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# Consensus of linear multi-agent systems with fully distributed control gains under a general directed graph

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*Abstract*— **In this paper, we study the leaderless consensus problem for general linear multi-agent systems under a general directed graph. A distributed consensus algorithm with gain adaption is proposed. A novel integral-type Lyapunov function is constructed to study the consensus convergence. The control gains are varying and updated adaptively by distributed adaptive laws. The proposed algorithms require no global information and thus can be implemented in a fully distributed manner.**

## I. Introduction

The consensus problem of multi-agent systems has received a great deal of attention in the last decade due to its potential applications in broad areas, including distributed computation [1], parameter estimation [2], optimization [3], and robotic networks [4]. The objective is to design distributed algorithms for multiple agents to achieve a common final state by interacting with their local neighbors. One important focus in the consensus problem is agent dynamics, including single and double integrators, general linear systems, and nonlinear systems. Most of the existing results have been mainly focused on agents with first-order or second-order integrators (see [5]–[8] and reference therein).

The present paper focuses on the consensus problem of multi-agent systems with general linear dynamics, which include single and double integrators as special cases. This problem has been studied previously based on relative state or output measurements with respect to the neighboring agents [9]–[18]. Specifically, the works in [9]–[12] rely on the assumption that the state matrix $A$ has no eigenvalue with positive real part, including the cases with $A$ being Hurwitz, neutrally stable, and marginally stable. The works in [13]–[19] consider the more general case that the agent's state-space realization $(A, B, C)$ is stabilizable and detectable. As stated in [14], this condition is actually necessary for the agents to achieve consensus.

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It is worth emphasizing that the extension of consensus algorithms from single integrators to general linear multi-agent systems is nontrivial. With a fair condition on the communication graph (e.g., the graph contains a directed spanning tree), single integrator agents are bound to achieve consensus. This however is no longer true with general linear agents. Indeed, even for the special case with $A$ having no eigenvalue with positive real part, the agents may still not achieve consensus (see Theorem 3 in [9] for a proof). In [13] and [15], distributed observer-type consensus algorithms are proposed by introducing a common scalar control gain. Besides the condition on the topology and the agents' dynamics, if the common control gain is above a certain bound, the agents will achieve consensus. However, this bound is contingent on global information which cannot be obtained in a distributed manner.

One approach to overcome the difficulty calls for the use of adaptive gain updating laws based on local information. Examples of this strategy include the distributed adaptive coordination algorithm for multiple nonlinear systems [20], [21] and general linear multi-agent systems [16]. Nevertheless, these results require a symmetric framework with an undirected graph. Recently, there have been attempts on the gain adaption under a directed graph [22]–[25]. In [22], the coordinated tracking problem is studied for multi-agent systems with first-order nonlinear dynamics. The authors construct an integral Lyapunov function and they utilize the property that the associated matrix (Laplacian matrix plus a diagonal matrix) for coordinated tracking problem is positive stable. However, the results in [22] cannot be used for leaderless consensus since the associated matrix (Laplacian matrix) is only semi-positive stable. [23] extends the results to general linear multi-agent systems. In [24], the consensus algorithm with gain adaption is proposed for second-order Lagrange systems. However, the final consensus state is stationary with a zero velocity. By establishing a connection between an undirected graph and a directed graph, the authors solve the consensus problem for second-order multi-agent systems with gain adaption in [25].

In this paper, we propose fully distributed consensus algorithms for general linear multi-agent systems which can be implemented in a fully distributed manner under a general directed graph. Our algorithms use relative state measurements, and we consider general linear agents where $(A, B)$ is stabilizable. Specifically, we begin with the algorithm using relative state measurements and heterogeneous constant gains, where we show that the agents achieve consensus if all the heterogeneous control gains are chosen large enough.

We then proposed a distributed consensus algorithm with gain adaption, in which the gains will always increase if they do not achieve consensus. The control gains in the proposed algorithm are varying and updated adaptively using only local information.

*Notations:* Let $\mathbf{1}_m$ and $\mathbf{0}_m$ denote, respectively, the $m \times 1$ column vector of all ones and all zeros. Let $\mathbf{0}_{m\times n}$ denote the $m \times n$ matrix with all zeros and $I_m$ denote the $m \times m$ identity matrix. Let $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote, respectively, the maximal and minimum eigenvalue of a square real matrix with real eigenvalues. Let $\sigma_{\max}(\cdot)$ denote the maximal singular value of a matrix. Let $\mathrm{diag}(z_1, \cdots, z_p)$ be the diagonal matrix with diagonal entries $z_1$ to $z_p$. Let $\mathrm{col}(z_1, \cdots, z_p)$ be the stacked vector of all vectors $z_1$ to $z_p$. For a complex number $\mu$, let $\mathscr{R}(\mu)$ be its real part and $\mathscr{I}(\mu)$ be its imaginary part. For a vector function $f(t) : \mathbb{R} \mapsto \mathbb{R}^n$, it is said that $f(t) \in \mathbb{L}_2$ if $\int_0^\infty f(\tau)^T f(\tau) \mathrm{d}\tau < \infty$ and $f(t) \in \mathbb{L}_\infty$ if for each element of $f(t)$, noted as $f_i(t)$, $\sup_t |f_i(t)| < \infty$, $i = 1, \ldots, n$. Throughout the paper, we use $\| \cdot \|$ to denote the Euclidean norm.

## II. Background and Problem Statement

We use a directed graph to describe the network topology between the $n$ agents. Let $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$ be a directed graph with the node set $\mathcal{V} \triangleq \{1, ..., n\}$ and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An edge $(i, j) \in \mathcal{E}$ denotes that agent $j$ can obtain information from agent $i$, but not vice versa. Here, node $i$ is the parent node while node $j$ is the child node. Equivalently, node $i$ is a neighbor of node $j$. The set of all neighbors of node $i$ is denoted as $\mathcal{N}_i$. A directed path from node $i$ to node $j$ is a sequence of edges of the form $(i, i_2)$, $(i_2, i_3)$, $\ldots$, $(i_k, j)$, in a directed graph. A directed graph is strongly connected if there exists a directed path from every node to every other node. A directed tree is a directed graph, where every node has exactly one parent except for one node, called the root, and the root has directed paths to every other node. A directed spanning tree of a directed graph is a direct tree that contains all nodes of the directed graph. A directed graph contains a directed spanning tree if there exists a directed spanning tree as a subset of the directed graph.

The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ associated with $\mathcal{G}$ is defined as $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. In this paper, self edges are not allowed, *i.e.,* $a_{ii} = 0$. The (nonsymmetric) Laplacian matrix $\mathcal{L}_A = [l_{ij}] \in \mathbb{R}^{n\times n}$ associated with $\mathcal{A}$ and hence $\mathcal{G}$ is defined as $l_{ii} = \sum_{j=1,j\neq i}^n a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$.

*Lemma 2.1:* [26], [27] Suppose that $\mathcal{G}$ is a directed graph of order $n$ and is strongly connected. There exists a vector $\xi \triangleq [\xi_1, \ldots, \xi_n]^T \in \mathbb{R}^n$ with $\sum_{i=1}^n \xi_i = 1$ and $\xi_i > 0$, $\forall i = 1, \ldots, n$, such that $\xi^T \mathcal{L}_A = 0$.

The following lemma establishes a connection between a strongly connected directed graph and an undirected graph, which was first proposed in [25].

*Lemma 2.2:* Suppose that $\mathcal{G}$ is a directed graph of order $n$ and is strongly connected. Define the matrix $\widehat{L} \triangleq \Xi\mathcal{L}_A + \mathcal{L}_A^T \Xi$, where $\Xi \triangleq \mathrm{diag}(\xi_1, \ldots, \xi_n)$ with $\xi_i$ defined as in Lemma 2.1. Then $\widehat{L}$ is the symmetric Laplacian matrix associated with an undirected graph. In addition, let $\varsigma \in \mathbb{R}^n$ be any positive vector. The following inequality holds

$$\min_{\substack{\vartheta^T\varsigma=0 \\ \vartheta^T\vartheta=1}} \vartheta^T \widehat{L} \vartheta > \frac{\lambda_2(\widehat{L})}{n}, \tag{1}$$

where $\lambda_2(\widehat{L})$ is the second smallest eigenvalue of $\widehat{L}$.

Using the properties of Kronecker product, we have the following result which will be used subsequently.

*Lemma 2.3:* Suppose that $U = [u_{ij}] \in \mathbb{R}^{n\times n}$ and $V = V^T \in \mathbb{R}^{p\times p}$. Let $S \in \mathbb{R}^{p\times p}$ be the unitary matrix such that $SVS^T \triangleq \mathrm{diag}(\lambda_1(V), \lambda_2(V), \cdots, \lambda_p(V))$ and let $x_i = [x_{i1}, x_{i2}, \ldots, x_{ip}]^T \in \mathbb{R}^p$, $i = 1, \ldots, n$. The following equality holds

$$x^T (U \otimes V) x = \sum_{k=1}^p \lambda_k(V) v_k^T U v_k,$$

where $x = \mathrm{col}(x_1, \ldots, x_n) \in \mathbb{R}^{np}$ and $v_k = [v_{k1}, \ldots, v_{kn}] \in \mathbb{R}^n$ is the stacked vector of the $k$th elements of all $Sx_i$, $i = 1, \ldots, n$.

## III. Consensus Algorithm with Relative State Measurements

In this section, we aim to design a fully distributed consensus algorithm for linear multi-agent systems where each agent simply chooses or updates its own control gain via the local relative state measurements with respective to its neighbors. We consider a group of $n$ agents where the dynamics of the agents is described by the following identical general linear equations

$$\dot{x}_i(t) = Ax_i + Bu_i, \qquad i = 1, \ldots, n. \tag{2}$$

where $x_i \in \mathbb{R}^p$ is the state of agent $i$, $u_i \in \mathbb{R}^m$ is the control input of agent $i$ which can only use local information from its neighbor agents, $A \in \mathbb{R}^{p\times p}$ and $B \in \mathbb{R}^{p\times m}$ are constant matrices.

For general linear multi-agent systems, the following consensus algorithms is proposed [10], [13]–[15]

$$u_i = \alpha K \sum_{j=1}^N a_{ij}(x_i - x_j), \tag{3}$$

where $\alpha$ is a positive constant representing the control gain and $K = -B^T P$ where $P > 0$ is the unique solution of the following control algebraic Riccati equation (ARE)

$$A^T P + PA - PBB^T P + I_p = 0. \tag{4}$$

Note that the above ARE has a unique solution if and only if $(A, B)$ is stabilizable [28]. Therefore, we have the following assumptions on the dynamics of (2).

*Assumption 3.1:* The pair $(A, B)$ is stabilizable.

Using (3) for (2), the agents achieve consensus if the underlying directed graph contains a directed spanning tree and the control gain is chosen such that

$$\alpha \geq \frac{1}{\min_{\mu_i \neq 0} \mathscr{R}(\mu_i)}, \tag{5}$$

where $\mu_i$ is the eigenvalue of the Laplaican matrix $\mathcal{L}_A$.

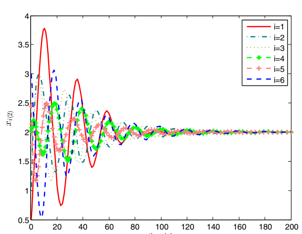
Note that in (3) all agents share the common control gain $\alpha$. However, such a design is not fully distributed because in a fully distributed context each agent simply chooses its own gains and these gains are generally not identical. Moreover, the common control gain $\alpha$ must be above a certain lower bound (see (5)), which is determined by the (nonsymmetric) Laplacian matrix. Such a requirement is also not fully distributed as global information is needed to determine the lower bound. One possible way is to assign each agent a constant gain and tune the gains according to each agent's local information and performance, i.e., increase the control gains of the agents that do not converge to their neighbors or even move far away from their neighbors during a period of time. The above principle works when the agents share a common control gain. But for the case with heterogeneous gains is much different as shown in the following example.

### A. Example

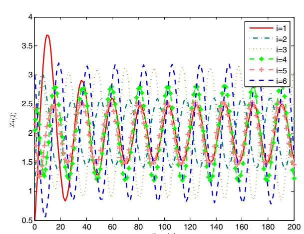
The dynamics of the agents are modeled as double integrators, and we have $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and thus $P = \begin{pmatrix} 1.7321 & 1 \\ 1 & 1.7321 \end{pmatrix}$. We consider the consensus problem for six agents with the following (nonsymmetric) Laplacian matrix

$$\begin{pmatrix} 0.15 & -0.1 & 0 & -0.05 & 0 & 0 \\ 0 & 0.3 & -0.15 & 0 & -0.15 & 0 \\ 0 & 0 & 0.1 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 0.15 & -0.15 & 0 \\ 0 & -0.15 & 0 & -0.15 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & -0.2 & 0.2 \end{pmatrix}.$$

Clearly, the associated graph is directed and contains a directed spanning tree. The initial states are chosen as $x_i(1) = 2i$ and $x_i(2) = 0.5i$, where $i = 1, \ldots, 6$.



(a) Velocities with a common gain $\alpha = 0.4$.



(b) Velocities with heterogeneous gains $\alpha_4 = 1.2$ and $\alpha_i = 0.4$, $i \neq 4$.

Fig. 1. The agents' velocities using a common control gain and heterogeneous control gain.

Due to the space limitation, we only show the second state of $x_i$, *i.e.,* the velocity of agent $i$. By simulation, we get that the lower bound using a common gain is nearly 0.176. Therefore, if we choose $\alpha = 0.4 > 0.176$, we can see from Fig. 1(a) that the agents achieve consensus. But if the control gains are heterogeneous, it is observed from Fig. 1(b) that the agents cannot achieve consensus even by increasing one of the control gains. Therefore, the case with heterogeneous control gains needs further investigation.

### B. Consensus algorithm with heterogeneous constant control gains

We begin with the problem by investigating the consensus for agents with heterogeneous constant control gains. Here we assume that the directed graph $\mathcal{G}$ is strongly connected and Assumption 3.1 holds. The consensus algorithm for the linear multi-agent systems is proposed as

$$u_i = \alpha_i K \sum_{j=1}^{N} a_{ij}(x_i - x_j), \tag{6}$$

where $\alpha_i$ is a positive constant and $K \in \mathbb{R}^{m \times p}$ is a constant matrix defined as in (3). In contrast to (3), here we allow that the agents have heterogeneous control gains. Actually, the heterogeneous gains make the consensus convergence more challenging since it is not clear how to use the eigenvalue analysis as in [10], [13]–[15] since there exist $n$ unknown variables ($\alpha_i$). On the other hand, since the underlying graph $\mathcal{G}$ is directed, due to the loss of symmetry, the Lyapunov analysis as in [16] which is only valid for undirected graphs cannot be directly used in our problem. Instead, we introduce the following novel Lyapunov function candidate

$$\begin{aligned} V =& \sum_{i=1}^{n} \alpha_i \xi_i \Big[\sum_{j=1}^{n} a_{ij}(x_i - x_j)\Big]^T P \sum_{j=1}^{n} a_{ij}(x_i - x_j) \\ =& [(\mathcal{L}_A \otimes I_p)x]^T (\Xi\Lambda \otimes P)(\mathcal{L}_A \otimes I_p)x, \end{aligned} \tag{7}$$

where $\xi_i$ is well defined as in Lemma 2.1 since $\mathcal{G}$ is strongly connected, $\Xi = \mathrm{diag}(\xi_1, \ldots, \xi_n)$, $\Lambda = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$, $x = \mathrm{col}(x_1, \ldots, x_n)$, and $P$ is defined as in (4). Here we integrate the graph information ($\xi_i$) and the control gains ($\alpha_i$) into the Lyapunov function candidate. Using (6), the closed-loop system of (2) can be written as

$$\dot{x} = (I_n \otimes A + \Lambda\mathcal{L}_A \otimes BK)x. \tag{8}$$

Then the derivative of $V(t)$ is given as

$$\begin{aligned} \dot{V}(t) &= 2[(\mathcal{L}_A \otimes I_p)x]^T (\Xi\Lambda\mathcal{L}_A \otimes P)\dot{x} \\ &= 2s^T (\Xi\Lambda^{-1} \otimes PA + \Xi\mathcal{L}_A \otimes PBK)s, \end{aligned} \tag{9}$$

where $s \triangleq (\Lambda\mathcal{L}_A \otimes I_p)x$. Note that $K = -B^T P$. From (9), we can obtain

$$\begin{aligned} \dot{V}(t) =& s^T \Big[\Xi\Lambda^{-1} \otimes PBB^T P\Big] s - s^T (\widehat{L} \otimes PBB^T P)s \\ & - s^T (\Xi\Lambda^{-1} \otimes I_p)s, \end{aligned} \tag{10}$$

where $\widehat{L}$ is defined as in Lemma 2.2 and we have used (4) to obtain the last equality. In (10), the first term is nonnegative and the second and third terms are nonpositive. We aim to derive conditions on $\alpha_i$ such that $\dot{V}(t)$ is negative definite. Since $\Xi\Lambda^{-1}$ is positive definite and $\widehat{L}$ is positive semidefinite, from the first glance, it seems impossible to make $\dot{V}(t)$ negative definite. However, note that $s$ is not arbitrary but associated with the Laplacian matrix $\mathcal{L}_A$. This fact leaves some hope for us. We next show a rigorous analysis on how to choose the control gains $\alpha_i$ such that $\dot{V}(t)$ is negative definite, where we will use the results in Lemma 2.2 and Lemma 2.3.

Note that $PBB^TP$ is symmetric positive semidefinite. There exists an unitary matrix $S$ such that $SPBB^TPS^T = \text{diag}(\lambda_1, \cdots, \lambda_p)$, where $\lambda_i \geq 0$, $i = 1, \ldots, p$, are $p$ eigenvalues of $PBB^TP$. Write $s$ as $s = [s_1, \ldots, s_n] \in \mathbb{R}^{np}$ with $s_i \in \mathbb{R}^p$. From the definition of $s$, we have $s_i = \alpha_i \sum_{j=1}^n a_{ij}(x_i - x_j)$. Let $v_k = [v_{k1}, \ldots, v_{kn}] \in \mathbb{R}^n$ be the stacked vector of the $k$th elements of all $Ss_i$, $i = 1, \ldots, n$. From Lemma 2.3, we have

$$s^T(\widehat{L} \otimes PBB^TP)s = \sum_{k=1}^{p} \lambda_k v_k^T \widehat{L} v_k. \tag{11}$$

Note that $Ss_i = \alpha_i \sum_{j=1}^n a_{ij}(Sx_i - Sx_j)$. we have

$$v_{ki} = \alpha_i \sum_{j=1}^{n} [(Sx_i)_k - (Sx_j)_k], \tag{12}$$

where $(Sx_i)_k$ denotes the $k$th element of the vector $Sx_i$, $k = 1, \ldots, p$. Define $\omega_k \triangleq [(Sx_1)_k, \ldots, (Sx_n)_k]^T \in \mathbb{R}^n$. From (12), we have $v_k = \Lambda \mathcal{L}_A \omega_k$. Therefore, $v_k^T \Lambda^{-1} \xi = \omega_k^T \mathcal{L}_A^T \xi = 0$. Since $\xi_i > 0$ and $\alpha_i > 0$, $\forall i = 1, \ldots, n$, the vector $\Lambda^{-1}\xi$ is positive. Under the condition that $\mathcal{G}$ is strongly connected, we can get from Lemma 2.2 that the $n$ eigenvalues of $\widehat{L}$ can be arranged as $0 = \lambda_1(\widehat{L}) < \lambda_2(\widehat{L}) \leq \cdots \leq \lambda_n(\widehat{L})$ and thus

$$v_k^T \widehat{L} v_k \geq \frac{\lambda_2(\widehat{L})}{n} v_k^T v_k. \tag{13}$$

From Lemma 2.3 and (11), we can obtain

$$\begin{aligned} s^T(\widehat{L} \otimes PBB^TP)s =& \sum_{k=1}^{p} \lambda_k v_k^T \widehat{L} v_k \\ \geq& \frac{\lambda_2(\widehat{L})}{n} \sum_{k=1}^{p} \lambda_k v_k^T v_k \\ =& \frac{\lambda_2(\widehat{L})}{n} s^T(I_p \otimes PBB^TP)s, \end{aligned} \tag{14}$$

where we have used (13) to obtain the inequality. Substituting (14) into (10), we obtain

$$\dot{V} \leq -\sum_{i=1}^{n} \Big[\frac{\lambda_2(\widehat{L})}{n} - \frac{\xi_i}{\alpha_i}\Big] s_i^T PBB^TP s_i - \sum_{i=1}^{n} \frac{\xi_i}{\alpha_i} s_i^T s_i.$$

We then have the following result.

*Theorem 3.2:* Suppose that the directed graph $\mathcal{G}$ is strongly connected and Assumption 3.1 holds. Using (6) for (2) with $K = -B^TP$ where $P > 0$ is the unique solution of the ARE (4), if the heterogeneous control gains are chosen such that

$$\alpha_i > \frac{n \max_i \xi_i}{\lambda_2(\widehat{L})}, \tag{15}$$

the agents achieve consensus exponentially.

As shown in (15), $\alpha_i$ should be above a certain lower bound which is determined by some global information ($\widehat{L}$ and $n$). But the positive side is that we allow all agents to have heterogeneous control gains, which implies that the principle to increase the control gains also works as long as the gains are chosen large enough. This fact inspires us to introduce an adaptive strategy for the control gains which will be discussed in the following section.

### *C. Consensus algorithm with heterogeneous varying control gains*

Here, we are ready to deal with the consensus problem for linear multi-agent systems with heterogeneous varying control gains. An intuitive algorithm is as follows

$$u_i = \alpha_i(t)\phi_i(v_i^TPv_i)K \sum_{j=1}^{n} a_{ij}(x_i - x_j), \tag{16}$$

$$\dot{\alpha}_i(t) = \gamma_i \varphi_i(\sum_{j=1}^{n} a_{ij}(x_i - x_j)), \tag{17}$$

where $\gamma_i$ is a positive constant, $v_i \triangleq \sum_{j=1}^n a_{ij}(x_i - x_j)$, $\phi_i(w)$ is continuous and monotonically increasing with respect to $w$ and satisfying $\phi_i(w) > 0$ when $w \geq 0$ to be determined later, and $\varphi_i(w)$ is continuous in $w$ which satisfies $\varphi_i(w) \geq 0$ and $\varphi_i(w) = 0$ if and only if $w = 0$. Here we assume that $\alpha_i(0) > 0$.

Under the condition that $\mathcal{G}$ is strongly connected, since $\alpha_i(t)$ and $\phi_i(v_i^TPv_i)$ is varying, we consider the following integral-type Lyapunov function

$$V(t) = \sum_{i=1}^{n} \alpha_i(t) \int_0^{v_i^TPv_i} \xi_i \phi_i(\tau) \mathrm{d}\tau. \tag{18}$$

Note from (17) that $\alpha_i(t) \geq \alpha_i(0) > 0$, and $P$ is positive definite. It implies that $V(t)$ will always be nonnegative and $V(t) = 0$ if and only if $\| \sum_{i=1}^n a_{ij}(x_i - x_j)\| = 0$, which also guarantees the consensus of all agents under a strongly connected directed graph. Therefore, $V(t)$ is a suitable Lyapunov function candidate. For conciseness, we denote by $\phi_i = \phi_i(v_i^TPv_i)$. Let $\Phi \triangleq \text{diag}\{\phi_1, \cdots, \phi_n\}$. The derivative of $V(t)$ is

$$\begin{aligned} \dot{V}(t) =& \sum_{i=1}^{n} \dot{\alpha}_i(t) \int_0^{v_i^TPv_i} \xi_i \phi_i(\tau) \mathrm{d}\tau \\ &+ 2[(\mathcal{L}_A \otimes I_p)x]^T (\Xi\Lambda\Phi \otimes P)(\mathcal{L}_A \otimes I_p)\dot{x}. \end{aligned} \tag{19}$$

Note that the time-varying vector $\Phi\Lambda\xi$ is always positive and $\phi_i(\omega)$ is continuous and monotonically increasing with respect to $\omega$. Following the same steps in Section III-B, we have

$$\begin{aligned} \dot{V}(t) \leq& \sum_{i=1}^{n} \xi_i \dot{\alpha}_i(t) \phi_i(v_i^TPv_i) v_i^TPv_i - \sum_{i=1}^{n} \xi_i \alpha_i(t) \phi_i v_i^T v_i \\ &- \sum_{i=1}^{n} \Big[\frac{\lambda_2(\widehat{L})}{n} \alpha_i^2(t) \phi_i^2 - \xi_i \alpha_i(t) \phi_i\Big] v_i^T PBB^TP v_i. \end{aligned} \tag{20}$$

The first term of (20) inspires us to introduce the function $\varphi_i(t, \omega)$ as $\varphi_i(t, \omega) = \omega^TPBB^TP\omega$, and thus the time-varying control gains are updated by $\dot{\alpha}_i(t) = \gamma_i v_i^T PBB^T Pv_i$. We have

$$\dot{V}(t) \leq -\sum_{i=1}^{n}\Big[\frac{\lambda_2(\widehat{L})}{n}\alpha_i^2(t)\phi_i^2 - \xi_i\alpha_i(t)\phi_i - \xi_i\gamma_i\phi_i v_i^T P v_i\Big] \cdot v_i^T PBB^T Pv_i - \sum_{i=1}^{n}\xi_i\alpha_i(t)\phi_i v_i^T v_i. \quad (21)$$

The challenge is how to derive an appropriate upper bound for the term $\xi_i\gamma_i\phi_i v_i^T Pv_i$. Note that $\phi_i$ is assumed to be a function of $v_i^T Pv_i$. Let $\phi_i(v_i^T Pv_i) = (c_{1i} + c_{i2}v_i^T Pv_i)^{r_i}$, where $c_{1i} > 0$, $c_{i2} > 0$, and $r_i \neq 1$ are positive constants[1]. Using Young's inequality, for positive real numbers $q_{i1}$ and $q_{i2}$ satisfying $\frac{1}{q_{i1}} + \frac{1}{q_{i2}} = 1$, we have

$$\xi_i\gamma_i\phi_i v_i^T Pv_i \leq \frac{(\xi_i\gamma_i)^{q_{i1}}}{q_{i1}(c_{i2}k_i)^{q_{i1}}} + \frac{k_i^{q_{i2}}\phi_i^{\frac{r_i+1}{r_i}q_{i2}}}{q_{i2}}, \quad (22)$$

where $k_i$ is a positive constant satisfying $\frac{k_i^{q_{i2}}}{q_{i2}} = \frac{\lambda_2(\widehat{L})}{2n}\alpha_i^2(0)$. Let $\frac{r_i+1}{r_i}q_{i2} = 2$. Since $r_i \neq 1$, we have $q_{i2} > 1$ and thus the Young's inequality is valid for $q_{i2}$ and $q_{i2}$ satisfying $\frac{1}{q_{i1}} + \frac{1}{q_{i2}} = 1$. In this case, the control algorithm (16) with (17) can be rewritten as

$$u_i = \alpha_i(t)(c_{1i} + c_{2i}v_i^T Pv_i)^{r_i} K \sum_{j=1}^{n} a_{ij}(x_i - x_j), \quad (23)$$

$$\dot{\alpha}_i(t) = \gamma_i v_i^T PBB^T Pv_i, \quad (24)$$

where $c_{1i} > 0$, $c_{i2} > 0$, $r_i \neq 1$, and $\gamma_i$ are positive constants. For simplicity, choose $r_i = 3$. We have

$$q_{i1} = 3, \quad q_{i2} = \frac{3}{2}, \quad k_i = \Big[\frac{3\lambda_2(\widehat{L})\alpha_i^2(0)}{4n}\Big]^{\frac{2}{3}}. \quad (25)$$

We then have the following main result.

*Theorem 3.3:* Suppose that the directed graph $\mathcal{G}$ is strongly connected. Using (23) and (24) for (2) with $K = -B^T P$ where $P > 0$ is the unique solution of the ARE (4), the following two statements hold.

(i) $\alpha_i(t)$ is monotonically increasing and will converge to a finite positive constant as $t \to \infty$, $\forall i = 1, \ldots, n$.

(ii) The agents achieve consensus asymptotically with a common varying velocity.

*Proof:* Consider the following Lyapunov function candidate

$$V_0(t) = V(t) + \sum_{i=1}^{n}\frac{\lambda_2(\widehat{L})\alpha_i(0)c_{1i}^{2r_i}}{8\gamma_i n}[\alpha_i - \bar{\alpha}]^2, \quad (26)$$

where $V(t)$ is defined as in (18), $r_i = 3$, and $\bar{\alpha}$ is a large constant satisfying

$$\bar{\alpha} > \frac{4n^2}{\lambda_2(\widehat{L})\min_i \alpha_i(0)c_{i1}^6} + \frac{32n^2 \max_i \gamma_i^3}{27\lambda_2^3(\widehat{L})\min_i \alpha_i^5(0)c_{1i}^6 c_{2i}^3}. \quad (27)$$

[1]Here we give an example of $\phi_i$ to make the proof clear. There are other choices of $\phi_i$, for example, $\phi_i(v_i^T Pv_i) = c_{1i} + c_{i2}(v_i^T Pv_i)^{r_i}$, with $c_{1i}$, $c_{i2}$, and $r_i \neq 1$ being positive constants. $r_i \neq 1$ is required for the use of Young's inequality.

From (21) and (22), the derivative of $V_0(t)$ is given as

$$\begin{aligned}\dot{V}_0(t) =& \dot{V}(t) + \sum_{i=1}^{n}\frac{\lambda_2(\widehat{L})\alpha_i(0)c_{1i}^6}{4n}[\alpha_i(t) - \bar{\alpha}]v_i^T PBB^T Pv_i \\ \leq& -\sum_{i=1}^{n}\Big[\frac{\lambda_2(\widehat{L})}{n}\alpha_i^2(t)\phi_i^2 - \xi_i\alpha_i(t)\phi_i - \frac{\lambda_2(\widehat{L})\alpha_i^2(0)}{2n}\phi_i^2 \\ &- \frac{\xi_i^3\gamma_i^3}{3c_{i2}^3k_i^3} - \frac{\lambda_2(\widehat{L})\alpha_i(0)c_{1i}^6}{4n}[\alpha_i(t) - \bar{\alpha}]\Big]v_i^T PBB^T Pv_i \\ &- \sum_{i=1}^{n}\xi_i\alpha_i(t)\phi_i v_i^T v_i.\end{aligned}$$

Note that $\alpha_i(t) \geq \alpha_i(0) > 0$, $\phi_i \geq c_{1i}^3$, and

$$\xi_i\alpha_i(t)\phi_i \leq \frac{n\xi_i^2}{\lambda_2(\widehat{L})} + \frac{\lambda_2(\widehat{L})}{4n}\alpha_i^2(t)\phi_i^2.$$

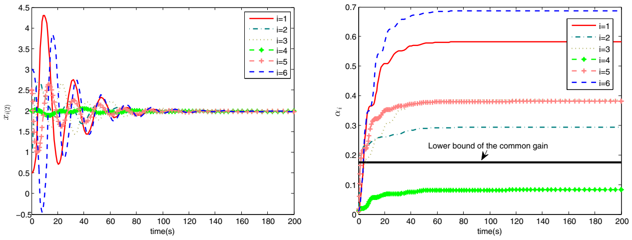
We have

$$\begin{aligned}\dot{V}_0(t) \leq& -\sum_{i=1}^{n}\Big[\frac{\lambda_2(\widehat{L})\alpha_i(0)c_{1i}^6}{4n}\bar{\alpha} - \frac{n\xi_i^2}{\lambda_2(\widehat{L})} - \frac{\xi_i^3\gamma_i^3}{3c_{2i}^3k_i^3}\Big] \\ &\cdot v_i^T PBB^T Pv_i - \sum_{i=1}^{n}\xi_i\alpha_i(t)\phi_i v_i^T v_i, \\ \leq& -\sum_{i=1}^{n}\xi_i\alpha_i(0)c_{1i}^3 v_i^T v_i,\end{aligned} \quad (28)$$

where we have used (27) and the fact that $PBB^T P$ is positive semidefinite to obtain the last inequality. Therefore, we have $V_0(t) \leq V_0(0)$ and thus $\sum_{j=1} a_{ij}(x_i - x_j), \alpha_i - \bar{\alpha} \in \mathbb{L}_\infty$. Since $\bar{\alpha}$ is a constant, we have $\alpha_i \in \mathbb{L}_\infty$. Also note that $\alpha_i$ is monotonically increasing. Therefore, all $\alpha_i$ will converge to some finite constants and thus i) holds. Note that $\dot{V}_0 = 0$ implies that $\sum_{j=1}^{n} a_{ij}(x_i - x_j) = 0$, $\forall i = 1, \ldots, n$. Then we can conclude from LaSalle's invariance principle that $\lim_{t\to\infty}\|\sum_{j=1}^{n} a_{ij}[x_i(t) - x_j(t)]\| = 0$, $\forall i = 1, \ldots, n$. Since $\mathcal{G}$ is strongly connected, we can get that the agents will achieve consensus and thus ii) holds. ■

Here we use the example in Section III-A to show the effectiveness of the proposed algorithm. We choose $r_i = 3$, $c_{1i} = 1$, $c_{2i} = 0.1$, and $\gamma_i = 0.03$. The initial values are chosen as $\alpha_i(0) = 0.01$. We can see from Fig. 2(a) that all agents achieve consensus. Fig. 2(b) shows the varying gains $\alpha_i(t)$, $i = 1, \ldots, n$, which converge to different constants. An incredible observation is that some $\alpha_i(t)$ is even smaller than the lower bound by using a common control gain. This is because that the case with heterogeneous gains are more feasible than the case with a common gain. Actually, the latter can be seen as a special case of heterogeneous gains. An intuitive future work is to find the best control gains in the sense of, for example, minimum energy of the whole system, with respect to the traditional LQR problem.

Another advantage of using varying heterogeneous control gains is that we allow uncertainties in the control input. For example, due to the disalignment of actuators attached to agent $i$, the control input might be $(1 + \delta_i)u_i$ instead of $u_i$. Then from the control input (23) and the preceding analysis, we can see that the uncertainties $\delta_i$ on $u_i$ will not effect the




(a) Agents' velocities. (b) Agents' control gains.

Fig. 2. The agents' velocities and control gains using (23) and (24).

consensus convergence if $\|\delta_i\| < 1$. Moreover, we also allow $\delta_i$ to be time-varying as long as its derivative are relatively small. On the other hand, uncertainties in the dynamics as in [17] are also deserved special attention and will be conducted in the future by combining the idea in the present paper.

*Remark 3.4:* Here we highlight the difference between our results and [23]. The work in [23] focuses on the leader-following tracking problem. The authors utilize the property that the associated matrix (Laplacian matrix plus a diagonal matrix) for the coordinated tracking problem is positive stable. In contrast, the associated matrix (Laplacian matrix) for the consensus problem studied in the current paper is only semi-positive stable. Therefore, the results in [23] cannot be used for the leaderless consensus problem. In fact, the leaderless problem is considered as future work in [23]. Note that the results in this paper are obtained under a strongly connected directed graph. By using the Perron-Frobenius form, all the results can be extended to the case where the directed graph has a directed spanning tree following similar steps to those in [24]. And the result in [23] becomes a special case of our results when there exists one agent that has no neighbors and has directed paths to all other agents.

## IV. Conclusions

In this paper we have studied the distributed consensus problem for general linear multi-agent systems under a general directed graph. Fully distributed consensus algorithms have been proposed using the relative state measurements. A notable feature of the proposed consensus algorithm is that the control gains are heterogeneous for each agent and can be obtained with only local information from the neighbors.

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