Maximal L^p-Regularity of Deterministic and Stochastic PDEs

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Contents

Introduction

Abstract

This thesis aims to explain and prove Theorem 1.1 from the paper 'Stochastic Maximal L^p-Regularity' by Jan van Neerven, Mark Veraar and Lutz Weis [43]. The theorem concerns 'Maximal L^p-regularity,' an important property in the study of nonlinear differential equations. We shall first cover the background needed to define and prove Maximal L^p-Regularity for deterministic PDEs; we will then move to Stochastic PDEs and prove the theorem from [43]. As background, we will outline a theory of randomised sums in Banach spaces (Chapters 2 and 6), a functional calculus based on the Cauchy integral formula (see Chapter 4), and stochastic integration of Banach-valued functions (Chapter 7).

Recommended Background

The paper [43] is written for the graduate, or even the professional research mathematician; this thesis aims to explain the same material to a late undergraduate or beginning graduate audience. However, the reader should have undergraduate or better experience with Banach space theory, measure theory and Lebesgue integration, and at least some familiarity with complex analysis and probability. An understanding of PDEs is recommended to motivate the results, but the thesis uses functional analytic techniques and no PDE familiarity is required to follow any discussions or proofs. Harmonic analysis techniques do come into play in the study of Maximal L^p-regularity, but these considerations are beyond this thesis' scope and are not required.

Relationship to the Literature

None of the mathematical results in this thesis are original - this thesis' contribution is to synthesise and simplify material from various sources, and present it in a more accessible form. The proof of the main theorem is adapted from [43] and the survey paper [44]. The primary sources for background are the lecture notes [29] and [41], and the preprints [23] and [24], with further sources referenced throughout the thesis.

Overview of the Thesis

We are going to study partial differential equations using functional analytic methods. How might such methods apply? Consider a linear PDE of the form:

$$\frac{\partial}{\partial t}u(t,x) + Au(t,x) = f(t,x), \quad t \geq 0, \quad x \in \mathbb{R}^n$$
$$u(0,x) = u_0(x)$$

with A some (usually second order) partial differential operator in the x coordinate. We identify u and f with functions $u(t) = (x \to u(t,x))$ and $f(t) = (x \to f(t,x))$, taking values in a suitable Banach space of functions X (such as $L^q(\mathbb{R}^n)$). We then identify the partial differential operator A with an unbounded operator $A : X \to X$. Under these identifications, we have converted our PDE into a Banach-space valued ODE, which we label (ACP) for 'Abstract Cauchy Problem':

$$\begin{aligned} \frac{d}{dt}u(t) + Au(t) = f(t), \quad t \geq 0 \\ u(0) = u_0. \end{aligned} \tag{ACP}$$

This is the connection by which functional analysis may describe PDEs.

In Chapter 1, we will see that an analogue of the L^p-spaces may be constructed for Banach-valued functions. With this in mind, we say that an operator A has *Maximal L^p-regularity* if, for every $f \in L^p(\mathbb{R}_+, X)$, the solution to (ACP) is differentiable, takes values in the domain of A, and satisfies

$$||\frac{d}{dt}u||_{L^p(\mathbb{R}_+,X)} + ||Au||_{L^p(\mathbb{R}_+,X)} \leq C||f||_{L^p(\mathbb{R}_+,X)}$$

This thesis' goal is to prove Maximal L^p-regularity for an abstract operator A that satisfies certain operator-theoretic conditions. This should be understood in the context of two other factors:

1. It is possible to verify (using techniques from PDE theory and harmonic analysis) that many relevant differential operators do satisfy these operator-theoretic conditions. See, for example, Chapters 10 and 14 of [29], Chapter 8 of [19], or [1].

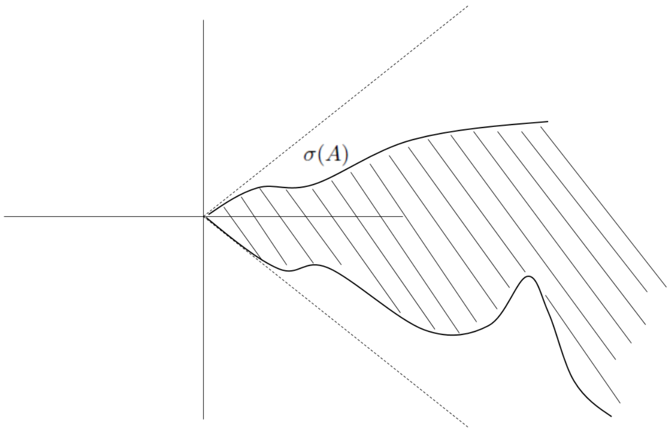
2. Maximal L^p-regularity may be used to study nonlinear PDEs through methods such as linearisation techniques and fixed point theorems. See, for example, 1.3 in [29], or [42].

These factors motivate Maximal L^p-regularity as a worthwhile area of study, and are themselves beyond the scope of this thesis.

Before proving Maximal L^p-regularity, we investigate what a solution to (ACP) might look like. If one were to pretend that (ACP) is a scalar ODE (treating u and f as scalar-valued functions and A as a complex number), then the unique solution would be given by the classical variation of constants formula:

$$u(t) := e^{-tA}u_0 + \int_0^t e^{-(t-s)A}f(s)ds$$

In fact, we may meaningfully define the above formula, and prove that it is indeed the unique solution to (ACP). In particular, we will define e^{-tA} using a functional calculus inspired by the Cauchy integral formula. We consider *sectorial* operators A, which (among other conditions) have spectrum $\sigma(A)$ contained in a sector $\Sigma_\theta = \{z \in \mathbb{C}, |\arg z| < \theta\}$.



Then, given a function φ that is bounded and holomorphic on a slightly larger sector Σ_Θ, we may define formally:

$$\varphi(A) := \frac{1}{2\pi i} \int_{\partial \Sigma_\Theta} \varphi(z)(z-A)^{-1} dz$$

We say that the operator A has a *McIntosh calculus*[1] of angle Θ if there exists $\Theta < \pi/2$ such that $\varphi(A)$ makes sense for all relevant functions φ. The construction is described in detail in Chapter 4.

One outcome of this construction is that we may define e^{-tA} via the McIntosh calculus. We will verify in Chapter 5 that this object behaves like a scalar exponential in many ways, and that in particular, the variation of constants formula is in fact the solution to (ACP). We will then show that proving Maximal L^p-regularity for A amounts to proving that the map

$$Kf = A \int_0^t e^{-(t-s)A} f(s) ds$$

defines a bounded operator $K : L^p(\mathbb{R}_+, X) \to L^p(\mathbb{R}_+, X)$. This map is a convolution; recall that the inverse Fourier transform of $\hat{f}\hat{g}$ is the convolution $f * g$ (for scalar functions). Motivated by this, we show that K may be written as $Kf = \mathcal{F}^{-1}[m\hat{f}]$ for a suitable Banach-space valued function m. Such a map is called a *Fourier multiplier*, and the classical Mihlin multiplier theorem relates the function m to boundedness of the corresponding Fourier multiplier. In Chapters 2 and 3 we will provide the background and proof of a Banach-valued multiplier theorem, proven by Weis in [48]. Armed with this theorem, we will obtain necessary and sufficient conditions for A to have (deterministic) Maximal L^p-regularity. In particular, we

[1] This construction stems from Alan McIntosh in [35] and is referred to in the literature as 'the H^∞-functional calculus.' In memory of Alan and his significant contribution to mathematics, this thesis shall instead use the term 'McIntosh calculus.'

will see that any operator A with a McIntosh calculus also has Maximal L^p-regularity. As mentioned, proving that any particular differential operator has a McIntosh calculus uses PDE and harmonic analysis techniques and is beyond the scope of the thesis.

In order to prove the Weis multiplier theorem, we will make extensive use of a concept called 'R-boundedness' that concerns randomly weighted sums of elements of Banach spaces. Thus even in this determinstic setting, we already make use of probability. This suggests that perhaps an analogue of Maximal L^p-regularity might transfer readily to the stochastic case, and indeed this is true. Chapters 6 and 7 construct a infinite dimensional Brownian motion (denoted $B_H(t)$) and develop a theory of Banach-valued stochastic integration, modelled on the Itô integral. We will prove an analogue of the Itô isometry with respect to a new norm called the 'γ-norm,' based on sums of elements in X randomly weighted by Gaussian variables.

With this construction in hand, we may write a Stochastic PDE:

$$dU(t) + AU(t)dt = F(t)dB_H(t)$$
$$U(0) = u_0$$

with the solution formally defined as the stochastic variation of constants formula:

$$U(t) := e^{-tA}u_0 + \int_0^t e^{-(t-s)A}F(s)dB_H(s)$$

In the stochastic case, Brownian motion is not Hölder continuous of parameter α for any $\alpha > 1/2$. This results in a loss of regularity, and we will see that stochastic Maximal L^p-regularity amounts to proving

$$||A^{\frac{1}{2}}u||_{L^p(\mathbb{R}_+,X)} \leq C||F||_{L^p(\mathbb{R}_+,X)}$$

Theorem 1.1 from [43] shows that the above estimate holds in the case $X = L^q(\mathcal{M})$ for any measure space $\mathcal{M}$, provided that $p \in (2,\infty)$, $q \in [2,\infty)$ and A has a McIntosh calculus. To prove the theorem we follow [44] and construct an extension of the McIntosh calculus, which this thesis calls the *R-McIntosh calculus*,[2].' This is defined for operator valued, bounded, holomorphic functions $\Phi : \Sigma_\Theta \to \mathcal{L}(X)$ and yields bounded operators $\Phi(A)$. Using the R-McIntosh calculus, we will show that the operator-valued function $z \to L_z$ defined by

$$L_zF = \int_0^t z^{\frac{1}{2}}e^{-(t-s)z}F(s)dB_H(s)$$

corresponds to a *bounded* operator $L_A(F)$ given by

$$F \to \int_0^t A^{\frac{1}{2}}e^{-(t-s)A}F(s)dB_H(s)$$

and the boundedness of this operator exactly ensures Stochastic Maximal L^p-Regularity. Thus the main theorem is proven.

[2]The literature refers to this as the operator-valued H^∞-calculus

Chapter 1

Calculus of Banach-Valued Functions

In order to understand the chapters that follow, we will need to understand how to differentiate and integrate Banach-valued functions. We will only need understand Banach-valued functions with scalar domains, and consequently the differentiation theory is very similar to the scalar case. A Banach-valued integration theory on the other hand, differs from the scalar Lebesgue integral in some unexpected ways, and so this shall make up the majority of the chapter.

1.1 Banach-Valued Differentiation

In this and later chapters, let X be a Banach space.

Here we are considering functions defined on $\mathbb{R}$ and $\mathbb{C}$, and so the derivative can be defined analogously to the scalar case without difficulty:

Definition 1.1. A function $f : \mathbb{R} \to X$ is *differentiable* at a point $t \in \mathbb{R}$ if

$$\lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

exists (as a limit with respect to the norm of X). We write the derivative of f at t as $f'(t)$. The function f is *differentiable* if $f'(t)$ exists for all $t \in \mathbb{R}$.

Definition 1.2. A function $f : \mathbb{C} \to X$ is *complex-differentiable* at a point $z \in \mathbb{C}$ if

$$\lim_{h \to 0} \frac{f(z+h) - f(t)}{h}$$

exists for all sequences $h_n \to 0$ in $\mathbb{C}$. A function f is *holomorphic* on a set $U \subset \mathbb{C}$ if it is complex-differentiable at every point in U.

Since the definition of differentiation is so similar to the scalar case, we will not prove various Banach analogues of differentiation formulae such as the product rule, the chain rule, Taylor's theorem, and various Complex Analysis results, and refer to such results in later chapters without comment.

1.2 Bochner Integration

1.2.1 Measurability

Let $(\mathcal{M}, \mathcal{A}, \mu)$ be a measure space. In this chapter, all norms are taken with respect to the Banach space X unless otherwise specified.

Definition 1.3. A function $f : \mathcal{M} \to X$ is called *simple* if it is a finite sum of functions of the form $\mathbb{1}_{A_n} x_n$, with $A_n \in \mathcal{A}$ such that $\mu(A_n) < \infty$ and $x_n \in X$.

Definition 1.4. A function $f : \mathcal{M} \to X$ is *measurable* if $f^{-1}(B)$ is measurable for every Borel set $B \subset X$.

In scalar measure theory, all measurable functions are the limit of simple functions. Unfortunately, in a Banach space this is not always true. For example, let X be a Banach space that is not separable. Then the identity map on X is clearly measurable, but there does not exist a sequence of simple functions converging pointwise to the identity. Indeed, suppose such a sequence I_n existed; then any point in X would be the limit of values taken by these functions I_n. But the range of the I_n is a countable set, contradicting the fact that X is not separable. To account for this, we must split the concept of measurability into three concepts. The first of these, measurability, was stated above in Definition 1.4. The second is:

Definition 1.5. A function $f : \mathcal{M} \to X$ is *strongly measurable* if there exists a sequence of simple functions $f_n : \mathcal{M} \to X$ such that $f_n(s) \to f(s)$ for almost every $s \in \mathcal{M}$.

Strong measurability is a strictly stronger requirement than measurability:

Proposition 1.6. *Any strongly measurable function $f : \mathcal{M} \to X$ is equal a.e. to a measurable function.*

Proof. Let A be a measurable set such that $f_n(s) \to f(s)$ on A and $\mu(A^c) = 0$. It suffices to show that $f^{-1}(U) \cap A \in \mathcal{A}$ for all open sets $U \subset X$. Take such a U and choose a sequence of simple functions converging pointwise to f. Let $d(x, U) := \inf_{y \in U} ||x - y||_X$. For $r > 0$, define

$$U_r := \{x \in U; d(x, U^c) > r\}$$

Then $f_n^{-1}(U^c) \in \mathcal{A}$ by the definition of a simple function as a sum of characteristic functions. Then we can write:

$$\begin{aligned}
f^{-1}(U) \cap A &= \{s \in \mathcal{M} \mid d(f(s), U^c) > \frac{1}{m} \text{ for some } m\} \cap A \\
&= \bigcup_{m \geq 1} \{s \in \mathcal{M} \mid \exists n \text{ s.t. } \forall k > n, d(f_k(s), U^c) > \frac{1}{m}\} \cap A \\
&= \bigcup_{m \geq 1} \left(\bigcup_{n \geq 1} \bigcap_{k \geq n} f_k^{-1}(U_{\frac{1}{m}}) \right) \cap A
\end{aligned}$$

This set is therefore a combination of countable unions and intersections, and therefore is also measurable, and so f is measurable on A. $\square$

There is a third, weaker form of measurability:

Definition 1.7. A function $f : \mathcal{M} \to X$ is *weakly measurable* if $s \to \langle f(s), x^* \rangle$ is measurable for all $x^* \in X^*$

Since $s \to \langle f(s), x^* \rangle$ are scalar functions, we may identify functions that are equal almost everywhere as in the scalar case. Measurable functions are weakly measurable, by continuity of the linear functionals x^* (and therefore by Proposition 1.6, strongly measurable functions are weakly measurable).

We state an important theorem that relates these three concepts of measurability:

Definition 1.8. Let $Y \subset X^*$, and define $S = \{x^* \in Y; ||x^*||_{X^*} = 1\}$. Then Y is a *norming subspace* for X if for all $x \in X$ $||x||_X = \sup_S |\langle x, x^* \rangle|$.

Theorem 1.9. Pettis Measurability Theorem
Let $(\mathcal{M}, \mathcal{A}, \mu)$ be a σ-finite measure space and X be a Banach space with a norming subspace $Y \subset X^*$. Then for any function $f : \mathcal{M} \to X$, the following are equivalent:
(1) f is strongly measurable.
(2) f is weakly measurable and separably valued, *meaning there exists a separable closed subspace $X_0 \subset X$ such that $f(s) \in X_0$ a.e.*
(3) f is separably valued and $s \to \langle f(s), x^* \rangle$ is measurable for all $x^* \in Y$.

Proof.
(1) $\implies$ (2): We have already seen that every strongly measurable function is weakly measurable. To show that f is separably valued, take a sequence (f_n) of simple functions converging to f, and let X_0 be the closed subspace spanned by the (countable) set of values taken by these simple functions. Then clearly X_0 is separable and $f(s) \in X_0$ for almost every $s \in \mathcal{M}$.

(2) $\implies$ (3): This is clear because $Y \subset X^*$.

(3) $\implies$ (1) : Let X_0 be the closed, separable space in which f takes values almost everywhere. Let $(x_n)_{n \in \mathbb{N}}$ be a dense sequence in X_0. Define $P_n : X_0 \to \{x_1, \dots x_n\}$ by:

$$P_n(x) = x_k, \text{ where } ||x - x_k|| \leq ||x - x_j|| \text{ for all } j \leq n.$$

If more than one x_k satisfies the above property, we choose the one with the least value of k. Note that since the $(x_n)_{n \in \mathbb{N}}$ is dense, $\lim_n P_n(x) = x$ for all $x \in X_0$. Now, since $f(s) \in X_0$ almost everywhere, there exists some measurable set A such that $f(A) \subset X_0$ and $\mu(A^c) = 0$. So we can choose some $x_0 \notin X_0$ and define $f_n : \mathcal{M} \to X$ via:

$$\begin{aligned} f_n(s) &:= P_n(f(s)) \text{ if } s \in A \\ &:= x_0 \qquad\quad \text{ if } s \in A^c \end{aligned}$$

Then for $0 < k \leq n$ we have

$$\begin{aligned} \{s \in \mathcal{M}; f_n(s) = x_k\} = A &\cap \{s \in \mathcal{M}; ||f(s) - x_k|| \leq ||f(s) - x_j|| \text{ for all } k \leq j \leq n\} \\ &\cap \{s \in \mathcal{M}; ||f(s) - x_k|| < ||f(s) - x_j|| \text{ for all } j < k\} \end{aligned}$$

Next we want to show that the other two sets on the right hand side are measurable sets in $(\mathcal{M}, \mathcal{A}, \mu)$. To do this we will use a lemma:

Lemma 1.10. *Let X_0 be a separable subspace of X, and Y be a linear subspace of X^* that is norming for X_0. Then Y contains a sequence (y_n) of unit vectors that are norming for X_0.*

Proof. Choose a dense sequence $(x_n)_{n\in\mathbb{N}} \subset X_0$. Since Y is norming, we can choose a sequence of unit vectors $(y_n)_{n\in\mathbb{N}} \subset Y$ such that $|\langle x_n, y_n\rangle| \geq (1-\epsilon_n)||x_n||$ for all n, with $\epsilon_n \to 0$. Then in fact this sequence $(y_n)_{n\in\mathbb{N}}$ is norming. To see this, for $x \in X_0$ and $\delta > 0$, choose N such that $\epsilon_N < \delta$ and $||x - x_N|| \leq \delta$. Then:

$$\begin{aligned}
(1-\delta)||x|| &\leq (1-\epsilon_N)||x|| \\
&\leq (1-\epsilon_N)||x_N|| + (1-\epsilon_N)||x - x_N|| \\
&\leq (1-\epsilon_N)||x_N|| + (1-\epsilon_N)\delta \\
&\leq |\langle x_N, y_N\rangle| + \delta \\
&\leq |\langle x - x_N, y_N\rangle| + |\langle x, y_N\rangle| + \delta \\
&\leq ||y_N|| \cdot ||x - x_N|| + |\langle x, y_N\rangle| + \delta \\
&\leq |\langle x, y_N\rangle| + 2\delta
\end{aligned}$$

Since this is true for arbitrary δ, we see that $||x|| \leq \sup_n |\langle x, y_n\rangle|$ and so $(y_n)_{n\in\mathbb{N}}$ is norming. □

With this lemma in hand, we note that our assumptions from (3) yield a space Y that is norming for X and therefore also norming for X_0. So by the lemma, we may choose a sequence of unit vectors $(y_n)_{n\in\mathbb{N}}$ in Y that are norming for X_0. Then for all $x \in X_0$, the function

$$\phi_x(s) = ||f(s) - x|| = \sup_n |\langle f(s) - x, y_n\rangle|$$

is measurable with respect to $(\mathcal{M}, \mathcal{A}, \mu)$ (by our assumption that f is weakly measurable). So this means our sets from before can be written as:

$$\{s \in \mathcal{M}; ||f(s) - x_k|| \leq ||f(s) - x_j|| \text{ for all } k \leq j \leq n\} = \{s \in \mathcal{M}; \phi_{x_k}(s) \leq \phi_{x_j}(s) \text{ for all } k \leq j \leq n\}$$
$$\{s \in \mathcal{M}; ||f(s) - x_k|| < ||f(s) - x_j|| \text{ for all } j < k\} = \{s \in \mathcal{M}; \phi_{x_k}(s) < \phi_{x_j}(s) \text{ for all } 1 < j < k\}$$

Therefore these sets are in $\mathcal{A}$. Therefore the functions f_n are measurable functions, and clearly they are also simple (since f_n has range consiting of n points).Finally, we note that outside the set A^c which has measure zero, we have:

$$||f_n(s) - f(s)||_X = ||P_n(f(s)) - f(s)||_X \to 0$$

Therefore $f_n \to f$ almost everywhere, and so f is strongly measurable. □

Corollary 1.11. *The three notions of measurability are equivalent on a separable Banach space. In particular, every continuous function is strongly measurable on a separable Banach space.*

Proof. We need only to show that on a separable Banach space X, there exists a norming subspace $Y \subset X^*$, and we may simply choose $Y = X^*$. This space is norming, indeed, choose a dense sequence $(x_n)_{n\in\mathbb{N}} \subset X$ and for each n choose x_n^* with norm one such that $\langle x_n, x_n^*\rangle = ||x_n||$.

Then for arbitrary x in X:

$$|\langle x, x_n^* \rangle| - ||x|| \leq |\langle x - x_n, x_n^* \rangle| + ||x_n|| - ||x||$$

and this clearly can be made arbitrarily small since $(x_n)_{n\in\mathbb{N}}$ is dense. $\square$

As we are about to see, we are concerned only with strong measurability when discussing Banach-valued integration. However, in our later work we will primarily study Banach spaces of the form $X = L^p(\mathcal{M})$, and these spaces are separable. Therefore we will treat continuous functions mapping into these spaces as strongly measurable without further comment.

1.2.2 Integration

We construct an integral operator acting on functions taking values in a Banach space. The process is motivated by the construction of the Lebesgue integral, and will be quite familiar.

Definition 1.12. A *simple function* $f : \mathcal{M} \to X$ is a function of the form

$$f = \sum_{n=1}^{N} \mathbb{1}_{A_n} x_n$$

for some $N \in \mathbb{N}$, $(x_n)_{n=1}^N \subset X$ and measurable sets $A_n \subset \mathcal{M}$ satisfying $\mu(A_n) < \infty$ for each n.

Definition 1.13. Define the Bochner integral of a simple function as:

$$\int \sum_{n=1}^{N} \mathbb{1}_{A_n} x_n d\mu = \sum_{n=1}^{N} \mu(A_n) x_n$$

The following facts are routine to check:

(1) This definition is independent of the representation of f.

(2) The Triangle Inequality: $\left\|\int f d\mu\right\| \leq \int ||f|| d\mu$

(3) Linearity: $\int f d\mu + \int g d\mu = \int f + g d\mu$ and $\int c f d\mu = c \int f d\mu$

Next, we extend the Bochner integral to strongly measurable functions:

Definition 1.14. A strongly measurable function $f : \mathcal{M} \to X$ is *Bochner integrable* (with respect to μ) if there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that

$$\int ||f - f_n|| d\mu \to 0$$

Note that $f - f_n$ is measurable (defined almost everywhere, following 1.6),and so by continuity of the norm, $s \to ||f(s) - f_n(s)||$ is measurable, and the integral above is well-defined.

Now, for simple functions $f_n \to f$ we have:

$$\left\|\int f_n d\mu - \int f_m d\mu\right\| \leq \int ||f_n - f_m|| d\mu \leq \int ||f - f_n|| d\mu + \int ||f - f_m|| d\mu$$

which goes to zero as $m, n \to \infty$, therefore $\left(\int f_n d\mu\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, and we can define:

$$\int f d\mu := \lim_{n\to\infty} \int f_n d\mu$$

Note that this limit does not depend on the approaching sequence. Indeed, take another sequence $\phi_n \to f$ almost everywhere. Then $f_n - \phi_n$ converges to 0 almost everywhere, and we can apply the same argument as above to see:

$$\left\|\int f_n d\mu - \int \phi_n d\mu\right\| = \left\|\int f_n - \phi_n d\mu\right\| \leq \int ||f - f_n||d\mu + \int ||f - \phi_n||d\mu \to 0$$

And so $(\int f_n d\mu)_{n\in\mathbb{N}}$ and $(\int \phi_n d\mu)_{n\in\mathbb{N}}$ converge to the same limit.

Next, we show that our definition of Bochner integrability gives us all the functions we would expect.

Proposition 1.15. *A strongly measurable function $f : \mathcal{M} \to X$ is Bochner integrable if and only if*

$$\int ||f||d\mu < \infty$$

Proof. If f is Bochner integrable, then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions converging pointwise a.e. to f, with $\int ||f - f_n||d\mu \to 0$. Choose some f_n such that $||f - f_n||$ is integrable, and we have:

$$\int ||f||d\mu \leq \int ||f - f_n||d\mu + \int ||f_n||d\mu < \infty$$

Conversely, let f be a strongly measurable function satisfying $\int ||f||d\mu < \infty$, and choose simple functions $f_n \to f$ pointwise almost everywhere. Then define:

$$\phi_n(s) := f_n(s)\mathbb{1}_{\{||f_n(s)|| \leq ||f(s)||\}}$$

Now $||\phi_n(s)|| \leq 2||f(s)||$, therefore $||\phi_n(s) - f(s)|| \leq 3||f(s)||$, but also $||\phi_n(s) - f(s)|| \to 0$ (each of these three statements holding a.e.). Therefore we can apply the dominated convergence theorem from scalar integration to see:

$$\int ||\phi_n(s) - f(s)||d\mu \to 0$$

Therefore f is Bochner integrable. ∎

1.2.3 Integration Results

We reprove various results from scalar integration theory in this new setting.

Theorem 1.16. Dominated Convergence Theorem:
Take a sequence of function $f_n : \mathcal{M} \to X$ that are Bochner integrable, converging to some f pointwise a.e. and satisfying $||f_n(s)|| \leq |g(s)|$ a.e. for some integrable function $g : \mathcal{M} \to \mathbb{R}$. Then f is Bochner integrable and

$$\lim_{n\to\infty} \int ||f_n - f|| d\mu = 0$$

which implies that

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$$

Proof. We have $||f_n(s) - f(s)|| \leq ||f_n(s)|| + ||f(s)|| \leq 2|g|$ almost everywhere, and converging pointwise to zero, therefore we may apply the scalar dominated convergence theorem to obtain the result. □

Theorem 1.17. *Let X, Y be Banach spaces, $f : \mathcal{M} \to X$ be Bochner integrable, and $T : X \to Y$ be a closed linear operator with domain $D \subset X$. Suppose that f takes values in D (a.e.), and that $Tf : \mathcal{M} \to Y$ is also Bochner integrable. Then f is integrable as a D-valued function, $\int f d\mu \in D$, and:*

$$T \int f d\mu = \int T f d\mu$$

Proof. First, we make the observation: if $f_1 : \mathcal{M} \to X$ and $f_2 : \mathcal{M} \to Y$ are Bochner integrable, then $f := (f_1, f_2) : \mathcal{M} \to X \times Y$ is Bochner integrable because $\int ||(f_1, f_2)|| d\mu = \int ||f_1|| + ||f_2|| d\mu < \infty$ and therefore we have:

$$\int f d\mu = \left(\int f_1 d\mu, \int f_2 d\mu \right)$$

Therefore the map $F : \mathcal{M} \to X \times Y$ given by $F(s) = (f(s), Tf(s))$ is Bochner integrable and takes its values in the graph $\Gamma(T)$. Therefore since T is closed and the Bochner integral is defined as a limit, $\int F d\mu$ is also in $\Gamma(T)$. This exactly tells us that

$$\left(\int f d\mu, \int T f d\mu \right) \in \Gamma(T)$$

which means that $\int f d\mu \in D$ and $T \int f d\mu = \int T f d\mu$. □

Note in particular that all bounded linear operators commute with the Bochner integral.

Theorem 1.18. Fubini's Theorem:
Let $(\mathcal{M}, \mathcal{A}, \mu)$ and $(\mathcal{N}, \mathcal{B}, \nu)$ be σ-finite measure spaces, and let $f : \mathcal{M} \times \mathcal{N} \to X$ be Bochner integrable. Then:
(1) For a.e. $t \in \mathcal{M}$, $s \to f(s,t)$ is Bochner integrable in $\mathcal{M}$,
(2) For a.e. $s \in \mathcal{N}$, $t \to f(s,t)$ is Bochner integrable in $\mathcal{N}$,
(3) We have:

$$\int_{\mathcal{M}\times\mathcal{N}} f(s,t)d(\mu \times \nu) = \int_{\mathcal{N}} \int_{\mathcal{M}} f(s,t)d\mu d\nu = \int_{\mathcal{M}} \int_{\mathcal{N}} f(s,t)d\nu d\mu$$

Proof. We use Proposition 1.15 to convert to the scalar case: if $f : \mathcal{M} \times \mathcal{N} \to X$ is Bochner integrable, then $(s,t) \to ||f(s,t)||$ is integrable. By the scalar Fubini's theorem, this means that $s \to ||f(s,t)||$ and $t \to ||f(s,t)||$ are integrable for fixed t, s respectively, which in turn means that $s \to f(s,t)$ and $t \to f(s,t)$ are integrable.

Next, for (arbitrary) $x^* \in X^*$ define $T_{x^*} f = \langle f, x^* \rangle$. Then, again by scalar Fubini:

$$\int_{\mathcal{M}\times\mathcal{N}} T_{x^*} f(s,t)d(\mu \times \nu) = \int_{\mathcal{N}} \int_{\mathcal{M}} T_{x^*} f(s,t)d\mu d\nu = \int_{\mathcal{M}} \int_{\mathcal{N}} T_{x^*} f(s,t)d\nu d\mu$$

By Theorem 1.17, this is equivalent to:

$$T_{x^*} \int_{\mathcal{M}\times\mathcal{N}} f(s,t)d(\mu \times \nu) = T_{x^*} \int_{\mathcal{N}} \int_{\mathcal{M}} f(s,t)d\mu d\nu = T_{x^*} \int_{\mathcal{M}} \int_{\mathcal{N}} f(s,t)d\nu d\mu$$

And this gives our result, since the Hahn-Banach theorem tells us that $T_{x^*}x = T_{x^*}y$ for all $x^* \in X^*$ if and only if $x = y$. $\square$

Lemma 1.19. *Take a Banach space X and a function $f : \mathcal{M} \to \mathcal{L}(X)$ that is Bochner integrable. Then for $x \in X$:*

$$\left(\int f(s)d\mu \right) x = \int f(s)xd\mu$$

Proof. Choose a sequence of simple functions $f_n \to f$ such that $\int ||f_n - f||_{\mathcal{L}(X)} d\mu \to 0$. Then $\int f_n d\mu \to \int f d\mu$ in $\mathcal{L}(X)$ norm, therefore $\left(\int f_n d\mu\right) x \to \left(\int f d\mu\right) x$ for each $x \in X$. On the other hand, since f_n is simple, we know that $\left(\int f_n d\mu\right) x = \int f_n(s) x d\mu$, and that:

$$\int ||f_n(s)x - f(s)x||_X d\mu \leq ||x||_X \int ||f_n(s) - f(s)||_{\mathcal{L}(X)} d\mu \to 0$$

And therefore $\int f(s)xd\mu = \lim_n \int f_n(s)xd\mu = \lim_n \left(\int f_n d\mu\right) x$. The result follows by uniqueness of limits. $\square$

Finally we point out, but do not prove, that a version of the Fundamental Theorem of Calculus still holds:

Theorem 1.20. Fundamental Theorem of Calculus:
Let $f : \mathbb{R} \to X$ be differentiable almost everywhere such that $f' \in L^1_{loc}(\mathbb{R}; X)$. Then for almost every $s < t \in \mathbb{R}$ we have

$$f(t) - f(s) = \int_s^t f'(s)ds$$

Conversely, for $a \in \mathbb{R}$ and $f \in L^1_{loc}(\mathbb{R}; X)$, define

$$g(t) := \int_a^t f(s)ds$$

Then g is differentiable and $g' = f$.

Proof. This is Lemma 2.50 in [23], as part of Section 2.3.c concerning differentiability of Banach-valued functions. □

The results in the section above hold for the Lebesgue integral and should be not be surprising to the reader. Therefore we shall often use these results in later chapters without comment or reference.

1.2.4 L^p Spaces

Two functions $f, g : \mathcal{M} \to X$ are called *equivalent* if they are equal almost everywhere. This defines an equivalence relation on the set of measurable functions $\mathcal{M} \to X$. We then have the definition:

Definition 1.21. For $p \in [1, \infty)$, the space $L^p(\mathcal{M}; X)$ is defined as the set of strongly measurable functions $f : \mathcal{M} \to X$ under the equivalence relation $f \sim g$ if $f = g$ a.e., such that these functions satisfy:

$$\int_{\mathcal{M}} ||f(\mu)||_X^p d\mu < \infty$$

We define $L^\infty(\mathcal{M}; X)$ as the set of strongly μ-measurable functions $f : \mathcal{M} \to X$ such that for any given f, there exist $M > 0$ such that $||f(s)||_X < M$ almost everywhere.

As in the scalar case, we endow these spaces with the norms:

$$||f||_{L^p(\mathcal{M};X)} := \left(\int_{\mathcal{M}} ||f||_X^p d\mu \right)^{\frac{1}{p}}$$
$$||f||_{L^\infty(\mathcal{M};X)} := \inf \{M \geq 0 \mid \mu\{s \text{ s.t. } ||f(s)||_X > M\} = 0\}$$

It is routine to show that these are in fact norms, and that $L^p(\mathcal{M}; X)$ and $L^\infty(\mathcal{M}; X)$ are complete and therefore Banach spaces.

Remark 1.22. Note in particular that, by definition, simple functions are dense in $L^p(\mathcal{M}; X)$ for all $p \in [1, \infty]$. We shall frequently use this fact in later chapters without comment.

Remark 1.23. Beyond our first discovery that the notion of measurability splits into three, our work on Banach-valued integration has not yielded surprising results. However, our scalar intuition fails in the case of Banach space duality. In particular, if we let $\frac{1}{p} + \frac{1}{p'} = 1$, the isomorphism

$$(L^p(\mathcal{M}, X))^* \simeq L^{p'}(\mathcal{M}, X^*)$$

does *not* hold for general Banach spaces. Understanding the reasons behind this is nontrivial, and beyond the scope of this thesis - such discussions may be found in Section 1.3 of [23]. For our purposes, we shall screen off such considerations by referencing the theorem:

Theorem 1.24. *Let X be a Banach space such that X^* is separable. Then for all $p, p' \in (1, \infty)$ such that $\frac{1}{p}+\frac{1}{p'} = 1$, the spaces $(L^p(\mathcal{M}, X))^*$ and $L^{p'}(\mathcal{M}, X^*)$ are isomorphic as Banach spaces. In particular, for any $q, q' \in (1, \infty)$ such that $\frac{1}{q} + \frac{1}{q'} = 1$, we have*

$$(L^p(\mathcal{M}, L^q(\mathcal{N})))^* \simeq L^{p'}(\mathcal{M}, L^{q'}(\mathcal{N}))$$

Proof. This may be obtained by combining Theorem 1.84 with Theorem 1.95 in [23]. It is important to note that the separability of X^* is merely a convenient condition that enables our work in later chapters, and the 'true' theory is found in the original text. □

We shall therefore take the dual of spaces of the form $L^p(\mathcal{M}, L^q(\mathcal{N}))$ as above without comment in later chapters. We conclude with a useful result on spaces of the form $L^p(\mathcal{M}; L^q(\mathcal{N}))$

Proposition 1.25 (Minkowski's Integral Inequality)**.** *Take $\mathcal{M}, \mathcal{N}$ measure spaces, $F : \mathcal{M} \times \mathcal{N} \to \mathbb{R}$ and $\alpha \geq 1$. Then we have*

$$\left(\int_{\mathcal{M}} \left|\int_{\mathcal{N}} F(\mu,\nu) d\mu\right|^{\alpha} d\nu\right)^{\frac{1}{\alpha}} \leq \int_{\mathcal{N}} \left(\int_{\mathcal{M}} |F(\mu,\nu)|^{\alpha} d\nu\right)^{\frac{1}{\alpha}} d\mu$$

Proof. This is a classical result, and may be found as (for example), Theorem 202 in [21]. □

Corollary 1.26. *Let $\mathcal{M}$ and $\mathcal{N}$ be measure spaces and $p > q$. Then given a function f in $L^q(\mathcal{N}; L^p(\mathcal{M}))$, identify f with the function $f : \mathcal{M} \to L^q(\mathcal{N})$ defined by $\mu \to f(\nu)(\mu)$. Then f is in $L^p(\mathcal{M}; L^q(\mathcal{N}))$ with $\|f\|_{L^p(\mathcal{M};L^q(\mathcal{N}))} \leq \|f\|_{L^q(\mathcal{N};L^p(\mathcal{M}))}$.*

Proof. Apply Minkowski's Integral Inequality with $F(\mu, \nu) = f(\mu)(\nu)$ and $\alpha = \dfrac{p}{q}$:

$$\begin{aligned}
\|f\|_{L^p(\mathcal{M};L^q(\mathcal{N}))} &= \left(\int_{\mathcal{M}} \left(\int_{\mathcal{N}} |f(\nu)(\mu)|^q d\nu\right)^{\frac{p}{q}} d\mu\right)^{\frac{1}{p}} \\
&\leq \left(\int_{\mathcal{N}} \left(\int_{\mathcal{M}} |f(\nu)(\mu)|^p d\nu\right)^{\frac{q}{p}} d\mu\right)^{\frac{1}{q}} \\
&= \|f\|_{L^q(\mathcal{N};L^p(\mathcal{M}))}
\end{aligned}$$

□

1.3 Further Reading

The short discussion on Banach-valued derivatives stems from the introductory sections of Chapter 2 in [10]. The work on measurability and integrability of Banach-valued functions follows ideas and results from Sections 1.1 and 1.2 in [23]. References to original papers (dating from the 1930s and later) may be found in the Notes for [23].

Chapter 2

R-Boundedness

Let $(r_n)_{n\in\mathbb{N}}$ be a set of independent, identically distributed variables on a probability space Ω, taking values ± 1 with equal probability $1/2$. Such variables are traditionally called 'Rademacher variables.' Then, given a sequence $(x_n)_{n=1}^{\infty}$ in a Banach space X, we may consider the 'randomised norm:'

$$\mathbb{E}\left\|\sum_{n=1}^{\infty} r_n x_n\right\|$$

Such norms will be used in upcoming chapters to understand a theory of stochastic integration of Banach-valued functions, and to prove Maximal Regularity in both the stochastic and the determinstic setting. In this chapter, we introduce and establish fundamental properties of this randomised norm.

2.1 The Khinchine-Kahane Inequality

The study of randomised norms is motivated by the very useful theorem:

Theorem 2.1. Khinchine-Kahane Inequality
Take a Banach space X. Then for all $N \in \mathbb{N}$, $(x_n)_{n=1}^{N} \subset X$ and $p, q \in [1,\infty)$ there exist constants $C_{p,q}$ and $C_{q.p}$ depending only on p, q and X such that

$$\frac{1}{C_{q,p}}\left(\mathbb{E}\left\|\sum_{n=1}^{N} r_n x_n\right\|_X^q\right)^{\frac{1}{q}} \leq \left(\mathbb{E}\left\|\sum_{n=1}^{N} r_n x_n\right\|_X^p\right)^{\frac{1}{p}} \leq C_{p,q}\left(\mathbb{E}\left\|\sum_{n=1}^{N} r_n x_n\right\|_X^q\right)^{\frac{1}{q}}$$

Proof. We will first prove two lemmas.

Lemma 2.2. Lévy's Inequality
Let $(Y_n)_{n=1}^{N}$ be independent symmetric *(that is, $Y_n \sim -Y_n$ for each n) random variables taking values in a Banach space X, each with mean zero. Let $S_k := \sum_{n=1}^{k} Y_n$. Then for all $\alpha \geq 0$ we have:*

$$\mathbb{P}\left(\max_{k\leq N} ||S_k|| > \alpha\right) \leq 2\mathbb{P}(||S_N|| > \alpha)$$

Proof. Fix $\alpha \geq 0$ and define $A_k \subset \Omega$ for each $k = 1, 2, \ldots N$ by:

$$A_k := \{||S_j|| \leq \alpha \text{ for } j < k \text{ and } ||S_k|| > \alpha\}$$

Then one can see that the $\{A_k\}_{k=1}^N$ are disjoint with union $A := \{\max_{k \leq N} ||S_k|| > \alpha\}$. Now:

$$S_N = S_k + \sum_{n=k+1}^{N} Y_n \text{ and } 2S_k - S_N = S_k - \sum_{n=k+1}^{N} Y_n$$

Since each Y_n has mean zero and the variables are symmetric, this means that $(Y_1, \ldots Y_k, S_N)$ and $(Y_1, \ldots Y_k, 2S_k - S_N)$ are identically distributed. Now, if $||S_k|| > \alpha$ then either $||S_N|| > \alpha$ or $||2S_k - S_N|| > \alpha$ (or both), and so:

$$\begin{aligned}
\mathbb{P}(A_k) &\leq \mathbb{P}(A_k \cap \{||S_N|| > \alpha\}) + \mathbb{P}(A_k \cap \{||2S_k - S_N|| > \alpha\}) \\
&= 2\mathbb{P}(A_k \cap \{||S_N|| > \alpha\})
\end{aligned}$$

And then, noting that the A_k cover $\{||S_N|| > \alpha\}$, we have:

$$\mathbb{P}(A) = \sum_{k=1}^{N} \mathbb{P}(A_k) \leq 2 \sum_{k=1}^{N} \mathbb{P}(A_k \cap \{||S_N|| > \alpha\}) = 2\mathbb{P}(||S_N|| > \alpha)$$

□

Lemma 2.3. *Take $x_1, \ldots x_n \in X$ and $\alpha > 0$. Then:*

$$\mathbb{P}\left(\left\|\sum_{n=1}^{N} r_n x_n\right\| > 2\alpha\right) \leq 4\ \mathbb{P}\left(\left\|\sum_{n=1}^{N} r_n x_n\right\| > \alpha\right)^2$$

Proof. Let us write $S_n := \sum_{n=1}^{N} r_n x_n$, and (as before) define $A_n \subset \Omega$ by:

$$A_n := \{||S_j|| \leq \alpha \text{ for } j < n \text{ and } ||S_n|| > \alpha\}$$

Then first I claim that:

$$\mathbb{P}(A_n \cap \{||S_N - S_{n-1}|| > \alpha\}) = \mathbb{P}\left(A_n \cap \{\|x_n + (S_N - S_n)\| > \alpha\}\right)$$

Notice that the right hand side is the intersection of independent events. Let us prove this:

$$\begin{aligned}
\mathbb{P}(A_n \cap \{||S_N - S_{n-1}|| > r\}) &= \mathbb{P}\left(A_n \cap \left\{\left\|\sum_{j=n}^{N} r_j x_j\right\| > r\right\}\right) \\
&= \mathbb{P}\left(A_n \cap \left\{\left\|r_n \sum_{j=n}^{N} r_j x_j\right\| > r\right\}\right) \\
&= \mathbb{P}\left(A_n \cap \left\{\left\|x_n + \sum_{j=n+1}^{N} r_n r_j x_j\right\| > r\right\}\right)
\end{aligned}$$

Noting that the $(r_j)_{j=n+1}^N$ are independent from r_n, we see that $||x_n + \sum_{j=n+1}^N r_n r_j x_j||$ and $||x_n + \sum_{j=n+1}^N r_j x_j||$ are identically distributed. So we have:

$$\begin{aligned}\mathbb{P}(A_n \cap \{||S_N - S_{n-1}|| > r\}) &= \mathbb{P}\left(A_n \cap \left\{\left\|x_n + \sum_{j=n+1}^N r_j x_j\right\| > r\right\}\right)\\ &= \mathbb{P}\left(A_n \cap \{\|x_n + (S_N - S_n)\| > r\}\right)\end{aligned}$$

By the same method one can show that $\mathbb{P}(||S_N - S_{n-1}|| > r) = \mathbb{P}\left(\|x_n + (S_N - S_n)\| > r\right)$.

Next, note that if $||S_{n-1}|| \leq \alpha$ and $||S_N|| > 2\alpha$ then $||S_N - S_{n-1}|| > \alpha$. Therefore we have, using our claim proven above:

$$\begin{aligned}\mathbb{P}(A_n \cap \{||S_N|| > 2\alpha\}) &\leq \mathbb{P}(A_n \cap \{||S_N - S_{n-1}|| > \alpha\})\\ &= \mathbb{P}(A_n \cap \{||x_n + S_N - S_n|| > \alpha\})\\ &= \mathbb{P}(A_n)\mathbb{P}(||x_n + S_N - S_n|| > \alpha)\\ &= \mathbb{P}(A_n)\mathbb{P}(||S_N - S_{n-1}|| > \alpha)\\ &\leq \mathbb{P}(A_n)\mathbb{P}(\max_{n-1\leq k} ||S_N - S_k|| > \alpha)\\ &\leq 2\mathbb{P}(A_n)\mathbb{P}(||S_N|| > \alpha).\end{aligned}$$

With the last inequality coming from Levy's Inequality (Lemma 2.2), modified slightly by changing the order of summation. Now, one can see that the $\{A_n\}_{n=1}^N$ are disjoint and cover the set $\{||S_N|| > \alpha\}$. Therefore we have from our work above and Lemma 2.2 a second time:

$$\begin{aligned}\mathbb{P}(||S_N|| > 2\alpha) &= \sum_{n=1}^N \mathbb{P}(A_n \cap \{||S_N|| > 2\alpha\})\\ &\leq 2\mathbb{P}(||S_N|| > \alpha)\sum_{n=1}^N \mathbb{P}(A_n)\\ &= 2\mathbb{P}(||S_N|| > \alpha)\mathbb{P}(\max_{n\leq N} ||S_n|| > \alpha)\\ &\leq 4\mathbb{P}(||S_N|| > \alpha)^2\end{aligned}$$

□

With these two lemmas in hand, we are now ready to prove the Khinchine-Kahane inequality. For reference, we restate it: for all $N \in \mathbb{N}$ and $x_1, \ldots x_N$ and $p, q \in [1, \infty)$ there exist constants $C_{p,q}$ and $C_{q.p}$ depending only on p, q and X such that

$$\frac{1}{C_{q,p}}\left(\mathbb{E}\left\|\sum_{n=1}^N r_n x_n\right\|_X^q\right)^{\frac{1}{q}} \leq \left(\mathbb{E}\left\|\sum_{n=1}^N r_n x_n\right\|_X^p\right)^{\frac{1}{p}} \leq C_{p,q}\left(\mathbb{E}\left\|\sum_{n=1}^N r_n x_n\right\|_X^q\right)^{\frac{1}{q}}$$

It is sufficient to prove the second inequality; the first comes from reversing the roles of p and q. Suppose we could prove the inequality for $q = 1$. Then, for any other $q > 1$, the result would follow by Hölder's inequality:

$$\left(\mathbb{E}\left\|\sum_{n=1}^{\infty} r_n x_n\right\|_X^p\right)^{\frac{1}{p}} \leq C_{p,1}\mathbb{E}\left\|\sum_{n=1}^{\infty} r_n x_n\right\|_X \leq C_{p,1}\mathbb{E}\left(\left\|\sum_{n=1}^{\infty} r_n x_n\right\|_X^q\right)^{\frac{1}{q}} \cdot \mathbb{E}\left(1^{q'}\right)^{\frac{1}{q'}}$$

Therefore it suffices to prove that for all $p \in (1, \infty)$:

$$\left(\mathbb{E}\left\|\sum_{n=1}^{\infty} r_n x_n\right\|_X^p\right)^{\frac{1}{p}} \leq C_{p,1}\mathbb{E}\left\|\sum_{n=1}^{\infty} r_n x_n\right\|_X$$

Fix N and $x_1, \ldots x_N$, write $S_N = \sum_{n=1}^N r_n x_n$, and let j be the unique integer such that $p \leq 2^j < 2p$. Then we may compute:

$$\begin{aligned}\mathbb{E}(||S_N||^p) &= \int_0^\infty \mathbb{P}(||S_N||^p > t)dt \\ &= \int_0^\infty \mathbb{P}(||S_N|| > t^{\frac{1}{p}})dt\end{aligned}$$

Change variables $t \to (2^j s)^p$, (which implies $dt \to 2^{jp} p s^{p-1} ds$):

$$= 2^{jp} p \int_0^\infty s^{p-1}\mathbb{P}(||S_N|| > 2^j s)ds$$

By repeatedly applying Lemma 2.3, we have $\mathbb{P}(||S_N|| > 2^j\alpha) \leq 4^{2^j-1}\mathbb{P}(||S_N|| > \alpha)^{2^j}$

$$\leq 2^{jp}4^{2^j-1}p \int_0^\infty s^{p-1}\mathbb{P}(||S_N|| > s)^{2^j}ds$$

Since $p \leq 2^j < 2p$:

$$\leq (2p)^p 4^{2p-1} p \int_0^\infty s^{p-1}\mathbb{P}(||S_N|| > s)^p ds$$

By Markov's inequality, for all $\alpha > 0$ we have $\alpha\mathbb{P}(||S_N|| > \alpha) \leq \mathbb{E}(||S_N||)$:

$$\begin{aligned}&\leq (2p)^p\mathbb{E}(||S_N||)^{p-1}4^{2p-1}p \int_0^\infty \mathbb{E}(||S_N||)^{p-1}\mathbb{P}(||S_N|| > s)ds \\ &= [(2p)^p 4^{2p-1}p] \times \mathbb{E}(||S_N||)^p\end{aligned}$$

Therefore $\mathbb{E}(||S_N||^p)^{\frac{1}{p}} \leq [(2p)^p 4^{2p-1}p]^{\frac{1}{p}}\mathbb{E}(||S_N||)$. This constant depends only on p, and this completes the proof. $\square$

Corollary 2.4. Kahane's Contraction Principle
For all $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ such that $|a_n| \leq 1$ for all n, and for all $p \in [1,\infty)$ we have

$$\mathbb{E}\left(\left\|\sum_{n=1}^{N} r_n a_n x_n\right\|^p\right)^{\frac{1}{p}} \leq 2\,\mathbb{E}\left(\left\|\sum_{n=1}^{N} r_n x_n\right\|^p\right)^{\frac{1}{p}}$$

Proof. First note that if each $a_n = \pm 1$ we trivially have $\mathbb{E}\left\|\sum_{n=1}^{N} r_n a_n x_n\right\| = \mathbb{E}\left\|\sum_{n=1}^{N} r_n x_n\right\|$.

Next, if each $a_n = 0$ or 1, we may write

$$\mathbb{E}\left\|\sum_{n=1}^{N} r_n a_n x_n\right\| \leq \frac{1}{2}\mathbb{E}\left\|\sum_{n=1}^{N} r_n x_n\right\| + \frac{1}{2}\mathbb{E}\left\|\sum_{n=1}^{N} r_n (2a_n - 1) x_n\right\| \leq \mathbb{E}\left\|\sum_{n=1}^{N} r_n x_n\right\|$$

by the previous case.

Next, if each $a_n \in [0,1]$ we write $a_n = \sum_{k=1}^{\infty} 2^{-k} a_{nk}$ with $a_{nk} = 0$ or 1. Then

$$\left\|\sum_{n=1}^{N} r_n a_n x_n\right\| \leq \sum_{k=1}^{\infty} 2^{-k}\left\|\sum_{n=1}^{N} r_n a_{nk} x_n\right\| \leq \left\|\sum_{n=1}^{N} r_n x_n\right\|$$

by the previous case. The result extends to each $a_n \in [-1,1]$ by the first case.

For complex a_n, we write $a_n = b_n + ic_n$, with $b_n, c_n \leq 1$ for each n. Then by the previous results

$$\left\|\sum_{n=1}^{N} r_n a_n x_n\right\| \leq \left\|\sum_{n=1}^{N} r_n b_n x_n\right\| + \left\|i\sum_{n=1}^{N} r_n c_n x_n\right\| \leq 2\left\|\sum_{n=1}^{N} r_n x_n\right\|$$

□

We can also extend Khinchine-Kahane to sums of Gaussian variables, which will be important when discussing the γ-norm and stochastic integration in Chapters 6-8.

Corollary 2.5. Khinchine-Kahane for Gaussian Variables
Take a Banach space X, and a sequence $(\gamma_n)_{n=1}^{\infty}$ of independent Gaussian variables with mean zero and variance 1. Then for all $N \in \mathbb{N}$, $x_1, \ldots x_N \in X$ and $p, q \in [1,\infty)$ there exist constants $C_{p,q}$ and $C_{q.p}$ depending only on p, q and X such that

$$\frac{1}{C_{q,p}}\left(\mathbb{E}\left\|\sum_{n=1}^{N} \gamma_n x_n\right\|_X^q\right)^{\frac{1}{q}} \leq \left(\mathbb{E}\left\|\sum_{n=1}^{N} \gamma_n x_n\right\|_X^p\right)^{\frac{1}{p}} \leq C_{p,q}\left(\mathbb{E}\left\|\sum_{n=1}^{N} \gamma_n x_n\right\|_X^q\right)^{\frac{1}{q}}$$

Proof. As in the Bernoulli case, we only prove the left inequality. Fix $k \in \mathbb{N}$ and define $\phi_{nk} := \frac{1}{\sqrt{k}}\sum_{j=1}^{k} r_{nk+j}$. Then for each k:

$$\begin{aligned}\left(\mathbb{E}\left\|\sum_{n=1}^{N}\phi_{nk}x_n\right\|_X^p\right)^{\frac{1}{p}} &= \left(\mathbb{E}\left\|\sum_{n=1}^{N}\sum_{j=1}^{k}r_{nk+j}\frac{x_n}{\sqrt{k}}\right\|_X^p\right)^{\frac{1}{p}}\\ &\leq C_{p,q}\left(\mathbb{E}\left\|\sum_{n=1}^{N}\sum_{j=1}^{k}r_{nk+j}\frac{x_n}{\sqrt{k}}\right\|_X^q\right)^{\frac{1}{q}}\\ &= C_{p,q}\left(\mathbb{E}\left\|\sum_{n=1}^{N}\phi_{nk}x_n\right\|_X^q\right)^{\frac{1}{q}}\end{aligned}$$

Then by the Central limit theorem, $(\phi_{1k},\dots\phi_{Nk}) \to (\gamma_1,\dots\gamma_N)$ in distribution as $k \to \infty$. It is possible to show this implies that

$$\left(\mathbb{E}\left\|\sum_{n=1}^{N}\phi_{nk}x_n\right\|_X^q\right)^{\frac{1}{q}} \to \left(\mathbb{E}\left\|\sum_{n=1}^{N}\gamma_n x_n\right\|_X^q\right)^{\frac{1}{q}}$$

A proof of this can be found in Lemma 3.13 and 3.14 of [41]. The proof is technical and uses techniques that are not extremely relevant to this thesis, so we shall omit it here. $\square$

2.2 R-Boundedness

Definition 2.6. A set $\mathcal{T} \subset \mathcal{L}(X,Y)$ is *R-bounded* if and only if for some $p \in [1,\infty)$, there exists a constant C (depending only on p, X, Y and $\mathcal{T}$) such that for all $N \in \mathbb{N}$, all $T_1,\dots T_m \in \mathcal{T}$ and all $x_1,\dots x_N \in X$, we have the estimate

$$\mathbb{E}\left\|\sum_{n=1}^{N}r_nT_nx_n\right\|_Y^p \leq C_p\ \mathbb{E}\left\|\sum_{n=1}^{N}r_nx_n\right\|_X^p$$

In this case, the inequality above will be true for all $p \in [1,\infty)$ (with a varying constant C_p). This follows from the Khinchine-Kahane inequality: suppose the statement is true for p, and take some other q, then:

$$\mathbb{E}\left\|\sum_{n=1}^{N}r_nT_nx_n\right\|_Y^q \lesssim \left(\mathbb{E}\left\|\sum_{n=1}^{N}r_nT_nx_n\right\|_Y^p\right)^{\frac{q}{p}} \lesssim \left(\mathbb{E}\left\|\sum_{n=1}^{N}r_nx_n\right\|_X^p\right)^{\frac{q}{p}} \lesssim \mathbb{E}\left\|\sum_{n=1}^{N}r_nx_n\right\|_X^q$$

Definition 2.7. Given an R-bounded set $\mathcal{T}$, we call its R-bound '$R_p(\mathcal{T})$' the smallest constant such that

$$\left(\mathbb{E}\left\|\sum_{n=1}^{N}r_nT_nx_n\right\|_Y^p\right)^{\frac{1}{p}} \leq R_p(\mathcal{T})\ \left(\mathbb{E}\left\|\sum_{n=1}^{N}r_nx_n\right\|_X^p\right)^{\frac{1}{p}}$$

for all $N \in \mathbb{N}$, all $T_1,\dots T_m \in \mathcal{T}$ and all $x_1,\dots x_N \in X$

Remark 2.8. The operators in $\mathcal{T}$ are uniformly bounded by $R_1(\mathcal{T})$, since we apply the above definition to a single x to obtain:

$$||Tx|| = \mathbb{E}\left(\|r_1 Tx\|_Y\right) \leq R_p(\mathcal{T})\ \mathbb{E}\left(\|r_1 x\|_X\right) = R_p(\mathcal{T})||x||$$

Proposition 2.9. *Basic facts about R-bounded sets:*

(1) If $\mathcal{T}' \subset \mathcal{T}$ then $\mathcal{T}'$ is also R-bounded, and $R_p(\mathcal{T}') \leq R_p(\mathcal{T})$, since every family $T_1, \ldots T_m \subset \mathcal{T}'$ is also in $\mathcal{T}$, and we have $\mathbb{E}\left(\|\sum_n r_n T_n x_n\|_Y^p\right) \leq R_p(\mathcal{T})^p\ \mathbb{E}\left(\|\sum_n r_n x_n\|_X^p\right)$.

(2) If $\mathcal{T}$ is R-bounded, then $\alpha\mathcal{T} = \{\alpha T \mid T \in \mathcal{T}\}$ is R-bounded and $R_p(\alpha\mathcal{T}) = |\alpha|^p R_p(\mathcal{T})$. Indeed, $\mathbb{E}\left(\|\sum_n r_n \alpha T_n x_n\|_Y^p\right) = |\alpha|\mathbb{E}\left(\|\sum_n r_n T_n x_n\|_Y^p\right) \leq |\alpha|^p R_p(\mathcal{T})^p\ \mathbb{E}\left(\|\sum_n r_n x_n\|_X^p\right)$.

(3) If $\mathcal{T}$ and $\mathcal{S}$ are R-bounded subets of $\mathcal{L}(X, Y)$, then $\mathcal{S} + \mathcal{T} := \{S + T; S \in \mathcal{S}, T \in \mathcal{T}\}$ is R-bounded with $R_p(\mathcal{T} + \mathcal{S}) \leq R_p(\mathcal{T}) + R_p(\mathcal{S})$. Indeed, verify this for $p = 1$:

$$\begin{aligned}
\mathbb{E}\left(\left\|\sum_{n=1}^m r_n(T_n + S_n)x_n\right\|_Y\right) &\leq \mathbb{E}\left(\left\|\sum_{n=1}^m r_n T_n x_n\right\|_Y + \left\|\sum_{n=1}^m r_n S_n x_n\right\|_Y\right) \\
&\leq R_1(\mathcal{T})\mathbb{E}\left(\left\|\sum_{n=1}^m r_n x_n\right\|_X\right) + R_1(\mathcal{S})\mathbb{E}\left(\left\|\sum_{n=1}^m r_n x_n\right\|_X\right) \\
&= [R_1(\mathcal{T}) + R_1(\mathcal{S})]\mathbb{E}\left(\left\|\sum_{n=1}^m r_n x_n\right\|_X\right)
\end{aligned}$$

The Khinchine-Kahane inequality gives us the result for general p.

(4) As a special case of (3), the set $\mathcal{T} \cup \mathcal{S}$ is R-bounded with $R_p(\mathcal{T} \cup \mathcal{S}) \leq R_p(\mathcal{T}) + R_p(\mathcal{S})$. This follows because the zero transformation can be included in any R-bounded set, and so any element of $\mathcal{T} \cup \mathcal{S}$ can be expressed as $T + 0$ or $0 + S$, therefore $\mathcal{T} \cup \mathcal{S} \subset \mathcal{T} + \mathcal{S}$.

(5) If $\mathcal{T}$ is an R-bounded subset of $\mathcal{L}(X, Y)$ and $\mathcal{S}$ is an R-bounded subset of $\mathcal{L}(Y, Z)$, then $\mathcal{S} \circ \mathcal{T} := \{S \circ T; S \in \mathcal{S}, T \in \mathcal{T}\}$ is R-bounded with $R_p(\mathcal{S} \circ \mathcal{T}) \leq R_p(\mathcal{S})R_p(\mathcal{T})$. Indeed, again verifying for $p = 1$:

$$\begin{aligned}
\mathbb{E}\left\|\sum_{n=1}^m r_n S_n T_n x_n\right\|_Z &\leq R_1(\mathcal{S})\mathbb{E}\left\|\sum_{n=1}^m r_n T_n x_n\right\|_Y \\
&\leq R_1(\mathcal{S})R_1(\mathcal{T})\mathbb{E}\left\|\sum_{n=1}^m r_n x_n\right\|_X
\end{aligned}$$

and moving to general p via the Khinchine-Kahane inequality.

We conclude this chapter with two useful methods by which we may construct R-bounded sets:

Definition 2.10. Given a set $\mathcal{T}$, define the convex hull and the absolute convex hull:

$$co(\mathcal{T}) := \left\{ \sum_{k=1}^{n} \lambda_k T_k \mid n \in \mathbb{N}, T_k \in \mathcal{T}, \lambda_k \in \mathbb{R}_+, \sum_{k=1}^{n} \lambda_k \leq 1 \right\}$$
$$absco(\mathcal{T}) := \left\{ \sum_{k=1}^{n} \lambda_k T_k \mid n \in \mathbb{N}, T_k \in \mathcal{T}, \lambda_k \in \mathbb{C}, \ \sum_{k=1}^{n} |\lambda_k| \leq 1 \right\}$$

Note that some definitions require these sums to equal 1, but we require only that the sums be less than or equal to 1. Note also the that closure of these sets will contain all infinite sums of the same form. Then we have the theorem:

Proposition 2.11. *Let $\mathcal{T} \subset \mathcal{L}(X,Y)$ be R-bounded. Then the convex hull of $\mathcal{T}$, the absolute convex hull of $\mathcal{T}$, and the closure of these sets with respect the strong operator topology are R-bounded. We have the inequalities:*

$$R(co(\mathcal{T})) \leq R\overline{(co(\mathcal{T}))}^s \leq R(\mathcal{T})$$
$$R(absco(\mathcal{T})) \leq R\overline{(absco(\mathcal{T}))}^s \leq 2R(\mathcal{T})$$

Proof. We will prove the result for $p = 1$, and the general case will follow from the Khinchine-Kahane inequality. First, examine the convex hull. Take an operator $P \in co(\mathcal{T}) = \sum_{k=1}^{n} p_k T_k$. Then we can plot the points $p_1, p_1 + p_2, \ldots \sum_{k=1}^{n} p_k$ on the interval $[0,1]$, and the distance between adjacent points will retrive the coefficients p_k. We could do the same thing for a second operator $Q = \sum_{k=1}^{m} q_k T'_k$, plotting both sets of points on $[0,1]$. Then the distance between adjacent points would provide a finer partition (λ_k), such that we could write $P = \sum_{i=k}^{K} \lambda_k T_k$ and $Q = \sum_{k=1}^{K} \lambda_k T'_k$. The preceding sentences serve as an argument for the fact that, given a finite family $(T_n)_{n=1}^{N}$ of operators in $co(\mathcal{T})$, we may choose positive numbers $(\lambda_k)_{k=1}^{K}$ that are independent of n and have sum less than or equal to 1, and operators $T_{kn} \in \mathcal{T}$ such that for each n we may write

$$T_n = \sum_{k=1}^{K} \lambda_k T_{kn}$$

Then, for fixed $x_1, \ldots x_N$ we have

$$\begin{aligned}
\mathbb{E} \left\| \sum_{n=1}^{N} r_n T_n x_n \right\|_Y &= \mathbb{E} \left\| \sum_{n=1}^{N} r_n \left(\sum_{k=1}^{K} \lambda_k T_{kn} \right) x_n \right\|_Y \\
&\leq \sum_{k=1}^{K} \lambda_k \mathbb{E} \left\| \sum_{n=1}^{N} r_n T_{kn} x_n \right\|_Y \\
&\leq \sum_{k=1}^{K} \lambda_k R_1(\mathcal{T}) \mathbb{E} \left\| \sum_{n=1}^{N} r_n x_n \right\|_X \\
&\leq R_1(\mathcal{T}) \mathbb{E} \left\| \sum_{n=1}^{N} r_n x_n \right\|_X
\end{aligned}$$

This proves the result for $co(\mathcal{T})$. To prove the result for $absco(\mathcal{T})$, take a family $(T_n)_{n=1}^N \subset absco(\mathcal{T})$ and write for each n, $T_n = \sum_{j_n=1}^{J_n} a_{j_n} T_{j_n}$. Then each a_{j_n} may be written as $a_{j_n} = \alpha_{j_n}\lambda_{j_n}$, such that for all j,n we have $|\alpha_{j_n}| = 1$, $\lambda_{j_n} \geq 0$ and $\sum_{j_n=1}^{J_n} \lambda_{j_n} \leq 1$. Therefore, by Kahane's Contraction Principle (Corollary 2.4) to move between the third and fourth lines, and the previous result for the convex hull to move between the fourth and fifth lines, we have

$$\begin{aligned}
\mathbb{E}\left\|\sum_{n=1}^N r_n T_n x_n\right\|_Y &= \mathbb{E}\left\|\sum_{n=1}^N r_n \left(\sum_{j_n=1}^{J_n} a_{j_n} T_{j_n}\right) x_n\right\|_Y \\
&= \mathbb{E}\left\|\sum_{n=1}^N \sum_{j_n=1}^{J_n} r_n \alpha_{j_n}\lambda_{j_n} T_{j_n} x_n\right\|_Y \\
&\leq 2\mathbb{E}\left\|\sum_{n=1}^N \sum_{j_n=1}^{J_n} r_n \lambda_{j_n} T_{j_n} x_n\right\|_Y \\
&\leq 2R_1(\mathcal{T})\mathbb{E}\left\|\sum_{n=1}^N r_n x_n\right\|_X
\end{aligned}$$

Finally, the result for the closures follows from the definition of the strong topology, since all operators present are applied to elements x_n of the relevant Banach space. □

Proposition 2.12. *Let $\mathcal{T}$ be an R-bounded set in $\mathcal{L}(X,Y)$ and $\mathcal{M}$ be a σ-finite measure space. For any strongly measurable function $\Phi : \mathcal{M} \to \mathcal{L}(X,Y)$ taking values in $\mathcal{T}$ and any $\phi \in L^1(\mathcal{M};\mathbb{C})$, we may construct the operator $T_{\Phi,\phi} \in \mathcal{L}(X,Y)$ defined by*

$$T_{\Phi,\phi}x := \int_{\mathcal{M}} \phi(s)\Phi(s)x d\mu$$

Then $\mathcal{T}' := \{T_{\Phi,\phi}$ s.t. $||\phi||_{L^1(\mathcal{M};\mathbb{C})} \leq 1\}$ is contained in $\overline{(absco(\mathcal{T}))}^s$ and thus by Proposition 2.11, the set $\mathcal{T}'$ is R-bounded with $R_p\mathcal{T}') \leq 2R_p(\mathcal{T})$.

Proof. Fix Φ and ϕ with $||\phi||_{L^1} \leq 1$. Then $T_{\Phi,\phi}$ (hereafter referred to as T) is in $\overline{(\text{absco}(\mathcal{T}))}^s$ if every strongly open set containing T also contains an element of $\overline{(\text{absco}(\mathcal{T}))}$. Such open sets are generated by sets of the form $U(T_0,x,\epsilon) := \{S \text{ s.t. } ||Sx - T_0x|| < \epsilon\}$. Therefore it is sufficient to show that for all $x \in X$ and $\epsilon > 0$ there exists an $S \in (\text{absco}(\mathcal{T}))$ with $||Sx - Tx|| < \epsilon$.

Given fixed $x \in X$ and $\epsilon > 0$, the function $\Psi(s) = \Phi(s)x$ is strongly measurable and in $L^\infty(\mathcal{M},Y)$ since the set $\mathcal{T}$ is uniformly bounded. Therefore there exist step functions Ψ_n that converge to Ψ in $L^\infty(\mathcal{M};Y)$ and we can choose n such that $||\Psi_n(s) - \Psi(s)||_{L^\infty(\mathcal{M};Y)} < \frac{\epsilon}{2}$. Having chosen this n, let $(P_j)_{j=1}^N$ be the (finite) set of level sets of Ψ_n (that is, subsets of $\mathcal{M}$ on which Ψ_n is constant), and note that these are disjoint and cover $\mathcal{M}$ a.e. Then we can choose an s_j in each P_j and we know that for almost every $s \in P_j$ we have:

$$\begin{aligned}
||\Phi(s)x - \Phi(s_j)x||_Y &\leq ||\Phi(s)x - \Psi_n(s)||_Y + ||\Psi_n(s) - \Psi_n(s_j)||_Y + ||\Psi_n(s_j) - \Phi(s_j)x||_Y \\
&\leq \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Now, define

$$S := \sum_{j=1}^{N} \left(\int_{P_j} \phi(s) d\mu \right) \Phi(s_j)$$

Then $S \in (\text{absco}(\mathcal{T}))$ since $\sum_{j=1}^{N} \left| \int_{P_j} \phi(s) d\mu \right| \leq \sum_{j=1}^{N} \int_{P_j} |\phi(s)| d\mu = 1$, and we have:

$$\begin{aligned}
||Tx - Sx|| &= \left\| \int_{\mathcal{M}} \phi(s)\Phi(s)x d\mu - \sum_{j=1}^{N} \left(\int_{P_j} \phi(s) d\mu \right) \Phi(s_j)x \right\| \\
&= \left\| \sum_{j=1}^{N} \left(\int_{P_j} \phi(s)[\Phi(s)x - \Phi(s_j)x] d\mu \right) \right\| \\
&\leq \sum_{j=1}^{N} \int_{P_j} |\phi(s)| \cdot \|\Phi(s)x - \Phi(s_j)x\| d\mu \\
&\leq \sum_{j=1}^{N} \int_{P_j} |\phi(s)| \epsilon d\mu \\
&\leq \epsilon
\end{aligned}$$

Therefore for all $T_{\Phi,\phi}$, $x \in X$ and $\epsilon > 0$ there exists an $S \in (\text{absco}(\mathcal{T}))$ with $||Tx - Sx|| \leq \epsilon$, which implies that every $T_{\Phi,\phi}$ is in $\overline{(\text{absco}(\mathcal{T}))}^{s}$ and so the set of $T_{\Phi,\phi}$ is R-bounded. □

2.3 Further Reading

This chapter is largely based on Chapter I.2 from [29], with the proof of the Khinchine-Kahane inequality adapted from Chapter 3 of [41]. The Khinchine-Kahane inequality was developed by J.P Kahane and the original proof and further discussion may be seen in [25]. Kahane's proof follows from the scalar case proven by A. Khinchine. Original proofs of the R-boundedness results outlined in this chapter (plus further result and generalisations) may be found in [8] and [48].

Chapter 3

Fourier Multipliers

The proof of determinstic Maximal L^p-Regularity in Chapter 5 will boil down to proving that a certain integral operator is bounded on $L^p(\mathbb{R}_+; X)$. In this chapter we develop the theory of such operators, culminating in the Weis Multiplier Theorem 3.16.

Let X be a Banach space, $\mathcal{S}(\mathbb{R}; X)$ be the space of Schwartz functions $\mathbb{R} \to X$ and $\mathcal{F}$ be the Fourier transform $\mathcal{S}(\mathbb{R}; X) \to \mathcal{S}(\mathbb{R}; X)$ defined by

$$[\mathcal{F}f](t) := \hat{f}(t) := \int_{-\infty}^{\infty} e^{-its} f(s) ds$$
$$[\mathcal{F}^{-1}\hat{f}](t) \ = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{its} \hat{f}(s) ds$$

Definition 3.1. A bounded, strongly measurable function $m : \mathbb{R} \to \mathcal{L}(X, Y)$ defines a map

$$T_m : \mathcal{S}(\mathbb{R}; X) \to L^\infty(\mathbb{R}; Y), \ \ T_m f := \mathcal{F}^{-1}[m\hat{f}]$$

Such a map is called an L^p*-Fourier multiplier* for $p \in (1, \infty)$ if there exists a constant C depending only on p, X and Y such that $||T_m f||_{L^p(\mathbb{R};Y)} \leq C||f||_{L^p(\mathbb{R};X)}$ for all $f \in \mathcal{S}(\mathbb{R}; X)$. In this case, T_m extends by the density of $\mathcal{S}(\mathbb{R}; X)$ to a bounded operator

$$T_m : L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; Y)$$

We define $M_p(X, Y) := \{m : \mathbb{R} \to \mathcal{L}(X, Y) \mid T_m \text{ is an } L^p\text{-Fourier multiplier}\}$, and we define

$$||m||_p := ||T_m||_{\mathcal{L}(L^p(\mathbb{R};X), L^p(\mathbb{R};Y))}$$

The reader may be familiar with the classical Mihlin's multiplier theorem from harmonic analysis:

Theorem 3.2. Mihlin Multiplier Theorem*:*
Let $m : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a bounded, a.e. differentiable function such that $tm'(t)$ is also bounded. Then m is a Fourier multiplier on $L^p(\mathbb{R}; \mathbb{R})$ for $p \in (1, \infty)$.

An extension to Hilbert spaces has been proven by Schwartz:

Theorem 3.3. Schwartz Multiplier Theorem*:*
Let H be a Hilbert space, and $m \in C^1(\mathbb{R}\backslash\{0\}, \mathcal{L}(H))$ such that $m(t)$ and $tm'(t)$ are bounded in $\mathcal{L}(H)$ on $\mathbb{R}\backslash\{0\}$. Then m is a Fourier multiplier on $L^p(\mathbb{R}; H)$ for $p \in (1, \infty)$.

Proofs of these theorems use on the Plancherel Theorem which asserts that m is a Fourier multiplier when $p = 2$, and then employ an interpolation argument (see, for example, [13] for the scalar case and [3] for the Hilbert-valued case). Unfortunately, this strategy breaks down once we consider a general Banach space because the Plancherel Theorem no longer holds. Therefore we attempt a new approach pioneered by Bourgain (see [5]), where we construct interesting Fourier multipliers out of simpler ones.

3.1 UMD Spaces, First Appearance

Definition 3.4. A Banach space X is a *UMD space* if the *Hilbert Transform*:

$$m_H(t) := \mathbb{1}_{(0,\infty)}(t) - \mathbb{1}_{(-\infty,0]}(t)$$

defines a Fourier multiplier on $L^p(\mathbb{R}; X)$ for each $p \in (1, \infty)$.

Remark 3.5. The letters 'UMD' stand for 'unconditional martingale difference,' and they stem from an equivalent definition of UMD spaces that we will see later in Section 7.2.3. The equivalence of these definitions is a deep and difficult result proven by Bourgain and Burkholder (see [4] and [6]).

Therefore we shall build our Fourier multiplier theory on UMD spaces. It can be proven using harmonic analysis techniques that this is not an unreasonable set of spaces to study, in particular we have the theorem:

Theorem 3.6. *Let $\mathcal{M}$ be a measure space and X be a closed subspace of $L^q(\mathcal{M})$ for $q \in (1, \infty)$. Then X is a UMD space.*

Proof. An example proof may be found as Theorem 3.14 in [29]. □

We also refer to another harmonic analysis result concerning UMD spaces which shall be needed in our work on deterministic maximal regularity.

Theorem 3.7. *Let X be a UMD space, $p \in (1, \infty), q \in [1, \infty)$ and $I_n := \{t \in \mathbb{R}; 2^{n-1} \leq |t| \leq 2^n\}$ for each $n \in \mathbb{Z}$. Then there exists a constant C depending only on X, p and q, such that for any $f \in L^p(\mathbb{R}; X)$ we have the following* 'Paley-Littlewood estimate'*:*

$$\frac{1}{C}||f||_{L^p(\mathbb{R},X)} \leq \mathbb{E}\left(\left\|\sum_{n\in\mathbb{Z}} r_n T_{\mathbb{1}_{I_n}} f\right\|^q_{L^p(\mathbb{R},X)}\right)^{\frac{1}{q}} \leq C||f||_{L^p(\mathbb{R},X)}$$

Proof. The proof of this statement is a difficult result by Bourgain using harmonic analysis techniques. A proof may be found in [5], over the interval $[0, 2\pi]$, and the result was extended by Zimmermann to $\mathbb{R}_+$ in [49]. □

3.2 Constructing Fourier Multipliers

In this section, we shall show that given a UMD Banach space X, we may construct non-trivial functions that generate Fourier multipliers out of the Hilbert transform.

Proposition 3.8. *Simple ways to construct Fourier multipliers:*

(1) Given $m_1, m_2 \in M_p(X,Y)$ and $c \in \mathbb{C}$, we have cm_1 and $m_1 + m_2 \in M_p(X,Y)$. We have $T_{cm_1} = cT_{m_1}$, and $T_{m_1+m_2} = T_{m_1} + T_{m_2}$. If $m_3 \in M_p(Y,Z)$ then $m_3 m_1 \in M_p(X,Z)$ and $T_{m_3 m_1} = T_{m_3} T_{m_1}$.

(2) If $m \in \mathcal{M}_p(X,Y)$ and $a \in \mathbb{R}$, then $m_a(t) := m(t-a) \in M_p(X,Y)$, and $||m_a||_p = ||m||_p$.

(3) Let m and $(m_n)_{n\in\mathbb{N}} \subset M_p(X,Y)$ such that $m_n(t)x \to m(t)x$ for all $x \in X$, and such that there exists C, D with $||m_n(t)||_{\mathcal{L}(X,Y)} \leq D$ and $||m_n||_p \leq C$ for all $t \in \mathbb{R}, n \in \mathbb{N}$. Then $m \in M_p(X,Y)$ and $||m||_p \leq C$.

Proof. (1) The first two results follow from the definition of a norm and the linearity of the Fourier transform. The third follows:

$$T_{m_1 m_3} f = \mathcal{F}^{-1}[m_1 m_3 \hat{f}] = \mathcal{F}^{-1}[m_1 \mathcal{F}\left(\mathcal{F}^{-1}[m_3 \hat{f}]\right)] = T_{m_1} T_{m_3} f$$

(2) We have

$$\begin{aligned} \mathcal{F}\left[e^{-ia(\cdot)} f\right](s) &= \int_{-\infty}^{\infty} e^{-ist} e^{-iat} f(t) dt \\ &= \hat{f}(s+a) \\ e^{iat}\mathcal{F}^{-1}[\hat{f}](t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ist} e^{iat} \hat{f}(s) ds \\ &= \mathcal{F}^{-1}[\hat{f}(\cdot - a)](t) \end{aligned}$$

Which means

$$\begin{aligned} &(T_{m_a} f)(t) = e^{iat} T_m \left[e^{-ias} f(s)\right](t) \\ \implies &||T_{m_a} f||_{L^p(\mathbb{R};Y)} \leq ||T_m||_{\mathcal{L}(L^p(\mathbb{R};X), L^p(\mathbb{R};Y))} \cdot ||f||_{L^p(\mathbb{R};X)} \\ \implies &||m_a||_p \leq ||m||_p \end{aligned}$$

Writing $m(t) = m_a(t+a)$ and repeating the above method yields equality of the norms.

(3) Given $f \in \mathcal{S}(\mathbb{R}; X)$, we have $m_n \hat{f} \to m\hat{f}$ pointwise, and:

$$\int_{-\infty}^{\infty} ||m_n(t)\hat{f}(t)||_Y dt \leq \int_{-\infty}^{\infty} D||\hat{f}(t)||_X dt < \infty$$

So $m_n \hat{f} \to m\hat{f}$ in $L^1(\mathbb{R}; Y)$ by the dominated convergence theorem. Therefore

$$\|(T_{m_n} f)(t) - (T_m f)(t)\|_Y \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} |e^{its}| \|m_n \hat{f}(s) - m\hat{f}(s)\|_Y ds \to 0 \text{ uniformly in } t$$

And so for all $t \in \mathbb{R}$ we have $||T_{m_n} f(t)||_Y \to ||T_m f(t)||_Y$. Thus by Fatou's Lemma (for scalar integrals):

$$\begin{aligned} ||T_m f||^p_{L^p(\mathbb{R};Y)} &= \int ||T_m f(t)||^p_Y dt \\ &\leq \liminf_{n\to\infty} \int ||T_{m_n} f(t)||^p_Y dt \\ &= \liminf_{n\to\infty} \|T_{m_n} f\|^p_{L^p(\mathbb{R};Y)} \\ &\leq C^p ||f||_{L^p(\mathbb{R},X)} \end{aligned}$$

□

From these methods, we can construct all characteristic functions out of the Hilbert transform $m_H(t) := \mathbb{1}_{(0,\infty)}(t) - \mathbb{1}_{(-\infty,0]}(t)$:

$$\begin{aligned} \mathbb{1}_{(0,\infty)} &= \frac{1+m_H}{2}, \\ \mathbb{1}_{(\infty,0]} &= \frac{1-m_H}{2} \\ \mathbb{1}_{(a,\infty)}(t) &= \mathbb{1}_{(0,\infty)}(t-a) \\ \mathbb{1}_{(\infty,b]}(t) &= \mathbb{1}_{(\infty,0]}(t-b) \\ \mathbb{1}_{(a,b]} &= \mathbb{1}_{(a,\infty)} \mathbb{1}_{(\infty,b]} \end{aligned}$$

Remark 3.9. If $m : \mathbb{R} \to \mathcal{L}(X,Y)$ takes a constant value $T \in \mathcal{L}(X,Y)$, then m is a Fourier Multiplier with $T_m = T$ (because T will commute with the Fourier transform). Therefore, from our work above, any simple function of the form $m = \sum_{i=1}^n \mathbb{1}_{(a_i,b_i]} T_i$ will be a Fourier multiplier.

Next, a more interesting way of constructing new Fourier multipliers:

Proposition 3.10. *Weighted averages of Fourier Multipliers:*

Let $S \subset \mathbb{R}$ be a measurable set, and index a family $\{m_s \in M_p(X,Y), s \in S\}$ such that (1) there exists $C \geq 0$ satisfying $||m_s||_p \leq C$ for all s, and (2) $(s,t) \to m_s(t)$ is in $L^\infty(S \times \mathbb{R}; \mathcal{L}(X,Y))$. Then we can take any function $\phi \in L^1(S,\mathbb{C})$ and speak of the 'weighted average' m_ϕ defined by

$$m_\phi(t) := \int_S m_s(t)\phi(s)ds$$

Then the claim of this proposition is that

$$T_{m_\phi} f = \int_S [T_{m_s} f]\phi(s)ds$$

And we have $m_\phi \in M_p(X,Y)$ and $||m_\phi||_p \leq C||\phi||_{L^1}$.

Proof. Note first that $m_\phi(t)$ is bounded in t, indeed:

$$\sup_t ||m_\phi(t)||_p \leq \sup_t \left(\sup_{s\in S} ||m_s(t)||_p \int_S |\phi(s)|ds \right) < \infty$$

Now, fix $f \in \mathcal{S}(\mathbb{R}; X)$ and define $[F(s)](t) = m_s(t)\hat{f}(t)\phi(s)$ for $s \in S$. Since $m_s(t)$ is bounded in t and s, $\hat{f}$ is bounded and ϕ is integrable, the function $F(s)$ is in $L^1(\mathbb{R}; Y)$ for each s, and F is in $L^1(S; L^1(\mathbb{R}; Y))$. Therefore:

$$m_\phi(t)\hat{f}(t) = \int_S m_s(t)\hat{f}(t)\phi(s)ds = \int_S [F(s)](t)ds \in L^1(\mathbb{R}; Y)$$

Since the inverse Fourier transform is bounded from $L^1(\mathbb{R}; Y)$ to $L^\infty(\mathbb{R}; Y)$, we may commute it past the integral and obtain:

$$T_{m_\phi} f = \mathcal{F}^{-1}[m_\phi \hat{f}] = \mathcal{F}^{-1}\left[\int_S F(s)ds\right] = \int_S \mathcal{F}^{-1}[F(s)]ds = \int_S T_{m_s} f\phi(s)ds$$

This proves the first part. Now, for fixed s, $T_{m_s} f\phi(s)$ is in $L^p(\mathbb{R}; Y)$ since $m_s \in \mathcal{M}_p(X, Y)$. By our assumption that the $||m_s||_p$ are uniformly bounded by C:

$$\begin{aligned}
\left\| \int_S T_{m_s} f\phi(s)ds \right\|_{L^p(\mathbb{R};Y)} &\leq \int_S ||T_{m_s} f\phi(s)||_{L^p(\mathbb{R};Y)}ds \\
&\leq \int_S |\phi(s)| \cdot C||f||_{L^p(\mathbb{R};X)}ds \\
&\leq C||f||_{L^p(\mathbb{R};Y)}||\phi||_{L^1(S;\mathbb{C})}
\end{aligned}$$

which means that $T_{m_\phi} f$ is in $L^p(\mathbb{R}; Y)$ with $||m_\phi|| \leq C||\phi||_{L^1(S;\mathbb{C})}$. The result follows for general $f \in L^p(\mathbb{R}; X)$ because $\mathcal{S}(\mathbb{R}; X)$ is dense. $\square$

This gives the major result:

Corollary 3.11. *Let X be a UMD space. If $m \in C^1(\mathbb{R}, \mathcal{L}(X, Y))$ has an integrable derivative and $\lim_{t\to-\infty} m(t) = 0$, then $m \in M_p(X, Y)$ and there exists C such that*

$$||m||_p \leq C \int_{-\infty}^{\infty} ||m'(t)||_{\mathcal{L}(X,Y)}dt$$

Proof. We may write, since $\lim_{t\to-\infty} m(t) = 0$

$$m(t) - m(u) = \int_u^t m'(s)ds \implies m(t) = \int_{-\infty}^t m'(s)ds = \int_{-\infty}^{\infty} m_s(t)h(s)ds$$

Where $h(s) = ||m'(s)||_{\mathcal{L}(X,Y)}$ is an L^1 function by our assumptions, and we define

$$m_s(t) := \mathbb{1}_{\{m'(s)\neq 0\}}(s)\frac{m'(s)}{||m'(s)||_{\mathcal{L}(X,Y)}}\mathbb{1}_{(s,\infty)}(t)$$

Then for fixed s, the function $t \to m_s(t)$ is in $M_p(X, Y)$ by Remark 3.9. We apply Proposition 3.10 to obtain the result. $\square$

3.3 R-Bounded Fourier Multipliers

Next, we combine the concepts of R-boundedness and Fourier multipliers. Corollary 3.11 and Proposition 2.12 are very similar, and indeed we can obtain:

Corollary 3.12. *Weighted averages of R-bounded Fourier multipliers*

Let $\mathcal{T} \subset \mathcal{L}(L^p(\mathbb{R};X), L^p(\mathbb{R};Y))$ be R-bounded, $S \subset \mathbb{R}$ be measurable and define

$$\mathcal{S}_{\mathcal{T}} := \{(s,t) \to m_s(t) \in L^\infty(S \times \mathbb{R}, \mathcal{L}(X,Y)) \mid \text{ for all } s \in S, m_s \in \mathcal{M}_p(X,Y) \text{ and } T_{m_s} \in \mathcal{T}\}$$

Then, as in Proposition 3.10 above, for $\phi \in L^1(S, \mathbb{C})$ define:

$$m_\phi(t) := \int_S m_s(t)\phi(s)ds$$

Then the set $\tilde{\mathcal{T}} := \{T_{m_\phi} \mid m \in S_{\mathcal{T}}, ||\phi||_{L^1} \leq 1\}$ is an R-bounded set of Fourier Multipliers with $R_p(\tilde{\mathcal{T}}) \leq 2R_p(\mathcal{T})$.

Proof. We know that $\tilde{\mathcal{T}}$ is a set of Fourier multipliers by Propositions 3.10 above, and we know that the set is R-bounded from Proposition 2.12. □

Using Corollary 3.12, we may construct R-bounded Fourier multipliers out of simple sets of such objects. The next proposition introduces these simple objects.

First, some definitions. Fix $p \in (1,\infty)$ and let $\mathcal{M}$ be a $\sigma-$finite measure space, then:
- For $\phi \in L^\infty(\mathcal{M};\mathbb{C})$, define the multiplication operator $M_\phi : L^p(\mathcal{M},X) \to L^p(\mathcal{M},X)$ by $(M_\phi f)(s) := \phi(s)f(s)$.
- Given $T \in \mathcal{L}(X,Y)$, define $\tilde{T} \in \mathcal{L}(L^p(\mathcal{M};X), L^p(\mathcal{M};Y))$ by $(\tilde{T}f)(s) := T(f(s))$.
- Let $\mathcal{T} \subset \mathcal{L}(X,Y)$ be R-bounded, and $\mathcal{I}$ be the set of all intervals in $\mathbb{R}$ (including intervals that are unbounded.

Then we have:

Proposition 3.13.

(1) The set of multiplication operators $\{M_\phi \text{ s.t. } ||\phi||_{L^\infty} \leq 1\}$ is R-bounded in $\mathcal{L}(L^p(\mathcal{M},X))$, with $R_p(\Phi) \leq 2$.

(2) The set $\tilde{\mathcal{T}} := \{\tilde{T} \mid T \in \mathcal{T}\}$ is R-bounded with $R_p(\tilde{\mathcal{T}}) \leq R_p(\mathcal{T})$.

(3) If X is a UMD space, then the set $\{T_{\mathbb{1}_I} \mid I \in \mathcal{I}\}$ is an R-bounded set of Fourier multiplier operators on $L^p(\mathbb{R};X)$.

(4) If X is a UMD space, then the set $\mathcal{S} = \{T_{T'\mathbb{1}_I} \mid I \in \mathcal{I}, T' \in \mathcal{T}\}$ is an R-bounded set of Fourier multiplier operators on $L^p(\mathbb{R};X)$, with $R_p(\mathcal{S}) \leq CR_p(\mathcal{T})$ for a constant C depending only on X and p.

Proof. (1) Choose sequences $(\phi_n)_{n=1}^N \subset L^\infty(\mathcal{M};\mathbb{C})$ and $(f_n)_{n=1}^N \subset L^p(\mathcal{M};X)$. Then, using Fubini's Theorem for the first equality and Kahane's contraction principle 2.4 for the second:

$$\begin{aligned}\mathbb{E}\left\|\sum_{n=1}^N r_n M_{\phi_n} f_n\right\|_{L^p(\mathcal{M};X)}^p &= \int_{\mathcal{M}}\left(\mathbb{E}\left\|\sum_{n=1}^N r_n\phi_n(\mu)f_n(\mu)\right\|_X^p\right)d\mu\\ &\le 2^p\int_{\mathcal{M}}\left(\mathbb{E}\left\|\sum_{n=1}^N r_n f_n(\mu)\right\|_X^p\right)d\mu\\ &= 2^p\mathbb{E}\left\|\sum_{n=1}^N r_n f_n\right\|_{L^p(\mathcal{M};X)}^p\end{aligned}$$

(2) Choose $(T_n)_{n=1}^N \in \mathcal{L}(X,Y)^N$ and $(f_n)_{n=1}^N \in L^p(\mathcal{M};X)^N$. Then, by Fubini's theorem:

$$\begin{aligned}\mathbb{E}\left\|\sum_{n=1}^N r_n \tilde{T}_n f_n\right\|_{L^p(\mathcal{M};Y)}^p &= \int_{\mathcal{M}}\left(\mathbb{E}\left\|\sum_{n=1}^N r_n T_n[f_n(\mu)]\right\|_Y^p\right)d\mu\\ &\le R_p^p(\mathcal{T})\int_{\mathcal{M}}\left(\mathbb{E}\left\|\sum_{n=1}^N r_n f_n(\mu)]\right\|_X^p\right)d\mu\\ &= R_p^p(\mathcal{T})\mathbb{E}\left\|\sum_{n=1}^N r_n f_n\right\|_{L^p(\mathcal{M};X)}^p\end{aligned}$$

(3) Consider the family $s \to M_s := M_{e^{is(\cdot)}}$ in $\mathcal{L}(L^p(\mathbb{R},X))$. This set is R-bounded from (1). We also take the Fourier multipliers $T_{\mathbb{1}_{(0,\infty)}}$ and $T_{\mathbb{1}_{(-\infty,0)}}$ (which exist because X is a UMD space) which are clearly an R-bounded set of two elements. Given $f \in \mathcal{S}(\mathbb{R},X)$, we have

$$[T_{\mathbb{1}_{(s,\infty)}}f](t) = e^{ist}[T_{\mathbb{1}_{(s,\infty)}}(e^{is(\cdot)}f)](t)$$

This tells us that $T_{\mathbb{1}_{(s,\infty)}} = M_s T_{\mathbb{1}_{(0,\infty)}} M_{-s}$, which means from part (4) of Proposition 2.9 that the set of operators $T_{\mathbb{1}_{(s,\infty)}}$ is R-bounded. Similarly, we have

$$\begin{aligned}T_{\mathbb{1}_{(-\infty,s)}} &= I - T_{\mathbb{1}_{(s,\infty)}} = M_s(I - T_{(s,\infty)})M_{-s} = M_s T_{\mathbb{1}_{(-\infty,0)}} M_{-s}\\ T_{\mathbb{1}_{(s,t)}} &= T_{\mathbb{1}_{(-\infty,t)}} T_{\mathbb{1}_{(s,\infty)}}\end{aligned}$$

And so $\mathcal{T}_{\mathcal{I}} := \{T_{\mathbb{1}_I} \mid I \in \mathcal{I}\}$ is an R-bounded subset of $\mathcal{L}(L^p(\mathbb{R},X))$.

(4) Clearly $T_{T'\mathbb{1}_I} = \tilde{T'} \circ T_{\mathbb{1}_I}$ is a Fourier multiplier because T' will commute with the Fourier transform. Results (3) and (4) of this Proposition yield that $\{T_{T'\mathbb{1}_I} \mid I \in \mathcal{I}, T' \in \mathcal{T}\}$ is R-bounded with $R_p(\sigma_{\mathcal{T},\mathcal{I}}) \le R_p(\sigma_{\mathcal{I}})R_p(\mathcal{T})$. □

We are ready to prove Theorem 3.15, one of the main outcomes of our exploration of R-boundedness and Fourier multipliers, and the key step in the proof of the Weis multiplier theorem.

Definition 3.14. Given an R-bounded set $\mathcal{T} \subset \mathcal{L}(X,Y)$, let S_ϕ be the set of functions $\phi \in L^1(\mathbb{R},\mathbb{C})$ with $||\phi||_{L^1} \le 1$, and S_ψ be the set of function $\psi \in L^\infty(\mathbb{R},\mathcal{L}(X,Y))$ taking values in

$\mathcal{T}$. Then define:

$$S(\mathcal{T}) := \{m \in C^1(\mathbb{R}, \mathcal{L}(X,Y));\ \lim_{t\to-\infty} m(t) = 0 \text{ and } m'(t) = \phi(t)\psi(t) \text{ for some } \phi \in S_\phi, \psi \in S_\psi\}$$

Theorem 3.15. *Let X have the UMD property. Then $\mathcal{S} := \{T_m : m \in S(\mathcal{T})\}$ is an R-bounded set of Fourier multiplier operators $L^p(\mathbb{R}, X) \to L^p(\mathbb{R}, Y)$ with $R_p(\mathcal{S}) \le CR_p(\mathcal{T})$ for some C depending only on X and p.*

Proof. Fix $m \in S(\mathcal{T})$ and choose ϕ, ψ such that $m'(s) = \phi(s)\psi(s)$ for all $s \in \mathbb{R}$. Then $m'(s)$ is integrable, and so by Corollary 3.11, the function m is a Fourier multiplier. To see that the set $\mathcal{S}$ is R-bounded, we note that for fixed t:

$$m(t) = \int_{-\infty}^{t} m'(s)ds = \int_{-\infty}^{\infty} \mathbb{1}_{\{s<t\}}\phi(s)\psi(s)ds := \int_{-\infty}^{\infty} m_s(t)\phi(s)ds$$

with $m_s(t) := \psi(s)\mathbb{1}_{\{s<t\}}$. Then for fixed s, T_{m_s} is in the set $\{T_{\mathbb{1}_{\mathcal{I}}T'}\ ;\ \mathcal{I}$ an interval of $\mathbb{R}, T' \in \mathcal{T}\}$, which is R-bounded by part d) of Proposition 3.13. The result follows by Corollary 3.12. □

3.4 Weis' Multiplier Theorem

We conclude this chapter with the statement and proof of the Weis multiplier theorem. This will be the key theorem that allows us to prove Maximal L^p-Regularity for determinstic PDEs in Chapter 5.

Theorem 3.16. *Let X and Y be UMD-spaces, and $p \in (1, \infty)$. Let $m \in C^1(\mathbb{R}\backslash\{0\}, \mathcal{L}(X,Y))$ be such that the sets:*

$$\mathcal{T}_1 := \{m(t), t \in \mathbb{R}\backslash\{0\}\}, \quad \mathcal{T}_2 := \{tm'(t), t \in \mathbb{R}\backslash\{0\}\}$$

are R-bounded. Then m is a Fourier multiplier and $||m||_p \le C \max\{R_p(\mathcal{T}_1), R_p(\mathcal{T}_2)\}$ with C depending only on X, Y and p.

Proof. The bulk of the proof involves the technical construction of a set $\{T_{m_n}, n \in \mathbb{Z}\}$ of R-bounded Fourier multiplier operators.

Following 2.9, we see that $\mathcal{T}_1 \cup \mathcal{T}_2 = \{m(t), tm'(t) \mid t \in \mathbb{R}\backslash\{0\}\}$ is R-bounded with $R_p(\mathcal{T}_1 \cup \mathcal{T}_2) \le R_p(\mathcal{T}_1) + R_p(\mathcal{T}_2)$. Let $\mathcal{T} := \text{absco}(\mathcal{T}_1 \cup \mathcal{T}_2)$. Then by Proposition 2.11, $\mathcal{T}$ is also R-bounded and $R_p(\mathcal{T}) \le 2(R_p(\mathcal{T}_1) + R_p(\mathcal{T}_2)) \le 4\max(R_p(\mathcal{T}_1), R_p(\mathcal{T}_2))$.

Choose a function $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ with supp $\varphi \subset \{t \in \mathbb{R}; 2^{-2} \le |t| \le 2\}$ such that $\varphi \equiv 1$ on $\{t \in \mathbb{R}; 2^{-1} \le |t| \le 1\}$. Let $C_\varphi := \sup\{|\varphi(t)|, |t\varphi'(t)|\}$. Then, for each $n \in \mathbb{Z}$, define:

$$\begin{aligned} \varphi_n(t) &= \varphi(2^{-n}t) & I_n &= \{t \in \mathbb{R}; 2^{n-1} \le |t| \le 2^n\} \\ m_n(t) &= \varphi_n(t)m(t) & J_n &= \{t \in \mathbb{R}; 2^{n-2} \le |t| \le 2^{n+1}\} \\ h_n(t) &= \frac{1}{t}\mathbb{1}_{J_n}(t) & \phi_n(t) &= 2^{-n}\varphi'(2^{-n}t)tm(t) + \varphi(2^{-n}t)tm'(t) \end{aligned}$$

Then $\varphi_n(t) \equiv 1$ on I_n. Indeed, $\varphi(t) \equiv 1$ on $\{t \in \mathbb{R}; 2^{-1} \le |t| \le 1\}$, therefore $\varphi_n(t) = \varphi(2^{-n}t) \equiv 1$ on $\{t \in \mathbb{R}; 2^{n-1} \le |t| \le 2^n\} = I_n$. By a similar argument, we see that $\operatorname{supp}\varphi_n \subset J_n$. Next, we calculate explicitly that $||h_n||_{L^1(\mathbb{R},\mathbb{R})} = 6\ln 2$ (this specific number is not important, we just need some constant). Indeed:

$$\int_{J_n} \frac{1}{|t|}dt = 2\int_{2^{n-2}}^{2^{n+1}} \frac{1}{t}dt = 2(\ln(2^{n+1}) - \ln 2^{n-2}) = 6\ln 2$$

Next, we note that ϕ_n takes values in an R-bounded set, specifically $\phi_n(t) \in 2C_\varphi\mathcal{T}$ for all t. Indeed, since $2^{-n}t\varphi'(2^{-n}t)$ and $\varphi(2^{-n}t)$ are less than C_φ, we have

$$\frac{2^{-n}t\varphi'(2^{-n}t)}{2C_\varphi} \text{ and } \frac{\varphi(2^{-n}t)}{2C_\varphi} \le \frac{1}{2}$$

And therefore:

$$\phi_n(t) = 2C_\varphi\left(\frac{2^{-n}t\varphi'(2^{-n}t)}{2C_\varphi}m(t) + \frac{\varphi(2^{-n}t)}{2C_\varphi}tm'(t)\right) \in 2C_\varphi\mathcal{T}$$

Next, recalling Definition 3.14 above, I claim that $m_n \in S(12\ln 2C_\varphi\mathcal{T})$ for all n. Indeed, by the product rule, (recalling that φ_n and φ_n' are supported on J_n):

$$\begin{aligned}
m_n'(t) &= 2^{-n}\varphi'(2^{-n}t)m(t) + \varphi(2^{-n}t)m'(t) \\
&= \frac{\mathbb{1}_{J_n}}{t}\left[2^{-n}\varphi'(2^{-n}t)tm(t) + \varphi(2^{-n}t)tm'(t)\right] \\
&= h_n(t)\phi_n(t) \\
&= \frac{h_n(t)}{6\ln 2}\cdot(6\ln 2\phi_n(t))
\end{aligned}$$

with $||\frac{h_n}{6\ln 2}||_{L^1(\mathbb{R},\mathbb{R})} = 1$ and ϕ_n taking values in $12\ln 2C_\varphi\mathcal{T}$. It is clear that $\lim_{t\to-\infty} m_n(t) = 0$, since m_n is supported on a finite set.

Therefore we can apply Theorem 3.15 to conclude that $\mathcal{S} := \{T_{m_n}; n \in \mathbb{Z}\}$ is an R-bounded set of Fourier multiplier operators from $L^p(\mathbb{R};X)$ to $L^p(\mathbb{R};Y)$, and $R_p(\mathcal{S}) \le CR_p(\mathcal{T}) \le 4C\max(R_p(\mathcal{T}_1), R_p(\mathcal{T}_2))$, with C depending only on X and p.

With this technical construction in hand, we can finish our proof. Take any function $f \in \mathcal{S}(\mathbb{R}, X)$ and since X is a UMD space, apply the Paley-Littlewood estimate (cited as Theorem 3.7) to $T_m f$:

$$||T_m f||^p_{L^p(\mathbb{R},Y)} \le C\mathbb{E}\left\|\sum_{n\in\mathbb{Z}} r_n T_{\mathbb{1}_{I_n}} T_m f\right\|^p_{L^p(\mathbb{R},Y)}$$

Now, note that for all f, $T_{\mathbb{1}_{I_n}}T_m = \mathcal{F}^{-1}[\mathbb{1}_{I_n}m\hat{f}] = T_{m_n}T_{\mathbb{1}_{I_n}}f$, since $m \equiv m_n$ on I_n. We then use the R-boundedness of the T_{m_n} and $T_{\mathbb{1}_{I_n}}$ families, to conclude:

$$
\begin{aligned}
||T_m f||^p_{L^p(\mathbb{R},Y)} &\lesssim \mathbb{E}\left\|\sum_{n\in\mathbb{Z}} r_n T_{\mathbb{1}_{I_n}} T_m f\right\|^p_{L^p(\mathbb{R},Y)} \\
&\lesssim \mathbb{E}\left\|\sum_{n\in\mathbb{Z}} r_n T_{m_n} T_{\mathbb{1}_{I_n}} f\right\|^p_{L^p(\mathbb{R},Y)} \\
&\leq R_p(\mathcal{S})\mathbb{E}\left\|\sum_{n\in\mathbb{Z}} r_n T_{\mathbb{1}_{I_n}} f\right\|^p_{L^p(\mathbb{R},X)} \\
&\leq C R_p(\mathcal{S})||f||_{L^p(\mathbb{R},X)}
\end{aligned}
$$

with the final constant C following from a second application of the Paley-Littlewood estimate Theorem 3.7.

Therefore T_m is bounded on Schwartz functions as an operator $L^p(\mathbb{R};X) \to L^p(\mathbb{R},Y)$, and thus it is a Fourier multiplier. □

Our goal was to prove Maximal L^p-regularity under sufficient conditions, but it is worth mentioning that in fact, a partial converse is true:

Theorem 3.17. *Let X and Y be UMD-spaces, and $p \in (1,\infty)$. Let $m \in C^1(\mathbb{R}\backslash\{0\}, \mathcal{L}(X,Y))$ be such that the operator T_m is a fourier multiplier. Then the set*

$$\mathcal{T}_1 := \{m(t), t \in \mathbb{R}\backslash\{0\}\},$$

is R-bounded. Note that we do not *have that $\mathcal{T}_2$ is R-bounded.*

Proof. The original proof is located in [9], and is reproduced as Theorem 3.13 in [29]. □

3.5 Further Reading

This section follows Chapter 3 of [29]. The original proof of the Weis multiplier theorem can be found in [48], and further discussion of Fourier multiplier theorems may be found in [17].

Chapter 4

The McIntosh Calculus

4.1 Basic Constructions

This section will introduce a functional calculus for certain unbounded operators, which shall be used (among other things) to define semigroups e^{-tA} that yield the solutions to Banach-valued ODEs. It is inspired by the classical Cauchy formula from complex analysis:

Theorem 4.1. Cauchy's Integral Formula
Let U be an open subset of $\mathbb{C}$, let $f : U \to \mathbb{C}$ be a holomorphic function and let γ be a non-self-intersecting closed curve in U that can be represented as a finite union of piecewise smooth paths $[0,1] \to \mathbb{C}$. Then for every a in the interior of γ, we have

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz$$

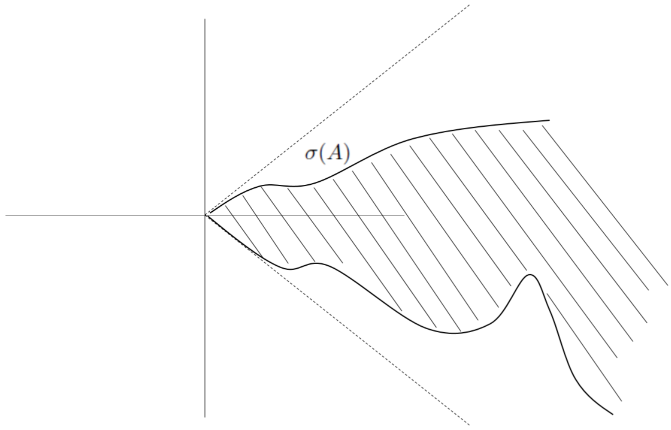
The Cauchy Integral formula also holds for a holomorphic function $f : \mathbb{C} \to X$ with X a separable Banach space. This follows from standard complex analysis, and we shall not include a proof here. Rather, we shall take this as our starting point and hope to define functions of an unbounded operator $A : X \to X$ using the resolvent $R(z, A) := (z - A)^{-1}$, and a formula defined formally as:

$$f(A) := \frac{1}{2\pi i} \int_\gamma f(z) R(z, A) dz$$

Of course, $R(z, A)$ does not exist for every $z \in \mathbb{C}$, and so this integrand is not yet well defined. Even if it is, this integral may not converge. We deal with these issues in this section, and in the process we restrict the class of both functions and operators that we consider.

Definition 4.2. The space $H^\infty(U)$ is the space of bounded holomorphic functions $f : U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$. It is endowed with the norm $||f||_{H^\infty(U)} := \sup_{z\in U} |f(z)|$.

Definition 4.3. An unbounded, closed, linear operator $A : X \to X$ with dense domain $D(A)$ is called *sectorial* of angle $0 < \theta < \frac{\pi}{2}$ if there exists a sector $\Sigma_\theta := \{z \neq 0 \in \mathbb{C}; |\arg z| < \theta\}$ such that $\sigma(A) \subset \overline{\Sigma_\theta}$, and for all $\theta' > \theta$ there exists a $C_{\theta'}$ such that $||zR(z, A)|| \leq C_{\theta'}$ for $z \notin \overline{\Sigma_{\theta'}}$.



Then for $\theta < \theta' < \frac{\pi}{2}$ and sectorial A, the integral:

$$f(A) = \frac{1}{2\pi i} \int_{\partial\Sigma_{\theta'}} f(z)R(z, A)dz$$

has a well-defined integrand for all $z \neq 0$, although we do not yet know whether this integral will converge.

Remark 4.4. Verifying that any particular operator is sectorial is beyond this thesis' scope. However, we include some remarks to indicate that sectorial operators are a reasonable object to study. The operator A is intended to represent a differential operator, usually of second order (as in the heat equation). The paper [1] describes general conditions that ensure that such operators are sectorial, which we outline briefly. Let M be a complex $n \times n$ matrix with L^∞ coefficients that satisfies:

$$\alpha|z|^2 \leq \operatorname{Re} Mz \cdot \bar{z} \text{ and } |Mz \cdot \bar{\zeta}| \leq \beta|z||\zeta|$$

for all z, ζ in $\mathbb{C}^n$ and some $0 < \alpha \leq \beta < \infty$. Then we can consider a general class of second-order differential operators acting on functions of $\mathbb{R}^n$, written as:

$$A := \operatorname{div}(M\nabla)$$

It is shown in chapter 4 of [1] that there exists $p_-(A)$ and $p_+(A)$ such that A is sectorial on the interval $(p_-(A), p_+(A))$, and *not* sectorial outside this interval. We have $p_-(A) = 1$ and $p_+(A) = \infty$ if $n = 1$ or 2, or if M takes real values, or if M has Hölder continuous coefficients of order λ for some $\lambda > 0$. For general M, we still have $p_-(A) < \frac{2n}{n+2}$ and $p_+(A) > \frac{2n}{n-2}$.

As another simple example, let A be an operator on $L^p(\mathcal{M})$ of the form $[Af](\mu) = a(\mu)f(\mu)$ for some L^∞ function a that is never equal to zero (a.e.). Then A is sectorial if and only if a takes values in the relevant sector (see 2.1.1 in [19]).

Remark 4.5. In this thesis, we will always assume that a sectorial operator is injective and has dense range, following [29]. This is a standard assumption, and one does not entail any serious loss of generality because a sectorial operator has restrictions that satisfy these properties. Formulating this rigorously requires operator theory that is beyond the scope of this thesis; the interested reader may consult Section 15 A of [29] and further references therein.

Definition 4.6. For some angle $\omega < \frac{\pi}{2}$, let $H_0^\infty(\Sigma_\omega)$ be the set of functions $f \in H^\infty(\Sigma_\omega)$ such that there exist $\epsilon, C > 0$ satisfying:

$$|f(z)| \leq C \left| \frac{z}{(1+z)^2} \right|^\epsilon \text{ for all } z \in \Sigma_\omega$$

This definition ensures that $|f(z)| \to 0$ sufficiently 'fast' as $|z| \to 0$ or $|z| \to \infty$.

Remark 4.7. We outline an exemplar method for integrating holomorphic functions over unbounded paths. Take $f \in H_0^\infty(\Sigma_\omega)$, and we want to consider the integral of $\frac{f(z)}{z}$ over $\partial\Sigma_{\theta'}$ for $\theta' < \omega$. We need to deal with the singularities that arise as $|z| \to 0$ and $|z| \to \infty$.

Let $B(0, r, R)$ be the set $\{z \in \mathbb{C};\ r < |z| < R\}$. Define the paths $\Gamma_1 := \partial(\Sigma_{\theta'} \cap B(0, r, R))$, $\Gamma_{2_r} := \Sigma_{\theta'} \cap \{z; |z| = r\}$ and $\Gamma_{2_R} := \Sigma_{\theta'} \cap \{z; |z| = R\}$. Then we may write

$$\int_{\partial\Sigma_\theta} \frac{f(z)}{z} dz = \lim_{\substack{r \to 0 \\ R \to \infty}} \left(\int_{\Gamma_1} \frac{f(z)}{z} dz - \int_{\Gamma_{2_r}} \frac{f(z)}{z} dz - \int_{\Gamma_{2_R}} \frac{f(z)}{z} dz \right)$$

The integral over Γ_1 is always zero, because we are integrating over a closed curve enclosing a region in which $\frac{f(z)}{z}$ is holomorphic. We show that the integrals over Γ_{2_r} and Γ_{2_R} are also zero in the limit of r and R:

First examine the integral over Γ_{2_r}. The length $|\Gamma_{2_r}|$ of this path is bounded by πr. Then we have, using our estimate on $|f(z)|$ from the assumption that $f \in H_0^\infty(\Sigma_\omega)$.

$$\left| \int_{\Gamma_{2_r}} f(z) dz \right| \leq |\Gamma_{2_r}| \cdot \sup_{z \in \Gamma_{2_r}} \left| \frac{f(z)}{z} \right| \leq C\pi r \cdot \frac{1}{r} \cdot \left(\frac{r^{-\epsilon}}{(1 + \frac{1}{r^2})^\epsilon} \right) \to 0 \text{ as } r \to 0$$

Therefore $\left| \int_{\Gamma_{2_r}} f(z) dz \right| \to 0$. We can perform a similar calculation as to see that the contribution from the path Γ_{2_R} goes to zero as $R \to \infty$. Therefore we have $\int_{\Sigma_\theta} \frac{f(z)}{z} dz = 0$.

We will use a method as described above several times in upcoming proofs. Sometimes small details will need to be changed, but none of these are unexpected or difficult. Therefore we shall reference 'the Cauchy integral formula' when discussing unbounded integrals, and this is to be understood with reference to this remark.

We are ready to construct the first stage of the McIntosh calculus:

Theorem 4.8. *Let A be a sectorial operator of angle θ on a Banach space X. Then for $\theta < \theta' < \omega$ and $f \in H_0^\infty(\Sigma_\omega)$, define*

$$f(A) := \frac{1}{2\pi i} \int_{\partial\Sigma_{\theta'}} f(z) R(z, A) dz$$

with $\partial\Sigma_{\theta'}$ oriented counterclockwise. Then:
(1) This integral is well-defined and $f(A) \in \mathcal{L}(X)$ for all $f \in H_0^\infty(\Sigma_\omega)$.
(2) This integral is independent of the particular choice of θ'.
(3) $f \to f(A)$ is a linear and multiplicative map.

Proof. (1) Since f is in $H_0^\infty(\Sigma_\omega)$ and A is sectorial, there exists C_1, C_2 and ϵ such that $|f(z)| \leq C_1 \left|\frac{z}{(1+z)^2}\right|^\epsilon$ and $||R(z,A)|| \leq C_2/|z|$ on $\partial\Sigma_{\theta'}$. This gives:

$$\int_{\partial\Sigma_{\theta'}} ||f(z)R(z,A)||dz \leq C_1C_2 \int_{\partial\Sigma_{\theta'}} \left|\frac{z}{(1+z)^2}\right|^\epsilon \frac{1}{|z|}dz$$

Next we may write

$$\begin{aligned}
\int_{\partial\Sigma_{\theta'}} \left|\frac{z}{(1+z)^2}\right|^\epsilon \frac{1}{|z|}dz &= \int_0^\infty \frac{1}{|1+te^{i\pm\theta}|^{2\epsilon}} \frac{1}{|te^{i\pm\theta}|^{1-\epsilon}}dt \\
&= \int_0^1 \frac{1}{|1+te^{i\pm\theta}|^{2\epsilon}} \frac{1}{t^{1-\epsilon}}dt + 2\int_1^\infty \frac{1}{|1+te^{i\pm\theta}|^{2\epsilon}} \frac{1}{t^{1-\epsilon}}dt
\end{aligned}$$

Since $\theta < \frac{\pi}{2}$, one can see geometrically that $|1+te^{i\pm\theta}| > \max(1,t)$ and so $\frac{1}{|1+te^{i\pm\theta}}| < \frac{1}{\min(1,\frac{1}{t})}$. Therefore we have

$$\begin{aligned}
\int_0^1 \frac{1}{|1+te^{i\pm\theta}|^{2\epsilon}} \frac{1}{t^{1-\epsilon}}dt + 2\int_1^\infty \frac{1}{|1+te^{i\pm\theta}|^{2\epsilon}} \frac{1}{t^{1-\epsilon}}dt &\leq \int_0^1 \frac{1}{t^{1-\epsilon}}dt + 2\int_1^\infty \frac{1}{t^{2\epsilon}}\frac{1}{t^{1-\epsilon}}dt \\
&= \frac{1}{\epsilon} + \frac{1}{\epsilon} \\
&< \infty
\end{aligned}$$

Therefore, by Proposition 1.15, this integral converges and is in $\mathcal{L}(X)$.

(2) This follows as in the scalar case. Suppose we choose $\theta < \theta' < \theta'' < \omega$. Then define $\Sigma_{(\theta',\theta'')} = \{z \ ; \theta' < |\arg z| < \theta''\}$, with the boundary oriented counterclockwise for the upper and lower half sectors respectively. Then we have

$$\frac{1}{2\pi i}\int_{\partial\Sigma_{\theta'}} f(z)R(z,A)dz - \frac{1}{2\pi i}\int_{\partial\Sigma_{\theta''}} f(z)R(z,A)dz = \frac{1}{2\pi i}\int_{\partial\Sigma_{(\theta',\theta'')}} f(z)R(z,A)dz$$

The function $f(z)R(z,A)$ is holomorphic on this domain, and then we can show that the integral is equal to zero by the Cauchy integral formula.

(3) Linearity is clear. To show multiplicativity, choose $\theta' < \theta'' < \omega$. Then for $f, g \in H_0^\infty(\Sigma_\omega)$ we have

$$\begin{aligned}
f(A)g(A) &= \left(\frac{1}{2\pi i}\right)^2 \left(\int_{\partial\Sigma_{\theta'}} f(z)R(z,A)dz\right)\left(\int_{\partial\Sigma_{\theta''}} g(\zeta)R(\zeta,A)d\zeta\right) \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\partial\Sigma_{\theta'}}\int_{\partial\Sigma_{\theta''}} f(z)R(z,A)g(\zeta)R(\zeta,A)d\zeta dz
\end{aligned}$$

Now, we have the 'resolvent equation' (noting that A commutes with its resolvent):

$$R(z,A)R(\zeta,A) = \frac{(z-A-\zeta+A)R(z,A)R(\zeta,A)}{z-\zeta} = \frac{R(\zeta,A)-R(z,A)}{z-\zeta}$$

And so we have

$$\begin{aligned} f(A)g(A) &= \left(\frac{1}{2\pi i}\right)^2 \int_{\partial\Sigma_{\theta'}} \int_{\partial\Sigma_{\theta''}} f(z)g(\zeta)\frac{R(\zeta,A)-R(z,A)}{z-\zeta} d\zeta dz \\ &= \frac{1}{2\pi i}\int_{\partial\Sigma_{\theta''}} g(\zeta)R(\zeta,A)\left[\frac{1}{2\pi i}\int_{\partial\Sigma_{\theta'}} \frac{f(z)}{z-\zeta}dz\right] d\zeta \\ &+ \frac{1}{2\pi i}\int_{\partial\Sigma_{\theta'}} f(z)R(z,A)\left[\frac{1}{2\pi i}\int_{\partial\Sigma_{\theta''}} \frac{g(\zeta)}{z-\zeta}d\zeta\right] dz \end{aligned}$$

Now, recalling that $\theta' < \theta''$ and $\arg z = \theta', \arg\zeta = \theta''$. we note that by the Cauchy integral formula,

$$\frac{1}{2\pi i}\int_{\partial\Sigma_{\theta'}} \frac{f(z)}{z-\zeta}dz = 0 \text{ and } \frac{1}{2\pi i}\int_{\partial\Sigma_{\theta''}} \frac{g(\zeta)}{z-\zeta}d\zeta = g(z)$$

We therefore conclude

$$f(A)g(A) = \frac{1}{2\pi i}\int_{\partial\Sigma_{\theta'}} f(z)R(z,A)g(z)dz = (f\cdot g)(A)$$

□

4.2 Extending the $H_0^\infty(\Sigma_\omega)$-Calculus.

We now understand a functional calculus on the restrictive class $H_0^\infty(\Sigma_\omega)$. Our next move is to extend this calculus to more general functions. We first prove some preparatory lemmas.

Lemma 4.9. *Let $(f_n)_{n\in\mathbb{N}}, f \in H^\infty(\Sigma_\omega)$ such that $f_n \to f$ pointwise and the f_n are uniformly bounded by some C. Then for all $g \in H_0^\infty(\Sigma_\omega)$, we have $(f_n\cdot g)(A) \to (f\cdot g)(A)$ in $\mathcal{L}(X)$.*

Proof. Since $f_n(z)g(z) \leq ||f_n||_{H^\infty(\Sigma_\omega)}g(z)$ for each n, we see that f_ng and fg are in $H_0^\infty(\Sigma_\omega)$, and we can apply the dominated convergence theorem to see:

$$(f_n\cdot g)(A) = \frac{1}{2\pi i}\int_{\partial\Sigma_{\theta'}} f_n(z)g(z)R(z,A)g(z)dz \to \frac{1}{2\pi i}\int_{\partial\Sigma_{\theta'}} f(z)g(z)R(z,A)g(z)dz = (f\cdot g)(A)$$

□

Lemma 4.10. *If $f(z) = \dfrac{z}{(\alpha-z)(\beta-z)}$ with $\alpha,\beta \notin \overline{\Sigma_\omega}$ then $f(A) = AR(\alpha,A)R(\beta,A)$*

Proof. First note that $R(\cdot,A) : \rho(A) \to \mathcal{L}(X)$ is holomorphic and Banach space valued, and so Cauchy's Theorem applies. Now, if $x \in \text{Ran}(A)$ then we can say $x = Ay$ and for $z \notin \Sigma_{\theta'}$ we have

$$||R(z,A)x|| = ||R(z,A)Ay|| = ||zR(z,A)y - y|| \leq C||y|| + ||y||$$

Therefore $R(z, A)x$ is bounded as $|z| \to 0$. Therefore given $\alpha \notin \overline{\Sigma_{\theta'}}$, the integral

$$\frac{1}{2\pi i}\int_{\partial\Sigma_{\theta'}} \frac{1}{\alpha - z}R(z, A)xdz$$

is a convergent Bochner integral, since we have a growth bound as $|z| \to 0$, and the integral decays like $\frac{1}{|z|^2}$ as $|z| \to \infty$.

Now, since $||R(z, A)x||$ is bounded, we define $B(0, \epsilon)$ to be the ball of radius ϵ centred at zero, and we have for $\epsilon < |\alpha|$:

$$\frac{1}{2\pi i}\int_{\partial\Sigma_{\theta'}} \frac{1}{\alpha - z}R(z, A)xdz = \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\epsilon)} \frac{1}{\alpha - z}R(z, A)xdz$$

But then by Cauchy's Theorem, there exists some small $0 < \delta < |\alpha|$ such that, $\frac{1}{\alpha - z}R(z, A)x$ is holomorphic on $\overline{B(0, \delta) \cap B(0, \epsilon)^c \cap (\Sigma_{\theta'})^c}$ for all $\epsilon < \delta$, and so:

$$\frac{1}{2\pi i}\int_{\partial\Sigma_{\theta'}} \frac{1}{\alpha - z}R(z, A)xdz = \frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} \frac{1}{\alpha - z}R(z, A)xdz$$

Reversing the direction of orientation gives

$$= \frac{-1}{2\pi i}\int_{\partial([\Sigma_{\theta'}\cup B(0,\delta)]^c)} \frac{1}{\alpha - z}R(z, A)xdz$$

We are now integrating over a domain on which $R(z, A)x$ is holomorphic, and this function decays with $|z|^2$, so by the Cauchy integral formula, we see that this integral is equal to $R(\alpha, A)x$.

Next, for $f(z) = \dfrac{z}{(\alpha - z)(\beta - z)}$, suppose $\alpha \neq \beta$. Then we may write

$$f(z) = \frac{1}{\alpha - \beta}\left(\frac{\alpha}{\alpha - z} - \frac{\beta}{\beta - z}\right)$$

And from our result above, we have for $x \in \mathrm{Ran}(A)$

$$\begin{aligned} f(A)x &= \frac{1}{2\pi i}\int_{\partial\Sigma_{\theta'}} \frac{1}{\alpha - \beta}\left(\frac{\alpha}{\alpha - z} - \frac{\beta}{\beta - z}\right)R(z, A)x \\ &= \frac{1}{\alpha - \beta}[\alpha R(\alpha, A) - \beta R(\beta, A)]x \\ &= \frac{1}{\alpha - \beta}A[R(\alpha, A) - R(\beta, A)]x \\ &= AR(\alpha, A)R(\beta, A)x \end{aligned}$$

Since $\mathrm{Ran}(A)$ is dense by Remark 4.5 and $f(A)$ is bounded, the lemma follows for all $x \in X$.

Now suppose $\alpha = \beta$. Choose a sequence $a_n \in (\overline{\Sigma_\omega})^c$ with $a_n \to \alpha$ such that $a_n \neq \alpha$ for all n. Define

$$g_n(z) = \frac{\sqrt{z}}{a_n - z} \text{ and } g(z) = \frac{\sqrt{z}}{\alpha - z}$$

Then g_n and g are in $H_0^\infty(\Sigma_\omega)$, and from Lemma 4.9 we have $f(A) = g^2(A) = \lim_n (g_n \cdot g)(A)$. Thus the case $\alpha \neq \beta$ applies and we have $(g_n \cdot g)(A) = AR(a_n, A)R(\alpha, A) \to AR(\alpha, A)^2$. This concludes the proof of the lemma. $\square$

Lemma 4.11. *Define* $\phi(z) := \dfrac{z}{(1+z)^2}$*, a function that is clearly in* $H_0^\infty(\Sigma_\omega)$ *for all* $\omega < \pi/2$*. Then* $\mathrm{Ran}(\phi(A))$ *is dense in* X*.*

Proof. The proof of this lemma involves several steps.

Step (1): From Lemma 4.10 we know that $\phi(A) = A(1+A)^{-2}$.

Step (2): I claim that $D(A) \cap \mathrm{Ran}(A) \subset \mathrm{Ran}(\phi(A))$. Indeed, if $x \in D(A) \cap \mathrm{Ran}(A)$ then, recalling our assumption that A is injective and thus A^{-1} is defined on $\mathrm{Ran}(A)$, we can define:

$$y := (1+A)^2 A^{-1} x = A^{-1}x + 2x + Ax$$

And then $\phi(A)y = x$. Therefore we are done if we can show that $D(A) \cap \mathrm{Ran}(A)$ is dense in X.

Step (3): Define

$$\phi_n(z) := \frac{n}{n+z} - \frac{1}{1+nz} = \left(\frac{n^2-1}{n}\right) \frac{z}{(n+z)(\frac{1}{n}+z)}$$

Then from Lemma 4.10, $\phi_n(A) = \left(\frac{n^2-1}{n}\right) A(n+A)^{-1}(\frac{1}{n}+A)^{-1}$. I claim that $\mathrm{Ran}(\phi_n(A)) \subset D(A) \cap \mathrm{Ran}(A)$. Indeed, we may write $\phi_n(A)$ in the equivalent forms:

$$\begin{aligned} (1) \quad \phi_n(A) &= \left(\frac{n^2-1}{n}\right)(n+A)^{-1}\left[A(\frac{1}{n}+A)^{-1}\right] \\ &= \left(\frac{n^2-1}{n}\right)(n+A)^{-1}\left[I - (\frac{1}{n}+A)^{-1}\right] \end{aligned}$$

and

$$(2) \quad \phi_n(A) = \left(\frac{n^2-1}{n}\right) A \left[(n+A)^{-1}(\frac{1}{n}+A)^{-1}\right]$$

In (1), we see that $I - (\frac{1}{n}+A)^{-1}$ is bounded and $(n+A)^{-1}$ maps into $D(A)$, so $\phi_n(A)$ maps into $D(A)$. Then (2) tells us that $\phi_n(A)$ maps into $\mathrm{Ran}(A)$, and we conclude that $\mathrm{Ran}(\phi_n(A)) \subset D(A) \cap \mathrm{Ran}(A)$ for each n.

Step (4): We show that $\phi_n(A)x \to x$ for all $x \in X$. We have by the resolvent equation:

$$\phi_n(A) = \left(\frac{n^2-1}{n}\right) A(n+A)^{-1}(\frac{1}{n}+A)^{-1} = n(n+A)^{-1} + \frac{1}{n}\frac{1}{n}(\frac{1}{n}+A)^{-1}$$

We will show that $n(n+A)^{-1}x \to x$ and $\frac{1}{n}(\frac{1}{n}+A)^{-1}x \to 0$. For $y \in D(A)$, we have

$$||n(n+A)^{-1}y - y|| = ||A(n+A)^{-1}y|| \leq ||R(-n, A)|| \cdot ||Ay||$$

And since A is sectorial, $||R(-n, A)|| \to 0$. Then, for general $x \in X$, we choose an approximating sequence y_k and, since $n(n+A)^{-1}$ is uniformly bounded in n, the result extends by density. To show that $\frac{1}{n}(\frac{1}{n}+A)^{-1}x \to 0$, for $y \in \text{Ran}(A)$ we may write $y = Az$ and

$$\begin{aligned}||\frac{1}{n}(\frac{1}{n}+A)^{-1}y|| &= \frac{1}{n}||(\frac{1}{n}+A)^{-1}Az|| \\ &\leq \frac{1}{n}||I - \frac{1}{n}(\frac{1}{n}+A)^{-1}|| \cdot ||z|| \\ &\to 0\end{aligned}$$

Noting that $||\frac{1}{n}(\frac{1}{n}+A)^{-1}||$ is uniformly bounded in n. The result again extends by density to general $x \in X$ (recalling from Remark 4.5 that A has dense range). Therefore $\phi_n(A)x \to x$.

Step (5): We have $\phi_n(A)x \to x$, therefore $\cup_n \text{Ran}(\phi_n(A))$ is dense. But each ϕ_n takes values in $D(A) \cap \text{Ran}(A)$ from Step (3), therefore $D(A) \cap \text{Ran}(A)$ is dense, and therefore by Step (2), $\text{Ran}(\phi(A))$ is dense in X. □

With these lemmas in hand, we are ready to extend the H_0^∞-calculus.

Definition 4.12. Let A be sectorial of angle θ, and $\theta < \omega$. For $f \in H_0^\infty(\Sigma_\omega)$, define

$$||f||_{H_A^\infty(\Sigma_v)} = ||f||_{H^\infty(\Sigma_\theta)} + ||f(A)||_{\mathcal{L}(X)}$$

Then we can define $H_A^\infty(\Sigma_\theta)$ as the class of functions $f \in H^\infty(\Sigma_\theta)$ such that there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $H_0^\infty(\Sigma_\theta)$ with $f_n \to f$ pointwise, and the $||f_n||_{H_A^\infty(\Sigma_\theta)}$ norms uniformly bounded.

Theorem 4.13. *Let A be a sectorial operator of angle θ on a Banach space X. Then the map $f \to f(A)$ defined on $H_0^\infty(\Sigma_\omega)$ in Theorem 4.8 has an extension defined on $H_A^\infty(\Sigma_\omega)$ that is linear and multiplicative.*

Proof. Take $f \in H_A^\infty(\Sigma_\theta)$ and a sequence $(f_n)_{n\in\mathbb{N}} \subset H_0^\infty(\Sigma_\theta)$ that converges pointwise to f, such that the norms $||f_n||_{H_A^\infty(\Sigma_\theta)}$ are uniformly bounded by C. Then we want to define for $x \in X$:

$$f(A)x := \lim_{n\to\infty} f_n(A)x$$

However, we don't know *a priori* that this limit exists, so we must take a slightly different approach. Using the function ϕ defined in Lemma 4.11 and the convergence result from Lemma 4.9, we see that for all $x \in X$:

$$f_n(A)(\phi(A)x) = (f_n \cdot \phi)(A)x \to (f \cdot \phi)(A)x$$

Therefore for all y in the dense subset $\text{Ran}(\phi(A))$, we see:

$$\lim_{n\to\infty} f_n(A)y = \lim_{n\to\infty} f_n(A)\phi(A)[\phi(A)^{-1}y] = (f \cdot \phi)(A)[\phi(A)^{-1}y]$$

And so this limit exists. We also have $||f(A)y|| \leq \sup_n ||f_n(A)|| \cdot ||y|| \leq C||y||$ and so $f(A)$ (defined on $\text{Ran}(\phi(A))$) is bounded in norm by C. Note that this definition is independent of the approximating sequence f_n.

Then for general $x \in X$, choose a sequence $y_k \to x$ and define $f(A)x = \lim_{k\to\infty} f(A)y_k$. This limit exists and is independent of the sequence y_k by the boundedness of $f(A)$ on $\mathrm{Ran}(\phi(A))$. We also retain our original 'desired' definition, $f(A)x = \lim_{n\to\infty} f_n(A)x$.

For $f \in H_0^\infty(\Sigma_\theta)$, choosing the stationary sequence $f_n = f$ for all n shows that this map is in fact an extension.

Linearity clearly follows from the limiting process. Multiplicativity also follows from the limiting process: take functions $f, g \in H_A^\infty(\Sigma_\theta)$, and sequences $f_n \to f$ and $g_n \to g$ in $H_0^\infty(\Sigma_\theta)$ both uniformly bounded by C. Then for arbitrary $x \in X$ and $\epsilon > 0$, choose N such that $||f(A)[g(A)x] - f_N(A)[g(A)x]|| \leq \epsilon$, $||g(A)x - g_N(A)x|| \leq \epsilon/C$, and $||(f_N \cdot g_N)(A)x - (f \cdot g)(A)x|| \leq \epsilon$. Then we have

$$\begin{aligned} ||f(A)g(A)x - (f\cdot g)(A)x|| &\leq ||f(A)[g(A)x] - f_N(A)[g(A)x]|| + ||f_N(A)[g(A)x] - f_N(A)[g_N(A)x]|| \\ &\quad + ||f_N(A)g_N(A)x - (f_N \cdot g_N)(A)x|| + ||(f_N \cdot g_N)(A)x - (f\cdot g)(A)x|| \\ &\leq \epsilon + C\frac{\epsilon}{C} + 0 + \epsilon \end{aligned}$$

And since ϵ was arbitrary, we have $f(A)g(A)x = (f\cdot g)(A)x$ for all x and thus the map $f \to f(A)$ is multiplicative. ☐

We now have a calculus that can apply to lots of functions. We include some useful properties of the calculus.

Proposition 4.14. *The function* $\dfrac{1}{\alpha - z}$ *is in* $H_A^\infty(\Sigma_\omega)$ *for all* $\alpha \notin \overline{\Sigma_\theta}$, *and* $\frac{1}{\alpha - A} = R(\alpha, A)$

Proof. Define for each n:

$$f_n(z) = \frac{-z}{(\alpha - z)(\frac{-1}{n} - z)} = \frac{1}{(\alpha - z)} + \frac{1}{n(\alpha - z)(\frac{-1}{n} - z)}$$

Then $f_n(z) \to \frac{1}{\alpha - z}$ pointwise and the $||f_n||_{H^\infty(\Sigma_\omega)}$ are uniformly bounded. By Lemma 4.10:

$$f_n(A) = -AR(\frac{-1}{n}, A)R(\alpha, A) = R(\alpha, A)[A(\frac{1}{n} + A)^{-1}]$$

which is a uniformly bounded sequence in $\mathcal{L}(X)$. Therefore $f_n \to f$ pointwise with uniformly bounded $H_A^\infty(\Sigma_\theta)$ norms, and so $f(A) = \lim_{n\to\infty} f(A)$ (in the strong topology). Finally we note as in Step (4) of the proof of Lemma 4.11, $A(\frac{1}{n} + A)^{-1}x \to x$ as $n \to \infty$, so $f_n(A) \to R(\alpha, A)$ (strongly). Therefore $f(A) = R(\alpha, A)$. ☐

We can apply the functional calculus to functions that decay only at ∞, provided that such functions are holomorphic near zero:

Proposition 4.15. *Let* $f \in H^\infty(\Sigma_\omega)$ *satisfy:*
(1) there exists $\delta > 0$ *such that* f *is holomorphic on a neighbourhood of* $B(0, \delta)$, *and*
(2) $|f(z)| \leq \dfrac{C}{(1 + |z|)^\epsilon}$ *for some* C, ϵ *and all* $z \in \Sigma_\omega$.
Then $f \in H_A^\infty(\Sigma_\omega)$ *and we have the formula*

$$f(A) = \frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'} \cup B(0,\delta))} f(z)R(z, A)dz$$

Proof. With ϕ as in Lemma 4.11, let $f_n(z) := f(z)\phi(z)^{\frac{1}{n}}$. Then each $f_n(z)$ is also holomorphic on a neighbourhood of $B(0,\delta)$. By a similar method to Lemma 4.10, we note that since $f_n(z)$ is continuous, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'})} f_n(z)R(z,A)dz &= \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\epsilon))} f_n(z)R(z,A)dz \\ &= \frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} f_n(z)R(z,A)dz\end{aligned}$$

with the second line following from the Cauchy Theorem. Then we have the estimate, using the fact that $||zR(z,A)|| < C'$ for some C' on this path:

$$\begin{aligned}||f_n(A)|| &= \frac{1}{2\pi}\left\|\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} f_n(z)R(z,A)dz\right\| \\ &\leq \frac{C'}{2\pi}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))}\frac{|f_n(z)|}{|z|}d|z| \\ &\leq \frac{C'}{2\pi}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))}\frac{|f(z)|}{|z|}d|z| \\ &\leq \frac{CC'}{2\pi}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))}\frac{1}{(1+|z|)^{\epsilon}|z|}d|z| \\ &< \infty\end{aligned}$$

Therefore we have that the f_n are uniformly bounded in H_A^∞ norm, and so $f \in H_A^\infty(\Sigma_\omega)$. The formula for $f(A)$ follows by the dominated convergence theorem. □

Proposition 4.16. *Let $(f_n)_{n=1}^\infty$ be a sequence in $H_A^\infty(\Sigma_\omega)$ such that $f_n \to f$ pointwise and the $||f_n||_{H_A^\infty(\Sigma_\omega)}$ are uniformly bounded by some C. Then f is in $H_A^\infty(\Sigma_\omega)$ and $f_n(A) \to f(A)$ strongly.*

Proof. Note that since $f_n \to f$, we have $f \in H^\infty(\Sigma_\omega)$. As in Lemma 4.11, define

$$\phi_m(z) := \frac{m}{m+z} - \frac{1}{1+mz}$$

Then $(f\phi_m)_{m=1}^\infty$ is also sequence in $H_0^\infty(\Sigma_\omega)$ that converges pointwise to f. To show that $||f\phi_m||_{H_A^\infty(\Sigma_\omega)}$ is uniformly bounded, note that for fixed m, $f_n\phi_m$ converges pointwise to $f\phi_m$, and so:

$$\begin{aligned}||(f\phi_m)(A)||_{\mathcal{L}(X)} &\leq \sup_n ||(f_n\phi_m)(A)||_{\mathcal{L}(X)} \\ &= \sup_n ||f_n(A)\phi_m(A)||_{\mathcal{L}(X)} \\ &\leq C||\phi_m(A)||_{\mathcal{L}(X)}\end{aligned}$$

And recall from Step (4) of Lemma 4.11 that $\phi_m(A) = mR(-m,A) + \frac{1}{m}R(\frac{-1}{m},A)$, and so $||\phi_m(A)|| \leq 2\sup_{z\in\Sigma_\omega^c}||zR(z,A)|| < \infty$. Therefore f is in $H_A^\infty(\Sigma_\omega)$. To show that $f_n(A) \to f(A)$ strongly, we note that $f_n(A)\phi_m(A)x = (f_n\phi_m)(A)x \to (f\phi_m)(A)x = f(A)\phi_m(A)x$, and so $f_n(A)y \to f(A)y$ for all y of the form $\phi_m(A)x$. We proved in Lemma 4.11 than $\phi_m(A)$ has dense range, and since the $||f_n(A)||$ are bounded, the result extends by density. □

Corollary 4.17. *The map $f \to f(A)$ is a closed operator (with domain $H_A^\infty(\Sigma_\omega)$) from $H^\infty(\Sigma_\omega)$ (with the $||\cdot||_{H^\infty(\Sigma_\omega)}$ topology) to $\mathcal{L}(X)$ (with the operator topology).*

Proof. Take some sequence $f_n \to f$ in $||\cdot||_{H^\infty(\Sigma_\omega)}$ norm such that $f_n(A)$ converges to some limit. Then the $||f_n||_{H^\infty(\Sigma_\omega)}$ norm and the $||f_n(A)||_{\mathcal{L}(X)}$ norm are both uniformly bounded, thus the $||f_n||_{H_A^\infty(\Sigma_\omega)}$ norms are uniformly bounded. We also have that $f_n \to f$ pointwise. Therefore, by Proposition 4.16, $f \in H_A^\infty(\Sigma_\omega)$ and $f_n(A) \to f(A)$ strongly, and therefore also in operator norm since we have assumed that $f_n(A)$ has a limit with respect to this norm. This exactly tells us that $f \to f(A)$ is closed. □

At this point we can state our main definition:

Definition 4.18. An sectorial operator A of angle θ has a *McIntosh calculus* of angle $\Theta > \theta$ if $H^\infty(\Sigma_\Theta) = H_A^\infty(\Sigma_\Theta)$, that is, if we can map any bounded analytic function f on Σ_Θ to an operator $f(A)$ as in Theorem 4.13.

Remark 4.19. This calculus is referred to as the H^∞-calculus in the literature. This thesis has chosen the term 'McIntosh calculus' to acknowledge the contribution of Alan McIntosh.

We have the following characterisation of operators with a McIntosh Calculus:

Proposition 4.20. *A sectorial operator A of angle θ has a McIntosh calculus of angle Θ if and only if there exists a C such that for all $f \in H_0^\infty(\Sigma_\Theta)$*

$$||f(A)||_{\mathcal{L}(X)} \leq C||f||_{H^\infty(\Sigma_\Theta)}$$

Proof. Suppose the inequality holds. As in Lemma 4.11, define

$$\phi_n(z) := \frac{n}{n+z} - \frac{1}{1+nz}$$

Then ϕ_n is in $H_0^\infty(\Sigma_\Theta)$ for all n. Then for $f \in H^\infty(\Sigma_\Theta)$, we have $f\phi_n \to f$ pointwise, implying that $||f\phi_n||_{H^\infty(\Sigma_\Theta)}$ is uniformly bounded, and then we clearly have:

$$\sup_n ||f_n||_{H_A^\infty(\Sigma_\Theta)} \leq \sup_n ||f_n||_{H^\infty(\Sigma_\Theta)} + \sup_n ||f_n(A)||_{\mathcal{L}(X)} = (C+1)\sup_n ||f||_{H^\infty(\Sigma_\Theta)} < \infty$$

Therefore $f \in H_A^\infty(\Sigma_\Theta)$ (and so $f(A)$ is well defined).

For the converse, we use the closed graph theorem since from Corollary 4.17, the map $f \to f(A)$ is closed. □

Remark 4.21. Verifying that any particular operator has a McIntosh calculus is beyond the scope of this thesis, but a large number of operators are known to have such a calculus. In particular, recall our discussion in Remark 4.4 of operators of the form $A = \mathrm{div}\,(M\nabla)$. Chapter 6 of [1] demonstrates that in fact, these operators have a McIntosh calculus whenever they are sectorial. One can say even more - the notions of sectoriality and McIntosh calculus are equivalent in all known examples of *differential* operators. There are however sectorial operators constructed using functional analytic methods that do not possess a McIntosh calculus. The interested reader is directed to Chapter 10 of [29] (and references therein) for such examples, as well as further demonstrations that various differential operators have a McIntosh calculus.

4.3 The McIntosh Calculus for Operator-Valued functions

We shall next outline the construction of a functional calculus for certain bounded holomorphic functions $F : \Sigma_\omega \to \mathcal{L}(X)$. This construction is identical to the preceding construction concerning scalar valued functions, and so the next two pages consist mainly of restating theorems in the new case. Proof and explanations point out only relevant differences.

Definition 4.22. Let $A : X \to X$ be sectorial of angle θ. Let $\mathcal{A}$ be the sub-algebra of $\mathcal{L}(X)$ consisting of all operators that commute with the resolvent $R(z, A)$ (for all z).

Define for $\theta < \omega < \pi/2$ the set $RH^\infty(\Sigma_\omega, \mathcal{A})$ to be the set of bounded holomorphic functions $F : \Sigma_w \to \mathcal{L}(X)$ such that:
(1) The function F takes values in $\mathcal{A}$.
(2) The set $\{F(z); z \in \Sigma_\omega\}$ is an R-bounded subset of $\mathcal{L}(X)$, with R-bound denoted by $||F||_{RH^\infty}$.

Then define $RH_0^\infty(\Sigma_w, \mathcal{A})$ as the functions satisfying for all $z \in \Sigma_\omega$ and for some $C, \epsilon > 0$:

$$||F(z)||_{\mathcal{L}(X)} \leq C \left| \frac{z}{(1 + |z|)^2} \right|^\epsilon$$

Then, analogously to Theorem 4.8, we have

Proposition 4.23. *Let $A : X \to X$ be sectorial of angle θ . Then for all $\theta < \theta' < w$ there is a linear and multiplicative map that is independent of θ', denoted by $\Phi_A : RH_0^\infty(\Sigma_w, \mathcal{A}) \to \mathcal{L}(X)$ and defined by*

$$\Phi_A(F) := F(A) := \frac{1}{2\pi i} \int_{\partial\Sigma_{\theta'}} F(z)R(z, A)dz$$

Proof. The proof proceeds as in Theorem 4.8. In order the prove that this map is multiplicative, we need the fact that all functions in $RH_0^\infty(\Sigma_\omega)$ commute with the resolvent. We do not use the R-boundedness condition at this stage (or anywhere else before Theorem 4.30). □

Lemma 4.24. *Let F_n, F be uniformly bounded in $RH^\infty(\Sigma_\omega)$ such that $F_n(z) \to F(z)$ for $z \in \Sigma_\omega$ in $\mathcal{L}(X)$ norm. Then for all $G \in RH_0^\infty(\Sigma_\omega)$ we have $\lim_{n\to\infty} \Phi_A(F_n \cdot G) = \Phi_A(F \cdot G)$ in $\mathcal{L}(X)$.*

Proof. This follows by the Dominated Convergence Theorem for operator valued functions, since $||F_n(z)G(z)R(z, A)|| \leq \sup_{n,z} ||F_n(z)|| \cdot ||G(z)R(z, A)||$:

$$\Phi_A(F_n \cdot G) = \frac{1}{2\pi i} \int_{\partial\Sigma_{\theta'}} F_n(z)G(z)R(z, A) \to \frac{1}{2\pi i} \int_{\partial\Sigma_{\theta'}} F(z)G(z)R(z, A)dz = \Phi_A(F \cdot G)$$

□

Remark 4.25. The proofs of Proposition 4.23 and Lemma 4.24 can be adapted show that for $f \in H_0^\infty(\Sigma_\omega)$ and $F \in RH_0^\infty(\Sigma_\omega)$, we have that $F(A)f(A) = (f \cdot F)(A)$ and $\lim_{n\to\infty} \Phi_A(F_n \cdot f) = \Phi_A(F \cdot f)$ in $\mathcal{L}(X)$.

Definition 4.26. Let A be sectorial of angle θ, and $\theta < \omega$. For $F \in RH_0^\infty(\Sigma_\omega)$, define

$$||F||_{RH_A^\infty(\Sigma_\omega)} = ||F||_{RH^\infty(\Sigma_\omega)} + ||F(A)||_{\mathcal{L}(X)}$$

Then we can define $RH_A^\infty(\Sigma_\omega)$ as the class of functions $F \in RH^\infty(\Sigma_\omega)$ such that there exists a sequence in $RH_0^\infty(\Sigma_\omega)$ with $F_n(z) \to F(z)$ pointwise in $\mathcal{L}(X)$ norm, and the $||F_n||_{RH_A^\infty(\Sigma_\omega)}$ norms uniformly bounded.

Proposition 4.27. *Let $A : X \to X$ be sectorial of angle θ. Then for all $\omega > \theta$ the map Φ_A defined in Proposition 4.23 extends to a map $\Phi_A : RH_A^\infty(\Sigma_\omega, \mathcal{A}) \to \mathcal{L}(X)$ that remains linear and multiplicative.*

Proof. The proof proceeds exactly as in Theorem 4.13. □

Definition 4.28. An sectorial operator A of angle θ has an *R-McIntosh calculus* of angle $\Theta > \theta$ if $RH^\infty(\Sigma_\Theta) = RH_A^\infty(\Sigma_\Theta)$.

Proposition 4.29. *As in Proposition 4.20, A has an* R-McIntosh calculus *if and only if there exists a C such that for all $F \in RH_0^\infty(\Sigma_\Theta)$:*

$$||F(A)||_{\mathcal{L}(X)} \leq C||F||_{RH^\infty(\Sigma_\omega)}$$

After retracing our steps, we come to an important and deep theorem that is difficult to prove:

Theorem 4.30. *Let $A : X \to X$ have a McIntosh calculus of angle Θ. Then A also has an R-McIntosh calculus of angle Θ.*

Proof. This proof is difficult and involves the concept of a γ-norm, which will be introduced in Chapter 6; thus we delay the proof until this time. This proof is also where we finally use the fact that for each F, the set $\{F(z); z \in \Sigma_\omega\}$ is R-bounded. □

4.4 Further Reading

This Chapter is adapted primarily from Chapter 9 of [29]; the reader may also consult chapters 2 and 5 from [19]. The original construction of the McIntosh calculus may be found in [35] and [11]. The discussion of the R-McIntosh calculus is adapted from Theorem 12.7 in [29]; ideas for an operator-valued functional calculus were first discussed in [32] and [31], and then expanded to the general case in [27].

Chapter 5

Deterministic Maximal L^p-Regularity

5.1 Analytic Semigroups and PDEs

5.1.1 Motivation

We want to study parabolic PDEs of the form

$$\frac{d}{dt}u(t,x) + Au(t,x) = f(t,x), \quad t \geq 0, \quad x \in \mathbb{R}^n$$
$$u(0,x) = u_0(x)$$

where $u, f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ are functions and A is some partial differential operator in the x coordinate. One method by such equations can be solved is to convert this partial differential equation into an ordinary differential equation of Banach-valued functions. We identify u and f with functions $u : \mathbb{R}_+ \to (x \to u(t,x))$ and $f : \mathbb{R}_+ \to (x \to f(t,x))$, where the functions $x \to u(t,x)$ and $x \to f(t,x)$ live in a suitable Banach space X. We then identify the partial differential operator A with an unbounded operator $A : X \to X$. Thus we have converted our PDE into the ODE:

$$\begin{aligned} \frac{d}{dt}u(t) + Au(t) &= f(t), \quad t \geq 0 \\ u(0) &= u_0. \end{aligned} \tag{IACP}$$

Such an ODE is called an *inhomogeneous abstract Cauchy problem.* We may also consider the *homogeneous abstract Cauchy problem* in the case $f \equiv 0$:

$$\begin{aligned} \frac{d}{dt}u(t) + Au(t) &= 0, \quad t \geq 0 \\ u(0) &= u_0. \end{aligned} \tag{HACP}$$

Examining equation (HACP), if we naively transfer across our intuition from the scalar case, we expect our solution to be of the form

$$u(t) = e^{-tA}u_0$$

If the operator A is sectorial, then we can make sense of the object e^{-tA} through the McIntosh calculus, and then we shall show that this is in fact the unique solution of (HACP).

5.1.2 Analytic Semigroups

Proposition 5.1. *Take a sectorial operator A of angle θ on a Banach space X. Then for all $\theta < \omega < \pi/2$, the function e^{-tz} (with fixed $t > 0$) is in $H_A^\infty(\Sigma_\omega)$ and for arbitrary fixed $\delta > 0$, we have*

$$e^{-tA} = \frac{1}{2\pi i} \int_{\partial(\Sigma_{\theta'} \cup B(0,\delta))} e^{-tz} R(z,A) dz$$

Moreover, the operators e^{-tA} are uniformly bounded in norm with respect to t.

Proof. The fact that $e^{-tz} \in H_A^\infty(\Sigma_\omega)$ follows from Proposition 4.15, because $z \to e^{-tz}$ is analytic in a neighbourhood of $B(0,\delta)$ for all δ, and clearly for any $\theta < \omega < \pi/2$ we have $|e^{-tz}| \leq \dfrac{C}{(1+|z|)^\epsilon}$ for all $z \in \Sigma_\omega$.

Next we want to show that the e^{-tA} are uniformly bounded in t. If we examine the integral

$$\int_{\partial(\Sigma_{\theta'} \cup B(0,\delta))} e^{-tz} R(z,A) dz$$

We see that the norm might grow arbitrarily large as $t \to 0$ because the integral does not decay 'fast' enough. To solve this problem, we note that by sectoriality, $\{||zR(z,A)||; z \in \partial(\Sigma_{\theta'} \cup B(0,\delta))\}$ is uniformly bounded by some C in z, but *also* in δ. In particular, it will still be true for arbitrarily *large* δ. Therefore, given fixed small $t > 0$, we may choose $\delta = \frac{1}{t}$ and write our integral as:

$$e^{-tA} = \frac{1}{2\pi i} \int_{\partial(\Sigma_{\theta'} \cup B(0,\frac{1}{t}))} e^{-tz} R(z,A) dz$$

Then we can uniformly bound $||e^{-tA}||$. To move from the second to the third line in the working below, we split the integral into two parts: the integral over the subset of $B(0,\frac{1}{t})$, and the integral over the straight rays of the segment.

$$\begin{aligned}
||e^{-tA}|| &\leq \frac{1}{2\pi} \int_{\partial(\Sigma_{\theta'} \cup B(0,\frac{1}{t}))} |e^{-tz}| \cdot ||R(z,A)|| d|z| \\
&\leq \frac{C}{2\pi} \int_{\partial(\Sigma_{\theta'} \cup B(0,\frac{1}{t}))} \frac{|e^{-tz}|}{|z|} d|z| \\
&\leq \left(t \cdot \frac{2\pi}{t}\right) + \int_{\frac{1}{t}}^{\infty} \frac{\left|\exp\left(-tse^{i\pm\theta'}\right)\right|}{|se^{i\pm\theta'}|} ds \\
&= 2\pi + 2\int_{\frac{1}{t}}^{\infty} e^{-ts\cos(\theta')} \frac{1}{s} ds \\
&= 2\pi + 2\int_{1}^{\infty} e^{-u\cos(\theta')} \frac{1}{u} du \\
&< \infty \text{ uniformly in } t
\end{aligned}$$

□

Proposition 5.2. *We have* $\lim_{t\to 0} e^{-tA}x = x$.

Proof. By the scalar Cauchy integral formula, we have for all $\delta > 0$ that

$$\frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} \frac{e^{-tz}}{z} x dz = x$$

Choose δ small enough that $|e^{-tz}| \le 1$ uniformly in t on $B(0,\delta)$. Then we have $|e^{-tz}| \le 1+e^{-\mathrm{Re} z}$ for all $t \ge 0$ on $\partial(\Sigma_{\theta'} \cup B(0,\delta))$.

Then, restricting for now to $x \in D(A)$, we have:

$$\begin{aligned} e^{-tA}x - x &\le \frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} e^{-tz}\left(R(z,A) - \frac{1}{z}\right) x dz \\ &= \frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} \frac{e^{-tz}}{z}\left(zR(z,A) - I\right) x dz \\ &= \frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} \frac{e^{-tz}}{z} R(z,A)Ax dz \end{aligned}$$

Now, we have:

$$\left\|\frac{e^{-tz}}{z}R(z,A)Ax\right\| \le \frac{C}{|z|^2}(1+e^{-\mathrm{Re} z})||Ax||dz$$

which is integrable. Therefore, since $e^{-tz} \to 1$ pointwise as $t \to 0$, we have by the dominated convergence theorem:

$$\lim_{t\to 0}\left(e^{-tA}x - x\right) = \frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} \frac{1}{z} R(z,A)Ax dz$$

But this is true for all $\delta > 0$, and we have:

$$\begin{aligned} \left\|\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} \frac{1}{z} R(z,A)Ax dz\right\| &\le C||Ax|| \int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} \frac{1}{|z|^2} d|z| \\ &\to 0 \text{ as } \delta \to \infty \end{aligned}$$

Therefore we have $e^{-tA}x - x \to 0$ for $x \in D(A)$. By the uniform boundedness of the e^{-tA} operators, the result extends to general $x \in X$ by density. □

Remark 5.3. Proposition 5.2 implies that $t \to e^{-tA}$ is strongly continuous, since for arbitrary $t > 0$ we have

$$\lim_{h\to 0} e^{-(t+h)A}x - e^{-tA}x = e^{-tA}\left(\lim_{h\to 0} e^{-hA}x - x\right)$$

We may therefore define $e^{0A} := I$, and we have a strongly continuous family $\{e^{-tA}; t \ge 0\}$ of uniformly bounded operators.

Remark 5.4. By the multiplicativity of the functional calculus, the operators e^{-tA} form a *semigroup*, in the sense that for all $t, s \in \mathbb{R}_+$, we have $e^{-(t+s)A} = e^{-tA}e^{-sA}$, thus these objects have an algebraic group structure but with no inverses (since these exponentials are only defined for positive t, s). For this reason we call the family $\{e^{-tA}, t > 0\}$ an *analytic semigroup*, and we say that A *generates* the semigroup.

Proposition 5.5. *For all $x \in D(A)$, $e^{-tA}x$ is in $D(A)$ and $Ae^{-tA}x = e^{-tA}Ax$. Moreover, $t \to e^{-tA}x$ is differentiable with derivative $-Ae^{-tA}x$.*

Proof. Take $x \in D(A)$, and write $Ax = y$. Then we have, noting that A commutes with its resolvent:

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} Ae^{-tz}R(z,A)xdz &= \frac{1}{2\pi i}\int_{\partial\Sigma_{\theta'}} e^{-tz}R(z,A)Axdz \\ &= \frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} e^{-tz}R(z,A)ydz \\ &= e^{-tA}y\end{aligned}$$

Therefore if $x \in D(A)$ then $e^{-tz}R(z,A)x \in D(A)$ for all $z \in \partial(\Sigma_{\theta'}\cup B(0,\delta))$, and $Ae^{-tz}R(z,A)x$ is Bochner integrable. Therefore by Theorem 1.17, (noting that a sectorial operator is closed by definition), we have

$$Ae^{-tA}x = A\frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} e^{-tz}R(z,A)xdz = \frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} Ae^{-tz}R(z,A)xdz = e^{-tA}Ax$$

Next we show differentiability. Examining the operator $\frac{d}{dt}$, we have for $x \in D(A)$:

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} \frac{d}{dt}e^{-tz}R(z,A)xdz &= \frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} -ze^{-tz}R(z,A)xdz \\ &= \frac{-1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} e^{-tz}(z-A+A)R(z,A)xdz \\ &= \frac{-1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} e^{-tz}xdz + \frac{-1}{2\pi i}A\int_{\partial\Sigma_{\theta'}} e^{-tz}R(z,A)xdz \\ &= -Ae^{-tA}x\end{aligned}$$

With the first integral evaluating to zero by the Cauchy integral formula. Therefore, applying Theorem 1.17 again, we see that $t \to e^{-tA}x$ is differentiable and $\frac{d}{dt}e^{-tA} = -Ae^{-tA}$. □

Next we describe an important property of analytic semigroups:

Definition 5.6. The semigroup $\{e^{-tA}; t \geq 0\}$ is said to be *exponentially stable* if there exists $C, \epsilon > 0$ such that for all $t \geq 0$:

$$||e^{-tA}|| \leq Ce^{-\epsilon t}$$

Remark 5.7. Analytic semigroups are not always exponentially stable. As a simple example, suppose there exists $x \in X$ such that $Ax = 0$. Then we have $R(z,A)x = \frac{x}{z}$, and so by the

Cauchy Integral Formula:

$$e^{-tA}x = \frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} e^{-tz}R(z,A)xdz = \frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} \frac{e^{-tz}}{z}xdz = x$$

and so we clearly have $||e^{-tA}|| \geq 1$ for all t. We can however, obtain exponentially stable semigroups in the following way:

Proposition 5.8. *If A is a sectorial operator of angle θ then for all $\epsilon > 0$ the operator $A+\epsilon I$ is also sectorial of angle at most θ, and $A+\epsilon I$ generates an exponentially stable analytic semigroup.*

Proof. We have $R(z, A+\epsilon I) = R(z-\epsilon, A)$, therefore the spectrum of $A+\epsilon I$ is equal to the spectrum of A translated to the right by ϵ. It is easy to see that the growth bound $||zR(z, A+\epsilon I)||$ still holds on Σ_θ. Therefore $A+\epsilon I$ is sectorial of angle at most θ.

Next, we see that

$$\begin{aligned} e^{-t(A+\epsilon I)} &= \frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} e^{-tz}R(z,A+\epsilon I)dz \\ &= \frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} e^{-tz}R(z-\epsilon,A)dz \end{aligned}$$

Change variables $z \to \zeta + \epsilon$, and note that by the Cauchy Integral formula, we may integrate over the same path as before:

$$\begin{aligned} &= \frac{1}{2\pi i}\int_{\partial(\Sigma_{\theta'}\cup B(0,\delta))} e^{-t\zeta}e^{-t\epsilon}R(\zeta,A)d\zeta \\ &= e^{-\epsilon t}e^{-tA} \\ \implies ||e^{-t(A+\epsilon)}|| &\leq e^{-\epsilon t}||e^{-t(A+\epsilon I)}|| \\ &\leq Ce^{-\epsilon t} \end{aligned}$$

where C is the uniform bound of $\{e^{-tA}; t \geq 0\}$. □

We conclude with an important lemma:

Lemma 5.9. *Given a sectorial operator A on a Banach space X of angle θ, and a point α such that $\operatorname{Re}\alpha < 0$, we have*

$$R(\alpha, A) = -\int_0^\infty e^{\alpha t}e^{-tA}dt$$

If $\operatorname{Re}\alpha = 0$ then the result still holds provided that $\{e^{-tA}; t \geq 0\}$ is exponentially stable.

Proof. We have

$$\int_0^\infty e^{\alpha t}e^{-tz}dt = \int_0^\infty e^{t(\alpha - z)}dt = \frac{-1}{\alpha - z}$$

Let $\Phi : H_A^\infty(\Sigma_\omega) \to \mathcal{L}(X)$ be the operator that maps a function f to its relevant operator $f(A)$.

Then we have, since the e^{-tA} are uniformly bounded:

$$\left\|\int_0^\infty \Phi(e^{\alpha t}e^{-tz})dt\right\| \leq \int_0^\infty \left|e^{\alpha t}\right| \cdot ||e^{-tA}||dt$$

If $\mathrm{Re}\,\alpha < 0$ or $\{e^{-tA}; t \geq 0\}$ is exponentially stable then this integral converges, therefore $\Phi(e^{\alpha t}e^{-tz})$ is Bochner integrable. The operator Φ is closed from Corollary 4.17, and so we may apply Theorem 1.17 to commute Φ past the integral, and Proposition 4.14 to recall that $R(\alpha, A) = \Phi\left(\frac{1}{\alpha - z}\right)$, and we have:

$$R(\alpha, A) = \Phi\left(\frac{1}{\alpha - z}\right) = \Phi\left(-\int_0^\infty e^{\alpha t}e^{-tz}dt\right) = -\int_0^\infty \Phi(e^{\alpha t}e^{-tz})dt = -\int_0^\infty e^{\alpha t}e^{-tA}dt$$

□

5.2 Analytic Semigroups as Solutions to ODEs

We now understand enough about analytic semigroups to use them to solve Banach-valued ODEs. We first turn to the homogeneous abstract Cauchy problem:

$$\begin{aligned} \frac{d}{dt}u(t) + Au(t) = 0, \quad t \geq 0 \\ u(0) = u_0. \end{aligned} \tag{HACP}$$

Definition 5.10. A function $u : \mathbb{R}_+ \to X$ is called a *classical solution* of (HACP) if u is differentiable, takes values in $D(A)$, and satisfies $u(0) = u_0$ and $\frac{d}{dt}u(t) + Au(t) = 0$ for each $t \geq 0$.

Theorem 5.11. *Let $A : X \to X$ be a sectorial operator of angle θ. Then, if the initial data u_0 is in $D(A)$, the function $u(t) = e^{-tA}u_0$ is the unique classical solution of (HACP).*

Proof. Propositions 5.2 and 5.5 tells us that $u(t) = e^{-tA}u_0$ is a classical solution to (HACP). To show uniqueness, suppose that there exists a second classical solution $v(t)$. Then by a product rule formula, we have for all $s < t \in \mathbb{R}_+$

$$\begin{aligned} \frac{d}{ds}e^{-(t-s)A}v(s) &= Ae^{-(t-s)A}v(s) + e^{-(t-s)A}v'(s) \\ &= Ae^{-(t-s)A}v(s) - e^{-(t-s)A}Av(s) \\ &= 0 \end{aligned}$$

Where in the third step we note that $e^{-(t-s)A}$ commutes with A by Proposition 5.5, because v takes values in $D(A)$. But this tells us that $e^{-(t-s)A}v(s)$ is constant with respect to s, implying that for all $t \in \mathbb{R}_+$ we have

$$e^{-(t-t)A}v(t) = e^{-(t-0)A}v(0) \implies v(t) = e^{-tA}v(0) = e^{-tA}u_0$$

And so $v(t) = u(t)$ and (HACP) has a unique classical solution. □

Now that we know we are looking in the right place, we expand our concept of solution:

Definition 5.12. A function $u : \mathbb{R}_+ \to X$ is a *mild solution* of (HACP) if u is of the form $e^{-tA}u_0$. A mild solution is therefore a classical solution if $u_0 \in D(A)$.

By construction, there is always a unique mild solution of an equation of the form (HACP), provided that A generates an analytic semigroup. We next move to the inhomogeneous case:

$$\begin{aligned} \frac{d}{dt}u(t) + Au(t) &= f(t), \quad t \geq 0 \\ u(0) &= u_0. \end{aligned} \tag{IACP}$$

Definition 5.13. Let $A : X \to X$ be a sectorial operator of angle θ. Then a function $u : \mathbb{R}_+ \to X$ is called a *strong solution* of (IACP) if:
(1) u is differentiable a.e.
(2) $u' \in L^1_{loc}(\mathbb{R}_+; X)$ and $u(t) = u(t_0) + \int_{t_0}^t u'(s)ds$ for all $t_0 < t$
(3) u takes values in $D(A)$ a.e.
(4) $u(0) = u_0$ and $u'(t) + Au(t) = f(t)$ for almost every t.

Theorem 5.14 (Variation of Constants:)**.** *There is at most one strong solution to (IACP), and if a strong solution $u(t)$ exists then it can be written as the* 'variation of constants' *formula:*

$$u(t) := e^{-tA}u_0 + \int_0^t e^{-(t-s)A}f(s)ds$$

Proof. Suppose there are two strong solutions, $u(t)$ and $v(t)$. Then $w(t) := u(t) - v(t)$ is a solution to the homogeneous ODE

$$\begin{aligned} \frac{d}{dt}u(t) + Au(t) &= 0, \quad t \geq 0 \\ u(0) &= 0 \end{aligned}$$

And we know from Theorem 5.11 that this implies that $w \equiv 0$, i.e. $u \equiv v$.

To show that a strong solution u satisfies the variation of constants formula, define $g(s) := e^{-(t-s)A}u(s)$. Then g is differentiable a.e., and we have by the product rule and the fact that u is a strong solution to (IACP):

$$\begin{aligned} \frac{d}{ds}g(s) &= Ae^{-(t-s)A}u(s) + e^{-(t-s)A}\frac{d}{ds}u(s) \\ &= Ae^{-(t-s)A}u(s) + e^{-(t-s)A}(f(s) - Au(s)) \\ &= e^{-(t-s)A}f(s) \end{aligned}$$

Note that these equalities hold almost everywhere, and that both sides of this equation are locally integrable. Integrate both sides of this equation and use the Fundamental Theorem of Calculus:

$$\begin{aligned}
\int_0^t \frac{d}{ds} g(s)ds &= \int_0^t e^{-(t-s)A} f(s)ds \\
g(t) - g(0) &= \int_0^t e^{-(t-s)A} f(s)ds \\
e^{-(t-t)A}u(t) - e^{-(t-0)A}u(0) &= \int_0^t e^{-(t-s)A} f(s)ds \\
\implies u(t) = e^{-tA}u_0 &+ \int_0^t e^{-(t-s)A} f(s)ds
\end{aligned}$$

□

As before, this motivates the definition:

Definition 5.15. Let $A : X \to X$ be a sectorial operator of angle θ. Then a function $u : \mathbb{R}_+ \to X$ is called a *mild solution* to (IACP) if u is of the form

$$u(t) := e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(s)ds$$

An equation of the form (IACP) has a unique mild solution provided that A generates an analytic semigroup and $f \in L^1_{\text{loc}}(\mathbb{R}_+; X)$.

We reference the following theorem relating mild and strong solutions:

Theorem 5.16. *A mild solution to (IACP) is also a strong solution if and only if u takes values in $D(A)$ almost everywhere and $Au \in L^1_{loc}(\mathbb{R}_+; X)$.*

Proof. The straightfoward proof can be found as Theorem 42.4 in [37]. Note in particular that if we know that u takes values in $D(A)$ a.e, then we also know that u is differentiable a.e. □

We have made significant progress regarding existence and uniqueness of solutions to these Banach-valued ODEs. We now examine the regularity of such solutions and define Maximal L^p-regularity.

5.3 Maximal L^p Regularity

We want to study regularity of equations of the form

$$u(t) := e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(s)ds$$

The left hand side is a sum of two terms, and questions about regularity and norm estimates may be answered by considering each term separately. Therefore we shall consider the case $u_0 = 0$ for the remainder of this chapter. This leads to the definition:

Definition 5.17. Let $A : X \to X$ be an unbounded operator that generates an analytic semigroup. We say that A has *maximal L^p-regularity* for $p \in (1, \infty)$ if, for all $f \in L^p(\mathbb{R}_+, X)$, the function

$$u_f(t) := \int_0^t e^{-(t-s)A} f(s) ds$$

is differentiable a.e., takes values in $D(A)$ a.e., and (most importantly) we have u'_f and Au_f belonging to $L^p(\mathbb{R}_+, X)$. This regularity is *maximal* in the sense that $f = u'_f + Au_f$ and so u'_f and Au_f cannot be 'more regular' than f itself. Note that it is sufficient to prove that only one of u'_f and Au_f is in $L^p(\mathbb{R}_+; X)$.

Remark 5.18. By Theorem 5.16, maximal regularity implies that there always exists a strong solution to the relevant inhomogeneous abstract Cauchy problem, provided $f \in L^p(\mathbb{R}_+; X)$.

Before we prove Maximal Regularity of A, some remarks:

Lemma 5.19. *The map $f \to u_f$ is bounded as a map $L^p(\mathbb{R}_+; X) \to L^p(\mathbb{R}_+; X)$ provided that $\{e^{-tA}; t \geq 0\}$ is exponentially stable.*

Proof. This follows from Young's inequality for convolutions: for measurable functions $f, g : \mathbb{R} \to \mathbb{R}$ and $p \in [1, \infty)$ we have

$$||f * g||_{L^p(\mathbb{R})} \leq ||f||_{L^1(\mathbb{R})} \cdot ||g||_{L^p(\mathbb{R})}$$

Applying this to move from the third to the fourth line, we have:

$$\begin{aligned}
||u_f||_{L^p(\mathbb{R}_+;X)} &= \left\| t \to \int_0^t e^{-(t-s)A} f(s) ds \right\|_{L^p(\mathbb{R}_+;X)} \\
&\leq \left\| t \to \int_0^t \left\| e^{-(t-s)A} \right\|_{\mathcal{L}(X)} \cdot \|f(s)\|_X \, ds \right\|_{L^p(\mathbb{R}_+)} \\
&= \left\| \left(s \to \mathbb{1}_{\mathbb{R}_+} \left\| e^{-sA} \right\|_{\mathcal{L}(X)} \right) * \left(s \to \mathbb{1}_{\mathbb{R}_+} \|f(s)\|_X \right)(t) \right\|_{L^p(\mathbb{R})} \\
&\leq \left\| t \to \left\| e^{-tA} \right\|_{\mathcal{L}(X)} \right\|_{L^1(\mathbb{R}_+)} \cdot ||f||_{L^p(\mathbb{R}_+;X)} \\
&\leq \left\| t \to Ce^{-\epsilon t} \right\|_{L^1(\mathbb{R}_+)} \cdot ||f||_{L^p(\mathbb{R}_+;X)}
\end{aligned}$$

where C is the uniform bound of $(e^{-tA})_{t \geq 0}$. □

This leads to the following important result concerning Maximal L^p-regularity:

Proposition 5.20. *If A has maximal L^p regularity and $\{e^{-tA}; t \geq 0\}$ is exponentially stable, then there exists a C depending only on p and X such that*

$$||\frac{d}{dt} u_f||_{L^p(\mathbb{R}_+,X)} + ||Au_f||_{L^p(\mathbb{R}_+,X)} \leq C||f||_{L^p(\mathbb{R}_+,X)}$$

Proof. Take an arbitrary sequence of functions $f_n \in L^p(\mathbb{R}_+; X)$ such that $f_n \to f$ and Au_{f_n} converges to some limit in L^p norm. Then, by Lemma 5.19, $u_{f_n} \to u_f$ in L^p norm. Since the operator A is closed, this implies that $Au_{f_n} \to Au_f$. But this tells us that the map $f \to Au_f$ is closed, which implies that it is also bounded by the closed graph theorem. Therefore there exists C' such that $||Au_f||_{L^p(\mathbb{R}_+,X)} \leq C'||f||_{L^p(\mathbb{R}_+,X)}$. The result follows using the fact that $\frac{d}{dt} u_f = f - Au$ and the triangle inequality. □

Remark 5.21. The estimate from Proposition 5.20 means that Maximal Regularity has many applications to existence and uniqueness of various PDEs, including non-autonomous evolution equations, quasi-linear PDEs, Volterra integral equations and the Navier Stokes equation. For more information concerning applications, refer to 1.3 in [29] and the references therein. We outline a simple illustrative example.

Define a family of unbounded operators $A_t : X \to X$ such that:
(1) The operators A_t have Maximal L^p-regularity and satisfy Proposition 5.20 for some constant C independent of t
(2) For each t, the domain $D(A_t)$ is some fixed D.
(3) The map $t \to A_t \in \mathcal{L}(D, X)$ is uniformly continuous.

Then we examine the non-autonomous Cauchy problem:

$$\begin{gathered} \frac{d}{dt}u(t) + A_t u(t) = f(t), \quad t \geq 0 \\ u(0) = u_0 \in D \end{gathered} \tag{NACP}$$

Fix $\tau > 0$. Restrict functions to the interval $[0, \tau]$ and define $F(u)$ to be the operator that maps a function $u(t) \in L^p([0, \tau]; D)$ to the solution of the autonomous problem:

$$\begin{gathered} v'(t) + A_0 v(t) = \phi(u, t) \\ v(0) = u_0 \end{gathered}$$

where $\phi(u, t) := [A_0 - A_t]u(t) + f(t) \in L^p([0, \tau]; X)$. Then one expects the solutions of (NACP) to be the fixed points of F. We can use the maximal L^p-regularity of the family A_t to show that F is a contraction on the space $L^p([0, \tau]; D)$, which ensures a unique solution candidate to (NACP) by Banach's fixed point theorem. Note that in this example we must also consider the intial data u_0 when we show Maximal L^p-regularity. It is possible to show that over a suitable subset $Y \subset X$ of intitial data points u_0, the operators A_t still satisfy

$$||A_t u||_{L^p([0,\tau];X)} \leq C||f||_{L^p([0,\tau];X)}$$

for some C independent of t, f *and* the initial data u_0. Dealing with such considerations is not too difficult, but is beyond the scope of this thesis - see 1.4 and N.1.4 in [29] for discussion of these factors.

Now, take u_α, u_β in $L^p([0, \tau]; D)$; then $F(u_\alpha) - F(u_\beta)$ is the solution of

$$\begin{gathered} v'(t) + A_0(t) = \phi(u_\alpha, t) - \phi(u_\beta, t) \\ v(0) = 0 \end{gathered}$$

Such an equation has a unique mild solution following Definition 5.15. By the Maximal L^p-regularity of A_0, we have C such that

$$||A_0[F(u_\alpha) - F(u_\beta)]||_{L^p([0,\tau];X)} \leq C||\phi(u_\alpha, t) - \phi(u_\beta, t)||_{L^p([0,\tau];X)}$$

Therefore we obtain:

$$\begin{aligned}
||F(u_\alpha) - F(u_\beta)||_{L^p([0,\tau];D)} &\leq ||A_0[F(u_\alpha) - F(u_\beta)]||_{L^p([0,\tau];X)} \\
&\leq C||\phi(u_\alpha, t) - \phi(u_\beta, t)||_{L^p([0,\tau];X)} \\
&= C||[A_0 - A_t](u_\alpha - u_\beta)||_{L^p([0,\tau];X)} \\
&\leq C \sup_{t\in[0,\tau]} ||A_0 - A_t||_{\mathcal{L}(D,X)}||u_\alpha - u_\beta||_{L^p([0,\tau];D)}
\end{aligned}$$

Where in the last line we used the fact that u_α and u_β take values in D. Therefore, since $t \to ||A(t)||_{\mathcal{L}(D,X)}$ is continuous, we may choose τ small enough (and depending only on C) such that

$$||F(u_\alpha) - F(u_\beta)||_{L^p([0,\tau];D)} \leq \alpha||u_\alpha - u_\beta||_{L^p([0,\tau];D)}$$

for some $\alpha < 1$. In this case there exists a unique fixed point corresponding to a solution to (NACP) over the time interval $[0, \tau]$. Then, since $t \to A_t$ is uniformly continuous, we can repeat the argument with initial data $u(0) = u(\tau)$ to extend this solution along any finite amount time.

5.4 The Deterministic Maximal Regularity Theorem

In this section, we prove Maximal L^p-Regularity for suitable sectorial operators.

Definition 5.22. A sectorial operator $A : X \to X$ of angle θ is *R-sectorial* if the set $\{zR(z, A) \mid z \notin \Sigma_{\theta'}\}$ is R-bounded for some $\theta < \theta' < \frac{\pi}{2}$.

Remark 5.23. Clearly R-sectoriality is a strictly stronger condition than sectoriality. In fact, it can be shown that if an operator A has a McIntosh calculus then it is R-sectorial (this is Theorem 12.8 in [29], originally proven in [27]), while the converse is false (e.g. Example 10.17 [29]). However, we know from Remark 4.21 that in fact most 'relevant' sectorial operators do have a McIntosh calculus, and so often we may prove R-sectoriality by proving that an operator has a McIntosh calculus.

Then our main theorem is:

Theorem 5.24. *Take a sectorial operator $A : X \to X$ such that $\{e^{-tA}; t \geq 0\}$ is exponentially stable and X is a UMD space. Then A has Maximal L^p Regularity if and only if A is R-sectorial.*

The heart of the proof is an application of Weis' multiplier theorem 3.16. We split the proof into several propositions.

Proposition 5.25. *For $\phi \in C_c(\mathbb{R}_+, D(A))$, define the operator:*

$$K\phi(t) := A \int_0^t e^{-(t-s)A}\phi(s)ds$$

Then K is a well-defined operator $C_c(\mathbb{R}_+, D(A)) \to L^\infty(\mathbb{R}_+, X)$.

Proof. For $\phi \in C_c(\mathbb{R}_+, D(A))$ we have $e^{-(t-s)A}\phi(s) \in D(A)$ for all s, and so the integrand of $\int_0^t Ae^{-(t-s)A}\phi(s)ds$ is well defined. Since ϕ is compactly supported, $s \to ||Af(s)||$ is bounded, and therefore:

$$\left\|\int_0^t e^{-(t-s)A} A\phi(s)ds\right\|_X \leq \sup_{s\in \mathrm{supp}\phi} \left\|e^{-sA}\right\|_{\mathcal{L}(X)} \sup_{s\in\mathrm{supp}\phi} \|Af(s)\|_X \int_0^t \mathbb{1}_{\mathrm{supp}\phi} dt$$

This is bounded uniformly in t. Therefore by Theorem 1.17, $\int_0^t Ae^{-(t-s)A}\phi(s)ds = A\int_0^t e^{-(t-s)A}\phi(s)ds$ and K is a well-defined operator. □

Proposition 5.26. *The operator A has Maximal L^p Regularity if and only if the operator K satisfies $||K\phi||_{L^p(\mathbb{R}_+;X)} \leq C||\phi||_{L^p(\mathbb{R}_+;X)}$ for some fixed C and for all $\phi \in C_c(\mathbb{R}_+, D(A))$.*

Proof. We clearly have $K\phi = Au_\phi$ for all $f \in C_c(\mathbb{R}_+, D(A))$. If K is unbounded as an operator from a subspace of $L^p(\mathbb{R}_+; X)$ to $L^p(\mathbb{R}_+; X)$, then we cannot have $||Au_f||_{L^p(\mathbb{R}_+;X)} \leq C||f||_{L^p(\mathbb{R}_+;X)}$ for any fixed C, and so by a converse of Proposition 5.20, A cannot have Maximal L^p Regularity.

Now, suppose that K is bounded in L^p norm on the space $C_c(\mathbb{R}_+, D(A))$, that is, there exists a C such that

$$||K\phi||_{L^p(\mathbb{R}_+;X)} \leq C||\phi||_{L^p(\mathbb{R}_+;X)}$$

for $\phi \in C_c(\mathbb{R}_+, D(A))$. Then we trivially have $||Au_\phi||_{L^p(\mathbb{R}_+;X)} \leq C||\phi||_{L^p(\mathbb{R}_+;X)}$on $C_c(\mathbb{R}_+, D(A))$. Take arbitrary $f \in L^p(\mathbb{R}_+; X)$. By Lemma 5.19, the map $f \to u_f$ is bounded in L^p norm. Therefore, by density of $C_c(\mathbb{R}_+, D(A))$ in $L^p(\mathbb{R}_+; X)$, we may write u_f as $u_f = \lim_n u_{\phi_n}$, with the ϕ_n a sequence in $C_c(\mathbb{R}_+, D(A))$ converging in L^p norm to f. Since the norms of Au_{ϕ_n} are controlled by norms of ϕ_n (by our assumption that K is bounded), the sequence $(Au_{\phi_n})_{n\in\mathbb{N}}$ is a Cauchy sequence converging to some limit L. Therefore we may take a subsequence such that $u_{\phi_n}(t) \to u_f(t)$ almost everywhere and $Au_{\phi_n}(t) \to L(t)$ almost everywhere. Using the fact that A is closed, we conclude that $u_f(t) \in D(A)$ and $L(t) = u_f(t)$ a.e. This exactly tells us that u_f takes values in $D(A)$ almost everywhere, and we have Au_f in $L^p(\mathbb{R}_+; X)$ by our approximation. Since f was arbitrary, we conclude that A has Maximal L^p Regularity. □

Proposition 5.27. *The operator K satisfies*

$$||K\phi||_{L^p(\mathbb{R}_+;X)} \leq C||\phi||_{L^p(\mathbb{R}_+;X)}$$

for $\phi \in C_c(\mathbb{R}_+, D(A))$ if and only if the set $\{tR(it, A), t > 0\}$ is R-bounded.

Proof. We will show that in fact K is of the form $K\phi = \mathcal{F}^{-1}[m\hat{\phi}]$ for some function $m : \mathbb{R}_+ \to \mathcal{L}(X)$. For $\phi \in C_c(\mathbb{R}_+, D(A))$ we have:

$$\begin{aligned}
\mathcal{F}[K\phi(t)] &:= \int_0^\infty e^{-it\xi} K\phi(\xi)d\xi \\
&= \int_0^\infty e^{-it\xi} A \int_0^\xi e^{-(\xi-s)A}\phi(s)dsd\xi \\
&= \int_0^\infty \int_0^\xi e^{-it\xi} Ae^{-(\xi-s)A}\phi(s)dsd\xi
\end{aligned}$$

Apply Fubini's Theorem:

$$= \int_0^\infty \int_s^\infty e^{-it\xi} A e^{-(\xi - s)A} \phi(s) d\xi ds$$

Change variables $\xi \to x := \xi - s$

$$\begin{aligned} &= \int_0^\infty \int_0^\infty e^{-it(x+s)} A e^{-xA} \phi(s) dx ds \\ &= \int_0^\infty e^{-its} A \left(\int_0^\infty e^{-itx} e^{-xA} dx \right) \phi(s) ds \end{aligned}$$

Use the result from Lemma 5.9 that $\int_0^\infty e^{-\lambda x} e^{-xA} dx = R(\lambda, A)$ (noting that $\operatorname{Re} \lambda$=0 and e^{-tA} is exponentially stable.)

$$\begin{aligned} &= \int_0^\infty e^{-its} A R(it, A) \phi(s) ds \\ &= A R(it, A) \hat{\phi}(t) \end{aligned}$$

Therefore we see that $\mathcal{F}[Kf(t)] = AR(it, A)\hat{\phi}(t)$ and so $Kf = \mathcal{F}^{-1}[m\hat{\phi}]$, where $m(t) := AR(it, A)$. In particular, recalling our notation from Chapter 3, the operator K may be written as T_m and the condition that $||K\phi||_{L^p(\mathbb{R}_+;X)} \leq C||\phi||_{L^p(\mathbb{R}_+;X)}$ is equivalent to saying that K is a Fourier multiplier.

Therefore we apply Weis' multiplier theorem 3.16. This tells us that K is a Fourier multiplier if the below sets are R-bounded:

$$\mathcal{T}_1 := \{m(t), t \neq 0\}, \quad \mathcal{T}_2 := \{tm'(t), t \neq 0\}$$

And conversely by Theorem 3.17, if K is bounded, then the set $\mathcal{T}_1$ is R-bounded. Then we can see:

$$\begin{aligned} m(t) = AR(it, A) = (it + A - it)(it - A)^{-1} = itR(it, A) - I \\ tm'(t) = -itAR(it, A)^2 = it(it - A)(it - A)^{-2} + t^2(it - A)^{-2} = itR(it, A) - [itR(it, A)]^2 \end{aligned}$$

And then, following the constructions outlined in 2.9, we see that the sets $\mathcal{T}_1$ and $\mathcal{T}_2$ are R-bounded if and only if the set $\{tR(it, A), t \neq 0\}$ is R-bounded. □

Therefore we see that the Maximal L^p Regularity of A is equivalent to the R-boundedness of the set $\{tR(it, A), t \neq 0\}$. Finally, we relate this equivalence to R-sectoriality.

Proposition 5.28. *Let A be sectorial of angle θ on a Banach space X. Then the set*

$$\{tR(it, A), t \neq 0\}$$

is R-bounded if and only if A is R-sectorial.

Proof. Recall that the R-sectoriality of A means that the set $\{zR(z, A) \mid z \notin \Sigma_{\theta'}\}$ is R-bounded for some $\theta < \theta' < \frac{\pi}{2}$. Clearly this implies that $\{tR(it, A), t \in \mathbb{R}_+\}$ is R-bounded. We prove the reverse implication. To do this, we separately show that the sets $\{zR(z,A) \mid \mathrm{Re} z < 0\}$ and $\{zR(z,A) \mid \theta' < |\arg z| \leq \frac{\pi}{2}\}$ are R-bounded (for some θ').

For $\{zR(z,A) \mid \mathrm{Re} z < 0\}$, we recall Poisson's formula: given a complex-valued function f that is bounded and holomorphic on the upper half plane, we have for $x \in \mathbb{R}, y > 0$:

$$f(x+iy) \int_{-\infty}^{\infty} \frac{yf(t)}{(t-x)^2+y^2} dt$$

This is a classical result and can be found in most books on Complex Analysis, see for example Theorem 16.7 of [2]. The proof for this holds equally for the $\mathcal{L}(X)$-valued holomorphic function $f(z) = -izR(-iz, A)$, and so we have for $z = a + bi$ with $a < 0$ and $b \in \mathbb{R}$:

$$zR(z,A) = f(iz) = \int_{-\infty}^{\infty} \frac{a}{\pi(a^2+(s-b)^2)} f(is) ds$$

It is easy to verify that $||\frac{a}{\pi(a^2+(s-b)^2)}||_{L^1(\mathbb{R})} = 1$, and therefore we have that $\{zR(z,A) \mid \mathrm{Re} z < 0\}$ is R-bounded by Proposition 2.12.

Next we will find θ' such that $\{zR(z,A) \mid \theta' < |\arg z| \leq \frac{\pi}{2}\}$ is R-bounded. Write the power series of $R(z,A)$ centred at it, then we have for z sufficiently close to it:

$$\begin{aligned} zR(z,A) &= z \sum_{m=0}^{\infty} \frac{(z-it)^m}{m!} \left[\frac{d^m}{dz^m} R(z,A) \right] (it) \\ &= z \sum_{m=0}^{\infty} \frac{(z-it)^m}{m!} R(it,A)^{m+1} (-1)^m m! \\ &= z \sum_{m=0}^{\infty} (it-z)^m R(it,A)^{m+1} \end{aligned}$$

Now, by our assumption, the set $\{tR(it,A); t \neq 0\}$ is R-bounded with R-bound (say) equal to C. Then choose $\theta < \theta' < \frac{\pi}{2}$ such that $\left| e^{i\theta'} - i \right| < \frac{1}{2C}$. Then for $\{zR(z,A) \mid \theta' < |\arg z| \leq \frac{\pi}{2}\}$ we can write $z = te^{i\pm\theta''}$ with $\theta' < \theta'' < \frac{\pi}{2}$ and we have:

$$zR(z,A) = e^{i\theta''} \sum_{m=0}^{\infty} \left(i - e^{i\theta''} \right)^m [tR(it,A)]^{m+1}$$

Then, observing results 2.9, we see that $\{[tR(it,A)]^{m+1}; t > 0\}$ has R-bound C^{m+1} for each m, and we may write:

$$\begin{aligned} R_1(\{zR(z,A); \theta' < |\arg z| \leq \frac{\pi}{2}\}) &= R_1 \left(\left\{ e^{i\theta''} \sum_{m=0}^{\infty} (i - e^{i\theta''})^m [tR(it,A)]^{m+1}; \theta' < \theta'' \leq \frac{\pi}{2} \right\} \right) \\ &\leq \sum_{m=0}^{\infty} \sup_{\theta' < \theta'' \leq \frac{\pi}{2}} |i - e^{i\theta''}|^m R_1(\{tR(it,A)]^{m+1}; t \neq 0\}) \\ &\leq \sum_{m=0}^{\infty} \left(\frac{1}{2C} \right)^m C^{m+1} \\ &= 2C \end{aligned}$$

In the moving from the first to the second line above, we have used an infinite analogue of property (3) from Proposition 2.9.

Therefore the sets $\{zR(z,A) \mid \mathrm{Re}z < 0\}$ and $\{zR(z,A) \mid \theta' < |\arg z| \leq \frac{\pi}{2}\}$ are R-bounded, and taking the union gives the R-sectoriality of A.

□

This completes the proof of the main theorem. Finally, we see that one may drop the assumption of exponential stability provided we only consider finite time.

Theorem 5.29. *Let A be a sectorial operator $A : X \to X$ and let $T > 0$ be arbitrary and fixed. Then A has Maximal L^p Regularity on the space $L^p((0,T],X)$ if A is R-sectorial.*

Proof. Suppose that A is R-sectorial. If e^{-tA} is exponentially stable, then the result follows from Theorem 5.24. Suppose that e^{-tA} is *not* exponentially stable, then, following our work in Proposition 1.8, one can consider the exponentially stable semigroup $e^{-t(A+\epsilon)}$ for arbitrary small $\epsilon > 0$. It is straightforward to verify that if A is R-sectorial then $A+\epsilon$ will be as well. Therefore we may apply Theorem 5.24 to the operator $A+\epsilon$ to conclude for any $f \in L^p(\mathbb{R}_+;X)$, the function $v(t) = \int_0^t e^{-(t-s)(A+\epsilon)} f(s)ds$ takes values in $D(A+\epsilon)$ and satisfies $(A+\epsilon)v \in L^p(\mathbb{R}_+;X)$.

Consider now the space $L^p((0,T];X)$ for arbitrary finite T. Choose arbitrary $f \in L^p((0,T];X)$, we want to show that u_f takes values in $D(A)$ a.e. and is in $L^p((0,T];X)$. Now, the function $g(s) = e^{-s\epsilon} f(s)$ is also in $L^p((0,T];X)$, and the maximal regularity of $A+\epsilon$ tells us that $v_g(t)$ takes values in $D(A)$ and is in $L^p(\mathbb{R}_+;X)$. Therefore we have

$$\begin{aligned} u_f(t) &= \int_0^t e^{-(t-s)A} f(s)ds \\ &= \int_0^t e^{(t-s)\epsilon} e^{-(t-s)(A+\epsilon)} f(s)ds \\ &= \int_0^t e^{t\epsilon} e^{-(t-s)(A+\epsilon)} e^{-s\epsilon} f(s)ds \\ &= e^{t\epsilon} v_g(t) \end{aligned}$$

We can see that $u(t)$ differs from $v(t)$ only by a scalar and therefore is in $D(A+\epsilon) = D(A)$, and Au differs from Av by a continuous variable and therefore is in the space $L^p((0,T];X)$. □

5.5 Further Reading

Analytic Semigroups are a well-known phenomenon, and theory may be seen in 3.6 of [37], 3.4 of [19], 9.9 in [29] or II.4.a in [14]. The books [14] and [37] in particular contain much more dicussion on semigroups (analytic and otherwise) for the interested reader. The discussion of Abstract Cauchy Problems is adapted from 4.2 in [37], and the discussion and proof of Maximal Regularity are from 1.3-1.5 in [29]. The relation to R-sectoriality is Theorem 2.20 from [29] and was originally shown in [48].

Chapter 6

The γ-Norm

6.1 Basic Definitions

6.1.1 The Space $\gamma_\infty(H, X)$

In this section, we will define a new norm for certain operators in $\mathcal{L}(H, X)$. This will be used in our work on stochastic integration and further study of the McIntosh calculus.

Definition 6.1. A *Gaussian Sequence* $(\gamma_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d normally distributed variables with mean zero and variance 1.

Definition 6.2. An bounded linear operator $T : H \to X$ is called *γ-summing* if

$$\sup \mathbb{E} \left\| \sum_{n=1}^{N} \gamma_n T h_n \right\|^2 < \infty$$

where the supremum is taken over finite orthonormal systems $\{h_1, \ldots h_N\}$ in H. We say that $T \in \gamma_\infty(H, X)$, and that

$$||T||_{\gamma_\infty(H,X)} := \left(\sup \mathbb{E} \left\| \sum_{n=1}^{N} \gamma_n T h_n \right\|^2 \right)^{\frac{1}{2}}$$

By the Khinchine-Kahane inequality for Gaussian variables (Corollary 2.5), we also have the equivalent norm for any $p \in [1, \infty)$

$$||T||_{\gamma^p_\infty(H,X)} := \left(\sup \mathbb{E} \left\| \sum_{n=1}^{N} \gamma_n T h_n \right\|^p \right)^{\frac{1}{p}}$$

Remark 6.3. By taking the supremum over 'systems' made up of a single element, we have that $||T||_{\mathcal{L}(H,X)} \leq ||T||_{\gamma_\infty(H,X)}$ for all T.

Proposition 6.4. *The space $\gamma_\infty(H, X)$ is a Banach space.*

Proof. The only nontrivial thing to show is that this space is complete. Take a Cauchy sequence $(T_n)_{n=1}^\infty$, then by Remark 6.3 above, $(T_n)_{n=1}^\infty$ is also a Cauchy sequence with respect to the $\mathcal{L}(X)$

norm. Thus T_n converges to some operator T in $\mathcal{L}(X)$ norm. We want to show that $T_n \to T$ in $\gamma_\infty(H,X)$ norm. Take $\epsilon > 0$, choose N such that $n, m > N \implies ||T_n - T_m||_{\gamma_\infty(H,X)} < \epsilon$. Take an orthonormal system $\{h_1, \dots h_k\}$, then by Fatou's lemma for scalar variables:

$$\begin{aligned}\mathbb{E}\left\|\sum_{j=1}^{k}\gamma_j(T_n - T)h_j\right\|^2 &\leq \liminf_{m\to\infty}\mathbb{E}\left\|\sum_{j=1}^{k}\gamma_j(T_n - T_m)h_j\right\|^2\\ &\leq \liminf_{m\to\infty}||T_n - T_m||^2_{\gamma_\infty(H,X)}\\ &< \epsilon^2\end{aligned}$$

Taking the supremum over orthonormal systems, we have that $T_n - T$ is in $\gamma_\infty(H,X)$, and so T is too, and $T_n \to T$ in $\gamma_\infty(H,X)$ norm. □

6.1.2 The Space $\gamma(H,X)$

Remark 6.5. Given any finite rank operator $T : H \to X$, there exists $N \in \mathbb{N}$, an orthonormal set $(h_n)_{n=1}^N \subset H$, and $(x_n)_{n=1}^N \subset X$ such that

$$T(h) = \sum_{n=1}^{N}[h, h_n]x_n$$

This can be straightforwardly shown using the Riesz representation theorem and a Gram-Schmidt orthonormalisation argument. In this case, one has:

$$||T||_{\gamma_\infty(H,X)} := \left(\mathbb{E}\left\|\sum_{n=1}^{N}\gamma_n x_n\right\|^2\right)^{\frac{1}{2}}$$

The right hand term is the γ_∞-norm of T over the set $\{h_1, \dots h_n\}$. It is clear that including any additional orthonormal vector h will not increase the norm because $Th = 0$. It is reasonably difficult but not illuminating to show that this norm is invariant over any other set $\{h_1', \dots h_n'\}$ that spans the same space as $\{h_1, \dots h_n\}$. Such a proof may be found in (for example) Section 3 of [40], and relies on the fact that orthogonal transformations preserve the expectation of finite-dimensional Gaussian vectors.

Unfortunately, finite rank operators are not dense in $\gamma_\infty(H,X)$, which leads to the following definition:

Definition 6.6. Define $\gamma(H,X)$ to be the completion of finite rank operators $T : H \to X$ under the $\gamma_\infty(H,X)$ norm. Note that $\gamma(H,X)$ is a Banach space. Define $\gamma^p(H,X)$ to be the same space under the (equivalent) $\gamma^p_\infty(H,X)$ norm. Operators in this space are traditionally called γ-*Radonifying* operators.

Clearly if $T \in \gamma(H,X)$ then $T \in \gamma_\infty(H,X)$ and $||T||_{\gamma(H,X)} = ||T||_{\gamma_\infty(H,X)}$. In general, $\gamma(H,X)$ is a proper, closed subspace of $\gamma_\infty(H,X)$. However, for our purposes we may reference the following theorem and this distinction becomes unimportant:

Theorem 6.7. *Let c_0 be the Banach space containing all infinite sequences $(c_n)_{n\in\mathbb{N}}$ of real numbers such that $c_n \to 0$. Let X be a Banach space that does* not *contain a closed subspace isomorphic to c_0. Then $\gamma_\infty(H,X) = \gamma(H,X)$. In particular, for $\mathcal{M}$ is a measure space and $q \in [1,\infty)$ the space $L^q(\mathcal{M})$ does not contain a closed subspace isomorphic to c_0 and thus $\gamma_\infty(H, L^q(\mathcal{M})) = \gamma(H, L^q(\mathcal{M}))$.*

This is Theorem 4.3 in [40], based on a result originally proven in [22] and [30]. The fact that $L^q(\mathcal{M})$ not contain a closed subspace isomorphic to c_0 can be seen in Corollary 7.5 and Example 7.6 in [24].

We have a convenient characterisation of the space $\gamma(H,X)$ when $X = L^q(\mathcal{M})$:

Proposition 6.8. *For $q \in [1,\infty)$ there is a canonical isometric isomorphism $\gamma^q(H;L^q(\mathcal{M})) \simeq L^q(\mathcal{M};\gamma^q(H;\mathbb{R}))$, and thus a Banach space isomorphism $\gamma(H;L^q(\mathcal{M})) \simeq L^q(\mathcal{M};H)$. The latter isomorphism is given by:*

$$f \in L^q(\mathcal{M};H) \to [h \to (f(\cdot),h)]$$

Proof. By the Riesz Representation theorem, $\gamma^q(H;\mathbb{R})$ is isomorphic to H. Take a *simple* function $f \in L^q(\mathcal{M};\gamma^q(H;\mathbb{R}))$. Then via a Gram-Schmidt process we can find some orthonormal set $(h_n)_{n=1}^N \subset H$ and disjoint sets $A_n \subset \mathcal{M}$ such that:

$$\begin{aligned} f(\mu) &= \sum_{n=1}^N c_n \mathbb{1}_{A_n}(\mu) h_n \\ \implies ||f||_{L^q}^q &= \sum_{n=1}^N c_n^q |A_n| \cdot ||h_n||_{\gamma^q(H;\mathbb{R})} \\ &= \sum_{n=1}^N c_n^q |A_n| \cdot \mathbb{E}(\gamma_n^q) \end{aligned}$$

Note that $\mathbb{E}(\gamma_n^q)$ is some fixed number depending only on q. Then we can associate the function f with the operator $F \in \gamma^q(H;L^q(\mathcal{M}))$ defined by $F(h) = (f(\cdot),h)$. We can formulate the γ^q-norm of F explicitly (recalling that the A_n are disjoint):

$$\begin{aligned} [F(h)](\mu) &= \sum_{n=1}^N c_n \mathbb{1}_{A_n}(\mu)(h,h_n) \\ ||F||_{\gamma^q(H;L^q(\mathcal{M}))}^q &= \mathbb{E}\left(\left\| \sum_{n=1}^N \gamma_n c_n \mathbb{1}_{A_n}(\mu) \right\|_{L^q}^q \right) \\ &= \mathbb{E} \int_{\mathcal{M}} \left[\sum_{n=1}^N \gamma_n c_n \mathbb{1}_{A_n}(\mu) \right]^q d\mu \\ &= \sum_{n=1}^N \mathbb{E} \int_{A_n} \gamma_n^q c_n^q d\mu \\ &= \sum_{n=1}^N c_n^q |A_n| \cdot \mathbb{E}(\gamma_n^q) \end{aligned}$$

Therefore an isometric isomorphism exists for simple functions. Since such functions are dense in both $\gamma^q(H; L^q(\mathcal{M}))$ and $L^q(\mathcal{M}; \gamma^q(H; \mathbb{R}))$, we have our result. The Banach space isomorphism $\gamma(H; L^q(\mathcal{M})) \simeq L^q(\mathcal{M}; H)$ follows via $\gamma^q(H; L^q(\mathcal{M})) \simeq \gamma(H; L^q(\mathcal{M}))$ from Khinchine-Kahane, and $L^q(\mathcal{M}; \gamma^q(H; \mathbb{R})) \simeq L^q(\mathcal{M}; H)$ by the Riesz representation theorem, and both isomorphisms produce equivalent norms. □

6.1.3 The Space $\gamma(L^2(\mathbb{R}_+); X)$

Definition 6.9. Let $\phi \in L^2(\mathbb{R}_+; X)$. Then we may define the bounded operator

$$T_\phi : L^2(\mathbb{R}_+) \to X, \quad T_\phi(g) := \int_0^\infty \phi(t)g(t)dt$$

We want to consider such operators in the context of the space $\gamma(L^2(\mathbb{R}_+); X)$.

Remark 6.10. In the case $X = L^q(\mathcal{M})$, we can relate the equivalence in Definition 6.9 to Proposition 6.8. Define $\phi : \mathbb{R}_+ \times \mathcal{M} \to \mathbb{R}$ such that $\phi_1(t) := \phi(t, \cdot)$ is in $L^2(\mathbb{R}_+; L^q(\mathcal{M}))$, and T_{ϕ_1} is in $\gamma(L^2(\mathbb{R}_+); L^q(\mathcal{M}))$ for some $q \in (1, \infty)$. Then we have

$$[T_{\phi_1} h](\mu) = \int_0^\infty \phi(t, \mu)h(t)dt = (\phi(\cdot, \mu), h)$$

But we saw in Proposition 6.8 that the isomorphism $\gamma(H; L^q(\mathcal{M})) \to L^q(\mathcal{M}; H)$ maps the operator $T_{\phi_1} h = (\phi(\cdot, \mu), h)$ to the function $\phi_2(\mu) := \phi(\cdot, \mu)$. Therefore we see that a function T_ϕ is in $\gamma(L^2(\mathbb{R}_+); L^q(\mathcal{M}))$ if and only if $\mu \to \phi(t, \mu)$ is in $L^q(\mathcal{M}; L^2(\mathbb{R}_+))$. We also see that definition 6.9 and Proposition 6.8 thus relate an operator T to the 'same' function of two variables.

Remark 6.11. The reader should take note that it is difficult to relate $\gamma(L^2(\mathbb{R}_+); X)$ with $L^2(\mathbb{R}_+; X)$ in general. Firstly, there is no general method to compare the L^2 norm of ϕ with the γ-norm of T_ϕ. Secondly, not every operator $T \in \gamma(L^2(\mathbb{R}_+); X)$ will be of the form T_ϕ for some $\phi \in L^2(\mathbb{R}_+; X)$, and not every T_ϕ will lie in the space $\gamma(L^2(\mathbb{R}_+); X)$. In particular, $\lim_n T_{\phi_n} \neq T_{\lim_n \phi_n}$ in general (regardless of which kind of limits we are taking). Nevertheless, we may obtain enough results to make progress.

Remark 6.12. For any rank one operator $Tf = (f, g)x$, we may define $\tilde{g} : \mathbb{R}_+ \to X$ by $\tilde{g}(t) = g(t)x$, and we have $T = T_{\tilde{g}}$. Therefore by linearity, any finite rank operator $T \in \gamma(L^2(\mathbb{R}_+); X)$ is of the form T_ϕ for some ϕ, and operators of the form T_ϕ are dense in $\gamma(L^2(\mathbb{R}_+); X)$. In fact, we can do better: choose *simple* functions $f_n \to \phi$ in $L^2(\mathbb{R}_+)$. Then, defining $\tilde{f}_n(t) = f_n(t)x$, we have

$$\begin{aligned}
||T_\phi - T_{\tilde{f}_n}||_{\gamma(L^2(\mathbb{R}_+);X)} &= ||h \to (h, \phi - f_n)x||_{\gamma(L^2(\mathbb{R}_+);X)} \\
&= \mathbb{E}\, \|\gamma_n \cdot ||\phi - f_n|| \cdot x\|_X \\
&= ||x||_X \cdot ||\phi - f_n||_{L^2(\mathbb{R}_+)} \\
&\to 0
\end{aligned}$$

Therefore by linearity, operators of the form T_f with f a simple function with one-dimensional range are dense in the space of rank one operators, and therefore dense in $\gamma(L^2(\mathbb{R}_+); X)$.

Lemma 6.13. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^2(\mathbb{R}_+;X)$ such that*
(1) $T_{f_n} \to T$ in γ-norm, and
(2) There exists a function $f \in L^2(\mathbb{R}_+;X)$ such that for all $x^* \in X^*$ we have $\langle f_n(t), x^*\rangle \to \langle f(t), x^*\rangle$ almost everywhere.
Then $T_f \in \gamma(L^2(\mathbb{R}_+);X)$ and $T = T_f$.

Proof. Fix x^* in X^*. Then we have for the adjoint operator $T^* : X^* \to L^2(\mathbb{R}_+)$:

$$||T^*x^*|| \leq ||T^*||_{\mathcal{L}(X^*;L^2(\mathbb{R}_+))} \cdot ||x^*|| = ||T||_{\mathcal{L}(L^2(\mathbb{R}_+);X)} \cdot ||x^*|| \leq ||T||_{\gamma(L^2(\mathbb{R}_+);X)} \cdot ||x^*||$$

with the last inequality following from Remark 6.3. Next, note that $\langle f_n, x^*\rangle = T^*_{f_n}x^*$. Indeed:

$$\langle T_{f_n}g, x^*\rangle = \left\langle \int_0^\infty f_n(t)g(t)dt, x^* \right\rangle = \int_0^\infty \langle f_n(t)g(t), x^*\rangle dt = (g, \langle f_n, x^*\rangle) = (g, T^*_{f_n}x^*)$$

So we have

$$||\langle f_n, x^*\rangle - T^*x^*||_{L^2(\mathbb{R}_+)} = ||T^*_{f_n}x^* - T^*x^*||_{L^2(\mathbb{R}_+)} \leq ||T_{f_n} - T||_{\gamma(L^2(\mathbb{R}_+);X)} \cdot ||x^*|| \to 0$$

Therefore $\langle f_n, x^*\rangle \to T^*x^*$ in $L^2(\mathbb{R}_+)$, and so a subsequence converges pointwise almost everywhere. But by our assumption, $\langle f_n, x^*\rangle \to \langle f, x^*\rangle$ almost everywhere, therefore $\langle f, x^*\rangle = T^*x^*$ for all x^*. Therefore $T^* = T^*_f$ and so $T = T_f$. $\square$

We conclude with two duality results.

Remark 6.14. Let $q, r \in (1,\infty)$, be such that $\frac{1}{q} + \frac{1}{r} = 1$. Let $\phi \in L^q(\mathcal{M}; L^2(\mathbb{R}_+))$, $\psi \in L^r(\mathcal{M}; L^2(\mathbb{R}_+))$. Then by duality, we have

$$|\langle \phi(t), \psi(t)\rangle| \leq ||\phi||_{L^q(\mathcal{M};L^2(\mathbb{R}_+))} \cdot ||\psi||_{L^r(\mathcal{M};L^2(\mathbb{R}_+))}$$

Identifying ϕ with $\Phi \in \gamma(L^2(\mathbb{R}_+); L^q(\mathcal{M}))$ and ψ with $\Psi \in \gamma(L^2(\mathbb{R}_+); L^r(\mathcal{M}))$ via the isomorphism from Proposition 6.8, we have:

$$|\langle \phi(t), \psi(t)\rangle| \leq C||\Phi||_{\gamma(L^2(\mathbb{R}_+);L^q(\mathcal{M}))} \cdot ||\Psi||_{\gamma(L^2(\mathbb{R}_+);L^r(\mathcal{M}))}$$

Remark 6.15. Let q, r be as above, but this time take $\phi' \in L^2(\mathbb{R}_+, L^q(\mathcal{M}))$, $\psi' \in L^2(\mathbb{R}_+, L^r(\mathcal{M}))$. Then we have

$$\begin{aligned}
\int_{\mathbb{R}_+} |\langle \phi'(t), \psi'(t)\rangle|\, dt &\leq \int_{\mathbb{R}_+}\int_{\mathcal{M}} |\phi(t,\mu)\psi(t,\mu)| d\mu dt \\
&= \int_{\mathcal{M}}\int_{\mathbb{R}_+} |\phi(t,\mu)\psi(t,\mu)| dt d\mu \\
&\leq ||\mu \to \phi'(t)(\mu)||_{L^q(\mathcal{M};L^2(\mathbb{R}_+))} \cdot ||t \to \psi'(t)(\mu)||_{L^r(\mathcal{M};L^2(\mathbb{R}_+))} \\
&\leq C||T_{\phi'}||_{\gamma(L^2(\mathbb{R}_+);L^q(\mathcal{M}))} \cdot ||T_{\psi'}||_{\gamma(L^2(\mathbb{R}_+);L^r(\mathcal{M}))}
\end{aligned}$$

The last line follows by Remark 6.14. One should note that all the right hand terms may be infinite (in which case the inequality still trivially holds).

6.2 The γ-Multiplier Theorem

In this section we will prove the important γ-Multiplier Theorem 6.24, which we shall use to prove that a McIntosh calculus implies an R-McIntosh calculus. We introduce some preparatory definitions and lemmas.

Definition 6.16. For $n \in \mathbb{N}, k \in \mathbb{Z}$, let $I_{n,k}$ be the interval $[2^{-k}n, 2^{-k}(n+1))$. Then define

$$\psi_{n,k}(t) := 2^{k/2}\left(\mathbb{1}_{[\frac{n}{2^k}, \frac{n+1/2}{2^k})}(t) - \mathbb{1}_{[\frac{n+1/2}{2^k}, \frac{n+1}{2^k})}(t)\right)$$

Then it is straightforward to verify that each $\psi_{n,k}$ is supported on $I_{n,k}$ and the set $\{\psi_{n,k}, n \in \mathbb{N}, k \in \mathbb{Z}\}$ is an orthonormal set in $L^2(\mathbb{R}_+)$. In fact, this set is an orthonormal basis for $L^2(\mathbb{R}_+)$ called the *Haar basis*. This is a classical result that can be found in many books, see for example Theorem 5.35 and Example 5.36 in [15].

Definition 6.17. An *unconditional decomposition* of a Banach space X is a family of bounded projections $(D_i)_{i=1}^{\infty}$ such that $D_iD_j \equiv 0$ for all $i \neq j$ and for all $\alpha_i \pm 1$ we have

$$\left\|\sum_{i=1}^{n} \alpha_i D_i x - x\right\|_X \to 0 \text{ as } n \to \infty$$

Order the Haar basis according to a single index i as $(\psi_i)_{i=1}^{\infty}$, and for $f \in L^2(\mathbb{R}_+; X)$, define the projection:

$$D_i f := \psi_i \int_0^{\infty} f(t)\psi(t)dt$$

It is trivial to verify that this is a projection, and that $D_iD_j \equiv 0$ for $i \neq j$. Then we have the deep theorem:

Theorem 6.18. *The projections $(D_i)_{i=1}^{\infty}$ are an unconditional decomposition of the space $L^2(\mathbb{R}_+; X)$ if and only if X is a UMD space.*

Proof. This is Theorem 4.29 in [23]. □

The above results allow us to prove the lemma:

Lemma 6.19. *Take a UMD space X. Let $f \in L^2(\mathbb{R}_+; X)$ be such that $T_f \in \gamma(L^2(\mathbb{R}_+); X)$. Then there exists simple functions $f_n \in L^2(\mathbb{R}_+; X)$ such that $T_{f_n} \to T_f$ in γ-norm, and $f_n(t) \to f(t)$ almost everywhere.*

Proof. Let D_i be the projection onto the Haar basis defined above. For $f \in L^2(\mathbb{R}_+; X)$, define:

$$P_n f := \sum_{i=1}^{n} D_i f = \sum_{i=1}^{n} \left(\int_0^{\infty} f(s)\psi_i(s)ds\right)\psi_i$$

Then by Theorem 6.18, we have $P_n f \to f$ in $L^2(\mathbb{R}_+; X)$. Next, define $\tilde{P}_n : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ as the analogous projection:

$$\tilde{P}_n g := \sum_{i=1}^{n} \left(\int_0^\infty g(s)\psi_i(s)ds \right) \psi_i$$

Then we have

$$\begin{aligned} T_{P_n f} g &= \int_0^\infty g(t) P_n f(t) dt \\ &= \sum_{i=1}^{n} \int_0^\infty g(t) \left(\int_0^\infty f(s)\psi_i(s)ds \right) \psi_i(t)dt \\ &= \int_0^\infty \sum_{i=1}^{n} \left(\int_0^\infty g(t)\psi_i(t)dt \right) \psi_i(s) f(s)ds \\ &= T_f \tilde{P}_n g \end{aligned}$$

Therefore $T_{P_n f} = T_f \tilde{P}_n$. Note that the operators $\tilde{P}_n$ are self-adjoint, and for all $h \in L^2(\mathbb{R}_+)$ we have $\tilde{P}_n h \to h$ in norm. I claim that for arbitrary $T \in \gamma(L^2(\mathbb{R}_+); X)$, we have $T\tilde{P}_n \to T$ in γ-norm. Indeed, take some rank one operator $T_{h_0}(h) = (h, h_0)x$ in $\gamma(L^2(\mathbb{R}_+); X)$. Then by the self-adjointness of I and $\tilde{P}_n$, we have

$$\begin{aligned} \left(T_{h_0}\tilde{P}_n - T_{h_0} \right)(h) &= ([\tilde{P}_n - I]h, h_0)x \\ &= (h, [\tilde{P}_n - I]h_0)x \\ \implies ||T_{h_0}\tilde{P}_n - T_{h_0}||_{\gamma(L^2(\mathbb{R}_+);X)} &= ||(\tilde{P}_n - I)h_0|| \cdot ||x|| \\ &\to 0 \end{aligned}$$

Therefore by the triangle inequality, the result is true for all finite rank operators, and by density the result is true for all operators in $\gamma(L^2(\mathbb{R}_+); X)$. In particular, we have that $T_{P_n f} = T_f \tilde{P}_n \to T_f$ in γ-norm.

Recalling that $P_n f \to f$ in $L^2(\mathbb{R}_+; X)$, let $(f_k)_{k\in\mathbb{N}}$ be a subsequence of $(P_n f)_{n\in\mathbb{N}}$ that converges to f pointwise almost everywhere (such a sequence can be proven to exist just as in the scalar case of $L^2(\mathbb{R}_+)$). Then by the work above, we also have $T_{f_n} \to T_f$ in γ-norm and we are done. □

Next we will need:

Lemma 6.20 (γ-Fatou Lemma).
Let T_n be a bounded sequence in $\gamma_\infty(H, X)$ that converges weakly to an operator T in $\mathcal{L}(H, X)$, that is, for all $h \in H$ and $x^* \in X^*$

$$\lim_{n\to\infty} \langle T_n h, x^* \rangle = \langle T h, x^* \rangle$$

Then $T \in \gamma_\infty(H, X)$ and for all $p \in [1, \infty)$ we have

$$||T||_{\gamma^p_\infty(H,X)} \le \liminf_{n\to\infty} ||T_n||_{\gamma^p_\infty(H,X)}$$

Proof. Take an orthonormal system $\{h_1, \dots h_k\}$ in H. Then the space $S = \text{span}\{Th_1, \dots Th_k\}$ is finite dimensional and therefore separable, thus by Lemma 1.10 there exists a sequence $(x_m^*)_{m\in\mathbb{N}}$ of unit vectors in X^* which is norming for S.

Now, for fixed $M \geq 1$, we have by Fatou's lemma for scalar functions:

$$\begin{aligned}\mathbb{E} \sup_{m\leq M} \left| \left\langle \sum_{j=1}^{k} \gamma_j T h_j x_m^* \right\rangle \right|^p &\leq \liminf_{n\to\infty} \mathbb{E} \sup_{m\leq M} \left| \left\langle \sum_{j=1}^{k} \gamma_j T_n h_j x_m^* \right\rangle \right|^p \\ &\leq \liminf_{n\to\infty} ||T_n||^p_{\gamma^p_\infty(H,X)}\end{aligned}$$

Then we may take $M \to \infty$, and the term on the left converges by the monotone convergence theorem (for scalar functions), thus we have

$$\mathbb{E} \left\| \sum_{j=1}^{k} \gamma_j T h_j \right\|^p \leq \liminf_{n\to\infty} ||T_n||^p_{\gamma^p_\infty(H,X)}$$

Since this is true for all orthonormal systems $\{h_1, \dots h_k\}$, it is true for the supremum over these, and the result follows. $\square$

Finally we introduce a Gaussian counterpart to R-boundedness:

Definition 6.21. A family of operators $\mathcal{T} \subset \mathcal{L}(X)$ is called *γ-bounded* if, for all finite sets $(x_1, \dots x_N) \subset X$, $(T_1, \dots T_N) \subset \mathcal{T}$, we have

$$\mathbb{E} \left\| \sum_{n=1}^{N} \gamma_n T_n x_n \right\|_X^p \leq C_p^p \, \mathbb{E} \left\| \sum_{n=1}^{N} \gamma_n x_n \right\|_X^p$$

The smallest C_p such that this is true is called $\gamma_p(\mathcal{T})$.

Proposition 6.22. *Any R-bounded family $\mathcal{T} \subset \mathcal{L}(X)$ is also γ-bounded and for all $p \in [1, \infty)$ we have $\gamma_p(\mathcal{T}) \leq R_p(\mathcal{T})$.*

Proof. Take independent sequences $(r_n)_{n\in\mathbb{N}}$ and $(\gamma_n)_{n\in\mathbb{N}}$, and denote $\mathbb{E}_r$ and $\mathbb{E}_\gamma$ as the relevant expectations. Take $(T_1, \dots T_N)$ and $(x_1, \dots x_N)$. Now, for fixed ω we have

$$\mathbb{E}_\gamma \left\| \sum_{n=1}^{N} \gamma_n T_n x_n \right\|_X^p = \mathbb{E}_\gamma \left\| \sum_{n=1}^{N} \gamma_n r_n(\omega) T_n x_n \right\|_X^p$$

Therefore we have, using Fubini's theorem:

$$\begin{aligned}\mathbb{E}_\gamma\left\|\sum_{n=1}^N \gamma_n T_n x_n\right\|_X^p &= \mathbb{E}_r\mathbb{E}_\gamma\left\|\sum_{n=1}^N \gamma_n r_n T_n x_n\right\|_X^p \\ &= \mathbb{E}_\gamma\mathbb{E}_r\left\|\sum_{n=1}^N \gamma_n r_n T_n x_n\right\|_X^p \\ &\leq R_p(\mathcal{T})^p\mathbb{E}_\gamma\mathbb{E}_r\left\|\sum_{n=1}^N \gamma_n r_n x_n\right\|_X^p \\ &= R_p(\mathcal{T})^p\mathbb{E}_\gamma\left\|\sum_{n=1}^N \gamma_n x_n\right\|_X^p\end{aligned}$$

□

Remark 6.23. It is worth mentioning that the converse holds (i.e. γ-boundedness implies R-boundedness) when X has *finite cotype*. This term is defined in Section 11 of [40]. For our purposes, we note that $L^q(\mathcal{M})$ has finite cotype for all $q \in [1,\infty)$ and thus γ-boundedness and R-boundedness are equivalent on these spaces. This is Theorem 8.10 in [24] or Proposition 2.6 in [40].

We now have the background needed to prove the γ-multiplier theorem:

Theorem 6.24 (γ-Multiplier Theorem)**.** *Let $F : \mathbb{R}_+ \to \mathcal{L}(X)$ be strongly measurable with γ-bounded range. Denote the the γ-bound of this range by $\gamma(F)$. Let $\phi \in L^2(\mathbb{R}_+; X)$ be such that $T_\phi \in \gamma(L^2(\mathbb{R}_+); X)$. Then the integral operator:*

$$T_{F\phi} : L^2(\mathbb{R}_+) \to X, \quad T_{F\phi}f = \int_0^\infty F(t)\phi(t)f(t)dt$$

is in $\gamma_\infty(L^2(\mathbb{R}_+); X)$, and for all $p \in [1,\infty)$ we have

$$||T_{F\phi}||_{\gamma^p_\infty(L^2(\mathbb{R}_+);X)} \leq \gamma_p(F)||T_\phi||_{\gamma^p(L^2(\mathbb{R}_+);X)}$$

Proof. First, let ϕ and F be simple, then without loss of generality we may choose a sequence $(x_n)_{n=1}^N \subset X$ and a sequence of $(B_n)_{n=1}^N \subset \mathcal{L}(X)$ and a *common* set $(A_n)_{n=1}^N$ of disjoint measurable subsets of $\mathbb{R}_+$ such that:

$$\phi(t) = \sum_{n=1}^N \frac{\mathbb{1}_{A_n}(t)}{\sqrt{|A_n|}}x_n, \quad F(t) = \sum_{n=1}^N \mathbb{1}_{A_n}(t)B_n \implies F(t)\phi(t) = \sum_{n=1}^N \frac{\mathbb{1}_{A_n}(t)}{\sqrt{|A_n|}}B_n x_n$$

Clearly $T_{F\phi} \in \gamma^p_\infty(L^2(\mathbb{R}_+); X)$, and we have

$$||T_{F\phi}||^p_{\gamma^p_\infty(L^2(\mathbb{R}_+);X)} = \mathbb{E}\left\|\sum_{j=1}^k \gamma_j B_n x_n\right\|^p \leq \gamma_p(F)^p\mathbb{E}\left\|\sum_{j=1}^k \gamma_j x_n\right\|^p = \gamma_p(F)^p||T_\phi||_{\gamma^p(L^2(\mathbb{R}_+);X)}$$

Next, retain our simple function ϕ but consider an arbitrary (strongly measurable) function F. Approximate this function with simple functions $F_k(t) \to F(t)$ a.e. such that each simple function takes values in the range of $F(t)$. This set is γ-bounded, and therefore bounded, thus we have for arbitrary $f \in L^2(\mathbb{R}_+)$:

$$\int_0^\infty ||F_k(t)\phi(t)f(t)||dt \leq \int_0^\infty \sup_t ||F_k(t)||_{\mathcal{L}(X)} \cdot ||\phi(t)f(t)||dt < \infty$$

And therefore by the Dominated Convergence Theorem, we have $T_{F_k\phi} \to T_{F\phi}$ strongly. This implies weak convergence and allows us to apply the γ-Fatou lemma to conclude that $T_{F\phi} \in \gamma^p_\infty(L^2(\mathbb{R}_+);X)$ and $||T_{F\phi}||_{\gamma^p_\infty(L^2(\mathbb{R}_+);X)} \leq \gamma_p(F)^p ||T_\phi||_{\gamma^p(L^2(\mathbb{R}_+);X)}$. Thus we have defined an operator $T_\phi \to T_{F\phi}$ for simple functions ϕ.

By Remark 6.12 above, operators of the form T_ϕ with ϕ a simple function are dense in the space $\gamma^p(L^2(\mathbb{R}_+);X)$. Therefore we may define the extension $\tilde{F}(\lim_{n\to\infty} T_{\phi_n}) = \lim_{n\to\infty} T_{F\phi_n}$ and we have $||\tilde{F}|| \leq \gamma_p(F)$.

It remains to show that this extension gives what we want, that is, for arbitrary $\psi \in L^2(\mathbb{R}_+;X)$ we have $\tilde{F}(T_\psi) = T_{F\psi}$. From Lemma 6.19, we can choose a sequence of simple functions $\phi_n \in L^2(\mathbb{R}_+;X)$ such that $T_{\phi_n} \to T_\psi$ in γ-norm and $\phi_n(t) \to \psi(t)$ almost everywhere. Then for all x^*, we have $\langle F(t)\phi_n(t), x^*\rangle \to \langle F(t)\psi(t), x^*\rangle$ almost everywhere. Clearly since we have F, F_k bounded and ψ in $L^2(\mathbb{R}_+;X)$, we also have $F_k\psi$ and $F\psi$ in $L^2(\mathbb{R}_+;X)$. Therefore by Lemma 6.13, we have that $\lim_{n\to\infty} T_{F\phi_n} = T_{F\psi}$. Therefore we conclude that $\tilde{F}(T_\psi) = \tilde{F}(\lim_{n\to\infty} T_{\phi_n}) = \lim_{n\to\infty} T_{F\phi_n} = T_{F\psi}$. □

This completes the proof of the γ-multiplier theorem. We now see an application, moving towards the extension of the McIntosh calculus discussed in section 4.3 of chapter 4.

6.3 The γ-norm and the McIntosh Calculus

Our understanding of γ-norms allows us to reference the following deep and surprising characterisation of the boundedness of the McIntosh calculus:

Definition 6.25. The Hilbert space $L^2(\mathbb{R}_+, \frac{dt}{t})$ is the space of measurable functions $f : \mathbb{R}_+ \to R$ such that

$$||f||^2_{L^2(\mathbb{R}_+,\frac{dt}{t})} := \int_0^\infty |f(t)|^2 \frac{dt}{t} < \infty$$

It is endowed with the inner product $(f,g) := \int_0^\infty f(t)g(t)\frac{dt}{t}$. All results from this chapter carry over, considering this space rather than $L^2(\mathbb{R}_+)$.

Theorem 6.26. *Let $A : X \to X$ be a sectorial operator of angle θ with a McIntosh calculus of angle Θ. Let φ be any function in $H_0^\infty(\Sigma_{\Theta'})$ for $\Theta' > \Theta$. Then for each $x \in X$, the function $\varphi_x(t) := \varphi(tA)x$ is in $L^2(\mathbb{R}_+;X,\frac{dt}{t})$ and defines an operator T_{φ_x} in $\gamma(L^2(\mathbb{R}_+,\frac{dt}{t});X)$. Moreover, there exists a C depending only on X, A and φ such that for all $x \in X$ we have*

$$\frac{1}{C}||x|| \leq ||T_{\varphi_x}||_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t});X)} \leq C||x||$$

Proof. This is Proposition 7.7 in [26], with the case $X = L^q(\mathcal{M})$ proven in [33]. □

Note that the above theorem implies a connection between the McIntosh calculus and the γ-norm which regrettably lies beyond the scope of thesis. The paper [26] would be an ideal starting point for the reader interested in pursuing this line of enquiry. In this thesis we content ourselves with proving our Theorem 4.30 from Chapter 4, relating the McIntosh calculus to the R-McIntosh calculus:

Theorem 4.30: Let $A : X \to X$ be sectorial of angle θ and have a McIntosh calculus of angle Θ. Then A also has an R-McIntosh calculus of angle Θ.

Proof. We are going to prove that for all $F \in RH_0^\infty(\Sigma_\Theta, \mathcal{A})$, there exists a C such that for all $x \in X$ and $x^* \in X^*$

$$|\langle F(A)x, x^*\rangle| \leq C||F||_{RH^\infty} \cdot ||x|| \cdot ||x^*||$$

which implies that

$$||F(A)||_{\mathcal{L}(X)} = \sup_{||x||=1, ||x^*||=1} |\langle F(A)x, x^*\rangle| \leq C||F||_{RH^\infty}$$

And so by Proposition 4.29 the operator A has an R-McIntosh calculus of angle Θ. First, we prove a preparatory lemma:

Lemma 6.27. *Let $A : X \to X$ be sectorial of angle θ. Choose $\theta < \alpha < \alpha' < \frac{\pi}{2}$, then for $s \in (0, 1)$ and z on $\partial\Sigma_\alpha$, define*

$$A^s R(z, A) := \frac{1}{2\pi i} \int_{\partial\Sigma_{\alpha'}} \frac{\lambda^s}{z - \lambda} R(\lambda, A) d\lambda$$

Then $A^s R(z, A)$ is a well-defined Bochner integral. Furthermore, for $\Theta > \theta$, take $F \in RH_0^\infty(\Sigma_\Theta, \mathcal{A})$. Then for any $\theta < \alpha < \Theta$ and any $s \in (0, 1)$, we have

$$F(A) = \frac{1}{2\pi i} \int_{\partial\Sigma_\alpha} z^{-s} F(z) A^s R(z, A) dz$$

Proof. We show that $A^s R(z, A)$ is a well-defined Bochner integral in the usual way (using the fact that $||\lambda R(\lambda, A)|| \leq C$ uniformly).

$$\begin{aligned} \int_{\partial\Sigma_{\alpha'}} \left\| \frac{\lambda^s}{z - \lambda} R(\lambda, A) \right\| d\lambda &= \int_0^\infty \left\| \frac{t^s e^{\pm is\alpha'}}{z - te^{\pm i\alpha'}} R(te^{\pm i\alpha'}, A) \right\| dt \\ &\leq C \int_0^\infty \frac{1}{t^{1-s} |z - te^{\pm i\alpha'}|} dt \\ &< \infty \end{aligned}$$

We want a specific bound on $||A^s R(z, A)||$. Take the upper ray of the contour, and consider:

$$\int_0^\infty \frac{1}{t^{1-s} |z - te^{i\alpha'}|} dt = \int_0^{|z|} \frac{1}{t^{1-s} |z - te^{i\alpha'}|} dt + \int_{|z|}^\infty \frac{1}{t^{1-s} |z - te^{i\alpha'}|} dt$$

Examine the first term on the right: $|z - e^{it\alpha'}|$ is bounded below since $z \in \partial\Sigma_\alpha$, therefore there exists a constant δ such that $|z - te^{i\alpha'}| > \delta|z|$ uniformly in t. This means that there exists a constant C_1 such that:

$$\int_0^{|z|} \frac{1}{t^{1-s}\,|z - te^{i\alpha'}|} dt < \frac{1}{\delta} \int_0^{|z|} \frac{1}{t^{1-s}\,|z|} dt \leq \frac{C_1}{|z|^{1-s}}$$

Now examine the second term, $\int_{|z|}^{\infty} \frac{1}{t^{1-s}|z-te^{i\alpha'}|} dt$. By a similar argument, there exists $\delta' > 0$ such that $|z - te^{i\alpha'}| > \delta' t$, and so there exists C_2 such that

$$\int_{|z|}^{\infty} \frac{1}{t^{1-s}\,|z - te^{i\alpha'}|} dt < \frac{1}{\delta'} \int_{|z|}^{\infty} \frac{1}{t^{1-s} t} dt \leq \frac{C_2}{|z|^{1-s}}$$

Therefore putting these together (and repeating these arguments on the lower ray) yields a constant C_3 with

$$||A^s R(z, A)|| \leq \frac{C_3}{|z|^{1-s}}$$

This tells us that:

$$\begin{aligned}
\int_{\partial\Sigma_\alpha} \left\|z^{-s} F(z) A^s R(z, A)\right\| dz &\leq \int_0^{\infty} t^{-s} \left\|F\left(e^{\pm i\alpha'}\right)\right\| \left\|A^{-s} R(te^{\pm is\alpha'}, A)\right\| dt \\
&\leq C \int_0^{\infty} t^{-s} \frac{t^{\epsilon}}{(1+t)^{2\epsilon}} \frac{1}{t^{1-s}} dt \\
&< \infty
\end{aligned}$$

And so the formula in the lemma is a convergent Bochner integral. It remains to show that this formula yields $F(A)$. We will reuse the construction from Lemma 4.11 of the functions:

$$\phi_n(z) := \frac{n}{n+z} - \frac{1}{1+nz} \implies \phi_n(A) = \left(\frac{n^2-1}{n}\right) A(n+A)^{-1}(\frac{1}{n} + A)^{-1}$$

Then we can define for $s \in (-1, 1)$:

$$A^s \phi_n(A) := \frac{1}{2\pi i} \int_{\partial\Sigma_\alpha} z^s \phi_n(z) R(z, A) dz$$

Then it is straightforward to show that this is a well-defined Bochner integral (using similar methods to the proof of $A^s R(z, A)$) above. It is also straightforward to show that $\phi_m(A)\phi_n(A) = A^{-s}\phi_m(A)A^s\phi_m(A)$ (adapting the proof of multiplicativity from Theorem 4.8).
Fix arbitrary m, and recall that $\phi_m(A)$ has dense range. Choose $x := \phi_m(A)y$ for $x, y \in X$,

$$\begin{aligned}
F(A)\phi_n(A)x &= F(A)\phi_n(A)\phi_m(A)y \\
&= F(A)[A^s\phi_m(A)][A^{-s}\phi_n(A)]y \\
&= [A^s\phi_m(A)]F(A)[A^{-s}\phi_n(A)]y
\end{aligned}$$

Thus we can say:

$$\begin{aligned}
F(A)\phi_n(A)x &= [A^s\phi_m(A)]F(A)[A^{-s}\phi_n(A)]y \\
&= \frac{1}{2\pi i}[A^s\phi_m(A)]\int_{\partial\Sigma_\alpha} F(z)z^{-s}\phi_n(z)R(z,A)ydz \\
&= \frac{1}{2\pi i}\int_{\partial\Sigma_\alpha} z^{-s}\phi_n(z)F(z)[A^s\phi_m(A)]R(z,A)ydz \\
&= \frac{1}{2\pi i}\int_{\partial\Sigma_\alpha} z^{-s}\phi_n(z)F(z)A^sR(z,A)xdz
\end{aligned}$$

We may then take the limit as $n \to \infty$, recalling that $\phi_n(A)x \to x$. On the right hand side, we use the Dominated Convergence Theorem, since $|\phi_n(z)| < 2$ uniformly in n, z and $\phi_n(z) \to 1$ for all z. This gives the result for all x in the (dense) range of $\phi_m(A)$ and the result extends. $\square$

With this lemma in hand, we return to proving that $|\langle F(A)x, x^*\rangle| \le C||F||_{RH^\infty} \cdot ||x|| \cdot ||x^*||$. Fix $\theta < \alpha < \Theta$, then for $x \in X$ and $x^* \in X^*$ we have from Lemma 6.27 in case $s = \frac{1}{2}$:

$$|\langle F(A)x, x^*\rangle| = \frac{1}{2\pi}\left|\int_{\partial\Sigma_\alpha}\left\langle z^{-\frac{1}{2}}F(z)A^{\frac{1}{2}}R(z,A)x, x^*\right\rangle dz\right|$$

Next, define

$$\psi_\alpha(z) := \frac{z^{\frac{1}{4}}}{\sqrt{e^{i\alpha} - z}} \in H_0^\infty(\Sigma_\Theta)$$

Then we have $\psi_\alpha^2 = \sqrt{z}(e^{i\alpha} - z)^{-1} \in H_0^\infty(\Sigma_\Theta)$, and by our definition in Lemma 6.27 we have $\psi_\alpha(A)^2 = \psi_\alpha^2(A) = A^{\frac{1}{2}}R(e^{i\alpha}, A)$. This also implies that $\psi_\alpha(\frac{1}{t}A)^2 = (\frac{1}{t})A^{\frac{1}{2}}R(e^{i\alpha}, \frac{1}{t}A) = \sqrt{t}A^{\frac{1}{2}}R(te^{i\alpha}, A)$. With this notation we can write:

$$\begin{aligned}
|\langle F(A)x, x^*\rangle| &\le \frac{1}{2\pi}\int_{\partial\Sigma_\alpha}\left|\left\langle z^{-\frac{1}{2}}F(z)A^{\frac{1}{2}}R(z,A)x, x^*\right\rangle\right| dz \\
&= \frac{1}{2\pi}\int_0^\infty\left|\left\langle F(te^{\pm i\alpha})A^{\frac{1}{2}}R(te^{\pm i\alpha}, A)x, x^*\right\rangle\right| \frac{dt}{\sqrt{t}} \\
&= \frac{1}{2\pi}\int_0^\infty\left|\left\langle F(te^{\pm i\alpha})\psi_{\pm\alpha}(\frac{1}{t}A)^2x, x^*\right\rangle\right| \frac{dt}{t} \\
&= \frac{1}{2\pi}\int_0^\infty\left|\left\langle F(te^{\pm i\alpha})\psi_{\pm\alpha}(\frac{1}{t}A)x, \psi_{\pm\alpha}(\frac{1}{t}A)^*x^*\right\rangle\right| \frac{dt}{t}
\end{aligned}$$

By γ-norm duality (Remarks 6.14 and 6.15):

$$\le C\left\|T_{F(te^{\pm i\alpha})\psi_{\pm\alpha}(\frac{1}{t}A)x}\right\|_{\gamma(L^2(\mathbb{R}_+);X)} \cdot \left\|T_{\psi_{\pm\alpha}(\frac{1}{t}A)^*x^*}\right\|_{\gamma(L^2(\mathbb{R}_+);X^*)}$$

Now, it is routine to verify that $\psi_{\pm\alpha}(\frac{1}{t}A)x$ is in $L^2(\mathbb{R}_+; X; \frac{dt}{t})$, which follows from the fact that ψ_α is in $H_0^\infty(\Sigma_\Theta)$. Theorem 6.26 tells us that since A has a McIntosh calculus, $T_{\psi_{\pm\alpha}(\frac{1}{t}A)x}$ is in the space $\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}); X)$. Therefore we may apply the γ-Multiplier Theorem 6.24:

$$|\langle F(A)x, x^*\rangle| \le C'||F||_{RH^\infty} \cdot \left\|T_{\psi_{\pm\alpha}(\frac{1}{t}A)x}\right\|_{\gamma(L^2(\mathbb{R}_+);X)} \cdot \left\|T_{\psi_{\pm\alpha}(\frac{1}{t}A)^*x^*}\right\|_{\gamma(L^2(\mathbb{R}_+);X^*)}$$

And by the γ-Norm Estimate from Theorem 6.26:

$$\leq C''||F||_{RH^\infty} \cdot ||x|| \cdot ||x^*||$$

Therefore we have our estimate, and A has an R-McIntosh calculus of angle Θ. □

6.4 Further Reading

A much more extensive discussion of the γ-norm is found in Chapter 9 from [24]. Basic constructions may be found in Section 9.1, and the γ-multiplier theorem is discussed in Section 9.4. The reader should be aware that this work is unpublished and not yet in a final draft. The γ-multiplier theorem was originally proven as Proposition 4.11 and Corollary 4.12 in [26]. Alternative discussions may be found in [40] (and references therein), and in Chapter 5 of [41]. The proof of the relationship between the McIntosh calculus and the R-McIntosh Calculus is drawn from Chapter 12 of [29] and originally proven in [27], though that proof does not use γ-norms.

Chapter 7

Stochastic Integration

In this chapter, we understand integration of Banach-valued functions against Brownian motion, as an extension of the Itô integral. This will allow us to solve stochastic partial differential equations through an analogue of the variation of constants formula. For background and context, we shall first briefly outline the main elements of the scalar theory of stochastic integration. Following standard practice, we often omit the ω variable (corresponding to the randomness) in this and the next chapter.

7.1 Background - Real-Valued Stochastic Integration

Definition 7.1. A *Brownian motion* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ is a family of real-valued random variables $B_t : \Omega \to \mathbb{R}$ indexed by $t \in \mathbb{R}_+$ and adapted to $(\mathcal{F}_t)_{t\geq 0}$ satisfying:
(1) $B_0(\omega) = 0$ for all ω.
(2) $B_t - B_s$ is independent of $\mathcal{F}_s$ for all $0 \leq s \leq t$.
(3) For all $0 \leq s \leq t$, $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$.
(4) $t \to B_t(\omega)$ is continuous for almost every $\omega \in \Omega$.

Standard texts on stochastic analysis provide explicit constructions of Brownian motion, ensuring that the definition above describes a process that actually exists. See for example, 3.4 in [38]. As in the construction of the Lebesgue integral, we next understand how to integrate simple functions against a Brownian motion.

Definition 7.2. A *simple adapted process* $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is a function of the form

$$f(t,\omega) = \sum_{n=1}^{N} \alpha_n(\omega)\mathbb{1}_{[s_n,t_n)}(t)$$

With each α_n a random variable in $L^2(\Omega)$ that is measurable with respect to $\mathcal{F}_{s_n}$.

The *stochastic integral* of an simple process $f(t,\omega)$ is a random variable $\int f(t)dB_t : \Omega \to \mathbb{R}$ defined by

$$\left(\int f(t)dB_t\right)(\omega) := \sum_{n=1}^{N} \alpha_n(\omega)(B_{t_n}(\omega) - B_{s_n}(\omega))$$

We will extend this integral to more general processes using a central result called the Itô Isometry. The result can be found in standard texts, but the proof is short and illuminating when compared to the Banach-valued results to follow, so we shall include a proof:

Proposition 7.3. Itô Isometry:
For f a simple adapted process, we have:

$$\mathbb{E}\left(\int f(t)dB_t\right)^2 = \mathbb{E}||f||^2_{L^2(\mathbb{R}_+)}$$

Proof. Without loss of generality, we can write

$$f(t,\omega) = \sum_{n=1}^{N} \alpha_n(\omega)\mathbb{1}_{(t_{n-1},t_n]}(t)$$

such that the intervals $(t_{n-1}, t_n]$ are disjoint and each α_n is measurable with respect to t_{n-1}. Then we have:

$$\begin{aligned}
\mathbb{E}\left(\int f(t)dB_t\right)^2 &= \mathbb{E}\left(\sum_{n=1}^{N} \alpha_n(B_{t_n} - B_{t_{n-1}})\right)^2 \\
&= \sum_{n=1}^{N} \mathbb{E}\left(\alpha_n^2[B_{t_n} - B_{t_{n-1}}]^2\right) + \sum_{n\neq m}^{N} \mathbb{E}\left(\alpha_m\alpha_n[B_{t_{n+1}} - B_{t_{n-1}}][B_{t_m} - B_{t_{m-1}}]\right)
\end{aligned}$$

Examine $\alpha_m\alpha_n[B_{t_{n+1}} - B_{t_n}][B_{t_{m+1}} - B_{t_m}]$ with $m \neq n$. If $m < n$ then α_m, α_n and $B_{t_{m+1}} - B_{t_m}$ are measurable with respect to $\mathcal{F}_{t_m}$, and so $B_{t_{n+1}} - B_{t_n}$ is independent of these variables, by property (2) in the definition of Brownian motion. Thus we have $\mathbb{E}(\alpha_m\alpha_n[B_{t_{n+1}} - B_{t_n}][B_{t_{m+1}} - B_{t_m}]) = \mathbb{E}(\alpha_m\alpha_n[B_{t_{m+1}} - B_{t_m}])\mathbb{E}(B_{t_{m+1}} - B_{t_m}) = 0$. The same argument works if $n < m$. Therefore we have, noting that α_n is measurable with respect to t_{n-1} for each n:

$$\begin{aligned}
\left\|\int f(t)dB_t\right\|^2_{L^2(\Omega)} &= \sum_{n=1}^{N} \mathbb{E}\left(\alpha_n^2[B_{t_n} - B_{t_{n-1}}]^2\right) \\
&= \sum_{n=1}^{N} \mathbb{E}\left(\alpha_n^2\right)\mathbb{E}\left([B_{t_n} - B_{t_{n-1}}]^2\right) \\
&= \mathbb{E}||f||^2_{L^2(\mathbb{R}_+)}
\end{aligned}$$

□

To extend the Itô integral to arbitrary processes, let $L^2_{\mathcal{F}}(\mathbb{R}_+ \times \Omega)$ be the space of $\mathcal{F}$-adapted L^2-integrable processes. Then:

Lemma 7.4. *The simple adapted processes are dense in $L^2_{\mathcal{F}}(\mathbb{R}_+ \times \Omega)$, that is, given any $f \in L^2_{\mathcal{F}}(\mathbb{R}_+ \times \Omega)$, there exists simple adapted processes with $\mathbb{E}||f_n - f||_{L^2_{\mathcal{F}}(\mathbb{R}_+)} \to 0$.*

Proof of this can also be found in standard texts, see for example, 6.6 in [38]. The proof is longer and less illuminating than that of the Itô isometry, and so we shall omit it here.

Armed with this lemma, given arbitrary $f \in L^2_{\mathcal{F}}(\mathbb{R}_+ \times \Omega)$ we may define:

$$\int f dB_t = \lim_{n\to\infty} \int f_n dB_t$$

where $(f_n)_{n=1}^{\infty}$ is any approximating sequence of simple adapted functions. This limit will always exist, and is independent of the approximating sequence. With this scalar background fresh in our minds, we move to the stochastic integration of Banach-valued functions.

7.2 Banach-Valued Stochastic Integration

In this chapter, we shall construct stochastic integrals that yield Banach-valued random variables. We shall construct a Hilbert-space-valued version of Brownian motion called H-cylindrical Brownian motion, which is the 'injected' into a Banach space via a function $F : \mathbb{R}_+ \to \mathcal{L}(H, X)$. The reader might wish to keep in mind the case $H = \mathbb{R}$, which corresponds to a one-dimensional Brownian motion and a function F that simply takes values in X.

7.2.1 H-Cylindrical Brownian Motion

Definition 7.5. A *H-isonormal process* is a bounded linear map $B : H \to L^2(\Omega)$ satisfying:
(1) for all $h \in H$, Bh is a Gaussian variable with mean zero.
(2) For all h_1, h_2, we have $\mathbb{E}(Bh_1 \cdot Bh_2) = (h_1, h_2)$

Remark 7.6. For any separable Hilbert space H, a H-isonormal process does exist. Indeed, let $(h_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H and γ_n be sequence of i.i.d Gaussian variables as in Chapter 6. Then $Bh := \sum_{n=1}^{\infty} \gamma_n(h, h_n)$ defines an H-isonormal process. Indeed, a convergent sum of independent Gaussian variables yields a Gaussian variable, and:

$$\begin{aligned}
\mathbb{E}(Bh_a \cdot Bh_b) &= \mathbb{E}\left(\sum_{n=1}^{\infty} \gamma_n(h_a, h_n) \sum_{m=1}^{\infty} \gamma_m(h_b, h_m)\right) \\
&= \sum_{n=1}^{\infty} \mathbb{E}(\gamma_n^2(h_a, h_n)(h_b, h_n)) + \sum_{n\neq m}^{\infty} \mathbb{E}(\gamma_n\gamma_m(h_a, h_n)(h_b, h_m)) \\
&= \sum_{n=1}^{\infty} (h_a, h_n)(h_b, h_n) \\
&= \left(h_a, \sum_{n=1}^{\infty} (h_b, h_n)h_n\right) \\
&= (h_a, h_b)
\end{aligned}$$

We can use this construction to define Brownian motion in a Hilbert space:

Definition 7.7. Let Ω be a probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$. An *$H$-cylindrical Brownian motion* B_H is an $L^2(\mathbb{R}_+; H)$-isonormal process such that $B_H(\mathbb{1}_{(s,t]}h)$ is $\mathcal{F}_t$-measurable and independent of $\mathcal{F}_s$ for all h and all $0 < s < t < \infty$. We then define notationally:

$$B_h(t) - B_h(s) := B_H(\mathbb{1}_{(s,t]}h)$$

Note that this means that $B_h(t) - B_h(s)$ is a Gaussian variable with mean zero and variance $||h||(t-s)$.

7.2.2 The Itô Integral for Deterministic Functions

Definition 7.8. Given a rank one operator $T : H \to X$ defined by $Th = (h, h_0)x_0$, we notate this as '$T = h_0 \otimes x_0$.'

Definition 7.9. A function $F : \mathbb{R}_+ \to \mathcal{L}(H, X)$ is called a *finite rank step function* if it is of the form

$$F(t) = \sum_{n=1}^{N} \mathbb{1}_{(s_n,t_n]} h_n \otimes x_n$$

For some $N \in \mathbb{N}$, some real numbers $0 \leq s_n < t_n < \infty$, and some sequences $(h_n)_{n=1}^N \subset H$ and $(x_n)_{n=1}^N \subset X$.

Let B_H be an H-cylindrical Brownian motion. Then the *Itô integral* of F against B_H is defined as:

$$\int_0^\infty F(t)dB_H(t) := \sum_{n=1}^{N} (B_{h_n}(t_n) - B_{h_n}(s_n))x_n$$

Then we can prove an analogue of the Itô isometry using the γ-norm defined in Chapter 6. Noting that any finite rank step function F will be in $L^2(\mathbb{R}_+; \mathcal{L}(H,X))$, we may define $T_F : L^2(\mathbb{R}_+; H) \to X$ by

$$T_F f := \int_0^\infty F(t)f(t)dt$$

Note that this is identical to the identification from Definition 6.9 in the case $H = \mathbb{R}$. In the general case, for a finite rank step function F as in Definition 7.9, we can formulate T_F explicitly as

$$T_F = \sum_{n=1}^{N} (\mathbb{1}_{(s_n,t_n]} h_n) \otimes x_n$$

Then we can prove an Itô isometry:

Theorem 7.10. Itô Isometry
Let X be a Banach space. Then for all finite rank step functions $F : \mathbb{R}_+ \to \mathcal{L}(H, X)$ and for all $p \in [1, \infty)$, we have:

$$\mathbb{E}\left(\left\|\int_0^\infty F(t)dB_H(t)\right\|_X^p\right) = ||T_F||^p_{\gamma^p(L^2(\mathbb{R}_+;H),X)}$$

Proof. Without loss of generality, we can write F as

$$F = \sum_{n=1}^{N} \mathbb{1}_{(t_{n-1},t_n]} \left(\sum_{j=1}^{k} h_j \otimes x_{j_n} \right) = \sum_{n=1}^{N} \sum_{j=1}^{k} \mathbb{1}_{(t_{n-1},t_n]} h_j \otimes x_{j_n}$$

with $0 \leq t_0 < \ldots < t_N < \infty$, and $\{h_1, ...h_k\}$ a common orthonormal set in H.

Define $\psi_n := \frac{1}{\sqrt{t_n - t_{n-1}}} \mathbb{1}_{(t_{n-1},t_n]} := c_n \mathbb{1}_{(t_{n-1},t_n]}$. Then the set $\{\psi_n\}_{n\in\mathbb{N}}$ is orthonormal in $L^2(\mathbb{R}_+)$, and the set $\{\psi_n h_j\}_{n\in\mathbb{N}, j\leq k}$ is orthonormal in $L^2(\mathbb{R}_+; H)$. Then we can write:

$$T_F = \sum_{n=1}^{N} \sum_{j=1}^{k} \psi_n h_j \otimes \frac{x_{jn}}{c_n}$$

By the definition of the γ-norm in Remark 6.5, we have:

$$||T_F||^p_{\gamma^p(L^2(\mathbb{R}_+;H),X)} = \mathbb{E} \left(\left\| \sum_{n,j} \gamma_{jn} \frac{x_{jn}}{c_n} \right\|^p \right)$$

On the other hand:

$$\begin{aligned} \mathbb{E} \left(\left\| \int_0^\infty F(t) dB_H(t) \right\|^p \right) &= \mathbb{E} \left(\left\| \sum_{n=1}^{N} \sum_{j=1}^{k} (B_{h_n}(t_n) - B_{h_n}(s_n)) x_{jn} \right\|^p \right) \\ &= \mathbb{E} \left(\left\| \sum_{n,j} \frac{(B_{h_n}(t_n) - B_{h_n}(s_n))}{\sqrt{t_n - t_{n-1}}} \frac{x_{jn}}{c_n} \right\|^p \right) \end{aligned}$$

By our defintion of H-cylindrical Brownian motion, $\dfrac{B_H(\mathbb{1}_{(t_{n-1},t_n]} h_j)}{\sqrt{t_n - t_{n-1}}}$ is a standard Gaussian variable, and thus we have equality. $\square$

Therefore if we take any sequence $(F_n)_{n\in\mathbb{N}}$ such that $(T_{F_n})_{n\in\mathbb{N}}$ converges to some limit T in γ-norm, then we know that $\int_0^\infty F(t) dB_H(t)$ will converge to some limit L in $L^p(\Omega; X)$. This motivates the following definition:

Definition 7.11. Let $p \in (1, \infty)$. A function $F : \mathbb{R}_+ \to \mathcal{L}(H, X)$ is said to be *stochastically integrable* if there exists a sequence of finite rank step functions $F_n : \mathbb{R}_+ \to \mathcal{L}(H, X)$ such that:

(1) F_n converges to F strongly in measure. By this we mean that for all $h \in H$ and all $\epsilon > 0$, the measure of $\{t, ||F_n(t)h - F(t)h|| > \epsilon\}$ goes to zero as n goes to infinity.

(2) The stochastic integrals $\int_0^\infty F_n(t) dB_H(t)$ converge to some limit L in $L^p(\Omega; X)$).

Then we define:

$$\int_0^\infty F(t) dB_H(t) := \lim_{n\to\infty} \int_0^\infty F_n(t) dB_H(t)$$

The above definition seems difficult to verify for any particular function F. Towards this, we reference the following theorem:

Theorem 7.12. *Take a function $F : \mathbb{R}_+ \to \mathcal{L}(H, X)$ such that for all $h \in H$, the map $t \to F(t)h$ is strongly measurable. Then F is stochastically integrable for all $p \in (1, \infty)$ if and only if:*
(1) The function $t \to F(t)^* x^*$ is in $L^2(\mathbb{R}_+; H)$ for all x^* in X^*, and
(2) There exists an operator $T \in \gamma(L^2(\mathbb{R}_+; H), X)$ such that for all $f \in L^2(\mathbb{R}_+; H)$ and x^* in X^* we have:

$$\langle Tf, x^* \rangle = \int_0^\infty \langle F(t)f(t), x^* \rangle dt$$

Furthermore, if such a T exists, then it is unique and we have for all $p \in (1, \infty)$:

$$\mathbb{E}\left(\left\| \int_0^\infty F(t) dB_H(t) \right\|_X^p \right) = ||T||^2_{\gamma^p(L^2(\mathbb{R}_+;H),X)}$$

Thus we say that T 'represents' *F.*

Proof. This difficult result is Theorem 6.17 in [41]. An original proof is Theorem 2.3 in [47]. □

This theorem allows us to verify whether specific functions are stochastically integrable. We illustrate two familiar cases.

Remark 7.13 (The Case $H = \mathbb{R}$.)**.** Given any function $F \in L^2(\mathbb{R}_+; X)$ such that $T_F \in \gamma(L^2(\mathbb{R}_+); X)$, we clearly have for all $f \in L^2(\mathbb{R}_+)$:

$$\langle T_F f, x^* \rangle = \left\langle \int_0^\infty F(t)f(t), x^* \right\rangle = \int_0^\infty \langle F(t)f(t), x^* \rangle dt$$

Therefore F is stochastically integrable. Morever, we know from the proof of Lemma 6.19 that there exist simple functions F_n such that $T_{F_n} \to T_F$ in γ-norm and $F_n(t) \to F(t)$ in L^2 (and therefore also in measure).This implies by the Itô isometry that $\int_0^\infty F_n(t) dB_H(t)$ converges. We can therefore define

$$\int F(t) dB_H(t) := \lim_{n \to \infty} \int F_n(t) dB_H(t)$$

Remark 7.14 (The Case $X = L^q(\mathcal{M})$)**.** Take $F : \mathbb{R}_+ \to \mathcal{L}(H, L^q(\mathcal{M}))$ such that $t \to F(t)^* g$ is in $L^2(\mathbb{R}_+; H)$ for all $g \in L^{q'}(\mathcal{M})$. Define the function $\psi : \mathcal{M} \to L^2(\mathbb{R}_+; H)$ by:

$$(\psi(\mu)(t), h) = [F(t)h](\mu)$$

Then, examining the right term in our condition of stochastic integrability in Theorem 7.12, with $g \in L^{q'}(\mathcal{M})$, we use Fubini's theorem:

$$\begin{aligned}\int_0^\infty \langle F(t)f(t), g \rangle dt &= \int_0^\infty \int_{\mathcal{M}} [F(t)f(t)](\mu) g(\mu) d\mu dt \\ &= \int_{\mathcal{M}} \left(\int_0^\infty (\psi(\mu)(t), f(t)) dt \right) g(\mu) d\mu \\ &= \langle T_\psi f, g \rangle\end{aligned}$$

where $T_\psi : L^2(\mathbb{R}_+, H) \to L^q(\mathcal{M})$ is defined by $T_\psi f = \int_0^\infty (\psi(\mu)(t), f(t)) dt$. But this is exactly the isomorphism from Proposition 6.8. Therefore we know that T_ψ (which 'represents' F)

is in $\gamma(L^2(\mathbb{R}_+; H), X)$, or equivalently F is stochastically integrable, if and only if ψ is in $L^q(\mathcal{M}; L^2(\mathbb{R}_+, H))$.

Next, we want to extend our theory such that we can integrate random functions $F : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, X)$. To do this, we will need two further preparatory concepts.

7.2.3 UMD Spaces, Second Appearance

Definition 7.15. Let X be a Banach space, $p \in (1, \infty), \Omega$ be a probability space and $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration. A family of random variables $D_n : \Omega \to X$ adapted to $\mathcal{F}_n$ is called an X-valued L^p *martingale difference sequence* if for each n we have D_n in $L^p(\Omega; X)$ and $\mathbb{E}(D_n \mid F_{n-1}) = 0$. Note that given a martingale M_n, the sequence $D_n := M_n - M_{n-1}$ defines a martingale difference sequence, hence the name

Then we have a definition of a UMD space:

Definition 7.16. A Banach space X is a UMD space if for some (equivalently all) $p \in (1, \infty)$ there exists a constant C (depending only on p and X) such that for all X-valued L^p martingale difference sequences $(D_n)_{n=1}^N$, and all signs $(a_n)_{n=1}^\infty \subset \{\pm 1\}^\mathbb{N}$, we have for all $N \in \mathbb{N}$:

$$\mathbb{E}\left(\left\|\sum_{n=1}^N a_n D_n\right\|^p\right) \leq C^p \mathbb{E}\left(\left\|\sum_{n=1}^N D_n\right\|^p\right)$$

Remark 7.17. The equivalence between a single value of p and all values of p with respect to the UMD property is a highly nontrivial result. A proof may be found as section 12.2 in [41], following an original proof from [6] and [7].

Remark 7.18. The inequality may also extend in the opposite direction, since:

$$\mathbb{E}\left(\left\|\sum_{n=1}^N D_n\right\|^p\right) = \mathbb{E}\left(\left\|\sum_{n=1}^N (a_n) a_n D_n\right\|^p\right) \leq C^p \mathbb{E}\left(\left\|\sum_{n=1}^N a_n D_n\right\|^p\right)$$

Remark 7.19. The reader may recall that in Chapter 3, we defined a UMD space as a Banach space on which the Hilbert transform was a Fourier multiplier. In fact, this definition and definition 7.16 are equivalent, a highly nontrivial result proven by Bourgain and Burkholder in [4] and [6].

7.2.4 Decoupling

The Itô Isometry proven in Theorem 7.10 will not hold in the case of random functions, because we take an expectation that considers the randomness of both the random function F and the H-cylindrical Brownian motion. To deal with this problem, we will 'decouple' these two random variables so that they may be treated independently.

We introduce and label some objects: Let $p \in (1, \infty)$ be fixed, X be a Banach space, and Ω be a probability space with a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of random variables in $L^p(\Omega)$ each with mean zero such that for all n, z_n is measurable with respect to $\mathcal{F}_n$ and independent of $\mathcal{F}_{n-1}$. Note that this implies that $(z_n)_{n\in\mathbb{N}}$ is an L^p martingale difference sequence.

Next, let $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be identical probability spaces and take the product space $\Omega_2 := (\Omega \times \tilde{\Omega}, \mathcal{F} \times \tilde{\mathcal{F}}, \mathbb{P} \times \tilde{\mathbb{P}})$). Then we can define:

$$Z_n(\omega, \tilde{\omega}) := z_n(\omega), \qquad \tilde{Z}_n(\omega, \tilde{\omega}) := z_n(\tilde{\omega})$$

Then for each n, Z_n and $\tilde{Z}_n$ are independent and identically distributed.

Finally, let $(y_n)_{n\in\mathbb{N}}$ be a sequence of X-valued random variables such that y_n is $\mathcal{F}_{n-1}$-measurable (and y_1 is constant almost surely). Such as sequence is called *$\mathcal{F}$-predictable.* Note that y_n is independent of z_n, but not z_{n-1}. Then define $Y_n : \Omega_2 \to X$ by $Y_n(\omega, \tilde{\omega}) := y_n(\omega)$. Then we have the theorem:

Theorem 7.20 (Decoupling)**.** *If X is a UMD space, then there exists a constant C depending only on X and p such that for all $N \in \mathbb{N}$:*

$$\mathbb{E}\left(\left\|\sum_{n=1}^{N} Z_n Y_n\right\|^p\right) \leq C^p \mathbb{E}\left(\left\|\sum_{n=1}^{N} \tilde{Z}_n Y_n\right\|^p\right)$$

Proof. For $n \leq N$, define:

$$D_{2n-1} := \frac{1}{2}(Z_n + \tilde{Z}_n)Y_n \text{ and } D_{2n} := \frac{1}{2}(Z_n - \tilde{Z}_n)Y_n$$

And define the filtration:

$$\mathcal{D}_{2n-1} := \sigma(\mathcal{F}_{n-1}, \tilde{\mathcal{F}}_{n-1}, Z_n + \tilde{Z}_n) \text{ and } \mathcal{D}_{2n} := \sigma(\mathcal{F}_n, \tilde{\mathcal{F}}_n)$$

Then I claim that D_n is a martingale difference sequence with respect to $\mathcal{D}_n$. Proving the theorem amounts to proving this claim, since:

$$\sum_{n=1}^{N} Z_n Y_n = \sum_{n=1}^{2N} D_n \text{ and } \sum_{n=1}^{N} \tilde{Z}_n Y_n = \sum_{n=1}^{2N} (-1)^{n+1} D_n$$

and we may apply the UMD property. Therefore it remains to prove the claim.

To do this, we first note that clearly D_n is $\mathcal{D}_n$-measurable for all n. Next, we have:

$$\begin{aligned}\mathbb{E}(D_{2n} \mid \mathcal{D}_{2n-1}) &= \mathbb{E}(\frac{1}{2}(Z_n - \tilde{Z}_n)Y_n \mid \sigma(\mathcal{F}_{n-1}, \tilde{\mathcal{F}}_{n-1}, Z_n + \tilde{Z}_n)) \\ &= \frac{1}{2}Y_n \mathbb{E}(Z_n - \tilde{Z}_n \mid \sigma(Z_n + \tilde{Z}_n))\end{aligned}$$

using the facts that Y_n is $\mathcal{F}_{n-1}$-measurable and $Z_n, \tilde{Z}_n$ are independent of $\mathcal{F}_{n-1}$ and $\tilde{\mathcal{F}}_{n-1}$. Then we note that since $Z_n, \tilde{Z}_n$ are i.i.d, by symmetry we must have $\mathbb{E}(Z_n \mid \sigma(Z_n + \tilde{Z}_n)) = \mathbb{E}(\tilde{Z}_n \mid \sigma(Z_n + \tilde{Z}_n))$. Therefore $\mathbb{E}(D_{2n} \mid \mathcal{D}_{2n-1}) = \frac{1}{2}Y_n\mathbb{E}(Z_n - \tilde{Z}_n \mid \sigma(Z_n + \tilde{Z}_n)) = 0$. Similarly, since Z_n and $\tilde{Z}_n$ are independent of $\mathcal{F}_{n-1}, \tilde{\mathcal{F}}_{n-1}$ and have mean zero, we have

$$\mathbb{E}(D_{2n-1} \mid \mathcal{D}_{2n-2}) = \frac{1}{2}Y_n \mathbb{E}(Z_n + \tilde{Z}_n \mid \mathcal{F}_{n-1}, \tilde{\mathcal{F}}_{n-1}) = 0$$

Therefore D_n is a martingale difference sequence with respect to $\mathcal{D}_n$, and this completes the proof. □

In this way we have 'decoupled' the random variables Z_n and Y_n. We are now ready to prove an Itô isometry for random functions.

7.2.5 The Itô Integral for Stochastic Functions

Definition 7.21. A *finite rank adapted step process* $F : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, X)$ is a function of the form

$$F(t,\omega) = \sum_{n=1}^{N} \mathbb{1}_{(s_n,t_n]}(t)\mathbb{1}_{A_n}(\omega)h_n \otimes x_n$$

for real numbers $0 < s_n < t_n < \infty$, sets $A_n \in \mathcal{F}_{s_n}$ and elements $(h_n)_{n=1}^N$ and $(x_n)_{n=1}^N$ in H and X respectively.

Then the stochastic integral with respect to a H-cylindrical Brownian motion is defined as:

$$\int_0^\infty F(t)dB_H(t) := \sum_{n=1}^{N} \mathbb{1}_{A_n}(B_{h_n}(t_n) - B_{h_n}(s_n))x_n$$

Then, by a similar method to before, we can prove an 'Itô isomorphism' between this integral and a random operator, $T_F : \Omega \to \gamma(L^2(\mathbb{R}_+; H), X)$ via:

$$T_F(w)(f) := \int_0^\infty F(t,w)f(t)dt$$

Note that this is now an isomorphism and not an isometry, but we shall have an equivalent norm and so all convergence results will still hold.

Theorem 7.22. Itô Isomorphism
Let X be a UMD space, and fix $p \in (1,\infty)$. For all finite rank adapted step processes F : $\mathbb{R}_+ \times \Omega \to \mathcal{L}(H, X)$, there exists a constant C depending only on p and X such that

$$\frac{1}{C^p}\mathbb{E}\left(\|T_F\|^p_{\gamma(L^2(\mathbb{R}_+;H),X)}\right) \leq \mathbb{E}\left(\left\|\int_0^\infty F(t)dB_H(t)\right\|^p\right) \leq C^p\mathbb{E}\left(\|T_F\|^p_{\gamma(L^2(\mathbb{R}_+;H),X)}\right)$$

Proof. We want two independent copies of our H-cylindrical Brownian motion. As in our decoupling result, take the product space $(\Omega \times \tilde{\Omega}, \mathcal{F} \times \tilde{\mathcal{F}}, \mathbb{P} \times \tilde{\mathbb{P}})$, and define (abusing notation slightly):

$$B_H(\omega,\tilde{\omega}) := B_H(\omega), \text{ and } \tilde{B}_H(\omega,\tilde{\omega}) := B_H(\tilde{\omega})$$

Similarly to the deterministic case, for fixed F write (without loss of generality):

$$F = \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{j=1}^{k} \mathbb{1}_{(t_{n-1},t_n]}\mathbb{1}_{A_{mn}}h_j \otimes x_{jmn}$$

With $0 \leq t_0 < \ldots < t_N < \infty$, the sets A_{mn} disjoint and measurable with respect to $\mathcal{F}_{t_{n-1}}$ and $\{h_1, ...h_k\}$ an orthonormal set. Then we want to apply our Decoupling Theorem 7.20. To begin, note that we have a bijection from $i = 1, \ldots Nk$ to $(j, n), j = 1, \ldots k$ and $n = 1, \ldots N$ via $i = k(n-1) + j$. This allows us to define:

$$Z_i := B_{h_j}(t_n) - B_{h_j}(t_{n-1}), \;\; \tilde{Z}_i := \tilde{B}_{h_j}(t_n) - \tilde{B}_{h_j}(t_{n-1}), \;\; Y_i(\omega) := \sum_{m=1}^{M} \mathbb{1}_{A_{mn}}(\omega) x_{jmn}$$

Under this notation scheme, we have:

$$\begin{aligned} \int_0^\infty F(t) dB_H(t) &:= \sum_{n,m,j} \mathbb{1}_{A_{mn}}(B_{h_j}(t_n) - B_{h_j}(t_{n-1})) x_{jmn} \\ &= \sum_{i=1}^{Nk} Z_i Y_i \\ \int_0^\infty F(t) d\tilde{B}_H(t) &= \sum_{i=1}^{Nk} \tilde{Z}_i Y_i \end{aligned}$$

Now, define the filtration $\mathcal{F}_i$ to be the σ-algebra generated by all $Z_{i'}$ such that $i' \leq i$. Then Z_i is clearly measurable with respect to $\mathcal{F}_i$. We also have that Z_i is independent of $\mathcal{F}_{i'}$ for all $i' < i$, since the $(t_{n-1}, t_n]$ are disjoint and the $(h_j)_{j=1}^K$ are orthonormal. Finally, one can see that Y_i is $\mathcal{F}_i$-predictable. Therefore we satisfy the requirements to apply Decoupling Theorem 7.20.

Define $\mathbb{E}_1$ and $\mathbb{E}_2$ to be the integrals with respect to the first and second coordinates of $\Omega \times \tilde{\Omega}$. Then we have by the Decoupling Theorem (with small amounts of abused notation):

$$\begin{aligned} \mathbb{E} \left\| \int_0^\infty F(t) dB_H(t) \right\|^p &= \mathbb{E}_1 \mathbb{E}_2 \left\| \int_0^\infty F(t) dB_H(t) \right\|^p \\ &\leq C^p \mathbb{E}_1 \mathbb{E}_2 \left\| \int_0^\infty F(t) d\tilde{B}_H(t) \right\|^p \end{aligned}$$

With the constant C independent of F. Now, by definition:

$$\mathbb{E}_2 \left\| \int_0^\infty F(t) d\tilde{B}_H(t) \right\|^p = \mathbb{E}_2 \left\| \sum_{i=1}^{Nk} \tilde{Z}_i Y_i \right\|^p$$

Here, each $\tilde{Z}_i$ is a Gaussian variable with mean zero, and, possibly moving a constant from Y_i to $\tilde{Z}_i$, we may consider each $\tilde{Z}_i$ to have variance 1. This allows us to apply the Khinchine-Kahane inequality for Gaussian sums to observe:

$$\mathbb{E}_1 \mathbb{E}_2 \left\| \int_0^\infty F(t) d\tilde{B}_H(t) \right\|^p \simeq \mathbb{E}_1 \left(\mathbb{E}_2 \left\| \int_0^\infty F(t) d\tilde{B}_H(t) \right\|^2 \right)^{\frac{p}{2}}$$

Then, because we are integrating over only the second coordinate of $\Omega \times \tilde{\Omega}$, the function $F(t, \omega)$ is deterministic with respect to $\mathbb{E}_2$, and the Itô isomorphism 7.10 from the deterministic section applies (for fixed ω):

$$\mathbb{E}_1\left(\mathbb{E}_2\left\|\int_0^\infty F(t)d\tilde{B}_H(t)\right\|^2\right)^{\frac{p}{2}} \leq C^p\mathbb{E}_1\left(||T_F||^p_{\gamma(L^2(\mathbb{R}_+;H);X)}\right)$$

The prove may be repeated reversing our inequalities to show that this is in fact an isomorphism. □

Therefore if we choose a sequence of simple functions F_n such that T_{F_n} converges in expectation to some (random) operator T, we know that the stochastic integrals of the F_n will converge to some limit L. This motivates a definition and theorem analogous to 7.11 and 7.12 above:

Definition 7.23. A function $F : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, X)$ is said to be L^p-*stochastically integrable* if there exists a sequence of finite rank adapted step processes $F_n : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, X)$ such that:
(1) F_n converges to F strongly in measure.
(2) The X-valued random variables $\int_0^\infty F_n(t)dB_H(t)$ converge to some limit L in $L^p(\Omega; X)$.
Then we define:

$$\int_0^\infty F(t)dB_H(t) := \lim_{n\to\infty}\int_0^\infty F_n(t)dB_H(t)$$

Theorem 7.24. *Take a process $F : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, X)$ such that for all $h \in H$, the map $(t, \omega) \to F(t, \omega)h$ is strongly measurable and adapted. Then F is L^p-stochastically integrable if and only if:*
(1) The function $t \to F(t)^*x^*$ is in $L^p(\Omega; L^2(\mathbb{R}_+; H))$ for all x^* in X^*, and
(2) There exists a random operator $T \in L^p(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$ such that for all $f \in L^2(\mathbb{R}_+; H)$ and x^* in X^* we have:

$$\langle Tf, x^*\rangle = \int_0^\infty \langle F(t)f(t), x^*\rangle dt \text{ almost surely}$$

Furthermore, if such a T exists, then it is unique and we have

$$\frac{1}{C^p}\mathbb{E}\left(||T||^p_{\gamma^p(L^2(\mathbb{R}_+;H),X)}\right) \leq \mathbb{E}\left(\left\|\int_0^\infty F(t)dB_H(t)\right\|^p_X\right) \leq C^p\mathbb{E}\left(||T||^p_{\gamma^p(L^2(\mathbb{R}_+;H),X)}\right)$$

Thus we say that T 'represents' *F.*

Proof. This is Theorem 13.7 in [41], originally proven in [46]. □

The theory outlined above provides a foundation for the study of Banach-valued stochastic integration. There is of course much yet to do when compared with the wealth of results seen in the case of scalar integration. In this thesis, we will content ourselves with this foundation, and move on to specific applications L^p-estimates relevant to Stochastic PDEs. We conclude with some results that will be needed for these applications.

7.3 Further Results

Remark 7.25. Let $F : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, X)$ be a stochastically integrable function, and let $G : \mathbb{R}_+ \to \mathcal{L}(X)$ be a bounded function. Then we want to understand stochastic integrals of the form

$$\int_0^\infty G(s)F(s)dB_H(s)$$

Such an integral should be thought of as $G(s)$ composed with $F(s)$ (rather than acting on $F(s)$). From this, we can obtain a stochastic version of the Fubini Theorem for finite rank adapted step processes:

Lemma 7.26 (Stochastic Fubini Theorem)**.** *Let $\mathcal{N}$ be a measure space and X be a UMD space. Take an L^1 function $G : \mathcal{N} \to \mathcal{L}(X)$ and a* finite rank adapted step process *$F : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, X)$. Then we have almost surely:*

$$\int_{\mathcal{N}} \int_0^\infty G(\nu)F(t)dB_H(t)d\nu = \int_0^\infty \int_{\mathcal{N}} G(\nu)F(t)d\nu dB_H(t)$$

Proof. Write F as:

$$\begin{aligned} F(t,\omega) &= \sum_{n=1}^N \mathbb{1}_{(s_n,t_n]}(t)\mathbb{1}_{A_n}(\omega)h_n \otimes x_n \\ \int_0^\infty F(t)dB_H(t) &= \sum_{n=1}^N \mathbb{1}_{A_n}(B_{h_n}(t_n) - B_{h_n}(s_n))x_n \end{aligned}$$

Now, note that following Remark 7.25, we have $G(\nu)(h \otimes x) := h \otimes (G(\nu)x)$. Therefore we have:

$$\begin{aligned} \int_{\mathcal{N}} \int_0^\infty G(\nu)F(t)dB_H(t)d\nu &= \int_{\mathcal{N}} \sum_{n=1}^N \mathbb{1}_{A_n}(B_{h_n}(t_n) - B_{h_n}(s_n))G(\nu)x_n d\nu \\ &= \sum_{n=1}^N \mathbb{1}_{A_n}(B_{h_n}(t_n) - B_{h_n}(s_n)) \left(\int_{\mathcal{N}} G(\nu)d\nu \right) x_n \\ &= \int_0^\infty \sum_{n=1}^N \mathbb{1}_{(s_n,t_n]}(t)\mathbb{1}_{A_n}(\omega)h_n \otimes \left(\int_{\mathcal{N}} G(\nu)x_n d\nu \right) dB_H(t) \\ &= \int_0^\infty \int_{\mathcal{N}} G(\nu)F(t)d\nu dB_H(t) \end{aligned}$$

□

It would be amiss at this point to not point out that the lemma above is in fact a first step to proving a general Stochastic Fubini Theorem for functions $\Phi : \mathcal{N} \times \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, X)$ (under suitable conditions on Φ). However, proving this theorem subtle and not necessary for this thesis; the interested reader is directed to [45].

Remark 7.27. Consider an integral of an adapted stochastic process in the case $X = L^q(\mathcal{M})$. By the canonical isomorphism that can be found as Proposition 6.8, any random operator $T_F : \Omega \to \gamma(L^2(\mathbb{R}_+; H); L^q(\mathcal{M}))$ may be identified with the random function $\tilde{F} : \Omega \to L^q(\mathcal{M}; L^2(\mathbb{R}_+; H))$ defined by (the below two definitions are equivalent):

$$(\tilde{F}(w)(\mu), f) = [T_F(\omega)f]\,(\mu) \text{ for } f \in L^2(\mathbb{R}_+; H).$$
$$(\tilde{F}(w)(\mu)(t), h) = [F(t,\omega)h](\mu) \text{ for } h \in H.$$

Thus the Itô isomorphism 7.22 has an equivalent formulation (for a C depending only on $p, q, \mathcal{M}, H$ and X:):

$$\frac{1}{C^p}\mathbb{E}\left(||\tilde{F}||^p_{L^q(\mathcal{M};L^2(\mathbb{R}_+;H))}\right) \leq \mathbb{E}\left(\left\|\int_0^\infty F(t)dB_H(t)\right\|^p_X\right) \leq C^p\mathbb{E}\left(||\tilde{F}||^p_{L^q(\mathcal{M};L^2(\mathbb{R}_+;H))}\right)$$

Remark 7.28. By Minkowski's integral inequality (Corollary 1.26), for $q \geq 2$ we also have the one-sided inequality:

$$\mathbb{E}\left(\left\|\int_0^\infty F(t)dB_H(t)\right\|^p\right) \leq C\mathbb{E}\,\|t \to F(t,\omega)\|^p_{L^2(\mathbb{R}_+;L^q(\mathcal{M};H))}$$

with $\hat{G}(\omega)(t)(\mu) := G(\omega)(\mu)(t)$. Or equivalently, since $L^q(\mathcal{M}; H) \simeq \gamma(H; L^q(\mathcal{M}))$, we have

$$\mathbb{E}\left(\left\|\int_0^\infty F(t)dB_H(t)\right\|^p\right) \leq C'\mathbb{E}\,\|t \to F(t,\omega)\|^p_{L^2(\mathbb{R}_+;\gamma(H;L^q(\mathcal{M})))}$$

7.4 Further Reading

Chapters 5,6, 12 and 13 of [41] provide a good, pedagogical introduction to the theory of Banach-valued stochastic integration. The survey paper [44] also covers this content with less detail but more scope. The original results stem from [47] (concerning deterministic functions) and [46] (concerning random functions), based on decoupling ideas from [16] and [36].

Chapter 8

Stochastic Maximal L^p-Regularity

8.1 Background

In Chapter 5, we saw that a deterministic PDE may be written as a Banach-valued ODE:

$$\frac{d}{dt}u(t) + Au(t) = f(t)$$
$$u(0) = u_0$$

with $A : X \to X$ an unbounded sectorial operator that generates an analytic semigroup e^{-tA}. Then the (mild) solution of this ODE is given by the variation of constants formula:

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(s)ds$$

To motivate the construction that follows, we ask: what would happen if the function f became random? Then the solution u would become a random process, and we would be solving a PDE involving stochastic processes. The Banach-valued ODE above turns out to be an ideal framework for considering stochastic partial differential equations. With this in mind, we aim to find a random process $U : \mathbb{R}_+ \times \Omega \to X$ that 'solves' the *stochastic abstract Cauchy problem*:

$$\begin{aligned} dU(t) + AU(t)dt &= F(t)dB_H(t) \\ U(0) &= u_0 \text{ a.s.} \end{aligned} \tag{SACP}$$

with $F : \mathbb{R}_+ \times \Omega \to \gamma(H, X)$ representing our external noise. However, by 'solve,' we mean nothing more than our definition from Chapter 5 concerning mild solutions:

Definition 8.1. A function $U : \mathbb{R}_+ \times \Omega \to X$ is a *mild solution* of SACP if for almost every $t \in \mathbb{R}_+$ and $\omega \in \Omega$ we have

$$U(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} F(s)dB_H(s)$$

In the definition above, we may examine regularity by separately considering each of the two terms on the right hand side. Therefore in later work we only consider the case $u_0 = 0$.

8.1.1 Fractional Powers and Stochastic Maximal L^p Regularity

Let $A : X \to X$ be a sectorial operator of angle θ. We are going to use the McIntosh calculus to define A^α for $\alpha \in (0, 1)$.

Recall the function $\phi \in H_0^\infty(\Sigma_\theta)$ defined in Lemma 4.11 by $\phi(z) = \frac{z}{(1+z)^2}$ and satisfying $\phi(A) = A(1+A)^{-2}$. Recall also from Lemma 4.11 that $D(A) \cap \mathrm{Ran}(A) \subset \mathrm{Ran}(\phi(A))$, and that $D(A) \cap \mathrm{Ran}(A)$ is dense in X. Since A is assumed to be injective, we have that $\phi(A)$ is also injective, and a partial inverse $\phi(A)^{-1}$ exists, defined on the dense subspace $D(A) \cap \mathrm{Ran}(A)$. Then we construct our fractional powers of operators using the function:

$$\phi_\alpha(z) = \frac{z^{1+\alpha}}{(1+z)^2} \in H_0^\infty(\Sigma_\theta)$$

Definition 8.2. For $\alpha \in (0, 1)$, define the operator $A^\alpha : X \to X$ with dense domain:

$$\begin{aligned} A^\alpha x &:= \phi(A)^{-1}\phi_\alpha(A)x \\ D(A^\alpha) &:= \{x \in X \text{ s.t. } \phi_\alpha(A)x \in D(A) \cap \mathrm{Ran}(A)\} \end{aligned}$$

Then we shall reference the following results:

Proposition 8.3. *For $\alpha \in (0, 1)$, the fractional power A^α satisfies:*
(1) A^α is a well-defined, closed, injective operator
(2) $D(A^\alpha) \supset D(A) \cap \mathrm{Ran}(A)$ and thus $D(A^\alpha)$ is dense
(3) A^α is sectorial of angle $\alpha\theta$

Proof. These are proven in 15 B, C in [29]. (1) and (2) follow from Theorem 15.8, and (3) is Theorem 15.16. $\square$

We also note that this definition of fractional powers agrees with the definition from Lemma 6.27, in the sense that for A^α defined as above, we have

$$A^\alpha R(z, A) = \frac{1}{2\pi i}\int_{\partial\Sigma_\omega} \frac{\lambda^\alpha}{z - \lambda} R(\lambda, A)d\lambda$$

Now that fractional powers are understood, we can define Maximal L^p-Regularity in the stochastic setting.

Definition 8.4. Denote $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H; L^q(\mathcal{M})))$ to be the closure of finite rank adapted step processes $F : \mathbb{R} \times \Omega \to \gamma(H; L^q(\mathcal{M}))$ with respect to the L^p norm.

Definition 8.5. We say that the operator A has *stochastic maximal L^p regularity* if for all $F \in L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; L^q(\mathcal{M}; H))$, the stochastic convolution:

$$U(t) = \int_0^t e^{-(t-s)A}F(s)dB_H(s)$$

takes values in $D(A^{\frac{1}{2}})$ almost surely, and we have the estimate

$$\mathbb{E}\left(||A^{\frac{1}{2}}U||^p_{L^p(\mathbb{R}_+;L^q(\mathcal{M}))}\right) \leq C^p\mathbb{E}\left(||F||^p_{L^p(\mathbb{R}_+;L^q(\mathcal{M};H))}\right)$$

Remark 8.6. In this case, the word 'Maximal' is a pragmatic choice that stems from the concrete operators that we have in mind. In particular, A is usually some second-order differential operator, in which case $A^{\frac{1}{2}}$ corresponds (roughly) to a first-order derivative. A solution $U(t)$ will not in general be 'better than' once-differentiable, since Brownian motion is not Hölder continuous of parameter α for any $\alpha > \frac{1}{2}$.

We give a simple example. Let A be the one-dimensional Laplacian Δ, let X be the Hilbert space $L^2(0,1)$ and let $F(s)$ constantly equal P, where P is the projection onto the constant unit function (defined by $P\varphi = (\varphi, 1) \cdot 1 \in L^2(0,1)$). Clearly we have $\Delta P\varphi = 0$ for all φ, and (as in Remark 5.7) this implies that $e^{-t\Delta}P\varphi = P\varphi$ for all $t \geq 0$ and all φ. But this implies that the solution to the relevant SPDE (with $U_0 = 0$) is given by:

$$U(t) = \int_0^t e^{-(t-s)\Delta} F(s) dB_H(s) = \int_0^t P dB_H(s)$$

which is a Brownian motion on $(0,1)$. If $U(t) \in D(\Delta^\alpha)$ for $\alpha > \frac{1}{2}$, then we have $U(t)$ in the Sobolev space $W^{2\alpha,2}(0,1)$. By the Sobolev embedding theorem, this implies that $U(t)$ is also in the Hölder space $C^\alpha(0,1)$, a contradiction since $U(t)$ is a Brownian motion on $(0,1)$. This simple example is meant to illuminate that, in general, we cannot expect better than a single space derivative in the stochastic PDE setting.

Remark 8.7. As in the determinstic case, stochastic maximal regularity may be used to prove existence and uniqueness of various stochastic PDEs such as time-dependent parabolic equations and the stochastic Navier Stokes equation. Such applications may be seen in (for example) the paper [42].

Remark 8.8. We will see that our main Theorem 8.15 works only in the case $p \in (2,\infty), q \in [2,\infty)$. This is not a flaw in the method of proof - it was proven in [28] that Maximal L^p-regularity does *not* hold in the other cases (and indeed, large parts of theory break down).

8.2 R-Boundedness of Stochastic Convolutions

Remark 8.9. To clean up notation, we use the symbols $\lesssim$ and $\simeq$, where '$a \lesssim b$' and '$a \simeq b$' are understood to mean 'there exists a constant C such that $a \leq Cb$' or '$\frac{1}{C}b \leq a \leq Cb$' respectively.

In this section, we prove a lemma that is fundamental to the upcoming proof of the Stochastic Maximal Regularity Theorem. This lemma is itself very lengthy and difficult to prove, hence it has been allocated its own section. It is not unreasonable for the casual reader to proceed directly to the following section wherein the main theorem is proven.

Let $\mathcal{K}$ be the set of absolutely continuous functions $k : \mathbb{R}_+ \to \mathbb{R}$ such that $\lim_{t\to\infty} k(t) = 0$ and

$$\int_0^\infty t^{\frac{1}{2}} |k'(t)| dt \leq 1$$

Given a finite rank adapted step process $F : \mathbb{R}_+ \times \Omega \to \gamma(H; L^q(\mathcal{M}))$, we can define the stochastic convolution $I(k)F : \mathbb{R}_+ \times \Omega \to L^q(\mathcal{M})$ by

$$(I(k)F)(t) := \int_0^t k(t-s)F(s)dB_H(s), t \geq 0$$

This integral is well-defined. Indeed, we may identify the process $s \to k(t-s)F(s)$ with the random operator $T : \Omega \to \mathcal{L}(L^2(\mathbb{R}_+; H), L^q(\mathcal{M}))$ given by

$$T(\omega)f := \int_0^t k(t-s)F(s,\omega)f(s)ds$$

This operator is finite rank, since F is finite rank, and so T takes values in $\gamma(L^2(\mathbb{R}_+; H), L^q(\mathcal{M}))$. Thus by Theorem 7.12, our function is stochastically integrable. Note that in the proof below, we will show that for any $k \in \mathcal{K}$ the stochastic convolution $I(k)$ is a bounded operator on finite rank adapted step processes, and therefore $I(k)$ extends by density to a bounded operator $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H; L^q(\mathcal{M}))) \to L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{M}))$. Then our lemma is:

Lemma 8.10. *The operators $I(k) : L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H; L^q(\mathcal{M}))) \to L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{M}))$ for $k \in \mathcal{K}$ form an R-bounded set provided that $p \in (2, \infty)$ and $q \in [2, \infty)$.*

Proof. We split this proof into steps.

Step 1

We will show that $\{I(k) \mid k \in \mathcal{K}\}$ is uniformly bounded.

Since, $\lim_{t\to\infty} k(t) = 0$ for any $k \in \mathcal{K}$ we can write by absolute continuity:

$$k(s) = -\int_s^\infty k'(v)dv$$

Then for F an arbitrary $\mathcal{F}$-adapted finite rank step process $t \geq 0$ we have:

$$\begin{aligned}
I(k)F(t) &= \int_0^t k(t-s)F(s)dB_H(s) \\
&= -\int_0^t \left(\int_{t-s}^\infty k'(v)dv\right) F(s)dB_H(s) \\
&= -\int_0^\infty \int_0^\infty k'(v)\mathbb{1}_{\{s\in(0,t)\}}\mathbb{1}_{\{t-s<v\}}F(s)dvdB_H(s)
\end{aligned}$$

Thinking of scalars as scalar-multiplying operators, applying the stochastic Fubini theorem from 7.26:

$$\begin{aligned}
&= -\int_0^\infty k'(v)\int_0^\infty \mathbb{1}_{\{s\in(0,t)\}}\mathbb{1}_{\{t-s<v\}}F(s)dB_H(s)dv \\
&= -\int_0^\infty \sqrt{v}k'(v)\left(\frac{1}{\sqrt{v}}\int_{(t-v)\vee 0}^t F(s)dB_H(s)\right)dv \\
&:= -\int_0^\infty \sqrt{v}k'(v)J(v)F(t)dv
\end{aligned}$$

Where $J(v)$ is defined for $v \in \mathbb{R}_+$ by

$$J(v)F(t) := \frac{1}{\sqrt{v}} \int_{(t-v)\vee 0}^{t} F(s) dB_H(s)$$

I claim that the $J(v)$ are uniformly bounded in v as operators $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H; L^q(\mathcal{M}))) \to L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{M}))$. We show this:

$$\begin{aligned} ||J(v)F||^p_{L^p(\mathbb{R}_+\times\Omega;L^q(\mathcal{M}))} &= \mathbb{E}\int_0^\infty \left\| \frac{1}{\sqrt{v}} \int_{(t-v)\vee 0}^{t} F(s)dB_H(s) \right\|^p_{L^q(\mathcal{M})} dt \\ &= v^{\frac{-p}{2}} \int_0^\infty \mathbb{E} \left\| \int_{(t-v)\vee 0}^{t} F(s)dB_H(s) \right\|^p_{L^q(\mathcal{M})} dt \end{aligned}$$

By the second inequality from Remark 7.28:

$$\begin{aligned} &\lesssim v^{\frac{-p}{2}} \int_0^\infty \mathbb{E} \left(\int_{(t-v)\vee 0}^{t} ||F(s)||^2_{\gamma(H,L^q(\mathcal{M}))} ds \right)^{\frac{p}{2}} dt \\ &= v^{\frac{-p}{2}} \int_0^\infty \mathbb{E} \left[\mathbb{1}_{(0,v)} * \left(s \to ||F(s)||^2_{\gamma(H,L^q(\mathcal{M}))} \right)(t) \right]^{\frac{p}{2}} dt \\ &= v^{\frac{-p}{2}} \mathbb{E} \left\| \mathbb{1}_{(0,v)} * \left(s \to ||F(s)||^2_{\gamma(H,L^q(\mathcal{M}))} \right) \right\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(\mathbb{R}_+)} \end{aligned}$$

By Young's Inequality for Convolutions (noting that $\frac{p}{2} > 1$):

$$\begin{aligned} &\leq v^{\frac{-p}{2}} ||\mathbb{1}_{(0,v)}||^{\frac{p}{2}}_{L^1(\mathbb{R}_+)} \cdot \mathbb{E} \left\| s \to ||F(s)||^2_{\gamma(H,L^q(\mathcal{M}))} \right\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(\mathbb{R}_+)} \\ &= v^{\frac{-p}{2}} \cdot v^{\frac{p}{2}} \mathbb{E}||F||^p_{L^p(\mathbb{R}_+;\gamma(H,L^q(\mathcal{M})))} \\ &= ||F||_{L^p(\mathbb{R}_+\times\Omega;\gamma(H,L^q(\mathcal{M})))} \end{aligned}$$

Therefore the $J(v)$ are uniformly bounded in v, say by M. This means:

$$\begin{aligned} ||I(k)F||^p_{L^p(\mathbb{R}_+\times\Omega;L^q(\mathcal{M}))} &= \left\| \int_0^\infty \sqrt{v}k'(v)J(v)F dv \right\|_{L^p(\mathbb{R}_+\times\Omega;L^q(\mathcal{M}))} \\ &\leq \int_0^\infty |\sqrt{v}k'(v)| \, \|J(v)F\|_{L^p(\mathbb{R}_+\times\Omega;L^q(\mathcal{M}))} dv \\ &\leq M \int_0^\infty |\sqrt{v}k'(v)| \cdot ||F||_{L^p(\mathbb{R}_+\times\Omega;\gamma(H,L^q(\mathcal{M})))} dv \\ &\leq M||F||_{L^p(\mathbb{R}_+\times\Omega;\gamma(H,L^q(\mathcal{M})))} \end{aligned}$$

With the final inequality coming from the bounded L^1 norm of $\sqrt{v}k'(v)$ (since $k \in \mathcal{K}$). Therefore the $I(k)$ are uniformly bounded. Note in particular that we have confirmed our claim that each $I(k)$ extends to a well-defined operator $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H; L^q(\mathcal{M}))) \to L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{M}))$.

Step 2

Suppose that we had shown that $\{J(v), v \in \mathbb{R}_+\}$ was R-bounded. Then recall Proposition 2.12 which states that, given an R-bounded set $\mathcal{T}$, a strongly measurable function Φ taking values in $\mathcal{T}$ and a real-valued L^1 function ϕ, we may define the integral

$$T_{\Phi,\phi}x := \int_{\mathcal{M}} \phi(s)\Phi(s)x d\mu$$

And the set $\mathcal{T}' := \{T_{\Phi,\phi} \text{ s.t. } ||\phi||_{L^1(\mathcal{M};\mathbb{C})} \leq 1\}$ is R-bounded with $R(\mathcal{T}') \leq 2R(\mathcal{T})$. We apply this theorem with $\phi(v) = \sqrt{v}k'(v)$ and $\Phi(v) = J(v)$ to conclude that the functions $I(k)$ defined by

$$I(k)F(t) = -\int_0^\infty \sqrt{v}k'(v)J(v)F(t)dv$$

form an R-bounded set $\{I(k) \mid k \in \mathcal{K}\}$. It remains to show that the set $\{J(v), v > 0\}$ is R-bounded (which will still take quite a while).

Step 3

We prove a preparatory lemma and refer to some known results from Harmonic Analysis:

Definition 8.11. Given two measure spaces $\mathcal{N}$ and $\mathcal{M}$, a *positive function* on $L^p(\mathcal{N}; L^q(\mathcal{M}))$ (with $p, q \in [1, \infty]$) is a function satisfying $f(\nu, \mu) \geq 0$ for all $\nu \in \mathcal{N}, \mu \in \mathcal{M}$. A *positive operator* is an operator $T : L^p(\mathcal{N}; L^q(\mathcal{M})) \to L^p(\mathcal{N}; L^q(\mathcal{M}))$ that maps positive functions to positive functions.

Lemma 8.12. *Let $T(v)$ be a strongly continuous one-parameter family of positive linear operators on $L^p(\mathcal{N}; L^q(\mathcal{M}))$. Suppose that the maximal function*

$$[T^*_\star(g)](\nu, \mu) := \sup_{v>0} |[T^*(v)g](\nu, \mu))|$$

is measurable and $L^{p'}(\mathcal{N}; L^{q'}(\mathcal{M}))$-bounded by $C \geq 0$ (with $\frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1$). Then for all $N \geq 1$, $\{\varphi_1, \dots \varphi_N\} \subset L^p(\mathcal{N}; L^q(\mathcal{M}))$ and $v_1, \dots v_N > 0$,

$$\left\| \sum_{n=1}^N T(v_n)|\varphi_n| \right\|_{L^p(\mathcal{N};L^q(\mathcal{M}))} \leq C \left\| \sum_{n=1}^N |\varphi_n| \right\|_{L^p(\mathcal{N};L^q(\mathcal{M}))}$$

Proof. Fix N. Let $l^p_N(X)$ be the space of sequences of N elements of a Banach space X endowed with the norm:

$$||(a_n)_{n=1}^N||_{l^p_N(X)} = \left(\sum_{n=1}^N ||a_n||^p_X \right)^{\frac{1}{p}}$$
$$||(a_n)_{n=1}^N||_{l^\infty_N(X)} = \max_n ||a_n||_X$$

Then we can consider $L^p(\mathcal{N}; l^1_N(L^q(\mathcal{M})))$ and we have:

$$\begin{aligned}\left\|\sum_{n=1}^{N} T(v_n)|\varphi_n|\right\|^p_{L^p(\mathcal{N};L^q(\mathcal{M}))} &= \int_{\mathcal{N}}\left(\int_{\mathcal{M}}\left|\sum_{n=1}^{N} T(v_n)|\varphi_n|(\nu,\mu)\right|^q d\mu\right)^{\frac{p}{q}} d\nu \\ &= \left\|(T(v_n)|\varphi_n|)_{n=1}^N\right\|_{L^p(\mathcal{N};L^q(\mathcal{M};l^1_N))}\end{aligned}$$

The space $L^p(\mathcal{N}; L^q(\mathcal{M}; l^1_N))$ is reflexive, and so we may take a supremum over g with norm 1 in the dual space $L^{p'}(\mathcal{N}; L^{q'}(\mathcal{M}; l^\infty_N))$, and we have (repeatedly using the relevant version of Hölder's inequality):

$$\begin{aligned}\left\|\sum_{n=1}^{N} T(v_n)|\varphi_n|\right\|_{L^p(\mathcal{N};L^q(\mathcal{M}))} &= \sup_g \int_{\mathcal{N}}\int_{\mathcal{M}}\left(\sum_{n=1}^{N} \cdot T(v_n)|\varphi_n|(\nu,\mu)\cdot g_n(\nu,\mu)\right) d\mu d\nu \\ &\leq \sup_g \int_{\mathcal{N}}\int_{\mathcal{M}} ||(|\varphi_n|(\nu,\mu))_{n=1}^N||_{l^1_N}\cdot ||([T^*(v_n)g_n](\nu,\mu))_{n=1}^N||_{l^\infty_N} d\mu d\nu \\ &\leq \sup_g \int_{\mathcal{N}}\left(||\varphi_n(\nu)||_{L^q(\mathcal{M};l^1_N)}\cdot ||[T^*(v_n)g_n](\nu)||_{L^{q'}(\mathcal{M};l^\infty_N)}\right) d\nu \\ &\leq ||(\varphi_n)_{n=1}^N||_{L^p(\mathcal{N};L^q(\mathcal{M};l^1_N))} \sup_g ||(T^*(v_n)g_n)_{n=1}^N||_{L^{p'}(\mathcal{N};L^{q'}(\mathcal{M};l^\infty_N))} \\ &= \sup_g ||(T^*(v_n)g_n)_{n=1}^N||_{L^{p'}(\mathcal{N};L^{q'}(\mathcal{M};l^\infty_N))}\left\|\sum_{n=1}^{N}|\varphi_n|\right\|_{L^p(\mathcal{N};L^q(\mathcal{M}))}\end{aligned}$$

It remains to show that $||(T^*(v_n)g_n)_{n=1}^N||_{L^{p'}(\mathcal{N};L^{q'}(\mathcal{M};l^\infty_N))}$ is uniformly bounded over g with norm 1. Using our definition of $T^*_\star$ and the assumption that this operator is bounded by C:

$$\begin{aligned}||(T^*(v_n)g_n)_{n=1}^N||_{L^{p'}(\mathcal{N};L^{q'}(\mathcal{M};l^\infty_N))} &= ||\max_n\ |T^*(v_n)g_n|\,||_{L^{p'}(\mathcal{N};L^{q'}(\mathcal{M}))} \\ &\leq ||\max_n\ T^*_\star|g_n|||_{L^{p'}(\mathcal{N};L^{q'}(\mathcal{M}))}\end{aligned}$$

Since $T^*_\star$ is a positive operator, we may commute it with the maximum:

$$\begin{aligned}&\leq ||T^*_\star(\max_n\ |g_n|)||_{L^{p'}(\mathcal{N};L^{q'}(\mathcal{M}))} \\ &\leq C||\max_n\ |g_n|||_{L^{p'}(\mathcal{N};L^{q'}(\mathcal{M}))} \\ &= C||(|g_n|)_{n=1}^N||_{L^{p'}(\mathcal{N};L^{q'}(\mathcal{M};l^\infty_N))} \\ &= C \text{ since } g \text{ has norm 1.}\end{aligned}$$

This completes the lemma. □

As well as this, we refer to the following results proven using Harmonic Analysis techniques which are beyond the scope of this thesis:

Definition 8.13. Given $f \in L^p(\mathbb{R}_+)$ define the *one-sided Hardy-Littlewood maximal function*:

$$M(f)(t) := \sup_{\delta>0} \frac{1}{\delta}\int_t^{t+\delta} |f(s)|ds$$

Then if we have $\varphi \in L^p(\mathbb{R}_+; L^q(\mathcal{M}))$ we can define

$$[\tilde{M}\varphi](t)(\mu) := \sup_{\delta>0} \frac{1}{\delta} \int_t^{t+\delta} |\varphi(s)(\mu)| ds$$

Then we will use the result:

Proposition 8.14 (Fefferman-Stein). *For all $p \in (1, \infty)$ and $q \in (1, \infty]$, the one sided Hardy-Littlewood maximal function $\tilde{M}$ is bounded on $L^p(\mathbb{R}_+, L^q(\mathcal{M}))$.*

Proof. This may be adapted from Theorem 4.6.6 in [18], moving from an ℓ^q estimate to an L^q estimate. □

Step 4

With these preparatory results in hand, we lay out the long proof that $\{J(v), v > 0\}$ is R-bounded. First some notation: take a sequence of Rademacher variables $r_n : \Omega' \to \{-1, 1\}$ notated by $r_n(\eta) = \pm 1$. Denoting $\mathbb{E}_r$ as the expectation with respect to Ω'. This is to distinguish from the stochastic process $F : \mathbb{R}_+ \times \Omega \to L^q(\mathcal{M})$ which has its random input variable denoted by ω.

Now, choose arbitrary $v_1, \dots v_N > 0$ and $F_1, \dots F_N \in L^p(\mathbb{R}_+ \times \Omega; \gamma(H, L^q(\mathcal{M})))$. Then the aim is to prove:

$$\mathbb{E}_r \left\| \sum_{n=1}^N r_n J(v_n) F_n \right\|^p_{L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{M}))} \lesssim_{p,q} \mathbb{E}_r \left\| \sum_{n=1}^N r_n F_n \right\|^p_{L^p(\mathbb{R}_+ \times \Omega; \gamma(H, L^q(\mathcal{M})))}$$

First, write out the $J(v_n)$ explicitly:

$$\begin{aligned}
\mathbb{E}_r \left\| \sum_{n=1}^N r_n J(v_n) F_n \right\|^p_{L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{M}))} &= \mathbb{E}_r \left\| \int_0^\infty \sum_{n=1}^N \frac{r_n}{\sqrt{v_n}} \mathbb{1}_{[t-v_n,t]} F_n(s) dB_H(s) \right\|^p_{L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{M}))} \\
&:= \mathbb{E}_r \left\| \int_0^\infty \Psi(\eta, s, t) dB_H(s) \right\|^p_{L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{M}))}
\end{aligned}$$

Where we define (for notational convenience)

$$\Psi(\eta, s, t) := \sum_{n=1}^N \frac{r_n(\eta)}{\sqrt{v_n}} \mathbb{1}_{[t-v_n,t]}(s) F_n(s)$$

Recall our assumption that $q \in [2, \infty)$. Then, by the inequality from Remark 7.27, for fixed (η, t) we have:

$$\mathbb{E} \left\| \int_0^\infty \Psi(\eta, s, t) dB_H(s) \right\|^p_{L^q(\mathcal{M})} \lesssim_{p,q} \mathbb{E} \left\| \Psi(\eta, s, t)(\mu) \right\|^p_{L^q(\mathcal{M}; L^2(\mathbb{R}_+; H))}$$

Where the right hand side is understood as the function $\mu \to \Psi(\eta, s, t)(\mu)$, and later sums may be understood as $\mu \to F(s)(\mu)$. For clarity we will not spell out these considerations. This implies:

$$\begin{aligned}\mathbb{E}_r \left\| \int_0^\infty \Psi(\eta,s,t)dB_H(s) \right\|^p_{L^p(\mathbb{R}_+\times\Omega;L^q(\mathcal{M}))} &= \mathbb{E}_r \int_0^\infty \mathbb{E} \left\| \int_0^\infty \Psi(\eta,s,t)dB_H(s) \right\|^p_{L^q(\mathcal{M}))} dt \\ &\lesssim_{p,q} \mathbb{E}_r \int_0^\infty \mathbb{E} \left\| \Psi(\eta,s,t)(\mu) \right\|^p_{L^q(\mathcal{M};L^2(\mathbb{R}_+;H))} dt \\ &= \mathbb{E} \int_0^\infty \mathbb{E}_r \left\| \Psi(\eta,s,t)(\mu) \right\|^p_{L^q(\mathcal{M};L^2(\mathbb{R}_+;H))} dt\end{aligned}$$

Next, for fixed (s,t) we have by the Khinchine-Kahane inequality :

$$\begin{aligned}\mathbb{E}_r \left\| \Psi(\eta,s,t)(\mu) \right\|^p_{L^q(\mathcal{M};L^2(\mathbb{R}_+;H))} &= \mathbb{E}_r \left\| \sum_{n=1}^N \frac{r_n}{\sqrt{v_n}} \mathbb{1}_{(t-v_n,t)}(s)F_n(s) \right\|^p_{L^q(\mathcal{M};L^2(\mathbb{R}_+;H))} \\ &\simeq_{p,q} \left(\mathbb{E}_r \left\| \sum_{n=1}^N \frac{r_n}{\sqrt{v_n}} \mathbb{1}_{(t-v_n,t)}(s)F_n(s) \right\|^q_{L^q(\mathcal{M};L^2(\mathbb{R}_+;H))} \right)^{p/q} \\ &= \left(\mathbb{E}_r \left\| \Psi(\eta,s,t) \right\|^q_{L^q(\mathcal{M};L^2(\mathbb{R}_+;H))} \right)^{p/q}\end{aligned}$$

Therefore we have:

$$\begin{aligned}\mathbb{E} \int_0^\infty \mathbb{E}_r \left\| \Psi(\eta,s,t \right\|^p_{L^q(\mathcal{M};L^2(\mathbb{R}_+;H))} dt &= \mathbb{E} \int_0^\infty \left(\mathbb{E}_r \left\| \Psi(\eta,s,t) \right\|^q_{L^q(\mathcal{M};L^2(\mathbb{R}_+;H))} \right)^{p/q} dt \\ &= \mathbb{E} \int_0^\infty \left(\int_{\mathcal{M}} \mathbb{E}_r \left\| \Psi(\eta,s,t)(\mu) \right\|^q_{L^2(\mathbb{R}_+;H)} d\mu \right)^{p/q} dt\end{aligned}$$

Use Khinchine-Kahane again to see that for fixed μ, we have $\mathbb{E}_r \left\| \Psi(\eta,s,t)(\mu) \right\|^q_{L^2(\mathbb{R}_+;H)} \simeq_q (\mathbb{E}_r \left\| \Psi(\eta,s,t)(\mu) \right\|^2_{L^2(\mathbb{R}_+;H)})^{q/2}$. Thus

$$\begin{aligned}\mathbb{E} \int_0^\infty \mathbb{E}_r \left\| \Psi(\eta,s,t) \right\|^p_{L^q(\mathcal{M};L^2(\mathbb{R}_+;H))} dt &\simeq_{p,q} \mathbb{E} \int_0^\infty \left(\int_{\mathcal{M}} \left(\mathbb{E}_r \left\| \Psi(\eta,s,t)(\mu) \right\|^2_{L^2(\mathbb{R}_+;H)} \right)^{q/2} d\mu \right)^{p/q} dt \\ &= \mathbb{E} \int_0^\infty \left(\int_{\mathcal{M}} \left(\int_0^\infty \mathbb{E}_r \left\| \Psi(\eta,s,t)(\mu) \right\|^2_H ds \right)^{q/2} d\mu \right)^{p/q} dt\end{aligned}$$

Now, evaluate $\mathbb{E}_r \left\| \Psi(\eta,s,t)(\mu) \right\|^2_H$ (for fixed s,μ,t), using the fact that the r_n are independent:

$$\begin{aligned}\mathbb{E}_r \left\| \Psi(\eta,s,t)(\mu) \right\|^2_H &= \mathbb{E}_r \left\| \sum_{n=1}^N \frac{r_n(\eta)}{\sqrt{v_n}} \mathbb{1}_{[t-v_n\vee 0,t]}(s)F_n(s)(\mu) \right\|^2_H \\ &= \sum_{n=1}^N \frac{1}{v_n} \mathbb{1}_{[t-v_n\vee 0,t]}(s)||F_n(s)(\mu)||^2_H\end{aligned}$$

Therefore we have:

$$
\begin{aligned}
&\mathbb{E}_r \left\| \sum_{n=1}^{N} r_n J(v_n) F_n \right\|^p_{L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{M}))} \\
&\lesssim_{p,q} \mathbb{E} \int_0^\infty \left(\int_{\mathcal{M}} \left(\int_0^\infty \sum_{n=1}^{N} \frac{1}{v_n} \mathbb{1}_{[t-v_n \vee 0, t]}(s) ||F_n(s)||_H^2 ds \right)^{q/2} d\mu \right)^{p/q} dt \\
&= \mathbb{E} \int_0^\infty \left(\int_{\mathcal{M}} \left(\sum_{n=1}^{N} \frac{1}{v_n} \int_{(t-v_n \vee 0)}^{t} ||F_n(s)(\mu)||_H^2 ds \right)^{q/2} d\mu \right)^{p/q} dt \\
&:= \mathbb{E} \left\| \sum_{n=1}^{N} T^*(v_n) \varphi(t,\mu) \right\|^{p/2}_{L^{p/2}(\mathbb{R}_+; L^{q/2}(\mathcal{M}))}
\end{aligned}
$$

Where $\varphi(t\mu) = ||F_n(t)(\mu)||_H^2$ and $T^*(v)$ acts on $L^{p/2}(\mathbb{R}_+; L^{q/2}(\mathcal{M}))$ via:

$$
[T^*(v)\varphi](t,\mu) := \frac{1}{v} \int_{(t-v\vee 0)}^{t} \varphi(s,\mu) ds
$$

If we define $2/p + 1/p^* = 1$ and $2/q + 1/q^* = 1$, then $T^*(v)$ is the adjoint of the operator on $L^{p^*}(\mathbb{R}_+; L^{q^*}(\mathcal{M}))$ given by

$$
[T(v)\psi](t,\mu) = \frac{1}{v} \int_t^{t+v} \psi(s,\mu) ds
$$

This operator is bounded by the one-sided Hardy-Littlewood maximal function $\tilde{M}$. We know however, by the Fefferman-Stein proposition 8.14, that the function $\tilde{M}$ is itself bounded, and so by Lemma 8.12 above we have:

$$
\mathbb{E} \left\| \sum_{n=1}^{N} T^*(v_n) \varphi(t,\mu) \right\|^{p/2}_{L^{p/2}(\mathbb{R}_+; L^{q/2}(\mathcal{M}))} \lesssim_{p,q} \mathbb{E} \left\| \sum_{n=1}^{N} \varphi(t,\mu) \right\|^{p/2}_{L^{p/2}(\mathbb{R}_+; L^{q/2}(\mathcal{M}))}
$$

And then we can reverse our steps:

$$\begin{aligned}
\mathbb{E}\left\|\sum_{n=1}^{N}\varphi(t,\mu)\right\|^{p/2}_{L^{p/2}(\mathbb{R}_+;L^{q/2}(\mathcal{M}))} &= \mathbb{E}\int_0^\infty \left(\int_{\mathcal{M}}\left(\sum_{n=1}^{N}||F_n(t,\mu)||_H^2\right)^{q/2} d\mu\right)^{p/q} dt \\
&= \mathbb{E}\int_0^\infty \left(\int_{\mathcal{M}}\left(\mathbb{E}_r\left\|\sum_{n=1}^{N} r_n F_n(t,\mu)\right\|_H^2\right)^{q/2} d\mu\right)^{p/q} dt \\
&\simeq \mathbb{E}\int_0^\infty \left(\int_{\mathcal{M}}\mathbb{E}_r\left\|\sum_{n=1}^{N} r_n F_n(t,\mu)\right\|_H^q d\mu\right)^{p/q} dt \\
&= \mathbb{E}\int_0^\infty \left(\mathbb{E}_r\left\|\sum_{n=1}^{N} r_n F_n(t)\right\|_{L^q(\mathcal{M};H)}^q\right)^{p/q} dt \\
&\simeq \mathbb{E}\int_0^\infty \mathbb{E}_r\left\|\sum_{n=1}^{N} r_n F_n(t)\right\|_{L^q(\mathcal{M};H)}^p dt \\
&= \mathbb{E}_r\left\|\sum_{n=1}^{N} r_n F_n\right\|_{L^p(\mathbb{R}_+\times\Omega;L^q(\mathcal{M};H))}^p \\
&\simeq \mathbb{E}_r\left\|\sum_{n=1}^{N} r_n F_n\right\|_{L^p(\mathbb{R}_+\times\Omega;\gamma(H,L^q(\mathcal{M})))}^p
\end{aligned}$$

With the last line using the canonical isomorphism from Proposition 6.8. We therefore have our inequality

$$\mathbb{E}_r\left\|\sum_{n=1}^{N} r_n J(v_n)F_n\right\|_{L^p(\mathbb{R}_+\times\Omega;L^q(\mathcal{M}))}^p \lesssim_{p,q} \mathbb{E}_r\left\|\sum_{n=1}^{N} r_n F_n\right\|_{L^p(\mathbb{R}_+\times\Omega;\gamma(H,L^q(\mathcal{M})))}^p$$

and conclude that $\{J(v), v>0\}$ is R-bounded. $\square$

Having proven this R-boundedness result, we are ready to prove the Stochastic Maximal L^p-Regularity Theorem using a McIntosh calculus argument.

8.3 The Maximal L^p Regularity Theorem

Theorem 8.15. *Let $p \in (2,\infty)$, $q \in [2,\infty)$. Suppose $A : L^q(\mathcal{M}) \to L^q(\mathcal{M})$ is a sectorial operator of angle θ that admits a McIntosh calculus of angle $\Theta < \frac{\pi}{2}$. Then A has stochastic maximal L^p-regularity.*

Proof. First note that by the canonical isomorphism from 6.8, we can identify a function $F \in L^p_{\mathcal{F}}(\mathbb{R}_+\times\Omega; L^q(\mathcal{M};H))$ with a function $F \in L^p_{\mathcal{F}}(\mathbb{R}_+\times\Omega;\gamma(H;L^q(\mathcal{M})))$. We are going to show that for all such F there exists a C such that

$$\left\|t \to \int_0^t A^{\frac{1}{2}}e^{-(t-s)A}F(s)dB_H(s)\right\| \le C^p||F||_{L^p_{\mathcal{F}}(\mathbb{R}_+\times\Omega;\gamma(H,L^q(\mathcal{M})))}$$

To do this we will construct a bounded map $F \to \int_0^t A^{\frac{1}{2}} e^{-(t-s)A} F(s) dB_H(s)$ using the R-McIntosh calculus. We split the proof into several steps.

Step 1

Define:

$$\tilde{A} : L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H; L^q(\mathcal{M}))) \to L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H; L^q(\mathcal{M})))$$

by $[(\tilde{A}F)(t,\omega)](h) := A([F(t,\omega)](h))$. The operator $\tilde{A}$ is well-defined for all $F \in L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H; D(A)))$.

Lemma 8.16. *The operator $\tilde{A}$ is sectorial of angle θ, has a McIntosh calculus of angle Θ, and the functional calculi of A and $\tilde{A}$ agree in the sense that*

$$[(f(\tilde{A})F)(t,\omega)](h) := f(A)([F(t,\omega)](h))$$

Proof. Proving this is conceptually easy, but takes a long time to explain technically; therefore this proof is presented in less detail than our other work. First, the resolvent $R(z, \tilde{A})$ may be defined analogously by $[(R(z,\tilde{A})F)(t,\omega)](h) := R(z,A)([F(t,\omega)](h))$ and exists if and only if $R(z,A)$ exists. Therefore A and $\tilde{A}$ have the same spectrum. Next we show that $||zR(z,\tilde{A})||$ remains bounded: for a finite rank adapted step process F, $zR(z,\tilde{A})F$ will also be a finite rank adapted step process, so we may choose an orthonormal set $\{h_1, \dots h_N\}$ such that

$$\begin{aligned}
\left\| [zR(z,\tilde{A})F](t,\omega) \right\|^2_{\gamma(H;L^q(\mathcal{M}))} &= \mathbb{E} \left\| \sum_{n=1}^N \gamma_n [zR(z,\tilde{A})F](t,\omega) h_n \right\|^2_{L^q(\mathcal{M})} \\
&= \mathbb{E} \left\| \sum_{n=1}^N \gamma_n zR(z,A)[F(t,\omega)h_n] \right\|^2_{L^q(\mathcal{M})} \\
&\le ||zR(z,A)||^2 \cdot \mathbb{E} \left\| \sum_{n=1}^N \gamma_n F(t,\omega) h_n \right\|^2_{L^q(\mathcal{M})} \\
&\le ||zR(z,A)||^2 ||F(t,\omega)||^2_{\gamma(H;L^q(\mathcal{M}))}
\end{aligned}$$

This inequality holds for all F by density, and thus we have for all $\theta < \theta'$ and $z \notin \Sigma_{\theta'}$:

$$\begin{aligned}
||zR(z,\tilde{A})||^p &= \sup_{||F||=1} \mathbb{E} \int_0^\infty \left\| [zR(z,\tilde{A})F](t,\omega) \right\|^p_{\gamma(H;L^q(\mathcal{M}))} dt \\
&\le \sup_{||F||=1} \mathbb{E} \int_0^\infty ||zR(z,A)||^p \cdot ||F(t,\omega)||^p_{\gamma(H;L^q(\mathcal{M}))} dt \\
&= ||zR(z,A)||^p \\
&\le C^p_{\theta'}
\end{aligned}$$

Where the last line follows from the sectoriality of A. Therefore $\tilde{A}$ is also sectorial of the same angle as A. The definition of $R(z,\tilde{A})$ implies that $[(f(\tilde{A})F)(t,\omega)](h) := f(A)([F(t,\omega)](h))$ on $H^\infty_0(\Sigma_v)$, and thus by limits, on all of $H^\infty_A(\Sigma_v)$. Finally, one can verify that $||f(\tilde{A})|| \le ||f(A)||$

by the same method as our demonstration above that $||zR(z,\tilde{A})|| \leq ||zR(z,A)||$, which tells us that $||f(\tilde{A})|| \leq C||f||_{H^\infty(\Sigma_\theta)}$ for all $f \in H^\infty(\Sigma_\theta)$, allowing us to conclude that $\tilde{A}$ has a McIntosh calculus of angle Θ. □

At this point we use Theorem 4.30 to conclude that the operator $\tilde{A}$ also has an R-McIntosh calculus of angle Θ. Our next few steps will move towards constructing a particular function $\tilde{L}_z \in RH^\infty(\Sigma_\theta, \mathcal{A})$.

Step 2

For a finite rank adapted step process $F : \mathbb{R}_+ \times \Omega \to \gamma(H; L^q(\mathcal{M}))$ and for $z \in \Sigma_\Theta$, define an operator L_z that maps F to a process $L_zF : \mathbb{R}_+ \times \Omega \to L^q(\mathcal{M})$ defined by

$$(L_zF)(t) := \int_0^t z^{\frac{1}{2}} e^{-z(t-s)} F(s) dB_H(s)$$

This integral is well defined, reasoning as in our justification that the integral operator $I(k)$ was well-defined in section 8.2. Specifically, we may identify $z^{\frac{1}{2}} e^{-z(t-s)} F(s)$ with the random operator $T : \Omega \to \mathcal{L}(L^2(\mathbb{R}_+; H), L^q(\mathcal{M}))$ given by

$$T(\omega)f := \int_0^t z^{\frac{1}{2}} e^{-z(t-s)} F(s,\omega) f(s) ds$$

This operator is finite rank, since F is finite rank, and so T takes values in $\gamma(L^2(\mathbb{R}_+; H), L^q(\mathcal{M}))$. Thus by Theorem 7.12, our the process $z^{\frac{1}{2}} e^{-z(t-s)} F(s)$ is stochastically integrable.

In fact, we can extend L_z to arbitrary random process:

Lemma 8.17. *The set $\{L_z; z \in \Sigma_\Theta\}$ is R-bounded.*

Proof. It is here that we apply the difficult Lemma 8.10 proven in the previous section. Using the substitution $t \to \dfrac{s}{\mathrm{Re}z}$, we obtain the bound:

$$\begin{aligned}
\int_0^\infty t^{\frac{1}{2}} |k_z'(t)| dt &= \int_0^\infty t^{\frac{1}{2}} | - z^{\frac{3}{2}} e^{-zt}| dt \\
&= \int_0^\infty \left(\frac{s}{\mathrm{Re}z}\right)^{\frac{1}{2}} | - z^{\frac{3}{2}}| \cdot e^{-s} \frac{ds}{\mathrm{Re}z} \\
&\leq \int_0^\infty \left(\frac{s}{\mathrm{Re}z}\right)^{\frac{1}{2}} \left(\frac{\mathrm{Re}z}{\cos\Theta}\right)^{\frac{3}{2}} \cdot e^{-s} \frac{ds}{\mathrm{Re}z} \\
&= \frac{1}{(\cos\Theta)^{\frac{3}{2}}} \int_0^\infty s^{\frac{1}{2}} e^{-s} ds \\
&= \frac{\sqrt{\pi}}{2(\cos\Theta)^{\frac{3}{2}}}
\end{aligned}$$

Recall that in Lemma 8.10 we define the set $\mathcal{K}$ as the set of absolutely continuous functions $k : \mathbb{R}_+ \to \mathbb{R}$ such that $\lim_{t\to\infty} k(t) = 0$ and

$$\int_0^\infty t^{\frac{1}{2}} |k'(t)| dt \leq 1$$

By the work above, we can choose a constant C_Θ depending only on Θ, and guarantee that $C_v k_z \in \mathcal{K}$ for all $z \in \Sigma_\Theta$. Therefore by our work in Lemma 8.10 the set $\{I(C_\Theta k_z) : z \in \Sigma_\Theta\}$ is R-bounded, and so the set $\{L_z : z \in \Sigma_\Theta\} = \{C_\Theta^{-1} I(C_\Theta k_z) : z \in \Sigma_\Theta\}$ is R-bounded. In particular, this means that each L_z is bounded, and so extends by density to a function $L_z : L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H; L^q(\mathcal{M}))) \to L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; L^q(\mathcal{M}))$. □

Step 3

Next, we want to identify L_z with an operator:

$$\tilde{L}_z : L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H; L^q(\mathcal{M}))) \to L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H; L^q(\mathcal{M})))$$

To do this, we can define an isometric embedding of $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; L^q(\mathcal{M}))$ into $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H; L^q(\mathcal{M})))$. Pick some arbitrary fixed unit vector $h_0 \in H$ and define the injection $\phi \to h_0 \otimes \phi$. Clearly $||\phi||_{L^q(\mathcal{M})} = ||h_0 \otimes \phi||_{\gamma(H;L^q(\mathcal{M}))}$. Then we can define $\tilde{L}_z$ via $\tilde{L}_z(F) = h_0 \otimes L_z(F)$, and we will have

$$||L_z F||_{L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H; L^q(\mathcal{M})))} = ||\tilde{L}_z F||_{L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H; L^q(\mathcal{M})))}$$

Note that this embedding is trivially injective and so an inverse exists on the embedded subspace. Note also the crucial fact that by the isometry and Step 2 above, the set $\{\tilde{L}_z; z \in \Sigma_w\}$ is R-bounded.

Step 4

We show that $\tilde{L}_z$ and $R(z, \tilde{A})$ commute, using the relationship between $f(A)$ and $f(\tilde{A})$ specified above in Step 1, here with $f(\zeta) = \dfrac{1}{z - \zeta}$. For arbitrary fixed $t \in \mathbb{R}_+$, $\omega \in \Omega$ and $h \in H$ (and recalling from Step 3 that $\tilde{L}_z(F) = h_0 \otimes L_z(F)$), we have:

$$[\tilde{L}_z R(z, \tilde{A}) F](t, \omega)(h) = [h, h_0] \int_0^t z^{\frac{1}{2}} e^{-z(t-s)} [R(z, \tilde{A}) F](s, \omega) dB_H(s, \omega)$$

One can verify by following definitions that this is:

$$= [h, h_0] \int_0^t z^{\frac{1}{2}} e^{-z(t-s)} R(z, A) F(s, \omega) dB_H(s, \omega)$$

Now, we clearly may move $R(z, A)$ out of the integral if F is an adapted step process, and since $R(z, A)$ is bounded, we may move it out for general F.

$$\begin{aligned} &= R(z, A)[h, h_0] \int_0^t z^{\frac{1}{2}} e^{-z(t-s)} F(s, \omega) dB_H(s, \omega) \\ &= R(z, A) \left([\tilde{L}_z F](t, \omega)(h) \right) \\ &= [R(z, \tilde{A}) \tilde{L}_z F](t, \omega)(h) \end{aligned}$$

Step 5

In Steps 2-4, we have shown that $\tilde{L}_z$ has R-bounded range and commutes with the resolvent $R(z,\tilde{A})$. Therefore $\tilde{L}_z \in RH^\infty(\Sigma_w,\mathcal{A})$, and so for $\theta < \theta' < \Theta$, the operator

$$F \to \int_{\partial\Sigma_{\theta'}} \tilde{L}_z R(z,\tilde{A}) F dz$$

is a well defined operator $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, L^q(\mathcal{M}))) \to L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, L^q(\mathcal{M})))$. The fact that the operator in Step 5 is well defined means that

$$F \to \Phi(\tilde{A})F := \int_{\partial\Sigma_{\theta'}} L_z R(z,\tilde{A}) F dz$$

is well defined as a bounded operator $\Phi(\tilde{A}) : L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H,X)) \to L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; X)$. Indeed, the embedding from Step 3 generates a closed subspace, and so $\int_{\partial\Sigma_{\theta'}} \tilde{L}_z R(z,\tilde{A})Fdz$ will still lie on this space. The inverse of the embedding (defined on the embedded subspace) retrieves $\int_{\partial\Sigma_{\theta'}} L_z R(z,\tilde{A})Fdz$ by linearity.

Step 6

Finally, we use Fubini's theorem to see that the bounded operator $\Phi(\tilde{A})$ is exactly that map that proves stochastic Maximal L^p-regularity. For a finite rank adapted step process:

$$\begin{aligned}
\Phi(\tilde{A})F(t) &= \int_{\partial\Sigma_{\theta'}} L_z R(z,\tilde{A}) F(t) dz \\
&= \int_{\partial\Sigma_{\theta'}} \int_0^t z^{\frac{1}{2}} e^{-z(t-s)} R(z,A) F(s) dB_H(s) dz
\end{aligned}$$

Applying the Stochastic Fubini Theorem 7.26:

$$\begin{aligned}
&= \int_0^t \int_{\partial\Sigma_{\theta'}} z^{\frac{1}{2}} e^{-z(t-s)} R(z,A) F(s) dz \; dB_H(s) \\
&= \int_0^t A^{\frac{1}{2}} e^{-(t-s)A} F(s) dB_H(s)
\end{aligned}$$

In other words, we have

$$\begin{aligned}
\left\| t \to \int_0^t A^{\frac{1}{2}} e^{-(t-s)A} F(s) dB_H(s) \right\|_{L^p_{\mathcal{F}}(\mathbb{R}_+\times\Omega; L^q(\mathcal{M}))} &= ||\Phi(\tilde{A})F||_{L^p_{\mathcal{F}}(\mathbb{R}_+\times\Omega;L^q(\mathcal{M}))} \\
&\leq C^p ||F||_{L^p_{\mathcal{F}}(\mathbb{R}_+\times\Omega;\gamma(H,L^q(\mathcal{M})))}
\end{aligned}$$

The result extends to all $F \in L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H; L^q(\mathcal{M})))$ by density, and we conclude that the operator A has stochastic maximal regularity. $\square$

8.4 Further Reading

A discussion of fractional operators may be found as chapter 15 of [29]. The Stochastic Maximal Regularity Theorem was first proven in [43]. Our proof of the R-boundedness of stochastic convolutions can be found in this paper. However, we use a different proof of the main theorem itself, adapted from [44]. Applications of Stochastic Maximal Regularity may be found in the paper [42].

Thankyou for taking the time to read my thesis.

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