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Revisiting Standing Waves on a Circular Path

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There are plenty of contributions in the physics literature on the subject of standing waves. What seems to be missing is an accessible, quantitative account of standing waves on a circular path at a level appropriate for undergraduate physics courses. In order to rectify this absence, a straightforward, concise, mathematical description of standing waves on a circular path is presented in terms of standard circular motion parameters.

Background

All textbooks on general physics and on wave motion cover the subject of standing waves to some degree. However, standing waves on circular paths barely get remarked on in physics textbooks. The most common allusion to such standing waves in the textbooks is done to provide a reason why the quantization of angular momentum applies to the Bohr model of the atom,[1–5] and without which this quantization condition is just a (unexplained) postulate. Less frequently mentioned are macroscopic cases, solid state physics examples, and optical "corrals."[6] In contrast, searching the internet for articles dealing with "standing waves in a circle" or "standing waves on circular paths" tends only to bring up references to the short papers by Meiners,[7] Vijay,[8] and Bloom & Bloom.[9] These three papers consider (non-equivalent mechanical) apparatuses designed to produce standing waves in (macroscopic) vibrating wire loops for the purpose of demonstrating standing wave patterns. (Videos of mechanically driven standing waves in circular objects are accessible from the internet.[10,11]) Nevertheless, the available physics literature does not appear to provide a mathematical description of standing waves on a circular path suitable for use in undergraduate teaching and this situation requires attention.

Traveling waves and standing waves in one dimension

Consider first the (linear) case of a one-dimensional, harmonic traveling wave pulse moving in a straight line. If the pulse is traveling to the right (along the positive *x*-axis) then it has a wave function, i.e., the mathematical function describing the wave disturbance at any position or time (denoted Ψ_R), given by:

$$\Psi_R = \Psi_0 \sin[k(x - vt)], \quad (1)$$

where Ψ_0 is the amplitude of the wave, v is its speed (assumed constant), t is time, and $k = (2\pi/\lambda)$ is the propagation constant (though many texts use the name "wave number"), with λ being the wavelength.[12] The wave function Ψ_R is a solution of the classical, one-dimensional wave equation. A one-dimensional, harmonic traveling wave pulse moving to the left (along the negative *x*-axis) with equal amplitude, speed, and wavelength to the wave described by Ψ_R has a wave function (denoted Ψ_L):

$$\Psi_L = \Psi_0 \sin[k(x + vt)]. \quad (2)$$

Notice that Eq. (2) is identical in its form to Eq. (1) except that the speed of the wave carries the opposite sign. It has been shown that when a wave pulse moving in the negative *x*-direction is depicted in the form given by Eq. (2), erroneous results can occur in some instances.[13] However the situation discussed below (i.e., generation of a standing wave) is not affected.

When two trains of traveling waves of equal amplitude, speed, and wavelength but moving in *opposite directions* are superimposed, a standing wave will be formed. This phenomenon is mathematically modeled by adding the equations describing the two traveling waves in accordance with the principle of linear superposition. If we add Eqs. (1) and (2) then we find:

$$\Psi_R + \Psi_L = \Psi_0 \sin[k(x - vt)] + \Psi_0 \sin[k(x + vt)] = 2\Psi_0 \sin(kx) \cos(kvt), \quad (3)$$

which is the expression for a standing wave. The resulting waveform will have points of zero disturbance (nodes) and points of maximum disturbance (antinodes), each respectively separated by half a wavelength. Its oscillation over time produces the well-known standing wave pattern (Fig. 1). This pattern does not progress but repeats itself as the positions of the nodes and antinodes do not change.[14]

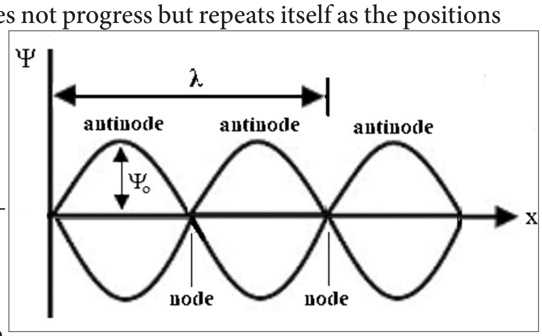

Fig. 1. Pattern of a standing wave over time.

A standing wave will have the same wavelength as the traveling wave trains from which it is formed. If these two traveling wave trains are unconstrained then there will be no restrictions on the wavelength of the standing wave. When boundary conditions apply (e.g., rigid walls, fixed ends), these conditions will determine the allowed wavelengths. If a traveling wave train of appropriate wavelength is constrained between two straight barriers, for example, a standing wave will form by interference of the incident and reflected waves with a node at each barrier. If the barriers are a distance L apart, then the boundary condition imposed requires the allowed wavelengths (λ) to be fixed according to:

$$n\lambda = 2L, \quad (4)$$

where $n = 1, 2, 3, \ldots$. If the standing wave has frequency f, where $f = (v/\lambda)$, then using Eq. (4) we find that the (harmonic) resonant frequencies (f_n) are:

$$f_n = (nv/2L). \quad (5)$$

All leading general physics textbooks provide these descriptions of (linear) traveling and standing waves.[15–19]

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Waves on a circular path

Consider now a harmonic traveling wave moving along a circular path. This is a *one-dimensional* problem with constraints, not a two-dimensional problem such as waves on a drum membrane. A wave's velocity, also called phase velocity, is defined as the velocity at which its crests and troughs progress.[20] (Note that the wave speed v is the magnitude of its velocity so that v may remain constant even though the velocity is not.) The relevant constraints are that the wave's velocity vector (whose direction is the wave's direction of propagation at a particular time) is tangential to a trajectory that

(i) is a fixed distance from a specified external point; and

(ii) has its initial and final points coincident.

The standard parameters for motion in a circle (see Fig. 2) may be employed in describing a traveling wave on a circular path. The constrained distance from a specified point is obviously the radius (r) from the center of the circular path (i.e., the trajectory).

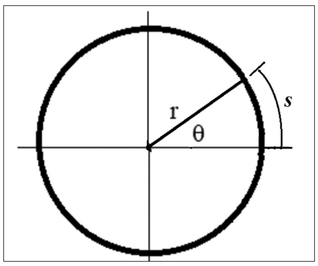

Fig. 2. Circular motion parameters.

A traveling wave moving along a circular path may be parametrized by the arc length traversed (s) from an initial point on the circle at any time t. A wave traveling anticlockwise with speed v is represented by the following wavefunction (denoted Ψ_A):

$$\Psi_A = \Psi_0 \sin\,[k(s - vt)], \tag{6}$$

where its form (i.e., with a negative sign) is the same as in Eq. (1) except that x has been replaced by s. Equation (6) may be re-expressed in terms of circular motion parameters:

$$\Psi_A = \Psi_0 \sin\,[kr(\theta - \theta' t)], \tag{7}$$

where the angle $\theta = s/r$ (radians) and $\theta' = (d\theta/dt) = v/r$, with the radial distance r assumed constant. Care needs to be taken here not to confuse the quantities used in describing the motion of a wave along a circular path with similar quantities typically used to describe the oscillation of a wave, e.g., frequency, period, etc.

A standing wave on a circular path is formed from the superposition of *two* traveling wave trains of equal amplitude, speed, and wavelength, one moving in a clockwise direction and the other anticlockwise.[21] Since Ψ_A represents a traveling wave moving anticlockwise, the wavefunction for a traveling wave moving clockwise (denoted Ψ_C) is:

$$\Psi_C = \Psi_0 \sin\,[kr(\theta + \theta' t)]. \tag{8}$$

Analogous to the case of Eqs. (1) and (2), if we add Eqs. (7) and (8), then we get the wave function of a standing wave on a circular path (denoted Ψ_S):

$$\Psi_S = \Psi_A + \Psi_C = 2\Psi_0 \sin\,(kr\theta)\cos\,(kr\theta' t). \tag{9}$$

The wavelength of a standing wave on a circular path (λ) is restricted by the constraints (i) and (ii) listed above to values given by:

$$n\lambda = 2\pi r, \tag{10}$$

where $n = 1, 2, 3, \ldots$, i.e., only an integral multiple of the wavelength is equal to the length of the circular path's circumference. The maximum wavelength is, therefore, equal to the circumference's length. Notice that this differs from the linear case in which the maximum allowed wavelength is *twice* the length from the initial point to the final point, e.g., $2L$ in Eq. (4). Figure 3 displays six examples of standing wave patterns ($n = 1, 2, 3, 4, 7$, and 10) oscillating about (blue colored) circles of fixed radii. The value of the integer n is the same as the number of complete oscillations in the standing wave pattern.

Just as in the linear case, the positions of the nodes and antinodes of a standing wave on a circular path do not change over time. The positions of the nodes are where $\Psi_S = 0$ so that $\sin\,(kr\theta) = 0$, i.e. $kr\theta = m\,\pi$, where m takes the integral values $0, 1, 2, 3, \ldots$. The nodes of the standing waves in Fig. 3 are obvious, e.g., $\theta = 0, \pi$ for $n = 1$; $\theta = 0, \pi/2, \pi, 3\pi/2$ for $n = 2$; etc. It can also be seen that the number of nodes in these standing wave patterns increases by two when n increases by one. Using

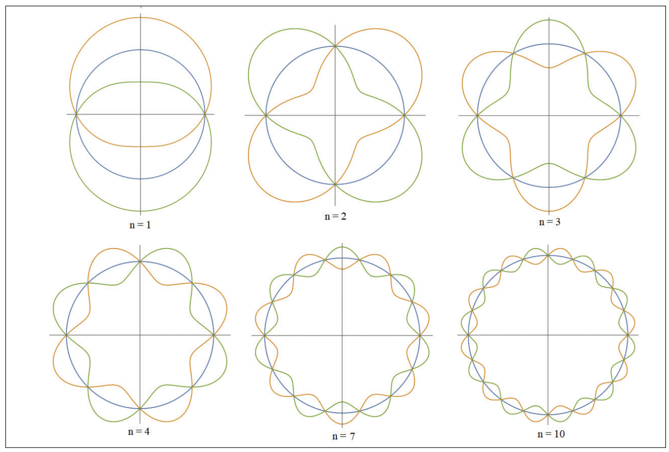

Fig. 3. Patterns of six standing waves on circular paths over time.[22]

Eq. (10), we find that the equation for the (harmonic) resonant frequencies (f_n) for a standing wave on a circular path is: $f_n = (nv/2\pi r)$. This differs in its form from the linear case [e.g., Eq. (5)] by a factor of two due to the maximum wavelength being equal to the circumference's length.

The most familiar theoretical application of standing waves on a circular path is in the Bohr model of the atom, descriptions of which are found in most textbooks that give an introduction to quantum physics. The restriction given by Eq. (10) provides a reason for the quantization of angular momentum in the model. The (empirically confirmed) de Broglie hypothesis declares that matter exhibits a duality of particle and wave natures. When an electron is depicted as a wave and the electron is bound inside an atom, then such a bound state may be described by a standing wave that "fits" within the atom, i.e., the electron standing wave is restricted by the boundary condition expressed by Eq. (10). The wavelength of the electron, according to the de Broglie hypothesis, is: $\lambda = h/p$, where

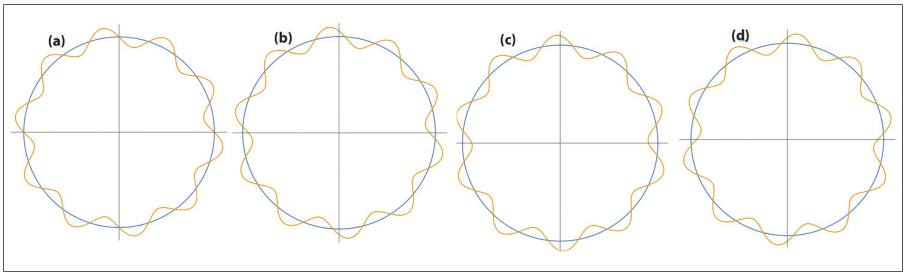

Fig. 4. Clockwise progression of a traveling wave train on a circular path.[25]

h is Planck's constant and p is the magnitude of the electron's momentum. Substituting (h/p) for λ in Eq. (10) yields $rp = nh/2\pi$, i.e., the angular momentum (rp) is quantized in integral multiples of $(h/2\pi)$. It is this result that leads directly to the Bohr model having stationary states with fixed electron orbital radii and discrete energy levels.[23] What is important to realize in this application (and is not clear in many texts) is that the conception of standing waves occupying electron orbits is *essential to understanding* how the quantitative details of the Bohr model emerge.[24]

Traveling vs. standing waves on circular paths

A single traveling wave train that moves completely around a circular path (clockwise or anticlockwise) is *not* a standing wave for it does not meet the conditions for the formation of a standing wave on a circular path. Instead, it is just a traveling wave that loops back on itself. Furthermore, the points of zero disturbance and maximum disturbance (nodes and antinodes) of a traveling wave on a circular path *do change their positions* over time. This is displayed in Fig. 4, which depicts four successive "snapshots" (a) – (d) of a traveling wave train looping clockwise around a circular path, where each "snapshot" is within the same period of oscillation. Clearly then, the superposition of a single traveling wave on a circular path with itself does not result in a standing wave.

Concluding remarks

We have seen that the standing wave on a circular path is a constrained one-dimensional problem in which the superposition of anticlockwise and clockwise traveling wave trains of equal amplitude, speed, and wavelength gives rise to the standing wave pattern. A rudimentary mathematical account suitable for use in undergraduate physics courses has been provided that describes such standing waves.

A practical exercise that students can perform on a benchtop is the generation of standing waves in a circular length of wire or a thin circular spring. The main pieces of apparatuses needed are a mechanical (vibration) driver and a strobe light. The number of nodes exhibited may be varied by changing the frequency of the driver. Instructions are available on the internet.[26, 27] Note, however, that the patterns of standing waves produced by these devices will not be identical to those of the standing waves on a circular path described above due to the mechanical driver being physically attached to the wire loop or circular spring.[28]

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References

1. W. Demtröder, *Atoms, Molecules and Photons: An Introduction to Atomic-, Molecular- and Quantum Physics*, 2nd ed. (Springer, Berlin, 2010), pp. 114–115.
2. S. T. Thornton and A. Rex, *Modern Physics for Scientists and Engineers*, 4th ed. (Brooks/Cole, Boston, 2013), p. 169.
3. J. E. House, *Fundamentals of Quantum Mechanics* (Elsevier Science & Technology, Amsterdam, 2017), pp. 17–18.
4. D. Young and S. Standler, *Cutnell and Johnson Physics* (Wiley, New York, 2018), p. 862.
5. A. J. Buffa, B. Lou, and J. D. Wilson, *College Physics Essentials Vol. 2. Electricity and Magnetism, Optics, Modern Physics*, 8th ed. (CRC Press, Boca Raton, FL, 2019), p. 688.
6. Y. Babayan et al., "Confining standing waves in optical corrals," *ASC Nano* **3** (3), 615–620 (2009).
7. H. F. Meiners, "Standing waves in a circle," *Am. J. Phys.* **33,** xiv (Oct. 1965).
8. Y. K. Vijay, "Mechanical oscillations in a circular loop," *Am. J. Phys.* **64** (8), 1077–1078 (Aug. 1996).
9. D. Bloom and D. W. Bloom, "Vibrating wire loop and the Bohr model," *Phys. Teach.* **41**, 292–294 (May 2003).
10. Videos of standing waves in a wire loop may be viewed at https://youtu.be/70OgGm2U9vU and https://www.youtube.com/watch?v=df3oJPi9_mg.
11. A video of standing waves in a flat spring steel loop may be viewed at https://www.youtube.com/watch?v=ZWIzD4EPXUw.
12. A. S. Narayanan and S. K. Saha, *Waves and Oscillations in Nature: An Introduction* (CRC Press, Boca Raton, FL, 2015), pp. 10–11 & 20.
13. J. Perkins and M. J. Ruiz, "A reliable wave convention for oppositely traveling waves," *Phys. Teach.* **56**, 622–623 (Dec. 2018).
14. See Ref. 12, pp. 29–30.
15. D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics*, 10th ed. (Wiley, Hoboken, NJ, 2013), Chaps. 16–17.
16. D. C. Giancoli, *Physics for Scientists and Engineers with Modern Physics*, 4th ed. (Pearson Prentice Hall, Upper Saddle River, NJ, 2014), Chap. 15.
17. H. D. Young and R. A. Freedman, *Sears and Zemansky's University Physics with Modern Physics*, 14th ed. (Pearson Education, London, 2016), Chap. 15.
18. R. A. Serway, J. W. Jewett Jr., K. Wilson, A. Wilson, and W. Rowlands, *Physics for Global Scientists and Engineers , Vol. 1,* 2nd ed. (Cengage Learning, Melbourne, 2017), Chaps. 17–18.
19. R. Shankar, *Fundamentals of Physics I: Mechanics, Relativity, and Thermodynamics*, expanded ed. (University Press, Yale, 2019), Chaps. 18–19.
20. Ref. 12, p. 20.
21. Ref. 9, p. 293.
22. Generated using Mathematica 11.1.
23. N. Zettili, *Quantum Mechanics: Concepts and Applications,* 2nd ed. (Wiley, Chichester UK, 2009), pp. 33–34.
24. K. R. Popper, *Quantum Theory and the Schism in Physics* (Unwin & Hyman, London, 1982), p. 48.
25. Generated using Mathematica 11.1.
26. "Standing Waves in a Circular Wire," http://www2.oberlin.edu/physics/catalog/demonstrations/waves/wireloop.html.
27. "7A50.40-Bohr Model-Vibrating Circular Wire," University of Iowa Physics, https://instructional-resources.physics.uiowa.edu/demos/7a5040-bohr-model-vibrating-circular-wire.
28. These differences are described in Ref. 9, p. 293.