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The One-Variable Fragment of $\mathbf{T}_{\rightarrow}$

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Abstract We show that there are infinitely many pairwise non-equivalent formulae in one propositional variable p in the pure implication fragment of the logic T of "ticket entailment" proposed by Anderson and Belnap. This answers a question posed by R. K. Meyer.

Keywords Relevant logic · Ticket entailment

1 Logical Preliminaries

In 1970 R. K. Meyer [6] posed the problem of determining the structure of the pure implication fragments of substructural logics restricted to formulae in one propositional variable. In particular, he asked how many equivalence classes of such formulae there are, where the equivalence of A and B is defined as the provability of theorems $A \rightarrow B$ and $B \rightarrow A$ by the canons of one logic or another. The one-variable fragment is an interesting abstraction of a propositional logic, as it consists in waiving all distinctions between the "content" of formulae, as represented by the different atoms, leaving only the "shape" given by the connectives. The most basic question is whether the number of equivalence classes is finite or infinite. In the years since 1970, Meyer's question has been answered for most logics in the class, but two logics have resisted: these are the Anderson-Belnap systems [1, 2] **T** of "ticket entailment" and **E** of entailment.

The logics in question are propositional systems whose formulae are built up from variables p, q, r, etc. by closing under fairly standard connectives such as $\neg$, $\rightarrow$ $\wedge$ and $\vee$. The logic of the extensional (additive or or truth-functional) connectives $\wedge$

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and $\vee$ does not concern us here, and is largely invariant over all the logics Meyer had in mind. The same goes for the negation connective $\neg$. Most of the salient differences between the logics emerge in their theories of the implication connective $\rightarrow$. The weakest logic we consider in this paper is **P−W**, whose theorems may be axiomatised with three axiom schemes

a1. $A \rightarrow A$
a2. $(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
a3. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$

and the standard rule of detachment or *modus ponens*. Adding an axiom of permutation

a4. $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$

or its special case the "assertion" axiom

a4′. $A \rightarrow ((A \rightarrow B) \rightarrow B)$

gives the pure implication fragment of linear logic [4] while leaving out permutation and instead adding "contraction"

a5. $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$

gives the logic $\mathbf{T}_{\rightarrow}$ which is our main focus below. Adding both contraction and permutation yields the relevant logic $\mathbf{R}_{\rightarrow}$. Adding contraction to get $\mathbf{T}_{\rightarrow}$ and then the "restricted assertion" axiom

a4″. $((A \rightarrow A) \rightarrow B) \rightarrow B$

yields Anderson and Belnap's logic $\mathbf{E}_{\rightarrow}$ which lies properly between $\mathbf{T}_{\rightarrow}$ and $\mathbf{R}_{\rightarrow}$. Strengthening $\mathbf{E}_{\rightarrow}$ (or indeed $\mathbf{T}_{\rightarrow}$) in another direction by adding a weakening postulate

a6. $A \rightarrow (B \rightarrow B)$

leads to $\mathbf{S4}_{\rightarrow}$, the pure (strict) implication fragment of the modal logic **S4**, while adding instead the weaker version

a6. $(A \rightarrow B) \rightarrow (C \rightarrow C)$

gives $\mathbf{S3}_{\rightarrow}$, the analogous part of the weaker, non-normal modal logic **S3**. Adding all of contraction, weakening and permutation gives $\mathbf{J}_{\rightarrow}$, the pure implication fragment of intuitionist logic. See Fig. 1 for an inclusion diagram.

It is easy to see that there are only two equivalence classes of formulae in one variable p in $\mathbf{J}_{\rightarrow}$, since it makes every such formula equivalent either to p or to $p \rightarrow p$. Meyer showed in his original paper [6] that in $\mathbf{R}_{\rightarrow}$ there are exactly 6 equivalence classes of one-variable formulae, and in 1976 Byrd [3] similarly showed that in $\mathbf{S4}_{\rightarrow}$ there are 9. On the other side of the street, we may observe the trivial fact that in linear logic, and of course in any weaker system such as **P−W**, there are infinitely many since the series

$$A_0 = p$$
$$A_{i+1} = p \rightarrow A_i$$

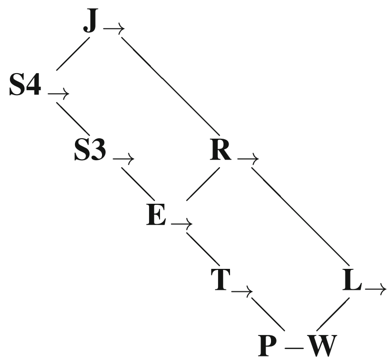


Fig. 1 Relationships between logical systems: rising lines indicate inclusion

does not terminate in equivalents in such contraction-free logics. More recently, Kowalski and Slaney showed [5] that the number of equivalence classes according to $\mathbf{S3}_{\rightarrow}$ is finite, though they were unable to calculate it exactly and we suspect it is large.

The purpose of the present paper is to show that there are infinitely many pairwise non-equivalent formulae in one variable in $\mathbf{T}_{\rightarrow}$. The proof has an algebraic flavour; we start with the algebras (more strictly, the relational structures) corresponding to the stronger logic $\mathbf{R}_{\rightarrow}$.

2 Church Monoids

By a *Church monoid* we mean a structure $\langle S, \leq, \circ, \rightarrow \rangle$ where S is a set partially ordered by $\leq$, and both $\circ$ and $\rightarrow$ are binary operations on S satisfying the postulates:

p1. $\exists a \forall b (a \circ b = b)$
p2. $(a \circ b) \circ c = a \circ (b \circ c)$
p3. $a \circ b = b \circ a$
p4. $a \leq a \circ a$
p5. If $a \leq b$ then $a \circ c \leq b \circ c$
p6. $a \circ b \leq c$ if and only if $a \leq b \rightarrow c$

As usual, we refer to the monoid identity element as e, and write

$$a^0 = e$$
$$a^{i+1} = a^i \circ a$$

to abbreviate iterations of an element a. The name "Church monoid" was introduced by Meyer and Routley [7] within an account of the more general class of Ackermann groupoids, which do not in general satisfy p2, p3 or p4. It is easy to see that in such structures, $a \leq b$ iff $e \leq a \rightarrow b$. For a logic with a conditional connective $\rightarrow$, interpretations are defined in the obvious way as homomorphisms from the formula algebra of the logic into the groupoid. Such a logic is *sound* for a structure iff its theorems are mapped into the positive cone (i.e. $\{x : e \leq x\}$) by every interpretation, and the logic corresponding to a class of such structures is the maximal one which is sound for every groupoid in the class. As is well known, Church monoids correspond in this way to the relevant logic $\mathbf{R}_{\rightarrow}$. Dropping the "square-increasing" postulate p4 results in models of the implicational part of linear logic, while other weakenings of the postulates lead to the algebraic counterparts of related substructural logics including $\mathbf{T}_{\rightarrow}$.

If a logic L is sound for an Ackermann groupoid G and two formulae A and B are mapped to different elements a and b by some interpretation in G, they cannot be provably equivalent in L: since $\leq$ is antisymmetric, either $a \not\leq b$ or $b \not\leq a$, so either $e \not\leq a \rightarrow b$ or $e \not\leq b \rightarrow a$. It follows trivially that if G is generated under the operation $\rightarrow$ by a single element, then the order of G is a lower bound on the number of pairwise non-equivalent formula in the one-variable implicational fragment of L.

There are many constructions for embedding Church monoids in larger ones with useful properties. In the present paper we consider only cases in which S is finite, $\leq$ is a total order and the positive cone is as large as possible, so start with any finite totally ordered commutative monoid such that the identity e of the monoid is the lowest element in the order. Note that p4 holds in virtue of p1 and p5. Add a zero element 0 below e and a top element T above all the rest. Set $0 \circ a = 0 = a \circ 0$ for all elements a, and $T \circ a = T = a \circ, T$ for all $a > 0$. Clearly, for any a and b, $a \rightarrow b$ is well defined as $\max\{x : x \circ a \leq b\}$, so we have a Church monoid. Now it is possible to extend the total order by adding finitely many more elements (we may call them the "large" elements) ordered above the "small" ones but below T. Let the smallest (lowest in the total order) of these new elements be m. Now to extend the monoid operation, where a and a' are nonzero small elements such that $a \leq a'$, and b and b' are large elements such that $b \leq b'$:

c1. $0 \circ b = 0 \circ b = 0$
c2. $e \circ b = b \circ e = b$
c3. $b \circ b' = b' \circ b = T$
c4. $a \circ b \in \{b, T\}$
c5. $a \circ b = b \circ a$
c6. $a^2 \circ b = a \circ b$
c7. If $a' \circ b' = b'$ then $a \circ b = b$

Note that there is some choice as to which elements $a \circ b$ are set to b and which are set to T, though not a completely free choice as the ordering postulate and associativity need to be preserved. We wish to prove that this construction yields a Church monoid. Satisfaction of all postulates except associativity is evident from the definition, so we prove only p2. Consider any three elements a, b and c: we need to show that $(a \circ b) \circ c = a \circ (b \circ c)$. Clearly this holds if any of the three is 0 or e, or if all three of them are small, or if any two of them are large. In the remaining case, two of them are small elements greater than e while the third is large. First suppose c is large (by commutativity, the case where a is large is the same) and without loss of generality suppose $a \leq b$. Since $b \leq a \circ b$, if $b \circ c = T$ then $(a \circ b) \circ c = T$, and $a \circ (b \circ c) = a \circ T = T$, so associativity holds in that case. If on the other hand $b \circ c = c$ then $b^2 \circ c = c$ by c6, so since $a \leq b$, $(a \circ b) \circ c = c$, and $a \circ (b \circ c) = a \circ c = c$ also. Lastly, suppose b is large while a and c are small, and let $c \leq a$. If $a \circ b = b$ then $b \circ c = b$ and $(a \circ b) \circ c = b \circ c = b$ and $a \circ (b \circ c) = a \circ b = b$. If $a \circ b = T$ then $(a \circ b) \circ c = T \circ c = T$ and $a \circ (b \circ c) \geq a \circ b = T$. There being no more cases to consider, associativity holds.

3 Ticket Groupoids

The next constructions consist of adding elements to totally ordered Church monoids in such a way as to preserve p1 and p6 of the Church monoid definition, together with the following weakened and adjusted versions of p2–p5:

p2′. $(a \circ b) \circ c \leq a \circ (b \circ c)$
p3′. $(a \circ b) \circ c \leq b \circ (a \circ c)$
p4′. $a \circ b \leq (a \circ b) \circ b$
p5′. If $a \leq b$ then $a \circ c \leq b \circ c$ and $c \circ a \leq c \circ b$

We need a name for these algebras. "$\mathbf{T}_{\rightarrow\circ}$ propositional structures" seems rather clumsy, so we dub them "ticket groupoids" as they correspond to the logic of ticket entailment.

Given a totally ordered Church monoid with e immediately above 0 as above, our first step is to insert a new left identity element t between 0 and e in the order, extending the $\circ$ operation appropriately. Note well that in the extended structure, the identity required for p1 is t, not e, and that since commutativity is no longer postulated, there is no right identity element. For any element x, we require $t \circ x = x$, so the cases in which t occurs on the left of $\circ$ are defined immediately. The new element t on the right of the $\circ$ operation, however, is less straightforward.

One approach is to choose a subset $\mathcal{O}$ of the elements, including 0 and t, to be designated as "open". Then the "interior" of an element x is defined as the greatest open element y such that $y \leq x$. We use the box familiar from modal logic for the interior operation:

$$\Box a = \max\{x : x \in \mathcal{O} \wedge x \leq a\}$$

Now where t is being added to a Church monoid obtained as above, with "small" and "large" elements, first we ensure that $a \circ b$ is large if a is large by requiring m, the smallest of the large elements, to be in $\mathcal{O}$ and setting

$$a \circ t = \Box a$$

for all elements a. There is some freedom in the choice of which elements are open, so long as the whole satisfies the constraints:

o1. $0 \in \mathcal{O}$
o2. $t \in \mathcal{O}$
o3. $m \in \mathcal{O}$

o4. If $T \in \mathcal{O}$ then $e \in \mathcal{O}$
o5. $\Box(a \circ b) \leq \Box a \circ \Box b$

Note that if $\Box a = t$ then

$$\begin{aligned}\Box a \circ \Box b &= t \circ \Box b \\ &= \Box b \\ &= \Box\Box b \\ &= \Box b \circ t \\ &= \Box b \circ \Box a\end{aligned}$$

The same holds if $\Box b = t$. If neither $\Box a$ nor $\Box b$ is t then they are both in the Church monoid, which is commutative, so $\circ$ is commutative on open elements. Hence

o5$'$. $\Box(a \circ b) \leq \Box b \circ \Box a$

holds if o5 does.

As before, $a \to b$ is defined on the new structure as the greatest x such that $x \circ a \leq b$. We may note that where neither a nor b is t, the value of $a \to b$ is unchanged by the addition of t.

As an aside, notice that the introduction of the modal box is the dual of that more usually found in relevant logic. In **E**, for example, the idea of an underlying modal character to the logic is as old as Anderson and Belnap's development of the system [1]. We are accustomed to think of $\Box a$ as $t \to a$ or some close relative such as $(a \to a) \to a$. In that case, $a \circ t$ is the dual "inverse" modality $\blacklozenge a$ satisfying the equivalence

$$\blacklozenge a \leq b \text{ iff } a \leq \Box b$$

In the ticket groupoids we are considering, however, it is $\Box a$ which is $a \circ t$ and which therefore plays the role of this black diamond. Well, it is *interesting* that $\mathbf{T}_{\to}$ allows this inverted construal of necessity, though further pursuit of the idea would take us beyond the concerns of the present paper.

Lemma 1 *The structure just defined with t is a ticket groupoid.*

Proof There is no difficulty in seeing that p1 holds as part of the construction, p6 holds by definition, and p5$'$ is obvious. For p4$'$, note that if $a = t$ then it amounts to square increasing $b \leq b \circ b$ which holds trivially. If $b = t$ then it is $\Box a \leq \Box\Box a$ which holds because $\Box a$ is open. If neither a nor b is t then they are part of a Church monoid in which p4$'$ holds. That leaves p2$'$ and p3$'$, which we treat together. In the case $a = t$ they both hold because $(a \circ b) \circ c$ evaluates to $b \circ c$, as do both $a \circ (b \circ c)$ and $b \circ (a \circ c)$. In the case $b = t$, both postulates amount to $\Box a \circ c \leq a \circ c$ which holds by p5$'$ because $\Box a \leq a$. In the case $c = t$, p2$'$ and p3$'$ amount to $\Box(a \circ b) \leq a \circ \Box b$ and $\Box(a \circ b) \leq b \circ \Box a$ respectively. These are easy consequences of o5 and o5$'$ respectively, since $\Box a \leq a$ and $\Box b \leq b$. In all other cases, none of the three elements a, b or c is t, so the associativity and commutativity of the Church monoid suffice. □

Next, to a totally ordered ticket groupoid constructed as above with identity t next to the bottom and with top element T we can add an element g immediately below T. g is open iff T is open. Extend $\circ$ to the new element as follows: $0 \circ g = g \circ 0 = 0$, $t \circ g = g$, $g \circ t = \Box g$, and for any $a > t$, $a \circ g = g \circ a = T$. Let $\to$ be defined by residuation as always. The constraint that if g is open then T is open is required so that the addition of g does not change the $\circ$ operation on existing elements, while the converse constraint that g is open if T is ensures that $\to$ is likewise unchanged.

Lemma 2 *The structure just defined with t and g is a ticket groupoid.*

Proof The construction does not disturb postulates p1, p5′ or p6. For the other postulates we need to consider additional cases in which one of the elements involved is g.

Consider p4′ first. The above argument that the postulate holds if either a or b is 0 or t still goes through in the presence of g. If one of a or b is g and the other is greater than t then both sides of the inequality go to T. If neither a nor b is g, then they lie within the ticket groupoid without g, so p4′ holds.

Postulates p2′ and p3′ are a little more complicated. Again, the argument in cases where one of the three elements is 0 or where one of a or b is t is not disturbed, and again if all three are greater than t and one of them is g then both $a \circ (b \circ c)$ and $b \circ (a \circ c)$ are T. Obviously, if g is not involved, then a, b and c lie within the existing ticket groupoid, so the only case requiring more argument is where c is t, either a or b is g and the other is strictly greater than t. Then $(a \circ b) \circ c$ is $\Box T$, so it is to be shown that for any $x > t$, both $\Box T \leq x \circ \Box g$ and $\Box T \leq g \circ \Box x$. If T is open, then both g and e are also open, so $\Box g = g$ and $t < \Box x$, so both $x \circ \Box g$ and $g \circ \Box x$ are T. If, on the other hand, T is not open, then $\Box g = \Box T$. But $T \circ x = T$, so

$$\begin{aligned}
\Box T &= \Box(T \circ x) \\
&\leq \Box T \circ \Box x \\
&\leq \Box T \circ x \\
&= \Box g \circ x
\end{aligned}$$

$\Box T \leq x \circ \Box g$ similarly. Hence in all cases, p2′ and p3′ hold. ☐

4 A Special Case

Let k be any integer greater than 1. We construct a ticket groupoid on the first $3k + 2$ natural numbers $0, \ldots, 3k + 1$. To accord with the construction in the last section, we name some of the numbers: $2k + 1$, $3k$ and $3k + 1$ shall be called m, g and T respectively. We also refer to numbers less than m as “small” and those in the range $m, \ldots, T$ as “large”.

The construction is done in stages. We begin with a totally ordered monoid defined on the numbers $2, \ldots, 2k$. The one we choose has $a \circ b = \max(a, b)$, thus identifying $\circ$ with the join operation $\vee$ of the total order considered as a lattice. This automatically makes the lowest element 2 the identity, and is obviously associative, commutative and square increasing (indeed, idempotent). Adding 0 and T as above results in a Church monoid. For example, Fig. 2 shows the algebra for $k = 3$:

Next it is necessary to add the numbers from m to $3k - 1$ and to extend the definition of $\circ$ to them. Note that c6 is guaranteed because $a^2 = a$ for any small a, so this may be done in any way that preserves ordering (c7) and the "easy" constraints c1–c5. For reasons which will become clear (see Lemma 3 below), we choose, for each nonzero small element a and large element b, to set $a \circ b = T$ if and only if $a \geq 2(T - b)$, and of course $a \circ b = b$ otherwise. In our running example of $k = 3$ this gives the algebra shown in Fig. 3.

It is easy to verify by observation that constraints c1–c7 hold.

Now to add the two missing numbers 1 (that is t) and g we need to choose the set $\mathcal{O}$ of open elements. We choose them to be 0 and all the odd numbers up to and including m. To verify that this is a legitimate choice, we must check the conditions o1–o5, of which only o5

$$\Box(a \circ b) \;\leq\; \Box a \circ \Box b$$

requires an argument. We proceed by cases. If either a or b is large, then $a \circ b$ is also large, so $\Box(a \circ b) = m$; but either $\Box a = m$ or $\Box b = m$, so $\Box a \circ \Box b$ is large, so $m \leq \Box a \circ \Box b$. If neither a nor b is large, either one of them is 0, in which case $\Box(a \circ b) = 0$ and the inequality holds, or else they are both nonzero. Without loss of generality, suppose $a \leq b$. Then $\Box a \leq \Box b$, so $\Box a \circ \Box b = \Box b$, and since $a \circ b = b$, $\Box(a \circ b) = \Box b$ also. Hence in all cases o5 holds.

Lemma 3 *The following three conditions hold:*

1. *If* $0 < a < m$ *and* a *is odd then* $t \rightarrow a = a + 1$*;*
2. *If* $0 < a < m$ *and* a *is even, then* $a \rightarrow g = g - \frac{a}{2}$*;*
3. *If* $m \leq b < T$ *then* $b \rightarrow b = 2(g - b) + 1$.

Proof We consider the three conditions separately.

1. Suppose a is small, nonzero and odd. $(a + 1) \circ t = \Box(a + 1) = a$, so $a + 1 \leq t \rightarrow a$, but $\Box(a + 2) = a + 2$, so $a + 2 \not\leq t \rightarrow a$. Hence $t \rightarrow a = a + 1$.

$\circ$	0	2	3	4	5	6	10
0	0	0	0	0	0	0	0
$e = 2$	0	2	3	4	5	6	10
3	0	3	3	4	5	6	10
4	0	4	4	4	5	6	10
5	0	5	5	5	5	6	10
6	0	6	6	6	6	6	10
$T = 10$	0	10	10	10	10	10	10

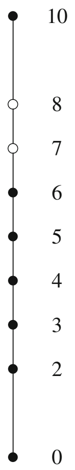
$\rightarrow$	0	2	3	4	5	6	10
0	10	10	10	10	10	10	10
2	0	2	3	4	5	6	10
3	0	0	3	4	5	6	10
4	0	0	0	4	5	6	10
5	0	0	0	0	5	6	10
6	0	0	0	0	0	6	10
10	0	0	0	0	0	0	10

Fig. 2 Example of the construction: basic Church monoid with $k = 3$

Fig. 3 Example $k = 3$: extended Church monoid, after adding large elements 7 and 8 but before adding t and g

$\circ$	0	2	3	4	5	6	7	8	10
0	0	0	0	0	0	0	0	0	0
$e = 2$	0	2	3	4	5	6	7	8	10
3	0	3	3	4	5	6	7	8	10
4	0	4	4	4	5	6	7	10	10
5	0	5	5	5	5	6	7	10	10
6	0	6	6	6	6	6	10	10	10
$m = 7$	0	7	7	7	7	10	10	10	10
8	0	8	8	10	10	10	10	10	10
$T = 10$	0	10	10	10	10	10	10	10	10

$\rightarrow$	0	2	3	4	5	6	7	8	10
0	10	10	10	10	10	10	10	10	10
2	0	2	3	4	5	6	7	8	10
3	0	0	3	4	5	6	7	8	10
4	0	0	0	4	5	6	7	7	10
5	0	0	0	0	5	6	7	7	10
6	0	0	0	0	0	6	6	6	10
7	0	0	0	0	0	0	5	5	10
8	0	0	0	0	0	0	0	3	10
10	0	0	0	0	0	0	0	0	10



2. Suppose for some $i > 0$, $a = 2i$ and $a < m$. We wish to show $a \rightarrow g = g - i$ or in other words $g - i$ is the greatest b such that $a \circ b \leq g$. By the construction, $a \circ b < T$ iff $a < 2(T - b)$, so we need that $g - i$ is the greatest b such that $2i < 2(T - b)$ which is to say $i < T - b$. Well, $i < T - (g - i)$ since $T - g = 1$. Moreover, $i \geq T - (g - i + 1)$ since $T - (g - i + 1) = i$.
3. Suppose $m \leq b \leq g$. By definition, $b \rightarrow b$ is the greatest x such that $x \circ b \leq b$. Obviously $b \rightarrow b < m$, since $m \circ b = T$. By stipulation, for small a, $a \circ b \leq b$ iff $a < 2(T - b)$, so we need to show that $2(g - b) + 1 < 2(T - b)$ but $2(g - b) + 2 \geq 2(T - b)$. Well, $g = T - 1$, so $2(g - b) + 2 = 2(T - b)$ as required.

□

By way of illustration, Fig. 4 shows the final algebra for our running example $k = 3$:

Lemma 4 *The structure just defined on* $0, \ldots, 3k + 1$ *is a ticket groupoid generated under the operation* $\rightarrow$ *by the single element* g.

Proof That the algebra is a ticket groupoid is immediate from Lemma 2 and the fact that its construction conforms to the given template. That it is generated by $\{g\}$

Fig. 4 Example $k = 3$: the final algebra

$\circ$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
$t = 1$	0	1	2	3	4	5	6	7	8	9	10
$e = 2$	0	1	2	3	4	5	6	7	8	10	10
3	0	3	3	3	4	5	6	7	8	10	10
4	0	3	4	4	4	5	6	7	10	10	10
5	0	5	5	5	5	5	6	7	10	10	10
6	0	5	6	6	6	6	6	10	10	10	10
$m = 7$	0	7	7	7	7	7	10	10	10	10	10
8	0	7	8	8	10	10	10	10	10	10	10
$g = 9$	0	7	10	10	10	10	10	10	10	10	10
$T = 10$	0	7	10	10	10	10	10	10	10	10	10

$\rightarrow$	0	1	2	3	4	5	6	7	8	9	10
0	10	10	10	10	10	10	10	10	10	10	10
1	0	2	2	4	4	6	6	10	10	10	10
2	0	0	2	3	4	5	6	7	8	8	10
3	0	0	0	3	4	5	6	7	8	8	10
4	0	0	0	0	4	5	6	7	7	7	10
5	0	0	0	0	0	5	6	7	7	7	10
6	0	0	0	0	0	0	6	6	6	6	10
7	0	0	0	0	0	0	0	5	5	5	10
8	0	0	0	0	0	0	0	0	3	3	10
9	0	0	0	0	0	0	0	0	0	1	10
10	0	0	0	0	0	0	0	0	0	0	10

follows from Lemma 3. The elements are generated from the outside in, starting with g and 1, and working in towards m. $3k$ is just g, and then for any i such that $0 \leq i < k$:

$$\begin{aligned} 2i + 1 &= (3k - i)\rightarrow(3k - i) \\ 2(i + 1) &= 1\rightarrow(2i + 1) \\ 3k - i - 1 &= 2(i + 1)\rightarrow g \end{aligned}$$

Clearly in this way all numbers in the range $1, \ldots, 3k$ are generated. Adding 0 and T is trivial: e.g. $g\rightarrow t = 0$ and $0\rightarrow t = T$. Hence the algebra is indeed generated from g using $\rightarrow$ alone. □

In the example of $k = 3$, the elements in the range $1, \ldots, 9$ are generated in the order 9, 1, 2, 8, 3, 4, 7, 5, 6 by following the recipe above.

Since the construction works for any finite k, the free ticket groupoid with one generator is evidently infinite.

Theorem 1 *There are infinitely many pairwise non-equivalent formulae in* $\mathbf{T}_{\rightarrow}$.

Proof It has long been known [7, 8] that the theorems of $\mathbf{T}_{\rightarrow}$ are exactly the formulae mapped into the positive cone by every homomorphism from the formula algebra into a ticket groupoid. Hence any formulae mapped to distinct elements of a ticket groupoid are non-equivalent in $\mathbf{T}_{\rightarrow}$. By Lemma 3, therefore, there are for any finite k more than $3k$ pairwise non-equivalent formulae, as the variable p can be assigned the value g. □

A simple sequence of implicational formulae which does not terminate in $\mathbf{T}_{\rightarrow}$ equivalents is generated thus:

$$\begin{aligned} A_0 &= p \\ A_{i+1} &= ((p\rightarrow p)\rightarrow(A_i\rightarrow A_i))\rightarrow p \end{aligned}$$

For any $k > i + 1$, A_i gets the value $g - i$, so $A_i\rightarrow A_i$ is evaluated as $2i + 1$ and $(p\rightarrow p)\rightarrow(A_i\rightarrow A_i)$ as $2(i + 1)$ sending A_{i+1} to $g - (i + 1)$. Hence $A_{i+1}\rightarrow A_i$ is a non-theorem, as it gets the value 0 on this interpretation.

5 Conclusion

This result closes one of the two main open problems concerning the number of one-variable formulae in implicational logics, leaving $\mathbf{E}_{\rightarrow}$ as the one major system for which the problem is unsolved. $\mathbf{E}_{\rightarrow}$ is now sandwiched between $\mathbf{S3}_{\rightarrow}$ above and $\mathbf{T}_{\rightarrow}$ below. The constructions of the present paper cannot be adapted in any obvious fashion for $\mathbf{E}_{\rightarrow}$, since it is essential to the latter that the removal of just the generator from any of its one-generator algebras leaves a Church monoid, so for $\mathbf{E}_{\rightarrow}$ there is no possibility of adding t as well. Moreover, since it forces

$$t\rightarrow a \;\leq\; a$$

there is no way either to exploit the "rôle reversal" of the dual modalities □ and ♦ which is a feature of the construction above.

We therefore leave the $\mathbf{E}_{\rightarrow}$ problem as future work, the solution to its companion $\mathbf{T}_{\rightarrow}$ problem sufficing for the present note.

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