widehat{\mathcal{H}}$ with input E_h, output U_h and state X_h, defined in (9.12), (9.13) and (9.14) respectively, is a nonlinear NI system with the storage function*

$$\widehat{V}_h(X_h) = \Sigma_{i=1}^N V_i(x_{h,i}) = \frac{1}{2} X_h^T K_h^{-1} X_h,$$

where $K_h = diag\{k_{h,1}, k_{h,2}, \cdots, k_{h,N}\}$. Here, e_i, u_i, $x_{h,i}$ and $V_i(x_{h,i})$ are the input, output, state and storage function of the i-th SISO HIGS $\mathcal{H}_i$, respectively, $i = 1, 2, \cdots, N$.

Proof. The proof follows directly from Lemma 9.2 and the nonlinear NI property of SISO HIGS of the form (9.1), as given in Theorem 9.1. Specifically, we have

$$\dot{\widehat{V}}_h(X_h) = \Sigma_{i=1}^N \dot{V}_i(x_{h,i}) \leq \Sigma_{i=1}^N e_i \dot{u}_i = E_h^T \dot{U}_h, \tag{9.18}$$

which satisfies Definition 3.1. $\square$

9.3.3 Stability of the interconnection of a MIMO linear NI system and a multi-HIGS

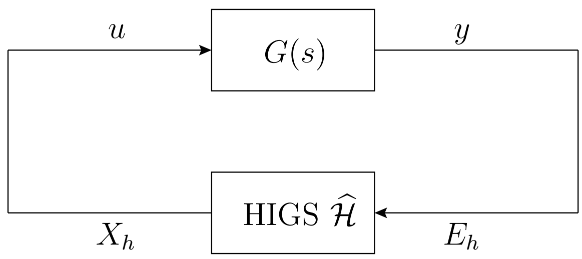


Figure 9.3: Closed-loop interconnection of a MIMO linear NI plant $G(s)$ and a multi-HIGS controller $\widehat{\mathcal{H}}$.

Lemma 9.3. *Consider the multi-HIGS of the form (9.15) shown in Fig. 9.2. If $\dot{\widehat{V}}_h(X_h) = E_h^T \dot{U}_h$, then $X_h = K_h E_h$.*

Proof. Considering (9.18), we have $\dot{\widehat{V}}_h(X_h) = E_h^T \dot{U}_h$ only if $\dot{V}_i(x_{h,i}) = e_i \dot{u}_i = e_i \dot{x}_{h,i}$ for all $i = 1, 2, \cdots, N$. For a HIGS of the form (9.1), $\dot{V}_i(x_{h,i}) = e_i \dot{x}_{h,i}$ implies $\frac{1}{k_{h,i}} x_{h,i} \dot{x}_{h,i} = e_i \dot{x}_{h,i}$. This holds if $\dot{x}_{h,i} = 0$ or $x_{h,i} = k_{h,i} e_i$. Consider the condition that $\dot{x}_{h,i} = 0$, in $\mathcal{F}_{1,i}$ mode, $\dot{x}_{h,i} = 0$ implies $e_i = 0$. According to (9.16), $e_i = 0$ implies that $x_{h,i} = 0$. In this case, $x_{h,i} = k_{h,i} e_i = 0$. In $\mathcal{F}_{2,i}$ mode, $x_{h,i} = k_{h,i} e_i$. Hence, $x_{h,i} = k_{h,i} e_i$ always holds in the case that $\dot{V}_i(x_{h,i}) = e_i \dot{u}_i = e_i \dot{x}_{h,i}$. Therefore, $\dot{\widehat{V}}_h(X_h) = E_h^T \dot{U}_h$ implies that $X_h = K_h E_h$. $\square$

Consider a MIMO linear NI system with the transfer function matrix $G(s)$ and the minimal realization

$$\dot{x} = Ax + Bu, \tag{9.19a}$$

$$y = Cx, \tag{9.19b}$$

where $u, y \in \mathbb{R}^N$ and $x \in \mathbb{R}^n$ are the input, output and state of the system, respectively. Here $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times N}$ and $C \in \mathbb{R}^{N\times n}$.

Theorem 9.4. *Consider the MIMO minimal linear NI system (9.19). There always exists a multi-HIGS $\widehat{\mathcal{H}}$, given in (9.15) and Fig. 9.2, such that the closed-loop interconnection of the system (9.19) and the multi-HIGS $\widehat{\mathcal{H}}$ as shown in Fig. 9.3 is asymptotically stable.*

Proof. Since the system (9.19) is minimal and NI, then according to Lemma 2.3, we have that $\det A \neq 0$ and there exists $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$ such that

$$AY + YA^T \leq 0, \qquad \text{and} \qquad B + AYC^T = 0. \tag{9.20}$$

Using Lyapunov's direct method, let the storage function of the closed-loop interconnection be

$$\begin{aligned} W(x, X_h) &= \frac{1}{2}x^T Y^{-1} x + \frac{1}{2}X_h^T K_h^{-1} X_h - X_h^T C x \\ &= \frac{1}{2}\begin{bmatrix} x^T & X_h^T \end{bmatrix} \begin{bmatrix} Y^{-1} & -C^T \\ -C & K_h^{-1} \end{bmatrix} \begin{bmatrix} x \\ X_h \end{bmatrix}. \end{aligned}$$

Using Schur complement theorem, $W(x, X_h) > 0$ for all $(x, X_h) \neq (0, 0)$ if

$$K_h^{-1} - CYC^T > 0. \tag{9.21}$$

Using (9.20), we have that $CYC^T = -CA^{-1}B = G(0)$, where $G(s) = C(sI - A)^{-1}B$ is the transfer function matrix of the system (9.19) Then, (9.21) can be written as

$$K_h^{-1} - G(0) > 0.$$

Since both Y^{-1} and K_h^{-1} are positive definite, the condition (9.21) is equivalent to

$$Y^{-1} - C^T K_h C > 0. \tag{9.22}$$

Take the time derivative of $W(x, X_h)$, we have

$$\begin{aligned} \dot{W}(x, X_h) &= x^T Y^{-1}\dot{x} + X_h K_h^{-1} \dot{X}_h - \dot{X}_h^T C x - X_h^T C \dot{x} \\ &= \left(x^T Y^{-1} - X_h C\right)\dot{x} + \dot{X}_h^T \left(K_h^{-1} X_h - Cx\right) \\ &= \left(x^T Y^{-1} - uC\right)\dot{x} + \dot{X}_h^T \left(K_h^{-1} X_h - E_h\right) \\ &= \left(x^T Y^{-1} + uB^T A^{-T} Y^{-1}\right)\dot{x} + \dot{X}_h^T \left(K_h^{-1} X_h - E_h\right) \\ &= \left(x^T A^T + uB^T\right)(A^{-T} Y^{-1})\dot{x} + \dot{X}_h^T \left(K_h^{-1} X_h - E_h\right) \\ &= \frac{1}{2}\dot{x}^T (A^{-T} Y^{-1} + Y^{-1} A^{-1})\dot{x} + \dot{X}_h^T \left(K_h^{-1} X_h - E_h\right), \end{aligned}$$

where $u = X_h$ and $E_h = y = Cx$ are also used. We have that $\dot{X}_h^T \left(K_h^{-1} X_h - E_h\right) = \dot{\hat{V}}_h(X_h) - E_h^T \dot{U}_h \leq 0$ and equality holds only if $X_h = K_h E_h$ according to Lemma 9.3. We also have that $\frac{1}{2}\dot{x}^T (A^{-T} Y^{-1} + Y^{-1} A^{-1})\dot{x} \leq 0$ because $A^{-T} Y^{-1} + Y^{-1} A^{-1} \leq 0$ according to (9.20). Therefore, $\dot{W}(x, X_h) \leq 0$ and the equality $\dot{W}(x, X_h) = 0$ holds only if $X_h = K_h E_h$ and $\dot{x}^T (A^{-T} Y^{-1} + Y^{-1} A^{-1})\dot{x} = 0$. We apply LaSalle's invariance principle in the following to prove that there exist $k_{h,i}$ and $\omega_{h,i}$ ($i = 1, 2, \cdots, N$) such that the closed-loop system is asymptotically stable. We only consider the case that $x \neq 0$. The function $\dot{W}(x, X_h)$ stays at zero only if $X_h \equiv K_h E_h$ and $\dot{x}^T(A^{-T}Y^{-1} + Y^{-1}A^{-1})\dot{x} \equiv 0$. The condition $X_h \equiv K_h E_h$ implies that

$$u \equiv K_h y \equiv K_h C x, \tag{9.23}$$

where the system settings $u = U_h = X_h$ and $E_h = y = Cx$ as shown in Fig. 9.3 are also used. In the sequel, we have that

$$\dot{x} \equiv Ax + Bu \equiv Ax + BK_hCx \equiv (A + BK_hC)x. \tag{9.24}$$

According to (9.20), we have that

$$A + BK_hC = A - AYC^TK_hC = AY(Y^{-1} - C^TK_hC),$$

which is nonsingular due to (9.22) and the non-singularity of the matrices A and Y. Therefore, according to (9.24), $\dot{x} \neq 0$ for any nonzero x. Also, $\dot{x}$ cannot remain a constant vector because

$$\ddot{x} = (A + BK_hC)\dot{x} \neq 0,$$

for any nonzero $\dot{x}$. The condition $\dot{x}^T(A^{-T}Y^{-1} + Y^{-1}A^{-1})\dot{x} \equiv 0$ implies that $\dot{x}$ must stay in the null space of $A^{-T}Y^{-1} + Y^{-1}A^{-1}$.

With the above information about $\dot{x}$ known in the case that $\dot{W}(x, X_h) \equiv 0$, we now prove that $\dot{W}(x, X_h)$ cannot remain zero forever. We first prove by contradiction that none of the single HIGS can stay in integrator mode unless both of the HIGS input and output remain zero. Suppose that there is a HIGS $\mathcal{H}_i$ staying in the integrator mode $\mathcal{F}_{1,i}$. Then we have that $\dot{x}_{h,i} = \omega_{h,i}e_i = \omega_{h,i}y_i$ according to (9.15), and $x_{h,i} = k_{h,i}y_i$ according to (9.23). Here, y_i denotes the i-th output element in y. Therefore, we have that

$$\dot{x}_{h,i} = \omega_{h,i}y_i = k_{h,i}\dot{y}_i \tag{9.25}$$

over a nonzero time interval $[t_a, t_b]$ where $t_a < t_b$. With $\omega_{h,i} > 0$ chosen, for nonzero $y_i(t_a)$, (9.25) implies that $y_i(t) = y_i(t_a)exp(\frac{\omega_{h,i}}{k_{h,i}}t)$ for $t \in [t_a, t_b]$. This means that the closed-loop system is unstable, which contradicts its Lyapunov stability proved above. For HIGS with an input $e_j = y_j$ that does not remain zero, it must stay in the gain mode $\mathcal{F}_{2,j}$. Now we prove that we can force it to exit the gain mode by choosing suitable HIGS parameters. Now we consider the condition $\omega_{h,j}e_j^2 > k_{h,j}e_j\dot{e}_j$ in $\mathcal{F}_{2,j}$. This condition cannot be satisfied for all HIGS $\mathcal{H}_j$ in gain mode over time via satisfying $e_j\dot{e}_j < 0$ because then $\dot{V}_{h,j} \leq e_j\dot{x}_{h,j} = k_{h,j}e_j\dot{e}_j < 0$, where $V_{h,j}$ is the storage function of the HIGS $\mathcal{H}_j$. Hence $y_j = e_j = \frac{1}{k_{h,j}} x_{h,j}$ will converge to zero. This implies that eventually $y = 0$, which is not the case considered here. Considering those e_j that eventually satisfy $e_j \dot{e}_j > 0$, since the trajectories of $\dot{e}_j$ and e_j in gain mode are independent of $\omega_{h,j}$, we can always choose sufficiently small $\omega_{h,j} > 0$ such that $\omega_{h,j} e_j^2 < k_{h,j} e_j \dot{e}_j$, in order to violate the condition $\mathcal{F}_{2,j}$ for some $\mathcal{H}_j$ in gain mode. These $\mathcal{H}_j$ will then enter integrator mode for at least some finite time. Thus, we can eventually force all HIGS to switch into integrator mode and have their inputs remaining zero, except for those staying in the gain mode by remaining in $\mathcal{F}_{2,j}$ via $\dot{e}_j \equiv 0$. In this case, $\dot{y} \equiv 0$, which implies that $\dot{x} \equiv 0$ according to observability. This contradicts with the fact that $\dot{x}$ cannot remain zero, which has been proved above. We conclude that $\dot{W}(x, X_h) \equiv 0$ will eventually be violated, and $W(x, X_h)$ will decrease monotonically until it reaches zero. $\square$

Remark 9.1. *The multi-HIGS model (9.15) allows the integrator frequencies $\omega_{h,i}$ to be zero. However, we need to choose strictly positive integrator frequencies in some cases. For example, a lossless NI plant $G(s)$ cannot be stabilized by a pure gain feedback. Hence, we need $\omega_{h,i} > 0$ at least for some i, or even for all i, to ensure the HIGS controllers will work properly. Similar remarks also apply to the results in Theorems 9.2 and 9.6.*

9.4 The cascade of two HIGS

In this section, the cascade of two HIGS elements is considered as a controller for SISO linear NI systems. We prove that the cascade of two HIGS is a nonlinear NI system and its interconnection with a linear NI system is asymptotically stable.

9.4.1 System description

We first provide a description for the cascaded HIGS. As shown in Fig. 9.4, it is a simple open-loop interconnection of two HIGS where the output of one HIGS is used as the output of the other HIGS.

Figure 9.4: A cascade of the HIGS $\mathcal{H}_1$ and the HIGS $\mathcal{H}_2$. The output x_1 of $\mathcal{H}_1$ is fed into $\mathcal{H}_2$ as its input e_2.

The system models of the two HIGS systems $\mathcal{H}_1$ and $\mathcal{H}_2$ are as follows:

$$\mathcal{H}_1 : \quad \begin{cases} \dot{x}_1 = \omega_1 e_1, & \text{if } (e_1, y_1, \dot{e}_1) \in \mathcal{F}_1 \\ x_1 = k_1 e_1, & \text{if } (e_1, y_1, \dot{e}_1) \in \mathcal{F}_2 \\ y_1 = x_1, \end{cases} \tag{9.26}$$

where $x_1, e_1, y_1 \in \mathbb{R}$ are the state, input and output of the system, respectively. Here, $\omega_1 \in [0, \infty)$ and $k_1 \in (0, \infty)$ are system parameters. Also, we have that

$$\mathcal{F} = \left\{ (e_1, y_1, \dot{e}_1) \in \mathbb{R}^3 | e_1 y_1 \geq \frac{1}{k_1} y_1^2 \right\}, \tag{9.27}$$

$$\mathcal{F}_1 = \mathcal{F} \backslash \mathcal{F}_2,$$

$$\mathcal{F}_2 = \left\{ (e_1, y_1, \dot{e}_1) \in \mathcal{F} | y_1 = k_1 e_1 \text{ and } \omega_1 e_1^2 > k_1 \dot{e}_1 e_1 \right\}. \tag{9.28}$$

In addition,

$$\mathcal{H}_2 : \quad \begin{cases} \dot{x}_2 = \omega_2 e_2, & \text{if} (e_2, y_2, \dot{e}_2) \in \widetilde{\mathcal{F}}_1 \\ x_2 = k_2 e_2, & \text{if} (e_2, y_2, \dot{e}_2) \in \widetilde{\mathcal{F}}_2 \\ y_2 = x_2, \end{cases} \tag{9.29}$$

where $x_2, e_2, y_2 \in \mathbb{R}$ are the state, input and output of the system, respectively. Here, $\omega_2 \in [0, \infty)$ and $k_2 \in (0, \infty)$ are system parameters. And we have that

$$\widetilde{\mathcal{F}} = \left\{ (e_2, y_2, \dot{e}_2) \in \mathbb{R}^3 | e_2 y_2 \geq \frac{1}{k_2} y_2^2 \right\}, \tag{9.30}$$

$$\widetilde{\mathcal{F}}_1 = \widetilde{\mathcal{F}} \backslash \widetilde{\mathcal{F}}_2,$$

$$\widetilde{\mathcal{F}}_2 = \left\{ (e_2, y_2, \dot{e}_2) \in \widetilde{\mathcal{F}} | y_2 = k_2 e_2 \text{ and } \omega_2 e_2^2 > k_2 \dot{e}_2 e_2 \right\}. \tag{9.31}$$

The interconnection of these two systems can be described by the equation

$$e_2 = y_1.$$

9.4.2 Nonlinear NI property of the cascade of two HIGS

The following lemma is required in the presentation of the main results.

Lemma 9.4. *Consider the HIGS of the form (9.1). Suppose* $e x_h = \frac{1}{k_h} x_h^2$ *over a time interval* $[t_a, t_b]$, *where* $t_a < t_b$, *then* $x_h = k_h e$ *for all* $t \in [t_a, t_b]$.

Proof. We have that $e x_h \equiv \frac{1}{k_h} x_h^2$ implies $x_h \equiv 0$ or $x_h \equiv k_h e$. We only consider the case $x_h \equiv 0$ in the $\mathcal{F}_1$ mode, because $x_h = k_h e$ always holds in the $\mathcal{F}_2$ mode. In the $\mathcal{F}_1$ mode, $u_h \equiv x_h \equiv 0$ implies that $\dot{x}_h \equiv 0$. This implies that $e \equiv 0$. Therefore, $x_h \equiv x_h \equiv k_h e \equiv 0$. □

Note that Lemma 9.4 is nontrivial since the equation $ex_h = \frac{1}{k_h}x_h^2$ has two solutions $x_h = 0$ and $x_h = k_h e$.

Theorem 9.5. *Consider two HIGS $\mathcal{H}_1$ and $\mathcal{H}_2$ having the system models (9.26) and (9.29), respectively. Suppose $k_2\omega_1 \leq k_1\omega_2$, then the cascade of $\mathcal{H}_1$ and $\mathcal{H}_2$ as shown in Fig. 9.4 is a nonlinear NI system with the Lyapunov storage function*

$$V(x_1, x_2) = ax_1^2 + \frac{k_2 - 2ak_1}{2k_1k_2^2}x_2^2, \tag{9.32}$$

satisfying

$$\dot{V}(x_1, x_2) \leq e_1\dot{x}_2. \tag{9.33}$$

Here, $0 < a < \frac{k_2}{2k_1}$ is a parameter. Moreover, if $\dot{V}(x_1, x_2) = e_1\dot{x}_2$ over a time interval $[t_a, t_b]$, where $t_a < t_b$, then for all $t \in [t_a, t_b]$ we have that $x_1 = k_1e_1$ and $x_2 = k_2e_2 = k_2k_1e_1$.

Proof. Since $0 < a < \frac{k_2}{2k_1}$, we have that $V(x_1, x_2)$ in (9.32) is positive definite. Because both $\mathcal{H}_1$ and $\mathcal{H}_2$ have two modes, i.e., integrator mode and gain mode, the cascaded system has four modes. We prove in the following that (9.33) holds in these four modes. Note that $e_2 = x_1$ will be used in the following.

Case 1. $\mathcal{H}_1$ in $\mathcal{F}_1$ and $\mathcal{H}_2$ in $\widetilde{\mathcal{F}}_1$. According to (9.27) and (9.30), we have that

$$e_1x_1 \geq \frac{1}{k_1}x_1^2, \tag{9.34}$$

$$x_1x_2 \geq \frac{1}{k_2}x_2^2 \implies x_1x_2 \leq k_2x_1^2, \tag{9.35}$$

where the deduction in (9.35) uses Lemma 9.1. Since $k_2\omega_1 \leq k_1\omega_2$ and $0 < a < \frac{k_2}{2k_1}$, we have that $2a\omega_1 - \omega_2 < 0$ and $k_2 - 2ak_1 > 0$. Taking the time derivative of $V(x_1, x_2)$, we get

$$\begin{aligned}
\dot{V}(x_1, x_2) - e_1\dot{x}_2 &= 2ax_1\dot{x}_1 + \frac{k_2 - 2ak_1}{k_1k_2^2}x_2\dot{x}_2 - e_1\dot{x}_2 \\
&= 2ax_1\omega_1e_1 + \frac{k_2 - 2ak_1}{k_1k_2^2}x_2\omega_2x_1 - \omega_2e_1x_1 \\
&= (2a\omega_1 - \omega_2)e_1x_1 + \frac{k_2 - 2ak_1}{k_1k_2^2}\omega_2x_1x_2 \\
&\leq (2a\omega_1 - \omega_2)\frac{1}{k_1}x_1^2 + \frac{k_2 - 2ak_1}{k_1k_2^2}\omega_2k_2x_1^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{k_1}x_1^2\left(2a\omega_1 - \omega_2 + \frac{k_2 - 2ak_1}{k_2}\omega_2\right) \\
&= \frac{2a}{k_1}x_1^2\left(\omega_1 - \frac{k_1}{k_2}\omega_2\right) \\
&\leq 0, \qquad (9.36)
\end{aligned}
$$

where the first inequality uses (9.34), (9.35) and also the fact that $2a\omega_1 - \omega_2 < 0$. This implies that $\dot{V}(x_1, x_2) - e_1\dot{x}_2 = 0$ in (9.36) is possible only if the equalities in (9.34) and (9.35) hold. If these equalities hold over the time interval $[t_a, t_b]$, then according to Lemmas 9.1 and 9.4, we have that $x_1 = k_1e_1$ and $x_2 = k_2e_2$ in $[t_a, t_b]$. That is $x_2 = k_1k_2e_1$.

Case 2. $\mathcal{H}_1$ in $\mathcal{F}_2$ and $\mathcal{H}_2$ in $\widetilde{\mathcal{F}}_2$. Taking the time derivative of $V(x_1, x_2)$, we get:

$$
\begin{aligned}
\dot{V}(x_1, x_2) - e_1\dot{x}_2 &= 2ax_1\dot{x}_1 + \frac{k_2 - 2ak_1}{k_1k_2^2}x_2\dot{x}_2 - e_1\dot{x}_2 \\
&= 2ak_1^2e_1\dot{e}_1 + \frac{k_2 - 2ak_1}{k_1k_2^2}k_1^2k_2^2e_1\dot{e}_1 - k_1k_2e_1\dot{e}_1 \\
&= e_1\dot{e}_1\left(2ak_1^2 + \frac{k_2 - 2ak_1}{k_1k_2^2}k_1^2k_2^2 - k_1k_2\right) \\
&= 0.
\end{aligned}
$$

In Case 2, $x_1 = k_1e_1$ and $x_2 = k_2e_2$ automatically holds.

Case 3. $\mathcal{H}_1$ in $\mathcal{F}_1$ and $\mathcal{H}_2$ in $\widetilde{\mathcal{F}}_2$. According to (9.27), in this case we have that

$$
e_1x_1 \geq \frac{1}{k_1}x_1^2 \implies e_1x_1 \leq k_1e_1^2, \qquad (9.37)
$$

where the deduction uses Lemma 9.1. Also, according to (9.31), we have that

$$
\omega_2e_2^2 > k_2\dot{e}_2e_2.
$$

We now take the time derivative of $V(x_1, x_2)$, which yields:

$$
\begin{aligned}
\dot{V}(x_1, x_2) - e_1\dot{x}_2 &= 2ax_1\dot{x}_1 + \frac{k_2 - 2ak_1}{k_1k_2^2}x_2\dot{x}_2 - e_1\dot{x}_2 \\
&= 2ax_1\omega_1e_1 + \frac{k_2 - 2ak_1}{k_1k_2^2}k_2x_1k_2\omega_1e_1 - e_1k_2\omega_1e_1 \\
&= \frac{\omega_1k_2}{k_1}x_1e_1 - \omega_1k_2e_1^2 \\
&\leq \frac{\omega_1k_2}{k_1}k_1e_1^2 - \omega_1k_2e_1^2 \\
&= 0, \qquad (9.38)
\end{aligned}
$$

where the inequality also uses (9.37). Therefore, $\dot{V}(x_1, x_2) - e_1\dot{x}_2 = 0$ in (9.38) over a time interval $[t_a, t_b]$ only if $x_1 = k_1e_1$ in $[t_a, t_b]$, according to Lemma 9.1. Also, as $\mathcal{H}_2$ is in $\widetilde{\mathcal{F}}_2$ mode, $x_2 = k_2e_2$ automatically holds. This implies that $x_2 = k_1k_2e_1$.

Case 4. $\mathcal{H}_1$ in $\mathcal{F}_2$ and $\mathcal{H}_2$ in $\widetilde{\mathcal{F}}_1$. According to (9.28) and (9.30), in this case we have that

$$\begin{aligned}
&\omega_1 e_1^2 > k_1\dot{e}_1e_1,\\
&x_1x_2 \geq \frac{1}{k_2}x_2^2 \implies x_1x_2 \leq k_2x_1^2 \implies e_1x_2 \leq k_1k_2e_1^2.
\end{aligned}$$

Taking the time derivative of $V(x_1, x_2)$ yields:

$$\begin{aligned}
\dot{V}(x_1, x_2) - e_1\dot{x}_2 &= 2ax_1\dot{x}_1 + \frac{k_2 - 2ak_1}{k_1k_2^2}x_2\dot{x}_2 - e_1\dot{x}_2\\
&= 2ak_1^2e_1\dot{e}_1 + \frac{k_2 - 2ak_1}{k_1k_2^2}x_2\omega_2k_1e_1 - e_1\omega_2k_1e_1\\
&< 2ak_1\omega_1e_1^2 + \frac{k_2 - 2ak_1}{k_1k_2^2}\omega_2k_1k_1k_2e_1^2 - \omega_2k_1e_1^2\\
&= 2ak_1e_1^2(\omega_1 - \omega_2\frac{k_1}{k_2})\\
&\leq 0.
\end{aligned}$$

In Case 4, $\dot{V}(x_1, x_2) < e_1\dot{x}_2$ and hence the case $\dot{V}(x_1, x_2) = e_1\dot{x}_2$ does not occur. We conclude that in all four cases, we have that $\dot{V}(x_1, x_2) \leq e_1\dot{x}_2$. Also, if $\dot{V}(x_1, x_2) = e_1\dot{x}_2$ over a time interval $[t_a, t_b]$, where $t_a < t_b$, then $x_1 = k_1e_1$ and $x_2 = k_2e_2 = k_1k_2e_1$ for all $t \in [t_a, t_b]$. □

9.4.3 Stability for the interconnection of an NI system and two cascaded HIGS

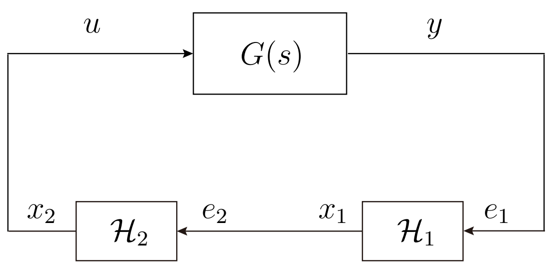


Figure 9.5: Closed-loop interconnection of a linear NI system and the cascade of two HIGS.

Theorem 9.6. *Consider the SISO minimal linear NI system (9.7). There exist* *two HIGS elements $\mathcal{H}_1$ and $\mathcal{H}_2$, given in (9.26) and (9.29), respectively, such that the interconnection of the system (9.7) and the cascade of $\mathcal{H}_1$ and $\mathcal{H}_2$ as shown in Fig. 9.5 is asymptotically stable.*

Proof. Since the system (9.7) is minimal and NI, then according to Lemma 2.3, we have that $\det A \neq 0$ and there exists $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$ such that

$$AY + YA^T \leq 0, \qquad \text{and} \qquad B + AYC^T = 0. \tag{9.39}$$

Using Lyapunov's direct method, let the storage function of the closed-loop interconnection be

$$\begin{aligned} W(x, x_1, x_2) &= \frac{1}{2}x^T Y^{-1} x + V(x_1, x_2) - Cxx_2 \\ &= \frac{1}{2}x^T Y^{-1} x + ax_1^2 + \frac{k_2 - 2ak_1}{2k_1k_2^2}x_2^2 - Cxx_2 \\ &= ax_1^2 + \frac{1}{2}\begin{bmatrix} x^T & x_2 \end{bmatrix} \begin{bmatrix} Y^{-1} & -C^T \\ -C & \frac{k_2-2ak_1}{k_1k_2^2} \end{bmatrix} \begin{bmatrix} x \\ x_2 \end{bmatrix}, \end{aligned} \tag{9.40}$$

where $V(x_1, x_2)$ as given in (9.32), is the storage function of the cascade of $\mathcal{H}_1$ and $\mathcal{H}_2$. Here, $a \in \mathbb{R}$ is a constant parameter that satisfies

$$0 < a < \frac{k_2}{2k_1}. \tag{9.41}$$

Using the Schur complement theorem, $W(x, x_1, x_2) > 0$ for all $(x, x_1, x_2) \neq (0, 0, 0)$ if

$$\frac{k_2 - 2ak_1}{k_1k_2^2} - CYC^T > 0. \tag{9.42}$$

Using (9.39), we have that $CYC^T = -CA^{-1}B = G(0)$, where $G(s) = C(sI - A)^{-1}B$ is the transfer function matrix of the system (9.7). Then, (9.42) becomes

$$\frac{k_2 - 2ak_1}{k_1k_2^2} > G(0). \tag{9.43}$$

There exist an a in the region (9.41) such that (9.43) holds if

$$\frac{k_2}{k_1k_2^2} > G(0).$$

This implies that

$$k_1k_2G(0) < 1.$$

According to Schur complement theorem, the positive definiteness of $W(x, x_1, x_2)$

in (9.40) also implies

$$Y^{-1} - \frac{k_1 k_2^2}{k_2 - 2ak_1} C^T C > 0,$$

which is equivalent to the condition (9.42). Considering (9.41), we have that

$$Y^{-1} - k_1 k_2 C^T C > Y^{-1} - \frac{k_1 k_2^2}{k_2 - 2ak_1} C^T C > 0. \tag{9.44}$$

Taking the time derivative of $W(x, x_1, x_2)$, we have

$$\begin{aligned}
\dot{W}(x, x_1, x_2) &= x^T Y^{-1} \dot{x} + \dot{V}(x_1, x_2) - C\dot{x}x_2 - Cx\dot{x}_2 \\
&= \left(x^T Y^{-1} - x_2 C\right) \dot{x} + \dot{V}(x_1, x_2) - e_1 \dot{x}_2 \\
&= \left(x^T Y^{-1} - uC\right) \dot{x} + \left(\dot{V}(x_1, x_2) - e_1 \dot{x}_2\right) \\
&= \left(x^T Y^{-1} + uB^T A^{-T} Y^{-1}\right) \dot{x} + \left(\dot{V}(x_1, x_2) - e_1 \dot{x}_2\right) \\
&= \left(x^T A^T + uB^T\right) (A^{-T} Y^{-1}) \dot{x} + \left(\dot{V}(x_1, x_2) - e_1 \dot{x}_2\right) \\
&= \frac{1}{2} \dot{x}^T (A^{-T} Y^{-1} + Y^{-1} A^{-1}) \dot{x} + \left(\dot{V}(x_1, x_2) - e_1 \dot{x}_2\right),
\end{aligned} \tag{9.45}$$

where $u = x_2$ and $e_1 = y = Cx$ are also used. According to Theorem 9.5, we have that for $k_1, k_2, \omega_1, \omega_2$ satisfying $k_2\omega_1 \le k_1\omega_2$, the inequality $\dot{V}(x_1, x_2) - e_1\dot{x}_2 \le 0$ always holds. Also, $\dot{x}^T(A^{-T}Y^{-1} + Y^{-1}A^{-1})\dot{x} \le 0$ because $A^{-T}Y^{-1} + Y^{-1}A^{-1} \le 0$, according to (9.39). Therefore, $\dot{W}(x, x_1, x_2) \le 0$. Using LaSalle's invariance principle, $\dot{W}(x, x_1, x_2)$ remains zero if both $\dot{x}^T(A^{-T}Y^{-1}+Y^{-1}A^{-1})\dot{x}$ and $\dot{V}(x_1, x_2) - e_1\dot{x}_2$ remain zero. According to Theorem 9.5, $\dot{V}(x_1, x_2) - e_1\dot{x}_2$ remains zero only if $x_1 \equiv k_1 e_1$ and $x_2 \equiv k_2 e_2 \equiv k_1 k_2 e_1$. That is

$$u \equiv k_1 k_2 C x. \tag{9.46}$$

We only consider the case that $x \neq 0$ in the following. This is because $x = 0$ implies that $x_2 = u = 0$ according to (9.46). Also, $x = 0$ implies $e_1 = y = 0$, which then implies $x_1 = 0$ according to (9.27). Here, the system settings of the interconnection in Fig. 9.5 are also used.

In this case of $x \neq 0$, the state equation (9.7a) of the system $G(s)$ becomes

$$\dot{x} \equiv Ax + Bu \equiv Ax + Bk_1 k_2 Cx = (A + k_1 k_2 BC)x.$$

According to (9.39),

$$A + k_1 k_2 BC = A - k_1 k_2 AYC^T C = AY(Y^{-1} - k_1 k_2 C^T C),$$

which is nonsingular due to (9.44) and the positive definiteness of the matrices A and Y. This implies $\dot{x} \neq 0$ for all $x \neq 0$. The condition $\dot{x}^T(A^{-T}Y^{-1}+Y^{-1}A^{-1})\dot{x} \equiv 0$ implies that $\dot{x}$ must stay in the null space of $A^{-T}Y^{-1} + Y^{-1}A^{-1}$. We now prove that $\dot{W}(x, x_1, x_2)$ cannot remain zero forever. First, we prove by contradiction that neither $\mathcal{H}_1$ nor $\mathcal{H}_2$ can stay in the integrator mode. Suppose $\mathcal{H}_1$ is in the integrator mode $\mathcal{F}_1$, then $\dot{x}_1 = \omega_1 e_1 = \omega_1 y$ according to (9.26). Also, since $x_1 = k_1 e_1 = k_1 y$, we have that

$$\dot{x}_1 = \omega_1 y = k_1 \dot{y}. \tag{9.47}$$

Since the system $G(s)$ is observable and x does not remain zero, then y does not remain zero. Choose $\omega_1 > 0$, (9.47) implies that for a nonzero time interval $[t_a, t_b]$ where $t_a < t_b$, we have

$$y(t) = y(t_a)exp(\frac{\omega_1}{k_1}t).$$

This contradicts the fact that the closed-loop interconnection is Lyapunov stable, as is shown by (9.45). Similarly, if $\mathcal{H}_2$ is in the integrator mode, then we have $\dot{x}_2 = \omega_2 e_2 = \omega_2 k_1 y$ and $x_2 = k_1 k_2 e_1 = k_1 k_2 y$. Following a similar analysis, this also leads to a contradiction. Then we conclude that both of the HIGS $\mathcal{H}_1$ and $\mathcal{H}_2$ are in the gain mode. We now prove that we can force them to eventually exit the gain mode. In this case, the condition $\omega_1 e_1^2 > k_1 e_1 \dot{e}_1$ is satisfied according to (9.31). This condition cannot always be satisfied over time via satisfying $e_1 \dot{e}_1 < 0$ because then the NI inequality $\dot{V}_1(x_1) \leq e_1 \dot{x}_1 = k_1 e_1 \dot{e}_1 < 0$ implies that x_1 converges to zero and so does y since $x_1 = k_1 y$. This is not the case considered here. Therefore, e_1 and $\dot{e}_1$ will eventually satisfy $e_1 \dot{e}_1 > 0$ at some time. Since the trajectory of $\dot{e}_1$ and e_1 is independent of ω_1, we can choose sufficiently small ω_1 such that $\omega_1 e_1^2 < k_1 e_1 \dot{e}_1$, in order to violate the condition $\mathcal{F}_2$. Then $\mathcal{H}_1$ will enter the integrator mode for at least some finite time. Following a similar analysis, we can choose suitable ω_2 to force $\mathcal{H}_2$ to eventually enter the integrator mode. As is proved above, the function $\dot{W}(x, x_1, x_2)$ cannot remain zero in the integrator mode. Therefore, $W(x, x_1, x_2)$ will decrease monotonically until it reaches zero. □

9.5 Experiment: using a multi-HIGS in the control of a MEMS nanopositioner

The NI properties of the HIGS elements shown in Theorems 9.1, 9.3, 9.5 and the stability results shown in Theorems 9.2, 9.4, 9.6 motivate a methodology of using HIGS in positive feedback to control flexible structures with colocated force actuators and position sensors. One example of such flexible structures is a MEMS nanopositioner. In this section, we apply this methodology experimentally on a 2-DOF MEMS nanopositioner, which can be modelled as a two-input two-output (TITO) linear NI system. Under the control of a TITO multi-HIGS controller in positive feedback, the MEMS nanopositioner can track a reference signal quickly and accurately.

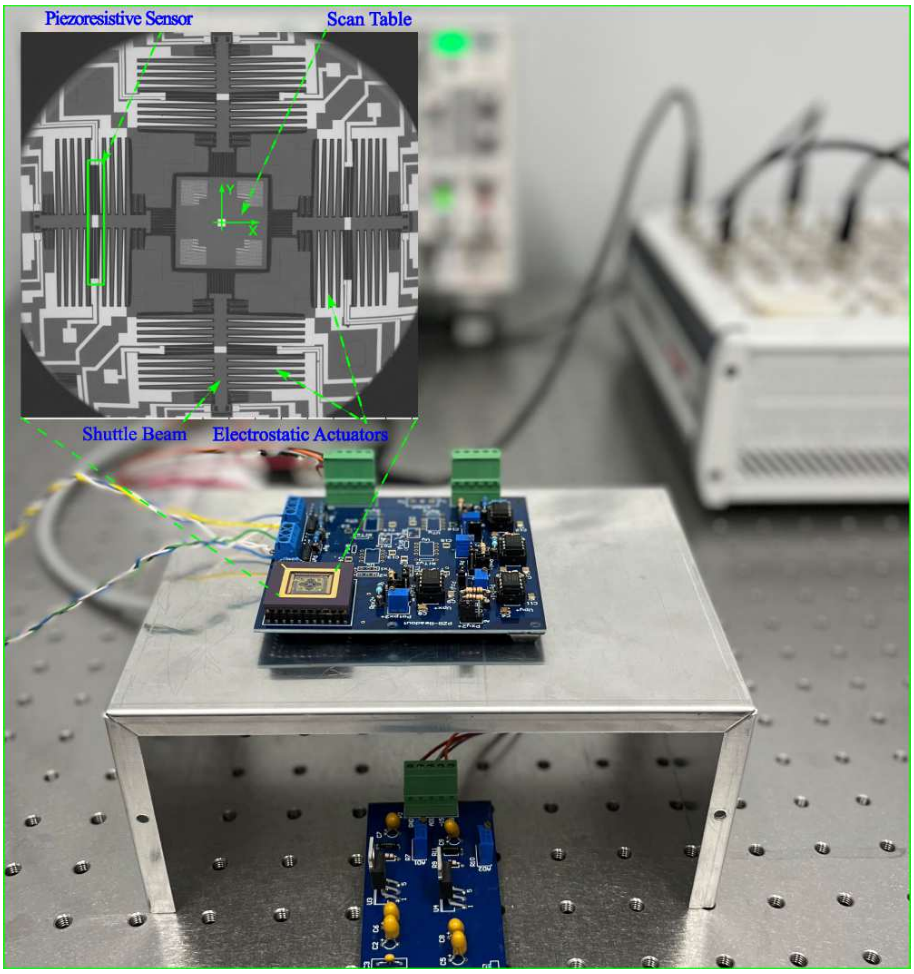


Figure 9.6: Experimental setup with the MEMS nanopositioner mounted on a custom-designed PCB. The close-up view shows the SEM image of the MEMS nanopositioner reported in [1].

9.5.1 2-DOF MEMS nanopositioner

A 2-DOF MEMS nanopositioner is a flexible structure with colocated force actuators and position sensors, which can be modelled as an NI system [1]. The nanopositioner features a stage at the center with dimensions of 1.8 mm × 1.8 mm. Four electrostatic comb-drives move the stage bidirectionally in X and Y directions. On-chip piezoresistive sensors measure lateral displacements of the stage. The close-up view in Fig. 9.6 depicts the scanning electron microscope (SEM) image of the MEMS nanopositioner. To alleviate the quadratic nonlinearity between the induced electrostatic force and the stage displacement, this device uses a bilateral actuation mechanism. A maximum linear displacement range of 13 μm can be achieved in each axis. The nanopositioner was previously designed and characterized in [1] and employed as the scanner stage of a video-rate atomic force microscope [149].

9.5.2 Frequency response

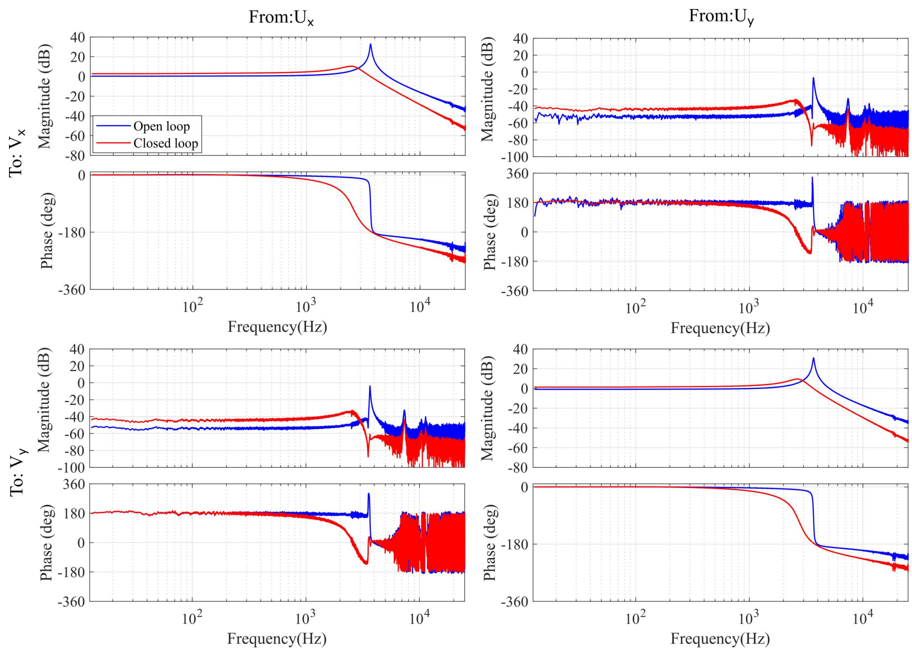


Figure 9.7: Frequency response of the MEMS nanopositioner in open loop and closed loop with the HIGS element in TITO configuration.

The TITO Frequency response function of the MEMS nanopositioner is obtained from actuator input to sensor output using an ONOSSOKI FFT Analyzer (CF-9400) with single channel excitation. For this purpose, a wideband chirp signal is applied to the actuators through a high-voltage amplifier with a gain of 20, and the frequency response of the 2-DOF nanopositioner is recorded up to 25 kHz. Fig. 9.7 shows the measured open-loop frequency response of the device. The fundamental resonance frequencies of the X and Y axis are 3665 Hz and 3680 Hz, respectively. We observe that the frequency responses of both axes of the MEMS nanopositioner are almost identical, with negligible cross-couplings at low frequencies. Typically, a flexure-guided nanopositioner with compatible collocated actuator-sensor pairs has NI property. Although the MEMS nanopositioner is NI in theory, it can be seen from Fig. 9.7 that some high-frequency dynamics slightly violate the conditions of the NI property. This is unavoidable due to the fabrication tolerances and signal conditioning in the read-out circuits, which cause the discrepancy from the ideal model. However, as discussed in [150], the 2-DOF MEMS nanopositioner can be considered an NI system for frequencies up to 3976 Hz. We show in the following that the stability results in Theorem 9.4 remain effective even in the presence of these spillover dynamics. The transfer function of the nanopositioner in TITO format can be described by

$$G(s) = \begin{bmatrix} G_{xx}(s) & G_{xy}(s) \\ G_{yx}(s) & G_{yy}(s) \end{bmatrix}$$

where $G_{ij}(s)$ denotes the transfer function from input j to the sensor output i. A minimal state-space realization of the transfer function matrix is constructed using the frequency response data (FRD) model obtained from the frequency response measurement and MATLAB system identification toolbox. Accordingly, the state-space model of the nanopositioner in lateral axes can be written as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{9.48}$$

where

$$A = \begin{bmatrix} 6989.99 & 22987.75 & 4565.46 & -1067.36 \\ -25554.29 & -7675.92 & -1058.081 & -4147.69 \\ 1399.21 & 505.53 & -7758.92 & 24470.97 \\ 569.57 & -1001.98 & -24272.18 & 7329.32 \end{bmatrix},$$

$$B = \begin{bmatrix} 157.81 & 24.58 \\ -197.95 & 12.53 \\ 33.58 & -89.97 \\ -18.69 & -86.94 \end{bmatrix},$$

$$C = \begin{bmatrix} -98.83 & -79.75 & -17.51 & -21.31 \\ 17.07 & -27.97 & -155.24 & 161.47 \end{bmatrix}.$$

As described in [151], the NI property of $G(s)$ can be assessed using the Hamiltonian method since $CB+B^TC^T > 0$. According to Theorem 1 in [151], the square transfer function $G(s)$ is NI if and only if the Hamiltonian matrix, N_0 described by (9.49), has no pure imaginary eigenvalues with odd multiplicity.

$$N_0 = \begin{bmatrix} A + BQ_0^{-1}CA & BQ_0^{-1}B^T \\ -A^TC^TQ_0^{-1}CA & -A^T - A^TC^TQ_0^{-1}B^T \end{bmatrix}, \tag{9.49}$$

where

$$Q_0 = -(CB + B^TC^T).$$

The eigenvalues of the Hamiltonian matrix in (9.49) for the state-space matrices of the system (9.48) are obtained as

$$\begin{bmatrix} \pm 1.068 \times 10^{8} \\ \pm 38.45 \\ -0.0043 \pm 41.33i \\ 4.68 \times 10^{-3} \\ \pm 9.56 \times 10^{4} i \end{bmatrix},$$

which reveals no pure imaginary eigenvalues, thus the transfer function matrix $G(s)$ with the state space realization of the form (9.48) is NI. It should be noted that the NI property of the nanopositioner can be also investigated using the measured frequency response through the eigenvalues of the matrix $j[G(j\omega)-G^{*}(j\omega)]$ [151]. The NI property of the identified system model obtained from the measured frequency response data implies that the NI property is satisfied over the frequency range of interest. Considering this, we aim to verify whether a HIGS controller can stabilize a TITO MEMS nanopositioner although its dynamics violate the NI property at high frequencies.

9.5.3 Controller design

According to stability results presented in Section 9.3, a stabilizing TITO multi-HIGS is required to satisfy the following conditions

$$K_h^{-1} - G(0) > 0.$$

Accordingly, K_h is determined by solving a feasibility problem formulated in MATLAB using the YALMIP toolbox [152] and solved with the MOSEK [153]. Since ω_h plays no role in stability analysis, it is tuned to achieve a desired performance level. Therefore, we have

$$K_h = \begin{bmatrix} 0.5617 & 0 \\ 0 & 0.6003 \end{bmatrix},$$
$$\omega_h = \begin{bmatrix} 1.1516 \times 10^{4} & 0 \\ 0 & 1.1560 \times 10^{4} \end{bmatrix} \text{(rad/s)}.$$

The closed-loop interconnection of the TITO multi-HIGS and the TITO MEMS nanopositioner is shown in Fig. 9.8, where r is the reference signal. The closed-loop frequency response of the MEMS nanopositioner in positive feedback with the TITO multi-HIGS is shown in Fig. 9.7. The frequency response is obtained using

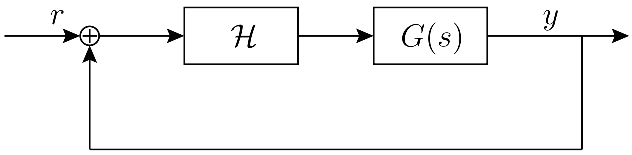


Figure 9.8: Closed-loop interconnection of a TITO multi-HIGS $\mathcal{H}$ and the TITO MEMS nanopositioner $G(s)$.

the frequency response data model of the nanopositioner and the describing function of the multi-HIGS.

9.5.4 Experiments

To assess the controller performance and the stability of the MEMS nanopositioner in closed-loop interconnection with the multi-HIGS element in TITO configuration, we implement the multi-HIGS controller in real time and perform closed-loop experiments in the time domain. Fig. 9.6 shows the experimental setup, including the MEMS nanopositioner mounted on a custom-designed PCB with actuation and sensing signal paths, a dSPACE rapid prototyping system, and high-gain voltage amplifiers.

The multi-HIGS element and the feedback loop were digitally implemented in a dSPACE rapid prototyping system with a sampling rate of 80 kHz. The X and Y axis sensor outputs were recorded in open loop and closed loop while a sequence of pulses with a frequency of 10 Hz and amplitude of 0.1 V was applied to the device as an external disturbance. Fig. 9.9 shows the nanopositioner sensor outputs in the X and Y axes, respectively. We observe that the closed-loop system with the multi-HIGS is asymptotically stable. This means that a MEMS nanopositioner can be stabilized by a multi-HIGS controller. These experimental results are consistent with our expectations. The reason that the out-of-bandwidth non-NI dynamics do not destabilize the closed-loop system is that at high frequencies, deviations from NI are too insignificant to cause instabilities. This property was observed and explained for the same plant in [150]. Furthermore, from Fig. 9.7, we observe that the magnitude of the frequency response of the MEMS nanopositioner is bounded below a certain level when the frequency is greater than 3976 Hz. Hence, the stability is guaranteed via the small-gain theorem; see [12, 154] for the stability of systems with "mixed" NI and small-gain properties. It can be seen in Fig. 9.9 that in comparison with the open-loop performance, applying the multi-HIGS in positive feedback improves the performance of the MEMS nanopositioner. From the close-up views, it is clear that fast settling time and reduced overshoot are achieved in both axes. To further improve the overshoot, ω_h can be further reduced, which in turn limits the closed-loop bandwidth.

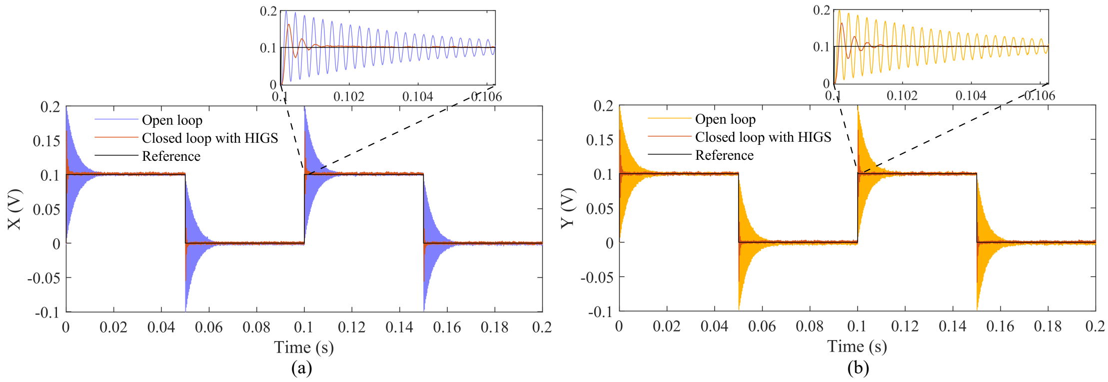


Figure 9.9: Time-domain response of the MEMS nanopositioner in X and Y axes in open loop and in positive feedback interconnection with the multi-HIGS element in TITO configuration in experiments.

9.6 Discrete-time HIGS

In this section, we use discrete-time HIGS as controllers for NI systems. We show that a discrete-time HIGS is an SANI system. Furthermore, we establish the following stability result: for any discrete-time NI system, there exists a discrete-time HIGS controller that ensures closed-loop asymptotic stability. An illustrative example is provided, where a ZOH sampled mass-spring system is asymptotically stabilized using a discrete-time HIGS controller. This section contributes in providing a specific digital control framework for physical systems with the NI property. The implementation process of a HIGS controller only involves selection of parameters in order to satisfy a simple condition. In comparison to the continuous-time design approach where a continuous-time controller is constructed based on the continuous-time model of the plant and subsequently discretized [155], the advantages of the framework in the present approach are two-fold: (i) the design and implementation processes are simpler; (ii) closed-loop stability is more rigorously guaranteed.

9.6.1 System description

Discrete-time HIGS were introduced in [145]. We adapt the model in [145] to fit the discrete-time system model (8.1) in the following:

$$\mathcal{H}: \begin{cases} x_h(k+1) = x_h(k) + \omega_h e(k), & \text{if } (x_h(k), e(k)) \in \mathcal{F} \\ x_h(k+1) = k_h e(k), & \text{if } (x_h(k), e(k)) \notin \mathcal{F} \\ y_h(k) = x_h(k+1). \end{cases} \tag{9.50}$$

Here, $e(k), x_h(k), y_h(k) \in \mathbb{R}$ are the system input, state and output, respectively. The constant parameters $\omega_h \geq 0$ and $k_h > 0$ are called the integrator frequency and the gain value, respectively. The HIGS is designed to operate under the sector constraint $(x_h(k), e(k)) \in \mathcal{F}$, where $\mathcal{F}$ is given by

$$\mathcal{F} = \{(x_h(k), e(k)) \in \mathbb{R}^2 | (x_h(k) + \omega_h e(k))e(k) \geq \frac{1}{k_h}(x_h(k) + \omega_h e(k))^2\}. \tag{9.51}$$

The HIGS is designed to operate primarily in the integrator mode if the input $e(k)$ leads to an output $y(k)$ within the sector $[0, k_h]$ under the integrator mode dynamics. Otherwise, the system operates in the gain mode so that $y_h(k) = k_h e(k)$, which automatically satisfies the sector constraint $[0, k_h]$. In what follows, we denote $e(k)$, $x_h(k)$ and $y_h(k)$ by e_k, $\widetilde{x}_k$ and $\widetilde{y}_k$ respectively for convenience.

9.6.2 SANI property of the discrete-time HIGS

We show in the following that the HIGS given in (9.50) is an SANI system.

Theorem 9.7. *The system given in (9.50) is an SANI system with the storage function*

$$\widetilde{V}(\widetilde{x}_k) = \frac{1}{2k_h}\widetilde{x}_k^2 \tag{9.52}$$

satisfying

$$\widetilde{V}(\widetilde{x}_{k+1}) - \widetilde{V}(\widetilde{x}_k) \leq e_k(\widetilde{x}_{k+1} - \widetilde{x}_k), \tag{9.53}$$

for any input e_k and state $\widetilde{x}_k$.

Proof. According to Definition 8.5 and Remark 8.3, the HIGS is an SANI system if it is NI from the input e_k to the state $\widetilde{x}_k$. Hence, we prove in the following that (9.53) is satisfied in both integrator mode and gain mode. Substituting (9.52) into (9.53) yields

$$\frac{1}{2k_h}\widetilde{x}_{k+1}^2 - \frac{1}{2k_h}\widetilde{x}_k^2 \leq e_k(\widetilde{x}_{k+1} - \widetilde{x}_k), \tag{9.54}$$

which is required to be satisfied in both modes.

Case 1. In the integrator mode, we have the state equation $\widetilde{x}_{k+1} = \widetilde{x}_k + \omega_h e_k$ and also $(\widetilde{x}_k, e_k) \in \mathcal{F}$. In this case, (9.54) becomes

$$2\widetilde{x}_k e_k \leq (2k_h - \omega_h)e_k^2, \tag{9.55}$$

which is always satisfied when $e_k = 0$. When $e_k \neq 0$, (9.55) can be rewritten as

$$2\frac{\widetilde{x}_k}{e_k} \leq 2k_h - \omega_h. \tag{9.56}$$

The condition $(\widetilde{x}_k, e_k) \in \mathcal{F}$ implies

$$\widetilde{x}_k^2 + (2\omega_h - k_h)\widetilde{x}_k e_k + (\omega_h - k_h\omega)e_k^2 \leq 0.$$

This implies that for $e_k \neq 0$,

$$\left(\frac{\widetilde{x}_k}{e_k}\right)^2 + (2\omega_h - k_h)\frac{\widetilde{x}_k}{e_k} + (\omega_h^2 - k_h\omega_h) \leq 0. \tag{9.57}$$

By solving (9.57), we have that operating in the integrator mode requires the HIGS input e_k and state $\widetilde{x}_k$ to satisfy

$$-\omega_h \leq \frac{\widetilde{x}_k}{e_k} \leq k_h - \omega_h.$$

Such a pair of $\widetilde{x}_k$ and e_k always satisfies (9.56).

Case 2. In the gain mode, we have that $\widetilde{x}_{k+1} = k_h e_k$ and $(\widetilde{x}_k, e_k) \notin \mathcal{F}$. In this case, (9.54) becomes

$$\widetilde{x}_k^2 - 2k_h \widetilde{x}_k e_k + k_h^2 e_k^2 \geq 0,$$

which always holds because

$$\widetilde{x}_k^2 - 2k_h \widetilde{x}_k e_k + k_h^2 e_k^2 = (\widetilde{x}_k - k_h e_k)^2 \geq 0.$$

Since condition (9.53) is satisfied in both modes, then the system (9.50) is an SANI system. □

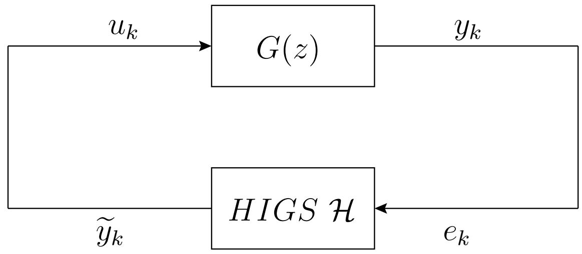


Figure 9.10: Closed-loop interconnection of a system with transfer function matrix $G(z)$ as given in (9.58) and the HIGS $\mathcal{H}$ given in (9.50).

9.6.3 Stability for the interconnection of a linear NI system and a HIGS

Motivated by the SANI property of the HIGS, we investigate whether a HIGS controller can be applied in the control of a minimal SISO linear NI system. Consider a SISO system with transfer function matrix $G(z)$ and a minimal realization

$$\Sigma\colon\ x_{k+1} = Ax_k + Bu_k, \tag{9.58a}$$

$$y_k = Cx_k, \tag{9.58b}$$

where $x_k \in \mathbb{R}^n$, $u_k, y_k \in \mathbb{R}^p$ are the system state, input and output, respectively. We show in the following that if the system Σ in (9.58) is NI, then there exists a HIGS controller $\mathcal{H}$ such that the positive feedback interconnection of Σ and $\mathcal{H}$ shown in Fig. 9.10 is asymptotically stable. The setting of the interconnection can be described as follows:

$$e_k = y_k;$$

$$u_k = \widetilde{y}_k.$$

This means the HIGS $\mathcal{H}$ takes the output of the system Σ as its input and feeds back its output to the system Σ as its input.

Theorem 9.8. *Suppose the SISO minimal system (9.58) with transfer function matrix $G(z)$ is NI and satisfies $\det(I - A) \neq 0$. Also, suppose the HIGS $\mathcal{H}$ of the form (9.50) satisfies that $0 < \omega_h \leq k_h < \frac{1}{G(1)}$. Then the closed-loop interconnection of $G(z)$ and $\mathcal{H}$ as shown in Fig. 9.10 is asymptotically stable.*

Proof. According to Theorem 8.1, the minimal system (9.58) is NI if and only if there exists a matrix $P = P^T > 0$ such that

$$A^T PA - P \leq 0, \quad \text{and} \quad C = B^T (I - A)^{-T} P.$$

We construct the following Lyapunov function for the closed-loop interconnection:

$$W(x_k, \widetilde{x}_k) = V(x_k) + \widetilde{V}(\widetilde{x}_k) - Cx_k\widetilde{x}_k = \frac{1}{2}x_k^T P x_k + \frac{1}{2k_h}x_k^2 - Cx_k\widetilde{x}_k.$$

Rewriting this as a quadratic form, we have that

$$W(x_k, \widetilde{x}_k) = \frac{1}{2}\begin{bmatrix} x_k^T & \widetilde{x}_k \end{bmatrix}\begin{bmatrix} P & -C^T \\ -C & \frac{1}{k_h} \end{bmatrix}\begin{bmatrix} x_k \\ \widetilde{x}_k \end{bmatrix}.$$

Using the Schur complement theorem, to ensure that $W(x_k, \widetilde{x}_k)$ is positive definite, we need

$$\frac{1}{k_h} - CP^{-1}C^T > 0. \tag{9.59}$$

Since $C = B^T (I - A)^{-T} P$, then (9.59) can be rewritten as

$$\frac{1}{k_h} - C(I - A)^{-1}B > 0,$$

which is satisfied because $G(1) = C(I - A)^{-1}B$ and

$$k_h G(1) < 1. \tag{9.60}$$

Note that $G(1) \neq 0$ according to the positive definiteness of P and the fact that C is not a zero row vector, which is guaranteed by the minimality of the system. We use Lyapunov's direct method [120] in the following. Taking the difference between $W(x_{k+1}, \widetilde{x}_{k+1})$ and $W(x_k, \widetilde{x}_k)$, we have

$$W(x_{k+1}, \widetilde{x}_{k+1}) - W(x_k, \widetilde{x}_k)$$

$$
\begin{aligned}
&= V(x_{k+1}) + \widetilde{V}(\widetilde{x}_{k+1}) - Cx_{k+1}\widetilde{x}_{k+1} - V(x_k) - \widetilde{V}(\widetilde{x}_k) + Cx_k\widetilde{x}_k \\
&\leq u_k(y_{k+1} - y_k) + e_k(\widetilde{x}_{k+1} - \widetilde{x}_k) - Cx_{k+1}\widetilde{x}_{k+1} + Cx_k\widetilde{x}_k \\
&= \widetilde{x}_{k+1}(e_{k+1} - e_k) + e_k(\widetilde{x}_{k+1} - \widetilde{x}_k) - e_{k+1}\widetilde{x}_{k+1} + e_k\widetilde{x}_k \\
&= 0
\end{aligned} \tag{9.61}
$$

which implies that the system is Lyapunov stable. Furthermore, $W(x_{k+1}, \widetilde{x}_{k+1}) - W(x_k, \widetilde{x}_k) = 0$ only if the inequality in (9.61) is an equality. That is

$$V(x_{k+1}) - V(x_k) = u_k(y_{k+1} - y_k); \tag{9.62}$$

$$\widetilde{V}(\widetilde{x}_{k+1}) - \widetilde{V}(\widetilde{x}_k) = e_k(\widetilde{x}_{k+1} - \widetilde{x}_k). \tag{9.63}$$

We prove in the following that (9.62) and (9.63) cannot hold together at all time indices k unless $(x_k, \widetilde{x}_k) = (0, 0)$. We consider the case that (9.62) and (9.63) hold for some index k and all future indices $k+1$, $k+2$, $\cdots$. When (9.63) holds, we have that

$$\frac{1}{2k_h}\widetilde{x}_{k+1}^2 - \frac{1}{2k_h}\widetilde{x}_k^2 = e_k(\widetilde{x}_{k+1} - \widetilde{x}_k). \tag{9.64}$$

We consider the following two cases, where the HIGS is assumed to work in the integrator mode and the gain mode, respectively.

Case 1. Integrator mode. In this case, $(\widetilde{x}_k, e_k) \in \mathcal{F}$ and

$$\widetilde{x}_{k+1} = \widetilde{x}_k + \omega_h e_k. \tag{9.65}$$

Substituting (9.65) in (9.64) yields

$$(\omega_h - 2k_h)e_k^2 + 2\widetilde{x}_k e_k = 0. \tag{9.66}$$

Case 1a. Suppose $e_k \neq 0$. Then we have $\widetilde{x}_k = (k_h - \frac{\omega_h}{2})e_k$, which can be substituted in the inequality in (9.51) and yields

$$(k_h + \frac{\omega_h}{2})e_k^2 \geq \frac{1}{k_h}(k_h + \frac{\omega_h}{2})^2 e_k^2.$$

This, after simplification, becomes

$$\omega_h^2 + 2k_h\omega_h \leq 0.$$

Considering the fact that $k_h > 0$ and $\omega > 0$, the above condition can never be satisfied. Hence, *Case 1a* can never happen.

Case 1b. Suppose $e_k = 0$. Then (9.66) is always satisfied. In this case, $(\widetilde{x}_k, e_k) \in \mathcal{F}$

implies that $\widetilde{x}_k = 0$. According to (9.65), we have that $\widetilde{x}_{k+1} = 0$ as well. The condition for $(\widetilde{x}_{k+1}, e_{k+1}) \in \mathcal{F}$ can be simplified to be

$$(k_h - \omega_h)e_{k+1}^2 \geq 0.$$

The fact that $k_h - \omega_h \geq 0$ guarantees that the next active mode is the integrator mode. Note that this condition is irrelevant to the HIGS input or state. Indeed, since *Case 1a* can never happen, then the system will fall in *Case 1b* for all future time indices $k+1$, $k+2$, $\cdots$. Following from a similar analysis, we have that

$$0 = e_k = e_{k+1} = e_{k+2} = \cdots, \tag{9.67}$$

and also

$$0 = \widetilde{x}_k = \widetilde{x}_{k+1} = \widetilde{x}_{k+2} = \cdots. \tag{9.68}$$

Since $u_k = \widetilde{y}_k = \widetilde{x}_{k+1}$, then according to (9.68) and (9.58a), we have that

$$x_{k+1} = Ax_k, \quad x_{k+2} = Ax_{k+1} = A^2 x_k, \quad \cdots \tag{9.69}$$

Since $e_k = y_k = Cx_k$, then according to (9.67) and (9.69), we have that

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_k = 0.$$

This implies that $x_k = 0$ due to the observability of $G(z)$. In this case, $(x_k, \widetilde{x}_k) = (0, 0)$. Hence, the closed-loop system is already in its equilibrium.

Case 2. Gain mode. In this case, $(\widetilde{x}_k, e_k) \notin \mathcal{F}$, and we have that

$$\widetilde{x}_{k+1} = k_h e_k. \tag{9.70}$$

Substituting (9.70) in (9.64), we have that

$$(\widetilde{x}_k - k_h e_k)^2 = 0.$$

That is

$$\widetilde{x}_k = k_h e_k = \widetilde{x}_{k+1}. \tag{9.71}$$

The condition $(\widetilde{x}_k, e_k) \notin \mathcal{F}$ implies that

$$(k_h + \omega_h)e_k^2 > 0.$$

This implies that $e_k \neq 0$. We only need consider the case that the HIGS operates in the gain mode for all future indices. This is because that under the constraints (9.63), if it enters the integrator mode, it will never exit the integrator mode, according to the analysis in *Case 1b*. Then it falls into *Case 1*. In the case that the system keeps operating in the gain mode, following from the same derivation as (9.71), we have that

$$\widetilde{x}_{k+1} = k_h e_{k+1} = \widetilde{x}_{k+2}. \tag{9.72}$$

Comparing (9.71), (9.72) and similar equations for future time indices, we have that

$$\widetilde{x}_k = k_h e_k = \widetilde{x}_{k+1} = k_h e_{k+1} = \widetilde{x}_{k+2} = k_h e_{k+2} = \cdots .$$

That is

$$e_k = e_{k+1} = e_{k+2} = \cdots .$$

This implies that

$$y_k = y_{k+1} = y_{k+2} = \cdots . \tag{9.73}$$

In this case, we have that

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k = Ax_k + B\widetilde{y}_k = Ax_k + B\widetilde{x}_{k+1} \\ &= Ax_k + Bk_h e_k = Ax_k + k_h BCx_k \\ &= (A + k_h BC)x_k. \end{aligned}$$

Similarly, we have

$$\begin{aligned} x_{k+2} &= (A + k_h BC)x_{k+1} = (A + k_h BC)^2 x_k, \\ &\vdots \\ x_{k+n-1} &= (A + k_h BC)^{n-1} x_k. \end{aligned}$$

According to (9.73), we have that

$$\begin{bmatrix} y_{k+1} - y_k \\ y_{k+2} - y_{k+1} \\ \vdots \\ y_{k+n} - y_{k+n-1} \end{bmatrix} = 0,$$

which implies

$$\begin{bmatrix} C \\ C(A+k_hBC) \\ \vdots \\ C(A+k_hBC)^{n-1} \end{bmatrix} (x_{k+1}-x_k) = 0. \tag{9.74}$$

We use eigenvector test to prove that observability of (A, C) implies that of $(A + k_hBC, C)$. Suppose $\eta \neq 0$ is a vector in the kernel of C; i.e., $C\eta = 0$. Then it is not an eigenvector of A; i.e., $A\eta \neq \lambda\eta$ for all scalars λ. Then η is not an eigenvector of $A + k_hBC$ as well because $(A + k_hBC)\eta = A\eta + k_hBC\eta = A\eta \neq \lambda\eta$ for all λ, considering $C\eta = 0$. Hence, $(A + k_hBC, C)$ is observable and (9.74) implies that $x_{k+1} = x_k$. That is, x_k is an eigenvector of $A + k_hBC$ with an eigenvalue $\lambda = 1$. This implies that

$$x_k = x_{k+1} = x_{k+2} = \cdots .$$

In this case, we also have that

$$\begin{aligned} x_k = x_{k+1} &= Ax_k + Bu_k = Ax_k + B\widetilde{y}_k = Ax_k + B\widetilde{x}_{k+1} \\ &= Ax_k + Bk_he_k. \end{aligned}$$

This implies that

$$x_k = k_h(I - A)^{-1}Be_k.$$

Also, we have that

$$e_k = Cx_k = k_hC(I - A)^{-1}Be_k. \tag{9.75}$$

Since we have $e_k \neq 0$ in *Case 2*, then (9.75) implies that

$$k_hC(I - A)^{-1}B = 1,$$

which is

$$k_hG(1) = 1.$$

This contradicts (9.60). To conclude, we have shown that if (9.62) and (9.63) hold together for all future time indices, then the HIGS cannot stay in the gain mode according to the analysis in *Case 2*. It will eventually switch to the integrator mode. Then, according to the analysis in *Case 1*, the HIGS will stay in the integrator mode. However, we have shown in *Case 1b* that this is only possible if the system is already at the equilibrium. In other words, if the system is not at the equilibrium, then (9.62) and (9.63) cannot hold together for all future indices, and $W(x_k, \widetilde{x}_k)$ will eventually decrease again until the system reaches its equilibrium. This means that the system is asymptotically stable. □

9.7 Illustrative example: using a discrete-time HIGS in the digital control of a mass-spring system

In this section, we demonstrate the feasibility of the proposed stability results in Theorem 9.8. Consider a mass-spring system shown in Fig. 9.11.

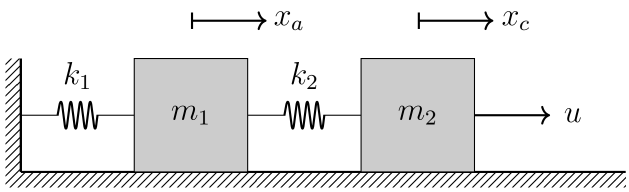


Figure 9.11: A mass-spring system with masses $m_1 = 0.04kg$, $m_2 = 0.02kg$ and spring constants $k_1 = 2N/m$ and $k_2 = 1N/m$. x_a and x_c denote the displacement of the masses m_1 and m_2, respectively. A force input u is applied on the mass m_2.

Sampling the system with the period $h = 0.04s$ using a ZOH device, we obtain the following discrete-time model (see also [112]).

$$\Sigma: \; x_{k+1} = Ax_k + Bu_k, \tag{9.76a}$$

$$y_k = Cx_k, \tag{9.76b}$$

where

$$A = \begin{bmatrix} \frac{1}{3}c_1 + \frac{2}{3}c_2 & \frac{1}{15}s_1 + \frac{1}{15}s_2 & \frac{1}{3}c_1 - \frac{1}{3}c_2 & \frac{1}{15}s_1 - \frac{1}{30}s_2 \\ -\frac{5}{3}s_1 - \frac{20}{3}s_2 & \frac{1}{3}c_1 + \frac{2}{3}c_2 & -\frac{5}{3}s_1 + \frac{10}{3}s_2 & \frac{1}{3}c_1 - \frac{1}{3}c_2 \\ \frac{2}{3}c_1 - \frac{2}{3}c_2 & \frac{2}{15}s_1 - \frac{1}{15}s_2 & \frac{2}{3}c_1 + \frac{1}{3}c_2 & \frac{2}{15}s_1 + \frac{1}{30}s_2 \\ -\frac{10}{3}s_1 + \frac{20}{3}s_2 & \frac{2}{3}c_1 - \frac{2}{3}c_2 & -\frac{10}{3}s_1 - \frac{10}{3}s_2 & \frac{2}{3}c_1 + \frac{1}{3}c_2 \end{bmatrix},$$

$$B = \begin{bmatrix} -\frac{2}{3}c_1 + \frac{1}{6}c_2 + \frac{1}{2} \\ \frac{10}{3}s_1 - \frac{5}{3}s_2 \\ -\frac{4}{3}c_1 - \frac{1}{6}c_2 + \frac{3}{2} \\ \frac{20}{3}s_1 - \frac{5}{3}s_2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix},$$

with

$$\begin{aligned} c_1 &= \cos(5h) = \cos(0.2);\ \ c_2 = \cos(10h) = \cos(0.4); \\ s_1 &= \sin(5h) = \sin(0.2);\ \ s_2 = \sin(10h) = \sin(0.4); \end{aligned}$$

Here, $x_k = \begin{bmatrix} x_{ak} & x_{bk} & x_{ck} & x_{dk} \end{bmatrix}^T \in \mathbb{R}^4$, $u_k, y_k \in \mathbb{R}$ are the state, input and output of the system, respectively. x_{ak} and x_{bk} are the displacement and velocity of the mass m_1 while x_{ck} and x_{dk} are the displacement and velocity of the mass m_2, respectively, at time step k. This system is NI according to Definition 8.1 with the storage function

$$V(x_k) = x_k^T P x_k,$$

where

$$P = \begin{bmatrix} 3 & 0 & -1 & 0 \\ 0 & 0.04 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.02 \end{bmatrix}.$$

We apply a HIGS controller of the form (9.50) in positive feedback with the plant (9.76). For the plant (9.76), we have that $G(1) = C(I-A)^{-1}B = \frac{3}{2}$. Hence, we choose the HIGS parameters to be $\omega_h = 0.1$, $k_h = 0.6$, which satisfies the condition $0 < \omega_h \leq k_h < \frac{1}{G(1)}$ as required in Theorem 9.8. A simulation is implemented with the initial values $x_0 = \begin{bmatrix} 3 & -2 & 5 & -1 \end{bmatrix}^T$. The state trajectories of the plant and the HIGS controller are shown in Fig. 9.12.

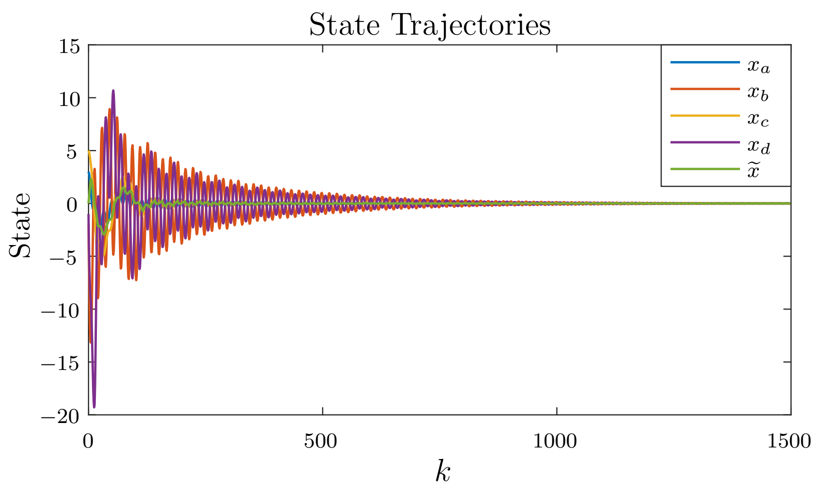


Figure 9.12: State trajectories of the plant (9.76) and the HIGS (9.50), which are interconnected in positive feedback. Starting from nonzero initial conditions, all the state variables converge to zero. The closed-loop system is asymptotically stable, which is consistent with our expectation according to Theorem 9.8.

9.8 Conclusion

In this chapter, it is shown that a single HIGS, a multi-HIGS, and the cascade of two HIGS are all nonlinear NI systems. Using nonlinear NI systems theory, these HIGS can be applied as controllers to stabilize linear NI systems. For a linear NI system with a minimal realization, there always exists a HIGS controller such that their positive feedback interconnection is asymptotically stable. We illustrate by a hardware experiment that HIGS can improve the control performance of a MEMS nanopositioner.

For the purpose of digital control, we also investigate the discrete-time HIGS. We show that a discrete-time HIGS has SANI property and can be used in the control of a discrete-time NI system. This result is then illustrated using an example where a discrete-time mass-spring system obtained using ZOH sampling is stabilized by a discrete-time HIGS.

Chapter 10

Conclusions and Future Work

10.1 Conclusions

We have provided several new results in NI systems theory in this thesis. A definition of NI systems property is given for MIMO nonlinear systems with free body motion. According to this definition, a system is said to be nonlinear NI if there exists a positive semidefinite storage function such that its time derivative is less than or equal to the inner product of the system input and the time derivative of the system output. We apply nonlinear OSNI controllers to achieve stability for nonlinear NI systems. Under some assumptions, the interconnection of a nonlinear NI system and a nonlinear OSNI system is proved to be asymptotically stable. Also, we investigated the conditions for linear and nonlinear systems to be state feedback equivalent to NI systems. A linear system with a minimal realization is found to be state feedback equivalent to an NI system if and only if there is a nonsingular output transform such that the resulting system is of relative degree less than or equal to two and is weakly minimum phase. The required state feedback control law is provided. We also provide conditions for state feedback equivalence to an OSNI system or an SSNI system. We provide sufficient conditions for an input-affine nonlinear system to be made nonlinear NI using state feedback control. We also provide the necessary and sufficient conditions for a particular normal form to be made nonlinear NI with a positive definite storage function. In both the cases of linear and nonlinear systems, we use the state feedback equivalence results to stabilize systems with NI uncertainties. We also provide control frameworks for the control of three types of networked nonlinear NI systems, including networked identical nonlinear NI systems, networked heterogeneous nonlinear NI systems, and networked heterogeneous nonlinear NI systems with free body motion. The information exchange between the plants is modelled using graphs. The nodes in the graph correspond to the plants, while the edges correspond to the controllers. OSNI controllers are applied, whose outputs are distributed to the plants as inputs via the Laplacian matrix or the incidence matrix, respectively, in different cases. We extend NI systems theory to switched systems. A switched system is said to be NI if each subsystem satisfies the NI property, and in addition, the storage functions of the subsystems in the switching sequence are nonincreasing. Under some assumptions, the interconnection of a switched NI system and a switched OSNI system is asymptotically stable. We provide a new definition for discrete-time NI systems property so that a ZOH sampled continuous-time NI system automatically satisfies the discrete-time NI systems definition. We also provide a definition of discrete-time OSNI systems, which is similar yet different to the continuous-time OSNI systems definition. We prove that under certain assumptions, the interconnection of a discrete-time NI system and a discrete-time step advanced OSNI system is asymptotically stable. Also, a similar stability result is proved for the interconnection of a discrete-time step advanced NI system and a discrete-time OSNI system. We investigate HIGS elements, as well as their variants, the multi-HIGS and the cascade of two HIGS. We show that all of these three types of HIGS elements are nonlinear NI systems and they can be applied to asymptotically stabilize NI plants. Also, we investigate discrete-time HIGS elements, which is shown to be an SANI system. It can be used in the digital control of a discrete-time NI system. We provide illustrative examples in all the chapters containing the results mentioned above. In particular, a hardware experiment was implemented where a multi-HIGS is applied on a MEMS nanopositioner in order to improve the control system performance.

10.2 Future work

The results in this thesis can be potentially extended in the following directions.

1. The networks considered in Chapter 6 are all static networks. However, in practice, the information flow in a network may inevitably experience link failure and creation. Also, time delays are not negligible in real-world networked systems. Therefore, it is important to investigate consensus problems for networked nonlinear NI systems with switching network topology and time delays. For the purpose of digital control, we can also consider the consensus problems for networked discrete-time NI systems based on the results in Chapter 8.
2. Although the discrete-time NI systems theory proposed in Chapter 8 is different from that given in [39], where a bilinear transform is used for the con-

version between continuous time and discrete time, it is also important to investigate the nonlinear generalization of the discrete-time NI systems theory given in [39]. The difficulty is that bilinear transform is a frequency domain approach and hence is not applicable to nonlinear systems.

3. We only apply the HIGS elements in Chapter 9 to control linear NI plants. Due to the nonlinear nature of many control systems, it is important to generalize the results in Chapter 9 to allow for nonlinear NI plants. Another possible area of future work is to investigate closed-loop stability when there are external disturbances acting on the system. In this case, instead of asymptotic stability, we would need to determine the ISS of the system.

Bibliography

[1] M. Maroufi and S. O. R. Moheimani, “A 2DOF SOI-MEMS nanopositioner with tilted flexure bulk piezoresistive displacement sensors,” *IEEE Sensors Journal*, vol. 16, no. 7, pp. 1908–1917, 2016.

[2] A. Lanzon and I. R. Petersen, “Stability robustness of a feedback interconnection of systems with negative imaginary frequency response,” *IEEE Transactions on Automatic Control*, vol. 53, no. 4, pp. 1042–1046, 2008.

[3] I. R. Petersen and A. Lanzon, “Feedback control of negative-imaginary systems,” *IEEE Control Systems Magazine*, vol. 30, no. 5, pp. 54–72, 2010.

[4] A. Preumont, *Vibration control of active structures: an introduction.* Springer, 2018, vol. 246.

[5] D. Halim and S. O. R. Moheimani, “Spatial resonant control of flexible structures-application to a piezoelectric laminate beam,” *IEEE Transactions on Control Systems Technology*, vol. 9, no. 1, pp. 37–53, 2001.

[6] H. R. Pota, S. O. R. Moheimani, and M. Smith, “Resonant controllers for smart structures,” *Smart Materials and Structures*, vol. 11, no. 1, pp. 1–8, 2002.

[7] J. M. Caicedo and S. J. Dyke, “Experimental validation of structural health monitoring for flexible bridge structures,” *Structural Control and Health Monitoring: The Official Journal of the International Association for Structural Control and Monitoring and of the European Association for the Control of Structures*, vol. 12, no. 3-4, pp. 425–443, 2005.

[8] K. Seto, “A structural control method of the vibration of flexible buildings in response to large earthquakes and strong winds,” in *Proceedings of 35th IEEE Conference on Decision and Control (CDC)*, vol. 1. IEEE, 1996, pp. 658–663.

[9] J. Fanson and T. Caughey, “Positive position feedback control for large space structures,” *AIAA journal*, vol. 28, no. 4, pp. 717–724, 1990.

[10] D. G. Wilson, R. D. Robinett, G. G. Parker, and G. P. Starr, "Augmented sliding mode control for flexible link manipulators," *Journal of Intelligent and Robotic Systems*, vol. 34, pp. 415–430, 2002.

[11] B. Bhikkaji and S. O. R. Moheimani, "Fast scanning using piezoelectric tube nanopositioners: A negative imaginary approach," in *2009 IEEE/ASME International Conference on Advanced Intelligent Mechatronics.* IEEE, 2009, pp. 274–279.

[12] S. K. Das, H. R. Pota, and I. R. Petersen, "Resonant controller design for a piezoelectric tube scanner: A mixed negative-imaginary and small-gain approach," *IEEE Transactions on Control Systems Technology*, vol. 22, no. 5, pp. 1899–1906, 2014.

[13] B. Brogliato, R. Lozano, B. Maschke, and O. Egeland, *Dissipative systems analysis and control: theory and applications.* Springer, London, 2007, vol. 2.

[14] A. Isidori, *Nonlinear control systems.* Springer Science & Business Media, 2013.

[15] H. K. Khalil, *Nonlinear systems.* Prentice Hall, Upper Saddle River, NJ, 2002, vol. 3.

[16] J.-N. Juang and M. Q. Phan, *Identification and control of mechanical systems.* Cambridge University Press, 2001.

[17] J. Xiong, I. R. Petersen, and A. Lanzon, "A negative imaginary lemma and the stability of interconnections of linear negative imaginary systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 10, pp. 2342–2347, 2010.

[18] M. A. Mabrok, A. G. Kallapur, I. R. Petersen, and A. Lanzon, "Generalizing negative imaginary systems theory to include free body dynamics: Control of highly resonant structures with free body motion," *IEEE Transactions on Automatic Control*, vol. 59, no. 10, pp. 2692–2707, 2014.

[19] Z. Song, A. Lanzon, S. Patra, and I. R. Petersen, "A negative-imaginary lemma without minimality assumptions and robust state-feedback synthesis for uncertain negative-imaginary systems," *Systems & Control Letters*, vol. 61, no. 12, pp. 1269–1276, 2012.

[20] I. R. Petersen, "Physical interpretations of negative imaginary systems theory," in *2015 10th Asian Control Conference (ASCC).* IEEE, 2015, pp. 1–6.

[21] M. A. Rahman, A. Al Mamun, K. Yao, and S. K. Das, "Design and implementation of feedback resonance compensator in hard disk drive servo system: A mixed passivity, negative-imaginary and small-gain approach in discrete

time," *Journal of Control, Automation and Electrical Systems*, vol. 26, no. 4, pp. 390–402, 2015.

[22] V. P. Tran, M. A. Mabrok, M. A. Garratt, and I. R. Petersen, "Hybrid adaptive negative imaginary-neural-fuzzy control with model identification for a quadrotor," *IFAC Journal of Systems and Control*, vol. 16, p. 100156, 2021.

[23] N. Nikooienejad and S. O. R. Moheimani, "Frequency domain-based integral resonant control design for a MEMS nanopositioner," in *2021 IEEE Conference on Control Technology and Applications (CCTA).* IEEE, 2021, pp. 874–879.

[24] ——, "Convex synthesis of SNI controllers based on frequency-domain data: MEMS nanopositioner example," *IEEE Transactions on Control Systems Technology*, vol. 30, no. 2, pp. 767–778, 2021.

[25] S. K. Das, H. R. Pota, and I. R. Petersen, "Multivariable negative-imaginary controller design for damping and cross coupling reduction of nanopositioners: a reference model matching approach," *IEEE/ASME Transactions on Mechatronics*, vol. 20, no. 6, pp. 3123–3134, 2015.

[26] M. Y.-Y. U. Haque, M. R. Islam, J. Hasan, and M. R. I. Sheikh, "Negative imaginary theory-based proportional resonant controller for voltage control of three-phase islanded microgrid," *Journal of Control, Automation and Electrical Systems*, vol. 32, pp. 214–226, 2021.

[27] P. Bhowmick and S. Patra, "On LTI output strictly negative-imaginary systems," *Systems & Control Letters*, vol. 100, pp. 32–42, 2017.

[28] P. Bhowmick and A. Lanzon, "Output strictly negative imaginary systems and its connections to dissipativity theory," in *2019 IEEE 58th Conference on Decision and Control (CDC).* IEEE, 2019, pp. 6754–6759.

[29] A. Lanzon, S. Patra, I. R. Petersen, and Z. Song, "A strongly strict negative-imaginary lemma for non-minimal linear systems," *Communications in Information and Systems*, vol. 11, no. 2, pp. 139–142, 2011.

[30] A. G. Ghallab, M. A. Mabrok, and I. R. Petersen, "Extending negative imaginary systems theory to nonlinear systems," in *2018 IEEE Conference on Decision and Control (CDC).* IEEE, 2018, pp. 2348–2353.

[31] D. Angeli, "Systems with counterclockwise input-output dynamics," *IEEE Transactions on Automatic Control*, vol. 51, no. 7, pp. 1130–1143, 2006.

[32] J. Wang, A. Lanzon, and I. R. Petersen, "Robust output feedback consensus for networked negative-imaginary systems," *IEEE Transactions on Automatic Control*, vol. 60, no. 9, pp. 2547–2552, 2015.

[33] ——, "Robust cooperative control of multiple heterogeneous negative-imaginary systems," *Automatica*, vol. 61, pp. 64–72, 2015.

[34] O. Skeik and A. Lanzon, "Robust output consensus of homogeneous multi-agent systems with negative imaginary dynamics," *Automatica*, vol. 113, p. 108799, 2020.

[35] O. Skeik, J. Hu, F. Arvin, and A. Lanzon, "Cooperative control of integrator negative imaginary systems with application to rendezvous multiple mobile robots," in *2019 12th International Workshop on Robot Motion and Control (RoMoCo)*. IEEE, 2019, pp. 15–20.

[36] J. Hu, B. Lennox, and F. Arvin, "Robust formation control for networked robotic systems using negative imaginary dynamics," *Automatica*, vol. 140, p. 110235, 2022.

[37] C. Li, J. Wang, J. Shan, A. Lanzon, and I. R. Petersen, "Robust cooperative control of networked train platoons: A negative-imaginary systems' perspective," *IEEE Transactions on Control of Network Systems*, vol. 8, no. 4, pp. 1743–1753, 2021.

[38] V. P. Tran, M. Garratt, and I. R. Petersen, "Switching time-invariant formation control of a collaborative multi-agent system using negative imaginary systems theory," *Control Engineering Practice*, vol. 95, p. 104245, 2020.

[39] A. Ferrante, A. Lanzon, and L. Ntogramatzidis, "Discrete-time negative imaginary systems," *Automatica*, vol. 79, pp. 1–10, 2017.

[40] M. Liu and J. Xiong, "Properties and stability analysis of discrete-time negative imaginary systems," *Automatica*, vol. 83, pp. 58–64, 2017.

[41] ——, "Bilinear transformation for discrete-time positive real and negative imaginary systems," *IEEE Transactions on Automatic Control*, vol. 63, no. 12, pp. 4264–4269, 2018.

[42] H.-J. Chen and A. Lanzon, "Closed-loop stability analysis of discrete-time negative imaginary systems," *Systems & Control Letters*, vol. 114, pp. 52–58, 2018.

[43] P. Bhowmick and A. Lanzon, "On discrete-time output negative imaginary systems," *IEEE Control Systems Letters*, vol. 6, pp. 1124–1129, 2021.

[44] S. Kurawa, P. Bhowmick, and A. Lanzon, "Negative imaginary theory for a class of linear time-varying systems," *IEEE Control Systems Letters*, vol. 5, no. 3, pp. 1001–1006, 2020.

[45] M. Liu and J. Xiong, "On non-proper negative imaginary systems," *Systems & Control Letters*, vol. 88, pp. 47–53, 2016.

[46] J. Xiong, A. Lanzon, and I. R. Petersen, "Negative imaginary lemmas for descriptor systems," *IEEE Transactions on Automatic Control*, vol. 61, no. 2, pp. 491–496, 2015.

[47] P. Bhowmick and A. Lanzon, "Applying negative imaginary systems theory to non-square systems with polytopic uncertainty," *Automatica*, vol. 128, p. 109570, 2021.

[48] P. Bhowmick and S. Patra, "On input-output negative-imaginary systems and an output strict negative-imaginary lemma," in *2016 Indian Control Conference (ICC)*. IEEE, 2016, pp. 176–181.

[49] A. Lanzon and P. Bhowmick, "Characterization of input–output negative imaginary systems in a dissipative framework," *IEEE Transactions on Automatic Control*, vol. 68, no. 2, pp. 959–974, 2022.

[50] K. Zhou and J. C. Doyle, *Essentials of robust control*. Prentice Hall, Upper Saddle River, NJ, 1998, vol. 104.

[51] R. A. Horn and C. R. Johnson, *Matrix analysis*. Cambridge university press, 2012.

[52] J. P. Hespanha, *Linear systems theory*. Princeton University Press, 2018.

[53] F. Zhang, *The Schur complement and its applications*. Springer Science & Business Media, 2006, vol. 4.

[54] A. Lanzon and H.-J. Chen, "Feedback stability of negative imaginary systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 11, pp. 5620–5633, 2017.

[55] D. S. Bernstein, *Matrix mathematics: theory, facts, and formulas*. Princeton University Press, 2009.

[56] W. M. Haddad and V. Chellaboina, *Nonlinear dynamical systems and control: a Lyapunov-based approach*. Princeton University Press, 2008.

[57] W. Ren and R. W. Beard, *Distributed consensus in multi-vehicle cooperative control*. Springer, 2008, vol. 27, no. 2.

[58] T. Gustafsson, "On the design and implementation of a rotary crane controller," *European Journal of Control*, vol. 2, no. 2, pp. 166–175, 1996.

[59] E. Pereira, S. S. Aphale, V. Feliu, and S. O. R. Moheimani, "Integral resonant control for vibration damping and precise tip-positioning of a single-link flexi-

ble manipulator," *IEEE/ASME Transactions on Mechatronics*, vol. 16, no. 2, pp. 232–240, 2010.

[60] P. C. Hughes, *Spacecraft attitude dynamics.* Courier Corporation, 2012.

[61] K. Shi, I. G. Vladimirov, and I. R. Petersen, "Robust output feedback consensus for networked identical nonlinear negative-imaginary systems," *IFAC-PapersOnLine*, vol. 54, no. 9, pp. 239–244, 2021.

[62] K. Shi, I. R. Petersen, and I. G. Vladimirov, "Output feedback consensus for networked heterogeneous nonlinear negative-imaginary systems with free-body motion," *IEEE Transactions on Automatic Control*, vol. 68, no. 9, pp. 5536–5543, 2023.

[63] P. Kokotovic and H. Sussmann, "A positive real condition for global stabilization of nonlinear systems," *Systems & Control Letters*, vol. 13, no. 2, pp. 125–133, 1989.

[64] A. Saberi, P. Kokotovic, and H. Sussmann, "Global stabilization of partially linear composite systems," *SIAM Journal on Control and Optimization*, vol. 28, no. 6, pp. 1491–1503, 1990.

[65] C. Byrnes, A. Isidori, and J. Willems, "Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 36, no. 11, pp. 1228–1240, 1991.

[66] C. I. Byrnes and A. Isidori, "Asymptotic stabilization of minimum phase nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 36, no. 10, pp. 1122–1137, 1991.

[67] G. Santosuosso, "Passivity of nonlinear systems with input-output feedthrough," *Automatica*, vol. 33, no. 4, pp. 693–697, 1997.

[68] W. Lin, "Feedback stabilization of general nonlinear control systems: a passive system approach," *Systems & Control Letters*, vol. 25, no. 1, pp. 41–52, 1995.

[69] Z.-P. Jiang, D. J. Hill, and A. L. Fradkov, "A passification approach to adaptive nonlinear stabilization," *Systems & Control Letters*, vol. 28, no. 2, pp. 73–84, 1996.

[70] M. Mabrok, A. G. Kallapur, I. R. Petersen, and A. Lanzon, "A generalized negative imaginary lemma and Riccati-based static state-feedback negative imaginary synthesis," *Systems & Control Letters*, vol. 77, pp. 63–68, 2015.

[71] K. Shi, I. R. Petersen, and I. G. Vladimirov, "Negative imaginary state feedback equivalence for systems of relative degree one and relative degree two,"

in *2021 60th IEEE Conference on Decision and Control (CDC)*. IEEE, 2021, pp. 3948–3953.

[72] ——, "Necessary and sufficient conditions for state feedback equivalence to negative imaginary systems," *IEEE Transactions on Automatic Control*, vol. 69, no. 7, pp. 4657–4672, 2024.

[73] V. Fomichev, A. Kraev, and A. Rogovskii, "Generalization of the notion of relative degree and its properties," *Differential Equations*, vol. 52, no. 8, pp. 1061–1071, 2016.

[74] A. Kraev, A. Rogovskii, and V. Fomichev, "On a generalization of relative degree," *Differential Equations*, vol. 50, no. 8, pp. 1122–1127, 2014.

[75] D. Hrovat, "Survey of advanced suspension developments and related optimal control applications," *Automatica*, vol. 33, no. 10, pp. 1781–1817, 1997.

[76] A. Isidori, E. Sontag, and M. Thoma, *Nonlinear control systems*. Springer, 1995, vol. 3.

[77] K. Shi, I. R. Petersen, and I. G. Vladimirov, "Making nonlinear systems negative imaginary via state feedback," *Automatica*, vol. 155, p. 111127, 2023.

[78] ——, "Negative imaginary state feedback equivalence for a class of nonlinear systems," *IFAC-PapersOnLine*, vol. 56, no. 1, pp. 288–293, 2023.

[79] E. D. Sontag *et al.*, "Smooth stabilization implies coprime factorization," *IEEE Transactions on Automatic Control*, vol. 34, no. 4, pp. 435–443, 1989.

[80] E. D. Sontag, "Input to state stability: Basic concepts and results," in *Nonlinear and Optimal Control Theory*. Springer, 2008, pp. 163–220.

[81] D. Liberzon, A. S. Morse, and E. D. Sontag, "Output-input stability and minimum-phase nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 3, pp. 422–436, 2002.

[82] L. Praly and Z.-P. Jiang, "Stabilization by output feedback for systems with ISS inverse dynamics," *Systems & Control Letters*, vol. 21, no. 1, pp. 19–33, 1993.

[83] D. Liberzon, *Switching in systems and control*. Springer Science & Business Media, 2003.

[84] E. D. Sontag and Y. Wang, "New characterizations of input-to-state stability," *IEEE Transactions on Automatic Control*, vol. 41, no. 9, pp. 1283–1294, 1996.

[85] R. M. Murray, "Recent research in cooperative control of multivehicle systems," *Journal of Dynamic Systems, Measurement, and Control*, vol. 129, no. 5, pp. 571–583, 2007.

[86] W. Ren, R. W. Beard, and E. M. Atkins, "A survey of consensus problems in multi-agent coordination," in *Proceedings of the 2005, American Control Conference, 2005.* IEEE, 2005, pp. 1859–1864.

[87] W. Ren, "Consensus strategies for cooperative control of vehicle formations," *IET Control Theory & Applications*, vol. 1, no. 2, pp. 505–512, 2007.

[88] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[89] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[90] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Stable flocking of mobile agents, part I: Fixed topology," in *42nd IEEE International Conference on Decision and Control (CDC)*, vol. 2. IEEE, 2003, pp. 2010–2015.

[91] R. O. Saber and R. M. Murray, "Flocking with obstacle avoidance: Cooperation with limited communication in mobile networks," in *42nd IEEE International Conference on Decision and Control (CDC)*, vol. 2. IEEE, 2003, pp. 2022–2028.

[92] W. Yu, W. Ren, W. X. Zheng, G. Chen, and J. Lü, "Distributed control gains design for consensus in multi-agent systems with second-order nonlinear dynamics," *Automatica*, vol. 49, no. 7, pp. 2107–2115, 2013.

[93] I. R. Petersen, "Negative imaginary systems theory and applications," *Annual Reviews in Control*, vol. 42, pp. 309–318, 2016.

[94] V. P. Tran, M. Garratt, and I. R. Petersen, "Formation control of multi-uavs using negative-imaginary systems theory," in *2017 11th Asian Control Conference (ASCC).* IEEE, 2017, pp. 2031–2036.

[95] Y. Qi, J. Wang, Q. Jia, and J. Shan, "Cooperative assembling using multiple robotic manipulators," in *2016 35th Chinese Control Conference (CCC).* IEEE, 2016, pp. 7973–7978.

[96] K. Shi, I. G. Vladimirov, and I. R. Petersen, "Robust output feedback consensus for networked heterogeneous nonlinear negative-imaginary systems," in *2020 Australian and New Zealand Control Conference (ANZCC).* IEEE, 2020, pp. 214–219.

[97] R. W. Brockett, "Hybrid models for motion control systems," in *Essays on Control.* Springer, 1993, pp. 29–53.

[98] R. A. DeCarlo, M. S. Branicky, S. Pettersson, and B. Lennartson, "Perspectives and results on the stability and stabilizability of hybrid systems," *Proceedings of the IEEE*, vol. 88, no. 7, pp. 1069–1082, 2000.

[99] K. S. Narendra, O. A. Driollet, M. Feiler, and K. George, "Adaptive control using multiple models, switching and tuning," *International journal of adaptive control and signal processing*, vol. 17, no. 2, pp. 87–102, 2003.

[100] Z. Sun, *Switched linear systems: control and design.* Springer Science & Business Media, 2006.

[101] I. Kolmanovsky and J. Sun, "A multi-mode switching-based command tracking in network controlled systems with pointwise-in-time constraints and disturbance inputs," in *2006 6th World Congress on Intelligent Control and Automation*, vol. 1. IEEE, 2006, pp. 199–204.

[102] E. Skafidas, R. J. Evans, A. V. Savkin, and I. R. Petersen, "Stability results for switched controller systems," *Automatica*, vol. 35, no. 4, pp. 553–564, 1999.

[103] D. A. Deenen, M. F. Heertjes, W. Heemels, and H. Nijmeijer, "Hybrid integrator design for enhanced tracking in motion control," in *2017 American Control Conference (ACC).* IEEE, 2017, pp. 2863–2868.

[104] M. Heertjes, N. Irigoyen Perdiguero, and D. Deenen, "Robust control and data-driven tuning of a hybrid integrator-gain system with applications to wafer scanners," *International Journal of Adaptive Control and Signal Processing*, vol. 33, no. 2, pp. 371–387, 2019.

[105] R. Goebel, R. G. Sanfelice, and A. R. Teel, "Hybrid dynamical systems," *IEEE Control Systems Magazine*, vol. 29, no. 2, pp. 28–93, 2009.

[106] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Control Systems Magazine*, vol. 19, no. 5, pp. 59–70, 1999.

[107] J. Zhao and D. J. Hill, "Dissipativity theory for switched systems," *IEEE Transactions on Automatic Control*, vol. 53, no. 4, pp. 941–953, 2008.

[108] ——, "Passivity and stability of switched systems: A multiple storage function method," *Systems & Control Letters*, vol. 57, no. 2, pp. 158–164, 2008.

[109] W. Chen and M. Saif, "Passivity and passivity based controller design of a class of switched control systems," *IFAC Proceedings Volumes*, vol. 38, no. 1, pp. 676–681, 2005.

[110] K. Shi, I. R. Petersen, and I. G. Vladimirov, "Nonlinear negative imaginary systems with switching," *IFAC-PapersOnLine*, vol. 56, no. 2, pp. 3936–3941, 2023.

[111] H. Lin and P. J. Antsaklis, "Stability and stabilizability of switched linear systems: a survey of recent results," *IEEE Transactions on Automatic control*, vol. 54, no. 2, pp. 308–322, 2009.

[112] K. J. Åström and B. Wittenmark, *Computer-controlled systems: theory and design.* Courier Corporation, 2013.

[113] L. Hitz and B. Anderson, "Discrete positive-real functions and their application to system stability," in *Proceedings of the Institution of Electrical Engineers*, vol. 116, no. 1. IET, 1969, pp. 153–155.

[114] M. De la Sen, "Preserving positive realness through discretization," *Positivity*, vol. 6, pp. 31–45, 2002.

[115] K. Premaratne and E. Jury, "Discrete-time positive-real lemma revisited: the discrete-time counterpart of the Kalman-Yakubovitch lemma," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 41, no. 11, pp. 747–750, 1994.

[116] Y. Kawano, A. Moreschini, and M. Cucuzzella, "Krasovskii passivity for sampled-data stabilization and output consensus," *arXiv preprint arXiv:2306.09706*, 2023.

[117] Y. Oishi, "Passivity degradation under the discretization with the zero-order hold and the ideal sampler," in *49th IEEE Conference on Decision and Control (CDC).* IEEE, 2010, pp. 7613–7617.

[118] C. I. Byrnes and W. Lin, "Losslessness, feedback equivalence, and the global stabilization of discrete-time nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 39, no. 1, pp. 83–98, 1994.

[119] K. Shi, I. R. Petersen, and I. G. Vladimirov, "Discrete-time negative imaginary systems from ZOH sampling," *To appear in the 26th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2024), arXiv preprint:2312.05419*, 2024.

[120] R. E. Kalman and J. E. Bertram, "Control system analysis and design via the "second method" of Lyapunov: II-discrete-time systems," *Journal of Basic Engineering*, vol. 82, pp. 394–400, 1960.

[121] W. Mei and F. Bullo, "Lasalle invariance principle for discrete-time dynamical systems: A concise and self-contained tutorial," *arXiv preprint arXiv:1710.03710*, 2017.

[122] R. H. Middleton, "Trade-offs in linear control system design," *Automatica*, vol. 27, no. 2, pp. 281–292, 1991.

[123] J. Freudenberg, R. Middleton, and A. Stefanpoulou, "A survey of inherent design limitations," in *Proceedings of the 2000 American Control Conference (ACC)*, vol. 5. IEEE, 2000, pp. 2987–3001.

[124] H. W. Bode *et al.*, *Network analysis and feedback amplifier design.* Van Nostrand, New York, 1945.

[125] I. M. Horowitz, *Synthesis of feedback systems.* Academic Press, New York, 1963.

[126] Q. Chen, C. Hollot, and Y. Chait, "Stability and asymptotic performance analysis of a class of reset control systems," in *Proceedings of the 39th IEEE Conference on Decision and Control (CDC)*, vol. 1. IEEE, 2000, pp. 251–256.

[127] Y. Li, K. H. Ang, and G. C. Chong, "PID control system analysis and design," *IEEE Control Systems Magazine*, vol. 26, no. 1, pp. 32–41, 2006.

[128] H. Hu, Y. Zheng, Y. Chait, and C. Hollot, "On the zero-input stability of control systems with Clegg integrators," in *Proceedings of the 1997 American Control Conference (ACC)*, vol. 1. IEEE, 1997, pp. 408–410.

[129] J. C. Clegg, "A nonlinear integrator for servomechanisms," *Transactions of the American Institute of Electrical Engineers, Part II: Applications and Industry*, vol. 77, no. 1, pp. 41–42, 1958.

[130] I. Horowitz and P. Rosenbaum, "Non-linear design for cost of feedback reduction in systems with large parameter uncertainty," *International Journal of Control*, vol. 21, no. 6, pp. 977–1001, 1975.

[131] Y. Chait and C. Hollot, "On Horowitz's contributions to reset control," *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal*, vol. 12, no. 4, pp. 335–355, 2002.

[132] L. Zaccarian, D. Nesic, and A. Teel, "First order reset elements and the Clegg integrator revisited," in *Proceedings of the 2005, American Control Conference, 2005.*, 2005, pp. 563–568 vol. 1.

[133] O. Beker, C. V. Hollot, and Y. Chait, "Plant with integrator: an example of reset control overcoming limitations of linear feedback," *IEEE Transactions on Automatic Control*, vol. 46, no. 11, pp. 1797–1799, 2001.

[134] G. Bartolini, "Chattering phenomena in discontinuous control systems," *International journal of systems science*, vol. 20, no. 12, pp. 2471–2481, 1989.

[135] M. Heertjes, S. Van Den Eijnden, B. Sharif, M. Heemels, and H. Nijmeijer, "Hybrid integrator-gain system for active vibration isolation with improved transient response," *IFAC-PapersOnLine*, vol. 52, no. 15, pp. 454–459, 2019.

[136] S. J. A. M. van den Eijnden, M. F. Heertjes, W. P. M. H. Heemels, and H. Nijmeijer, "Hybrid integrator-gain systems: A remedy for overshoot limitations in linear control?" *IEEE Control Systems Letters*, vol. 4, no. 4, pp. 1042–1047, 2020.

[137] S. Van den Eijnden, Y. Knops, and M. Heertjes, "A hybrid integrator-gain based low-pass filter for nonlinear motion control," in *2018 IEEE Conference on Control Technology and Applications (CCTA)*, 2018, pp. 1108–1113.

[138] K. Gruntjens, M. Heertjes, S. Van Loon, N. Van De Wouw, and W. Heemels, "Hybrid integral reset control with application to a lens motion system," in *2019 American Control Conference (ACC)*, 2019, pp. 2408–2413.

[139] A. S. P., "HIGS-based skyhook damping design of a multivariable vibration isolation system," Master's thesis, Eindhoven University of Technology, 2020.

[140] R. de Baaij, "Non-linear dynamic control for positive-real systems," Master's thesis, February 2021. [Online]. Available: http://essay.utwente.nl/86049/

[141] K. S. A. Hebers, "Hybrid integrator-gain based notch filter design," Master's thesis, Eindhoven University of Technology, 2020.

[142] S. Van Loon, K. Gruntjens, M. F. Heertjes, N. van de Wouw, and W. Heemels, "Frequency-domain tools for stability analysis of reset control systems," *Automatica*, vol. 82, pp. 101–108, 2017.

[143] S. J. A. M. Van Den Eijnden, M. F. Heertjes, W. P. M. H. Maurice Heemels, and H. Nijmeijer, "Frequency-domain tools for stability analysis of hybrid integrator-gain systems," in *2021 European Control Conference (ECC)*, 2021, pp. 1895–1900.

[144] D. A. Deenen, B. Sharif, S. van den Eijnden, H. Nijmeijer, M. Heemels, and M. Heertjes, "Projection-based integrators for improved motion control: Formalization, well-posedness and stability of hybrid integrator-gain systems," *Automatica*, vol. 133, p. 109830, 2021.

[145] B. Sharif, D. W. Alferink, M. F. Heertjes, H. Nijmeijer, and W. Heemels, "A discrete-time approach to analysis of sampled-data hybrid integrator-gain systems," in *2022 IEEE 61st Conference on Decision and Control (CDC)*. IEEE, 2022, pp. 7612–7617.

[146] K. Shi, N. Nikooienejad, I. R. Petersen, and S. O. R. Moheimani, "A negative imaginary approach to hybrid integrator-gain system control," in *2022 IEEE 61st Conference on Decision and Control (CDC)*. IEEE, 2022, pp. 1968–1973.

[147] ——, "Negative imaginary control using hybrid integrator-gain systems: Application to MEMS nanopositioner," *IEEE Transactions on Control Systems Technology*, vol. 32, no. 3, pp. 718–730, 2024.

[148] K. Shi and I. R. Petersen, "Digital control of negative imaginary systems: A discrete-time hybrid integrator-gain system approach," in *2024 European Control Conference (ECC)*. IEEE, 2024, pp. 2611–2616.

[149] N. Nikooienejad, M. Maroufi, and S. O. R. Moheimani, "Iterative learning control for video-rate atomic force microscopy," *IEEE/ASME Transactions on Mechatronics*, vol. 26, no. 4, pp. 2127–2138, 2021.

[150] N. Nikooienejad and S. O. R. Moheimani, "Convex synthesis of SNI controllers based on frequency-domain data: MEMS nanopositioner example," *IEEE Transactions on Control Systems Technology*, vol. 30, no. 2, pp. 767–778, 2022.

[151] M. A. Mabrok, A. G. Kallapur, I. R. Petersen, and A. Lanzon, "Spectral conditions for negative imaginary systems with applications to nanopositioning," *IEEE/ASME Transactions on Mechatronics*, vol. 19, no. 3, pp. 895–903, 2013.

[152] J. Löfberg, "Yalmip : A toolbox for modeling and optimization in matlab," in *In Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004.

[153] M. ApS, *The MOSEK optimization toolbox for MATLAB manual. Version 9.0.*, 2019. [Online]. Available: http://docs.mosek.com/9.0/toolbox/index.html

[154] S. Patra and A. Lanzon, "Stability analysis of interconnected systems with "mixed" negative-imaginary and small-gain properties," *IEEE Transactions on Automatic Control*, vol. 56, no. 6, pp. 1395–1400, 2011.

[155] D. Nešić, A. R. Teel, and P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations," *Systems & Control Letters*, vol. 38, no. 4-5, pp. 259–270, 1999.