From Homological Algebra to Topology
via *Type B Zigzag Algebra* and Heisenberg Algebra.

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For my father and mother.

廿八数至寒窗毕，
数算真相美妙哉，
同调代数论拓扑，
世界万千皆弦数。

Twenty-eight years worth of maths,
Truth and beauty lie within,
Topos carol algebras echo,
Tick-tocking string whirring gravity.

Berpuluh tahun mengejar kejayaan,
Ilmu hisab betapa benar dan cantik,
Huruf tanda mengukur garisan,
Daya bersatu dawai berdetik.

Declaration

The work in this thesis is my own except where otherwise stated.

Kie Seng, Nge

Acknowledgements

"We acknowledge and celebrate the First Australians on whose lands the Australian National University (ANU) operates, and pay our respect to the elders of the Ngambri and Ngunnawal people, past, present, and emergent."

First and foremost, I would like to express my greatest gratitude to my beloved supervisor, Anthony Michael Licata or simply Tony, a more familiar name known by his collaborators, his colleagues, his students, and his family. I thank Tony for generously accepting me as his first, if not, the second graduate student. I am grateful that he brought me into the field of representation theory which has vast connections to other fields. Moreover, he shaped me into an independent and thoughtful mathematician through fortnightly critical dialogues with him. As a caring supervisor, he often cares of his students' welfare. Whenever I reached a bottleneck, he could try his best to give the most useful advices, which I appreciated very much. Thank you for the conversations we had. Thank you, Tony, as my teacher[1].

Second, I would like to thank my PhD[2]-mate, Edmund Xian Chen Heng. PhD[3] is usually completed singly as the topic might get specific, but I'm lucky to have met him in my PhD journey. We started our first bachelor degree together at the ANU. We were not as close back then, not until he chose to start his PhD with my supervisor. In the first two years of PhD, we shared the same learning and conference experience before we broke off pursuing our own final project topics. In particular, two of the conferences we participated in have resulted in a joint paper on type B zigzag algebra and its sequel to 2-category that constitute the first two chapters in this thesis. I find him particularly easy to communicate with, especially since we both come from the same country, Malaysia. Whenever there are two clear paths we can proceed in doing maths, without a second thought, we would pick the distinct path with each other. However, it has always been pleasant[4] and fruitful discussions with him; there are countless days where we discussed maths at the office of Mathematical Science Institute (MSI) until the sun set and walked home together in the dark. I have learnt a lot from him perceiving maths from his perspective; he possessed good qualities that make him suitable to be a mathematics researcher. Congratulation to Heng that he got a postdoctorate offer from Institut des Hautes Études Scientifiques. Good luck to Heng in his future endeavour and undertakings!

Not forgetting, Heng and I would like to thank Peter McNamara for suggesting the construction of the type B zigzag algebra during the Kiola Conference 2019. We would also like to thank Hoel Queffelec and Daniel Tubbenhauer for their helpful comments on the draft(s) of our earlier version of the paper [HN19]. We thank Lars Thorge Jensen, who was willing to share with us his Master thesis work on proving the faithfulness of the type A 2-braid group when we met at the Mooloolaba Conference 2020. I personally want to thank Bryan Bailing Wang, who allowed me to learn symplectic geometry in his reading classes and gave me opportunities to share my research interest with his students. He even encouraged his previous PhD student, Shuaige Qiao, to work with me on my "ill-posed" question on type B Floer cohomology. When Qiao returned to China after graduation, we endeavoured to communicate over Zoom, hoping to produce some new results despite the VPN problem and unsynchronised internet speed. This, in fact, is a fun collaboration between two different mathematics backgrounds where I learn how to communicate with a person who is not so familiar with representation theory. Although by the thesis submission date, Qiao and I haven't quite reached our dream goal due to my inadequate knowledge in geometry, we have made decent progress in understanding the subject of Floer cohomology[5] of Type A Milnor fibre; some different insights from [HN19]

[1] Chinese say, "A day as a teacher, a lifetime as a father (一日为师，终身为父)"; forever respect your teacher as your father.
[2] Doctor of Philosophy.
[3] However, permanent head damage is the phrase that netizen refer to whenever they find PhD's are hard to get along with.
[4] I found it amusing on one occasion that the customers around us kept turning their heads to us while we were discussing maths on the restaurant's dining table with a mix of English and Chinese phrases that they knew but didn't quite understand.
[5] It is a folklore that the study of Floer (co)homology is the study of every mathematical fields.

have been presented in this thesis. Qiao is now taking up a postdoctoral position in China. I wish her all the best in her future career!

Before jumping to the next section, I would like to take this opportunity to thank the other members of the MSI who impacted me with their knowledge of maths. Thank you, Uri Onn, Asilata Bapat, Anand Deopurkar, Vigleik Angeltveit, Amnon Neeman, James Borger, and Joan Licata. I also thank mathematics researcher Joshua Jeishing Wen, who is willing to share his research on wreath Macdonald polynomial with me through emails even though we had never met each other in person before. Thanks to Ivo De Los Santos Vekemans for organising the homological algebra graduate reading seminar, which forms the basis of my research afterwards. Thank you, Bradley Windelborn, for many heart-to-heart talks opposite my MSI desk, though you have left early for your new PhD life in New Zealand.

Inevitably, homesickness is a part of the PhD journey when it comes to studying in the land, not where the home is based. Certain nostalgic elements make me feel at home; badminton is certainly the first on the list. I am grateful to gather a band of badminton fanatics, "Pattern More Than Badminton," who love badminton as crazy as me and can play badminton with me anytime[6] at any place[7]. A big shout-out to Guang Hui Wong, Shao Qi Lim, ZunMing Lim, Chang Yong Ting, Wei Ric Chan, Kevin Yu Shen Ling, Heng Kok Hong, EuiYong Hong, Anderson Li Yang Kho, Wei Shen Oh, Sharon Kai Yun Ngu, Qishang Chua, Shuwen Hua, Susumu Okamoto, Ayako Ueki, Muhammad Iqbal Nur Wahid, Han Fey Yap, Hai Zhu, Hsun-Ming Hu, Bin Guo, Jiazhen Xu, Song Peng, Yifan Zhu, Ron Hu, Nutthadech Banditakkarakul, Shichao Liu, Fuyao Liu, Weiwei Fan, Muhammad Mus'ab Bin Yusof, Mohd Emmier Isfahan Bin Mohd Norzam, Mikiko Sasaki, Hohin Mo, Joshua Zhen Ting, Poey Maribbay, Wanqi Yao, Wang Yui Lo, Jacky Ka-Him Yuen, Quentin Laurentius, Ee Keat Beh, Huynh Thuy Khanh Nguyen, Jayesh Pinidiyapathirage, Andi Wang as well as the coach, Shuen Joe and many others. Second, I am thankful to have weekly bondings as well as sharings on zillions of issues and ideas with fellow Malaysian postgraduate friends. I value my friendship with Danny Ka Po Tee, Bryan Pi-Ern Tee, Choon Kong Lai, Joshua Zheyan Soo, Wei Wen Wong, and Anson Wee Ooi. Third, I appreciate the love and care I received from the Canberra Chinese Methodist Church. Special thanks to my sisters-in-christ, Yan Li and Mei Shy Lui, as well as my brothers-in-christ, Andrew Chu King Yao, Dongrui Gao, and Siang Bing Ting, who always provided me with the transport to church and encouraged me when I was down. Thanks to all other church members who always keep me in their prayers during my most stressful time. Fourth, I thank the Foochow aunt and uncle who cordially invite me to their house to enjoy their home-cooked food on several occasions. As a patron, I value the commitment of Madam Lu Malaysian restaurant for offering the best service and food to fulfil my craving for the taste of home.

Last but not least, a huge thank you to my family, especially to my father and my mother, who fully support me in pursuing my PhD path without a single complaint. They had sacrificed their time without my accompany for ten years ever since I left high school. Thank you, Pa and Ma. At the same time, I congratulate my brother on submitting his Honours thesis in Germany just before I did.

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Soli Deo gloria.

𝄞*Gloria Patri* 𝄀

Sincerely,
Kie Seng, Nge (继胜, 倪)

[6] Literally, somebody in the group spontaneously asked me for a singles game while writing this sentence.
[7] I recall that we played on the street behind the building of College of Business and Economics when we were only allowed a certain period of outdoor exercises in the COVID-19 lockdown period.
[8] The most catastrophic bushfire season ever experienced in the history of Australia.
[9] In the midst of thesis writing and two years of Chinese New Year not able to have a reunion with family.
[10] Canberra's COVID-19 lockdown legend 'Ken Behrens'.

Abstract

We construct a faithful categorical action of the type B braid group on the bounded homotopy category of finitely generated projective modules over a finite dimensional algebra which we call the type B zigzag algebra. This categorical action is closely related to the action of the type B braid group on curves on the disc. Thus, our exposition can be seen as a type B analogue of the work of Khovanov-Seidel [KS02]. Moreover, we relate our topological (respectively categorical) action of the type B Artin braid group to their topological (respectively categorical) action of the type A Artin braid group.

Then, we prove Rouquier's conjecture [Rou06, Conjecture 9.8] on the faithfulness of Type B 2-braid group on Soergel category following the strategy used by [Jen14] with the diagrammatic tools from [EW16a].

In the final part of the thesis, we produce a graded Fock vector in the Laurent ring $\mathbb{Z}[t, t^{-1}]$ for each crossingless matching using Heisenberg algebra. We conjecture that the span of such vectors forms a Temperley-Lieb representation, and hence, a new presentation of Jones polynomial can be obtained.

Preface

" Categorification is the modern approach to natural science. "

In search of the truth of life, humans have been unceasingly employing mathematics to explain the mechanics of the universe [11]. Galileo, in the 17th century, famously stated that the universe is written in this grand book, which stands continually open before our eyes, communicating in mathematical language. Dating back to the days of ancient Greece, a clear example is written in the dialogue *Timaues*, where Plato made a direct metaphysical one-to-one correspondence between five Platonic solids (tetrahedron, cube, octahedron, dodecahedron, and icosahedron) and five basic elements (earth, air, fire, water and quintessence[12]) to classify matters using geometry. Those assignments seem to be wrong in many ways, but could there be a merit to the idea? (What about the symmetry of the solids? The McKay correspondence[13].) Another philosopher, Pythagoras, is perhaps, the earliest human being who believes that everything in the world is made up of numbers (or in the modern view, the world is digital), according to Aristotle. No one then could expect that the every-high-school-students-must-know formula — $a^2 + b^2 = c^2$ — formed the cornerstone of modern differential geometry when it was first established as Pythagoras' theorem on right-angled triangles[14]. It determines what form of geometry a space could possibly exhibit; we now know it is either Euclidean, hyperbolic, or spherical.

For a long time, we thought that we were creatures living in a flat Euclidean setting. It was not until 1915 that Einstein found out that the spacetime where we inhabit in is actually a stitch of various geometries, using his widely celebrated theory of general relativity, and astoundingly, gravity can be described as the physical manifestation of a mathematical object known as the curvature! This is one of the greatest triumphs of modern science in the twentieth century; it replaces Newton's law of universal gravitation that fails to explain minute anomalies in the orbit of Mercury. Since then, people have started the journey of searching the grand quest - a theory of everything - in order to unify the four fundamental natural forces, namely the gravity, the strong force, the weak force, and the electromagnetic force. The last three mentioned forces are quantum mechanics in nature and are successfully unified under the main attribution to Yang-Mills theory through the representation theory of $SU(3) \times SU(2) \times U(1)$, that is, the Standard Model. On the other hand, general relativity deals with objects of gigantic scale, for example, Mercury, Sun, and a black hole. Hence, it is unclear how an atomic (negligibly small relative to that of celestial bodies) mass of an electron or photon can even interact with the theory.

Driven by this mission, Crane [Cra95] reinterpreted quantum mechanics in a general relativistic context by probing into the algebraic structure governing the quantum theory of gravity and conjectured that quantum gravity is a 2-category which emerges as a "categorification" of the 1-category (modular tensor category) [Cra91] formulating Chern-Simons-Witten theory [Wit89]. To make sense of categorification mathematically, Crane[15] worked together with Frenkel[16] to outline a combinatorial method to enhance an algebraic structure with categorically higher morphisms and, what is more, the original algebraic structure can be retained by the process of decategerification, forgetting the newly-introduced morphisms in the categorified algebraic structure [CF94]. It is in this author's birth year 1994 when the idea of categorification [CF94] was offi-

[11] Interested readers can get a flavour of the modern views on the mathematical universe from [Teg08; Sch20; Pen05].
[12] Plato remarked, "...the God used for arranging the constellations on the whole heaven."
[13] Please see related references [McK80; Ste08; Nak99] on classification by the symmetry of Platonic solids.
[14] Pythagoras' theorem was proved twice in *Euclid's Elements* [Euc56, (Book I, Proposition 47), (Book VI, Proposition 31)].
[15] His earlier ideas [CY98; Cra94; BC98; Cra06; CC05; Cra07; Cra97] related to (combinatorial/diagrammatic) categorification rooted from approaches to quantum gravity.
[16] Frenkel's students wrote about his fundamental work on categorification [Dun+14] which later blossomed into an active field of research in higher representation theory through him.

cially conveyed in precise mathematical terms and illustrated in the example of categorification of Lusztig's idempotented version of quantum algebra $U_q(\mathfrak{sl}_2)$. At a generic value of the quantum parameter, this was done by Lauda [Lau10; Lau11; Kho+12; Lau12], although the case at the root of unity still remains open today. Some recent work has been done along this line; Khovanov (a math descendant of Frenkel) laid the first path [Kho16], suggesting ways to categorify at a prime root of unity, which was then picked up by Qi (a math descendant of Khovanov) et al. [Qi13; Qi14; KQ15; EQ16b; EQ16a; QS17]. For more discussion on the subject of categorification, the readers are welcome to refer to [LS22; Maz12; Sav15; Sav19b; LM14; BD98; BHW10; BD01; BL11].

To date, categorification has proven fruitful in various research fields such as representation theory, algebraic topology, link homology, mathematical physics, et cetera. For instance, it provides a framework for Kontsevich's homological mirror symmetry [Kon95], which was announced at the International Congress of Mathematicians, Zurich, in 1994. Motivated by the mirror symmetry in string theory, Kontsevich conjectured that the derived Fukaya category of a Calabi-Yau manifold (deemed a symplectic manifold) is equivalent to the derived category of coherent sheaves on its dual manifold (deemed a complex algebraic variety). A gentle introduction to homological mirror symmetry and mirror symmetry can be found in [Boc21] and [Hor+03; CK99; Qui15], respectively. Around the same time, as noted in [KT12], Arnold [Arn95] suggested the realisation of symplectomorphism by the monodromy of Milnor fibres. This was then picked up by Kontsevich and Donaldson, where they categorified it to the monodromy action on the Fukaya category of Milnor fibres.

Consequently, powered by homological mirror symmetry, Khovanov, together with Seidel and Thomas (both math descendants of Donaldson), initiated the study of categorical braid group action on the derived category of Fukaya category of type A Milnor fibres. This was done in relation to the categorical braid group action on the bounded derived category of coherent sheaves on the minimal resolution of higher dimensional type A Kleinian singularity [ST01] and the categorical braid group action on the bounded derived category $D^b(\mathscr{A}\text{-}g_r\text{mod})$ of graded modules over the type A zigzag algebra $\mathscr{A}$ [KS02]. Here, we refer to the complex d-dimensional type A_n Kleinian singularities [Slo80a; Slo80b] as the variety defined by the zero set of the functions

$$g(x_0, x_1, \ldots, x_d) = x_0^2 + x_1^2 + \cdots + x_{d-1}^2 + x_d^{n+1} \tag{0.0.1}$$

while the type A_n Milnor fibre, by a theorem of Milnor [Mil68], as the intersection of the closed ball $\overline{B}^{2d+2}(\epsilon)$ of sufficiently small radius $\epsilon > 0$ around the origin in $\mathbb{C}^{d+1}$ and the symplectic manifold defined by the zero set of the deformation of the type A_n singularities

$$g(x_0, x_1, \ldots, x_d) = x_0^2 + x_1^2 + \cdots + x_{d-1}^2 + h_w(x_d) \tag{0.0.2}$$

where $h_w(z) = z^{n+1} + w_n z^n + \cdots + w_1 z + w_0 \in \mathbb{C}[z]$ for $w = (w_0, w_1, \ldots, w_n)$ in the interior $B^{2d+2}(\delta)$ of $\overline{B}^{2d+2}(\delta)$ for sufficiently small $\delta > 0$ such that the intersection of the symplectic manifold and the boundary $S^{2d+1}(\epsilon)$ of $\overline{B}^{2d+2}(\epsilon)$ remains transverse. Moreover, the zero set of the function h_w is referred to as the bifurcation diagram by Arnold [Arn72; AGV12a; Arn+98]. Since $\mathscr{A}$ is of finite homological dimension [KS02, Proposition 2.1], the bounded derived category $D^b(\mathscr{A}\text{-}g_r\text{mod})$ of graded modules over $\mathscr{A}$ is isomorphic to the bounded homotopy category $\text{Kom}^b(\mathscr{A}\text{-p}_r\text{g}_r\text{mod})$ of the graded projective modules over $\mathscr{A}$.

This project leads to [KS02] the faithfulness of braid group representations in low dimensional topology due to categorification. Seidel [Sei99] first took the initiative to define a type A_n chain of Lagrangian spheres from a set of generating geodesics in bifurcation diagrams associated with the type A_n singularities. Moreover, the symplectomorphisms defined by such Lagrangian spheres satisfy the braid relations in Artin's $(n+1)$-strand braid group, or the type A_n braid group. He has also shown that those bifurcation diagrams are topologically punctured discs $\mathbb{D}^A$ [KS02, Lemma 6.10], and subsequently, the Milnor fibres are Lefschetz fibrations over the punctured discs $\mathbb{D}^A$. Recall that the Dehn twists, which generate the mapping class group of punctured discs [FM12], also form the type A_n braid group. This way, Seidel and Khovanov employed certain bimodule functors in the bounded homotopy category $\text{Kom}^b(\mathscr{A}_n\text{-p}_r\text{g}_r\text{mod})$ of graded projective modules over the Fukaya algebra of type A_n chain of Lagrangian spheres, that is the geometric interpretation of type A_n zigzag algebra. Following this, they formed a correspondence between geodesics connecting the punctures in $\mathbb{D}^A$ and minimal complexes in $\text{Kom}^b(\mathscr{A}_n\text{-p}_r\text{g}_r\text{mod})$. In particular, they proved that the topological braid group action intertwines with the categorical bimodule braid group action, so that the dimension of graded module homomorphism space agreed with the graded curve intersection number (inspired by graded Lagrangian manifold [Sei00]). The faithfulness of the graded topological braid group action on a certain

covering space of the punctured disc and following that, the faithfulness of graded symplectic braid group action on the Fukaya category of the Milnor fibre follows from the unique bigrading in $\mathrm{Kom}^b(\mathscr{A}_n\text{-p}_r\text{g}_r\text{mod})$.

Symplectic geometry	Low-dimensonal topology	Homological Algebra
bifurcation diagram	$(n+1)$-punctured disc	bounded homotopy category
exact Lagrangian isotopy	curve isotopy	complex homotopy
Fukaya algebra	disc bounding combinatorics	zigzag algebra
Lagrangian sphere	geodesics	projective modules($p_r$mod)
symplectomorphism	Dehn twist	bimodule functor
Lagrangian intersection	curve intersection	$p_r$mod morphism
vanishing cycle intersection form	curves intersection numbers	Cartan matrix
Floer cohomology dimension	bigraded intersection number	$p_r$mod morphism dimension
graded Lagrangian Grassmannian	graded projective space	graded projective modules

Table 1: List of Comparison of Terminology in type A setting.

See Table 1 for a comparison of the terminology appearing in [KS02]. A few categorification results follow: the categorical braid group representation on $\mathrm{Kom}^b(\mathscr{A}_n\text{-p}_r\text{g}_r\text{mod})$ categorifies the classical reduced Burau representation [KT08; AK09], and the dimension of graded endomorphism algebra of projective modules categorifies the intersection form of vanishing cycles in type A singularities, which is also the Cartan matrix of type A Dynkin diagram. It is known that Burau representation of the type A braid group is equivalent to the homological representation[17] of the mapping class group of the punctured disc [KT08] and is not faithful for $n \geq 5$ [Moo91; LP93; Big99]. By lifting the punctured disc to a bigraded covering space inspired by graded Lagrangian Grassmannian, the graded mapping class group of punctured disc acts faithfully on the bigraded covering space. Similarly, the type A braid group injects into the graded symplectic mapping class group of Milnor fibre because the graded dimension of Floer cohomology is related to the dimension of graded module homomorphisms, which is also linked to the graded curve intersection numbers.

In Chapter 1 of this thesis, we generalise the results of the type A braid group in [KS02] to that of the non-simply-laced type B braid group. We focus on the result in [HN19] where we utilise the strength of degree 2 field extension to build a new zigzag algebra, as mentioned in the title, the type B zigzag algebra to categorify the intersection form of vanishing (hemi)cycles in type B singularities and the type B non-symmetric integral Cartan matrix (see Corollary 1.8.4) under the interpretation of dimension of graded modules over different fields (see Corollary 1.8.2). Inspired by the folding of Dynkin diagrams[18], at Kiola Conference 2019, Heng[19] and the author defined a type B zigzag algebra $\mathscr{B}_n$ which, after a degree two scalar extension, is isomorphic to $\mathscr{A}_n \cong \mathbb{C} \otimes_{\mathbb{R}} \mathscr{B}_n$ (see Proposition 1.4.1). Subsequently, we defined the bounded homotopy category $\mathrm{Kom}^b(\mathscr{B}_n\text{-}p_r g_r mod)$ and the type B braid bimodule functors acting on the category. The type B braid bimodule functors indeed defined the type B generalised braid group action on $\mathrm{Kom}^b(\mathscr{B}_n\text{-}p_r g_r mod)$ (see Proposition 1.3.9 and 1.3.12). As in [KS02], the bimodules we used (see Proposition 2.6.7) provided a functor realisation to the type B Temperley-Lieb algebra [Gre98].

Echoing the philosophy in the thesis title, the author is inclined to extend the story to the topology analogous to what Khovanov and Seidel did. In the 1980s, Slodowy [Slo80b; Slo80a] introduced simple singularities associated to inhomogeneous Dynkin diagrams using the Kleinian singularities together with certain automorphisms on them. In particular, a type B simple singularity is defined as the type A singularity associated with a $\mathbb{Z}/2\mathbb{Z}$-automorphism on it. While studying critical points of differential functions on manifolds with boundary, Arnold also gave a definition for type B simple singularity on surfaces [Arn78; AGV12b; AGV12a]. On the complex surface $x^2+y^{2n}=0$, it is the involution $(x,y) \mapsto (x,-y)$. After a generic deformation of the complex surface, we were bound to consider the quotient space of the $2n$-punctured disc $\mathbb{D}^A_{2n}$ by a π-rotation. Unsurprisingly, after some trials and errors, we worked out a similar map L_B (ref. the

[17] As an aside, homological representation refers to the study of the representation on the first fundamental group of the punctured disc. On the contrary, homological algebra refers to the study of homology on chain complexes of modules.

[18] With a hint from Peter McNamara, University of Queensland.

[19] Heng is being acknowledged in the Acknowledgements section.

left map in Theorem 0.0.1), which to curves in the quotient disc $\mathbb{D}^B_{n+1}$ associates complexes in $\mathrm{Kom}^b(\mathscr{B}_n\text{-}p_rg_rmod)$. In addition, the type B braid group $\mathcal{A}(B_n)$ appears in at least two ways as a subgroup of the mapping class groups of punctured disc; one is a classical injection $\mathcal{A}(B_n) \rightarrowtail \mathcal{A}(A_n)$ into type A braid group $\mathcal{A}(A_n)$ and the other one $\mathcal{A}(B_n) \hookrightarrow \mathcal{A}(A_{2n-1})$ with $\mathcal{A}(A_{2n-1})$ is given by Crisp [Cri99] from the Dynkin diagram viewpoint or Birman and Hilden [BH73a] from the topological viewpoint. The latter helps in connecting the categorical algebraic construction and the topological setting between type A and type B, providing us with the following big intertwining picture:

Theorem 0.0.1 (= Proposition 1.5.5)**.** *The following diagram is commutative, and the four maps on the square are $\mathcal{A}(B_n)$-equivariant:*

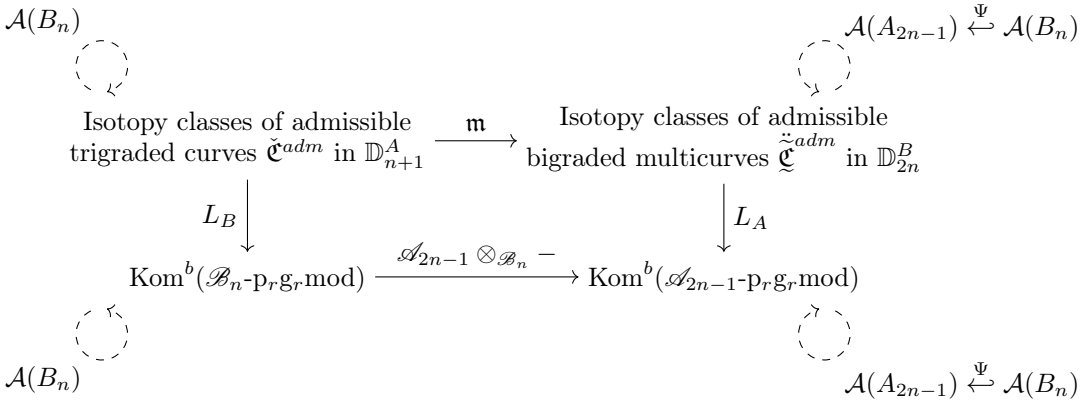


Figure 0.0.0.1: The commutative diagram of Theorem 0.0.1. The first row is from Proposition 1.2.10 and the second row is from Theorem 1.4.3.

It follows from the commutative diagram and the faithful type A categorical action that we indeed construct a faithful type B categorical action.

Actually, we could have done the same as Khovanov and Seidel where the faithfulness result relies on the intersection numbers, and we did! The author introduced the trigraded covering space for $\mathbb{D}^B_{n+1}$ (in Section 1.2.4) and graded intersection number for curves in $\mathbb{D}^B_{n+1}$ motivated by the homomorphism space of corresponding module complexes (in Section 1.2.5). Not included in [HN19], the author just established that the type B zigzag algebra $\mathscr{B}_n$, in fact, sit as an invariant algebra inside the type A zigzag algebra $\mathscr{A}_{2n-1}$ under the $\mathbb{Z}/2\mathbb{Z}$-action of the product group of $\mathbb{Z}/2\mathbb{Z}$-complex conjugation and the $\mathbb{Z}/2\mathbb{Z}$-automorphism of the type A Dynkin diagram.

Theorem 0.0.2 (=Theorem 1.8.1)**.** *The type B zigzag algebra $\mathscr{B}_n$ is the $\mathbb{R}$-algebra of the invariant subalgebra, in other words,*

$$\mathscr{B}_n \cong_{\mathbb{R}} \mathscr{A}^{inv}_{2n-1}$$

under the $\mathbb{Z}/2\mathbb{Z}$-action of the product group of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ where the first group factor is coming from the symplectic consideration and the second group factor is coming from the algebraic consideration.

This serves as another algebraic interpretation of definition of the type B zigzag algebra which seems rather ad hoc at first sight. This result came out from the author's effort in finding the geometric meaning of the type B zigzag algebra in collaboration with Qiao[20].

Practising the chant of categorification, the bimodule category $(\mathscr{A}_n, \mathscr{A}_n)$-bimod and similarly, $(\mathscr{B}_n, \mathscr{B}_n)$-bimod can be interpreted as a 2-category where the object is the single index n, the morphisms are bimodules, and the 2-morphisms are the bimodule homomorphisms. Since the Temperley-Lieb algebra is a quotient algebra of Hecke algebra and the bimodule functors in $\mathrm{Kom}^b(\mathscr{A}_n\text{-p}_r\text{g}_r\text{mod})$ and $\mathrm{Kom}^b(\mathscr{B}_n\text{-}\ p_rg_rmod)$ categorify a (quotient of) Temperley-Lieb algebra of an appropriate type, it is natural to look into their relations with the category of Soergel bimodules. Originally, Soergel [Soe07] first introduced the category of Soergel bimodules through the combinatorics of Harish-Chandra modules [Soe92] to categorify the Hecke algebra. This fact here was first conjectured by Soergel (he verified it for finite Weyl group over field of certain characteristic)

[20] Qiao is being acknowledged in the Acknowledgements section.

and fully verified through a collective work [EK10; Eli11; Eli16a; Eli16c; EW16a] mainly from Elias (a math descendant of Khovanov) and Williamson (a math descendent of Soergel) using diagrammatic category description which finally led them to positivity conjecture of the Kazhdan-Lusztig polynomial[21] [EW14]. A nice introduction to Soergel bimodules can be found in the textbook [Eli+20] and Libedinsky's note [Lib19].

In Chapter 2 of this thesis, we upgrade the faithfulness result in Chapter 1 to a faithful 2-categorical braid group action on the Soergel category. Building on his work [CR08] with Chuang, Rouquier constructed Rouquier complexes [Rou06] which he called the 2-braid group. He conjectured [Rou06, Conjecture 9.8] that the 2-braid group categorifies the generalised braid group where he outlined a proof for the type A case in [Rou06, Remark 9.9]. This proof for the type A braid group was carried out by Jensen (a math descendant of Williamson) in his Master thesis [Jen14] and subsequently extended in [Jen17] using combinatorial methods. Since we have type B zigzag algebra at our disposal, the content of Chapter 2 illustrates the use of the same technique in [Jen14] to construct a monoidal functor from the diagrammatic Soergel category [EW16a] to our bimodule category $(\mathscr{B}_n, \mathscr{B}_n)$-bimod and hence, the faithfulness result is carried across to the 2-category, thereby proving the Rouquier conjecture for type B:

Theorem 0.0.3 (=Corollary 2.7.4)**.** *The type B 2-braid group decategorifies to the type B braid group.*

In Chapter 3 of this thesis, we finally consider the wreath product zigzag algebra analogue of [KS02], which is inspired by another 2-category, the Heisenberg category. In 2012, Licata (a math descendant of Frenkel; my supervisor) and Cautis (a math descendant of Harris) defined a 2-category called Heisenberg category [CL12], which categorifies Heisenberg algebra and, together with Sussan (a math descendant of Frenkel), constructed a braid group action on that category [CLS14]. The bimodule functors employd in the braid group action there was regarded as a wreath analogue of the bimodule functor considered in [KS02]. Please refer [Kho14; LS12; LS13; LS15] for related definitions of Heisenberg category, [KLM19; Cau+18b; Cau+18a] for algebras arising from Heisenberg category and paper series [Sav17; RS17; BSW20a; BSW21; Sav19a; BSW20b; BSW22; MS18; QSY18; RS15a; RS15b; SY15; LS21] by Savage (a math descendant of Frenkel) et al. on recent developments of Heisenberg category in more general settings.

However, it is the conjectural braid group action emerging from the categorified vertex operator [CL14] with a similar line of argument generating braid group action from exponential of quantum algebra Chevalley generators by Lusztig [Lus10] that is related to the link homology. We are currently working on a thick/idempotented calculus for the type A quantum lattice Heisenberg category in order to fully develop the homological algebra part of the category [Pen71; Cvi08; Gon20; KS14; AW17a; AW17b; AW17c]. Even on the chain group level, it heavily uses combinatorial gadgets like symmetric functions [Rut48; Mac15; Mac98; Sta99; Egg19; Pro07], tableaux identities [Ful97; Sag01; FH91; GW09; Tun85], q-calculus [Sta97; AE04; HW08], basic hypergeometric series [Ern12], and many more that the author is still searching...

The final question we will answer in this chapter is whether a chain complex can be associated to a multicurve analogue of [KS02] via Heisenberg categorification. In the last chapter, we give a partial solution to the problem posed by my supervisor, Licata. Instead of a chain complex, we associate a set of chain groups to a crossingless matching without differential (in Section 3.6). In other words, we construct a graded Fock vector living in Fock space representation of Heisenberg algebra over the Laurent ring $\mathbb{Z}[t, t^{-1}]$ given a crossingless matching. This is done by using combinatorial tools like Gaussian polynomial [AE04] and partition statistics from Kostka polynomial [Hag08].

We conjecture that the span of the graded Fock vector coming from crossingless matchings forms a Temperley-Lieb representation (see Conjecture 3.6.5). Since every link can be represented by a braid plat closure [Bir74], we may potentially obtain a new presentation of the Jones polynomial [Jon85; Jon91] from the above Temperley-Lieb representation lying in the Fock space representation of Heisenberg algebra (see Conjecture 3.6.6). This is done through Kauffman's diagrammatic braid group representation into Temperley-Lieb algebra [Kau87; Kau13] inspired by state models [Kau83].

Conjecture 0.0.4 (=Conjecture 3.6.5 and 3.6.6)**.** *The span of the graded Fock vector coming from crossingless matchings forms a Temperley-Lieb representation and hence, produces a new presentation of the Jones polynomial in Fock space representation.*

The entire construction above admits a 2-category description where the endomorphism of certain modules can be related to Khovanov arc algebra [Kho02], which in fact, is the original motivation of the project.

[21] Kazhdan-Lusztig polynomial is the coefficient of Kazhdan-Lusztig basis element expressed in terms of the basis elements indexed by Coxeter group generators [KL79].

Contents

Chapter 1

The Type B Zigzag Algebra.

1.1 Background and Outline of the Chapter

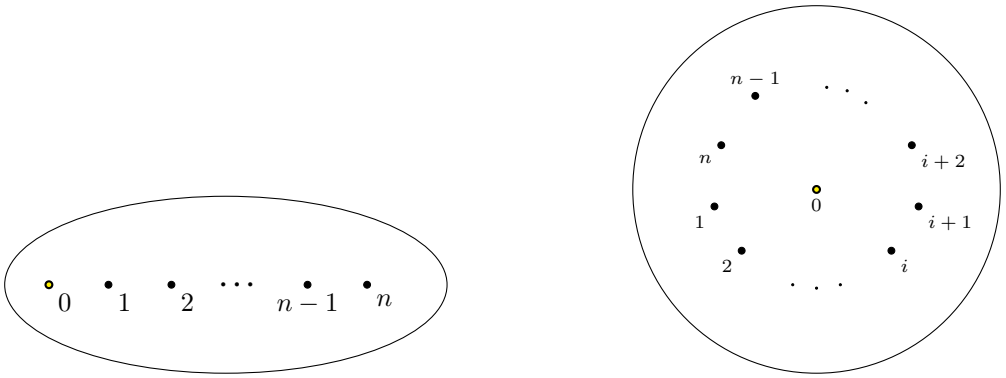
Eager readers can skip this section for the subsequent result sections. This section consists of part of the introduction in [HN19]. The background here provides a detailed comparison between the existing literature for the recent development of the subject after the publication of [KS02] and a more concise summary of the chapter, which was not covered in the preface. We also inform some upcoming works inspired by the work of this chapter.

In the seminal work [KS02], Khovanov-Seidel introduced a categorical action of the type A_m Artin braid group $\mathcal{A}(A_m)$, where the group acts faithfully by exact autoequivalences on $\mathrm{Kom}^b(\mathscr{A}_m\text{-p}_r\text{g}_r\text{mod})$, the homotopy category of projective (graded) modules over the type A_m zigzag algebra $\mathscr{A}_m$. Moreover, they showed that this braid group action on $\mathrm{Kom}^b(\mathscr{A}_m\text{-p}_r\text{g}_r\text{mod})$ is fundamentally related to the mapping class group action on curves on the punctured disc. More precisely, they constructed a map L_A that associates complexes of projective $\mathscr{A}_m$-modules to isotopy classes of curves in the disc, and showed that L_A intertwines the categorical action of $\mathcal{A}(A_m)$ on complexes with the mapping class action of $\mathcal{A}(A_m)$ on curves. Furthermore, the geometric intersection number between two curves c_1 and c_2 can be computed from the dimension of their corresponding total Hom space $\mathrm{HOM}^*(L_A(c_1), L_A(c_2))$. This may be seen as a bridge connecting two fundamental appearances of the same group: the former as the Artin group associated to the type A Coxeter group, and the latter as the mapping class group of the punctured disc.

Another family of Artin groups which also appear as mapping class groups, are the type B Artin groups $\mathcal{A}(B_n)$. To this end, Gadbled-Thiel-Wagner developed a type B analogue of the Khovanov-Seidel construction in [GTW17]. However, their story is more accurately stated using extended affine type A Artin groups $\widehat{\mathcal{A}}(\hat{A}_{n-1})$. Although the groups $\mathcal{A}(B_n)$ and $\widehat{\mathcal{A}}(\hat{A}_{n-1})$ are indeed isomorphic, they have rather different origins; both algebraically and topologically. When interpreted as Artin groups, they are given by different Artin presentations corresponding to different Coxeter diagrams. Although they are both mapping class groups of an $(n+1)$-punctured disc (fixing one of the punctures), their corresponding natural affine configurations of the disc are different (see Figure 1.1.0.1).

The goal of the present chapter is to develop a proper (non-simply-laced) type B analogue of the constructions given by Khovanov-Seidel and Gadbled-Thiel-Wagner. To this end, we introduce a (finite dimensional) quotient of a quiver algebra $\mathscr{B}_n$ over $\mathbb{R}$, which we call the type B_n zigzag algebra. Unlike in type A and extended affine type A, the root system of type B is no longer simply-laced. As a result, the definition of the type B zigzag algebra is somewhat subtle – the indecomposable projective $\mathscr{B}_n$-module whose class in the Grothendieck group is a short simple root will only have the structure of a $\mathbb{R}$-vector space, while all the other indecomposable projective $\mathscr{B}_n$-modules whose classes are long simple roots will actually be $\mathbb{C}$-vector spaces. Note that this is somewhat reminiscent of the following non-simply-laced extension in quiver theory: by studying representations of K-species [Gab73] instead of quivers (the base field are allowed to be different at each vertex), the finite type K-species are characterised by *all* (including non-simply-laced) Dynkin diagrams [DR75, Theorem B]. The relevance of $\mathscr{B}_n$ to Coxeter theory is provided by the following theorem:

Theorem 1.1.1 (= Theorem 1.3.13 and Theorem 1.4.6)**.** *The homotopy category of projective graded modules $Kom^b(\mathscr{B}_n\text{-}p_r g_r mod)$ carries a faithful (weak) action of the type B_n Artin group $\mathcal{A}(B_n)$.*



(a) For type B. (b) For extended affine type A.

Figure 1.1.0.1: Two different affine configurations of the $(n+1)$-punctured disc corresponding to the action of $\mathcal{A}(B_n)$ and $\widehat{\mathcal{A}}(\hat{A}_{n-1})$, where the puncture labelled '0' is fixed.

Similar to the works of Khovanov-Seidel and Gadbled-Thiel-Wagner, we establish the following results that relate the categorical notions to low-dimensional topology:

Theorem 1.1.2 (= Theorem 1.5.7 and Proposition 1.7.3)**.** *There exists a map L_B that associates complexes in $Kom^b(\mathscr{B}_n\text{-}p_r g_r mod)$ to curves in the $(n+1)$-punctured disc. This map L_B is $\mathcal{A}(B_n)$-equivariant, intertwining the $\mathcal{A}(B_n)$-action on curves and the $\mathcal{A}(B_n)$-action on complexes in $Kom^b(\mathscr{B}_n\text{-}p_r g_r mod)$. Moreover, the (trigraded) intersection number between two curves c_1 and c_2 is given by the Poincaré polynomial of the total Hom space between $L_B(c_1)$ and $L_B(c_2)$.*

One main feature of our work contrasting that of Gadbled–Thiel–Wagner is an explicit realisation of a well-known connection between type B and type A. To explain this, recall that the type B_n Artin group is known to be a (proper) subgroup of the type A_{n-1} Artin group: algebraically the embedding is induced from a folding of Coxeter diagrams [Bri73]; topologically the embedding is obtained by lifting through the double-branched cover of a $(n+1)$-punctured disc $\mathbb{D}^B_{n+1}$ by a $2n$-punctured disc $\mathbb{D}^A_{2n}$ [BH73b]. The topological interpretation induces a $\mathcal{A}(B_n)$-equivariant map $\mathfrak{m}$ that takes curves in the $(n+1)$-punctured disc to *multicurves* in the $2n$-punctured disc, defined by taking the preimage of the covering map. Our work includes a categorical interpretation of this map $\mathfrak{m}$, given by a scalar extension functor:

Theorem 1.1.3 (= Proposition 1.4.1 and Theorem 1.5.1)**.** *The type B zigzag algebra $\mathscr{B}_n$ (over $\mathbb{R}$) algebra is isomorphic to Khovanov-Seidel type A zigzag algebra $\mathscr{A}_{2n-1}$ after extending scalars to $\mathbb{C}$, namely*

$$\mathbb{C} \otimes_{\mathbb{R}} \mathscr{B}_n \cong \mathscr{A}_{2n-1} \quad \text{as } \mathbb{C}\text{-algebras.}$$

This induces a scalar extension functor $\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} -$, which renders the diagram in Figure 1.1.0.2 commutative, with all four maps on the square $\mathcal{A}(B_n)$-equivariant.

We would like to mention here that the construction of the type B zigzag algebra in this chapter can be easily modified to allow for other *Lie-type Dynkin diagrams*, particularly for types C, F_4, and G_2 (where for label 6 one uses a field extension of degree 3 instead). Together with the simply-laced constructions [HK01; LQ21], this (only) covers all of the Lie-type Dynkin diagrams. To the best of our knowledge, there is no easy generalisation of this construction via (finite-dimensional) algebras that encapsulates all *Coxeter diagrams*; not even for the finite types H and $I_2(k)$. A different construction via algebra objects in fusion categories that allows for arbitrary Coxeter diagrams will be provided in Heng's PhD thesis.

Finally, recall that the type A_n zigzag algebra has a geometric origin: it is quasi-isomorphic as a differential graded algebra (dga) with zero differential to the dga associated to an A_n-chain of spherical objects [ST01]. In particular, it is quasi-isomorphic to the Fukaya A_∞-algebra of the Donaldson-Fukaya category corresponding to the Milnor fibre of type A singularities [Sei08]. Type B singularities have also been studied; from Arnold's symplectic point of view [Arn+98; AGV12b] as boundary singularities and from the algebraic geometry point of view by Slodowy [Slo80a; Slo80b] as simple singularities associated with a $\mathbb{Z}/2\mathbb{Z}$-group action. We expect our type B zigzag algebra to have a similar geometric origin as in the type A case. This is an ongoing work of the author with Qiao.

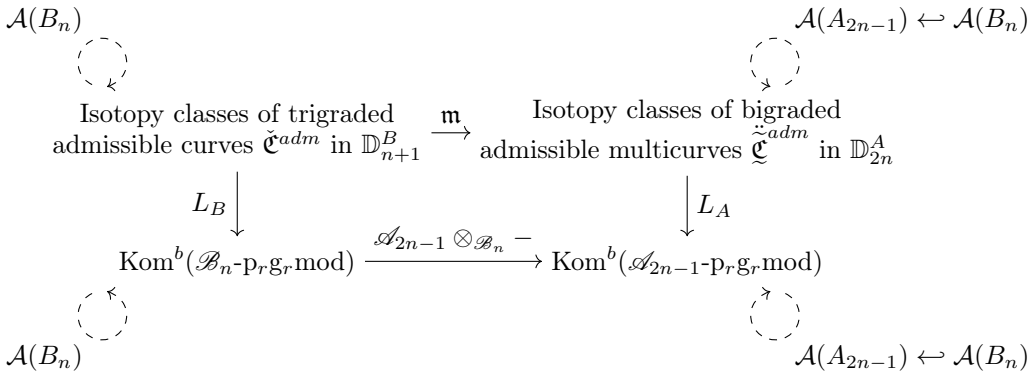


Figure 1.1.0.2: The commutative diagram of Theorem 1.1.3. The map $\mathfrak{m}$ is obtained by lifting curves in $\mathbb{D}^B_{n+1}$ to multicurves in $\mathbb{D}^A_{2n}$ through the double-branched cover $\mathbb{D}^A_{2n} \twoheadrightarrow \mathbb{D}^B_{n+1}$. The map L_B (resp. L_A) is as in Theorem 1.1.2 (resp. [KS02, Theorem 4.3]).

Outline of the chapter

Section 1.2 contains all the topological side of this chapter. We describe the double-branched cover of $\mathbb{D}^B_{n+1}$ by $\mathbb{D}^A_{2n}$, which induces an injection of groups $\Psi : \mathcal{A}(B_n) \hookrightarrow \mathcal{A}(A_{2n-1})$. Within this section, we also make explicit the definition of curves and admissible curves, and also introduce the notion of trigraded curves – the type B analogue of the bigraded curves for type A. The construction of the $\mathcal{A}(B_n)$-equivariant map $\mathfrak{m}$, which lifts trigraded curves to bigraded multicurves can be found in Section 1.2.8. We also introduce the notion of a trigraded intersection number of trigraded curves – the type B analogue of bigraded intersection number of bigraded curves, and relate the two through the map $\mathfrak{m}$.

Section 1.3 is where the zigzag algebras are defined. We start by recalling the type A_{2n-1} zigzag $\mathbb{C}$-algebra $\mathscr{A}_{2n-1}$ as defined in [KS02] (with a minor change in grading) and describe the $\mathcal{A}(A_{2n-1})$-action on $\mathrm{Kom}^b(\mathscr{A}_{2n-1}\text{-p}_r\text{g}_r\text{mod})$. We then define our type B_n zigzag $\mathbb{R}$-algebra $\mathscr{B}_n$ and describe the corresponding $\mathcal{A}(B_n)$-action on $\mathrm{Kom}^b(\mathscr{B}_n\text{-p}_r\text{g}_r\text{mod})$. We end this section by describing the functor realisation of the type B_n Temperley-Lieb algebra given by certain $(\mathscr{B}_n, \mathscr{B}_n)$-bimodules.

In Section 1.4, we develop the algebraic analogue of $\mathfrak{m}$, given by an extension of scalar functor $\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} -$. In particular, we show that there is an injection $\mathscr{B}_n \hookrightarrow \mathscr{A}_{2n-1}$ of $\mathbb{R}$-algebras, which induces an extension of scalar functor $\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} - : \mathscr{B}_n\text{-mod} \to \mathscr{A}_{2n-1}\text{-mod}$ and thus on their homotopy categories. We will then show that the functor $\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} -$ is $\mathcal{A}(B_n)$-equivariant, with the $\mathcal{A}(B_n)$-action on $\mathrm{Kom}^b(\mathscr{A}_{2n-1}\text{-p}_r\text{g}_r\text{mod})$ induced by the injection Ψ as defined in Section 1.2. This in turn, allows us to deduce the faithfulness of the $\mathcal{A}(B_n)$ categorical action by using the faithfulness of the $\mathcal{A}(A_{2n-1})$ categorical action proven in [KS02].

Section 1.5 is where we prove the remaining pieces of Theorem 1.1.3. We recall the $\mathcal{A}(A_m)$-equivariant map L_A that associates admissible complexes to curves as in [KS02] and construct the type B analogue map L_B in a similar fashion. To show that L_B is $\mathcal{A}(B_n)$-equivariant and that the diagram in Theorem 1.1.3 commutes, we will begin with the latter, which will allow us to derive the former.

Section 1.6 contains the decategorified version of the main theorem (see Theorem 1.6.3 for the corresponding diagram). Namely, just as the $\mathcal{A}(A_m)$ action on $\mathrm{Kom}^b(\mathscr{A}_m\text{-p}_r\text{g}_r\text{mod})$ categorifies the Burau representation (which can be described as a representation on the first homology of an explicit covering space of $\mathbb{D}^A_{2n}$), we show that the categorical action of $\mathcal{A}(B_n)$ on $\mathrm{Kom}^b(\mathscr{B}_n\text{-p}_r\text{g}_r\text{mod})$ categorifies a representation on (a submodule of) the first homology of an explicit covering space of $\mathbb{D}^B_{n+1}$.

Section 1.7 is where we relate the trigraded intersection numbers of (admissible) curves and the Poincaré polynomial of the homomorphism spaces of their corresponding complexes.

Section 1.8 is where we prove the type B zigzag algebra as an invariant algebra over $\mathbb{R}$ of the type A zigzag algebra (see Proposition 1.4.1). Moreover, we show that the endomorphism space of projective modules over the type B zigzag algebra categorifies the type B intersection form (see Corollary 1.8.2) and the type B Cartan matrix (see Corollary 1.8.4).

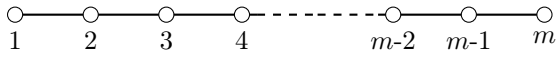
1.2 Artin Groups of Type B_n and Type A_{2n-1} as Mapping Class Groups

In this section, we will first describe type A and type B Artin groups using generators and relations. After that, we associate these two Artin groups to mapping class groups of surfaces. We then introduce trigraded curves and trigraded intersection numbers as trigraded analogues of bigraded curves and bigraded intersection numbers in [KS02, Section 3b]. Finally, we construct a $\mathcal{A}(B_n)$-equivariant lift of the isotopy classes of trigraded curves to the isotopy classes bigraded multicurves.

1.2.1 Artin groups by generators and relation

An *Artin group* associated to a Coxeter graph Γ is a group defined by generators and relations according to the data of the graph Γ. In this chapter, we shall only concern ourselves with the Artin groups associated to the type A and type B Coxeter graphs. As such, we shall explicitly define them below, and refer the reader to [KT08] and [BB05] for a more extensive theory on Artin groups.

For $m \geq 2$, the type A_m Artin group $\mathcal{A}(A_m)$ associated to the type A_m Coxeter graph



is the group generated by

$$\sigma_1^A, \sigma_2^A, \ldots, \sigma_m^A$$

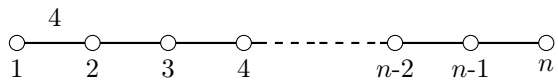
subject to the relations

$$\sigma_j^A \sigma_k^A = \sigma_k^A \sigma_j^A, \qquad \text{for } |j-k| > 1; \tag{1.2.1}$$

$$\sigma_j^A \sigma_{j+1}^A \sigma_j^A = \sigma_{j+1}^A \sigma_j^A \sigma_{j+1}^A, \qquad \text{for } j = 1, 2, \ldots, m-1. \tag{1.2.2}$$

Note that $\mathcal{A}(A_m)$ is the usual $(m+1)$-strand braid group $\mathcal{B}r_{m+1}$.

For $n \geq 2$, the type B_n Artin group $\mathcal{A}(B_n)$ associated to the type B_n Coxeter graph



is the group generated by

$$\sigma_1^B, \sigma_2^B, \ldots, \sigma_n^B$$

subject to the relations

$$\sigma_1^B \sigma_2^B \sigma_1^B \sigma_2^B = \sigma_2^B \sigma_1^B \sigma_2^B \sigma_1^B; \tag{1.2.3}$$

$$\sigma_j^B \sigma_k^B = \sigma_k^B \sigma_j^B, \qquad \text{for } |j-k| > 1; \tag{1.2.4}$$

$$\sigma_j^B \sigma_{j+1}^B \sigma_j^B = \sigma_{j+1}^B \sigma_j^B \sigma_{j+1}^B, \qquad \text{for } j = 2, 3, \ldots, n-1. \tag{1.2.5}$$

1.2.2 Mapping class groups of discs with marked points

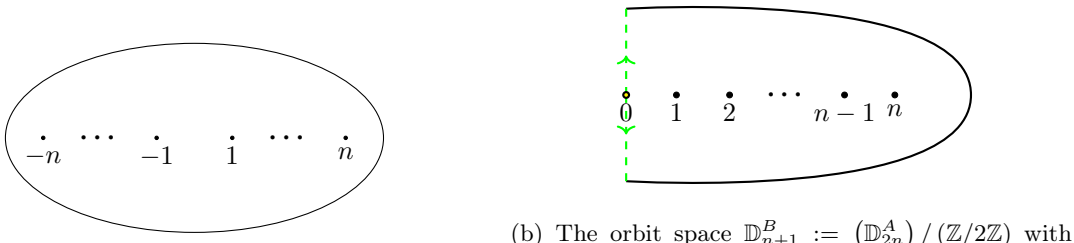
Suppose $\mathcal{S}$ is a compact, connected, oriented surface, possibly with boundary $\partial\mathcal{S}$, and $\Delta \subset \mathcal{S} \backslash \partial\mathcal{S}$ a finite set of marked points. We denote such a surface as $(\mathcal{S}, \Delta)$ and we will just write $\mathcal{S}$ if the associated Δ is clear from the context. Let $\Delta^{id} \subset \Delta$ be a subset. Denote by $\mathrm{Diff}(\mathcal{S}, \partial\mathcal{S}; \Delta^{id})$ as the group of orientation-preserving diffeomorphisms $f : \mathcal{S} \to \mathcal{S}$ with $f|_{\partial\mathcal{S} \cup \Delta^{id}} = id$ and $f(\Delta) = \Delta$. If $\Delta^{id} = \emptyset$, then we write $\mathrm{Diff}(\mathcal{S}, \partial\mathcal{S}) := \mathrm{Diff}(\mathcal{S}, \partial\mathcal{S}; \emptyset)$ for simplicity. We then define the mapping class group $\mathrm{MCG}(\mathcal{S}, \Delta^{id})$ of the surface $\mathcal{S}$ with a set Δ of marked points fixing elements in Δ^{id} pointwise by $\mathrm{MCG}(\mathcal{S}, \Delta^{id}) := \pi_0\big(\mathrm{Diff}(\mathcal{S}, \partial\mathcal{S}; \Delta^{id})\big)$. In a similar fashion, if $\Delta^{id} = \emptyset$, we denote the mapping class group of $\mathcal{S}$ by $\mathrm{MCG}(\mathcal{S}) := \mathrm{MCG}(\mathcal{S}, \emptyset)$. We will just write $\mathrm{MCG}(\mathcal{S})$ if those conditions are clear from the context. The elements of $\mathrm{MCG}(\mathcal{S})$ are called *mapping*

classes. We shall see that both Artin groups from Section 1.2.1 appear as mapping class groups, where we refer the reader to [FM12] for a more detailed exposition on this.

Branched covering of $\mathbb{D}^B_{n+1}$ by $\mathbb{D}^A_{2n}$

Consider the following closed disc $\mathbb{D}^A_{2n} := \{z \in \mathbb{C} : \|z\| \leq n+1\}$ embedded in $\mathbb{C}$, equipped with the set $\Delta := \{-n, \dots, -1, 1, \dots, n\}$ of $2n$ marked points, as drawn in Figure 1.2.2.1a.

Let $r : \mathbb{D}^A_{2n} \to \mathbb{D}^A_{2n}$ be the half rotation of the disc $\mathbb{D}^A_{2n}$ defined by $r(x) = -x$ for $x \in \mathbb{D}^A_{2n}$. Consider the group $\mathcal{R} \cong \mathbb{Z}/2\mathbb{Z}$ generated by r and its action on $\mathbb{D}^A_{2n}$. It is clear that each $x \in \mathbb{D}^A_{2n} \backslash \{0\}$ has a neighbourhood U_x such that $r(U_x) \cap U_x = \emptyset$. In this way, the quotient map $q_{br} : \mathbb{D}^A_{2n} \to \mathbb{D}^A_{2n}/(\mathbb{Z}/2\mathbb{Z})$ to its orbit space is a normal branched covering with branched point $\{0\}$ [Pie96]. From now on, we will denote $\mathbb{D}^B_{n+1}$ as the orbit space $\left(\mathbb{D}^A_{2n}\right) / \left(\mathbb{Z}/2\mathbb{Z}\right)$, and $\Lambda = \{[0], [1], [2], \cdots, [n]\}$ as the set of $n+1$ marked points in $\mathbb{D}^B_{n+1}$. To simplify notation and to help us picture the orbit space $\mathbb{D}^B_{n+1}$, to each equivalence class in $\mathbb{D}^B_{n+1}$ we always pick the element with positive real part as the representative whenever possible (i.e. as long as the equivalence class does not contain points on the imaginary line). This way, we shall abuse notation and denote the set of marked points Λ as $\{0, 1, 2, \cdots, n\}$. Figure 1.2.2.1b illustrates how we will be picturing $\mathbb{D}^B_{n+1}$, where the two oriented green lines are identified.



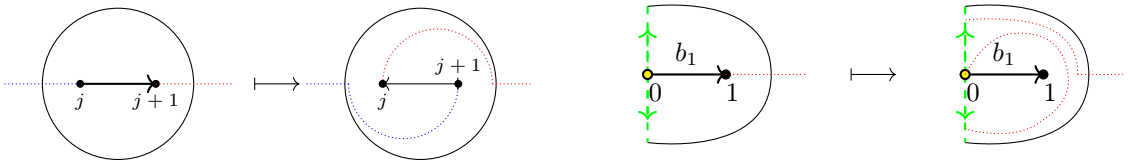
(a) The disc $\mathbb{D}^A_{2n}$ with marked points Δ.

(b) The orbit space $\mathbb{D}^B_{n+1} := \left(\mathbb{D}^A_{2n}\right) / \left(\mathbb{Z}/2\mathbb{Z}\right)$ with marked points Λ.

Figure 1.2.2.1: The affine configurations of the two discs.

Artin groups as mapping class groups

By construction, the marked points on $\mathbb{D}^A_{2n}$ and $\mathbb{D}^B_{n+1}$ are subsets of $\mathbb{Z}$. Therefore, we enumerate the marked points on the disc by increasing sequences of points. Let ϱ_j (resp. b_j) be the horizontal curve connecting the j-th marked point and $(j+1)$-th marked point in $\mathbb{D}^A_{2n}$ (resp. $\mathbb{D}^B_{n+1}$) for $1 \leq j \leq 2n$ (resp. $1 \leq j \leq n+1$).



(a) A half twist t^A_j (similarly t^B_j).

(b) A full twist $(t^B_{b_1})^2$.

Figure 1.2.2.2: The twists in $\mathbb{D}^A_{2n}$ and $\mathbb{D}^B_{n+1}$.

The group $\mathcal{A}(A_{2n-1})$ is isomorphic to the mapping class group $\mathrm{MCG}(\mathbb{D}^A_{2n})$ of a closed disc $\mathbb{D}^A_{2n}$ with $2n$ marked points. The generator σ^A_j corresponds to the half twist $[t^A_{\varrho_j}]$ along the arc ϱ_j. Here, $t^A_{\varrho_j}$ is a diffeomorphism in $\mathbb{D}^A_{2n}$ rotating a small open disc enclosing the j-th and $(j+1)$-th marked points anticlockwise by an angle of π, permuting the two enclosed marked points, whilst leaving all other marked points fixed; see Figure 1.2.2.2a.

Similarly, the group $\mathcal{A}(B_n)$ is isomorphic to the mapping class group $\mathrm{MCG}(\mathbb{D}^B_{n+1}, \{0\})$ of a closed disc $\mathbb{D}^B_{n+1}$ with $n+1$ marked points, fixing the point $\{0\}$ pointwise. The generator σ^B_1 corresponds to the full twist $[(t^B_{b_1})^2]$ along the arc b_1, and for $2 \leq j \leq n$, each generator σ^B_j correspond to the half twist $[t^B_{b_j}]$ along

the arc b_j. Here, $t^B_{b_j}$ is a diffeomorphism in $\mathbb{D}^B_{n+1}$ rotating a small open disc enclosing the j-th and $(j+1)$-th marked points by an angle of π anticlockwise as illustrated in Figure 1.2.2.2a. As a result, it interchanges the j-th and $(j+1)$-th marked points and leaves the other points fixing pointwise. Consequently, $(t^B_{b_1})^2$ is a diffeomorphism rotating a small open disc enclosing the marked points 0 and 1 anticlockwise by an angle of 2π leaving all the marked points fixed, as shown in Figure 1.2.2.2b.

Injection of MCG $\left(\mathbb{D}^B_{n+1}, \{0\}\right)$ into MCG $\left(\mathbb{D}^A_{2n}\right)$

A diffeomorphism f^B in $\mathrm{Diff}(\mathbb{D}^B_{n+1}, \{0\})$ can be lifted to a unique fiber-preserving diffeomorphism f^A in $\mathrm{Diff}(\mathbb{D}^A_{2n})$ via the branched covering map q_{br}. Similarly, an isotopy in $\mathbb{D}^B_{n+1}$ can be lifted to an isotopy in $\mathbb{D}^A_{2n}\backslash\{0\}$. As such, we have a well-defined map Ψ on the mapping class groups from $\mathrm{MCG}\left(\mathbb{D}^B_{n+1}, \{0\}\right) \to \mathrm{MCG}\left(\mathbb{D}^A_{2n}\right)$ defined by lifting the mapping class of f^B to the mapping class of f^A. More concretely, using the standard presentation of the groups, Ψ is given by σ^B_1 mapping to σ^A_n and σ^B_j mapping to $\sigma^A_{n+j-1}\sigma^A_{n-(j-1)}$ for $j \geq 2$. In fact, the image of the map Ψ is generated by fiber-preserving mapping classes in $\mathrm{MCG}_p\left(\mathbb{D}^A_{2n}\right)$. By [BH73b, Theorem 1], we know that any fibre-preserving diffeomorphism f^A which is isotopic to the identity possesses a fiber-preserving isotopy to the identity, which can then be projected to $\mathbb{D}^B_{n+1}$ to get the isotopy $f^B \simeq id$. Therefore, we have the following well-known result:

Proposition 1.2.1. *The homomorphism $\Psi : MCG\left(\mathbb{D}^B_{n+1}, \{0\}\right) \to MCG\left(\mathbb{D}^A_{2n}\right)$ defined by*

$$\Psi([t^B_{b_i}]) = \begin{cases} [t^A_{b_n}], & \text{for } i = 1; \\ \left[t^A_{b_{n+i-1}} t^A_{b_{n-(i-1)}}\right], & \text{for } i \geq 2 \end{cases} \qquad \text{is injective.}$$

1.2.3 Curves and geometric intersection numbers

Here we collect the definitions of curves and geometric intersection numbers as defined in [KS02, Section 3a]. Let $(\mathcal{S}, \Delta)$ be a surface with marked points as in Section 1.2.2. A *curve* c in $(\mathcal{S}, \Delta)$ is a subset of $\mathcal{S}$ that is either a simple closed curve in the interior $\mathcal{S}^o := \mathcal{S}\backslash(\partial\mathcal{S} \cup \Delta)$ of $\mathcal{S}$ and essential (non-nullhomotopic in $\mathcal{S}^o$), or the image of an embedding $\gamma : [0,1] \to \mathcal{S}$ which is transverse to the boundary $\partial\mathcal{S}$ of $\mathcal{S}$ with its endpoint lying in $\partial\mathcal{S} \cup \Delta$, that is, $\gamma^{-1}(\partial\mathcal{S} \cup \Delta) = \{0, 1\}$. In this way, our defined curves are smooth and unoriented. A *multicurve* in $(\mathcal{S}, \Delta)$ is the union of a finite collection of disjoint curves in $(\mathcal{S}, \Delta)$. We say two curves c_0 and c_1 are *isotopic* if there exists an isotopy in $\mathrm{Diff}(\mathcal{S}, \partial\mathcal{S}; \Delta)$ deforming one into the other, denoted by $c_0 \simeq c_1$. Note that the points on $\partial\mathcal{S} \cup \Delta$ cannot move during an isotopy. Therefore, we can partition all curves in $(\mathcal{S}, \Delta)$ into isotopy classes of curves. Two multicurves $\mathfrak{c}_0, \mathfrak{c}_1$ are isotopic if they have the same number of disjoint curves, and each curve in $\mathfrak{c}_0$ is isotopic to one and only one curve in $\mathfrak{c}_1$. Two curves c_0, c_1 are said to have *minimal intersection* if given two intersection points $z_- \neq z_+$ in $c_0 \cap c_1$, the two arcs $\alpha_0 \subset c_0$, $\alpha_1 \subset c_1$ with endpoints $z_- \neq z_+$ such that $\alpha_0 \cap \alpha_1 = \{z_-, z_+\}$ do not form an empty bigon (the bigon contains no marked points) unless z_-, z_+ are marked points. Two multicurves $\mathfrak{c}_0, \mathfrak{c}_1$ are said to have *minimal intersection* if any two curves $c_0 \subseteq \mathfrak{c}_0$ and $c_1 \subseteq \mathfrak{c}_1$ have minimal intersection (see Figure 1.2.3.1).

(a) (b) (c) (d)

Figure 1.2.3.1: The dotted curves and solid curves belong to different multicurves. The multicurves in (A) and (B) do not have minimal intersection, whereas the multicurves in (C) and (D) do.

Observe that given two arbitrary curves c_0, c_1 in $(\mathcal{S}, \Delta)$, we can always find a curve $c_1' \simeq c_1$ such that c_0 and c_1' have minimal intersection. Given two curves c_0 and c_1 in $(\mathcal{S}, \Delta)$, we define the *geometric intersection numbers* $I(c_0, c_1) \in \frac{1}{2}\mathbb{Z}$ as follows:

$$I(c_0, c_1) = \begin{cases} 2, & \text{if } c_0, c_1 \text{ are closed and isotopic;} \\ |(c_0 \cap c_1')\backslash\Delta| + \frac{1}{2}|(c_0 \cap c_1') \cap \Delta|, & \text{if } c_0 \cap c_1' \cap \partial\mathbb{D} = \emptyset. \end{cases} \tag{1.2.6}$$

By [KS02, Lemma 3.2] and [KS02, Lemma 3.3], the definition is indeed independent of the choice of c_1'. Moreover, note that the definition above doesn't depend on the orientation of $\mathcal{S}$ and is symmetric. We extend the definition of geometric intersection numbers for multicurves by just adding up the geometric intersection numbers of each pair of curves $c_0 \subseteq \mathfrak{c}_0$ and $c_1 \subseteq \mathfrak{c}_1$.

1.2.4 Trigraded curves in $\mathbb{D}^B_{n+1}$

In this subsection, we shall extend the notion of bigraded curves and bigraded intersection numbers defined in [KS02, Section 3d] to *trigraded curves* and *trigraded intersection numbers* (see also Section 1.2.7).

Let us remind the reader that we equipped the disc $\mathbb{D}^A_{2n}$ with the set of marked points $\Delta = \{-n, \dots, -1, 1, \dots, n\}$ and the disc $\mathbb{D}^B_{n+1}$ with the set of marked points $\Lambda = \{0, 1, \dots, n\}$. Consider another set of marked points $\Delta_0 = \Delta \cup \{0\}$ in the disc $\mathbb{D}^A_{2n}$. Fix the notation as follows: $\mathfrak{D}^B_\Lambda := PT\left(\mathbb{D}^B_{n+1} \setminus \Lambda\right)$, and $\mathfrak{D}^A_{\Delta_0} := PT\left(\mathbb{D}^A_{2n} \setminus \Delta_0\right)$ where $PT(\cdot)$ denotes the real projectivisation of the tangent bundle of the respective discs by taking an oriented trivialisation of its tangent bundle. We can then identify $\mathfrak{D}^A_{\Delta_0} \cong \mathbb{R}\mathrm{P}^1 \times \left(\mathbb{D}^A_{2n} \setminus \Delta_0\right)$. In $\mathbb{D}^A_{2n} \setminus \Delta_0$, pick a small loop λ_j winding positively around the puncture $j \in \Delta_0$. In this way, the classes $[point \times \lambda_j]$ and $[\mathbb{R}\mathrm{P}^1 \times point]$ form a basis of $H_1(\mathfrak{D}^A_{\Delta_0}; \mathbb{Z})$. From [Hat02, The Universal Coefficient Theorem for Cohomology], we can then build the covering space $\widetilde{\mathfrak{D}}^A_{\Delta_0}$ of $\mathfrak{D}^A_{\Delta_0}$ classified by the cohomology class $C_0 \in H^1(\mathfrak{D}^A_{\Delta_0}; \mathbb{Z} \times \mathbb{Z})$ defined as follows:

$$C_0([point \times \lambda_0]) = (0, 0); \tag{1.2.7}$$

$$C_0([point \times \lambda_j]) = (-2, 1), \quad \text{for } j = -n, \dots, -1, 1, \dots, n; \tag{1.2.8}$$

$$C_0([\mathbb{R}\mathrm{P}^1 \times point]) = (1, 0). \tag{1.2.9}$$

In fact, $\widetilde{\mathfrak{D}}^A_{\Delta_0}$ is a covering for $\mathfrak{D}^B_\Lambda$ with group of deck transformation $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, as explained in the following lemma.

Lemma 1.2.2.

1. *Under the action of the rotation group $\mathcal{R}$ generated by the half rotation r, the quotient map $q : \mathbb{D}^A_{2n} \setminus \Delta_0 \to \mathbb{D}^B_{n+1} \setminus \Lambda$ is a normal covering space with deck transformation group $\mathcal{R} \cong \mathbb{Z}/2\mathbb{Z}$.*

2. *The composite $\widetilde{\mathfrak{D}}^A_{\Delta_0} \xrightarrow{\mathfrak{p}} \mathfrak{D}^A_{\Delta_0} \xrightarrow{\mathfrak{q}} \mathfrak{D}^B_\Lambda$ is a normal covering, where $\mathfrak{q}$ is the normal covering map induced by the quotient map q on the disc component and the identity map on the $\mathbb{R}P^1$ component.*

3. *The group of deck transformations for the covering space $\widetilde{\mathfrak{D}}^A_{\Delta_0} \xrightarrow{\mathfrak{q}\circ\mathfrak{p}} \mathfrak{D}^B_\Lambda$ is $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

Proof. The proofs of (1) and (2) are straightforward and we leave them to the reader.

We will now prove (3). Since the covering $\mathfrak{q} \circ \mathfrak{p}$ is normal, its deck transformation group G is given by $G \cong \frac{\pi_1\left(\mathfrak{D}^B_\Lambda\right)}{(\mathfrak{q}_* \circ \mathfrak{p}_*)\left(\pi_1\left(\widetilde{\mathfrak{D}}^A_{\Delta_0}\right)\right)}$. Observe that we have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(\widetilde{\mathfrak{D}}^A_{\Delta_0}) & \xrightarrow{\mathfrak{p}_*} & \pi_1(\mathfrak{D}^A_{\Delta_0}) & \xrightarrow{C_0} & \mathbb{Z} \times \mathbb{Z} & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle \mathfrak{q}_*} & & \downarrow{\scriptstyle \widetilde{\mathfrak{q}_*}} & & \\
1 & \longrightarrow & \pi_1(\widetilde{\mathfrak{D}}^A_{\Delta_0}) & \xrightarrow{\mathfrak{q}_* \circ \mathfrak{p}_*} & \pi_1(\mathfrak{D}^B_\Lambda) & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & {\scriptstyle \lambda_0 \mapsto 1}\downarrow & & \downarrow & & \\
1 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & = & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array},$$

where $\mathbb{Z}\times\mathbb{Z}$ is the deck transformation group of the covering $\mathfrak{p}$, $\mathbb{Z}/2\mathbb{Z}$ is the deck transformation group of the covering $\mathfrak{q}$, and $\widetilde{\mathfrak{q}_*}$ is the map induced by $\mathfrak{q}_*$. We shall show that the rightmost column of short exact sequence is left-split; namely we shall construct a map $\widetilde{\varphi}: G\to\mathbb{Z}\times\mathbb{Z}$ such that $\widetilde{\varphi}\circ\widetilde{\mathfrak{q}_*}=\mathrm{id}$:

$$1\longrightarrow\mathbb{Z}\times\mathbb{Z}\xrightarrow{\widetilde{\mathfrak{q}_*}}G\longrightarrow\mathbb{Z}/2\mathbb{Z}\longrightarrow 1\ ,$$
$$\widetilde{\varphi}$$

which shows that $G\cong\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ as required.

We shall first define a map $\varphi:\pi_1(\mathfrak{D}^B_\Lambda)\to\pi_1(\mathfrak{D}^A_{\Delta_0})$ and show that $C_0\circ\varphi$ factors uniquely through the quotient G, which we will define to be our map $\widetilde{\varphi}:G\to\mathbb{Z}\times\mathbb{Z}$. We pick loops $\lambda_i\subset\mathbb{D}^A_{2n}\setminus\Delta_0$ and $\ell_i\subset\mathbb{D}^B_{n+1}\setminus\Lambda$ as in Figure 1.2.4.1.

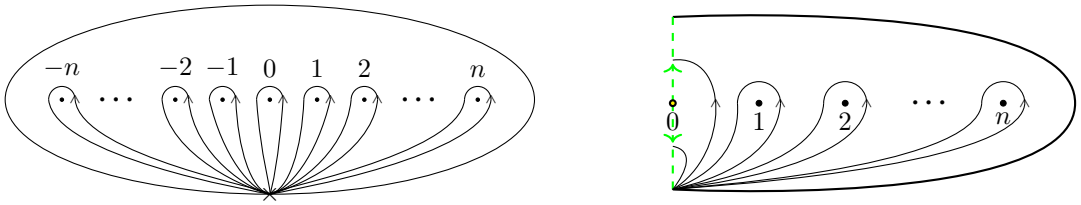


Figure 1.2.4.1: The loops chosen for the fundamental groups of $\mathbb{D}^A_{2n}\setminus\Delta_0$ (left) and $\mathbb{D}^B_{n+1}\setminus\Lambda$ (right).

The induced map $q_*:\pi_1(\mathbb{D}^A_{2n}\setminus\Delta_0)\to\pi_1(\mathbb{D}^B_{n+1}\setminus\Lambda)$ on the fundamental groups satisfies

$$q_*([\lambda_j])=\begin{cases}[\ell_0\circ\ell_0], & \text{for } j=0;\\ [(\ell_0\ell_1\cdots\ell_{j-1})\ell_{|j|}(\ell_{j-1}^{-1}\cdots\ell_1^{-1}\ell_0^{-1})], & \text{for } -n\le j\le -1;\\ [\lambda_j], & \text{for } 1\le j\le n.\end{cases}$$

Now define $\varphi:\pi_1(\mathfrak{D}^B_\Lambda)\to\pi_1(\mathfrak{D}^A_{\Delta_0})$ by sending

$$\begin{cases}[point\times\ell_0]\mapsto[point\times\lambda_0];\\ [point\times\ell_j]\mapsto[point\times\lambda_j], & \text{for all } j\in\{1,...,n\},\ \text{and};\\ [\mathbb{R}\mathrm{P}^1\times point]\mapsto[\mathbb{R}\mathrm{P}^1\times point].\end{cases}$$

We claim that $C_0\circ\varphi\circ\mathfrak{q}_*=C_0$. Firstly, note that $C_0\circ\varphi\circ\mathfrak{q}_*([point\times\lambda_0])=0=C_0([point\times\lambda_0])$, and $\varphi\circ\mathfrak{q}_*([\mathbb{R}\mathrm{P}^1\times point])=[\mathbb{R}\mathrm{P}^1\times point]$ by construction. Moreover, for $j\in\{-n,...,-1,1,...,n\}$,

$$(\varphi\circ\mathfrak{q}_*)[point\times\lambda_j]=$$
$$\begin{cases}[point\times\lambda_0\lambda_1\cdots\lambda_{j-1}][point\times\lambda_{|j|}][point\times\lambda_0\lambda_1\cdots\lambda_{j-1}]^{-1}, & \text{for } -n\le j\le -1;\\ [point\times\lambda_j], & \text{for } 1\le j\le n.\end{cases}$$

Since C_0 maps to $\mathbb{Z}\times\mathbb{Z}$, which is abelian, we have that, from (1.2.8),

$$(C_0\circ\varphi\circ\mathfrak{q}_*)[point\times\lambda_j]=C_0[point\times\lambda_{|j|}]=C_0[point\times\lambda_j].$$

This shows that $C_0\circ\varphi\circ\mathfrak{q}_*$ and C_0 agrees on all generators of $\pi_1(\mathfrak{D}^A_{\Delta_0})$, and so are equal. This implies that

$$(C_0\circ\varphi\circ\mathfrak{q}_*\circ\mathfrak{p}_*)\left(\pi_1(\widetilde{\mathfrak{D}}^A_{\Delta_0})\right)=(C_0\circ\mathfrak{p}_*)\left(\pi_1(\widetilde{\mathfrak{D}}^A_{\Delta_0})\right)=0.$$

As such, $C_0\circ\varphi$ factors uniquely through the quotient G, and we denote this map by $\widetilde{\varphi}:G\to\mathbb{Z}\times\mathbb{Z}$. By definition, $\widetilde{\mathfrak{q}_*}$ is uniquely determined by the images of $[point\times\lambda_1],[\mathbb{R}P^1\times point]\in\pi_1(\mathfrak{D}^A_{\Delta_0})$ under $\mathfrak{q}_*$. It is easy to see now that $\widetilde{\varphi}\circ\widetilde{\mathfrak{q}_*}=\mathrm{id}$ by the construction of $\widetilde{\varphi}$. Q.E.D.

Note that every $f\in\mathrm{Diff}(\mathbb{D}^B_{n+1},\{0\})$ preserves the class C_0 and therefore can be lifted to a unique equivariant diffeomorphism $\check{f}$ of $\widetilde{\mathfrak{D}}^A_{\Delta_0}$ that acts trivially on the fibre of $\widetilde{\mathfrak{D}}^A_{\Delta_0}$over all points in $T_z\mathbb{D}^B_{n+1}$ for

$z \in \partial\mathbb{D}^B_{n+1}$. Furthermore, every curve c in $\mathbb{D}^B_{n+1}$ admits a canonical section $s_c : c\backslash\Lambda \to \mathfrak{D}^A_{\Delta_0}$ defined by $s_c(z) = T_z c$. We define a *trigrading* of c to be a lift $\check{c}$ of s_c to $\widetilde{\mathfrak{D}}^A_{\Delta_0}$ and a *trigraded curve* to be a pair $(c, \check{c})$ consisting a curve and its trigrading; we will often just write $\check{c}$ instead of $(c, \check{c})$ when the context is clear. We denote the $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-action on $\widetilde{\mathfrak{D}}^A_{\Delta_0}$ by χ^B. On top of that, we can easily extend the notion of isotopy to the set of trigraded curves where χ^B and $\mathrm{MCG}\left(\mathbb{D}^B_{n+1}, \{0\}\right)$ have induced actions on the set of isotopy classes of trigraded curves. In particular, for $[f] \in \mathcal{A}(B_n) \cong \mathrm{MCG}\left(\mathbb{D}^B_{n+1}, \{0\}\right)$ and a trigraded curve $\check{c}$, $[\check{f}](\check{c}) := \check{f} \circ \check{c} \circ f^{-1} : f(c) \setminus \Lambda \to \widetilde{\mathfrak{D}}^A_{\Delta_0}$.

Lemma 1.2.3.

1. *A curve c admits a trigrading if and only if it is not a simple closed curve.*
2. *The $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-action on the set of isotopy classes of trigraded curves is free. Equivalently, a trigraded curve $\check{c}$ is never isotopic to $\chi(r_1, r_2, r_3)\check{c}$ for any $(r_1, r_2, r_3) \neq 0$.*

Proof. This is essentially the same proof as in [KS02, Lemma 3.12 and 3.13]. Q.E.D.

Lemma 1.2.4.

1. *Let c be a curve in $\mathbb{D}^B_{n+1}$ which joins two points of $\Lambda\backslash\{0\}$, $t_c \in MCG\left(\mathbb{D}^B_{n+1}, \{0\}\right)$ the half twist along it, and $\check{t}_c$ its preferred lift to $\widetilde{\mathfrak{D}}^A_{\Delta_0}$. Then, $\check{t}_c(\check{c}) = \chi^B(-1, 1, 0)\check{c}$ for any trigrading $\check{c}$ of c.*
2. *Let c be a curve in $\mathbb{D}^B_{n+1}$ which joins two points of Λ with one of them being $\{0\}$, $t_c \in \mathcal{G}^B$ the full twist along it, and $\check{t}_c$ its preferred lift to $\widetilde{\mathfrak{D}}^A_{\Delta_0}$. Then, $\check{t}_c(\check{c}) = \chi^B(-1, 1, 1)\check{c}$ for any trigrading $\check{c}$ of c.*

Proof. The proof of (1) is as in [KS02, Lemma 3.14]. We will now prove (2). Let $\beta : [0, 1] \to \mathbb{D}^B_{n+1} \setminus \Lambda$ be an embedded vertical smooth path from a point $\beta(0) \in \partial\mathbb{D}^B_{n+1}$ to the fixed point $\beta(1) \in c$ of t_c. Note that we have $\check{t}_c(\check{c}) = \chi(r_1, r_2, r_3)\check{c}$ as $t_c(c) = c$. Consider the closed path $\kappa : [0, 2] \to \mathfrak{D}^A_{\Delta_0}$ given by

$$\kappa(t) = \begin{cases} Dt_c(\mathbb{R}\beta'(t)), & \text{if } t \leq 1; \\ \mathbb{R}\beta'(2-t), & \text{if } t \geq 1, \end{cases}$$

where $\mathbb{R}\beta'(s) \subset T_{\beta(s)}\mathbb{D}^A_{2n}$. The above situation is illustrated in Figure 1.2.4.2. Then, we compute $(r_1, r_2, r_3) = C([\kappa]) = C([\mathbb{RP}^1 \times \mathit{points}]) + C([\mathit{points} \times \lambda_0] + C([\mathit{points} \times \lambda_j]) = (-1, 1, 1)$. Note that $[\kappa]$ only picks up one copy of $[\mathbb{RP}^1 \times \mathit{points}]$ due to the nature of the disc $\mathbb{D}^B_{n+1}$. Q.E.D.

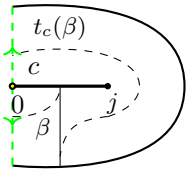


Figure 1.2.4.2: The action of full twist around curve joining $\{0\}$ and another point in Λ.

1.2.5 Local index and trigraded intersection numbers

Mimicking the definition of local index for bigraded curves in [KS02, pg. 225], we shall define the local index of an intersection between two trigraded curves. Suppose $(c_0, \check{c}_0)$ and $(c_1, \check{c}_1)$ are two trigraded curves, and $z \in \mathbb{D}^B_{n+1} \setminus \partial\mathbb{D}^B_{n+1}$ is a point where c_0 and c_1 intersect transversally. Take a small circle $\ell \subset \mathbb{D}^B_{n+1} \setminus \Lambda$ around z and an embedded arc $\alpha : [0, 1] \to \ell$ which moves clockwise around ℓ such that $\alpha(0) \in c_0$ and $\alpha(1) \in c_1$ and $\alpha(t) \notin c_0 \cup c_1$ for all $t \in (0, 1)$. If $z \in \Delta$, then α is unique up to a change of parametrisation, otherwise, there are two choices which can be told apart by their endpoints. Then, take a smooth path $\kappa : [0, 1] \to \mathfrak{D}^B_\Lambda$ with $\kappa(t) \in \left(\mathfrak{D}^B_\Lambda\right)_{\alpha(t)}$ for all t, going from $\kappa(0) = T_{\alpha(0)}c_0$ to $\kappa(1) = T_{\alpha(1)}c_1$ such that $\kappa(t) = T_{\alpha(t)}\ell$ for every t.

One can take κ as a family of tangent lines along α which are all transverse to ℓ. After that, lift κ to a path $\check{\kappa} : [0,1] \to \check{\mathfrak{D}}^B_\Lambda$ with $\check{\kappa}(0) = \check{c}_0(\alpha(0))$; subsequently, there exists some $(\mu_1, \mu_2, \mu_3) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ such that

$$\check{c}_1(\alpha(1)) = \chi^B(\mu_1, \mu_2, \mu_3)\check{\kappa}(1), \tag{1.2.10}$$

as $\check{c}_1(\alpha(1))$ and $\check{\kappa}(1)$ are the lift of the same point in $\mathfrak{D}^B_\Lambda$. To this end, we define the local index of $\check{c}_0, \check{c}_1$ at z as

$$\mu^{trigr}(\check{c}_0, \check{c}_1; z) = (\mu_1, \mu_2, \mu_3) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

It is easy to see that the definition is independent of all the choices made.

The local index has a nice symmetry property similar to [KS02, pg. 227], namely:

Lemma 1.2.5. *If $(c_0, \check{c}_0)$ and $(c_1, \check{c}_1)$ are two trigraded curves such that c_0 and c_1 have minimal intersection, then*

$$\mu^{trigr}(\check{c}_1, \check{c}_0; z) = \begin{cases} (1,0,0) - \mu^{trigr}(\check{c}_0, \check{c}_1; z), & \text{if } z \notin \Delta; \\ (0,1,0) - \mu^{trigr}(\check{c}_0, \check{c}_1; z), & \text{if } z \in \Delta \backslash \{0\}; \\ (1,0,1) - \mu^{trigr}(\check{c}_0, \check{c}_1; z), & \text{if } z \in \{0\}. \end{cases}$$

Proof. The first two formulae are essentially the same as in [KS02, pg. 227] and can be proven in a similar fashion, which we omit the details. The third formula can be verified using Figure 1.2.5.1. Q.E.D.

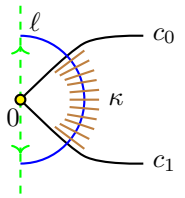


Figure 1.2.5.1: Two curves c_0, c_1 intersecting at $\{0\}$.

Let $\check{c}_0$ and $\check{c}_1$ be two trigraded curves that do not intersect at $\partial\mathbb{D}^B_{n+1}$. Pick a curve $c_1' \simeq c_1$ which intersects minimally with c_0. Then, by Lemma 1.2.3, c_1' has a unique trigrading $\check{c}_1'$ of c_1' so that $\check{c}_1' \simeq \check{c}_1$. We define the *trigraded intersection number* $I^{trigr}(\check{c}_0, \check{c}_1) \in \mathbb{Z}[q_1^{\pm 1}, q_2 \pm 1, q_3]/\langle q_3^2 - 1\rangle$ of $\check{c}_0$ and $\check{c}_1$ as follows:

- if $\check{c}_1 \simeq \chi(r_1, r_2, r_3)\check{c}_0$ with $(r_1, r_2, r_3) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $c_0 \cap c_1 \cap \{0\}$ non-empty, then

$$I^{trigr}(\check{c}_0, \check{c}_1) = q_1^{r_1} q_2^{r_2} q_3^{r_3}(1 + q_2); \tag{1.2.11}$$

- otherwise

$$\begin{aligned} I^{trigr}(\check{c}_0, \check{c}_1) = (1+q_3)(1+q_1^{-1}q_2) &\sum_{z \in (c_0 \cap c_1') \backslash \Delta} q_1^{\mu_1(z)} q_2^{\mu_2(z)} q_3^{\mu_3(z)} \\ + (1+q_3) &\sum_{z \in (c_0 \cap c_1') \cap \Delta \backslash \{0\}} q_1^{\mu_1(z)} q_2^{\mu_2(z)} q_3^{\mu_3(z)} \\ + (1+q_1^{-1}q_2q_3) &\sum_{z \in (c_0 \cap c_1') \cap \{0\}} q_1^{\mu_1(z)} q_2^{\mu_2(z)} q_3^{\mu_3(z)}. \end{aligned} \tag{1.2.12}$$

The fact that this definition is independent of the choice of c_1' and is an invariant of the isotopy classes of $(\check{c}_0, \check{c}_1)$ follows similarly as in the case of ordinary geometric intersection numbers.

Lemma 1.2.6. *The trigraded intersection number has the following properties:*

(T1) For any $f \in Diff\left(\mathbb{D}^B_{n+1}, \{0\}\right)$, $I^{trigr}(\check{f}(\check{c}_0), \check{f}(\check{c}_1)) = I^{trigr}(\check{c}_0, \check{c}_1)$.

(T2) For any $(r_1, r_2, r_3) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$,
$I^{trigr}(\check{c}_0, \chi(r_1, r_2, r_3)\check{c}_1) = I^{trigr}(\chi(-r_1, -r_2, r_3)\check{c}_0, \check{c}_1) = q_1^{r_1} q_2^{r_2} q_3^{r_3} I^{trigr}(\check{c}_0, \check{c}_1)$.

(T3) *If c_0, c_1 are not isotopic curves with $\{0\}$ as one of its endpoints, $c_0 \cap c_1 \cap \partial\mathbb{D}^B_{n+1} = \emptyset$, and $I^{trigr}(\check{c}_0, \check{c}_1) = \sum_{r_1,r_2,r_3} a_{r_1,r_2,r_3} q_1^{r_1} q_2^{r_2} q_3^{r_3}$, then $I^{trigr}(\check{c}_1, \check{c}_0) = \sum_{r_1,r_2,r_3} a_{r_1,r_2,r_3} q_1^{-r_1} q_2^{1-r_2} q_3^{r_3}$.*

If c_0, c_1 are isotopic curves with $\{0\}$ as one of its endpoints, $c_0 \cap c_1 \cap \partial\mathbb{D}^B_{n+1} = \emptyset$, then $I^{trigr}(\check{c}_1, \check{c}_0) = I^{trigr}(\check{c}_0, \check{c}_1)$.

Proof. For (T1) and (T2) these can be proven using a simple topological argument which we omit. For (T3), this is a consequence of Lemma 1.2.5. We point out that the term $(1+q_1^{-1}q_2q_3)$ in the definition of trigraded intersection numbers for two curves that intersect at the point $\{0\}$ is essential for property (T3). Q.E.D.

1.2.6 Admissible curves and normal form in $\mathbb{D}^B_{n+1}$

Following [KS02, Section 3b and 3e], we introduce the notion of (trigraded) admissible curves in $\mathbb{D}^B_{n+1}$ and their normal forms. Other than the extra consideration of trigradings, the main difference lies in the normal forms of (trigraded) admissible curves in the region containing the marked point 0; see Figure 1.2.6.4.

We fix the set of basic curves $b_1, \ldots, b_n$ and choose vertical curves $d_1, \ldots, d_n$ as in Figure 1.2.6.1, which divide $\mathbb{D}^B_{n+1}$ into regions $D_0, \ldots, D_{n+1}$. Note that unlike in [KS02], none of our basic curves touches the boundary of the disc $\mathbb{D}^B_{n+1}$.

A curve c is called ***admissible*** if it is equal to $f(b_j)$ for some $0 \leq j \leq n$ and some diffeomorphisms $f \in \mathrm{Diff}\left(\mathbb{D}^B_{n+1}, \{0\}\right)$. Note that the endpoints of c must then lie in $\{0, \ldots, n\}$; conversely, every curves which start and end at $\{0, \ldots, n\}$ are admissible. Moreover, the two (distinct) orbits $\mathcal{O}([b_1])$ and $\mathcal{O}([b_2])$ under the action of $\mathcal{A}(B_n) \cong MCG(\mathbb{D}^B_{n+1}, \{0\})$ partition the set of isotopy classes of admissible curves.

If an admissible curve c in its isotopy class has minimal intersection with all the d_j's among its other representatives, then we say that c is in ***normal form***. A normal form of c is always achievable by performing an isotopy.

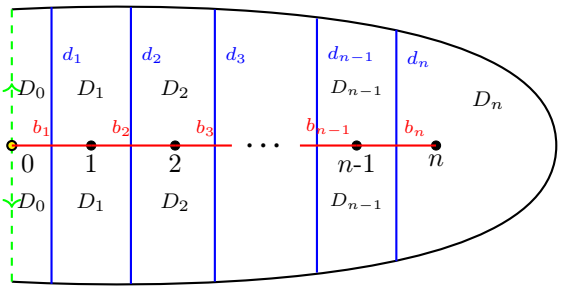


Figure 1.2.6.1: The curves b_i and d_i in the aligned configuration with regions D_i.

Let c be an admissible curve in normal form. We use the same classification as in [KS02, Section 3e] to group every connected components of $c \cap D_j$ into finitely many types. For $1 \leq j \leq n$, the classification is exactly the same as in [KS02, Section 3e]: there are six types for the case $1 \leq j \leq n-1$ as depicted in Figure 1.2.6.2; whereas for $j = n$, there are two types as shown in Figure 1.2.6.3. At $j = 0$, we have two possible types as depicted in Figure 1.2.6.4, where they are drawn slightly different due to the nature of D_0; compare Type 2' in Figure 1.2.6.2 and Type 2" in Figure 1.2.6.4[1]. Note that an admissible curve c intersecting all the d_j transversely with each connected component of $c \cap D_j$ belonging to Figure 1.2.6.2, Figure 1.2.6.3, and Figure 1.2.6.4 is already in normal form.

For the rest of this section, c will be an admissible curve in normal form. We call the intersections of c with the curves d_i ***crossings*** and denote them $cr(c) = c \cap (d_0 \cup d_1 \cup \ldots \cup d_{n-1})$. Those intersections $c \cap d_j$ are called ***j-crossings of c***. For $0 \leq j \leq n$, the connected components of $c \cap D_j$ are called ***segments*** of c. If the endpoints of a segment are both crossings, then it is ***essential***.

[1] Technically, the difference between Type 2' and Type 2" lies in the their trigradings when we consider trigraded curves later on.

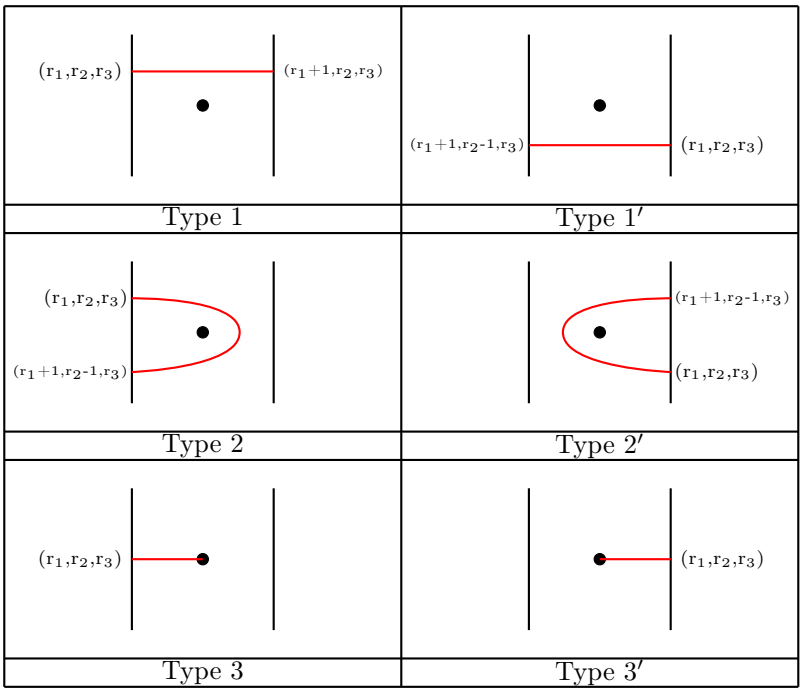


Figure 1.2.6.2: The six possible types of connected components $c \cap D_j$, for c in normal form and $1 \leq j < n$.

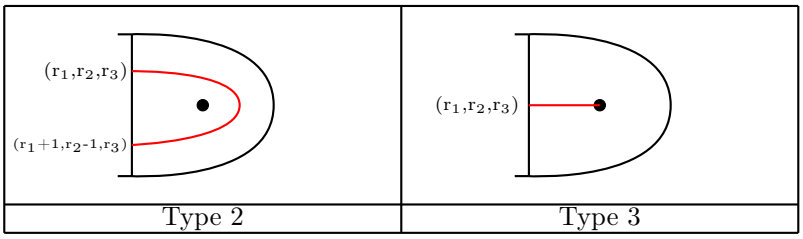


Figure 1.2.6.3: The two possible types of connected components $c \cap D_n$.

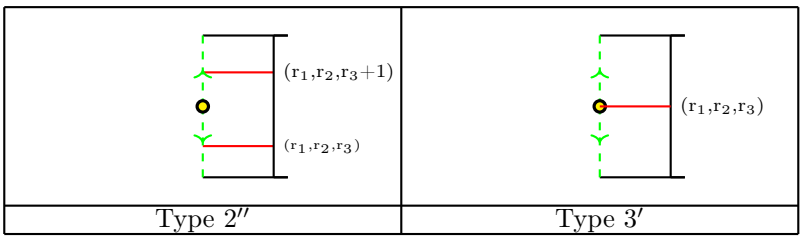


Figure 1.2.6.4: The two possible types of connected components $c \cap D_0$.

Now, we will study the action of half twist $t^B_{b_k}$ on normal forms. In gerenal, $t^B_{b_k}(c)$ won't be in normal form even though c is a normal form. Nonetheless, $t^B_{b_k}(c)$ has minimal intersection with all d_j for $j \neq k$. In order to get $t^B_{b_k}(c)$ into a normal form, one just need to isotope it so that its intersections with d_k is minimal. The same argument used in [KS02, Proposition 3.17] gives us the following analogous result:

Proposition 1.2.7.

1. *The normal form of $t^B_{b_k}(c)$ coincides with c outside of $D_{k-1} \cup D_k$. The curve $t^B_{b_k}(c)$ can be brought into normal form by an isotopy inside $D_{k-1} \cup D_k$.*

2. *Suppose that $t^B_{b_k}(c)$ is in normal form. There is a natural bijection between j-crossings of c and the j-crossings of $t^B_{b_k}(c)$ for $j \neq k$. There is a natural bijection between connected components of intersections of c and $t^B_{b_k}(c)$ inside $D_{k-1} \cup D_k$.*

A connected component of $c \cap (D_{j-1} \cup D_j)$ is called ***j-string of*** c. Denote by $st(c, j)$ the set of j-string of c. In addition, we define a ***j-string*** as a curve in $D_{j-1} \cup D_j$ which is a j-string of c for some admissible curve c in normal form.

Two j-strings are isotopic (equivalently belong to the same isotopy class) if there exists a deformation of one into the other via diffeomorphisms f of $D' = D_{j-1} \cup D_j$ which fix d_{j-1} and d_{j+1} as well as preserves the marked points in D' pointwise , that is, $f(d_{j-1}) = d_{j-1}, f(d_{j+1}) = d_{j+1}$, and $f|_{\Delta \cap D'} = id$.

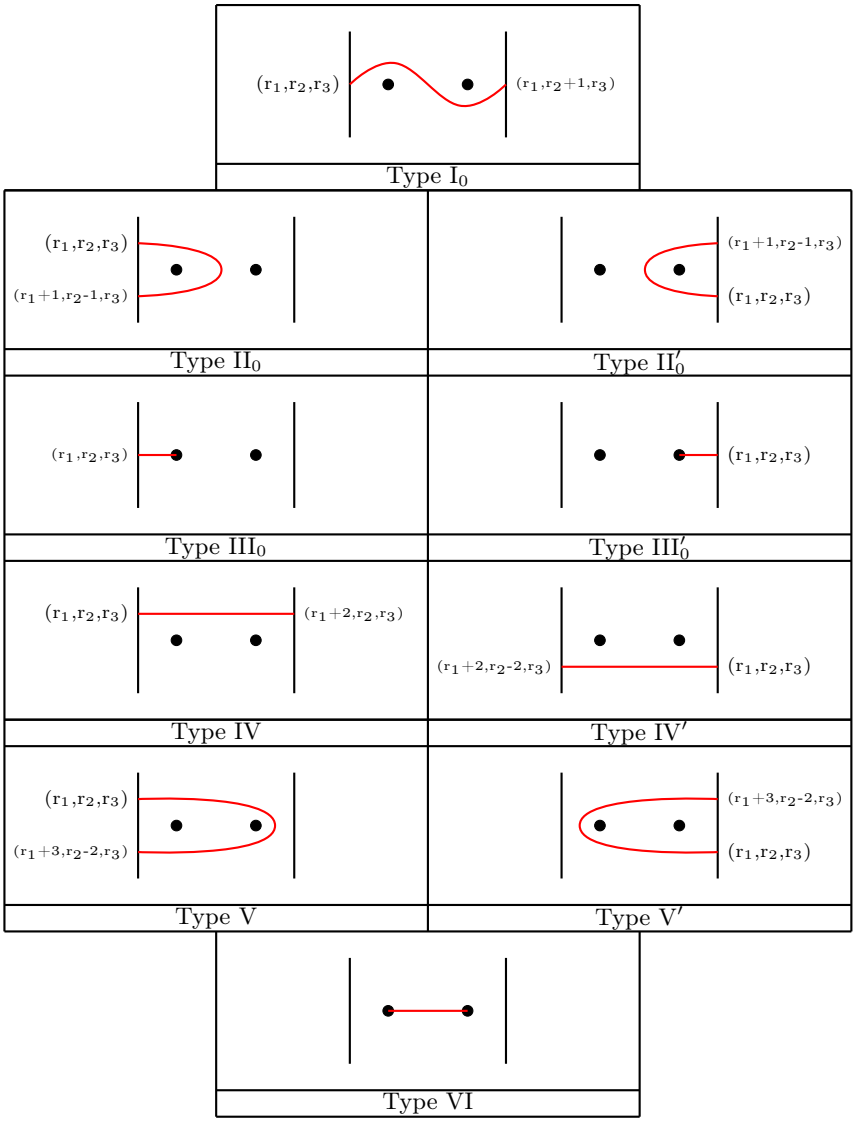


Figure 1.2.6.5: The isotopy classes of j-strings, for $1 < j < n$.

For $1 < j < n$, isotopy classes of j-strings can be divided into types as follows: there are five infinite families $I_w, II_w, II'_w, III_w, III'_w (w \in \mathbb{Z})$ and five exceptional types IV, IV', V, V' and VI (see Figure 1.2.6.5). When $j = n$, there is a similar list, with two infinite families and two exceptional types (see Figure 1.2.6.6). The rule for obtaining the $(w + 1)$-th from the w-th is by applying $t^B_{b_j}$. For 1-string, there are instead four infinite-family types: $II'_w, II'_{w+\frac{1}{2}}, III'_w, III'_{w+\frac{1}{2}} (w \in \mathbb{Z})$ and two exceptional types V'' and VI (see Figure 1.2.6.7). As for segments of curves, these are drawn slightly different due to the nature of the disc; compare Type V' in Figure 1.2.6.5 and type V'' in Figure 1.2.6.7. Note that for 1-strings, the rule for

obtaining the $(w+1)$-th from the w-th is instead by applying $(t^B_{b_1})^2$.

Based on our definition, j-strings are assumed to be in normal form. As before, we can define *crossings* and *essential segments* of j-string as in the case for admissible curves in normal form and denote the set of crossings of a j-string g by $cr(g)$.

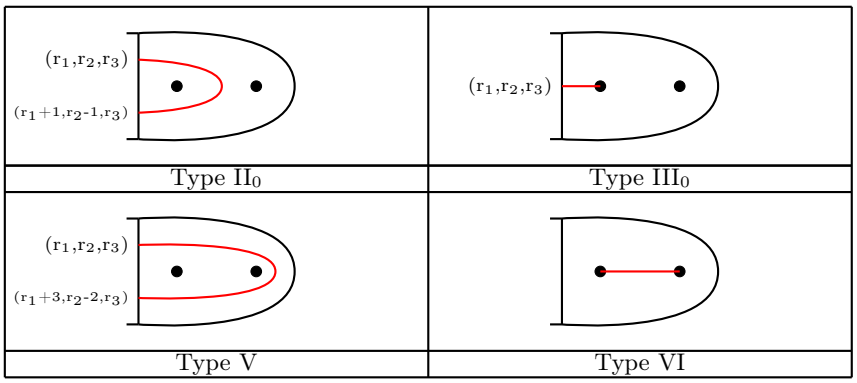


Figure 1.2.6.6: The isotopy classes of n-strings.

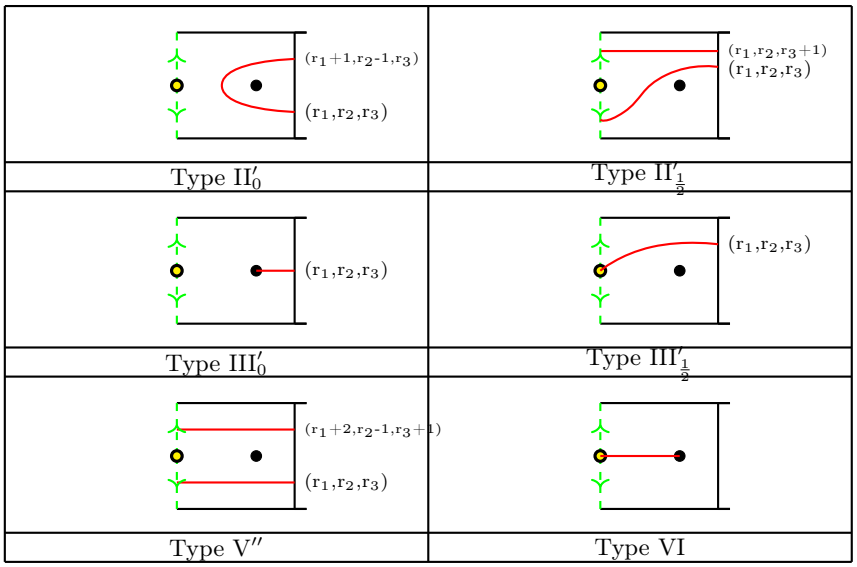


Figure 1.2.6.7: The isotopy classes of 1-strings.

Now, let us adapt the discussion to trigraded curves. Choose trigradings $\check{b}_j, \check{d}_j$ of b_j, d_j for $1 \leq j \leq n$, such that

$$I^{trigr}(\check{d}_j, \check{b}_j) = (1+q_3)(1+q_1^{-1}q_2), \qquad I^{trigr}(\check{b}_j, \check{b}_{j+1}) = 1+q_3. \tag{1.2.13}$$

These conditions determine the trigradings uniquely up to an overall shift $\chi^B(r_1, r_2, r_3)$.

Suppose $\check{c}$ is a trigrading of an admisible curve c in normal form. If $a \subset c$ is a connected component of $c \cap D_j$ for some j, and $\check{a}$ is $\check{c}|_{a \setminus \Lambda}$, then $\check{a}$ is evidently determined by a together with the local index $\mu^{trigr}(\check{d}_{j-1}, \check{a}; z)$ or $\mu^{trigr}(\check{d}_j, \check{a}; z)$ at any point $z \in (d_{j-1} \cup d_j) \cap a$. Moreover, if there is more than one such point, the local indices determine each other.

In Figure 1.2.6.2, Figure 1.2.6.3, and Figure 1.2.6.4, we classify the types of pair $(a, \check{a})$ with the local indices. For instance, consider the Type $1(r_1, r_2, r_3)$ with $(k-1)$-crossing $z_0 \in d_{k-1} \cap a$ and k-crossing $z_1 \in d_k \cap a$. We have that the local index at z_0 and at z_1 are $\mu^{trigr}(\check{d}_{k-1}, \check{a}; z_0) = (r_1, r_2, r_3)$ and $\mu^{trigr}(\check{d}_k, \check{a}; z_1) = (r_1+1, r_2, r_3)$ respectively.

We recall from Section 1.2.4, that there is a unique lift $\check{f}$ that is the identity on the boundary for any diffeomorphism $f \in \mathrm{Diff}\left(\mathbb{D}^B_{n+1}, \{0\}\right)$ to a diffeomorphism of $\widetilde{\mathfrak{D}}^A_{\Delta_0}$. Similarly, we can lift mapping classes in

$\mathbb{D}^B_{n+1}$ to mapping classess in $\widetilde{\mathfrak{D}}^A_{\Delta_0}$. Denote by $\check{t}^A_{b_j}$ the canonical lift of the twist $t^A_{b_j}$ along the curve b_j in $\mathbb{D}^A_{2n}$.

Proposition 1.2.8. *By considering $\widetilde{\mathfrak{D}}^A_{\Delta_0}$ as a subset of $\widetilde{\mathfrak{D}}^A_{\Delta}$, the map $\psi : \mathcal{A}(B_n) \to MCG\left(\widetilde{\mathfrak{D}}^A_{\Delta_0}\right)$ defined by*

$$\psi(\sigma^B_i) = \begin{cases} [\check{t}^A_{b_n}|_{\widetilde{\mathfrak{D}}^A_{\Delta_0}}], & \text{for } i = 1; \\ [\check{t}^A_{b_{n+i-1}} \circ \check{t}^A_{b_{n-(i-1)}}|_{\widetilde{\mathfrak{D}}^A_{\Delta_0}}], & \text{for } i \geq 2, \end{cases}$$

is an injective group homomorphism.

Proof. This can be proven similar to Proposition 1.2.1. Q.E.D.

A crossing of c will be also a *crossing of* $\check{c}$, and we denote the set of crossings of $\check{c}$ by $cr(\check{c})$. Note that as set, $cr(\check{c}) = cr(c)$. However, a crossing of $\check{c}$ comes with a local index in $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Moreover, to each crossing y of $\check{c}$ we assign a 4-tuple (y_0, y_1, y_2, y_3) where y_0 denotes the index of the vertical curve which contains the crossing $y \in d_{y_0} \cap c$, and (y_1, y_2, y_3) is the local index (μ_1, μ_2, μ_3) of the crossing y.

We define the *essential segments* of $\check{c}$ as the essential segments of c together with the trigradings which can be obtained from local indices assigned to the ends of the segments.

We also define a *j-string of* $\check{c}$ as a connected component of $\check{c} \cap (D_{j-1} \cup D_j)$ together with the trigrading induced from $\check{c}$. Denote the set of j-string of $\check{c}$ by $st(\check{c}, j)$.

On top of that, we define a trigraded j-string as a trigraded curve in $D_{j-1} \cup D_j$ that is a connected component of $\check{c} \cap (D_{j-1} \cup D_j)$ for some trigraded curve $\check{c}$.

In Figure 1.2.6.5, Figure 1.2.6.6, and Figure 1.2.6.7 we depict the isotopy classes of trigraded j-strings. Since j-strings of type VI do not intersect with $d_{j-1} \cup d_{j+1}$, we say that a trigraded j-string $\check{g}$ with the underlying j-string g of type VI has type $VI(r_1, r_2, r_3)$ if $\check{g} = \chi^B(r_1, r_2, r_3)\check{b}_j$.

The next crucial lemma is the type B analogue of [KS02, Lemma 3.20], allowing the computation of trigraded intersection numbers between $\check{b}_j$ and any given trigraded curve.

Lemma 1.2.9. *Let $(c, \check{c})$ be a trigraded curve. Then, $I^{trigr}(\check{b}_j, \check{c})$ can be computed by adding up contributions from each trigraded j-string of $\check{c}$. For $j > 0$, the contributions are listed in the following table:*

$I_0(0,0,0)$	$II_0(0,0,0)$	$II'_0(0,0,0)$	$III_0(0,0,0)$
$q_1 + q_2 + q_2q_3 + q_1q_3$	$q_1 + q_2 + q_2q_3 + q_1q_3$	$1 + q_1q_2^{-1} + q_3 + q_1q_2^{-1}q_1q_3$	$q_2 + q_2q_3$

$III'_0(0,0,0)$	IV	IV'	V	V'	$VI(0,0,0)$
$1 + q_3$	0	0	0	0	$1 + q_2 + q_3 + q_2q_3$

and the remaining ones can be computed as follows: to determine the contribution of a component of type, say, $I_u(r_1, r_2, r_3)$, one takes the contribution of $I_0(0,0,0)$ and multiplies it by $q_1^{r_1}q_2^{r_2}q_3^{r_3}(q_1^{-1}q_2)^u$. For $j = 0$, the relevant contributions are

$II'_0(0,0,0)$	$II'_{\frac{1}{2}}(0,0,0)$	$III'_0(0,0,0)$	$III'_{\frac{1}{2}}(0,0,0)$	V''	$VI(0,0,0)$
$1 + q_3 + q_1q_2^{-1} + q_1q_2^{-1}q_3$	$1 + q_3 + q_1^{-1}q_2 + q_1^{-1}q_2q_3$	$1 + q_3$	$q_1^{-1}q_2 + q_3$	0	$1 + q_2$

and the remaining ones can be computed as follows: to determine the contribution of a component of type, say, $II'_u(r_1, r_2, r_3)$, one takes the contribution of $II'_0(0,0,0)$ and multiplies it by $q_1^{r_1}q_2^{r_2}q_2^{r_2}(q_1^{-1}q_2q_3)^u$.

Proof. Apply Lemma 1.2.4 as well as $(T2)$ and $(T3)$ of Lemma 1.2.6. Q.E.D.

1.2.7 Bigraded curves and bigraded multicurves in $\widetilde{\mathfrak{D}}^A_{\Delta_0}$

We briefly remind the reader the definition of a bigraded curve in $\widetilde{\mathfrak{D}}^A_{\Delta_0}$; refer to [KS02, Section 3d] for a more detailed construction. Consider the projectivisation $\mathfrak{D}^A_{\Delta} := PT\left(\mathbb{D}^A_{2n} \setminus \Delta\right)$ of the tangent bundle of $\mathbb{D}^A_{2n} \setminus \Delta$.

The covering $\widetilde{\mathfrak{D}}^A_\Delta$ of $\mathfrak{D}^A_\Delta$ is classified by the cohomology class $C \in H^1(\mathfrak{D}^A_\Delta; \mathbb{Z}\times\mathbb{Z})$ defined as follows:

$$C([point \times \lambda_i]) = (-2,1), \quad \text{for } i = -n, \cdots, -1, 1, \cdots, n; \tag{1.2.14}$$

$$C([\mathbb{RP}^1 \times point]) = (1,0). \tag{1.2.15}$$

A bigrading of a curve $c \in \mathbb{D}^A_{2n}$ is a lift $\ddot{c}$ of s^A_c to $\widetilde{\mathfrak{D}}^A_\Delta$ where $s^A_c : c \setminus \Delta \to \mathfrak{D}^A_\Delta$ is the canonical section given by $s^A_c(z) = T_z c$. A *bigraded curve* is a pair $(c, \ddot{c})$, where sometimes we abbreviate as $\ddot{c}$.

A *bigraded multicurve* $\ddot{\mathfrak{c}}$ is a union of a finite collection of disjoint bigraded curves. There is an obvious notion of isotopy for bigraded multicurves.

1.2.8 Lifting of trigraded curves to bigraded multicurves

Our goal is to define a map $\mathfrak{m} : \check{\mathfrak{C}} \to \ddot{\underset{\sim}{\mathfrak{C}}}$ from the set $\check{\mathfrak{C}}$ of isotopy classes of trigraded curves to the set $\ddot{\underset{\sim}{\mathfrak{C}}}$ of isotopy classes of bigraded multicurves. Let c be a curve in $\mathbb{D}^B_{n+1}$ with trigrading $\check{c}$. First consider the case when $c \cap \{0\} = \emptyset$. Recall the map $q_{br} : \mathbb{D}^A_{2n} \to \mathbb{D}^B_{n+1}$ as defined in Section 1.2.2. Then, $q^{-1}_{br}(c)$ has two connected components in $\mathbb{D}^A_{2n}$; denote them as $\widetilde{c}, \underset{\sim}{c}$, such that $\widetilde{c} \setminus \Delta = \pi^A \circ \mathfrak{p} \circ \check{c}(c \setminus \Lambda)$ and $\underset{\sim}{c} \setminus \Delta = \pi^A \circ \mathfrak{p} \circ \chi^B(0,0,1)\check{c}(c \setminus \Lambda)$. Define $\ddot{\widetilde{c}} : \widetilde{c} \setminus \Delta \to \widetilde{\mathfrak{D}}^A_\Delta$ as $\ddot{\widetilde{c}} := \widetilde{\mathscr{F}} \circ \check{c} \circ q_{br}|_{\widetilde{c}\setminus\Delta}$; similarly $\ddot{\underset{\sim}{c}} : \underset{\sim}{c} \setminus \Delta \to \widetilde{\mathfrak{D}}^A_\Delta$ is defined by $\ddot{\widetilde{c}} := \widetilde{\mathscr{F}} \circ \chi^B(0,0,1)\check{c} \circ q_{br}|_{\widetilde{c}\setminus\Delta}$ where $\widetilde{\mathscr{F}} : \widetilde{\mathfrak{D}}^A_{\Delta_0} \to \widetilde{\mathfrak{D}}^A_\Delta$ is the unique map induced by the inclusion $\mathscr{F} : \mathfrak{D}^A_{\Delta_0} \to \mathfrak{D}^A_\Delta$. It is easy to check that these are indeed bigradings of their respective curves. On the other hand, if c contains 0 as one of its endpoints,we define $\widetilde{\underset{\sim}{c}} := \widetilde{c \setminus \{0\}} \amalg \{0\} \amalg \underbrace{c \setminus \{0\}}$, which is just a single connected component. Furthermore, the $\ddot{\underset{\sim}{c}}$ is defined to be the unique continuous extension of $\ddot{\widetilde{c \setminus \{0\}}} \amalg \underbrace{\ddot{c \setminus \{0\}}}$, which is again an easy verification that it is a bigrading of $\widetilde{\underset{\sim}{c}}$.

In total, we define the map $\mathfrak{m} : \check{\mathfrak{C}} \to \ddot{\underset{\sim}{\mathfrak{C}}}$ as follows: for a trigraded curve $(c, \check{c})$ in $\widetilde{\mathfrak{D}}^A_{\Delta_0}$,

$$\mathfrak{m}((c,\check{c})) := \begin{cases} (\widetilde{c}, \ddot{\widetilde{c}}), & \text{if } c \text{ has } \{0\} \text{ as one of its endpoints;} \\ (\widetilde{c}, \ddot{\widetilde{c}}) \amalg (\underset{\sim}{c}, \ddot{\underset{\sim}{c}}), & \text{otherwise.} \end{cases}$$

Due to the isotopy lifting property of the space, $\mathfrak{m}$ is well-defined on the isotopy classes of trigraded curves.

Recall the natural induced action of $\mathcal{A}(B_n) \cong \mathrm{MCG}(\mathbb{D}^B_{n+1}, \{0\})$ on $\check{\mathfrak{C}}$ given in the paragraph before Lemma 1.2.3. Since $\mathcal{A}(A_{2n-1}) \cong \mathrm{MCG}(\mathbb{D}^A_{2n})$ acts on $\ddot{\underset{\sim}{\mathfrak{C}}}$, there exists an induced action of $\mathcal{A}(B_n)$ on $\ddot{\underset{\sim}{\mathfrak{C}}}$ through the injection Ψ as given in Proposition 1.2.1.

Proposition 1.2.10. *The map $\mathfrak{m} : \check{\mathfrak{C}} \to \ddot{\underset{\sim}{\mathfrak{C}}}$ from isotopy classes of trigraded curves in $\widetilde{\mathfrak{D}}^A_{\Delta_0}$ to isotopy classes of bigraded multicurves in $\widetilde{\mathfrak{D}}^A_\Delta$ is $\mathcal{A}(B_n)$-equivariant:*

$$\begin{array}{ccc} \mathcal{A}(B_n) & & \mathcal{A}(A_{2n-1}) \overset{\Psi}{\hookleftarrow} \mathcal{A}(B_n) \\ \circlearrowleft & & \circlearrowleft \\ \text{Isotopy classes } \check{\mathfrak{C}} \text{ of trigraded curves in } \mathbb{D}^B_{n+1} & \xrightarrow{\ \mathfrak{m}\ } & \text{Isotopy classes } \ddot{\underset{\sim}{\mathfrak{C}}} \text{ of bigraded multicurves in } \mathbb{D}^A_{2n} \end{array}$$

Proof. This follows directly from the definition $\mathfrak{m}$ and the actions. Q.E.D.

1.2.9 Bigraded intersection number and bigraded admissible multicurves in $\widetilde{\mathfrak{D}}^A_\Delta$

The local index for bigraded curves in $\widetilde{\mathfrak{D}}^A_\Delta$ is defined in the same spirit as the local index for trigraded curves in $\widetilde{\mathfrak{D}}^A_{\Delta_0}$. For a more detailed explanation, we refer the reader to [KS02, Section 3d].

We recall from [KS02, Section 3d] that the bigraded intersection number of two bigraded curves $\ddot{c}_0, \ddot{c}_1$

that do not intersect at $\partial\mathbb{D}^A_{2n}$ is defined by

$$I^{bigr}(\ddot{c}_0,\ddot{c}_1) = (1+q_1^{-1}q_2)\left(\sum_{z\in(c_0\cap c_1')\setminus\Delta} q_1^{\mu_1(z)}q_2^{\mu_2(z)}\right) + \left(\sum_{z\in(c_0\cap c_1')\cap\Delta} q_1^{\mu_1(z)}q_2^{\mu_2(z)}\right). \tag{1.2.16}$$

We extend the definition of bigraded intersection number of bigraded curves to bigraded multicurves by adding up the bigraded intersection numbers of each pair of bigraded curves.

To talk about bigraded admissible curves in $\mathbb{D}^A_{2n}$ and their normal forms, we need to fix a set of basic bigraded curves. To do so, first recall the set of trigraded basic curves $(b_j,\check{b}_j)$ and the set of trigraded vertical curves $(d_j,\check{d}_j)$ as defined in Section 1.2.6. Denote this set of basic trigraded curves as $\check{\mathfrak{B}}$. Consider, for each $(c,\check{c})\in\check{\mathfrak{B}}$, its lift to bigraded multicurves in $\mathbb{D}^A_{2n}$:

$$\mathfrak{m}(c,\check{c}) = \begin{cases} (\widetilde{c},\ddot{\widetilde{c}}), & \text{if } (c,\check{c}) = (b_1,\check{b}_1);\\ (\widetilde{c},\ddot{\widetilde{c}})\amalg(\underset{\sim}{c},\ddot{\underset{\sim}{c}}), & \text{otherwise,}\end{cases}$$

where $\widetilde{c}$ denotes the curve whose points have positive real parts; so points in $\underset{\sim}{c}$ have negative real parts. We shall fix the set of bigraded basic curves $(\varrho,\ddot{\varrho}_j)$ and bigraded vertical curves $(\theta_j,\ddot{\theta}_j)$ as follows:

- choose $(\theta_n,\ddot{\theta}_n) := (\widetilde{d}_1,\ddot{\widetilde{d}}_1)$;
- choose $(\theta_{n+j-1},\ddot{\theta}_{n+j-1}) := (\widetilde{d}_j,\ddot{\widetilde{d}}_j)$ and $(\theta_{n-j+1},\ddot{\theta}_{n-j+1}) := (\underset{\sim}{d}_j,\ddot{\underset{\sim}{d}}_j)$, for $2\le j\le n$;
- choose $(\varrho_{n+j-1},\ddot{\varrho}_{n+j-1}) := (\widetilde{b}_j,\ddot{\widetilde{b}}_j)$ and $(\varrho_{n-j+1},\ddot{\varrho}_{n-j+1}) := (\underset{\sim}{b}_j,\ddot{\underset{\sim}{b}}_j)$, for $2\le j\le n$;
- choose $(\varrho_n,\ddot{\varrho}_n) := (\widetilde{b}_1,\ddot{\widetilde{b}}_1)$.

The following figure illustrates the (underlying) basic curves ϱ_j and vertical curves θ_j chosen:

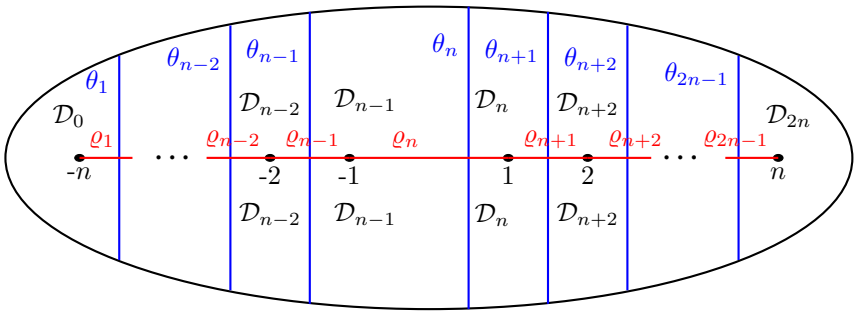


Figure 1.2.9.1: The basic curves θ_i and ϱ_i in the aligned configuration with regions $\mathcal{D}_i$ for $\mathbb{D}^A_{2n}$.

Remark 1.2.11. *We warn the reader of two slight differences here in comparison to [KS02]: the underlying set of basic curves $\{\varrho_j\}$ chosen are the same as in [KS02] minus the one curve connected to the boundary of the disc; moreover, the bigradings of the basic curves and veritical curves are also slightly different. Compare our equations (1.2.18) and (1.2.19) to the defining equations in [KS02, pg. 232].*

Lemma 1.2.12. *The bigradings we choose for the set of basic curves and vertical curves in $\mathbb{D}^A_{2n}$ satisfy the following by properties:*

$$I^{bigr}(\ddot{\theta}_j,\ddot{\varrho}_j) = 1+q_1^{-1}q_2, \qquad \text{for } 1\le j\le 2n-1; \tag{1.2.17}$$
$$I^{bigr}(\ddot{\varrho}_j,\ddot{\varrho}_{j+1}) = 1, \qquad \text{for } n\le j\le 2n-2; \tag{1.2.18}$$
$$I^{bigr}(\ddot{\varrho}_j,\ddot{\varrho}_{j-1}) = 1, \qquad \text{for } 2\le j\le n. \tag{1.2.19}$$

Proof. This follows immediately from the construction. Q.E.D.

Similar to curves in $\mathbb{D}^B_{n+1}$, a curve c in $\mathbb{D}^A_{2n}$ is called ***admissible*** if $c = f(\varrho_j)$ for some $-n \leq j \leq n$ and some $f \in \mathrm{Diff}\left(\mathbb{D}^A_{2n}, \partial\mathbb{D}^A_{2n}\right)$. Note that unlike in [KS02], admissible curves in $\mathbb{D}^A_{2n}$ will not touch the boundary of the disc (none of our basic curves ϱ_j does). An admissible curve c in its isotopy class that has minimal intersection with all the θ_j's among its other representatives is said to be in ***normal form***. We define crossings, essential segments, j-strings and bigraded j-strings in a similar fashion to the trigraded case (see Section 1.2.6). In particular, given a j-crossing x of a bigraded curve $\ddot{c}$, we fix $x_0 := j$ and (x_1, x_2) is the local index (μ_1, μ_2) of the crossing y. All these notions can be extended to that for multicurves and bigraded multicurves naturally.

Suppose c_0 and c_1 are two admissible curves in $\mathbb{D}^B_{n+1}$ intersecting at $z \in \mathbb{D}^B_{n+1}$. If $z = 0$, we require $c_0 \not\simeq c_1$. Their preimage $q_{br}^{-1}(c_0)$ and $q_{br}^{-1}(c_1)$ in $\mathbb{D}^A_{2n}$ under the map $q_{br} : \mathbb{D}^A_{2n} \to \mathbb{D}^B_{n+1}$ would then intersect minimally. However, if $c_0 \cap c_1 \cap \{0\} \neq \emptyset$, and $c_0 \simeq c_1$, they will not intersect minimally as illustrated below:

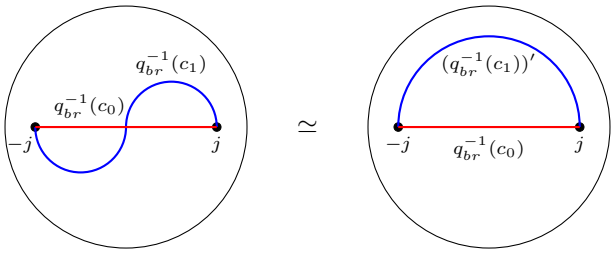


Figure 1.2.9.2: The preimages $q_{br}^{-1}(c_0), q_{br}^{-1}(c_1)$ in $\mathbb{D}^A_{2n}$.

As such, we obtain the following proposition:

Proposition 1.2.13. *Let $\check{c}_0$ and $\check{c}_1$ be two trigraded curves intersecting at $z \in \mathbb{D}^B_{n+1}$, with local index $\mu^{trigr}(\check{c}_0, \check{c}_1, z) = (r_1, r_2, r_3)$. If $c_0 \cap c_1 \cap \{0\} \neq \emptyset$, we require $c_0 \not\simeq c_1$. If $z \neq 0$, further suppose that $\mathfrak{m}(c_0, \check{c}_0) = (\widetilde{c_0}, \widetilde{\check{c}_0}) \amalg (\underset{\sim}{c_0}, \underset{\sim}{\check{c}_0})$ and $\mathfrak{m}(c_1, \check{c}_1) = (\widetilde{c_1}, \widetilde{\check{c}_1}) \amalg (\underset{\sim}{c_1}, \underset{\sim}{\check{c}_1})$ such that $\widetilde{c_0} \cap \widetilde{c_1} = \widetilde{z}$ and $\underset{\sim}{c_0} \cap \underset{\sim}{c_1} = \underset{\sim}{z}$. Then*

$$\begin{cases} \mu^{bigr}(\ddot{\widetilde{c}}_0, \ddot{\widetilde{c}}_1, \widetilde{z}) = (r_1, r_2) = \mu^{bigr}(\ddot{\underset{\sim}{c}}_0, \ddot{\underset{\sim}{c}}_1, \underset{\sim}{z}), & \text{for } z \neq 0; \\ \mu^{bigr}(\ddot{\widetilde{c}}_0, \ddot{\underset{\sim}{c}}_1, 0) = (r_1, r_2), & \text{for } z = 0. \end{cases}$$

Furthermore, this proposition allows us to relate trigraded intersection numbers and bigraded intersection number in the following way.

Corollary 1.2.14. *For any trigraded admissible curves $(c_0, \check{c}_0)$ and $(c_1, \check{c}_1)$*

$$I^{trigr}(\check{c}_0, \check{c}_1)|_{q_3=1} = I^{bigr}\left(\mathfrak{m}(\check{c}_0), \mathfrak{m}(\check{c}_1)\right).$$

In particular, $\frac{1}{2} I^{trigr}(\check{c}_0, \check{c}_1)|_{q_1=q_2=q_3=1} = I(\mathfrak{m}(c_0), \mathfrak{m}(c_1))$, i.e. $\frac{1}{2} I^{trigr}(\check{c}_0, \check{c}_1)|_{q_1=q_2=q_3=1}$ counts the geometric intersection number of the lifts of c_0 and c_1 in $\mathbb{D}^A_{2n}$ under the map $\mathfrak{m}$.

Proof. The case when $c_0 \not\simeq c_1$ in $\mathbb{D}^B_{n+1}$ or when at least one of c_0 and c_1 does not have its endpoint at $\{0\}$ follows directly from Proposition 1.2.13. The other case follows from a direct computation. The last statement relating trigraded intersection number and geometric intersection number follows from the property of bigraded intersection number (see [KS02, pg.227, property (B1)]). Q.E.D.

We shall abuse notation and also allow $\mathfrak{m}$ to lift crossings of a trigraded admissible curve $(c, \check{c})$ to crossings of the bigraded admissible multicurves $\mathfrak{m}((c, \check{c})) = (\widetilde{c}, \ddot{\widetilde{c}}) \amalg (\underset{\sim}{c}, \ddot{\underset{\sim}{c}})$. Suppose z is a j-crossing of c, for $j > 1$. Then, $q_{br}^{-1}(z) = \{\widetilde{z}, \underset{\sim}{z}\}$ where $\widetilde{z} \in \widetilde{c}$ and $\underset{\sim}{z} \in \underset{\sim}{c}$. If z is a 1-crossing of c, then we also have $q_{br}^{-1} = \{\widetilde{z}, \underset{\sim}{z}\}$; in this case we shall pick $\widetilde{\underset{\sim}{z}}$ to be the unique element in $\{\widetilde{z}, \underset{\sim}{z}\} \cap \theta_n$. So, if z is a j-crossing of c with $\mu^{trigr}(\dot{d}_k, \check{c}, z) = (r_1, r_2, r_3)$, We define $\mathfrak{m}(z) = \{\widetilde{z}, \underset{\sim}{z}\}$, both $\widetilde{z}$ and $\underset{\sim}{z}$ with local index (r_1, r_2) for $j > 1$; otherwise $\mathfrak{m}(z) = \{\widetilde{\underset{\sim}{z}}\}$ for $j = 1$, with local index (r_1, r_2) by Proposition 1.2.13. Let $\check{h}$ be a connected subset of $\check{c}$ within some connected region given by unions of D_j's, equipped with the trigrading given by local indices of crossings of $\check{h}$ induced from crossings of $\check{c}$. We define $\mathfrak{m}(\check{h})$ to consist of $q_{br}^{-1}(\check{h})$, with bigradings given by local indices of crossings of $q_{br}^{-1}(\check{h})$ induced from crossings of $\mathfrak{m}(\check{c})$.

1.3 Type A_{2n-1} and Type B_n Zigzag Algebras

In this section, we recall the construction of the type A_m zigzag algebra $\mathscr{A}_m$ as given in [KS02] (with slight change in gradings) and recall the $\mathcal{A}(A_m)$ action on the bounded homotopy category $\mathrm{Kom}^b(\mathscr{A}_m\text{-p}_r\text{g}_r\text{mod})$ of complexes of projective graded modules over $\mathscr{A}_m$. We then construct a type B_n zigzag algebra $\mathscr{B}_n$, following a similar construction, and show that $\mathcal{A}(B_n)$ acts on $\mathrm{Kom}^b(\mathscr{B}_n\text{-p}_r\text{g}_r\text{mod})$. We shall assume that the reader is familiar with projective modules over finite-dimensional (graded) algebras, and refer the unfamiliar readers to [SY11, Chapter 1]. Note that throughout this whole paper, all complexes are bounded.

1.3.1 Type A_m zigzag algebra $\mathscr{A}_m$

Consider the following quiver Γ_m:

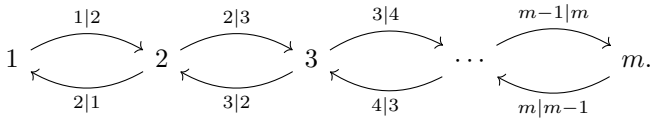


Figure 1.3.1.1: The quiver Γ_m.

We can take its path algebra $\mathbb{C}\Gamma_m$ over $\mathbb{C}$; $\mathbb{C}\Gamma_m$ is the $\mathbb{C}$-vector space spanned by the set of all paths in Γ_m, with multiplication given by concatenation of paths (the multiplication is zero if the endpoints do not agree). We denote the constant path at each vertex j by e_j.

In this chapter, we will only consider m odd, and the grading we put on $\mathbb{C}\Gamma_m$ will be slightly different to [KS02]; we set

- the degree of $(j|j-1)$ is 1 for $j > \frac{m+1}{2}$ and 0 for $j \leq \frac{m+1}{2}$,
- the degree of $(j|j+1)$ is 1 for $j < \frac{m+1}{2}$ and 0 for $j \geq \frac{m+1}{2}$, and
- the degree of $e_j = 0$, for all j,

where the grading is extended to all paths via multiplication. In this way, this path algebra is $\mathbb{Z}$-graded with the grading shift denoted by $\{-\}$ and unital with a family of pairwise orthogonal primitive central idempotent e_j summing up to the unit element.

Let $\mathscr{A}_m$ be the quotient of the path algebra of the quiver Γ_m by the relations:

$$(j|j+1|j) = (j|j-1|j),$$
$$(j-1|j|j+1) = 0 = (j+1|j|j-1),$$

for all $2 \leq j \leq m-1$. It is easy to see that these relations are homogenous with respect to the above grading, so that $\mathscr{A}_m$ is a $\mathbb{Z}$-graded algebra. As a $\mathbb{C}$-vector space, it has dimension $4m-2$ with the following basis $\{e_1, \ldots, e_m, (1|2), \ldots, (m-1|m), (2|1), \ldots (m|m-1), (1|2|1), \ldots, (m|m-1|m)\}$.

The indecomposable projective, $\mathbb{Z}$-graded $\mathscr{A}_m$-modules are denoted by $P_j^A := \mathscr{A}_m e_j$, and we denote the (additive) category of projective, graded $\mathscr{A}_m$-modules by $\mathscr{A}_m\text{-}p_r g_r\text{mod}$. We recall the following results from [KS02]:

Theorem 1.3.1. *For each j, the following complex of graded $(\mathscr{A}_m, \mathscr{A}_m)$-bimodule*

$$\mathcal{R}_j := 0 \to P_j^A \otimes_{\mathbb{C}} {}_jP^A \xrightarrow{\beta_j} \mathscr{A}_m \to 0,$$

with $\mathscr{A}_m$ in cohomological degree 0 is invertible in the homotopy category $Kom^b((\mathscr{A}_m, \mathscr{A}_m)\text{-}bimod)$ of graded $(\mathscr{A}_m, \mathscr{A}_m)$-bimodules and satisfies the following relations:

$$\mathcal{R}_j \otimes \mathcal{R}_k \cong \mathcal{R}_k \otimes \mathcal{R}_j, \quad \text{for } |j-k| > 1;$$
$$\mathcal{R}_j \otimes \mathcal{R}_{j+1} \otimes \mathcal{R}_j \cong \mathcal{R}_{j+1} \otimes \mathcal{R}_j \otimes \mathcal{R}_{j+1}.$$

Proof. See [KS02, Proposition 2.4 and Theorem 2.5]. Q.E.D.

Proposition 1.3.2. *There is a (weak) $\mathcal{A}(A_m)$-action on $Kom^b(\mathscr{A}_m\text{-}p_r g_r mod)$, where each standard generator σ_j^A of $\mathcal{A}(A_m)$ acts on a complex $M \in Kom^b(\mathscr{A}_m\text{-}p_r g_r mod)$ via $\mathcal{R}_j$:*

$$\sigma_j^A(M) := \mathcal{R}_j \otimes_{\mathscr{A}_m} M.$$

Proof. See [KS02, Proposition 2.7]. Q.E.D.

We will abuse notation and use σ_j^A in place of $\mathcal{R}_j$ whenever the context is clear.

1.3.2 Type B_n zigzag algebra $\mathscr{B}_n$

Consider the following quiver Ω_n:

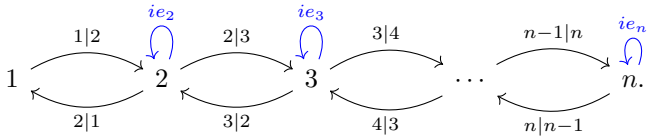


Figure 1.3.2.1: The quiver Ω_n.

Take its path algebra $\mathbb{R}\Omega_n$ over $\mathbb{R}$ and consider the two gradings on $\mathbb{R}\Omega_n$ given as follows:

(i) the first grading is defined following the convention in [KS02] where we set

- the degree of $(j+1|j)$ to be 1 and $(j|j+1)$ to be 0 for all $1 \leq j \leq n-1$, and
- the degree of e_j, ie_j (blue paths in Figure 2.6.1.1) to be 0 for all $1 \leq j \leq n$,

extending to all paths.

(ii) the second grading is a $\mathbb{Z}/2\mathbb{Z}$-grading defined by setting

- the degree of ie_j as 1 for all $1 \leq j \leq n$, and
- the degree of all other paths in Figure 2.6.1.1 and the constant paths as zero,

extending again to all paths.

We denote a shift in the $\mathbb{Z}$-grading by $\{-\}$ and a shift in the $\mathbb{Z}/2\mathbb{Z}$-grading by $\langle - \rangle$.

We are now ready to define the zigzag algebra of type B_n:

Definition 1.3.3. *The zigzag path algebra of B_n, denoted by $\mathscr{B}_n$, is the quotient algebra of the path algebra $\mathbb{R}\Omega_n$ modulo the usual zigzag relations given by*

$$(j|j-1)(j-1|j) = (j|j+1)(j+1|j), \tag{1.3.1}$$
$$(j-1|j)(j|j+1) = 0 = (j+1|j)(j|j-1), \tag{1.3.2}$$

for $2 \leq j \leq n-1$; in addition to the relations

$$(ie_j)(ie_j) = -e_j, \qquad \text{for } j \geq 2; \tag{1.3.3}$$
$$(ie_{j-1})(j-1|j) = (j-1|j)(ie_j), \qquad \text{for } j \geq 3; \tag{1.3.4}$$
$$(ie_j)(j|j-1) = (j|j-1)(ie_{j-1}), \qquad \text{for } j \geq 3; \tag{1.3.5}$$
$$(1|2)(ie_2)(2|1) = 0; \tag{1.3.6}$$
$$(ie_2)(2|1|2) = (2|1|2)(ie_2). \tag{1.3.7}$$

We shall also denote the (non-trivial) loop on vertex j by $X_j := (j|j \pm 1)(j \pm 1|j)$. The relations above are homogeneous with respect to both the $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ gradings, so $\mathscr{B}_n$ is a bigraded algebra.

Proposition 1.3.4. *As a $\mathbb{R}$-vector space, $\mathscr{B}_n$ has dimension $8n-6$.*

Proof. Using the relations (1.3.1) to (2.6.5), the vector subspace (over $\mathbb{R}$) spanned by paths that do not pass through the vertex 1 is isomorphic to $\mathscr{A}_{n-1}$ viewed as a $\mathbb{R}$-vector space, which has $\mathbb{R}$-dimension $2(4(n-1)-2) = 8n-12$. Combined with the remaining relations, one can check that the remaining paths that passes through the vertex 1 (modulo relations) are exactly $e_1, (1|2), (2|1), (1|2)(ie_2), (ie_2)(2|1)$ and X_1. Hence, $\dim_{\mathbb{R}}(\mathscr{B}_n) = 8n-12+6 = 8n-6$; in particular, the set $\{e_1, \dots, e_n, ie_2, \dots, ie_n, (1|2), \dots, (n-1|n), (2|1), \dots, (n|n-1), (ie_2)(2|1), (1|2)(ie_2), (ie_2)(2|3), \dots, (ie_{n-1})(n-1|n), (3|2)(ie_2), \dots, (n|n-1)(ie_{n-1}), X_1, \dots, X_n, (ie_2)X_2, \dots, (ie_n)X_n\}$ forms a $\mathbb{R}$-basis of $\mathscr{B}_n$. Q.E.D.

The indecomposable (left) projective, bigraded $\mathscr{B}_n$-modules are given by $P_j^B := \mathscr{B}_n e_j$, and we denote the (additive) category of projective, bigraded $\mathscr{B}_n$-modules by $\mathscr{B}_n$-$p_r g_r$mod. For $j=1$, P_j^B is naturally a $(\mathscr{B}_n, \mathbb{R})$-bimodule; there is a natural left $\mathscr{B}_n$-action given by multiplication of the algebra and the right $\mathbb{R}$-action induced by the left (commutative) $\mathbb{R}$-action. But for $j \geq 2$, we shall endow P_j^B with a right $\mathbb{C}$-action.

To this end, let us view $\mathbb{C}$ as a $\mathbb{Z}/2\mathbb{Z}$-graded algebra over $\mathbb{R}$ by endowing the reals with degree 0 and the complex imaginary i with degree 1 over $\mathbb{Z}/2\mathbb{Z}$ and extend linearly. Note also that (2.6.3) in Definition 1.3.3 is analogous to the relation satisfied by the complex imaginary number i. We define a right $\mathbb{C}$-action on P_j^B by $p*(a+ib) = ap + bp(ie_j)$ for $p \in P_j^B, a+ib \in \mathbb{C}$. It follows from the definition that this right action restricted to $\mathbb{R}$ agrees with the natural right (and left) $\mathbb{R}$-action. This makes P_j^B into a bigraded $(\mathscr{B}_n, \mathbb{C})$-bimodule for $j \geq 2$. Dually, we shall define ${}_jP^B := e_j\mathscr{B}_n$, where we similarly consider it as a bigraded $(\mathbb{R}, \mathscr{B}_n)$-bimodule for $j=1$ and as a bigraded $(\mathbb{C}, \mathscr{B}_n)$-bimodule for $j \geq 2$.

Proposition 1.3.5. *Denote ${}_jP_k^B := {}_jP^B \otimes_{\mathscr{B}_n} P_k^B$. We have that*

$$
{}_jP_k^B \cong \begin{cases}
{}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}, & \text{as bigraded } (\mathbb{C},\mathbb{C})\text{-bimodules, for } j,k \in \{2,\dots,n\} \text{ and } k-j=1;\\
{}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}\{1\}, & \text{as bigraded } (\mathbb{C},\mathbb{C})\text{-bimodules, for } j,k \in \{2,\dots,n\} \text{ and } j-k=1;\\
{}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}} \oplus {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}\{1\}, & \text{as bigraded } (\mathbb{C},\mathbb{C})\text{-bimodules, for } j=k=2,3,\dots,n;\\
{}_{\mathbb{R}}\mathbb{C}_{\mathbb{C}}, & \text{as bigraded } (\mathbb{R},\mathbb{C})\text{-bimodules, for } j=1 \text{ and } k=2;\\
{}_{\mathbb{C}}\mathbb{C}_{\mathbb{R}}\{1\}, & \text{as bigraded } (\mathbb{C},\mathbb{R})\text{-bimodules, for } j=2 \text{ and } k=1;\\
{}_{\mathbb{R}}\mathbb{R}_{\mathbb{R}} \oplus {}_{\mathbb{R}}\mathbb{R}_{\mathbb{R}}\{1\}, & \text{as bigraded } (\mathbb{R},\mathbb{R})\text{-bimodules, for } j=k=1.
\end{cases}
$$

Proof. The case where $j,k \in \{2,...,n\}$ follows as in type A. By identifying ${}_jP_k$ as the $\mathbb{R}$-vector subspace of $\mathscr{B}_n$ spanned by paths starting at vertex j and ending at vertex k, we see that ${}_1P_2$ has basis $\{(1|2), (1|2)(ie_2)\}$; ${}_2P_1$ has basis $\{(2|1), (ie_2)(2|1)\}$ and ${}_1P_1$ has basis $\{e_1, X_1\}$. The fact that the bimodule and bigrading structures agree follows from the definition and is left as a simple exercise to the reader. Q.E.D.

Remark 1.3.6. *Note that all the bigraded bimodules in Proposition 1.3.5 can be restricted to bigraded $(\mathbb{R},\mathbb{R})$-bimodules by identifying ${}_{\mathbb{R}}\mathbb{C}_{\mathbb{R}} \cong \mathbb{R} \oplus \mathbb{R}\langle 1\rangle$. For example, ${}_1P_2^B$ restricted to an $(\mathbb{R},\mathbb{R})$-bimodule is generated by $(1|2)$ and $(1|2)ie_2$, and it is isomorphic to $\mathbb{R} \oplus \mathbb{R}\langle 1\rangle \cong {}_{\mathbb{R}}\mathbb{C}_{\mathbb{R}}$.*

Lemma 1.3.7. *Denote $\mathbb{K}_1 := \mathbb{R}$ and $\mathbb{K}_j := \mathbb{C}$ when $j \geq 2$. The maps*

$$\beta_j : P_j^B \otimes_{\mathbb{K}_j} {}_jP^B \to \mathscr{B}_n \text{ and } \gamma_j : \mathscr{B}_n \to P_j^B \otimes_{\mathbb{K}_j} {}_jP^B\{-1\}$$

defined by:

$$\beta_j(x \otimes y) := xy,$$

$$
\gamma_j(1) := \begin{cases}
X_j \otimes e_j + e_j \otimes X_j + (j+1|j) \otimes (j|j+1) & \\
\quad +(-ie_{j+1})(j+1|j) \otimes (j|j+1)(ie_{j+1}), & \text{for } j=1;\\
X_j \otimes e_j + e_j \otimes X_j + (j-1|j) \otimes (j|j-1) + (j+1|j) \otimes (j|j+1), & \text{for } 1<j<n;\\
X_j \otimes e_j + e_j \otimes X_j + (j-1|j) \otimes (j|j-1), & \text{for } j=n
\end{cases}
$$

are $(\mathscr{B}_n, \mathscr{B}_n)$*-bimodule maps.*

Proof. It is obvious from the definition that the β_j are maps of $(\mathscr{B}_n, \mathscr{B}_n)$-bimodules for all j. The fact that γ_j is a $(\mathscr{B}_n, \mathscr{B}_n)$-bimodule map also follows from a tedious check on each basis element, which we shall omit and leave it to the reader. Q.E.D.

Definition 1.3.8. *Define the following complexes of bigraded* $(\mathscr{B}_n, \mathscr{B}_n)$*-bimodules:*

$$R_j := (0 \to P_j^B \otimes_{\mathbb{K}_j} {}_jP^B \xrightarrow{\beta_j} \mathscr{B}_n \to 0), \text{ and}$$
$$R_j' := (0 \to \mathscr{B}_n \xrightarrow{\gamma_j} P_j^B \otimes_{\mathbb{K}_j} {}_jP^B\{-1\} \to 0).$$

for each $1 \leq j \leq n$*, with both* $\mathscr{B}_n$ *in cohomological degree 0,* $\mathbb{K}_1 = \mathbb{R}$ *and* $\mathbb{K}_j = \mathbb{C}$ *for* $j \geq 2$*.*

Proposition 1.3.9. *We have the following isomorphisms in the homotopy category* $Kom^b((\mathscr{B}_n, \mathscr{B}_n)\text{-}bimod)$ *of complexes of projective graded* $(\mathscr{B}_n, \mathscr{B}_n)$*-bimodules:*

$$R_j \otimes R_j^{'} \cong_{\mathscr{B}_n} \cong R_j^{'} \otimes R_j; \tag{1.3.8}$$
$$R_j \otimes R_k \cong R_k \otimes R_j, \quad \text{for } |k-j| > 1; \tag{1.3.9}$$
$$R_j \otimes R_{j+1} \otimes R_j \cong R_{j+1} \otimes R_j \otimes R_{j+1}, \quad \text{for } j \geq 2. \tag{1.3.10}$$

Proof. These relations can be verified similarly as in [KS02, Theorem 2.5]. Q.E.D.

1.3.3 Adjunctions and Dehn Twist

To show the last type B_n relation (the 4-braiding relation), we shall introduce a larger family of invertible complexes that will aid us in our calculation. This construction mirrors the notion of Dehn twists in topology and uses the theory on adjunctions (we highly recommend [Kho10] for an amazing exposition on expressing adjunctions using planar diagrammatics). Throughout this section we will denote $\mathbb{K}_1 := \mathbb{R}$ and $\mathbb{K}_j := \mathbb{C}$ for $j \geq 2$.

Definition 1.3.10. *Let* $X \in Kom^b((\mathscr{B}_n, \mathbb{K}_j)\text{-}bimod)$ *and* $X^\ell, X^r \in Kom^b((\mathbb{K}_j, \mathscr{B}_n)\text{-}bimod)$ *such that* $X^\ell \otimes_{\mathscr{B}_n} -$ *and* $X^r \otimes_{\mathscr{B}_n} -$ *are left and right adjoints of* $X \otimes_{\mathbb{K}_j} -$ *respectively. We define the twist of* X *as the complex of* $(\mathscr{B}_n, \mathscr{B}_n)$*-bimodule*

$$\sigma_X := cone\left(X \otimes_{\mathbb{K}_j} X^r \xrightarrow{\varepsilon} \mathscr{B}_n\right),$$

with ε *the counit of the adjunction* $X \dashv X^r$*. Similarly, the dual twist of* X *is given by*

$$\sigma_X' := cone\left(\mathscr{B}_n \xrightarrow{\eta} X \otimes_{\mathbb{K}_j} X^\ell\right)$$

with η *the unit of the adjunction* $X \vdash X^\ell$*. The twist* σ_X *is said to be spherical if the twist and dual twist are inverses of each other, namely* $\sigma_X \otimes_{\mathscr{B}_n} \sigma_X' \cong \mathscr{B}_n \cong \sigma_X' \otimes_{\mathscr{B}_n} \sigma_X$*.*

One can verify from the definition of the adjunctions that the twist (resp. dual twist) is uniquely defined up to isomorphism, i.e. $X \cong Y$ implies $\sigma_X \cong \sigma_Y$ (resp. $\sigma_X' \cong \sigma_Y'$). On the other hand, the shift functors $[1], \{1\}$ and $\langle 1 \rangle$ are autoequivalences, so we also have that $\sigma_X = \sigma_{X[r]\{s\}\langle t \rangle}$. More generally, given a pair of adjunctions (X, X^ℓ, X^r) on X with $Y \cong \Sigma \otimes_{\mathscr{B}_n} X$ where Σ an invertible object in $\mathrm{Kom}^b((\mathscr{B}_n, \mathscr{B}_n)\text{-mod})$, we also have a pair of adjunctions (Y, Y^ℓ, Y^r) on Y given by $Y^\ell := X^\ell \otimes_{\mathscr{B}_n} \Sigma^{-1}$, and $Y^r := X^r \otimes_{\mathscr{B}_n} \Sigma^{-1}$. Furthermore, the twists and dual twists are related by: $\sigma_Y \cong \Sigma \otimes_{\mathscr{B}_n} \sigma_X \otimes_{\mathscr{B}_n} \Sigma^{-1}$, and $\sigma_Y' \cong \Sigma \otimes_{\mathscr{B}_n} \sigma_X' \otimes_{\mathscr{B}_n} \Sigma^{-1}$.

Lemma 1.3.11. *The functor* $P_j^B \otimes_{\mathbb{K}_j} -$ *is a left adjoint of* ${}_jP^B \otimes_{\mathscr{B}_n} -$ *and a right adjoint of* ${}_jP^B\{-1\} \otimes_{\mathscr{B}_n} -$*.*

Proof. To show that the $P_j^B \otimes_{\mathbb{K}_j} -$ is a left adjoint to ${}_jP^B \otimes_{\mathscr{B}_n} -$, take the counit to be the functor induced by $P_j^B \otimes_{\mathbb{K}_j} {}_jP^B \xrightarrow{\beta_j} \mathscr{B}_n$ and the unit is instead induced by $\mathbb{K}_j \xrightarrow{\varphi} {}_jP^B \otimes_{\mathscr{B}_n} P_j^B$, where φ is defined by $\varphi(1) = e_j \otimes_{\mathscr{B}_n} e_j$. To show ${}_jP^B\{-1\} \otimes_{\mathscr{B}_n} -$ is a left adjoint to $P_j^B \otimes_{\mathbb{K}_j} -$, take the counit to be the functor

induced by ${}_jP^B \otimes_{\mathscr{B}_n} P_j^B\{-1\} \xrightarrow{\varphi'} \mathbb{K}_j$ where φ' is defined by $\varphi'(e_j \otimes e_j) = 0, \varphi'(X_j \otimes e_j) = 1$ (note that $X_j \otimes e_j = e_j \otimes X_j$), and the unit is instead induced by $\mathscr{B}_n \xrightarrow{\gamma_j} P_j^B \otimes_{\mathbb{K}_j} {}_jP^B\{-1\}$. We leave the verification of the conditions required to the reader. Q.E.D.

Using this, we shall now prove the last type B_n relation required:

Proposition 1.3.12. *We have the following isomorphism of $Kom^b((\mathscr{B}_n, \mathscr{B}_n)$-bimod):*

$$R_2 \otimes_{\mathscr{B}_n} R_1 \otimes_{\mathscr{B}_n} R_2 \otimes_{\mathscr{B}_n} R_1 \cong R_1 \otimes_{\mathscr{B}_n} R_2 \otimes_{\mathscr{B}_n} R_1 \otimes_{\mathscr{B}_n} R_2.$$

Proof. We shall drop the tensor products for the sake of readability: $R_2R_1R_2R_1 \cong R_1R_2R_1R_2$. Using Proposition 1.3.9, note that this relation is equivalent to $(R_2R_1R_2)R_1(R_2'R_1'R_2') \cong R_1$. By the adjunctions shown in Lemma 1.3.11, note that R_1 and R_1' are by definition the same as the twist $\sigma_{P_1^B}$ and dual twist $\sigma'_{P_1^B}$ of P_1^B respectively. As such, we get that

$$\sigma_{R_2R_1R_2(P_1^B)} \cong (R_2R_1R_2)\sigma_{P_1^B}(R_2'R_1'R_2') \cong \sigma_{P_1^B}.$$

It is now sufficient to show $R_2R_1R_2(P_1^B)$ and P_1^B are isomorphic in $\mathrm{Kom}^b((\mathscr{B}_n, \mathbb{R})$-bimod) up to cohomological or internal gradings shifts. This is shown in the following series computation (note that we have omitted the cohomological grading since it does not matter).

$$R_2(P_1^B) = 0 \to P_2^B \otimes_{\mathbb{C}\ 2} P_1^B \to P_1^B \to 0 \cong 0 \to P_2^B\{1\} \xrightarrow{(2|1)} P_1^B \to 0 \qquad \text{(by } \textit{Proposition } 1.3.5\text{)}$$

$$\begin{aligned}
R_1R_2(P_1^B) &\cong R_1\left(0 \to P_2^B\{1\} \xrightarrow{(2|1)} P_1^B \to 0\right) \\
&= \mathrm{cone}\left\{\begin{array}{ccc} P_1^B\{1\} \otimes_{\mathbb{R}\ 1} P_2^B & \xrightarrow{\mathrm{id}\otimes(2|1)} & P_1^B \otimes_{\mathbb{R}\ 1} P_1^B \\ \downarrow & & \downarrow \\ P_2^B\{1\} & \xrightarrow{(2|1)} & P_1^B \end{array}\right\} \\
&\cong \mathrm{cone}\left\{\begin{array}{ccc} P_1^B\{1\} \oplus P_1^B\{1\}\langle 1\rangle & \xrightarrow{\left[\begin{smallmatrix}0 & 0\\ \mathrm{id} & 0\end{smallmatrix}\right]} & P_1^B \oplus P_1^B\{1\} \\ \downarrow{\scriptstyle [(1|2)\ (1|2)i]} & & \downarrow{\scriptstyle [\mathrm{id}\ X_1]} \\ P_2^B\{1\} & \xrightarrow{(2|1)} & P_1^B \end{array}\right\} \qquad \text{(by } \textit{Proposition } 1.3.5\text{)} \\
&\cong 0 \to P_1^B\{1\}\langle 1\rangle \xrightarrow{(1|2)i} P_2^B\{1\} \to 0 \\
R_2R_1R_2(P_1^B) &\cong \mathrm{cone}\left\{\begin{array}{ccc} P_2^B \otimes_{\mathbb{C}\ 2} P_1^B\{1\}\langle 1\rangle & \longrightarrow & P_2^B \otimes_{\mathbb{C}\ 2} P_2^B\{1\} \\ \downarrow & & \downarrow \\ P_1^B\{1\}\langle 1\rangle & \xrightarrow{(1|2)i} & P_2^B\{1\} \end{array}\right\} \\
&\cong \mathrm{cone}\left\{\begin{array}{ccc} P_2^B\{2\}\langle 1\rangle & \xrightarrow{\left[\begin{smallmatrix}0\\ \mathrm{id}\end{smallmatrix}\right]} & P_2^B\{1\} \oplus P_2^B\{2\}\langle 1\rangle \\ \downarrow{\scriptstyle (2|1)} & & \downarrow{\scriptstyle [\mathrm{id}\ X_2 i]} \\ P_1^B\{1\}\langle 1\rangle & \xrightarrow{(1|2)i} & P_2^B\{1\}. \end{array}\right\} \qquad \text{(by } \textit{Proposition } 1.3.5\text{)} \\
&\cong P_1^B\{1\}\langle 1\rangle
\end{aligned}$$

Q.E.D.

Theorem 1.3.13. *We have a (weak) $\mathcal{A}(B_n)$-action on $Kom^b(\mathscr{B}_n\text{-}p_rg_rmod)$, where each standard generator σ_j^B for $j \geq 2$ of $\mathcal{A}(B_n)$ acts on a complex $M \in Kom^b(\mathscr{B}_n\text{-}p_rg_rmod)$ via R_j, and σ_1^B acts via $R_1\langle 1\rangle$:*

$$\sigma_j^B(M) := R_j \otimes_{\mathscr{B}_n} M, \text{ and } (\sigma_j^B)^{-1}(M) := R_j' \otimes_{\mathscr{B}_n} M,$$

$$\sigma_1^B(M) := R_1\langle 1\rangle \otimes_{\mathscr{B}_n} M, \text{ and } (\sigma_1^B)^{-1}(M) := R_1'\langle 1\rangle \otimes_{\mathscr{B}_n} M.$$

Proof. This follows directly from Proposition 1.3.9 and Proposition 1.3.12, where the required relations still hold with the extra third grading shift $\langle 1\rangle$ on R_1 and R_1'. Q.E.D.

From now on, we will abuse notation and use σ_j^B and $(\sigma_j^B)^{-1}$ in place of R_j and R_j' (with an extra grading shift $\langle 1\rangle$ for $j = 1$) respectively whenever it is clear from the context what we mean.

1.4 Relating Categorical Type B_n and Type A_{2n-1} actions

In Section 1.2, we have defined $\mathfrak{m}$ that lifts isotopy classes of trigraded curves in $\mathbb{D}_{n+1}^B$ to isotopy classes of bigraded multicurves in $\mathbb{D}_{2n}^A$. Furthermore, we showed that the map $\mathfrak{m}$ is equivariant under the $\mathcal{A}(B_n)$-action. In this section, we shall develop the algebraic version of this story. We will first relate our type B_n zigzag algebra $\mathscr{B}_n$ to the type A_{2n-1} zigzag algebra $\mathscr{A}_{2n-1}$ by showing that $\mathbb{C} \otimes_{\mathbb{R}} \mathscr{B}_n \cong \mathscr{A}_{2n-1}$ as graded $\mathbb{C}$-algebras (forgetting the $\mathbb{Z}/2\mathbb{Z}$ grading in $\mathscr{B}_n$). Through this, we have an injection $\mathscr{B}_n \hookrightarrow \mathbb{C} \otimes_{\mathbb{R}} \mathscr{B}_n \cong \mathscr{A}_{2n-1}$ as graded $\mathbb{R}$-algebras. Thus, we can relate the two categories $\mathrm{Kom}^b(\mathscr{B}_n\text{-p}_r\text{g}_r\text{mod})$ and $\mathrm{Kom}^b(\mathscr{A}_{2n-1}\text{-p}_r\text{g}_r\text{mod})$ through an extension of scalar $\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} -$. We end this section by showing that the functor $\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} -$ is $\mathcal{A}(B_n)$-equivariant, which also allows us to deduce that the $\mathcal{A}(B_n)$-action on $\mathrm{Kom}^b(\mathscr{B}_n\text{-}p_rg_rmod)$ is faithful.

Let Q be a left $\mathbb{C}$-module. Throughout this section, we shall denote ${}_{\bar{\mathbb{C}}}Q$ to be the left $\mathbb{C}$-module with a deformed left action, given by multiplication its with complex conjugate:

$$a(c) = \bar{a}c. \tag{1.4.1}$$

Similarly for Q a right $\mathbb{C}$-module, we use $Q_{\bar{\mathbb{C}}}$ to denote the right $\mathbb{C}$-module with the deformed action.

Proposition 1.4.1. *Consider the graded $\mathbb{R}$-algebra $\ddot{\mathscr{B}}_n$, where $\ddot{\mathscr{B}}_n$ is just $\mathscr{B}_n$ without the $\mathbb{Z}/2\mathbb{Z}$-grading $\langle -\rangle$. The $\mathbb{Z}$-graded $\mathbb{C}$-algebras $\mathbb{C} \otimes_{\mathbb{R}} \ddot{\mathscr{B}}_n$ and $\mathscr{A}_{2n-1}$ are isomorphic.*

Proof. Note that as $\mathbb{C}$-vector space, we have the following decomposition:

$$\begin{aligned}\mathbb{C} \otimes_{\mathbb{R}} \ddot{\mathscr{B}}_n &\cong \bigoplus_{j=2}^{n} \mathbb{C} \otimes_{\mathbb{R}} \left({}_jP_j^B \ \oplus {}_jP_{(j-1)}^B \oplus {}_{(j-1)}P_j^B\right) \oplus (\mathbb{C} \otimes_{\mathbb{R}} {}_1P_1^B) \\ &\cong \bigoplus_{j=2}^{n} (\mathbb{C} \otimes_{\mathbb{R}} {}_jP_j^B) \oplus \bigoplus_{j=2}^{n} (\mathbb{C} \otimes_{\mathbb{R}} {}_jP_{(j-1)}^B) \oplus \bigoplus_{j=2}^{n} (\mathbb{C} \otimes_{\mathbb{R}} {}_{(j-1)}P_j^B) \oplus (\mathbb{C} \otimes_{\mathbb{R}} {}_1P_1^B).\end{aligned}$$

Firstly, for each $j \geq 2$, note that ${}_jP_j^B$ is itself a $\mathbb{C}$-algebra with unit $e_j \otimes 1$. Moreover, after tensoring with $\mathbb{C}$ over $\mathbb{R}$, $\mathbb{C} \otimes_{\mathbb{R}} {}_jP_j^B$ has idempotent $\nu_j := \frac{1}{2}(1 \otimes e_j + i \otimes ie_j)$. We shall define a $\mathbb{C}$-linear morphism $\Phi : \mathbb{C} \otimes_{\mathbb{R}} \ddot{\mathscr{B}}_n \to \mathscr{A}_{2n-1}$ by specifying the images of the basis elements of $\mathbb{C} \otimes_{\mathbb{R}} \ddot{\mathscr{B}}_n$. It will be easy to see that Φ is grading preserving and we leave the routine check that Φ is an algebra morphism to the reader. For $j = 1$,

$$\mathbb{C} \otimes_{\mathbb{R}} {}_1P_1^B \to {}_nP_n^A$$
$$\begin{cases} 1 \otimes \frac{1}{2}X_1 & \mapsto X_n; \\ 1 \otimes e_1 & \mapsto e_n. \end{cases}$$

For $2 \leq j \leq n$,

1. $\mathbb{C}\otimes_{\mathbb{R}} {}_jP_j^B \to {}_{n-j+1}P_{n-j+1}^A \oplus {}_{n+j-1}P_{n+j-1}^A$
$$\begin{cases}(1\otimes X_j)\nu_j & \mapsto X_{n-j+1};\\ (1\otimes X_j)(1\otimes e_j-\nu_j) & \mapsto X_{n+j-1};\\ \nu_j & \mapsto e_{n-j+1};\\ (1\otimes e_j-\nu_j) & \mapsto e_{n+j-1},\end{cases}$$

2. $\mathbb{C}\otimes_{\mathbb{R}} {}_{j-1}P_j^B \to {}_{n-j+2}P_{n-j+1}^A \oplus {}_{n+j-2}P_{n+j-1}^A$
$$\begin{cases}\big(1\otimes(j-1|j)\big)\nu_j & \mapsto \big((n-j+2)\mid(n-j+1)\big);\\ \big(1\otimes(j-1|j)\big)(1\otimes e_j-\nu_j) & \mapsto \big((n+j-2)\mid(n+j-1)\big),\end{cases}$$

3. $\mathbb{C}\otimes_{\mathbb{R}} {}_jP_{j-1}^B \to {}_{n-j+1}P_{n-j+2}^A \oplus {}_{n+j-1}P_{n+j-2}^A$
$$\begin{cases}v_j(1\otimes(j|j-1)) & \mapsto \big((n-j+1)\mid(n-j+2)\big);\\ (1\otimes e_j-v_j)(1\otimes(j|j-1)) & \mapsto \big((n+j-1)\mid(n+j-2)\big).\end{cases}$$

It is easy to see that this map is surjective and $\dim_{\mathbb{C}}(\mathbb{C}\otimes_{\mathbb{R}}\ddot{\mathscr{B}}_n)=\dim_{\mathbb{C}}(\mathscr{A}_{2n-1})$. Q.E.D.

Let $i:\ddot{\mathscr{B}}_n\hookrightarrow\mathbb{C}\otimes_{\mathbb{R}}\ddot{\mathscr{B}}_n$ be the canonical injection of $\mathbb{Z}$-graded $\mathbb{R}$-algebras. Proposition 1.4.1 allows us to view $\ddot{\mathscr{B}}_n$ as a $\mathbb{Z}$-graded subalgebra over $\mathbb{R}$ of $\mathscr{A}_{2n-1}$ through $\Phi\circ i$. Thus, every $\mathscr{A}_{2n-1}$-module can be restricted to a $\ddot{\mathscr{B}}_n$-module. In particular, $\mathscr{A}_{2n-1}$ as a $\mathbb{Z}$-graded $(\mathscr{A}_{2n-1},\mathscr{A}_{2n-1})$-bimodule can be restricted to a $\mathbb{Z}$-graded $(\mathscr{A}_{2n-1},\ddot{\mathscr{B}}_n)$-bimodule. This gives us an extension of scalar functor $\mathscr{A}_{2n-1}\otimes_{\ddot{\mathscr{B}}_n}-$, sending $\mathbb{Z}$-graded $\ddot{\mathscr{B}}_n$-modules to $\mathbb{Z}$-graded $\mathscr{A}_{2n-1}$-modules.

Let $\mathfrak{F}$ denote the functor which forgets the $\mathbb{Z}/2\mathbb{Z}$-grading of the bigraded $\mathscr{B}_n$-modules. We define

$$\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n}-:=\mathscr{A}_{2n-1}\otimes_{\ddot{\mathscr{B}}_n}(\mathfrak{F}(-)).$$

The proposition below identifies the indecomposable projectives under the functor $\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n}-$.

Proposition 1.4.2. *Recall the deformed action of* $\mathbb{C}$ *given in* (1.4.1). *We have the following isomorphisms of* $\mathbb{Z}$*-graded bimodules:*

$$\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n}P_1^B\cong(P_n^A)_{\mathbb{R}},\qquad \text{as } \mathbb{Z}\text{-graded } (\mathscr{A}_{2n-1},\mathbb{R})\text{-bimodules; and}$$
$$\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n}P_j^B\cong\Big(P_{n-(j-1)}^A\Big)_{\bar{\mathbb{C}}}\oplus P_{n+(j-1)}^A,\qquad \text{as } \mathbb{Z}\text{-graded } (\mathscr{A}_{2n-1},\mathbb{C})\text{-bimodules.}$$

Proof. Define $\Phi_1:\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n}P_1^B\to(P_n^A)_{\mathbb{R}}$ and $\Phi_j:\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n}P_j^B\to\Big(P_{n-(j-1)}^A\Big)_{\bar{\mathbb{C}}}\oplus P_{n+(j-1)}^A$ as the maps given on the basis elements by $a\otimes b\mapsto a\Phi(1\otimes b)$ and extend linearly. It is easy to check that Φ_1 is a graded $(\mathscr{A}_{2n-1},\mathbb{R})$-bimodule morphism and Φ_j is a graded $(\mathscr{A}_{2n-1},\mathbb{C})$-bimodule morphism; the only detail that one should be aware of is that Φ_j maps into $\Big(P_{n-(j-1)}^A\Big)_{\bar{\mathbb{C}}}\oplus P_{n+(j-1)}^A$ instead of $P_{n-(j-1)}^A\oplus P_{n+(j-1)}^A$. The fact that they are isomorphisms follows easily from looking at the dimensions. Q.E.D.

It follows from the above proposition that $\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n}-$ sends projectives to projectives. Therefore $\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n}-$ extends to a functor from $\mathrm{Kom}^b(\ddot{\mathscr{B}}_n\text{-p}_r\text{g}_r\text{mod})$ to $\mathrm{Kom}^b(\mathscr{A}_{2n-1}\text{-p}_r\text{g}_r\text{mod})$. We will denote the functor

$$\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n}-:=\mathscr{A}_{2n-1}\otimes_{\ddot{\mathscr{B}}_n}(\mathfrak{F}(-)):\mathrm{Kom}^b(\mathscr{B}_n\text{-p}_r\text{g}_r\text{mod})\to\mathrm{Kom}^b(\mathscr{A}_{2n-1}\text{-p}_r\text{g}_r\text{mod}).$$

Recall the injection $\Psi:\mathcal{A}(B_n)\to\mathcal{A}(A_{2n-1})$ as defined in Proposition 1.2.1; the image of standard generators are explicitly given by:

$$\Psi(\sigma_j^B)=\begin{cases}\sigma_n^A, & \text{for } j=1;\\ \sigma_{n-(j-1)}^A\sigma_{n+(j-1)}^A, & \text{otherwise.}\end{cases}$$

We have previously shown that $\mathcal{A}(B_n)$ acts on $\mathrm{Kom}^b(\mathscr{B}_n\text{-p}_r\text{g}_r\text{mod})$ and similarly $\mathcal{A}(A_{2n-1})$ acts on $\mathrm{Kom}^b(\mathscr{A}_{2n-1}\text{-p}_r\text{g}_r\text{mod})$. Through Ψ, we have an induced action of $\mathcal{A}(B_n)$ on $\mathrm{Kom}^b(\mathscr{A}_{2n-1}\text{-p}_r\text{g}_r\text{mod})$. We will now prove the algebraic version of Proposition 1.2.10.

Theorem 1.4.3. *For all $1 \leq j \leq n$, we have an isomorphism $\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n}\sigma_j^B \cong \Psi(\sigma_j^B)_{\mathscr{B}_n}$ in $Kom^b((\mathscr{A}_{2n-1},\mathscr{B}_n)\text{-}bimod)$. In particular, the functor $\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n} -$ is $\mathcal{A}(B_n)$-equivariant:*

$$\begin{array}{ccc} \mathcal{A}(B_n) & & \mathcal{A}(A_{2n-1}) \overset{\Psi}{\hookleftarrow} \mathcal{A}(B_n) \\ \circlearrowleft & & \circlearrowleft \\ Kom^b(\mathscr{B}_n\text{-}p_rg_rmod) & \xrightarrow{\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n} -} & Kom^b(\mathscr{A}_{2n-1}\text{-}p_rg_rmod), \end{array}$$

i.e. for any $\sigma \in \mathcal{A}(B_n)$ and any complex $C \in Kom^b(\mathscr{B}_n\text{-}p_rg_rmod)$,

$$\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n}(\sigma\otimes_{\mathscr{B}_n} C) \cong \Psi(\sigma)\otimes_{\mathscr{A}_{2n-1}}(\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n} C).$$

Before we start with the proof, we will need the following lemma:

Lemma 1.4.4. *Recall the deformed $\mathbb{C}$-action given in (1.4.1). We have that*

$$\mathbb{C}\otimes_{\mathbb{R}}{}_1P^B \cong ({}_nP^A)_{\mathscr{B}_n}, \qquad \text{as } \mathbb{Z}\text{-graded } (\mathbb{C},\mathscr{B}_n)\text{-bimodules;} \tag{1.4.2}$$

$${}_jP^B \cong \left({}_{n+(j-1)}P^A\right)_{\mathscr{B}_n}, \qquad \text{as } \mathbb{Z}\text{-graded } (\mathbb{C},\mathscr{B}_n)\text{-bimodules;} \tag{1.4.3}$$

$${}_jP^B \cong {}_{\bar{\mathbb{C}}}\left({}_{n-(j-1)}P^A\right)_{\mathscr{B}_n}, \qquad \text{as } \mathbb{Z}\text{-graded } (\mathbb{C},\mathscr{B}_n)\text{-bimodules;} \tag{1.4.4}$$

$${}_{\mathbb{C}}\mathbb{C}\otimes_{\bar{\mathbb{C}}}\mathbb{C}_{\mathbb{C}} \cong {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}, \qquad \text{as } \mathbb{Z}\text{-graded } (\mathbb{C},\mathbb{C})\text{-bimodules.} \tag{1.4.5}$$

Proof. We will only define the maps; the proofs that they are isomorphisms respecting the required structures follows from a simple verification.

For (1.4.2) take the morphism $\phi_1 : \mathbb{C}\otimes_{\mathbb{R}}{}_1P^B \to ({}_nP^A)_{\mathscr{B}_n}$ as the restriction of Φ constructed in the proof of Proposition 1.4.1. Note that ϕ_1 does indeed map into $({}_nP^A)_{\mathscr{B}_n}$ since

$$\Phi(c\otimes b) = \Phi(c\otimes e_1b) = \Phi((1\otimes e_1)(c\otimes b)) = \Phi(1\otimes e_1)\Phi(c\otimes b) = e_n\Phi(c\otimes b).$$

For (1.4.3) and (1.4.4), consider the morphisms

$$\begin{aligned} \phi_{+j} : {}_jP^B &\to \left({}_{n+(j-1)}P^A\right)_{\mathscr{B}_n} \\ b &\mapsto e_{n+(j-1)}\Phi(1\otimes b), \end{aligned} \qquad \text{and} \qquad \begin{aligned} \phi_{-j} : {}_jP^B &\to {}_{\bar{\mathbb{C}}}\left({}_{n-(j-1)}P^A\right)_{\mathscr{B}_n} \\ b &\mapsto e_{n-(j-1)}\Phi(1\otimes b). \end{aligned}$$

Finally for (1.4.5), consider the morphism $c : {}_{\mathbb{C}}\mathbb{C}\otimes_{\bar{\mathbb{C}}}\mathbb{C}_{\mathbb{C}} \to \mathbb{C}$ uniquely defined by $1\otimes 1 \mapsto 1$. Q.E.D.

Proof of Theorem 1.4.3. We shall show the required statement by showing it for each generator, namely $\Psi(\sigma_j^B)_{\mathscr{B}} \cong \mathscr{A}\otimes_{\mathscr{B}}\sigma_j^B$ as complexes of $(\mathscr{A},\mathscr{B})$-bimodules for each j.

Let $j \geq 2$. Using the relevant isomorphisms in Lemma 1.4.4, we have the following chain of bimodule isomorphisms:

$$\begin{aligned} &\left(P^A_{n-(j-1)}\otimes_{\mathbb{C}}\left({}_{n-(j-1)}P^A\right)_{\mathscr{B}}\right)\oplus\left(P^A_{n+(j-1)}\otimes_{\mathbb{C}}\left({}_{n+(j-1)}P^A\right)_{\mathscr{B}}\right) \\ &\cong \left(P^A_{n-(j-1)}\otimes_{\bar{\mathbb{C}}}\left({}_{n-(j-1)}P^A\right)_{\mathscr{B}}\right)\oplus\left(P^A_{n+(j-1)}\otimes_{\mathbb{C}}\left({}_{n+(j-1)}P^A\right)_{\mathscr{B}}\right) \quad \text{(by (1.4.5))} \\ &\cong \left(\left(P^A_{n-(j-1)}\right)_{\bar{\mathbb{C}}}\otimes_{\mathbb{C}}{}_jP^B\right)\oplus\left(P^A_{n+(j-1)}\otimes_{\mathbb{C}}{}_jP^B\right) \quad \text{(by (1.4.3) and (1.4.4))} \\ &\cong \left(\left(P^A_{n-(j-1)}\right)_{\bar{\mathbb{C}}}\oplus P^A_{n+(j-1)}\right)\otimes_{\mathbb{C}}{}_jP^B. \end{aligned}$$

Using Proposition 1.4.2, we have that

$$\left(\left(P^A_{n-(j-1)}\right)_{\bar{\mathbb{C}}}\oplus P^A_{n+(j-1)}\right)\otimes_{\mathbb{C}}{}_jP^B \cong \mathscr{A}\otimes_{\mathscr{B}}P^B_j\otimes_{\mathbb{C}}{}_jP^B.$$

Let us denote the composition of this chain of isomorphisms by

$$\Xi : \left(P^A_{n-(j-1)} \otimes_{\mathbb{C}} \left({}_{n-(j-1)}P^A\right)_{\mathscr{B}}\right) \oplus \left(P^A_{n+(j-1)} \otimes_{\mathbb{C}} \left({}_{n+(j-1)}P^A\right)_{\mathscr{B}}\right) \xrightarrow{\cong} \mathscr{A} \otimes_{\mathscr{B}} P^B_j \otimes_{\mathbb{C}} {}_jP^B.$$

Since we have that

$$\begin{aligned}\Psi(\sigma^B_j)_{\mathscr{B}} &= \left(\sigma^A_{n-(j-1)}\sigma^A_{n+(j-1)}\right)_{\mathscr{B}} \\ &= \left(P^A_{n-(j-1)} \otimes_{\mathbb{C}} \left({}_{n-(j-1)}P^A\right)_{\mathscr{B}}\right) \oplus \left(P^A_{n+(j-1)} \otimes_{\mathbb{C}} \left({}_{n+(j-1)}P^A\right)_{\mathscr{B}}\right) \xrightarrow{\left[\beta^A_{n-(j-1)}\ \beta^A_{n+(j-1)}\right]} \mathscr{A}_{\mathscr{B}}\end{aligned}$$

and

$$\begin{aligned}\mathscr{A} \otimes_{\mathscr{B}} \sigma^B_j &= \mathscr{A} \otimes_{\mathscr{B}} \left(P^B_j \otimes_{\mathbb{C}} {}_jP^B \xrightarrow{\beta^B_j} \mathscr{B}\right) \\ &= \mathscr{A} \otimes_{\mathscr{B}} P^B_j \otimes_{\mathbb{C}} {}_jP^B \xrightarrow{\mathrm{id}\otimes_{\mathscr{B}}\beta^B_j} \mathscr{A} \otimes_{\mathscr{B}} \mathscr{B},\end{aligned}$$

all that is left is to show that the following diagram commutes:

$$\begin{array}{ccc}\left(P^A_{n-(j-1)} \otimes_{\mathbb{C}} \left({}_{n-(j-1)}P^A\right)_{\mathscr{B}}\right) \oplus \left(P^A_{n+(j-1)} \otimes_{\mathbb{C}} \left({}_{n+(j-1)}P^A\right)_{\mathscr{B}}\right) & \xrightarrow{\left[\beta^A_{n-(j-1)} \quad \beta^A_{n+(j-1)}\right]} & \mathscr{A}_{\mathscr{B}} \\ \downarrow \Xi & & \downarrow \cong \\ \mathscr{A} \otimes_{\mathscr{B}} P^B_j \otimes_{\mathbb{C}} {}_jP^B & \xrightarrow{\mathrm{id}\otimes_{\mathscr{B}}\beta^B_j} & \mathscr{A} \otimes_{\mathscr{B}} \mathscr{B},\end{array}$$

which we leave for the reader to verify.

The proof for $j = 1$ is simpler and follows from a similar argument. Q.E.D.

Remark 1.4.5. *Define $U_j := P^B_j \otimes {}_jP^B$ and $\mathcal{U}_j := P^A_j \otimes {}_jP^A$. Proposition 1.4.2 implies that*

$$\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} U_j \cong \begin{cases}(\mathcal{U}_n)_{\mathscr{B}_n}, & \text{for } j = 1; \\ \left(\mathcal{U}_{n-(j-1)} \oplus \mathcal{U}_{n+(j-1)}\right)_{\mathscr{B}_n}, & \text{otherwise.}\end{cases}$$

When $n = 2$, this also relates our bimodules to the ones given in [MT19], where $\mathcal{U}_2 = \Theta_t$ and $\mathcal{U}_1 \oplus \mathcal{U}_3 = \Theta_s$ in [MT19, Example 2.12] for the A_3 graph (up to a difference in grading).

We may now use this relation to deduce that the categorical action of $\mathcal{A}(B_n)$ is faithful:

Theorem 1.4.6. *The (weak) action of $\mathcal{A}(B_n)$ on the category $Kom^b(\mathscr{B}_n\text{-}p_r g_r mod)$ given in Theorem 1.3.13 is faithful.*

Proof. Assume that we are given $\sigma \in \mathcal{A}(B_n)$ such that $\sigma(C) \cong C$ for all $C \in \mathrm{Kom}^b(\mathscr{B}_n\text{-}\mathrm{p_r g_r mod})$. We will show that this implies σ is the identity. In particular, take $C = \oplus^n_{j=1} P^B_j$ so that we have

$$\sigma\left(\oplus^n_{j=1} P^B_j\right) \cong \oplus^n_{j=1} P^B_j.$$

Applying the functor $\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} -$, we obtain

$$\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} \sigma\left(\oplus^n_{j=1} P^B_j\right) \cong \mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} \left(\oplus^n_{j=1} P^B_j\right) \cong \oplus^{2n-1}_{j=1} P^A_j. \tag{1.4.6}$$

Applying Theorem 1.4.3 to the LHS of (1.4.6), we get

$$\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} \sigma\left(\oplus^n_{j=1} P^B_j\right) \cong \Psi(\sigma)\left(\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} \left(\oplus^n_{j=1} P^B_j\right)\right) \cong \Psi(\sigma)\left(\oplus^{2n-1}_{j=1} P^A_j\right). \tag{1.4.7}$$

Combining the two equations (1.4.6) and (1.4.7) above we deduce that

$$\Psi(\sigma)\left(\oplus^{2n-1}_{j=1} P^A_j\right) \cong \oplus^{2n-1}_{j=1} P^A_j.$$

Since it was shown in [KS02, Corollary 1.2] that the type A categorical action is faithful, we conclude that $\Psi(\sigma) = \mathrm{id}$.

Suppose $\Psi(\sigma) \neq \mathrm{id}$, then $\Psi(\sigma)$ must contain at least one term of $P_i \otimes {}_jP$ in the non-zeroth cohomological degree of its complex. It is not hard to deduce that $\Psi(\sigma)(P_j) = P_j$ for all j given $\Psi(\sigma)\left(\oplus_{j=1}^{2n-1} P_j^A\right) \cong \oplus_{j=1}^{2n-1} P_j^A$ and $\Psi(\sigma)$ is an indecomposable complex. If $P_i \otimes {}_jP$ only existed in the zeroth cohomological, then $\Psi(\sigma)$ is no longer an indecomposable complex which contradicts the fact that autoequivalences send indecomposable complexes to indecomposable complexes. Since $\Psi(\sigma)$ is a bounded complex, without loss of generality, we pick j in the bimodule term $P_i \otimes {}_jP$ appearing in the greatest non-zeroth cohomological degree. Then, $\Psi(\sigma)$ acting on P_j will produce an extra copy of shifted P_j contradicting the assumption.

But Ψ is injective, so we must have that $\sigma = \mathrm{id}$ as required. Q.E.D.

1.5 Main Theorem

Let us state the main theorem that we aim to prove:

Theorem 1.5.1. *The diagram in Figure 1.5.0.1 is commutative, where the maps $\mathfrak{m}, L_B, L_A$ and $\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} -$ are all $\mathcal{A}(B_n)$-equivariant.*

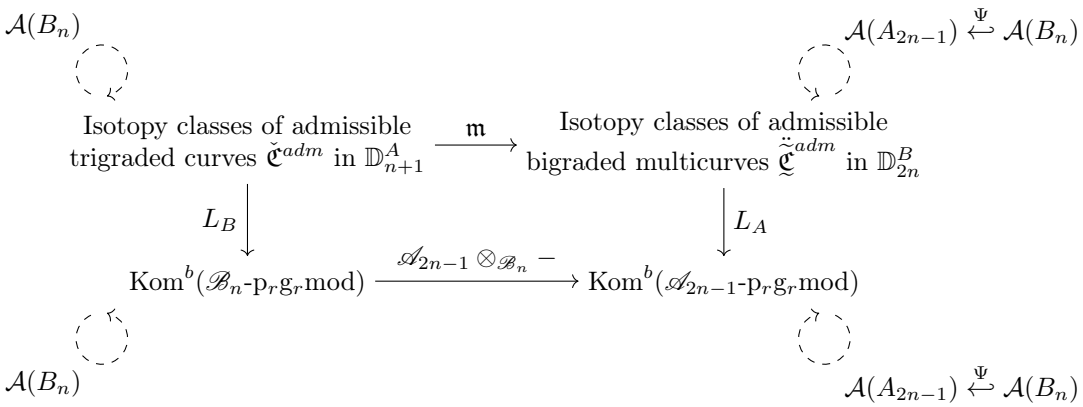


Figure 1.5.0.1: The commutative diagram of Theorem 1.5.1. The first row is from Proposition 1.2.10 and the second row is from Theorem 1.4.3.

In Section 1.2, we introduced and showed that $\mathfrak{m}$ is $\mathcal{A}(B_n)$-equivariant. In Section 1.4, we showed that the functor $\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} -$ is also $\mathcal{A}(B_n)$-equivariant. Thus, the missing pieces are:

1. the definitions of the maps L_B and L_A;
2. the fact that the maps L_B and L_A are $\mathcal{A}(B_n)$-equivariant; and
3. the commutativity of the diagram.

We shall start by recalling the definition of L_A from [KS02, Section 4a] and their result that L_A is $\mathcal{A}(A_{2n-1})$-equivariant. We then define L_B in Section 1.5.2, where the (technical) proof that the diagram commutes and that L_B is $\mathcal{A}(B_n)$-equivariant are given in Section 1.5.3.

1.5.1 Complexes associated to admissible multi-curves (Type A)

Here we state the constructions and results shown in [KS02, Section 4], which can be easily extended to admissible multicurves. Note that we added a subscript L_A instead of L used in [KS02] to differentiate between type A and type B later on. Let $\ddot{c}$ be a bigraded admissible curve in normal form. We associate

to $\ddot{c}$ an object $L_A(\ddot{c})$ in the category $\mathrm{Kom}^b(\mathscr{A}_{2n-1}\text{-p}_r\text{g}_r\text{mod})$. Start by defining $L_A(\ddot{c})$ as a bigraded $\mathscr{A}_{2n-1}$-module:

$$L_A(\ddot{c}) = \bigoplus_{x \in cr(\ddot{c})} P(x), \tag{1.5.1}$$

where $P(x) = P^A_{x_0}[-x_1]\{x_2\}$, with $[-]$ denoting the cohomological degree shift; see the paragraph after Remark 1.2.11 for the definition of (x_0, x_1, x_2). For every $x, y \in cr(\ddot{c})$ define $\partial_{yx} : P(x) \to P(y)$ by the following rules:

- If x and y are the endpoints of an essential segment and $y_1 = x_1 + 1$, then
  1. If $x_0 = y_0$ (then it must be that $x_2 = y_2 + 1$) then
  $$\partial_{yx} : P(x) \to P(y) \cong P(x)[-1]\{1\}$$
  is the multiplication on the right by $X_{x_0} \in \mathscr{A}_{2n-1}$.
  2. If $x_0 = y_0 \pm 1$, then ∂_{yx} is the right multiplication by $(x_0|y_0) \in \mathscr{A}_{2n-1}$;
- otherwise $\partial_{yx} = 0$.

We define the differential ∂ as $\partial := \sum_{x,y} \partial_{yx}$. See [KS02, Lemma 4.1] for a proof that this defines a complex. Moreover, it follows easily that

$$L_A(\chi_A(r_1, r_2)\ddot{c}) \cong L_A(\ddot{c})[-r_1]\{r_2\}. \tag{1.5.2}$$

For $\ddot{g}$ a bigraded j-string of $\ddot{c}$, we can also assign a complex $L_A(\ddot{g})$ to $\ddot{g}$, where as a bigraded abelian group, $L_A(\ddot{g}) = \bigoplus_{x \in cr(g)} P^A(x)$ and the differentials are obtained from essential segments of $\ddot{g}$ the same way as for admissible curves. We can easily extend this to define $L_A(\ddot{h})$ for $h \subseteq c$ a connected subset of c such that $h = \cup g_{\alpha,j}$ with each $g_{\alpha,j}$ some bigraded j-string of c. The following theorem is proven in [KS02, Theorem 4.3]:

Theorem 1.5.2. *For a braid $\sigma \in \mathcal{A}(A_m)$ and a bigraded admissible curve $\ddot{c}$ in $\mathbb{D}^A_{m+1}$, we have $\sigma L_A(\ddot{c}) \cong L_A(\sigma(\ddot{c}))$ in the category $Kom^b(\mathscr{A}_m\text{-}p_r g_r mod)$, i.e. L_A is $\mathcal{A}(A_m)$-equivariant.*

We extend L_A to admissible multicurves as follows: given bigraded multicurves $\coprod_j \ddot{c}_j$,

$$L_A\left(\coprod_j \ddot{c}_j\right) := \bigoplus_j L_A(\ddot{c}_j).$$

It follows easily that this defines a complex, and both (1.5.2) and Theorem 1.5.2 still hold for admissible multicurves.

1.5.2 Complexes associated to admissible curves (Type B)

Consider a trigraded admissible curve $\check{c}$. We associate to $\check{c}$ an object $L_B(\check{c})$ in the category $\mathrm{Kom}^b(\mathscr{B}_n\text{-}p_r g_r mod)$. Start by defining $L_B(\check{c})$ as a trigraded $\mathscr{B}_n$-module:

$$L_B(\check{c}) = \bigoplus_{y \in cr(\check{c})} P(y)$$

where $P(y) = P^B_{y_0}[-y_1]\{y_2\}\langle y_3\rangle$ (see the second paragraph after Proposition 1.2.8 in Section 1.2.6 for definition of y_0, y_1, y_2 and y_3).

We now define maps $\partial_{zy} : P(y) \to P(z)$ for each $y, z \in cr(\check{c})$ using the following rules (note that these are ***not*** the differentials yet):

- If y and z are the endpoints of an essential segment in D_j for $j \geq 1$ and $z_1 = y_1 + 1$, then

1. If $y_0 = z_0$ (then also $y_2 = z_2 + 1$ and $y_3 = z_3$), then
$$\partial_{zy} : P(y) \to P(z) \cong P(y)[-1]\{1\}$$
is the right multiplication by the element $X_{y_0} \in \mathscr{B}_n$.
2. If $y_0 = z_0 \pm 1$, then ∂_{zy} is the right multiplication by $(y_0|z_0) \in \mathscr{B}_n$.

- Otherwise, $\partial_{yz} = 0$.

We will modify some of these maps before using them as differentials. Define the following equivalence relation on the set of 1-crossings:

$$y \sim y' \iff y \text{ and } y' \text{ are connected by an essential segment in } D_0.$$

Consider the partitioning of the set of 1-crossings using the equivalence relation above. Referring to the possible normal forms in D_0 given by Figure 1.2.6.4, every equivalence classes under this relation consists of either one element (Type $3'$) or two elements (Type $2''$). For each equivalence class $[y]$ of 1-crossings, we modify the some of the maps given previously by the following rule:

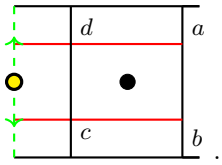
- If $[y] = \{y\}$, we modify nothing;
- otherwise, $[y] = \{y, y'\}$ has two distinct 1-crossings. Note that at least one of the 1-crossings must be an endpoint of some essential segment in D_1 (otherwise c is clearly not be admissible). Up to relabelling, we may assume that y is always a 1-crossing that is an endpoint of an essential segment γ in D_1. Note that γ cannot have both y and y' as endpoints (this will imply that c is a simple closed curve, which is not admissible). As such, let us label the other endpoint of γ connecting y as $z \neq y'$. Consider the two possible cases for z:
  1. z is a 2-crossing:
     (a) if $y_1 = z_1 + 1$, then we have that $\partial_{yz} : P(z) \to P(y)$ is given by the right multiplication by $(2|1)$. We modify $\partial_{y'z} : P(z) \to P(y') \cong P(y)\langle 1\rangle$ (which was necessarily 0 previously) so that it is now the right multiplication by $-i(2|1)$;
     (b) otherwise, we have instead $z_1 = y_1 + 1$. In this case, $\partial_{zy} : P(y) \to P(z)$ is given by the right multiplication by $(1|2)$. We modify $\partial_{zy'} : P(y') \cong P(y)\langle 1\rangle \to P(z)$ (which was necessarily 0 previously) so that it is now the right multiplication by $(1|2)i$.
  2. z is a 1-crossing:
     (a) if $y_1 = z_1 + 1$, we modify nothing;
     (b) otherwise, we have instead $z_1 = y_1 + 1$. In this case $\partial_{zy} : P(y) \to P(z)$ is given by the right multiplication by X_1. Once again consider the two possible cases of the equivalence class $[z]$:
        (i) If $[z] = \{z\}$, we modify nothing;
        (ii) otherwise $[z] = \{z, z'\}$ with $z \neq z'$. We then modify $\partial_{z'y'} : P(y)\langle 1\rangle \cong P(y') \to P(z') \cong P(z)\langle 1\rangle$ (which was necessarily 0 previously) so that it is now the right multiplication by X_1;

  Similarly, if the other 1-crossing y' in $[y]$ is also an endpoint of an essential segment in D_1 (distinct from γ), we shall repeat the same process above for y'.

Finally, we define the differential as $\partial = \sum_{x,y \in cr(\check{c})} \partial_{xy}$, where ∂_{xy} are the *modified* version above.

Lemma 1.5.3. *$(L_B(\check{c}), \partial)$ is a complex of projective graded $\mathscr{B}_n$-modules with a grading-preserving differential.*

Proof. For $x, y, z \in cr(\check{c})$ with $x_0, y_0, z_0 \geq 2$, the same argument as in the type A shows that the product of $\partial_{zy}\partial_{yx} : P_x \to P_z$ is always 0. The only occurrence of $\partial_{zy}\partial_{yx} \neq 0$ is when $\partial_{zy} = \partial_{ac}, \partial_{ad}$ and $\partial_{yx} = \partial_{db}, \partial_{cb}$ with a, b, c, d the crossings of the following type of 1-string labelled below:



Note that the two non-zero composition $\partial_{ad}\partial_{db}$ and $\partial_{ac}\partial_{cb}$ always occur as a pair. Moreover, we see that their sum is equal to 0: $\partial_{ad}\partial_{db} + \partial_{ac}\partial_{cb} = X_2 i - X_2 i = 0$, thus showing that $\partial^2 = 0$ as required. Q.E.D.

Lemma 1.5.4. *For any triple (r_1, r_2, r_3) of integers and any trigraded admissible curve $\check{c}$ we have:*

$$L_B(\chi(r_1, r_2, r_3)\check{c}) \cong L_B(\check{c})[-r_1]\{r_2\}\langle r_3\rangle.$$

Proof. This follows directly from the definition. Q.E.D.

1.5.3 Some rather technical results

This subsection is where we complete the proof of Theorem 1.5.1 by proving that the diagram commutes (Proposition 1.5.5) and that L_B is $\mathcal{A}(B_n)$-equivariant (Theorem 1.5.7). The proof of these two statements, first of which is technical, will occupy the rest of this section.

Proposition 1.5.5. *The diagram in Figure 1.5.0.1 commutes, i.e. for each trigraded admissible curve $\check{c}$ in $\mathbb{D}^B_{n+1}$ we have that $\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} L_B(\check{c}) \cong L_A(\mathfrak{m}(\check{c}))$ in $Kom^b(\mathscr{A}_{2n-1}\text{-}p_r g_r mod)$.*

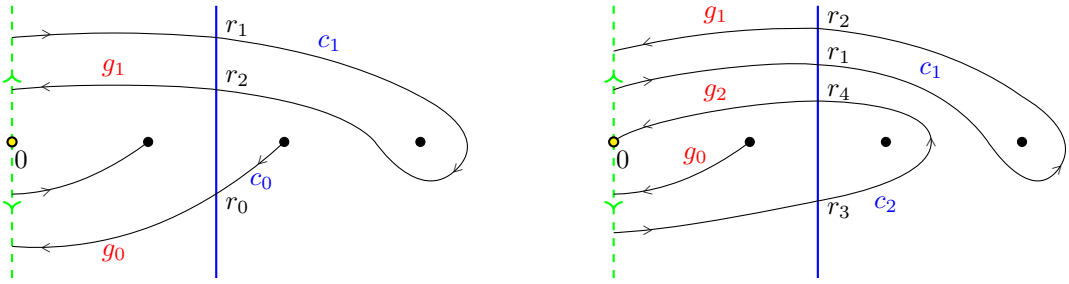
Remark 1.5.6. *The theorem is stated in the context of homotopy category since the group actions of $\mathcal{A}(A_{2n-1})$ and $\mathcal{A}(B_n)$ live there. In fact, we will show that $\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} L_B(\check{c}) \cong L_A(\mathfrak{m}(\check{c}))$ in the category $Com^b(\mathscr{B}_n\text{-}p_r g_r mod)$ of bounded complexes (no homotopy will be required).*

Proof of Proposition 1.5.5. Let x be any j-crossing of $\check{c}$. If $j \geq 2$, we have that $\mathfrak{m}(x)$ consists of a $(n+(j-1))$-crossing $\widetilde{x}$ and a $(n-(j-1))$-crossing $\underset{\sim}{x}$ of $\mathfrak{m}(\check{c})$; if $j = 1$, $\mathfrak{m}(x)$ consists of a single n-crossing $\widetilde{\underset{\sim}{x}}$ of $\mathfrak{m}(\check{c})$. In either cases, we have isomorphisms $\Phi_j : \mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} P^B(x) \to P^A(\widetilde{x}) \oplus P^A(\underset{\sim}{x}) = P^A_{n+(j-1)} \oplus P^A_{n-(j-1)}$ or $\Phi_1 : \mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} P^B(x) \to P^A(\widetilde{\underset{\sim}{x}}) = P^A_n$ given in the proof of Proposition 1.4.2. Putting together these isomorphisms for each crossing x of $\check{c}$, we obtain a cohomological and internal grading preserving isomorphism of $\mathscr{A}_{2n-1}$-modules between the underlying modules of $\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} L_B(\check{c})$ and $L_A(\mathfrak{m}(\check{c}))$; denote this isomorphism by η. Denoting the complexes $\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} L_B(\check{c})$ as (Q, δ) and $L_A(\mathfrak{m}(\check{c}))$ as (Q', δ') (so η is an isomorphism from Q to Q'), it follows that η induces an isomorphism of complexes:

$$(Q, \delta) \cong (Q', \delta_0),$$

with $\delta_0 = \eta\delta\eta^{-1}$. We now aim to show that $(Q', \delta_0) \cong (Q', \delta')$ in $\mathrm{Kom}^b(\mathscr{A}_{2n-1}\text{-}\mathrm{p_r g_r mod})$. In fact, we will show that they are isomorphic in the ordinary category $\mathrm{Com}^b(\mathscr{A}_{2n-1}\text{-}\mathrm{p_r g_r mod})$ of complexes in $\mathscr{A}_{2n-1}\text{-}\mathrm{p_r g_r mod}$. Before we proceed with the proof, we will need to introduce some (substantial amount of) notation.

(Slicing c) Recall that c, being admissible, must have both of its end points at two *distinct* marked points, so at least one of its end points is not 0 and let m be such an end point. Orient the curve c so that it starts from m and ends at its other end point. Following this orientation, we can slice c into distinct connected components $c_j \subset c \cap (\cup_{j\geq 2} D_j)$ and $g_{j'} \subset c \cap (D_0 \cup D_1)$ (note that $g_{j'}$ are the 1-strings of c) enumerated as follows: following the orientation of c, g_j denotes the $(j-1)$-th 1-string component of c; whereas c_i is the component of c that has a total of i 1-strings components before c_i. In other words, if $m \neq 1$, we start from c_0 to g_0 to c_1 and so on; otherwise $m = 1$ and we start from g_0 to c_0 to g_1 and so on. Note that if c has no 1-string component, $c = c_0$. Following the same orientation, we shall also enumerate the 2-crossings r_t of c (if any), starting the enumeration with $t = 0$ if $m = 1$; otherwise we start with $t = 1$. The figure below illustrates two examples with different starting point m.



(a) A slicing with starting point $m = 2 \neq 1$.

(b) A slicing with starting point $m = 1$.

Figure 1.5.3.1: Examples of slicings of curves.

Now consider the following subsets of (graded) crossings of $\mathfrak{m}(\check{c})$:

$$\begin{cases} C_j := \mathfrak{m}(\check{c}_j) \cap (\bigcup_i \ddot{\theta}_i); \\ \bar{C}_j := \mathfrak{m}(\check{c}_j) \cap (\bigcup_{i \neq n-1,n+1} \ddot{\theta}_i); \\ G_{j'} := \mathfrak{m}(\check{g}_{j'}) \cap (\ddot{\theta}_{n-1} \cup \ddot{\theta}_n \cup \ddot{\theta}_{n+1}); \\ \bar{G}_{j'} := \mathfrak{m}(\check{g}_{j'}) \cap \ddot{\theta}_n; \text{ and} \\ R_j := \mathfrak{m}(\check{r}_j). \end{cases} \tag{1.5.3}$$

Note that by definition, the subsets of crossings $\bar{C}_i$, R_j and $\bar{G}_k$ are pairwise disjoint, and $\left(\coprod_j \bar{C}_j\right) \amalg \left(\coprod_j R_j\right) \amalg \left(\coprod_j \bar{G}_j\right)$ is the set of all crossings of $\mathfrak{m}(\check{c})$. On the other hand, C_j and $G_{j'}$ contains all the crossings of $\mathfrak{m}(c_j)$ and $\mathfrak{m}(g_{j'})$ respectively, which may contain common crossings R_i.

For K a subset of crossings of $\mathfrak{m}(\check{c})$, we define $Q'_K := \bigoplus_{x \in K} P^A(x) \subseteq Q'$. If K is empty, Q'_K will be the 0 module by convention. Using this, we can decompose Q' as follow: when $m \in \Lambda \setminus \{0, 1\}$, we have

$$Q' = \underbrace{Q'_{\bar{C}_0}}_{Q'_{C_0}} \oplus \overbrace{\underbrace{Q'_{R_0}}_{Q'_{G_0}} \oplus Q'_{\bar{G}_0} \oplus \underbrace{Q'_{R_1}}_{Q'_{G_1}}}^{Q'_{G_0}} \oplus \underbrace{Q'_{\bar{C}_1} \oplus Q'_{R_2}}_{Q'_{C_1}} \oplus \dots \quad ; \tag{1.5.4}$$

whereas when $m = 1$, we have instead

$$Q' = \underbrace{Q'_{\bar{G}_0}}_{Q'_{G_0}} \oplus \overbrace{\underbrace{Q'_{R_1}}_{Q'_{C_1}} \oplus Q'_{\bar{C}_1} \oplus \underbrace{Q'_{R_2}}_{Q'_{C_2}}}^{Q'_{C_1}} \oplus \underbrace{Q'_{\bar{G}_2} \oplus Q'_{R_3}}_{Q'_{G_2}} \oplus \dots \quad . \tag{1.5.5}$$

In general, given a decomposition of modules $M = \oplus_{i \in Y} M_{K_i}$ and a complex (M, ∂), we can then write ∂ as a block matrix. We will use the notation $\partial_{K_i K_j}$ to denote the block of ∂ that maps from M_{K_j} to M_{K_i}, where we use the shorthand notation ∂_{K_i} for the block of ∂ that maps from M_{K_i} to itself. We will also use the notation $\partial_{\oplus_{i \in X \subseteq Y} K_i}$ for the block of ∂ that maps from $\oplus_{i \in X} M_{K_i}$ to itself. To illustrate, consider the decomposition of Q' as in (1.5.4) and let (Q', ∂) be a cochain complex with differential ∂. We will then write the differential ∂ as the matrix

$$\begin{bmatrix} \partial_{\bar{C}_0} & \partial_{R_0\bar{C}_0} & \partial_{\bar{G}_0\bar{C}_0} & \partial_{R_1\bar{C}_0} & \partial_{\bar{C}_1\bar{C}_0} & \cdots \\ \partial_{\bar{C}_0 R_0} & \partial_{R_0} & \partial_{\bar{G}_0 R_0} & \partial_{R_1 R_0} & \partial_{\bar{C}_1 R_0} & \cdots \\ \partial_{\bar{C}_0\bar{G}_0} & \partial_{R_0\bar{G}_0} & \partial_{\bar{G}_0} & \partial_{R_1\bar{G}_0} & \partial_{\bar{C}_1\bar{G}_0} & \cdots \\ \partial_{\bar{C}_0 R_1} & \partial_{R_0 R_1} & \partial_{\bar{G}_0 R_1} & \partial_{R_1} & \partial_{\bar{C}_1 R_1} & \cdots \\ \partial_{\bar{C}_0\bar{C}_1} & \partial_{R_0\bar{C}_1} & \partial_{\bar{G}_0\bar{C}_1} & \partial_{R_1\bar{C}_1} & \partial_{\bar{C}_1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where the blocks ∂_{C_j} corresponding to the summands Q'_{C_j} are the blocks in red and similarly blocks ∂_{G_j} corresponding to the summands Q'_{G_j} are the blocks in blue.

After our lengthy notational digression, let us return to the proof and analyse the difference between the matrices of the two differentials δ_0 and δ' corresponding to the two possible decomposition of Q' as in (1.5.4) and (1.5.5). By looking at how the components c_i and g_j are connected, it follows that the non-zero entries of the matrix of δ_0 are all contained in the block matrices $(\delta_0)_{C_k}$ and $(\delta_0)_{G_k}$ for all k; similarly the connection between the components $\mathfrak{m}(c_i)$ and $\mathfrak{m}(g_j)$ dictates that the non-zero entries of the matrix of δ' are all contained in the block matrices δ'_{C_k} and δ'_{G_k} for all k. One can show from a direct computation that

$$(\delta_0)_{C_k} = \delta'_{C_k}, \quad \text{for all } k. \tag{1.5.6}$$

So between the differentials δ_0 and δ', only the block matrices $(\delta_0)_{G_k}$ and δ'_{G_k} may differ. As such, if we were in the case where c has no 1-string, i.e. $c = c_0$, then we are done. In the rest of the proof, we shall assume otherwise and treat the rest of the cases.

As we will only need to focus on the module summand Q'_{G_j} later on, for each j let us simplify both the decompositions of Q' in (1.5.4) and (1.5.5) into

$$Q' = Q'_{V_j} \oplus \underbrace{Q'_{R_j} \oplus Q'_{\bar{G}_j} \oplus Q'_{R_{j+1}}}_{Q'_{G_j}} \oplus Q'_{W_j}, \tag{1.5.7}$$

where Q'_{V_j} (resp. Q'_{W_j}) consists of all the module summands before Q'_{R_j} (resp. after $Q'_{R_{j+1}}$) for both decompositions (1.5.4) and (1.5.5). From here on we will use this simplified decomposition for the matrix of any differential on Q'.

Let $g_0, g_1, ..., g_{s-1}$ be the 1-strings in c. To show $(Q', \delta_0) \cong (Q', \delta')$, by (1.5.6) and (1.5.7) it is sufficient to construct a cohomological and internal grading preserving isomorphism of modules $\mu_j : Q' \to Q'$ for each $0 \leq j \leq s-1$, so that we have an induced chain of isomorphisms in $\mathrm{Com}^b(\mathscr{A}_{2n-1}\text{-p}_r\text{g}_r\text{mod})$:

$$(Q', \delta_0) \cong (Q', \delta_1) \cong \cdots \cong (Q', \delta_{s-1}) \cong (Q', \delta_s) = (Q', \delta');$$

with $\delta_{j+1} := \mu_j \delta_j \mu_j^{-1}$, where each δ_j for $1 \leq j \leq s$ satisfies the following property:

$$\begin{cases} (\delta_j)_{G_{j-1}} = \delta'_{G_{j-1}}, & \text{and} \\ (\delta_j)_{XY} = (\delta_{j-1})_{XY}, & \text{for all } X, Y \in \{V_{j-1}, G_{j-1}, W_{j-1}\} \text{ such that } (X, Y) \neq (G_{j-1}, G_{j-1}). \end{cases} \tag{*}$$

In other words, each μ_j will be constructed in a way that the conjugation of δ_j by μ_j^{-1} *only* alters the differential component $(\delta_j)_{G_j}$, so that $(\delta_{j+1})_{G_j} = \mu_j(\delta_j)_{G_j}\mu_j^{-1} = \delta'_{G_j}$ without affecting the rest of the differential components. In particular, if δ_ℓ satisfy (*) for all $1 \leq \ell \leq j-1$, then we have that $(\delta_j)_{G_j} = (\delta_0)_{G_j}$. Moreover, property (*) will guarantee that $\delta_s = \delta'$.

What remains is to define the required $\mu_j : Q' \to Q'$. We will define μ_j according to the type of 1-string $\check{g}_j$. Within each possible type of 1-string $\check{g}_j$, we will show the following:

1. For $j = 0$, we show that we can always construct μ_0 to get $(Q'_0, \delta_0) \cong (Q'_1, \delta_1)$ such that $\delta_1 = \mu_0\delta_0\mu_0^{-1}$ satisfies (*).
2. For $j \geq 1$, we show that given $(Q', \delta_0) \cong \cdots \cong (Q'_j, \delta_j)$ with $\delta_1, ..., \delta_j$ satisfying (*) for $j \geq 1$, we can construct $\mu_j : Q' \to Q'$ such that $\delta_{j+1} = \mu_j\delta_j\mu_j^{-1}$ satisfies (*).

By an induction on the total number of 1-strings components (over j), we can always construct a chain of isomorphisms μ_j with property (*) and hence complete the proof of this theorem.

The rest of the proof is an extensive case-by-case analysis. The list below shows that given each type of $\check{g}_j$, we can construct $\mu_j : Q' \to Q'$ satisfying (*). We shall start with the simple cases: types IV, III'_k and II'_k, followed by the types V″, $\mathrm{III}'_{k+\frac{1}{2}}$ and $\mathrm{II}'_{k+\frac{1}{2}}$ that requires some further analysis; refer to Figure 1.2.6.7 for the list of possible types of 1-strings. Within the list, we will omit the gradings when writing out the modules in Q' as it will be obvious from construction that μ_j preserves both the cohomological and internal gradings. We will also use solid arrows for differentials and dashed arrows for the isomorphism $\mu_j : Q' \to Q'$.

1. $\check{g}_j$ is of Type VI:
   This case is only possible when $j = 0$ and $g_j = c$. But in this case (Q', δ_0) is given by $0 \to P_n^A \to 0$, which is already the complex (Q', δ') as required ($\delta_0 = 0 = \delta'$).

2. $\check{g}_j$ is of Type III'_k for $k \in \mathbb{Z}$:
   The case when $k = 0$ is straightforward, where we just pick μ_j to be the identity.

   Now consider the case when $k > 0$. If $j = 0$, we get that $(\delta_j)_{G_j} = (\delta_0)_{G_j}$. For $j \geq 1$, δ_j satisfies (*) by our inductive assumption, so we have that $(\delta_j)_{G_j} = (\delta_0)_{G_j}$. Thus for all j, we can draw the part of (Q', δ_j) that contains $(Q'_{G_i}, (\delta_j)_{G_j}) = (Q'_{G_j}, (\delta_0)_{G_j})$ as follows:

$$\begin{array}{ccccccccccc}
P_n^A & \xrightarrow{2X_n} & P_n^A & \longrightarrow \cdots \longrightarrow & P_n^A & \xrightarrow{2X_n} & P_n^A & \xrightarrow{F} & P_{n-1}^A & \hookleftarrow & \nabla \\
\oplus & & \oplus & & \oplus & & \oplus & \searrow\swarrow & \oplus & \nearrow\searrow & \\
P_n^A & \xrightarrow{2X_n} & P_n^A & \longrightarrow \cdots \longrightarrow & P_n^A & \xrightarrow{2X_n} & P_n^A & \longrightarrow & P_{n+1}^A & &
\end{array} \qquad = (Q', \delta_j),$$

   where $F = \begin{bmatrix} (n|n-1) & -(n|n-1)i \\ (n|n+1) & (n|n+1)i \end{bmatrix} = \begin{bmatrix} (n|n-1) & 0 \\ 0 & (n|n+1) \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$ and where ∇ denotes the rest of the complex (Q', δ_j) containing the modules complement to Q'_{G_j}. In particular, for $j = 0$, ∇ is the part of (Q', δ_0) that contains the module Q'_{W_0}; if $j \geq 1$, then this case is only possible when $j = s - 1$ and ∇ is the part of (Q', δ_0) that contains the module $Q'_{V_{s-1}}$. Nevertheless, the construction of μ_j below depends only on the form above, so the construction will work for all j. Denote

$$M := \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \tag{1.5.8}$$

   and I as the 2×2 identity matrix. We define $\mu_j|_{Q'_{G_j}}$ to be the following dashed map, with μ_j acting as the identity map on the rest of the modules in ∇:

$$\begin{array}{ccccccccccc}
P_n^A & \xrightarrow{2X_n} & P_n^A & \longrightarrow \cdots \longrightarrow & P_n^A & \xrightarrow{2X_n} & P_n^A & \xrightarrow{F} & P_{n-1}^A & \hookleftarrow & \nabla \\
\oplus & & \oplus & & \oplus & & \oplus & \searrow\swarrow & \oplus & \nearrow\searrow & \\
P_n^A & \xrightarrow{2X_n} & P_n^A & \longrightarrow \cdots \longrightarrow & P_n^A & \xrightarrow{2X_n} & P_n^A & \longrightarrow & P_{n+1}^A & & \\
{\scriptstyle 2^{k-1}M}\downarrow & & \downarrow{\scriptstyle 2^{k-2}M} & & {\scriptstyle 2M}\downarrow & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle I} & & \\
P_n^A & \xrightarrow{X_n} & P_n^A & \longrightarrow \cdots \longrightarrow & P_n^A & \xrightarrow{X_n} & P_n^A & \xrightarrow{(n|n-1)} & P_{n-1}^A & \hookleftarrow & \nabla \\
\oplus & & \oplus & & \oplus & & \oplus & & \oplus & \nearrow\searrow & \\
P_n^A & \xrightarrow{X_n} & P_n^A & \longrightarrow \cdots \longrightarrow & P_n^A & \xrightarrow{X_n} & P_n^A & \xrightarrow{(n|n+1)} & P_{n+1}^A & &
\end{array} \qquad \begin{array}{l} = (Q', \delta_j) \\ \\ \\ \\ =: (Q', \delta_{j+1}). \end{array}$$

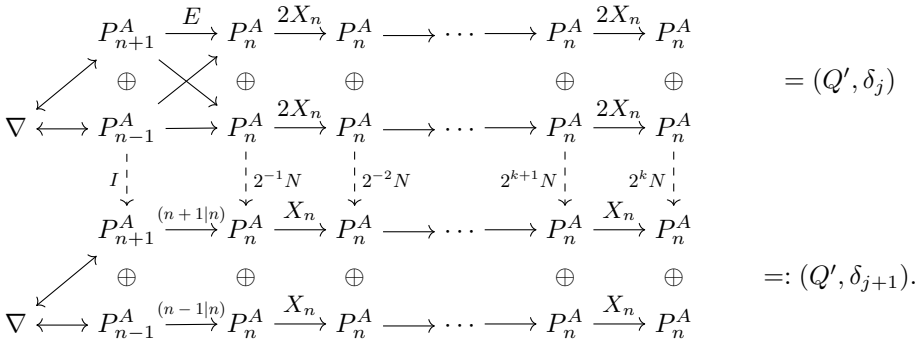
   The arrows in the last two rows shows the differential component $(\delta_{j+1})_{G_j}$ in δ_{j+1}, induced by the conjugation of μ_j^{-1}. Hence, the required condition (*) follows directly.
   Now consider when $k < 0$. As before, we have that $(\delta_j)_{G_j} = (\delta_0)_{G_j}$ for all j, so the analysis of the part of (Q', δ_j) that contains $(Q'_{G_j}, (\delta_j)_{G_j})$ will be the same. Similarly the construction of μ_i below will work for both cases. We draw the part of (Q', δ_j) that contains $(Q'_{G_j}, (\delta_j)_{G_j}) = (Q'_{G_j}, (\delta_0)_{G_j})$ as follows:

$$\begin{array}{ccccccccccc}
 & & P_{n+1}^A & \xrightarrow{E} & P_n^A & \xrightarrow{2X_n} & P_n^A & \longrightarrow \cdots \longrightarrow & P_n^A & \xrightarrow{2X_n} & P_n^A \\
 & \swarrow\nearrow & \oplus & \searrow\swarrow & \oplus & & \oplus & & \oplus & & \oplus \\
\nabla & \hookleftarrow & P_{n-1}^A & \longrightarrow & P_n^A & \xrightarrow{2X_n} & P_n^A & \longrightarrow \cdots \longrightarrow & P_n^A & \xrightarrow{2X_n} & P_n^A
\end{array} \qquad = (Q', \delta_j)$$

   with $E = \begin{bmatrix} -(n+1|n)i & (n-1|n)i \\ (n+1|n) & (n-1|n) \end{bmatrix}$ and where ∇ denotes the rest of the complex (Q', δ_j) containing the modules complement to Q'_{G_j}. Denote

$$N := \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \tag{1.5.9}$$

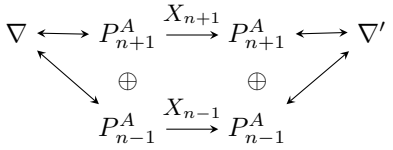
and I as the 2×2 identity matrix. Note that $N\begin{bmatrix} -(n+1|n)i & (n-1|n)i \\ (n+1|n) & (n-1|n) \end{bmatrix} = 2\begin{bmatrix} (n+1|n) & 0 \\ 0 & (n-1|n) \end{bmatrix}$. We define $\mu_j|_{Q'_{G_j}}$ to be the following dashed map, with μ_j the identity map on all modules in ∇:



The arrows in the last two rows shows the differential component $(\delta_{j+1})_{G_j}$ in δ_{j+1}, induced by the conjugation of μ_j^{-1}. It follows directly that the required condition (*) is satisfied.

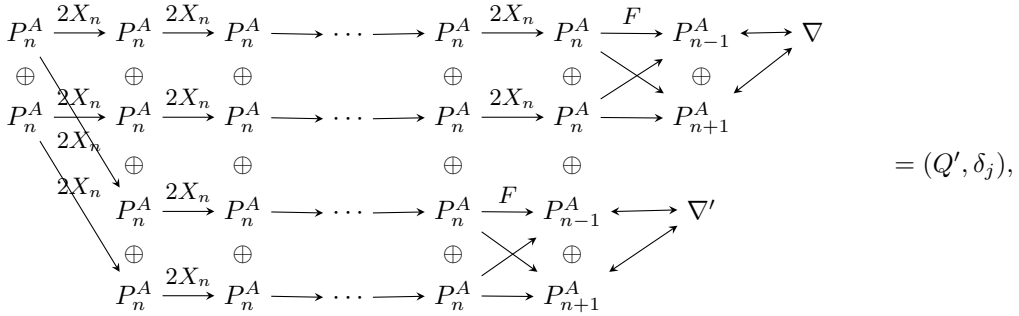
3. $\check{g}_j$ is of Type II'_k for $k \in \mathbb{Z}$:
As before, we have that $(\delta_j)_{G_j} = (\delta_0)_{G_j}$ for all j by our inductive assumption, so the part of (Q', δ_j) that contains $(Q'_{G_j}, (\delta_j)_{G_j})$ will be of the same form. As such, the construction of μ_j below will work for all j.

We shall start with $k = 0$. We draw the part of (Q', δ_j) that contains $(Q'_{G_j}, (\delta_j)_{G_j}) = (Q'_{G_j}, (\delta_0)_{G_j})$ as follows:

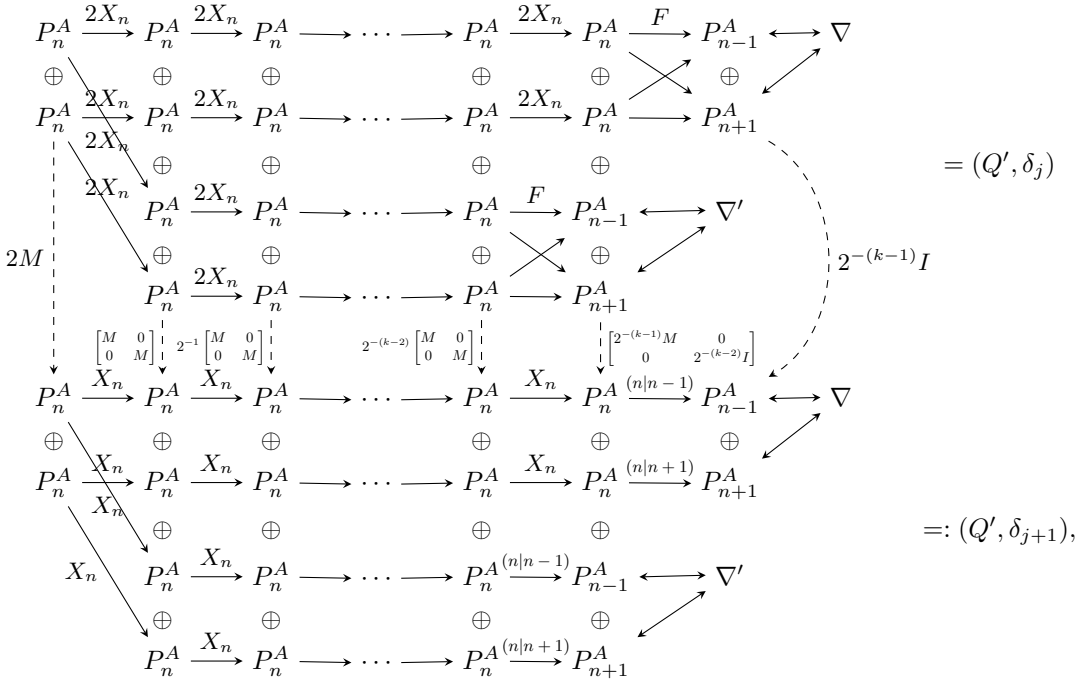


where either ∇ or ∇' is the part of (Q', δ_j) that contains the module Q'_{V_j}, whereas the other contains Q'_{W_j}. However, in this case we already have that $(\delta_j)_{G_j} = \delta'_{G_j}$, thus we just choose μ_j to be the identity map.

For $k > 0$, the part of (Q', δ_j) containing $(Q'_{G_j}, (\delta_j)_{G_j}) = (Q'_{G_j}, (\delta_0)_{G_j})$ is as follows:



where either ∇ or ∇' is the part of (Q', δ_j) that contains the module Q'_{V_j} and the other contains Q'_{W_j}. We define $\mu_j|_{Q'_{G_j}}$ to be the following dashed map:



where M is as in (1.5.8). For the rest of the modules in Q', μ_j sends v to $2^{-(k-1)}v$ (resp. v to $2^{-(k-2)}v$) for any v belonging to the modules in ∇ (resp. ∇'). The black arrows in the last four rows show the differential component $(\delta_{j+1})_{G_j}$ in δ_{j+1}, induced by the conjugation of μ_j^{-1}. It is easy to see that the required condition (*) is satisfied.

The construction for $k < 0$ is similar, using the map N (from (1.5.9)) in place of M.

4. $\check{g}_j$ is of Type V'':
Recall the definitions of g_j, c_j and r_j from the paragraph (**Slicing** *c*) and recall the subsets of crossings of $\mathfrak{m}(\check{c})$ as defined in (1.5.3). Let h_j be the connected component of $(c \setminus g_j) \cup (g_j \cap d_2)$ that contains the point m, so that g_j intersects h_j at the point r_j. To illustrate, $h_0 = c_0$ and $h_1 = c_0 \cup g_0 \cup c_1$ in Figure 1.5.3.1a; whereas $h_0 = \emptyset$, $h_1 = g_0 \cup c_1$, and $h_2 = g_0 \cup c_1 \cup g_1 \cup c_2$ in Figure 1.5.3.1b. Let $\mathfrak{m}(\check{h}_j) = \widetilde{h}_j \coprod \underset{\sim}{h}_j$ and $\mathfrak{m}(r_j) = \widetilde{r}_j \coprod \underset{\sim}{r}_j$ so that the curves of $\mathfrak{m}(\check{g}_j)$ and $\widetilde{h}_j$ intersect at the point $\widetilde{r}_j \in \theta_{n+1}$ and the curves of $\mathfrak{m}(\check{g}_j)$ and $\underset{\sim}{h}_j$ intersect at the point $\underset{\sim}{r}_j \in \theta_{n-1}$.

Now recall the decomposition of Q' given in (1.5.7). By definition, $\{\widetilde{r}_j, \underset{\sim}{r}_j\}$ is the subset of crossings $R_j \subseteq G_j$. We get that

$$P^A(\widetilde{r}_j) \oplus P^A(\underset{\sim}{r}_j) = Q'_{R_j}.$$

First consider the case when $j = 0$. Then $g_j = g_0$ is of this type only when $m \in \Lambda \setminus \{0, 1\}$ since $g_0 \cap \{1\} = \emptyset$, so we have $Q'_{V_0 \oplus R_0} = Q'_{C_0}$. Furthermore, (1.5.6) implies that $(\delta_0)_{C_0} = \delta'_{C_0}$, giving us

$$(Q'_{V_0} \oplus Q'_{R_0}, (\delta_0)_{V_0 \oplus R_0}) = (Q'_{C_0}, (\delta_0)_{C_0}) = (Q'_{C_0}, \delta'_{C_0}) = L_A(\mathfrak{m}(\check{c}_0))$$

with the last equality following from the definition of $(Q', \delta') = L_A(\mathfrak{m}(\check{c}))$. By definition of h_j, it follows that $c_0 = h_0$, so we can conclude that

$$(Q'_{V_0} \oplus Q'_{R_0}, (\delta_j)_{V_0 \oplus R_0}) = L_A(\mathfrak{m}(\check{h}_0)) = L_A(\widetilde{h_0}) \oplus L_A(\underset{\sim}{h_0}),$$

where the last equality follows from the fact that $\mathfrak{m}(\check{h}_0) = \widetilde{h_0} \coprod \underset{\sim}{h_0}$.

Now consider when $j \geq 1$. In this case δ_j satisfies property (*) by our induction hypothesis, allowing us to conclude that

$$(\delta_j)_{V_j} = \delta'_{V_j}, \quad (\delta_j)_{V_j R_j} = \delta'_{V_j R_j}, \text{ and } (\delta_j)_{R_j V_j} = \delta'_{R_j V_j}.$$

Thus we have that $(\delta_j)_{V_j\oplus R_j} = \delta'_{V_j\oplus R_j}$, giving us

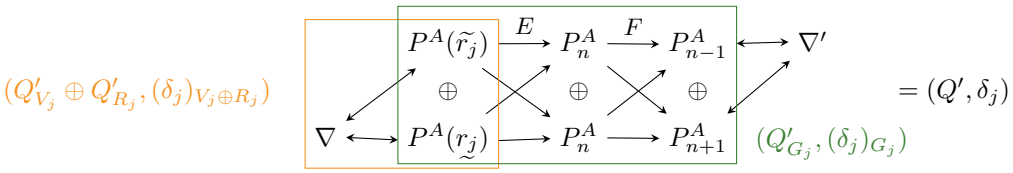
$$(Q'_{V_j} \oplus Q'_{R_j}, (\delta_j)_{V_j\oplus R_j}) = (Q'_{V_j} \oplus Q'_{R_j}, \delta'_{V_j\oplus R_j}) = L_A(\mathfrak{m}(\check{h}_j)) = L_A(\widetilde{h_j}) \oplus L_A(\underset{\sim}{h_j}),$$

where the second equality follows from the definition of $(Q', \delta') = L_A(\mathfrak{m}(\check{c}))$ and the third equality follows from the fact that $\mathfrak{m}(\check{h}_j) = \widetilde{h_j} \coprod \underset{\sim}{h_j}$.

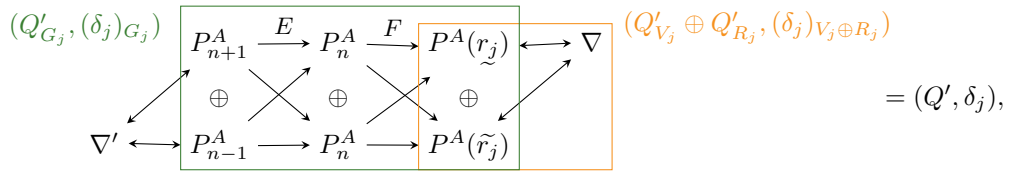
Thus for all j, we obtain

$$(Q'_{V_j} \oplus Q'_{R_j}, (\delta_j)_{V_j\oplus R_j}) = L_A(\widetilde{h_j}) \oplus L_A(\underset{\sim}{h_j}). \tag{1.5.10}$$

Furthermore, note that $\widetilde{h_j}$ and $\underset{\sim}{h_j}$ contain the points $\widetilde{r}_j$ and $\underset{\sim}{r_j}$ respectively, so $L_A(\widetilde{h_j})$ contains $P^A(\widetilde{r_j})$ as a submodule and $L_A(\underset{\sim}{h_j})$ contains $P^A(\underset{\sim}{r_j})$ as a submodule. Let us now understand the relation between $(Q'_{G_j}, (\delta_j)_{G_j})$, $(Q'_{V_j} \oplus Q'_{R_j}, (\delta_j)_{V_j\oplus R_j})$ and (Q', δ_j). As before our inductive hypothesis gives us $(\delta_j)_{G_j} = (\delta_0)_{G_j}$ for all j. So the part of (Q', δ_j) that contains $(Q'_{G_j}, (\delta_j)_{G_j})$ will be the same for all j, and it has either of the following two forms:

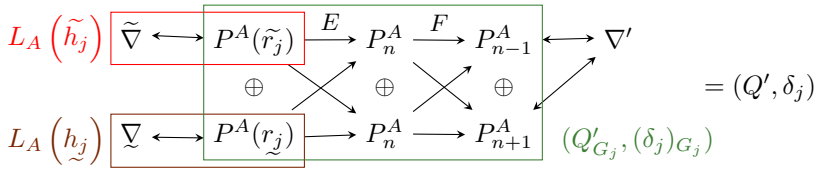


or

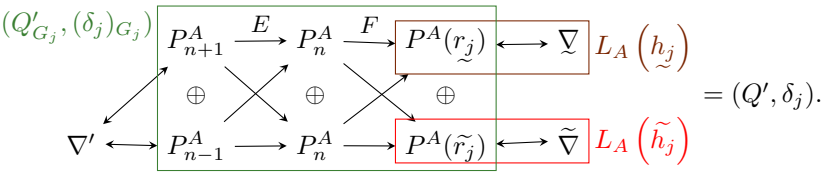


where ∇ denotes the rest of the complex (Q', δ_j) that contains the module Q'_{V_j} and ∇' denotes the rest of the complex (Q', δ_j) that contains the module Q'_{W_j}.

Using (1.5.10), the part of (Q', δ_j) that contains $(Q'_{G_j}, (\delta_j)_{G_j})$, $L_A(\widetilde{h_j})$ and $L_A(\underset{\sim}{h_j})$ has either of the following two forms:

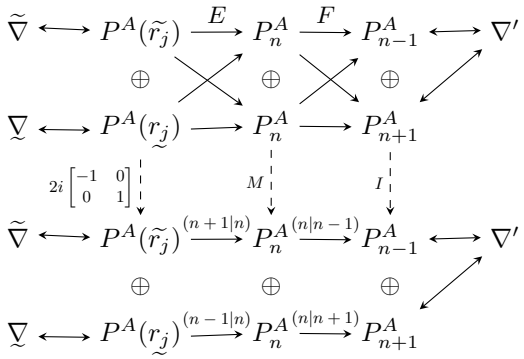


or



Thus for all j, we conclude that (Q', δ_j) must be of the above two possible forms. It is now sufficient to give a construction of μ_j for each of the two possible forms.

We begin with the construction of μ_j for the first possible form of (Q', δ_j). Firstly, recall M from (1.5.8) and note that $ME = 2i\begin{bmatrix} (n+1|n) & 0 \\ 0 & (n-1|n) \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. We define $\mu_j|_{Q'_{G_j}}$ to be the following dashed map:



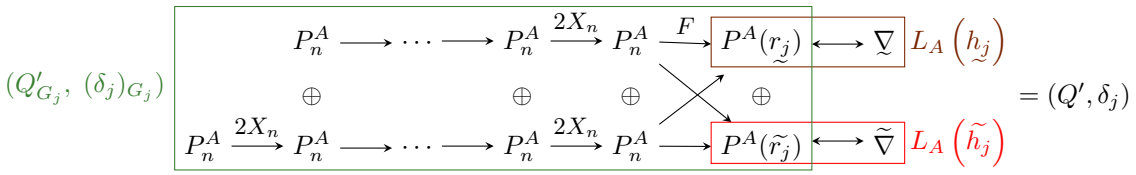
For the rest of the modules in Q', we define μ_j as the identity for modules contained in ∇', and μ_j sends v to $-2iv$ (resp. v to $2iv$) for any v belonging to the modules in $\widetilde{\nabla}$ (resp. $\underset{\sim}{\nabla}$). The black arrows in the last two rows shows the differential component $(\delta_{j+1})_{G_j}$ in δ_{j+1}, induced by the conjugation of μ_j^{-1}. It is easy to see that δ_{j+1} does indeed satisfy the required property (*).

The construction of μ_j for the second form is similar, changing $\mu_j|_{P_n^A \oplus P_n^A}$ to N from (1.5.9) instead of M.

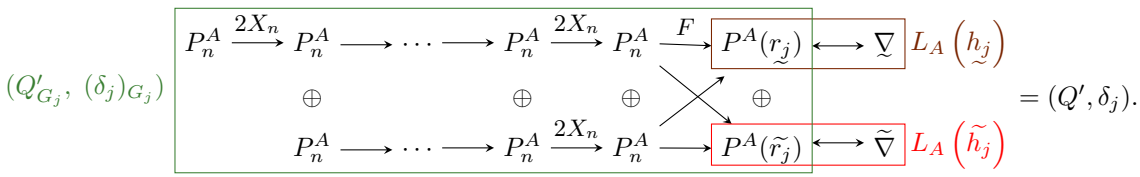
5. $\check{g}_j$ is of Type $\mathrm{III}'_{k+\frac{1}{2}}$ for $k \in Z$:
Note that the case for $k = 0$ is straightforward: μ_j is just the identity. We will provide the analysis and construction of μ_j for $k > 0$ and $k < 0$.

As in other types, the equations $(\delta_j)_{G_j} = (\delta_0)_{G_j}$ holds for all j by the induction hypothesis. Using the same argument in Type V″, one can show that (1.5.10) hold for all j in this type as well. So the part of (Q', δ_j) that contains $(Q'_{G_j}, (\delta_j)_{G_j})$, $L_A(\widetilde{h_j})$ and $L_A(\underset{\sim}{h_j})$ will be of the same form for all j. It follows that the construction of μ_j below will work for all j.

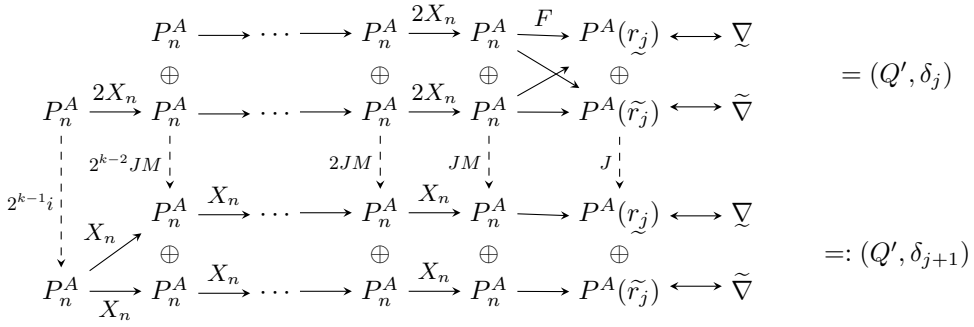
Let us start with $k > 0$. Using the same notation as in the analysis of Type V″ we can draw the part of (Q', δ_j) that contains $(Q'_{G_j}, (\delta_j)_{G_j}) = (Q'_{G_j}, (\delta_0)_{G_j})$, $L_A(\widetilde{h_j})$ and $L_A(\underset{\sim}{h_j})$ as either of the two forms:



or

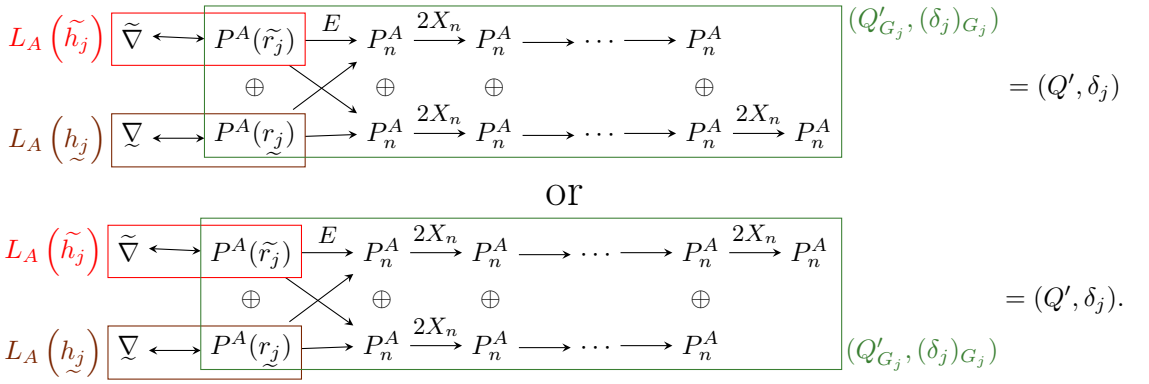


We will construct μ_j for the first form; the second form is a mirrored construction. Define $\mu_j|_{Q'_{G_j}}$ to be the following dashed map:

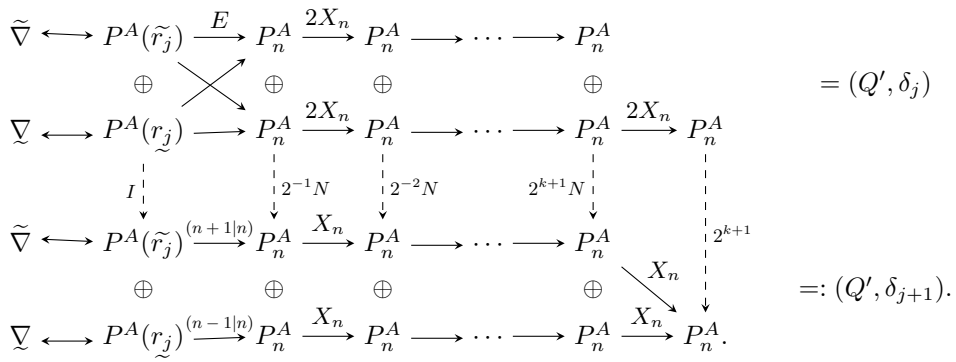


with $J = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. For the rest of the modules in Q', μ_j sends v to $-v$ (resp. v to v) for any v belonging to the modules in $\underset{\sim}{\nabla}$ (resp. $\widetilde{\nabla}$). The black arrows in the last two rows shows the differential component $(\delta_{j+1})_{G_j}$ in δ_{j+1}, induced by the conjugation of μ_j^{-1}. It is easy to see that μ_j does indeed satisfy the required property (*).

Similarly for $k < 0$, we draw the part of (Q', δ_j) that contains $(Q'_{G_j}, (\delta_j)_{G_j}) = (Q'_{G_j}, (\delta_0)_{G_j})$, $L_A(\widetilde{h}_j)$ and $L_A(\underset{\sim}{h}_j)$ as either of the two forms:



Once again we construct μ_j for the first form; the second form is a mirrored construction. Note that $NE = 2\left[\begin{smallmatrix} (n+1|n) & 0 \\ 0 & (n-1|n) \end{smallmatrix}\right]$. We define $\mu_j|_{Q'_{G_j}}$ to be the following dashed map, with μ_j acting as the identity on the rest of the modules in both $\widetilde{\nabla}$ and $\underset{\sim}{\nabla}$:

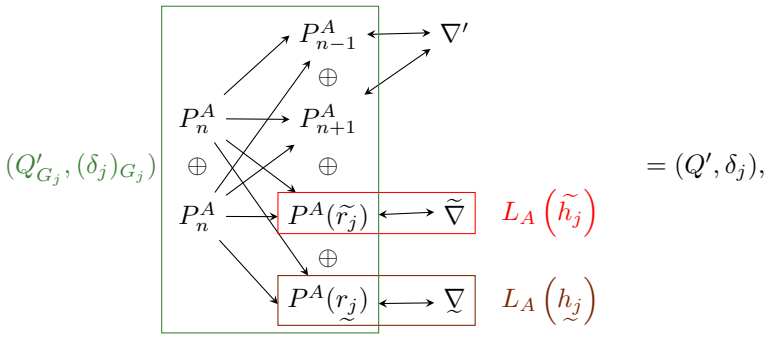


The black arrows in the last two rows shows the differential component $(\delta_{j+1})_{G_j}$ in δ_{j+1}, induced by the conjugation of μ_j^{-1}. The required condition (*) follows directly.

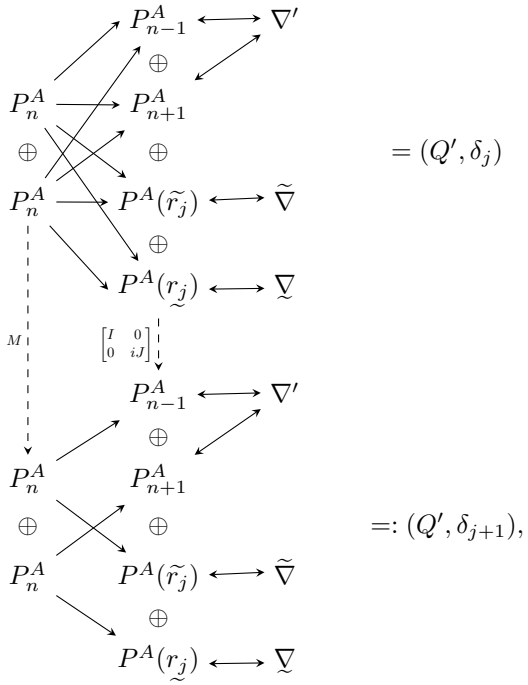
6. $\check{g}_j$ is of Type $\mathrm{II}'_{k+\frac{1}{2}}$ for $k \in \mathbb{Z}$:
As in other types, the equations $(\delta_j)_{G_j} = (\delta_0)_{G_j}$ holds for all j by the induction hypothesis. Using the same argument in Type V″, one can show that (1.5.10) hold for all j in this type as well. So the part

of (Q', δ_j) that contains $(Q'_{G_j}, (\delta_j)_{G_j})$, $L_A(\widetilde{h_j})$ and $L_A(\underset{\sim}{h_j})$ will be of the same form for all j. It follows that the construction of μ_j below will work for all j.

Let us start with $k = 0$. With the same notation as in the analysis of Type V″, we can draw the part of (Q', δ_j) that contains $(Q'_{G_j}, (\delta_j)_{G_j}) = (Q'_{G_j}, (\delta_0)_{G_j})$, $L_A(\widetilde{h_j})$ and $L_A(\underset{\sim}{h_j})$ as follows:

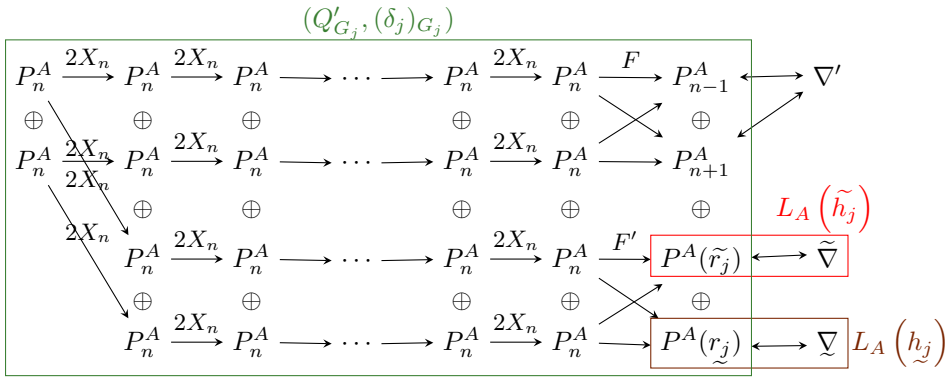


where the first map is given by $\left[\begin{smallmatrix} F \\ F' \end{smallmatrix}\right]$, with F' defined by $\left[\begin{smallmatrix} -i & 0 \\ 0 & i \end{smallmatrix}\right] F' = \left[\begin{smallmatrix} {\scriptstyle (n|n+1)} & 0 \\ 0 & {\scriptstyle (n|n-1)} \end{smallmatrix}\right] M$. We define $\mu_j|_{Q'_{G_j}}$ to be the following dashed map:

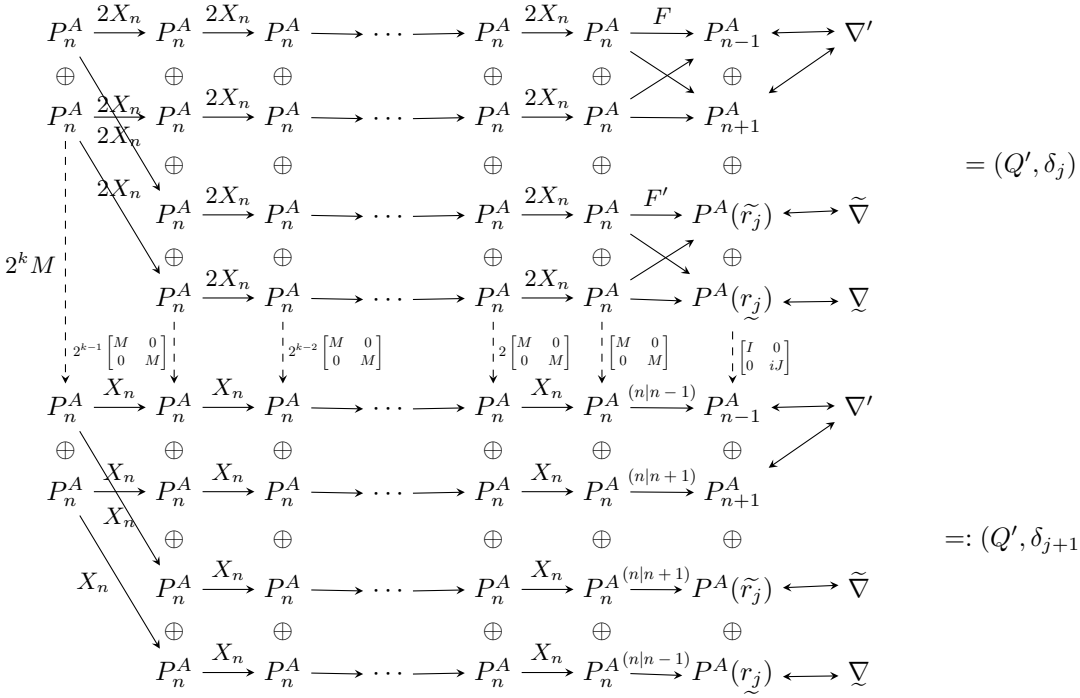


where $J = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, μ_j is the identity on all modules contained in ∇' and μ_j sends v to $-iv$ (resp. iv) for any v belonging to the modules in $\widetilde{\nabla}$ (resp. $\underset{\sim}{\nabla}$). The black arrows in the last four rows shows the differential component $(\delta_{j+1})_{G_j}$ in δ_{j+1}, induced by the conjugation of μ_j^{-1}. Thus, the required condition (*) is easily satisfied.

For $k > 0$, we can draw the part of (Q', δ_j) that contains $(Q'_{G_j}, (\delta_j)_{G_j}) = (Q'_{G_j}, (\delta_0)_{G_j})$, $L_A(\widetilde{h_j})$ and $L_A(\underset{\sim}{h_j})$ as follows:



Similarly, we define $\mu_j|_{Q'_{G_j}}$ to be the following dashed map:



where μ_j is the identity on all modules contained in ∇' and μ_j sends v to $-iv$ (resp. iv) for any v belonging to the modules in $\widetilde{\nabla}$ (resp. $\underset{\sim}{\nabla}$). The black arrows in the last four rows shows the differential component $(\delta_{j+1})_{G_j}$ in δ_{j+1}, induced by the conjugation of μ_j^{-1}. Thus, the required condition (*) is easily satisfied. The construction for $k < 0$ is similar, using the map N in place of M.

This completes the list of μ_j for all possible types of 1-string $\check{g}_j$ for all j, and thus completing the proof. Q.E.D.

Using this result, we can now deduce a type B_n version of Theorem 1.5.2:

Theorem 1.5.7. *For $\sigma^B \in \mathcal{A}(B_n)$ and an admissible trigraded curve $\check{c}$ in $\mathbb{D}^B_{n+1}$, we have that*

$$\sigma^B(L_B(\check{c})) \cong L_B(\sigma^B(\check{c})),$$

in the category of $Kom^b(\mathscr{B}_n\text{-}p_r g_r mod)$, i.e. the map L_B is $\mathcal{A}(B_n)$-equivariant.

Proof. Let $\check{c}$ be a trigraded curve in $\mathbb{D}^B_{n+1}$ and σ^B be an element of $\mathcal{A}(B_n)$. By Proposition 1.5.5, the diagram in Figure 1.5.0.1 commutes. Together with the three other maps being equivariant, we can conclude that

$$\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} \left(L_B(\sigma^B(\check{c}))\right) \cong \mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} (\sigma^B(L_B(\check{c}))). \tag{1.5.11}$$

Recall that the functor $\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n} -$ was defined as $\mathscr{A}_{2n-1}\otimes_{\ddot{\mathscr{B}}_n}\mathfrak{F}(-)$ (see paragraph after Proposition 1.4.1 for the definitions). Since $\mathscr{A}_{2n-1}\cong\mathbb{C}\otimes_{\mathbb{R}}\ddot{\mathscr{B}}_n$ as graded $\mathbb{C}$-algebras, (1.5.11) together with the fact that both categories $\mathrm{Kom}^b(\ddot{\mathscr{B}}_n\text{-}p_rg_rmod)$ and $\mathrm{Kom}^b(\mathscr{A}_{2n-1}\text{-}p_rg_rmod)$ are Krull-Schmidt implies that

$$L_B(\sigma^B(\check{c}))\cong\sigma^B(L_B(\check{c}))\langle s\rangle, \tag{1.5.12}$$

with $s=0$ or 1.

We aim to show that $s=0$ for all cases. First consider the case when $c\cap\{0\}=\emptyset$. As $\check{c}\simeq\chi(r_1,r_2,r_3)\beta(\check{b}_2)$ and $P_2^B\cong P_2^B\langle1\rangle$, it follows easily that $(L_B(\check{c}))\langle0\rangle\cong(L_B(\check{c}))\langle1\rangle$, and so we are done. Now consider when c has one of its end points at 0. Note that it is sufficient to prove the statement for $\sigma^B=\sigma_j^B$ for each j. We assign to each complex C in $\mathrm{Kom}^b(\mathscr{B}_n\text{-}\mathrm{p}_r\mathrm{g}_r\mathrm{mod})$ an element of $\mathbb{Z}/2\mathbb{Z}$ denoted by $\mathrm{sgn}(C)$, by taking the sum of the third grading $\langle-\rangle$ over all modules P_1 in C. One can easily show that sgn is invariant under isomorphisms in $\mathrm{Kom}^b(\mathscr{B}_n\text{-}\mathrm{p}_r\mathrm{g}_r\mathrm{mod})$, where using Lemma 1.5.8 below, we get that s must be 0 as required. Q.E.D.

Lemma 1.5.8. *For any trigraded curve $\check{c}$ with one of its endpoint at $\{0\}$ and any generating braid σ_j^B, we have $sgn\left(\sigma_j^B(L_B(\check{c}))\right)=sgn(L_B(\sigma_j^B(\check{c})))$.*

Proof. For $j\geq2$, it is clear that $\mathrm{sgn}\left(\sigma_j^B(L_B(\check{c}))\right)=\mathrm{sgn}\left(L_B(\check{c})\right)=\mathrm{sgn}(L_B(\sigma_j^B(\check{c})))$.

Now fix $j=1$. First consider the case when $\check{c}$ is of type VI, i.e. $\check{c}=\chi^B(r_1,r_2,r_3)\check{b}_1$. Then $\sigma_1^B(\check{c})=\chi^B(r_1-1,r_2+1,r_3+1)\check{b}_1$ by Lemma 1.2.4(2). So, $L_B(\sigma_1^B(\check{c}))=P_1^B[-r_1+1]\{r_2+1\}\langle r_3+1\rangle$. Besides,

$$\begin{aligned}\sigma_1^B(L_B((\check{c})))&\cong\left(P_1^B[1]\{1\}\oplus P_1^B[1]\xrightarrow{[X_1\ \ id]}P_1^B\right)[-r_1]\{r_2\}\langle r_3+1\rangle\\&\cong P_1^B[-r_1+1]\{r_2+1\}\langle r_3+1\rangle.\end{aligned}$$

Thus, $\mathrm{sgn}\left(\sigma_j^B(L_B(\check{c}))\right)=r_3+1=\mathrm{sgn}(L_B(\sigma_j^B(\check{c})))$.

Otherwise, we analyse sgn based on the number of 2-crossing in $\check{c}$. Note that, for 1-strings $\check{g}$,

$$\mathrm{sgn}\left(L_B(\sigma_1^B(\check{g}))\right)=\begin{cases}\mathrm{sgn}\left(L_B(\check{g})\right), & \text{when } \check{g} \text{ is of type } II'_w,\ II'_{w+\frac{1}{2}},\ III'_{w+\frac{1}{2}} \text{ and } V'';\\ \mathrm{sgn}\left(L_B(\check{g})\right)+1, & \text{when } \check{g} \text{ is of type } III'_w.\end{cases}$$

Note that σ_1^B won't change the number of 2-crossings of $\check{c}$ and as $\check{c}\cap\{0\}=\{0\}$, $\check{c}$ contains 1-string of type III'_w if and only if $\check{c}$ has even number of 2-crossings. Since $\mathrm{sgn}\left(L_B(\sigma_1^B(\check{c}))\right)$ can be computed by summing over all 1-strings of $\check{c}$, we conclude that

$$\mathrm{sgn}\left(L_B(\sigma_1^B(\check{c}))\right)=\begin{cases}\mathrm{sgn}\left(L_B(\check{c})\right), & \text{if } \check{c} \text{ has odd number of 2-crossings;}\\ \mathrm{sgn}\left(L_B(\check{c})\right)+1, & \text{if } \check{c} \text{ has even number of 2-crossings.}\end{cases}$$

On the other hand, note that $\check{c}$ and $\sigma_1^B(\check{c})$ both have an odd number of 1-crossings. Moreover, $\mathrm{sgn}(C\langle1\rangle)=\mathrm{sgn}(C)+1$ if and only if C has an odd number of underlying P_1^B's. As such, we have that

$$\mathrm{sgn}\left(\sigma_1^B(L_B(\check{c}))\right)=\begin{cases}\mathrm{sgn}\left(L_B(\check{c})\right), & \text{if } L_B(\check{c}) \text{ has odd number of modules } P_2^B;\\ \mathrm{sgn}\left(L_B(\check{c})\right)+1, & \text{if } L_B(\check{c}) \text{ has even number of modules } P_2^B,\end{cases}$$

since $\sigma_1^B=R_1\langle1\rangle$, ${}_1P^B\otimes_{\mathscr{B}_n}P_1^B\cong\mathbb{R}\oplus\mathbb{R}\{1\}$, ${}_1P^B\otimes_{\mathscr{B}_n}P_2^B\cong\mathbb{R}\oplus\mathbb{R}\langle1\rangle$ and ${}_1P^B\otimes_{\mathscr{B}_n}P_j^B=0$ for all $j\geq3$ (see Proposition 1.3.5 and Remark 1.3.6). Thus, we get $\mathrm{sgn}\left(\sigma_1^B(L_B(\check{c}))\right)=\mathrm{sgn}\left(L_B(\sigma_1^B(\check{c}))\right)$. Q.E.D.

1.6 Categorification of Homological Representations

In this section, we shall relate the categorical representations of type A_{2n-1} and type B_n Artin groups to their representations on the first homology of surfaces.

Throughout this section we will use $\mathcal{K}_A := \mathrm{Kom}^b(\mathscr{A}_{2n-1}\text{-p}_r\text{g}_r\text{mod})$ and $\mathcal{K}_B := \mathrm{Kom}^b(\mathscr{B}_n\text{-p}_r\text{g}_r\text{mod})$ as shorthand notations. We will also use $\mathcal{Z}_A$, $\mathcal{Z}_{B,s}$ and $\mathcal{Z}_{B,r}$ to denote the rings $\mathbb{Z}[q^{\pm 1}]$, $\mathbb{Z}[q^{\pm 1}, s]/\langle s^2-1\rangle$ and $\mathbb{Z}[q^{\pm 1}, r]/\langle r^2-1\rangle$ respectively. We denote the Grothendieck group of $\mathcal{K}_A$ and $\mathcal{K}_B$ as $K_0(\mathcal{K}_A)$ and $K_0(\mathcal{K}_B)$ respectively; recall that they are the abelian groups freely generated by the isomorphism classes of objects, quotient by the relation $[\mathrm{cone}(A \xrightarrow{f} B)] = [B] - [A]$.

1.6.1 Representations on Grothendieck groups

First, consider the Grothendieck group $K_0(\mathcal{K}_A)$. The functor $\{1\}$ makes $K_0(\mathcal{K}_A)$ into a $\mathcal{Z}_A$-module defined by $[X\{1\}] = q[X]$. Note that $K_0(\mathcal{K}_A)$ is isomorphic to the split Grothendieck group $K_0^{\oplus}(\mathscr{A}_{2n-1}\text{-}p_r g_r\text{mod})$ (see [Ros15, Theorem 1.1]), so $K_0(\mathcal{K}_A)$ is a free module over $\mathcal{Z}_A$ of rank $2n-1$, generated by $\{[P_j^A] \mid 1 \leq j \leq 2n-1\}$. The action of $\mathcal{A}(A_{2n-1})$ on $\mathcal{K}_A$ preserves cones; namely for all $\sigma \in \mathcal{A}(A_{2n-1})$,

$$\sigma(\mathrm{cone}(A \xrightarrow{f} B)) \cong \mathrm{cone}(\sigma(A) \xrightarrow{\sigma(f)} \sigma(B)).$$

Moreover, the action commutes with the grading shift functor, so we have an induced $\mathcal{Z}_A$-linear representation of $\mathcal{A}(A_{2n-1})$ on $K_0(\mathcal{K}_A)$, which we denote by $\rho_{KA} : \mathcal{A}(A_{2n-1}) \to \mathrm{Aut}_{\mathcal{Z}_A}(K_0(\mathcal{K}_A))$.

Now consider the Grothendieck group $K_0(\mathcal{K}_B)$. The functors $\{1\}$ and $\langle 1\rangle$ make $K_0(\mathcal{K}_B)$ into a module over $\mathcal{Z}_{B,s}$. As $P_j^B\langle 1\rangle \cong P_j^B$ for all $j \geq 2$, we have that $s[P_j] = [P_j]$ for all $j \geq 2$. It is easy to see now that

$$K_0(\mathcal{K}_B) \cong \mathcal{Z}_{B,s} \oplus (\mathcal{Z}_{B,s}/\langle s-1\rangle)^{\oplus n-1}$$

as $\mathcal{Z}_B$-modules, generated by $\{[P_j^B] \mid 1 \leq j \leq n\}$. As before, the action of $\mathcal{A}(B_n)$ on $\mathcal{K}_B$ preserves cones and it commutes with the grading shift functors, so we have an induced $\mathcal{Z}_{B,s}$-linear representation of $\mathcal{A}(B_n)$ on $K_0(\mathcal{K}_B)$, which we denote by $\rho_{KB} : \mathcal{A}(B_n) \to \mathrm{Aut}_{\mathcal{Z}_{B,s}}(K_0(\mathcal{K}_B))$.

1.6.2 Homological representations

It is well-known that the reduced Burau representation of $\mathcal{A}(A_{2n-1})$ can be realised as a homological representation (see for example [KT08, Theorem 3.7 and Theorem 3.9]). Nonetheless, we shall spell out the construction here as it will shed some light on the construction of the homological representation for type B_n and also clarify the relationship between them.

Consider the covering space $\mathcal{D}_{2n}$ classified by the cohomology class $C_{\mathcal{D}} \in H^1(\mathbb{D}_{2n}^A \setminus \Delta, \mathbb{Z})$ defined by $[\lambda_k] \mapsto 1$, for all $k \in \Delta = \{-n, ..., -1, 1, ..., n\}$, where each λ_k is a closed loop around the puncture k. It is easy to see that the space $\mathcal{D}_{2n}$ is homotopy equivalent to the infinite graph $\Gamma_{\mathbb{Z}}$ given below:

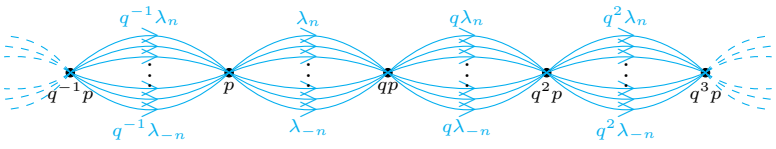


Figure 1.6.2.1: The infinite graph $\Gamma_{\mathbb{Z}}$ homotopy equivalent to $\mathcal{D}_{2n}$.

The action of $\mathcal{A}(A_{2n-1})$ on $\mathbb{D}_{2n}^A \setminus \Delta$ lifts to an action on $\mathcal{D}_{2n}$ that commutes with the deck transformation group $\mathbb{Z}$, so it induces a $\mathbb{Z}[\mathbb{Z}]$-linear action of $\mathcal{A}(A_{2n-1})$ on $H_1(\mathcal{D}_{2n}, \mathbb{Z})$, which we denote by $\rho_{RHA} : \mathcal{A}(A_{2n-1}) \to \mathrm{Aut}_{\mathbb{Z}[\mathbb{Z}]}(H_1(\mathcal{D}_{2n}, \mathbb{Z}))$.

Now let $\Delta_0 := \Delta \cup \{0\}$ and consider the covering space $\mathscr{D}_{2n+1}$ of $\mathbb{D}_{2n}^A \setminus \Delta_0$ classified by the cohomology class $C_{\mathscr{D}} \in H^1(\mathbb{D}_{2n}^A \setminus \Delta_0, \mathbb{Z})$ defined by

$$[\lambda_j] \mapsto \begin{cases} 1, & \text{for } j \neq 0; \\ 0, & \text{for } j = 0, \end{cases}$$

where each λ_j is a closed loop around the puncture j. Note that the composition of coverings

$$\mathscr{D}_{2n+1} \to \mathbb{D}^A_{2n} \setminus \Delta_0 \to \mathbb{D}^B_{n+1} \setminus \Lambda$$

is a normal covering space of $\mathbb{D}^B_{n+1} \setminus \Lambda$, with its group of deck transformation isomorphic to $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (cf. Lemma 1.2.2).

Let l_j be a closed loop around each puncture $j \in \Lambda$ of $\mathbb{D}^B_{n+1}$. Note that in $H_1(\mathbb{D}^A_{2n} \setminus \Delta_0, \mathbb{Z})$ we have that

$$\begin{cases} [\lambda_0] = [l_0] + r[l_0]; & \\ [\lambda_{-j}] = r[l_j], & \text{for } j > 0; \\ [\lambda_j] = [l_j], & \text{for } j > 0. \end{cases} \tag{1.6.1}$$

As such, the space $\mathscr{D}_{2n+1}$ is homotopy equivalent to the space given below:

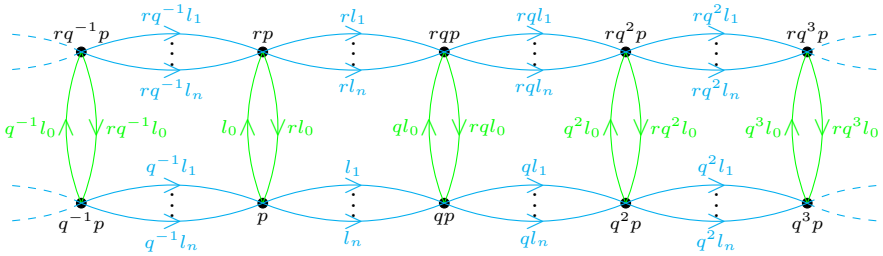


Figure 1.6.2.2: The infinite graph $\Gamma_{\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}$ homotopy equivalent to $\mathscr{D}_{2n+1}$.

The action of $\mathcal{A}(B_n)$ on $\mathbb{D}^B_{n+1} \setminus \Lambda$ lifts to an action on $\mathscr{D}_{2n+1}$ that commutes with deck transformation group $\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}] \cong \mathcal{Z}_{B,r}$, so it induces a $\mathcal{Z}_{B,r}$-linear action on $H_1(\mathscr{D}_{2n+1}, \mathbb{Z})$, which we denote by $\widetilde{\rho_{RHB}} : \mathcal{A}(B_n) \to \mathrm{Aut}_{\mathcal{Z}_{B,r}}(H_1(\mathscr{D}_{2n+1}, \mathbb{Z}))$.

1.6.3 Relating the representations on Grothedieck groups and homology groups

It has been shown in [KS02, Section 2e.1] that the induced action of $\mathcal{A}(A_{2n-1})$ on the Grothendieck group $K_0(\mathcal{K}_A)$ is isomorphic to the reduced Burau representation, but we shall spell it out here before dealing with the type B case. Recall that $\mathscr{H}_n$ and $K_0(\mathcal{K}_B)$ are modules over $\mathcal{Z}_{B,r}$ and $\mathcal{Z}_{B,s}$ respectively, whereas $H_1(\mathcal{D}_{2n}, \mathbb{Z})$ and $K_0(\mathcal{K}_A)$ are modules over $\mathcal{Z}_A$.

Proposition 1.6.1. *The two $\mathcal{Z}_A$-linear representations $\rho_{KA} : \mathcal{A}(A_{2n-1}) \to Aut_{\mathcal{Z}_A}(K_0(\mathcal{K}_A))$ and $\rho_{RHA} : \mathcal{A}(A_{2n-1}) \to Aut_{\mathbb{Z}[\mathbb{Z}]}(H_1(\mathcal{D}_{2n}, \mathbb{Z}))$ are isomorphic. In particular, the categorical action of $\mathcal{A}(A_{2n-1})$ on $\mathcal{K}_A$ categorifies the reduced Burau representation.*

Proof. One can check that $H_1(\mathcal{D}_{2n}, \mathbb{Z})$ is a free module over $\mathbb{Z}[\mathbb{Z}] \cong \mathcal{Z}_A$ of rank $2n-1$, with basis $\{[\gamma_1], ..., [\gamma_{2n-1}]\}$ defined by:

$$[\gamma_j] := \begin{cases} [\lambda_{-1}] - [\lambda_1], & \text{for } j = n; \\ (-1)^{n-j}([\lambda_{-n+j-1}] - [\lambda_{-n+j}]), & \text{for } j \leq n-1; \\ (-q)^{n-j}([\lambda_{-n+j}] - [\lambda_{-n+j+1}], & \text{for } j \geq n+1. \end{cases} \tag{1.6.2}$$

Similarly, $\{P^A_1, P^A_2, ..., P^A_{2n-1}\}$ is a $\mathcal{Z}_A$-basis for $K_0(\mathcal{K}_A)$.

Under both of these bases, both ρ_{KA} and ρ_{RHA} are given by the same matrices for each generator of $\mathcal{A}(A_{2n-1})$ as follows:

$$\sigma^A_1 \mapsto \begin{bmatrix} -q & -q & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{bmatrix}, \qquad \sigma^A_{2n-1} \mapsto \begin{bmatrix} I_{n-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -q & -q \end{bmatrix},$$

$$\sigma_j^A \mapsto \begin{cases} \begin{bmatrix} I_{j-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -q & -q & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-j-1} \end{bmatrix}, & \text{for } 2 \leq j \leq n-1; \\ \begin{bmatrix} I_{j-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -q & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-j-1} \end{bmatrix}, & \text{for } j = n; \\ \begin{bmatrix} I_{j-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -q & -q & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-j-1} \end{bmatrix}, & \text{for } n+1 \leq j \leq 2n-2. \end{cases}$$

It follows that ρ_{KA} and ρ_{RHA} are isomorphic representations. Q.E.D.

Proposition 1.6.2. *Under the identification $\mathcal{Z}_{B,s} \cong \mathcal{Z}_{B,r}$ given by $s \mapsto -r$ and $q \mapsto q$, the $\mathcal{Z}_{B,s}$-linear representation $\rho_{KB} : \mathcal{A}(B_n) \to Aut_{\mathcal{Z}_{B,s}}(K_0(\mathcal{K}_B))$ is isomorphic to a $\mathcal{Z}_{B,s}$-linear subrepresentation of $\widetilde{\rho_{RHB}} : \mathcal{A}(B_n) \to Aut_{\mathcal{Z}_{B,s}}(H_1(\mathscr{D}_{2n+1}, \mathbb{Z}))$.*

Proof. Consider the sub $\mathcal{Z}_{B,r}$-module $\mathscr{H}_n \subseteq H_1(\mathscr{D}_{2n+1}, \mathbb{Z})$ generated by $\{[\xi_1], ..., [\xi_n]\}$, where:

$$[\xi_j] = \begin{cases} (1-q)[l_0] - (1-r)[l_1] & \text{for } j = 1; \\ (-q)^{1-j}(1-r)([l_{j-1}] - [l_j]) & \text{for } j \geq 2. \end{cases}$$

Note that $[\xi_j] = -r[\xi_j]$ for all $j \geq 2$, so that $\mathscr{H}_n \cong \mathcal{Z}_{B,r} \oplus (\mathcal{Z}_{B,r}/\langle r+1 \rangle)^{\oplus n-1}$ as $\mathcal{Z}_{B,r}$-modules.

It is easy to verify on the generators that $\mathscr{H}_n \subseteq H_1(\mathscr{D}_{2n+1}, \mathbb{Z})$ is closed under the action of $\mathcal{A}(B_n)$, so we obtain a $\mathcal{Z}_{B,r}$-linear subrepresentation on $\mathscr{H}_n$, which we denote by $\rho_{RHB} : \mathcal{A}(B_n) \to \mathrm{Aut}_{\mathcal{Z}_{B,r}}(\mathscr{H}_n)$.

Using the set of generators $\{P_1^B, P_2^B, ..., P_n^B\}$ for $\mathcal{K}_B$, ρ_{KB} is given by the following matrices for each generator of $\mathcal{A}(B_n)$:

$$\rho_{KB}(\sigma_1^B) = \begin{bmatrix} -sq & -(1+s) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{bmatrix}; \qquad \rho_{KB}(\sigma_n^B) = \begin{bmatrix} I_{n-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -q & -q \end{bmatrix};$$

$$\rho_{KB}(\sigma_j^B) = \begin{bmatrix} I_{j-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -q & -q & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-j-1} \end{bmatrix}, \qquad \text{for } j \neq 1, n.$$

One can check that using the set of generators $\{[\xi_1], ..., [\xi_n]\}$ for $\mathscr{H}_n$, ρ_{RHB} is defined by associating the above matrices to the generators of $\mathcal{A}(B_n)$ with $s = -r$. Thus, under the identification $\mathcal{Z}_{B,s} \cong \mathcal{Z}_{B,r}$ given by $s \mapsto -r$ and $q \mapsto q$, the two representations are isomorphic. Q.E.D.

Denote $ev_{\pm 1} : \mathcal{Z}_{B,s} \to \mathcal{Z}_A$ as the $\mathbb{Z}[q^{\pm 1}]$-linear evaluation maps defined by $s \mapsto \pm 1$. Throughout the rest of this section, we shall view $H_1(\mathcal{D}_{2n}, \mathbb{Z})$ as a $\mathcal{Z}_{B,s}$-module through ev_{-1} and $K_0(\mathcal{K}_A)$ as a $\mathcal{Z}_{B,s}$-module through ev_1.

The functor $\mathscr{A}_{2n-1} \otimes_{\mathscr{B}_n} -$ as in Proposition 1.4.2 preserves cones, hence it induces a map on the

Grothendieck groups $K_0(\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n}-):K_0(\mathcal{K}_B)\to K_0(\mathcal{K}_A)$, given by

$$K_0(\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n}-)([P_j^B])=\begin{cases}[P_n^A], & \text{for } j=1;\\ [P_{n-(j-1)}^A]+[P_{n+(j-1)}^A], & \text{otherwise.}\end{cases}$$

On the other hand, the natural inclusion map $\iota:\mathbb{D}_{2n}^A\setminus\Delta_0\to\mathbb{D}_{2n}^A\setminus\Delta$ induces a map on the homology $\iota:H_1(\mathbb{D}_{2n}^A\setminus\Delta_0,\mathbb{Z})\to H_1(\mathbb{D}_{2n}^A\setminus\Delta,\mathbb{Z})$ that sends

$$[\lambda_j]\mapsto\begin{cases}0, & \text{for } j=0;\\ [\lambda_j], & \text{otherwise.}\end{cases}$$

Thus, ι lifts uniquely to $\widetilde{\iota}:\mathscr{D}_{2n+1}\to\mathcal{D}_{2n}$, which induces a map on the homology $\widetilde{\iota}:H_1(\mathscr{D}_{2n+1},\mathbb{Z})\to H_1(\mathcal{D}_{2n},\mathbb{Z})$. We shall now show a "decategorified" analogue of Theorem 1.5.1:

Theorem 1.6.3. *Consider the action of $\mathcal{A}(B_n)$ on $\mathscr{H}_n$ and $K_0(\mathcal{K}_B)$ given by $\rho_{RHB}=\widetilde{\rho_{RHB}|_{\mathscr{H}_n}}$ and ρ_{KB} respectively as in Proposition 1.6.2; similarly consider the action of $\mathcal{A}(B_n)\overset{\Psi}{\hookrightarrow}\mathcal{A}(A_{2n-1})$ on $H_1(\mathcal{D}_{2n},\mathbb{Z})$ and $K_0(\mathcal{K}_A)$ given by ρ_{RHA} and ρ_{KA} respectively as in Proposition 1.6.1. Then there exists $\mathcal{Z}_{B,s}$-linear isomorphisms Θ_A and Θ_B such that following diagram is commutative with all four maps $\mathcal{Z}_{B,s}$-linear and $\mathcal{A}(B_n)$-equivariant:*

$$\begin{array}{ccc}
\mathcal{A}(B_n) & & \mathcal{A}(A_{2n-1})\overset{\Psi}{\hookleftarrow}\mathcal{A}(B_n)\\
\circlearrowright & & \circlearrowright\\
\mathscr{H}_n & \xrightarrow{\quad\widetilde{\iota}\quad} & H_1(\mathcal{D}_{2n},\mathbb{Z})\\
\cong\big\downarrow\Theta_B & & \cong\big\downarrow\Theta_A\\
K_0(\mathcal{K}_B) & \xrightarrow{K_0(\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n}-)} & K_0(\mathcal{K}_A)\\
\circlearrowright & & \circlearrowright\\
\mathcal{A}(B_n) & & \mathcal{A}(A_{2n-1})\overset{\Psi}{\hookleftarrow}\mathcal{A}(B_n)
\end{array}.$$

Proof. We use isomorphisms Θ_A and Θ_B which identify the respective generators chosen in the proof of Proposition 1.6.1 and Proposition 1.6.2 respectively.

The functor $\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n}-$ commutes with the grading functor, so $K_0(\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n}-)$ is $\mathcal{Z}_{B,s}$ linear. The fact that it is $\mathcal{A}(B_n)$-equivariant follows from Theorem 1.4.3. On the other hand, it follows immediately from the construction of the covering spaces $\mathscr{D}_{2n+1}$ and $\mathcal{D}_{2n}$ that $\widetilde{\iota}$ is a $\mathcal{Z}_{B,s}$-linear map and is $\mathcal{A}(B_n)$-equivariant.

Using (1.6.1) and (1.6.2), the restriction of $\widetilde{\iota}$ to $\mathscr{H}_n$ is given by

$$\widetilde{\iota}([\xi_j])=\begin{cases}[\gamma_n], & \text{for } j=1;\\ [\gamma_{n-(j-1)}]+[\gamma_{n+(j-1)}], & \text{for } j\geq 2.\end{cases}$$

It now follows immediately that $\Theta_A\circ\widetilde{\iota}=K_0(\mathscr{A}_{2n-1}\otimes_{\mathscr{B}_n}-)\circ\Theta_B$. Q.E.D.

1.7 Trigraded Intersection Numbers, Graded Dimensions of Homomorphism Spaces

In this section, we shall relate the trigraded intersection number and the Hom spaces between the corresponding complexes. Throughout this section, we will fix the following shorthand notations: $\mathcal{K}_B:=\mathrm{Kom}^b(\mathscr{B}_n\text{-}\mathrm{p_r g_r mod})$, $\mathcal{K}_A:=\mathrm{Kom}^b(\mathscr{A}_{2n-1}\text{-}\mathrm{p_r g_r mod})$, $\mathscr{B}m:=\mathscr{B}_n\text{-mod}$ and $\mathscr{A}m:=\mathscr{A}_{2n-1}\text{-mod}$.

For $V=\oplus_{(r,s)\in\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}V_{(r,s)}\{r\}\langle s\rangle$ a $(\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})$-graded $\mathbb{R}$-vector space, we denote its *bigraded dimension* as

$$\mathrm{bigrdim}(V):=\sum_{(r,s)\in\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}\dim(V_{(r,s)})q_2^rq_3^s.$$

Recall that for each pair of objects $(C^*, \partial_C), (D^*, \partial_D)$ in $\mathcal{K}_B$, one can consider the internal (bigraded) Hom complex $\mathrm{HOM}^*_{\mathcal{K}_B}(C, D)$ defined as follows: for each cohomological degree $s_1 \in \mathbb{Z}$,

$$\mathrm{HOM}^{s_1}_{\mathcal{K}_B}(C, D) := \bigoplus_{\substack{(s_2,s_3)\in\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z},\\ m+n=s_1}} \mathrm{Hom}_{\mathscr{B}m}(C^m, D^n\{s_2\}\langle s_3\rangle)\{-s_2\}\langle s_3\rangle$$

is a $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{R}$-vector space and the differentials are given by

$$d(f) = \partial_D \circ f - (-1)^{s_1} f \circ \partial_C,$$

for each $f \in \mathrm{HOM}^{s_1}_{\mathcal{K}_B}(C, D)$. It follows that each $H^n\left(\mathrm{HOM}^*_{\mathcal{K}_B}(C, D)\right)$ is a $(\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})$-graded $\mathbb{R}$-vector space. We define the *Poincaré polynomial* $\mathfrak{P}(C, D) \in \mathbb{Z}[q_1, q_1^{-1}, q_2, q_2^{-1}, q_3]/\langle q_3^2 - 1\rangle$ of $\mathrm{HOM}^*_{\mathcal{K}_B}(C, D)$ as

$$\mathfrak{P}(C, D) := \sum_{s_1\in\mathbb{Z}} q_1^{s_1} \mathrm{bigrdim}_{\mathbb{R}}\left(H^{s_1}\left(\mathrm{HOM}^*_{\mathcal{K}_B}(C, D)\right)\right).$$

Lemma 1.7.1. *For any trigraded admissible curve $\check{c}$, the following internal Hom complexes are quasi-isomorphic:*

$$(HOM^*_{\mathcal{K}_B}(P^B_j, L_B(\check{c})), d^*_C) \cong \bigoplus_{\check{g}\in st(\check{c},j)} (HOM^*_{\mathcal{K}_B}(P^B_j, L_B(\check{g})), d^*_G),$$

for all $1 \le j \le n$ and $(s_1, s_2, s_3) \in \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/2$.

Proof. To simplify notation, denote $(C^*, \partial^*_C) := L_B(\check{c})$ and $(G^*, \partial^*_G) := \bigoplus_{\check{g}\in st(\check{c},j)} L_B(\check{g})$. Note that G^* can be obtained from C^* by discarding the modules P^B_k in $L_B(\check{c})$ for $|k - j| > 1$. In particular, for all $m \in \mathbb{Z}$, $C^m = G^m \oplus U^m$ where U^m consists of all indecomposable P^B_k in C^m with $|k - j| > 1$. Using the decomposition above, let us write $\partial^m_C : C^m = G^m \oplus U^m \to G^{m+1} \oplus U^{m+1} = C^{m+1}$ as:

$$\partial^m_C = \begin{bmatrix} \tau^m & * \\ * & * \end{bmatrix}$$

so that $\tau^m : G^m \to G^{m+1}$. Also note that the differential $\partial^m_G : G^m \to G^{m+1}$ can be obtained from τ^m by modifying the differentials $P^B(x) \xrightarrow{\partial_{yx}} P^B(y)$ to 0 whenever x and y are crossings of two different j-strings of $\check{c}$. Since

$$\bigoplus_{(s_1,s_2,s_3)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}} \mathrm{Hom}_{\mathscr{B}m}(P^B_j, P^B_k[s_1]\{s_2\}\langle s_3\rangle) = 0, \tag{1.7.1}$$

for all k such that $|j - k| > 1$, it follows that

$$\bigoplus_{(s_1,s_2,s_3)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}} \mathrm{Hom}_{\mathscr{B}m}(P^B_j, U^{s_1}\{s_2\}\langle s_3\rangle) = 0, \tag{1.7.2}$$

and thus

$$\mathrm{HOM}^m_{\mathcal{K}_B}(P^B_j, C^*) = \mathrm{HOM}^m_{\mathcal{K}_B}(P^B_j, G^*), \tag{1.7.3}$$

for each $m \in \mathbb{Z}$ as underlying graded vector space. Moreover, we know that $d^m_G = (\partial^m_G \circ -)$ and $d^m_C = (\partial^m_C \circ -)$ by definition of the HOM complex. But (1.7.2) allows us to conclude that $d^m_C = (\tau^m \circ -)$. Therefore to prove the proposition, it is sufficient to show that $d^m_C = (\tau^m \circ -)$ and $d^m_G = (\partial^m_G \circ -)$ have isomorphic kernels and isomorphic images for each $m \in \mathbb{Z}$. For the rest of the proof, let $m \in \mathbb{Z}$ be arbitrary.

Let us first consider the simple case when $j \neq 2$. We claim that $d^m_C = d^m_G$. Note that when $j \neq 2$, ∂_{yx} in τ^m that are modifed to 0 in ∂^m_G are always right multiplication by loops X_{j-1} or X_{j+1}. But for such maps, the corresponding induced maps on the HOM complex $(\partial_{yx} \circ -)$ are always 0, so $(\tau^m \circ -) = (\partial^m_G \circ -)$ as required.

Now let us consider the case when $j = 2$. The types of maps $\partial_{yx} : P(x) \to P(y)$ in τ^m that are modifed to 0 in ∂^m_G are of the following types:

(i) $\partial_{yx} = X_1$ or X_3;

(ii) $\partial_{yx} = (1|2)i$ or $\partial_{yx} = -i(2|1)$.

Moreover, ∂_{yx} of type (ii) does not exist in ∂_G^m by definition of L_B. By the same argument in the case $j \neq 2$, the induced differential in the HOM complex by ∂_{yx} of type (i) is 0. So can relate d_C^m and d_G^m as follows:

$$d_C^m = (\tau^m \circ -) = (\partial_G^m \circ -) + (\delta \circ -) = d_G^m + (\delta \circ -), \tag{1.7.4}$$

where $\delta := \sum \partial_{yx}$, summing over all ∂_{yx} in τ^m that are of type (ii).

Before we analyse the kernel and image of both d_C^m and d_G^m, we shall consider a decomposition of G^m and G^{m+1} using τ^m. Denote $\mathfrak{G}^m$ and $\mathfrak{G}^{m+1}$ as the subset of all crossings of $\check{c}$ such that $G^m = \bigoplus_{z\in\mathfrak{G}^m} P(z)$ and $G^{m+1} = \bigoplus_{z\in\mathfrak{G}^{m+1}} P(z)$. We shall reorganise the direct summands of G^m and G^{m+1} in the following way:

1. Set

   - $\alpha = 1$,
   - $\epsilon := \delta$,
   - $X := \mathfrak{G}^m$,
   - $H^m := G^m$,
   - $Y = \mathfrak{G}^{m+1}$, and
   - $H^{m+1} = G^{m+1}$.

2. If $\epsilon = 0$, then skip to step (3); otherwise let ∂_{yx} be one of the summands in δ. Consider the smallest subset $X' \subseteq X$ and $Y' \subseteq Y$ such that

   - $x \in X'$,
   - $y \in Y'$, and
   - $\partial_{zw} = 0, X_1$ or X_3 whenever $w \in (X')^c, z \in Y'$ or $w \in X', z \in (Y')^c$.

   We organise the direct summands of H^m in the following way:

   $$H^m = Q_\alpha^m \oplus \left(\bigoplus_{x\in(X')^c} P(x) \right), \text{ and } H^{m+1} = Q_\alpha^{m+1} \oplus \left(\bigoplus_{y\in(Y')^c} P(y) \right).$$

   where $Q_\alpha^m := \bigoplus_{x\in X'} P(x)$ and $Q_\alpha^{m+1} := \bigoplus_{y\in Y'} P(y)$. Let $e = \sum \partial_{yx}$, summing over all $\partial_{yx} = (1|2)i, -i(2|1)$ with $x \in X'$ and $y \in Y'$.
   Redefine

   - $\alpha := \alpha + 1$,
   - $\epsilon := \epsilon - \gamma$,
   - $H^m := \bigoplus_{x\in(X')^c} P(x)$, and
   - $H^{m+1} := \bigoplus_{y\in(Y')^c} P(y)$.

   Repeat step (2).

3. If $H^m \neq 0$, then set $Q_\alpha^m := H^m$, else if $H^{m+1} \neq 0$, then set $Q_\alpha^{m+1} := H^{m+1}$.

4. Output $G^m = \bigoplus_{s\in S} Q_s^m$ and $G^{m+1} = \bigoplus_{s'\in S'} Q_{s'}^m$ with the appropriate index sets $S = \{1, ..., M\}$ and $S' = \{1, ..., M'\}$.

Now consider τ^m and ∂_G^m as block matrices corresponding to the decomposition obtained above:

$$\tau^m = [(\tau^m)_{s',s}]_{(s',s)\in S'\times S}, \quad \partial_G^m = [(\partial_G^m)_{s',s}]_{(s',s)\in S'\times S}.$$

Note that by the construction of the decomposition we have that the block $(\tau^m)_{s',s}$ only have entries X_1, X_3 or 0 for all $s \neq s'$. On the HOM complexes, the decompositions also give us

$$\mathrm{HOM}^m_{\mathcal{K}_B}(P^B_j, C^*) = \mathrm{HOM}^m_{\mathcal{K}_B}(P^B_j, G^*) = \bigoplus_{s\in S} \mathrm{Hom}_{\mathscr{B}m}(P^B_j, Q^m_s), \text{ and}$$
$$\mathrm{HOM}^{m+1}_{\mathcal{K}_B}(P^B_j, C^*) = \mathrm{HOM}^{m+1}_{\mathcal{K}_B}(P^B_j, G^*) = \bigoplus_{s'\in S'} \mathrm{Hom}_{\mathscr{B}m}(P^B_j, Q^{m+1}_{s'}).$$

Similarly consider the two differentials d^m_C and d^m_G written as block matrices corresponding to the decompositions: $d^m_C = [(d^m_C)_{s',s}]_{(s',s)\in S'\times S}$, $d^m_G = [(d^m_G)_{s',s}]_{(s',s)\in S'\times S}$. The construction of the decomposition guarantees the property that $(d^m_C)_{s',s} = (d^m_G)_{s',s} = 0$ whenever $s \neq s'$ (recall that the induced maps $(X_1 \circ -)$ and $(X_3 \circ -)$ are 0). So to show that $d^m_C = (\tau^m \circ -)$ and $d^m_G = (\partial^m_G \circ -)$ have isomorphic images and isomorphic kernels, it is now sufficient to show them for each block $(d^m_C)_{s',s} = ((\tau^m)_{s',s} \circ -)$ and $(d^m_G)_{s',s} = ((\partial^m_G)_{s',s} \circ -)$ where $s = s'$.

To simplify notation, for the rest of this proof we shall drop the subscript s and denote:

$$Q^m := Q^m_s; \quad Q^{m+1} := Q^{m+1}_s; \quad d_C := (d^m_C)_{s,s}; \quad d_G := (d^m_G)_{s,s}; \quad \tau := \tau^m_{s,s}; \text{ and } \partial_G := (\partial^m_G)_{s,s}.$$

We shall now look at the possible types of maps $\tau : Q^m \to Q^{m+1}$ which gives us all possible $d_C = (\tau \circ -)$, where $d_G = (\partial_G \circ -)$ can be obtained by $d_G = (\tau \circ -) - (\delta \circ -)$ (following from (1.7.4)).

If $d_C = d_G$, i.e. τ has no entry of type (ii) so that $\delta = 0$, then there is nothing left to show. Otherwise, τ contains at least one entry ∂_{yx} of type (ii). The two possibilities of ∂_{yx} of type (ii) are $(1|2)i$ and $-i(2|1)$. We will only explicitly show the classification method used to obtain all possible types of $Q^m \xrightarrow{\tau} Q^{m+1}$ when $\partial_{yx} = (1|2)i$, where the same method can be applied to the case when $\partial_{yx} = -i(2|1)$.

So let us consider the case when τ has an entry with $\partial_{yx} = (1|2)i$. Recall that by the definition of L_B, for any ∂_{yx} of type (ii), there must be a corresponding 1-crossing x' of $\check{c}$ such that

- x' and x are connected by an essential segment in D_0, and
- x' and y are endpoints of an essential segment of $\check{c}$ in D_1.

So in the case $\partial_{yx} = (1|2)i$, x' and y are connected through an essential segment in D_1 of type 1 (refer to Figure 1.2.6.2) and the map $\partial_{yx'} : P(x') \to P(y)$ is the right multiplication by $(1|2)$. By the construction of Q^m and Q^{m+1}, Q^m must then at least contain the direct summands $P(x)$ and $P(x')$ and Q^{m+1} must at least contain the direct summand $P(y)$, so the differential $Q^m \xrightarrow{\tau} Q^{m+1}$ must contain at least two entries $\partial_{yx} = (1|2)i$ and $\partial_{yx'} = (1|2)$. Thus the crossings x, x' and y must be contained in the corresponding partial curve of $\check{c}$ below:

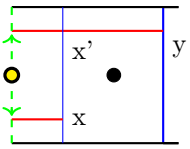


Figure 1.7.0.1: The crossings x, x' and y of the partial curve of $\check{c}$.

As seen from Figure 1.7.0.1 above, x and y are the only crossings that can be in a another distinct essential segment of $\check{c}$.

Let us now first consider the subcase when x is not in another distinct essential segment. If y is also not in another distinct essential segment, then we have that $Q^m \xrightarrow{\tau} Q^{m+1}$ is of the form:

$$\begin{array}{ccc} P(x') & \xrightarrow{(1|2)} & P(y) \\ \oplus & \nearrow_{(1|2)i} & \\ P(x) & & \end{array} .$$

If instead y is part of another essential segment of $\check{c}$ with its other endpoint some crossing w, then the essential segment must be in D_2. Since $P(y)$ is a direct summand of Q^{m+1}, w must have the property $w_1 = y_1 - 1$ so that $P(w)$ is an entry of Q^m and ∂_{yw} is an entry of τ. The only two such possibilities are:

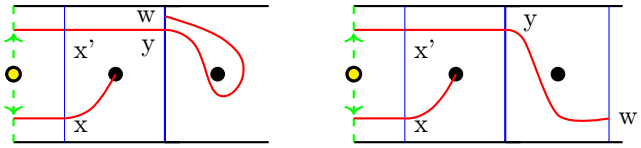


Figure 1.7.0.2: The two possible essential segments from y.

Now note that if w is a 2-crossing (left picture in Figure 1.7.0.2, then one sees that w can not be connected to any other crossing z through another distinct essential segment in $\check{c}$ with $P(z)$ a direct summand of Q^{m+1}; if instead w a 3-crossing (right picture in Figure 1.7.0.2, then the only possibility for $P(z)$ to be a direct summand of Q^{m+1} is when w is also a 3-crossing, with w and z endpoints of an essential segment of $\check{c}$ in D_3 of type 2, giving us $\partial_{wz} = X_3$. Recall the chosen decomposition of G^m and G^{m+1}, where $Q^{m+1} \subseteq G^{m+1}$ corresponds to the smallest subset of crossings in $\mathfrak{G}^{m+1}$ which contains y, with the property that maps between the direct summands of the decompositions of G^m and G^{m+1} are either 0 or X_1 or X_3. Thus $P(z)$ must be excluded from Q^{m+1}. We can therefore conclude that for the subcase when x is not connected to any other distinct essential segments, we have 3 possible forms for $Q^m \xrightarrow{\tau} Q^{m+1}$:

$$\begin{array}{ccc} P(x') & \xrightarrow{(1|2)} & P(y) \\ \oplus & \nearrow_{(1|2)i} & \\ P(x) & & \end{array} \quad \text{or} \quad \begin{array}{ccc} P(w) & & \\ \oplus & \searrow^{X_2 \text{ or } (3|2)} & \\ P(x') & \xrightarrow{(1|2)} & P(y) \\ \oplus & \nearrow_{(1|2)i} & \\ P(x) & & \end{array} .$$

To analyse the maps d_C and d_G, let us identify the morphism spaces as

$$\begin{aligned} \mathrm{HOM}^m_{\mathcal{K}_B}(P_2^B, G^*) &= \bigoplus_{(s,t)\in\mathbb{Z}\times\mathbb{Z}/\mathbb{Z}} \mathrm{Hom}_{\mathscr{B}^m}(P_2^B, (P(x')\oplus P(x)\oplus P(w))\{s\}\langle t\rangle)\{s\}\langle t\rangle \\ &\cong (\mathbb{R}\{(2|1)\}\langle x_3\rangle \oplus \mathbb{R}\{i(2|1)\}\langle x_3+1\rangle)\,\{x_2+1\} \\ &\quad \oplus (\mathbb{R}\{(2|1)\}\langle x'_3\rangle \oplus \mathbb{R}\{i(2|1)\}\langle x'_3+1\rangle)\,\{x'_2\} \\ &\quad \oplus Z, \end{aligned}$$

where $Z = 0$ for the first type of $Q^m \xrightarrow{\tau} Q^{m+1}$, and

$$\begin{aligned} \mathrm{HOM}^{m+1}_{\mathcal{K}_B}(P_2^B, G^*) &= \bigoplus_{(s,t)\in\mathbb{Z}\times\mathbb{Z}/\mathbb{Z}} \mathrm{Hom}_{\mathscr{B}^m}(P_2^B, P(y)\{s\}\langle t\rangle)\{s\}\langle t\rangle \\ &\cong (\mathbb{R}\{X_2\}\langle y_3\rangle \oplus \mathbb{R}\{X_2 i\}\langle y_3+1\rangle)\,\{y_2+1\} \\ &\quad \oplus (\mathbb{R}\{\mathrm{id}\}\langle y_3\rangle \oplus \mathbb{R}\{i\}\langle y_3+1\rangle)\,\{y_2\}. \end{aligned}$$

Using this identification, we can write d_C and d_G as the corresponding matrices:

$$d_C = \begin{bmatrix} 1 & 0 & 0 & -1 & e \\ 0 & 1 & 1 & 0 & f \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad d_G = \begin{bmatrix} 1 & 0 & 0 & 0 & e \\ 0 & 1 & 0 & 0 & f \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where d_G is obtained from d_C by removing maps that were induced by $(1|2)i$. It follows that d_C and d_G have the same image and have isomorphic kernels.

Now consider the other subcase where x is in another essential segment of $\check{c}$ with its other endpoint some crossing y'. Note that since x is already part of an essential segment in D_0, the essential segment connecting

x and y' can only be in D_1. As before, we must have $y'_1 = x_1 - 1$ so that $P(y')$ is a direct summand of Q^{m+1} and that $\partial_{y'x}$ is an entry of τ. Furthermore, if x and y' is connected by the essential segment of Type 2 in Figure 1.2.6.2, then $\partial_{y'x} = X_1$. Therefore such $P(y')$ is excluded from Q^{m+1}. Collecting the results, the only possible essential segment connecting x and y' with $y'_1 = x_1 - 1$ and $\partial_{y'x} \neq X_1$ is the essential segment of Type 1. Thus the crossings x, x', y and y' must be contained in the corresponding partial curve of $\check{c}$ below:

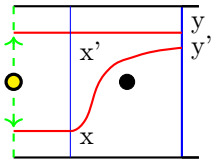


Figure 1.7.0.3: The crossings x, x', y and y' when x is in another essential segment.

The same analysis in the previous subcase on the possible essential segments connected to y can be applied similarly to the crossings y and y' here. Thus we conclude that for this subcase, $Q^m \xrightarrow{\partial^m_C} Q^{m+1}$ is equal to one of the following 6 types (there are 3 possible combinations of X_2 and $(3|2)$ maps in the rightmost diagram):

$$\begin{array}{ccc} P(x') & \xrightarrow{(1|2)} & P(y) \\ \oplus & \searrow\nearrow_{(1|2)i} & \oplus \\ P(x) & \xrightarrow[(1|2)]{} & P(y') \end{array} \quad , \quad \begin{array}{ccc} P(z) & & \\ \oplus & \searrow^{X_2 \text{ or } (3|2)} & \\ P(x') & \xrightarrow{(1|2)} & P(y) \\ \oplus & \searrow\nearrow_{(1|2)i} & \oplus \\ P(x) & \xrightarrow[(1|2)]{} & P(y') \end{array} \quad \text{or} \quad \begin{array}{ccc} P(z) & & \\ \oplus & \searrow^{X_2 \text{ or } (3|2)} & \\ P(x') & \xrightarrow{(1|2)} & P(y) \\ \oplus & \searrow\nearrow_{(1|2)i} & \oplus \\ P(x) & \xrightarrow[(1|2)]{} & P(y') \\ \oplus & \nearrow_{X_2 \text{ or } (3|2)} & \\ P(z') & & \end{array}$$

swapping x with x' (and correspondingly y with y') if necessary. Let us again identify the morphism spaces as:

$$\begin{aligned} \mathrm{HOM}^m_{\mathcal{K}_B}(P_2^B, G^*) &= \bigoplus_{(s,t)\in\mathbb{Z}\times\mathbb{Z}/\mathbb{Z}} \mathrm{Hom}_{\mathscr{B}m}(P_2^B, (P(x') \oplus P(x) \oplus P(z))\{s\}\langle t\rangle)\{s\}\langle t\rangle \\ &\cong (\mathbb{R}\{(2|1)\}\langle x_3\rangle \oplus \mathbb{R}\{i(2|1)\}\langle x_3+1\rangle)\,\{x_2+1\} \\ &\quad \oplus (\mathbb{R}\{(2|1)\}\langle x'_3\rangle \oplus \mathbb{R}\{i(2|1)\}\langle x'_3+1\rangle)\,\{x'_2\} \\ &\quad \oplus Z, \end{aligned}$$

where $Z = 0$ for the first type of $Q^m \xrightarrow{\tau} Q^{m+1}$, and

$$\begin{aligned} \mathrm{HOM}^{m+1}_{\mathcal{K}_B}(P_2^B, G^*) &= \bigoplus_{(s,t)\in\mathbb{Z}\times\mathbb{Z}/\mathbb{Z}} \mathrm{Hom}_{\mathscr{B}m}(P_2^B, P(y)\{s\}\langle t\rangle)\{s\}\langle t\rangle \\ &\cong (\mathbb{R}\{X_2\}\langle y_3\rangle \oplus \mathbb{R}\{X_2 i\}\langle y_3+1\rangle)\,\{y_2+1\} \\ &\quad \oplus (\mathbb{R}\{X_2\}\langle y'_3\rangle \oplus \mathbb{R}\{X_2 i\}\langle y'_3+1\rangle)\,\{y'_2+1\} \\ &\quad \oplus (\mathbb{R}\{\mathrm{id}\}\langle y_3\rangle \oplus \mathbb{R}\{i\}\langle y_3+1\rangle)\,\{y_2\} \\ &\quad \oplus (\mathbb{R}\{\mathrm{id}\}\langle y'_3\rangle \oplus \mathbb{R}\{i\}\langle y'_3+1\rangle)\,\{y'_2\} \\ &= (\mathbb{R}\{X_2\}\langle y_3\rangle \oplus \mathbb{R}\{X_2 i\}\langle y_3+1\rangle)\,\{y_2+1\} \\ &\quad \oplus (\mathbb{R}\{X_2\}\langle y'_3\rangle \oplus \mathbb{R}\{X_2 i\}\langle y'_3+1\rangle)\,\{y'_2+1\} \\ &\quad \oplus V. \end{aligned}$$

Writing d_C and d_G as the corresponding matrix, we get

$$d_C = \begin{bmatrix} 1 & 0 & 0 & -1 & e' \\ 0 & 1 & 1 & 0 & f' \\ 0 & -1 & 1 & 0 & g' \\ 1 & 0 & 0 & 1 & h' \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad d_G = \begin{bmatrix} 1 & 0 & 0 & 0 & e' \\ 0 & 1 & 0 & 0 & f' \\ 0 & 0 & 1 & 0 & g' \\ 0 & 0 & 0 & 1 & h' \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which also have the same image and same kernel. Thus for $\partial_{yx} = (1|2)i$, all possible cases of d_C and d_G have isomorphic images and isomorphic kernels as required.

Applying the same classification method to the case when $\partial_{yx} = -i(2|1)$, the posssible types of $Q^m \xrightarrow{\tau} Q^{m+1}$ is given by:

$$\begin{array}{ccc} & & P(y) \\ & \overset{-i(2|1)}{\nearrow} & \oplus \\ P(x) & \xrightarrow{(2|1)} & P(y') \end{array} \quad \text{or} \quad \begin{array}{ccc} & & P(y) \\ & \overset{-i(2|1)}{\nearrow} & \oplus \\ P(x) & \xrightarrow{(2|1)} & P(y') \\ & \underset{X_2 \text{ or } (2|3)}{\searrow} & \oplus \\ & & P(z) \end{array}$$

when y is not part of another distinct essential segment of $\check{c}$, and

$$\begin{array}{ccc} P(x') & \xrightarrow{(2|1)} & P(y) \\ \oplus & \overset{-i(2|1)}{\nearrow} & \oplus \\ P(x) & \xrightarrow{(2|1)} & P(y') \end{array} \quad , \quad \begin{array}{ccc} P(x') & \xrightarrow{(2|1)} & P(y) \\ \oplus & \overset{-i(2|1)}{\nearrow} & \oplus \\ P(x) & \xrightarrow{(2|1)} & P(y') \\ & \underset{X_2 \text{ or } (2|3)}{\searrow} & \oplus \\ & & P(z) \end{array} \quad \text{or} \quad \begin{array}{ccc} & & P(z') \\ & \overset{X_2 \text{ or } (2|3)}{\nearrow} & \downarrow \oplus \\ P(x') & \xrightarrow{(2|1)} & P(y) \\ \oplus & \overset{-i(2|1)}{\nearrow} & \oplus \\ P(x) & \xrightarrow{(2|1)} & P(y') \\ & \underset{X_2 \text{ or } (2|3)}{\searrow} & \oplus \\ & & P(z) \end{array}$$

when y is part of another distinct essential segment of $\check{c}$, where as before we swap y with y' (correspondingly x with x') if neccessary. By identifying the morphism spaces and comparing the corresponding matrices of d_C and d_G as before, it follows that d_C and d_G have isomorphic images and isomorphic kernels. This covers all cases of δ_C and δ_G, completing the proof. Q.E.D.

Lemma 1.7.2. *The Poincaré polynomial $\mathfrak{P}(P_j^B, L_B(\check{g}))$ of $HOM^*_{\mathcal{K}_B}(P_j^B, L_B(\check{g}))$ is equal to the trigraded intersection number $I^{trigr}(\check{b}_j, \check{g})$ for any trigraded j-string $\check{g}$.*

Proof. This follows exactly as in [KS02, Lemma 4.11]. Q.E.D.

Proposition 1.7.3. *For any σ and τ in $\mathcal{A}(B_n)$, and any $1 \leq j, k \leq n$, the Poincaré polynomial of*

$$HOM^*_{\mathcal{K}_B}(\sigma(P_j^B), \tau(P_k^B))$$

is equal to the trigraded intersection number $I^{trigr}(\check{\sigma}(\check{b}_j), \check{\tau}(\check{b}_k))$.

Proof. By Lemma 1.2.9, we get that $I^{trigr}(\check{b}_j, \check{c}) = \sum_{\check{g} \in st(\check{c}, j)} I^{trigr}(\check{b}_j, \check{g})$. Using Lemma 1.7.1, we instead get that $\mathfrak{P}(P_j^B, L_B(\check{c})) = \sum_{\check{g} \in st(\check{c}, j)} \mathfrak{P}(P_j^B, L_B(\check{g}))$. By Lemma 1.7.2, each $\mathfrak{P}(P_j^B, L_B(\check{g})) = I^{tri}(\check{b}_j, \check{g})$, thus we can conclude that $I^{trigr}(\check{b}_j, \check{c}) = \mathfrak{P}(P_j^B, L_B(\check{c}))$. The proposition now follows from the fact that the categorical action of $\mathcal{A}(B_n)$ respects morphism spaces and similarly the topological action of $\mathcal{A}(B_n)$ respects trigraded intersection number. Q.E.D.

Remark 1.7.4. *Note that we can also use Proposition 1.7.3 to prove the faithfulness of the $\mathcal{A}(B_n)$ categorical action. The proof is similar to [KS02, the paragraph before Section 5] modulo the center of $\mathcal{A}(B_n)$, which is an easy check that elements of the centre act by shifting degrees and therefore are not isomorphic to the identity functor.*

1.8 What is actually Type B Zigzag Algebra?

1.8.1 Type B zigzag algebra as a subalgebra of type A zigzag algebra

One might wonder if the the Type B zigzag algebra has any geometric incarnation. Is Type B zigzag algebra "Fukaya algebra" in some Fukaya category? Here is one answer to it on the cohomology level. As in [Sei08], we will consider the Floer cohomology of type A Milnor fibre in complex coefficient. Then, the type A zigzag algebra we considering would be over $\mathbb{C}$:

$$A_n \cong \bigoplus_{i,j=1}^{n} HF^*(L_i, L_j; \mathbb{C}). \tag{1.8.1}$$

by associating

$$\begin{aligned} e_i &\mapsto HF^0(L_i, L_i);\\ X_i := (i|i+1|i) = (i|i-1|i) &\mapsto HF^n(L_i, L_i);\\ (i|i-1) &\mapsto HF^0(L_i, L_{i-1}), i \leq n;\\ (i|i-1) &\mapsto HF^n(L_i, L_{i-1}), i > n;\\ (i|i+1) &\mapsto HF^0(L_i, L_{i+1}), i \geq n;\\ (i|i+1) &\mapsto HF^n(L_i, L_{i+1}), i < n. \end{aligned}$$

Some words of caution here: This isomorphism assume that the A_∞-algebra of Floer cochain complexes underlying the right hand side is formal (see [ST01; Sei08] for more treatment), type A zigzag algebra in the left hand side (1.8.1) has to be regarded as a differential graded algebra, and although the associations on the right hand side are vector spaces, they are of dimension one.

Consider the type A_{2n-1} Milnor fibre

$$M^d_{A_{2n-1}} = \left\{(x_0, \ldots, x_d) \in \mathbb{C}^{d+1} \mid x_0^2 + \ldots + x_{d-1}^2 + x_d^{2n} = 0\right\}.$$

The $\mathbb{Z}/2\mathbb{Z}$-action on the last coordinate $x_d \mapsto -x_d$ of type A Milnor fibre induces a $\mathbb{Z}/2\mathbb{Z}$-action on the Lagrangian spheres

$$\begin{aligned} &L_n \mapsto L_n;\\ \text{For } 2 \leq j \leq n,&\\ &L_{n-j+1} \mapsto L_{n+j-1};\\ &L_{n+j-1} \mapsto L_{n-j+1}. \end{aligned}$$

Please refer to [KS02] for the construction of Lagrangian spheres from the bifurcation diagram of Milnor fibre which is proven to form a disc as the topological model for the mapping class group to act on. Note that if you consider the symplectic Dehn twist generated by the disjoint union of Lagrangian spheres which are in the same orbit under the action above, they forms the type B symplectomorphism on the Type A Milnor fibre. This is the [BH73a] embedding on $\mathcal{A}(B_n)$ in $\mathcal{A}(A_{2n-1})$. Based on [Sei99], the symplectic Dehn twist defined is dependent on the Lagrangian spheres, but not its orientation.

On the cohomology level, we could also consider a conjugation action $i \mapsto -i$ on the coefficient field, that is given by the non-trivial element $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$.

Theorem 1.8.1. *The type B zigzag algebra $\mathscr{B}_n$ is the $\mathbb{R}$-algebra of the invariant algebra*

$$\mathscr{A}_{2n-1}^{inv} \cong HF^{inv}\left(\bigoplus_{i=1}^{2n-1} L_i, \bigoplus_{j=1}^{2n-1} L_j\right)$$

under the $\mathbb{Z}/2\mathbb{Z}$-action of the product group of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ where the first group factor is coming from the symplectic consideration and the second group factor is coming from the algebraic consideration as explained in the preceding paragraph;

$$\mathscr{B}_n \cong_{\mathbb{R}} \mathscr{A}_{2n-1}^{inv}.$$

Proof. We split $\mathscr{A}_{2n-1} \cong HF\left(\bigoplus_{i=1}^{2n-1} L_i, \bigoplus_{j=1}^{2n-1} L_j; \mathbb{C}\right) = HF(L_n, L_n) \oplus \bigoplus_{i,j=2}^{n} HF(L_{n-i+1}, L_{n+j-1})$. We then consider the invariant part of the $\mathbb{Z}/2\mathbb{Z}$-action from the $\mathbb{Z}/2\mathbb{Z}$-symplectic action and from the $\mathbb{Z}/2\mathbb{Z}$-algebraic action.

Then, the $\mathbb{R}$-linear morphism $\Phi_{\mathbb{R}} : \mathscr{B}_n \to \mathscr{A}_{2n-1}$ defined as follow:

$$\begin{aligned} e_1 &\mapsto e_n, \\ X_1 &\mapsto X_n; \end{aligned}$$

For $2 \leq j \leq n$,

$$\begin{aligned} e_j &\mapsto e_{n-j+1} + e_{n+j-1}; \\ X_j &\mapsto X_{n-j+1} + X_{n+j-1}; \\ ie_j &\mapsto ie_{n-j+1} - ie_{n+j-1}; \\ iX_j &\mapsto iX_{n-j+1} - iX_{n+j-1}; \\ (j-1|j) &\mapsto ((n-j+2)|(n-j+1)) + ((n+j-2)|(n+j-1)); \\ (j-1|j)i &\mapsto i((n-j+2)|(n-j+1)) - i((n+j-2)|(n+j-1)); \\ (j|j-1) &\mapsto ((n-j+1)|(n-j+2)) + ((n+j-1)|(n+j-2)); \\ i(j|j-1) &\mapsto i((n-j+1)|(n-j+2)) - i((n+j-1)|(n+j-2)). \end{aligned}$$

It is not hard to verify that $\Phi_{\mathbb{R}}$ is a grading preserving $\mathbb{R}$-algebra morphism (It is actually implicit in the proof of Proposition 1.4.1).

This map is particularly injective as the images of the map are linearly independent. The diagonal $\mathbb{Z}/2\mathbb{Z}$-action of the product group on the Type A Fukaya algebra is

$$\begin{aligned} e_n &\mapsto e_n, \\ X_n &\mapsto X_n; \end{aligned}$$

For $2 \leq j \leq n$,

$$\begin{aligned} e_{n-j+1} + e_{n+j-1} &\mapsto e_{n+j-1} + e_{n-j+1}; \\ X_{n-j+1} + X_{n+j-1} &\mapsto X_{n+j-1} + X_{n-j+1}; \\ ie_{n-j+1} - ie_{n+j-1} &\mapsto ie_{n+j-1} - ie_{n-j+1} = -(ie_{n-j+1} - ie_{n+j-1}); \\ iX_{n-j+1} - iX_{n+j-1} &\mapsto iX_{n+j-1} - iX_{n-j+1} = -(iX_{n-j+1} - iX_{n+j-1}); \\ ((n-j+2)|(n-j+1)) + ((n+j-2)|(n+j-1)) &\mapsto ((n+j-2)|(n+j-1)) + ((n-j+2)|(n-j+1)); \\ i((n-j+2)|(n-j+1)) - i((n+j-2)|(n+j-1)) &\mapsto i((n+j-2)|(n+j-1)) - i((n-j+2)|(n-j+1)); \\ ((n-j+1)|(n-j+2)) + ((n+j-1)|(n+j-2)) &\mapsto ((n+j-1)|(n+j-2)) + ((n-j+1)|(n-j+2)); \\ i((n-j+1)|(n-j+2)) - i((n+j-1)|(n+j-2)) &\mapsto i((n+j-1)|(n+j-2)) - i((n-j+1)|(n-j+2)). \end{aligned}$$

The basis is notably invariant under the $\mathbb{Z}/2\mathbb{Z}$-action. Hence, $\mathscr{B}_n$ sits as an invariant $\mathbb{R}$-algebra inside $\mathscr{A}_{2n-1}$. Q.E.D.

1.8.2 Categorifying intersection form and Cartan matrix

In [Arn78], Arnold studied the degenerate critical points of functions on a manifold with boundary. Here, a manifold with boundary is a smooth manifold with a fixed smooth hypersurface. More accounts on the subject can be found in [Arn+98] and [AGV12b]. One theorem Arnold achieved in [Arn78] is that the simple crititcal points of functions on the boundary of a manifold with boundary is classified by the simple Lie algebras B_n, C_n, and F_4. The Type B_n singularities is the function

$$g(x_0, x_1, \ldots, x_d^n) = x_0^2 + x_1^2 + \ldots + x_{d-1}^2 + x_d^n, \tag{1.8.2}$$

on manifold with boundary $x_d = 0$. The diffeomorphisms of a manifold with boundary are the diffeomorphisms of the manifold that fix the boundary. Let $\widehat{\mathbb{C}}^{d+1} \cong \mathbb{C}^{d+1}$ be the double branched covering of the space $\mathbb{C}^{d+1}$ along the hyperplane $\{x_d = 0\} \cong \mathbb{C}^d$ defined by, on coordinate,

$$x_0 = \hat{x}_0, x_1 = \hat{x}_1, \ldots, x_d = \hat{x}_d^2,$$

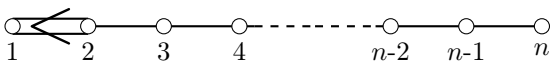
Under the natural involution $(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_d) \mapsto (\hat{x}_0, \hat{x}_1, \ldots, -\hat{x}_d)$, the function $g(x_0, x_1, \ldots, x_d^n)$ induces a function $\hat{g}(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_d) = g(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_d^2)$ on the space $\widehat{\mathbb{C}}^{d+1}$ which is invariant under the action of the group $\mathbb{Z}/2\mathbb{Z}$. A locally analytic automorphisms of the space $\mathbb{C}^{d+1}$ preserving the subspace $\mathbb{C}^d$ induces a locally analytic automorphisms of the space $\widehat{\mathbb{C}}^{d+1}$ which commutes with the action of the group $\mathbb{Z}/2\mathbb{Z}$. As a result, a singularity of a function on a manifold with boundary is equivalent to a singularity of a function on its covering which is invariant under the invariant. So, studying (1.8.2) is equivalent to study

$$\hat{g}(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_d) = \hat{x}_0 + \hat{x}_1, \ldots + \hat{x}_d^{2n}, \tag{1.8.3}$$

on $\widehat{\mathbb{C}}^{d+1}$. Then, the $(d-1)$-th homology of the non-singular level manifold

$$\widehat{V}_\epsilon = \{x \in \mathbb{C}^n \mid \hat{g} = \epsilon, \|x\| \leq \delta\}, \tag{1.8.4}$$

contain a subspace H^- which is anti-invariant under the involution. The basis of H^- which are the long vanishing cycles corresponding to the preimage of the vanishing cycles constructed by the paths joining the critical values of the perturbed g and the short vanishing cycles corresponding to the preimage of the vanishing hemicycles constructed by paths joining the critical values of the perturbed $g|_{x_n=0}$ The Type B intersection form is defined using the type B Dynkin diagram. Following [Bou02, Chapter VI], the type B root system has the length of the long root is $\sqrt{2}$ of the length of the short root with the associated Dynkin diagram



where the inequality sign is oriented to the shorter root. Now, the the long vanishing cycles corresponding to the long roots have self-intersection number -4 and the short vanishing cycle corresponding to the short root has self intersection number -2. The intersection number of distinct vanishing cycles corresponding to the edges joining two roots is two times the multiplicity of the edge which is 2 if both roots are long and equals to the multiplicity of the edge which 2 if one root is short and the other is long. In short, the type B intersection form is

$$\begin{pmatrix} -2 & 2 & 0 & & \cdots & 0 \\ 2 & -4 & 2 & 0 & \cdots & 0 \\ 0 & 2 & -4 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & 0 & 2 & -4 & 2 \\ 0 & 0 & \cdots & 0 & 2 & -4 \end{pmatrix}.$$

On the other hand, the type B Cartan matrix ([Bou02, Chapter IV.1.3], [BB05, Chapter 4.3]) is

$$\begin{pmatrix} 2 & -1 & 0 & & \cdots & 0 \\ -2 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Corollary 1.8.2. *The type B zigzag algebra categorify the intersection form of the type B boundary singularities.*

Proof. By (1.3.5), the path-length-graded dimension matrix associated to the Type B zigzag algebra when $t = -1$ is

$$\begin{pmatrix} gr dim_{\mathbb{R}}\, {}_1P_1 & gr dim_{\mathbb{R}}\, {}_2P_1 & gr dim_{\mathbb{R}}\, {}_3P_1 & \cdots & gr dim_{\mathbb{R}}\, {}_nP_1 \\ gr dim_{\mathbb{R}}\, {}_1P_2 & gr dim_{\mathbb{R}}\, {}_2P_2 & gr dim_{\mathbb{R}}\, {}_3P_2 & \cdots & gr dim_{\mathbb{R}}\, {}_nP_2 \\ gr dim_{\mathbb{R}}\, {}_1P_3 & gr dim_{\mathbb{R}}\, {}_2P_3 & gr dim_{\mathbb{R}}\, {}_3P_3 & \cdots & gr dim_{\mathbb{R}}\, {}_nP_3 \\ \vdots & & & \ddots & \vdots \\ gr dim_{\mathbb{R}}\, {}_1P_n & gr dim_{\mathbb{R}}\, {}_2P_n & gr dim_{\mathbb{R}}\, {}_3P_n & \cdots & gr dim_{\mathbb{R}}\, {}_nP_n \end{pmatrix} |_{t=-1}$$

$$= \begin{pmatrix} 1+t^2 & 2t & 0 & & \cdots & 0 \\ 2t & 2(1+t^2) & 2t & 0 & \cdots & 0 \\ 0 & 2t & 2(1+t^2) & 2t & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & 0 & 2t & 2(1+t^2) & 2t \\ 0 & 0 & \cdots & 0 & 2t & 2(1+t^2) \end{pmatrix} |_{t=-1}$$

$$= \begin{pmatrix} 2 & -2 & 0 & & \cdots & 0 \\ -2 & 4 & -2 & 0 & \cdots & 0 \\ 0 & -2 & 4 & -2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & 0 & -2 & 4 & -2 \\ 0 & 0 & \cdots & 0 & -2 & 4 \end{pmatrix}$$

which is the minus of the Type B intersection form. Q.E.D.

Remark 1.8.3. *This is in accordance with the idea the singularities corresponding to the Lie algebras B_n arises as an invariant germ of a function relative to the linear action of a finite group G on the space $\mathbb{C}^n$. It would be best if the algebraic action can come from geometry as well, perhaps one could naturally consider the Pin structure on the Lagrangian submanifold. In particular, Seidel considered, in his book, a $\mathbb{Z}/2\mathbb{Z}$-action on Fukaya category which induces a non-trivial action on the Pin structure of the Lagrangian submanifold. There we have a pair of isomorphisms bearing the identities $f \circ f' = -e$ where f and f' are isomorphisms transforming the Lagrangian submanifold with its Pin structure and $-e$ is the nontrivial element in the Ker(Pin $\to O_n$). The conjugation action on the cohomology seems more natural on K-theory. Since there is a Chern character map from the K-theory to cohomology theory, are there any connection between them? Note that the category of vector bundle and category of coherent sheaves are equivalent. Otherwise, we need to make sense of a negative of a generator in Floer cohomology, in order to give a geometric interpretation.*

Corollary 1.8.4. *The type B zigzag algebra categorify the type B Cartan matrix.*

Proof. By Proposition 1.3.5, the path-length-graded left module dimension matrix associated to the Type

B zigzag algebra when $t=-1$ is

$$\begin{pmatrix} g_r dim_{\mathbb{R}}\ {}_1P_1 & g_r dim_{\mathbb{C}}\ {}_2P_1 & g_r dim_{\mathbb{C}}\ {}_3P_1 & \cdots & g_r dim_{\mathbb{C}}\ {}_nP_1 \\ g_r dim_{\mathbb{R}}\ {}_1P_2 & g_r dim_{\mathbb{C}}\ {}_2P_2 & g_r dim_{\mathbb{C}}\ {}_3P_2 & \cdots & g_r dim_{\mathbb{C}}\ {}_nP_2 \\ g_r dim_{\mathbb{R}}\ {}_1P_3 & g_r dim_{\mathbb{C}}\ {}_2P_3 & g_r dim_{\mathbb{C}}\ {}_3P_3 & \cdots & g_r dim_{\mathbb{C}}\ {}_nP_3 \\ \vdots & & & \ddots & \vdots \\ g_r dim_{\mathbb{R}}\ {}_1P_n & g_r dim_{\mathbb{C}}\ {}_2P_n & g_r dim_{\mathbb{C}}\ {}_3P_n & \cdots & g_r dim_{\mathbb{C}}\ {}_nP_n \end{pmatrix}|_{t=-1}$$

$$= \begin{pmatrix} 1+t^2 & t & 0 & & \cdots & 0 \\ 2t & 1+t^2 & t & 0 & \cdots & 0 \\ 0 & t & 1+t^2 & t & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & 0 & t & 1+t^2 & t \\ 0 & 0 & \cdots & 0 & t & 1+t^2 \end{pmatrix}|_{t=-1}$$

$$= \begin{pmatrix} 2 & -1 & 0 & & \cdots & 0 \\ -2 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

which is the minus of the type B Cartan matrix. Q.E.D.

Remark 1.8.5. *There is a subtle difference between the choice of Type B Cartan matrix above and in the next chapter (see Theorem 2.7.2), but they "play" the same number game in [BB05].*

Chapter 2

The Faithfulness of the Type B 2-Braid Group.

2.1 Background and Outline of the Chapter

From the first chapter, we know that a categorical braid group action realised by Khovanov-Seidel [KS02] for type A braid group uses complexes of bimodules over a zigzag algebra. Following their work, Licata-Queffelec [LQ21] similarly defined the simply-laced type ADE zigzag algebras, which provides categorical actions for the corresponding generalised braid groups. In our previous chapter, we suggested a suitable candidate for the type B zigzag algebra, which similarly allows a categorical action of the type B generalised braid group. Note that in all of the above cases, the faithfulness of the braid group actions is known.

Another way to construct the categorical actions of Artin groups is through the theory of Soergel bimodules. This was introduced by Rouquier in [Rou06] as the *2-braid groups*, where the braid generators act on the category of Soergel bimodules via tensoring with the Rouquier complexes. Rouquier also conjectured that this categorification is faithful, which was later proven by Brav-Thomas [BT11] for type ADE, and then generalised to all finite types by Jensen [Jen14; Jen17]. In type A, the categorical actions via zigzag algebras can be described as a "quotient" of the ones via Soergel bimodules, as shown in the work of Jensen in his Master thesis [Jen14]. We shall briefly describe this relationship here. Firstly, note that the zigzag algebra for type A leads to a categorification of (a further quotient of) the type A Temperley-Lieb algebra, using certain (monoidal, additive) subcategory of $\mathscr{A}$-bimodules [KS02]. Just as Temperley-Lieb algebras are quotients of Hecke algebras (categorified by Soergel bimodules), this relation can be understood categorically through a full and essentially surjective (quotient) functor from the category of Soergel bimodules to this subcategory of $\mathscr{A}$-bimodules, as described in [Jen14]. Our construction of the type B zigzag algebra $\mathscr{B}$ also fits into this whole construction, where a certain subcategory of $\mathscr{B}$-bimodules categorifies (a further quotient of) the type B Temperley-Lieb algebra and is also a quotient category of Soergel bimodules. This provides an alternate proof to Rouquier's conjecture for the type B case, in addition to serving as a piece of evidence for the type B zigzag algebra constructed being a correct candidate.

Outline of the chapter

In Section 2.2, we recall the definition of Soergel bimodules. We first define Coxeter systems by generators and relations as well as the corresponding generalised braid groups. A combinatorial tool known as the two-coloured quantum number is introduced before defining a realisation of the Coxeter system. Demazure surjectivity is mentioned for general theory to hold and for the purpose of computation later.

Section 2.3 is where we introduce the Temperley-Lieb category, and subsequently, its souped-up version, the two-coloured Temperley-Lieb 2-category. We also provide a recursive formula for the Jones-Wenzl projectors and the two-coloured Jones-Wenzl projectors.

In Section 2.4, we recall the whole definition of diagrammatic category for Soergel bimodules from [EW16a] where they proved the equivalence between the diagrammatic category with the original algebraic

combinatorial Soergel bimodule category introduced by Soergel himself.

Section 2.5 is where we recall the construction of 2-braid groups introduced by Rouquier [Rou06].

To make this chapter self-contained, in Section 2.6, we extract the algebraic construction of the type B 2-braid group from Section 1.3.2 in Chapter 1.

Section 2.7 is where we present the faithfulness of the type B 2-braid group by constructing a monoidal functor from the diagrammatic Soergel bimodule homotopy category to type B bimodule homotopy category.

2.2 Soergel Bimodules

2.2.1 Coxeter systems by generators and relations and its corresponding generalised braid groups

Recall from [KT08] that a *Coxeter matrix* $M = (m_{s,t})_{s,t\in S}$ is a symmetric matrix where S is a finite set, $m_{s,s} = 1$ for every s in S and $m_{s,t} \in \{2, 3, \cdots, \infty\}$. Equivalently, m can be represented by a *Coxeter graph*, or *Coxeter diagram* with its node set as S and its edges determined by the unordered pairs s, t such that $m_{s,t} \geq 3$. For edges with $m_{s,t} \geq 4$, we will label them by that number where the label will be dropped if $m_{s,t} = 3$. When s and t commute, we have $m_{s,t} = 2$, and they are not connected in the Coxeter graph.

To every Coxeter matrix M, we can associate a group W_M with presentation: the generators being the elements of S and the relations being $(st)^{m_{s,t}} = e$, for all s, t in S such that $m_{s,t} \neq \infty$ where e is the identity element in W_M. By convention, $m_{s,s} = 1$ Subsequently, we will call the pair (W_M, S), a Coxeter system where W_M is a group, or more specifically a Coxeter group, with a set S of its generators subject to the following relations:

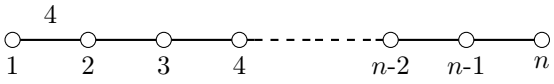
$$s^2 = e \qquad \text{for all } s \in S, \tag{2.2.1}$$

$$\text{(Braid relation)} \qquad \underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{m_{st}} \qquad \text{for all } s, t \in S, \tag{2.2.2}$$

where $2 \leq m_{s,t} < \infty$ and $m_{s,t} = m_{t,s}$. If $m_{s,t} = \infty$, then the braid relation is omitted. The rank of (W, S) is defined to be the cardinality of S . We might use the words *index* and *colour* to refer to an element $s \in S$. An *expression* for an element $w \in W$ is a finite sequence of indices, denoted by $\underline{w}$.

If we now remove the relation (2.2.1), then the group $\mathcal{A}(W_M)$ generated by S and the braid relations (2.2.2) is called the *generalised braid group*[1] associated to W_M. In which case, there is a surjective map $\pi : \mathcal{A}(W_M) \to W_M$.

Example 2.2.1. *For $n \geq 2$, the type B_n braid group $\mathcal{A}(B_n)$ associated to the type B_n Coxeter graph*



or the type B_n Coxeter matrix

$$\begin{bmatrix} 1 & 3 & 2 & 2 & \cdots & 2 \\ 3 & 1 & 3 & 2 & & \vdots \\ 2 & 3 & 1 & 3 & \ddots & 2 \\ 2 & 2 & 3 & 1 & \ddots & 2 \\ \vdots & & \ddots & \ddots & \ddots & 3 \\ 2 & \cdots & 2 & 2 & 3 & 1 \end{bmatrix}$$

is generated by

$$\sigma_1^B, \sigma_2^B, \ldots, \sigma_n^B$$

subject to the relations

$$\sigma_1^B\sigma_2^B\sigma_1^B\sigma_2^B = \sigma_2^B\sigma_1^B\sigma_2^B\sigma_1^B; \tag{2.2.3}$$

$$\sigma_j^B\sigma_k^B = \sigma_k^B\sigma_j^B \qquad \text{for } |j-k| > 1; \tag{2.2.4}$$

$$\sigma_j^B\sigma_{j+1}^B\sigma_j^B = \sigma_{j+1}^B\sigma_j^B\sigma_{j+1}^B \qquad \text{for } j = 2, 3, \ldots, n. \tag{2.2.5}$$

[1] Also known as *Artin group* or *Artin-Tits group*.

2.2.2 Two-coloured quantum numbers

Remember that the quantum numbers, as defined by $[m] = [m]_q := \frac{q^m - q^{-m}}{q - q^{-1}}$ for $m \in \mathbb{Z}_{\geq 0}$. In particular, $[2]_q = q + q^{-1}, [1]_q = 1$, and $[0]_q = 0$. The subscript q will be suppressed in general, but one point to bear in mind is that if $q = e^{i\theta}$ is a root of unity, we have a familiar number $[2]_q^2 = 4\cos^2\theta$ from the representation theory of Coxeter graphs.

Let δ be an indeterminate, such that the ring $\mathbb{Z}[\delta]$ is a subring of $\mathbb{Z}[q, q^{-1}]$ under the specialization $\delta \mapsto [2]$. Next, we introduced the subring $\mathbb{Z}[x, y]$ of $\mathbb{Z}[\delta]$ where the corresponding subring $\mathbb{Z}[x, y]$ can be identified with the subring $\mathbb{Z}[\delta^2]$ via $xy = \delta^2$. Subsequently, the *two-coloured quantum numbers* is defined to be the elements in $\mathbb{Z}[x, y]$ inductively as follows:

(i) $[0]_x = [0]_y = 0$, $[1]_x = [1]_y = 1$, and $[2]_x = x, [2]_y = y$;

(ii) $[2]_x[m]_y = [m + 1]_x + [m - 1]_x$;

(iii) $[2]_y[m]_x = [m + 1]_y + [m - 1]_y$.

For m odd, since $[m]_x = [m]_y$, we shorten the notation to $[m]$.

We also have the following two-coloured analogue of quantum number equations: for $m \geq a$,

$$[m - a]_y + [m + 1]_x[a]_y = [m]_y[a + 1]_x \text{ and } [m - a]_x + [m + 1]_y[a]_x = [m]_x[a + 1]_y. \tag{2.2.6}$$

For instance, we have $[2]_x = x$ and $[2]_y = y$; $[3]_x = xy - 1 = [3]_y$; $[4]_x = x^2y - 2x$ and $[4]_y = xy^2 - 2y$; $[5]_x = x^2y^2 - 3xy + 1 = [5]_y$; $[6]_x = x^3y^2 - 4x^2y + 2x$ and $[6]_x = x^2y^3 - 4xy^2 + 2y$.

2.2.3 Definition of Soergel bimodules

Realisation of Coxeter System

Let $\Bbbk$ be a commutative ring and (W, S) be a Coxeter system. A *realisation* of (W, S) over $\Bbbk$ is a free, finite rank $\Bbbk$-module $\mathfrak{h}$, together with a choice of simple co-roots $\{\alpha_s^\vee \mid s \in S\} \subset \mathfrak{h}$ and roots $\{\alpha_s \mid s \in S\} \subset \mathfrak{h}^* = \mathrm{Hom}_\Bbbk(\mathfrak{h}, \Bbbk)$ having a Cartan matrix $(a_{s,t})_{s,t\in S}$ where $a_{s,t} := \langle \alpha_s^\vee, \alpha_t \rangle$ satisfying:

(i) $a_{s,s} = 2$ for all $s \in S$;

(ii) for any $s, t \in S$ with $m_{st} < \infty$, let $x = a_{s,t}$ and $y = a_{t,s}$, then $[m_{st}]_x = [m_{st}]_y = 0$;

(iii) the assignment $s(v) := v - \langle v, \alpha_s \rangle \alpha_s^\vee$ for all $v \in \mathfrak{h}$ yields a representation of W.

We call a realization *symmetric* if $a_{s,t} = a_{t,s}$ for all $s, t \in S$.

A realization is *balanced* if for each $s, t \in S$ with $m_{st} < \infty$, one has $[m_{st} - 1]_x = [m_{st} - 1]_y = -1$ for m_{st} odd or $[m_{st} - 1] = 1$ for m_{st} even.

Remark 2.2.2. *Note that we use a slightly different definition of a balanced realization from [Eli16b; EW16b] because our circle in the Temperley-Lieb algebra (refer Section 2.3.1) evaluates to δ instead of -δ.*

Given a realization $(\mathfrak{h}, \{\alpha_s^\vee\}, \{\alpha_s\})$ of (W, S) over $\Bbbk$, we can look at its *Cartan matrix* $(\langle \alpha_s^\vee, \alpha_t \rangle)_{s,t\in S}$ which is symmetric if and only if the realization is. Conversely, suppose we start with a matrix $(a_{s,t})_{s,t\in S}$ such that $a_{s,s} = 2$. We can construct the free $\Bbbk$-module $\mathfrak{h} = \bigoplus_{s\in S} \Bbbk\alpha_s^\vee$, and define the roots $\alpha_s \in \mathfrak{h}^*$ by the formula $\langle \alpha_s^\vee, \alpha_t \rangle = a_{s,t}$. If this yields a realization of (W, S), then the matrix $(a_{s,t})_{s,t\in S}$ is the *Cartan matrix* for (W, S) over $\Bbbk$. We call a realization minimal if the basis $\{\alpha_s^\vee\}$ of $\mathfrak{h}$ can be recontructed from its Cartan matrix from the above way.

With such a realization $(\mathfrak{h}, \{\alpha_s^\vee\}, \{\alpha_s\})$ of (W, S), we define $R := \bigoplus_{k\geq 0} S^k(\mathfrak{h}^*)$ to be the $\mathbb{Z}$-graded $\Bbbk$-symmetric algebra on $\mathfrak{h}^*$ with $\deg \mathfrak{h}^* = 2$. We consider the action of W on $\mathfrak{h}^*$ via its geometric representation, that is, for all,$\sigma_s(\gamma) = \gamma - \langle \alpha_s^\vee, \gamma \rangle \alpha_s$, and subsequently, extend it to an action on the whole R by graded automorphisms. In one way, we can see R as the algebra of regular functions on $\mathfrak{h}$.

Example 2.2.3. *Let (W, S) be a Coxeter system of finite rank. Let $\Bbbk = \mathbb{R}$ and $\mathfrak{h} = \bigoplus_{s\in S} \mathbb{R}\alpha_s^\vee$. Consider the following realization by defining the elements $\{\alpha_s\} \subset \mathfrak{h}^*$ by*

$$\langle \alpha_s^\vee, \alpha_t \rangle = -2cos\left(\frac{\pi}{m_{s,t}}\right) \tag{2.2.7}$$

with the convention that $m_{s,s} = 1$ and $\pi/\infty = 0$. We get the geometric representation in [Hum90] Moreover, this is a balanced symmetric realization of (W_M, S).

Example 2.2.4. *Consider the geometric representation of the type B Coxeter graph, as defined in [BB05]. Let $\Bbbk = \mathbb{R}$. For $m_{s,t} = 4$, we define $a_{s,t} = -2$ and $a_{t,s} = -1$ whereas for $m_{s,t} = 3$, we define $a_{s,t} = -1 = a_{t,s}$. This is a balanced non-symmetric realization of the type B_n Coxeter system.*

Demazure Surjectivity and Soergel bimodules

For $n \in \mathbb{Z}$, we denote the grading shift by (n) such that given a $\mathbb{Z}$-graded (bi)module $M = \bigoplus_i M^i$, the shifted object is defined by the formula $M(n)^i = M^{i+n}$. We wish to view the category of (R, R)-bimodules, denoted by (R, R)-*bimod*, as the graded, abelian, monoidal category of $\mathbb{Z}$-graded (R, R)-bimodules that are finitely generated as left and as right R-modules with grading-preserving bimodule homomorphisms as morphism. For any two graded (R, R)-bimodules M and N, we denote the graded Hom space by $\mathrm{Hom}^*(M, N) = \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}_{(R,R)-bimod}(M, N(k))$.

Fix a $s \in S$, Let $R^s \subseteq R$ be the invariant subring under the action of s, and we will extend the evaluation map $\langle \alpha_s^\vee, \cdot \rangle : \mathfrak{h}^* \to \Bbbk$ to the *Demazure operator* $\partial_s : R \to R^s(-2)$, by the formula

$$\partial_s(f) = \frac{f - \sigma_s(f)}{\alpha_s}. \tag{2.2.8}$$

Observe that both the numerator and denominator is s-antiinvariant, so that the fraction will lie in the subring R^s of s-invariants. But, in order for the fraction to be well-defined, we still need Demazure Surjectivity on our realization, or rather on the base ring $\Bbbk$, so that $\Bbbk$ contains certain algebraic integers. Since the base rings we considered in this chapter are $\mathbb{R}$ and $\mathbb{C}$, we have the following assumption:

Assumption 2.2.5 (Demazure Surjectivity)**.** *For all $s \in S$, the map $\alpha_s : \mathfrak{h} \to \Bbbk$ and the the map $\partial_s : \mathfrak{h}^* \to \Bbbk$ are surjective.*

As long as Demazure Surjectivity holds, $\alpha_s \neq 0$, so that there exist some $\delta \in \mathfrak{h}^*$ such that $\langle \alpha_s^\vee, \delta \rangle = 1$ and $\sigma_s(\delta) = \delta - \alpha_s \neq \delta$. The well-definedness of Demazure Operator will be answered by following lemma:

Lemma 2.2.6. *Any element $f \in R$ can be written uniquely as $f = g\delta + h$ for $g, h \in R^s$.*

Proof. Please refer to [EW16b, Claim 3.9]. Q.E.D.

Using this lemma, we could could have just defined $\partial_s(f) := g$. One can also see that the above definition is well-defined irrespective of the choice of δ by a similar argument.

What is interesting about Lemma 2.2.6 is that it also shows R is free of rank 2 over R^s with basis $\{1, \delta\}$.

For all s, we define the (R, R)-bimodule $B_s := R \otimes_{R^s} R(1)$ Given an expression $\underline{w} = s_1 s_2 \cdots s_k$, the corresponding *Bott-Samelson bimodule* is the tensor product

$$B_{\underline{w}} := B_{s_1} \otimes_R B_{s_2} \otimes_R \cdots \otimes B_{s_k} \cong R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} R \otimes_{R^{s_3}} \cdots \otimes_{R^{s_k}} R(k).$$

The category of Bott-Samelson bimodules, denoted by $\mathbb{BS}$-bimod, is the full monoidal subcategory of (R, R)-bimod generated by B_s for $s \in S$. Take the additive closure of $\mathbb{BS}$-bimod and then its Karoubi envelope, we form the category of Soergel bimodules, denoted by $\mathbb{S}$-bimod.

2.3 Temperley-Lieb 2-Category

2.3.1 The (uncoloured) Temperley-Lieb category

We define *(uncoloured) Temperley-Lieb category* $\mathcal{TL}$ as having coefficients which lie in $\mathbb{Z}[\delta]$. It is a monoidal category with objects $n \in \mathbb{N}$ which can be seen as n points on a line. Its morphisms from n to m are the $\mathbb{Z}[\delta]$-module spanned by crossingless matchings with n points on bottom to m points on top. The multiplication is given by vertical concatenation of diagrams, and resolving any circles with the scalar δ. Note that $\mathrm{End}(n) = TL_n$ is called the uncolored Temperley-Lieb algebra on n strands with the identity element $\mathbb{1}_n$.

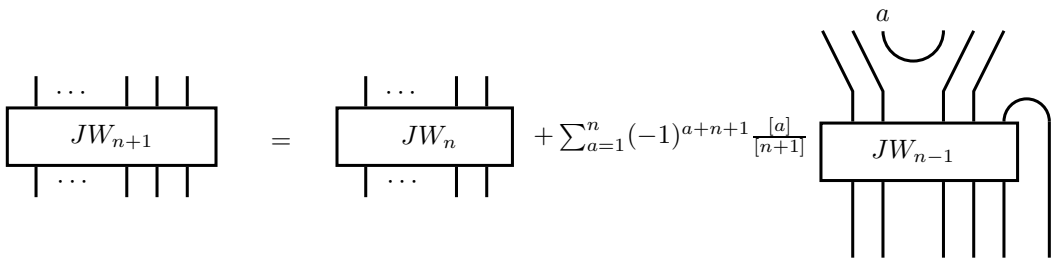
Jones-Wenzl Projectors

Consider the representation of the quantum group $U_q(\mathfrak{sl}_2)$ with irreducible representations V_k having highest weight q^k. In [FK97], the category $\mathcal{TL}$ governs the representations of quantum group $U_q(\mathfrak{sl}_2)$ by serving as the algebra of intertwining operators of tensor products of the fundamental representation V_1. As a result, after doing the base change to $\mathbb{Q}(q)$ under the map $\delta \mapsto q + q^{-1}$, the category $\mathcal{TL}$ is equivalent to the full subcategory of $U_q(\mathfrak{sl}_2)$-representations, such that $\mathrm{Hom}_{\mathcal{TL}}(n,m) \otimes_{\mathbb{Z}} \mathbb{Q}(q) = \mathrm{Hom}(V_1^{\otimes n}, V_1^{\otimes m})$. Since every $V_1^{\otimes n}$ contains an indecomposable representation V_n with multiplicity one as one of its summands, there exists a canonical idempotent $JW_n \in \mathrm{End}_{\mathcal{TL}}(n) \otimes \mathbb{Q}(q) = \mathrm{End}_{U_q(\mathfrak{sl}_2)}(V_1^{\otimes n})$ called the *Jones-Wenzl projectors* projecting from $V_1^{\otimes n}$ to V_n. Performing another base change $\delta \mapsto [2]_q$ again, we can equally use quantum numbers to express them, provided certain n-th quantum numbers appearing as quantum binomial coefficients are invertible.

Proposition 2.3.1. *The Temperley-Lieb algebra TL_n, after extension of scalers, contains canonical idempotents which project $V_1^{\otimes n}$ to each isotypic component. Given a choice of primitive idempotents, TL_n contains maps which realize the isomorphisms between the different irreducible summands of the same isotypic component. These maps can be defined in any extension of $\mathbb{Z}[\delta]$ where the quantum numbers $[2], [3], \ldots, [n]$ are invertible.*

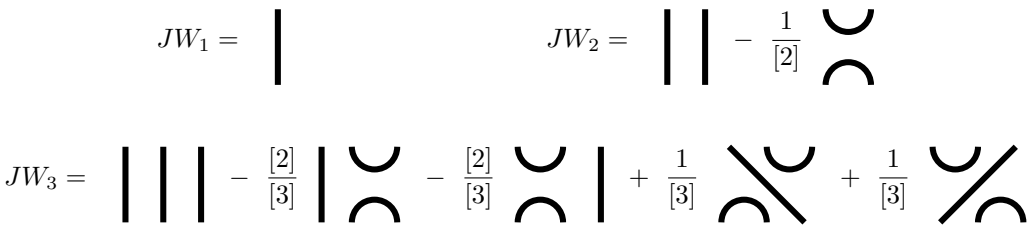
Proof. Please refer to [Eli16b, Prop 4.2]. Q.E.D.

For more properties of Jones-Wenzl projectors, please refer to [Eli16a; Mor17; Eli11]. But, we will write down one recursive formula which allows us to get our desired Jones-Wenzl projectors which sums over the possible positions of cups:



(2.3.1)

Example 2.3.2.



2.3.2 The two-coloured Temperley-Lieb 2-category

We define *two-coloured Temperley-Lieb 2-category* $2\mathcal{TL}$ as having coefficients which lie in $\mathbb{Z}[x, y]$. This category $2\mathcal{TL}$ has two objects: red and blue. It has two generating 1-morphisms: a map from red to blue and a map from blue to red. The 2-morphisms are the $Z[x, y]$-module spanned by appropriately-coloured crossingless matchings. The multiplication is given by vertical concatenation of diagrams, and replacing any closed component (i.e. circle) with a scalar indeterminate; a circle with red (resp. blue) interior evaluates to x (resp. y).

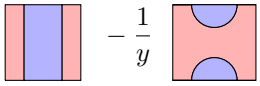
We can obtain the usual uncoloured Temperley-Lieb category $\mathcal{TL}$ by letting $x = y = \delta$ and removing the colours. On the other hand, any crossingless matchings in JW_{m-1} will divide the planar strip into $m + 1$ regions which can be coloured alternatingly across strands with two different colours, say red and blue.

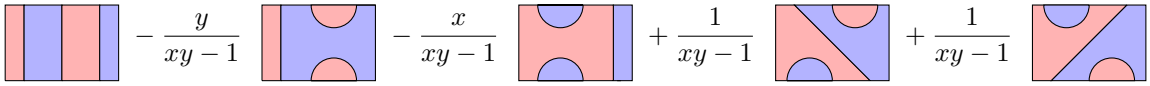
Two-coloured Jones-Wenzl Projectors

The corresponding two-coloured Temperley-Lieb algebra in $2\mathcal{TL}$ also contains Jones-Wenzl projectors, but with each coming in two flavours with different coefficients depending on whether it is left-blue-aligned or left-red-aligned. The other flavour can be acquired by switching the colours as well as switching x and y.

Example 2.3.3.

$JW_1 =$ 

$JW_2 =$ 

$JW_3 =$ 

As dicussed before, the fact that $[m]_x = [m]_y = 0$ guaranteed that JW_{m-1} is well-defined, that is, the denominators $[m-1]_x, [m-1]_y$ are invertible.

Rotation of Jones-Wenzl Projectors

A *rotation* of JW_{m-1} by a single strand is obtained by cupping on the leftmost bottom and capping on the rightmost top. It is known that a rotation of JW_{m-1} by two strands will produce JW_{m-1}. If a rotation of an uncolored morphism JW_{m-1} by a single strand returns JW_{n-1}, then JW_{m-1} is said to be *rotation-invariant*. Note that the following property is provided by a balanced realization.

Proposition 2.3.4. *Rotating the left-blue-aligned Jones-Wenzl projectors JW_{m-1} by one strand yields the left-red-aligned Jones-Wenzl projectors if and only if $[m-1] = 1$ for m even or $[m-1]_x = [m-1]_y = -1$ for m odd.*

Proof. See [Eli16b] for more information. Q.E.D.

This property of rotational invariant is to ensure the well-definedness of certain Soergel graphs whose definition will be made explicit in the next section.

2.4 Diagrammatic Category for Soergel Bimodules

2.4.1 Soergel graphs

A *planar graph in the strip* is a finite graph $(\Gamma, \partial\Gamma)$ embedded in $(\mathbb{R} \times [0, 1], \mathbb{R} \times \{0, 1\})$. In other words, all vertices of the graph Γ lie in the interior $\mathbb{R} \times (0, 1)$, and removing the vertices we have a 1-manifold with boundary whose intersection with $\mathbb{R} \times \{0, 1\}$ is precisely its boundary. In particular, this allows edges that

connect two vertices, edges that connect a vertex to the boundary, edges that connect two points on the boundary, and edges that form circles (closed 1-manifolds embedded in the plane).

The *boundary* $\mathbb{R}\times\{0,1\}$ consists of two components, the *top boundary* $\mathbb{R}\times\{1\}$ and the *bottom boundary* $\mathbb{R}\times\{0\}$. A *boundary edge* is a local segment of an edge which hits the boundary; there is one such boundary edge for every point on the boundary of the graph. A connected component of a graph with boundary might just be referred as *component*.

A *Soergel graph* is an isotopy class of a planar graph in the strip, though the isotopy we considered must preserve the top and bottom boundary. The edges in this graph are labelled/coloured by an element s of S. The edge labels meeting the boundary provide two sequences of colours labelling the top and bottom boundary of the Soergel graph. The vertices in this graph are of three types:

(i) Univalent vertices (dots);

(ii) Trivalent vertices connecting three edges of the same colour;

(iii) $2m_{st}$-valent vertices connecting edges which alternate in colour between two elements s,t of S with $m_{st}<\infty$.

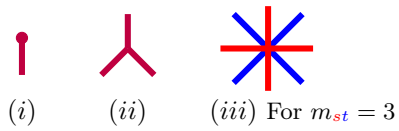


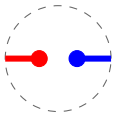
Figure 2.4.1.1: The vertices in a Soergel graph.

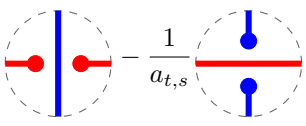
A *region* of the graph is a connected component component of the complement of the Soergel graph in $\mathbb{R}\times[0,1]$. We may place a decoration of boxes each labelled with a homogeneous polynomial f in R inside every region of the Soergel graph; they can be regarded as 0-valent vertices with labels. A Soergel graph has a *degree*, which accumulates $+1$ for each dot (i), -1 for each trivalent vertex (ii), 0 for each $2m_{st}$-valent vertex (iii), and the degree of each polynomial.

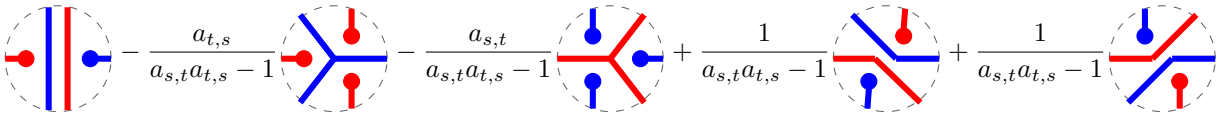
2.4.2 Jones-Wenzl projectors as Soergel graphs

Continuing the spirit of Section 2.3.2, we do the following construction. Fix two indices $s,t\in S$ and a 2-coloured crossingless matching. We produce a Soergel graph on the planar disc by first deformation retracting each region into a tree composed out of trivalent and univalent vertices and colouring these trees appropriately. With this process, we will always obtain a Soergel graph of degree $+2$. We also specialise $\mathbb{Z}[x,y]$ to $\Bbbk$ under the map sending $x\mapsto a_{s,t}$ and $y\mapsto a_{t,s}$. We then call the linear combination of Soergel graphs associated to a Jones-Wenzl projector, the *Jones-Wenzl morphism*. As usual it comes in two colour-flavours.

Example 2.4.1.

$JW_1 =$ 

$JW_2 =$ 

$JW_3 =$ 

2.4.3 Soergel bimodules as a diagrammatic category

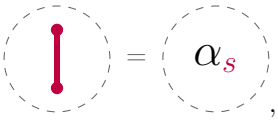
We should stress that the diagrammatic category we described here are not in complete generality, but rather suits our purpose for type A and B Coxeter system. Let $\mathcal{D}_S$, or simply $\mathcal{D}$ denote the $\Bbbk$-linear monoidal category defined as follows;

i. Objects: finite sequences $\underline{w} = s_{i_1} s_{i_2} \dots s_{i_k}$, sometimes denoted $B_{\underline{w}}$, with a monoidal structure given by concatenation.

ii. Homomorphisms: the Hom space $Hom_{\mathcal{D}}(\underline{w}, \underline{v})$ is the free $\Bbbk$-module generated by Soergel graphs with bottom boundary $\underline{w}$ and top boundary $\underline{v}$ modulo the relations listed below. Hom spaces will be graded by the degree of the Soergel graphs, and all the relations below are homogeneous.

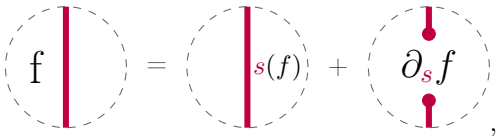
(a) The polynomial relations:

- the barbell relation



(2.4.1)

- the polynomial forcing relation



(2.4.2)

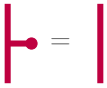
(b) The one colour relations:

- the needle relation

(2.4.3)

- the Frobenius relations

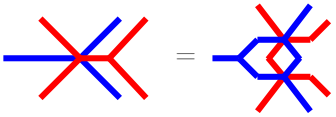
general associativity (2.4.4)



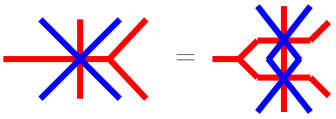
general unit (2.4.5)

(c) The two colour relations:

- the two colour associativity



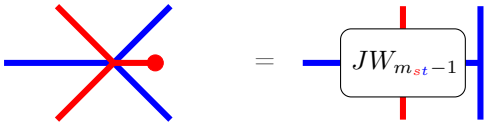
if $m_{st} = 3$ (2.4.6)



if $m_{st} = 4$ (2.4.7)

- the dot-crossing relations

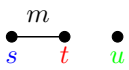
if $m_{st} = 2$ (2.4.8)



if $m_{st} = 3$ (2.4.9)

if $m_{st} = 4$ (2.4.10)

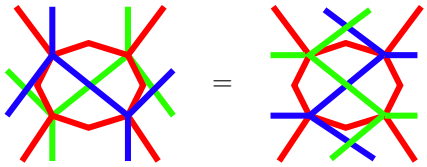
(d) The three colour relations or "Zamolodzhiko" relations:

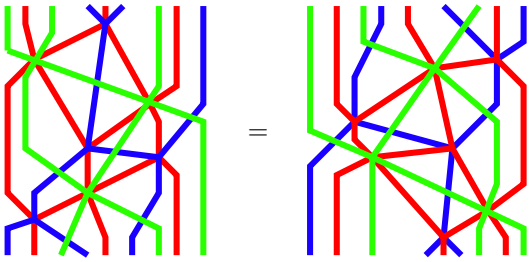
- Coming from a finite parabolic subgroup of rank 3 of type $A_1 \times I_2(m)$, that is, the Coxeter graph of the parabolic subgroup genetated by (s,t,u) is 

$(B_2 \times A_1)$  if $m_{st} = 4$ (2.4.11)

$(A_2 \times A_1)$  if $m_{st} = 3$ (2.4.12)

$(A_1 \times A_1 \times A_1)$  if $m_{st} = 2$ (2.4.13)

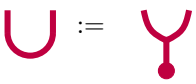
(A_3)  (2.4.14)

(B_3)  (2.4.15)

This concludes the definition of $\mathcal{D}$.

Note that the one colour relations from (2.4.4) to (2.4.5) encode the data of B_s being a Frobenius object. For practical reason, we will provide those one colour relations required for a Frobenius object. These relations are expressed in term of cups and caps defined as follows:

The definition of cap 

The definition of cup 

The trivalent vertices give multiplication or comultiplication while the dots provide unit or counit. To obtain (2.4.4), we can rotate (2.4.16) (resp. (2.4.17)) using a cup (resp. a cap) and apply (2.4.22) (resp. (2.4.21)).

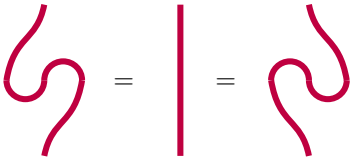
On the other hand, (2.4.21) to (2.4.24) records the cyclicity of all morphisms. Therefore, (2.4.5) can be replaced by (2.4.18) and (2.4.19). Here is the list of them:

the associativity of the multiplication. (2.4.16)

the coassociativity of the comultiplication. (2.4.17)

the enddot as counit for the comultiplication. (2.4.18)

the startdot as unit for the multiplication. (2.4.19)



the self-biadjointness of B_s. (2.4.20)

the mult. is a "half rotation" of the comult. (2.4.21)

the comult. is a half "rotation" of the mult. (2.4.22)

the counit is a "rotation" of the unit (2.4.23)

the unit is a "rotation" of the counit (2.4.24)

This marks the end of the supplementary one colour relations.

Consider a monoidal functor $\mathcal{F} : \mathcal{D} \to \mathbb{BS}$-bimod defined by sending the sequences $\underline{w}$ in $\mathcal{D}$ to the Bott-Samelson bimodules $B_{\underline{w}}$ in $\mathbb{BS}$-bimod on generators. On morphism, it is given by the table belows:

	deg 1	$\longmapsto$	$B_s \to R$	$f \otimes g \mapsto fg$
	deg 1	$\longmapsto$	$R \to B_s$	$1 \mapsto \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$
	deg -1	$\longmapsto$	$B_s \to B_s B_s$	$1 \otimes 1 \mapsto 1 \otimes 1 \otimes 1$
	deg -1	$\longmapsto$	$B_s B_s \to B_s$	$1 \otimes g \otimes 1 \mapsto \partial_s g \otimes 1$
	deg f	$\longmapsto$	$R \to R$	$1 \mapsto f$
	deg 0	$\longmapsto$	$\underbrace{B_s B_t \cdots}_{m_{st}} \xrightarrow{i \circ p(B_{st})} \underbrace{B_t B_s \cdots}_{m_{st}}$	

Table 2.1: The correspondence between generating morphisms in $\mathcal{D}$ and $\mathbb{BS}$-bimod.

Note that the bimodule homomorphism for $2m_{st}$-valent vertex is not presented because its formula is difficult and unenlightening to write down in general. Suppose $B_{s,t}$ is the indecomposable Soergel bimodule indexed by the longest element of the finite rank two parabolic subgroup generated by s and t. Then, $B_{s,t}$ appears as a direct summand of both $B_s B_t \cdots$ and $B_t B_s \cdots$ with multiplicity one. In principal, the $2m_{st}$-valent vertex is just the projection and inclusion of this indecomposable summand $B_{s,t}$.

By construction, this functor is essentialy surjective, so it suffices to show that it is fully faithful. By a Light Leaves Basis theorem from Libedinsky [Lib08], the collection of the morphisms in the right hand side of Table 2.1 forms a basis of Bott-Samelson bimodule homomorphisms. As a result, $\mathcal{F}$ is full. It can be argued that the graded dimensions of the homomorphism spaces in $\mathcal{D}$ and $\mathbb{BS}$-bimod coincide, and therefore $\mathcal{F}$ induces an isomorphism on Hom spaces as a consequence of being a surjection betweeen graded vector spaces of the same finite dimension in each graded component. Finally, $\mathcal{F}$ is fully-faithful, as desired. Since idempotent completion preserves equivalences, we get the following theorem:

Theorem 2.4.2. *Consider the Karoubi envelope $Kar(\overline{\mathcal{D}})$ of the graded additive closure $\overline{\mathcal{D}}$ of $\mathcal{D}$. Then, the two idempotent completions $Kar(\overline{\mathcal{D}})$ and $\mathbb{S}$-bimod are equivalent as monoidal categories.*

Proof. [EW16b, Theorem 6.28] Q.E.D.

2.5 The 2-Braid Group

Let (M, e) be a monoid with unit e and $\mathcal{C}$ be a category. A *weak action* of M on $\mathcal{C}$ consists of a family of functors $\{F_f : \mathcal{C} \to \mathcal{C} \mid f \in M\}$ and a natural isomorphism $c_e : F_e \xrightarrow{\cong} id$ from F_e to the identity functor such that $F_{gh} \cong F_g \circ F_h$ for all $g, h \in M$.

An *action* of M on $\mathcal{C}$ is a weak action of M on $\mathcal{C}$ together with natural isomorphisms of functors $c_{f,g} : F_f F_g \xrightarrow{\cong} F_{fg}$ for all $f, g \in M$ such that $c_{f,e}$ and $c_{e,f}$ are induced by c_e and such that the following diagram commutes:

$$\begin{array}{ccc} F_{ghk} & \xrightarrow{\cong} & F_{gh} \circ F_k \\ \downarrow\cong & & \downarrow\cong \\ F_g \circ F_{hk} & \xrightarrow{\cong} & F_g \circ F_h \circ F_k \end{array}$$

An action of M on $\mathcal{C}$ is called *faithful*, if the functors F_g and F_h are not isomorphic for $g \neq h \in G$. An action of a group G on $\mathcal{C}$ is an action of the underlying unital monoid on $\mathcal{C}$.

We define the elementary Rouquier complexes corresponding to a simple reflection $s \in S$ as in [Rou06]:

$$F_s := 0 \to B_s \xrightarrow{\bullet} R(1) \to 0, \text{ and}$$

$$E_s = F_{s^{-1}} := 0 \to R(-1) \xrightarrow{\bullet} B_s \to 0$$

with both B_s in cohomological degree 0.

Rouquier proves the following result:

Proposition 2.5.1. *Suppose $s \neq t \in S$ with $m_{st} \leq \infty$. Then, we obtain the following morphisms:*

$$F_s \otimes E_s \cong R \cong E_s \otimes F_s;$$
$$\underbrace{F_s \otimes F_t \otimes F_s \otimes \cdots}_{m_{st}\, terms} \cong \underbrace{F_t \otimes F_s \otimes F_t \otimes \cdots}_{m_{st}\, terms}$$

in the bounded homotopy category $K^b(\mathbb{S}\text{-}bimod)$ of Soergel bimodules. Moreover, the map $s \mapsto F_s$ extends to a morphism from $\mathcal{A}(W)$ to the group of isomorphism classes of invertible objects of $K^b(\mathbb{S}\text{-}bimod)$. The composition with the group homomorphism into the group of isomophism classes of autoequivalences on $K^b(\mathbb{S}\text{-}bimod)$ defines a weak action of $B_{(W,S)}$ on $K^b(\mathbb{S}\text{-}bimod)$.

Moreover, the weak action of $B_{(W,S)}$ can be upgraded to an action on $K^b(\mathbb{S}\text{-bimod})$ as explained in [Rou06, Thm 9.5].

This implies that we have a monoidal functor from the strict monoidal category with set of objects $B_{(W,S)}$, with only arrows the identity mapds and with tensor product given by multiplication to the strict monoidal category of endofunctors of $K^b(\mathbb{S}\text{-bimod})$. We then define the *2-braid group*, denoted by 2-$\mathcal{B}r$ as the full monoidal subcategory of $K^b(\mathbb{S}\text{-bimod})$ generated by E_s and F_s for $s \in S$. Observe that the set of isomorphism classes of objects in 2-$\mathcal{B}r$, denoted by Pic(2-$\mathcal{B}r$), forms a group called the *Picard group* of the monoidal category 2-$\mathcal{B}r$ under tensor product composition. Finally, Rouquier conjectures the following:

Conjecture 2.5.2. *Pic(2-$\mathcal{B}r$) is isomorphic to $B_{(W,S)}$.*

Jensen proved this conjecture for type A Coxeter group in his master's thesis following the work of [KS02] and for all arbitrary finite type Coxeter groups in [Jen17] inspired by the work of [BT11].

2.6 Categorified Action of Type B

For completeness of this chapter, we recall some essential construction and result from Chapter 1, but the grading we employed here is the path length grading. Proof of statements will follow similarly.

2.6.1 Type B_n zigzag algebra $\mathscr{B}_n$

Consider the following quiver Q_n:

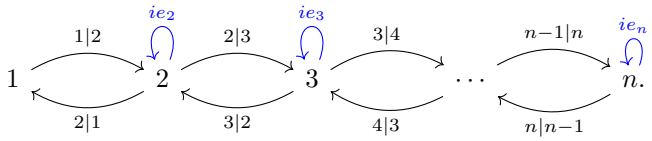


Figure 2.6.1.1: The quiver Q_n.

Take its path algebra $\mathbb{R}Q_n$ over $\mathbb{R}$ and consider the path gradings on $\mathbb{R}Q_n$ denoted by $(-)$ where the "imaginary" path ie_j has grading 0. For the purpose of this chapter, (1) is a grading shift down by 1.

We are now ready to define the zigzag algebra of type B_n:

Definition 2.6.1. *The zigzag path algebra of B_n, denoted by $\mathscr{B}_n$, is the quotient algebra of the path algebra $\mathbb{R}Q_n$ modulo the usual zigzag relations given by*

$$(j|j-1)(j-1|j) = (j|j+1)(j+1|j) \qquad (=: X_j); \tag{2.6.1}$$
$$(j-1|j)(j|j+1) = 0 = (j+1|j)(j|j-1); \tag{2.6.2}$$

for $2 \leq j \leq n-1$, in addition to the relations

$$(ie_j)(ie_j) = -e_j, \qquad \text{for } j \geq 2; \tag{2.6.3}$$
$$(ie_{j-1})(j-1|j) = (j-1|j)(ie_j), \qquad \text{for } j \geq 3; \tag{2.6.4}$$
$$(ie_j)(j|j-1) = (j|j-1)(ie_{j-1}), \qquad \text{for } j \geq 3; \tag{2.6.5}$$
$$(1|2)(ie_2)(2|1) = 0, \tag{2.6.6}$$
$$(ie_2)X_2 = X_2(ie_2). \tag{2.6.7}$$

Since the relations are all homogeneous with respect to the given gradings, $\mathscr{B}_n$ is also a bigraded algebra. As a $\mathbb{R}$-vector space, $\mathscr{B}_n$ has dimension $8n-6$, with basis $\{e_1, \ldots, e_n, ie_2, \ldots, ie_n,$ $(1|2), \ldots, (n-1|n), (2|1), \ldots (n|n-1), (ie_2)(2|1), (1|2)(ie_2), (ie_2)(2|3), \ldots, (ie_{n-1})(n-1|n), (3|2)(ie_2),$ $\ldots, (n|n-1)(ie_{n-1}), (1|2|1), \ldots, (2n-1|2n-2|2n-1), (ie_2)(2|1|2), \ldots, (ie_n)(n|n-1|n)\}$.

The indecomposable (left) projective $\mathscr{B}_n$-modules are given by $P_j^B := \mathscr{B}_n e_j$. For $j = 1$, P_j^B is naturally a $(\mathscr{B}_n, \mathbb{R})$-bimodule; there is a natural left $\mathscr{B}_n$-action given by multiplication of the algebra and the right $\mathbb{R}$-action induced by the natural left $\mathbb{R}$-action. But for $j \geq 2$, we shall endow P_j^B with a right $\mathbb{C}$-action. Note that (2.6.3) is the same relation satisfied by the complex imaginary number i. We define a right $\mathbb{C}$-action on P_j^B by $p * (a+ib) = ap + bp(ie_j)$ for $p \in P_j^B, a+ib \in \mathbb{C}$. Further note that this right action restricted to $\mathbb{R}$ agrees with the natural right (and left) $\mathbb{R}$-action. This makes P_j^B into a $(\mathscr{B}_n, \mathbb{C})$-bimodule for $j \geq 2$. Dually, we shall define ${}_jP^B := e_j\mathscr{B}_n$, where we similarly consider it as a $(\mathbb{R}, \mathscr{B}_n)$-bimodule for $j = 1$ and as a $(\mathbb{C}, \mathscr{B}_n)$-bimodule for $j \geq 2$.

It is easy to check that we have the following isomorphisms of $\mathbb{Z}$-graded bimodules:

Proposition 2.6.2 (= Proposition 1.3.5)**.**

$${}_jP_k^B := {}_jP^B \otimes_{\mathscr{B}_n} P_k^B \cong \begin{cases} {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}} & \text{as } (\mathbb{C},\mathbb{C})\text{-}g_r\text{bimod, for } j,k \in \{2,\ldots,n\} \text{ and } k-j=1; \\ {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}(-1) & \text{as } (\mathbb{C},\mathbb{C})\text{-}g_r\text{bimod, for } j,k \in \{2,\ldots,n\} \text{ and } j-k=1; \\ {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}} \oplus {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}(-2) & \text{as } (\mathbb{C},\mathbb{C})\text{-}g_r\text{bimod, for } j=k=2,3,\ldots,n; \\ {}_{\mathbb{R}}\mathbb{C}_{\mathbb{C}} & \text{as } (\mathbb{R},\mathbb{C})\text{-}g_r\text{bimod, for } j=1 \text{ and } k=2; \\ {}_{\mathbb{C}}\mathbb{C}_{\mathbb{R}}(-1) & \text{as } (\mathbb{C},\mathbb{R})\text{-}g_r\text{bimod, for } j=2 \text{ and } k=1; \\ {}_{\mathbb{R}}\mathbb{R}_{\mathbb{R}} \oplus {}_{\mathbb{R}}\mathbb{R}_{\mathbb{R}}(-2) & \text{as } (\mathbb{R},\mathbb{R})\text{-}g_r\text{bimod, for } j=k=1. \end{cases}$$

Note that all the graded bimodules in Proposition 2.6.2 can be restricted to a $(\mathbb{R},\mathbb{R})$-bimodule. By identifying ${}_{\mathbb{R}}\mathbb{C}_{\mathbb{R}} \cong \mathbb{R} \oplus \mathbb{R}$ as $\mathbb{Z}/2\mathbb{Z}$-graded $(\mathbb{R},\mathbb{R})$-bimodules, the bimodules in (2.6.2) restricted to just the $\mathbb{R}$ actions are also isomorphic as bigraded ($\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$) $(\mathbb{R},\mathbb{R})$-bimodule. For example, ${}_1P_2^B$ as a $(\mathbb{R},\mathbb{R})$-bimodule is generated by $(1|2)$ and $(1|2)i$, so it is isomorphic to $\mathbb{R} \oplus \mathbb{R} \cong {}_{\mathbb{R}}\mathbb{C}_{\mathbb{R}}$.

Lemma 2.6.3 (= Lemma 1.3.7)**.** *Denote $k_j := \mathbb{R}$ when $j = 1$ and $k_j := \mathbb{C}$ when $j \geq 2$. The maps*

$$\beta_j : P_j^B \otimes_{k_j} {}_jP^B \to \mathscr{B}_n \text{ and } \gamma_j : \mathscr{B}_n \to P_j^B \otimes_{k_j} {}_jP^B(2)$$

defined by:

$$\beta_j(x \otimes y) := xy,$$
$$\gamma_j(1) := \begin{cases} X_j \otimes e_j + e_j \otimes X_j + (j+1|j) \otimes (j|j+1) & \\ \quad +(-ie_{j+1})(j+1|j) \otimes (j|j+1)(ie_{j+1}), & \text{for } j = 1; \\ X_j \otimes e_j + e_j \otimes X_j + (j-1|j) \otimes (j|j-1) + (j+1|j) \otimes (j|j+1), & \text{for } 1 < j < n; \\ X_j \otimes e_j + e_j \otimes X_j + (j-1|j) \otimes (j|j-1), & \text{for } j = n, \end{cases}$$

are $(\mathscr{B}_n, \mathscr{B}_n)$-bimodule maps.

Proof. It is obvious that β_j are for all j. The fact that γ_j is a $(\mathscr{B}_n, \mathscr{B}_n)$-bimodule map also follows from a tedious check on each basis element, which we shall omit and leave it to the reader. Q.E.D.

Definition 2.6.4. *Define the following complexes of graded $(\mathscr{B}_n, \mathscr{B}_n)$-bimodules:*

$$R_j := (0 \to P_j^B \otimes_{\mathbb{F}_j} {}_jP^B \xrightarrow{\beta_j} \mathscr{B}_n \to 0), \text{ and}$$
$$R_j' := (0 \to \mathscr{B}_n \xrightarrow{\gamma_j} P_j^B \otimes_{\mathbb{F}_j} {}_jP^B\langle 2 \rangle \to 0).$$

for each $j \in \{1, 2, \cdots, n\}$, with both $\mathscr{B}_n$ in cohomological degree 0, $\mathbb{F}_1 = \mathbb{R}$ and $\mathbb{F}_j = \mathbb{C}$ for $j \geq 2$.

Proposition 2.6.5. *There are isomorphisms in the homotopy category, $Kom^b((\mathscr{B}_n, \mathscr{B}_n)\text{-}bimod)$, of complexes of projective graded $(\mathscr{B}_n, \mathscr{B}_n)$-bimodules:*

$$R_j \otimes R_j^{'} \cong \mathscr{B}_n \cong R_j^{'} \otimes R_j; \tag{2.6.8}$$
$$R_j \otimes R_k \cong R_k \otimes R_j, \quad \text{for } |k-j| > 1; \tag{2.6.9}$$
$$R_j \otimes R_{j+1} \otimes R_j \cong R_{j+1} \otimes R_j \otimes R_{j+1}, \quad \text{for } j \geq 2; \tag{2.6.10}$$
$$R_2 \otimes_{\mathscr{B}_n} R_1 \otimes_{\mathscr{B}_n} R_2 \otimes_{\mathscr{B}_n} R_1 \cong R_1 \otimes_{\mathscr{B}_n} R_2 \otimes_{\mathscr{B}_n} R_1 \otimes_{\mathscr{B}_n} R_2. \tag{2.6.11}$$

Proof. Please refer to Proposition 1.3.9 and Proposition 1.3.12. Q.E.D.

Theorem 2.6.6. *(=Theorem 1.3.13) We have a (weak) $\mathcal{A}(B_n)$-action on $Kom^b(\mathscr{B}_n\text{-}p_rg_rmod)$, where each standard generator σ_j^B for $j \geq 2$ of $\mathcal{A}(B_n)$ acts on a complex $M \in Kom^b(\mathscr{B}_n\text{-}p_rg_rmod)$ via R_j, and σ_1^B acts via $R_1\langle 1 \rangle$:*

$$\sigma_j^B(M) := R_j \otimes_{\mathscr{B}_n} M, \text{ and } \sigma_j^{B^{-1}}(M) := R_j' \otimes_{\mathscr{B}_n} M,$$
$$\sigma_1^B(M) := R_1\langle 1 \rangle \otimes_{\mathscr{B}_n} M, \text{ and } \sigma_1^{B^{-1}}(M) := R_1'\langle 1 \rangle \otimes_{\mathscr{B}_n} M.$$

Proof. This follows directly from Proposition 1.3.9 and Proposition 1.3.12, where the required relations still hold with the extra third grading shift $\langle 1 \rangle$ on R_1 and R_1'. Q.E.D.

2.6.2 Functor realisation of the type B Temperley-Lieb algebra

In [KS02, section 2b], Khovanov-Seidel showed that the $(\mathscr{A}_m, \mathscr{A}_m)$-bimodules $\mathcal{U}_j := P_j^A \otimes_{\mathbb{C}} {}_jP^A$ provides a functor realisation of the type A_m Temperley-Lieb algebra. We will see that we have a similar type B_n analogue of this.

Recall that the type B_n Temperley-Lieb algebra $TL_v(B_n)$ over $\mathbb{Z}[v, v^{-1}]$ can be described explictly as (see [Gre98, Proposition 1.3]) the algebra generated by $E_1, ..., E_n$ with relation

$$E_j^2 = vE_j + v^{-1}E_j;$$
$$E_jE_k = E_kE_j, \quad \text{if } |j-k| > 1;$$
$$E_jE_kE_j = E_j, \quad \text{if } |j-k| = 1 \text{ and } j, k > 1;$$
$$E_jE_kE_jE_k = 2E_jE_k, \quad \text{if } \{j, k\} = \{1, 2\}.$$

To match with the above, for this subsection only, we shall adopt the path length grading on our algebra $\mathscr{B}_n$ instead, where we insist that the blue paths ie_j in Figure 2.6.1.1 have length 0. We shall denote this path length grading shift by $(-)$ to avoid confusion. Define $\mathcal{U}_j := P_j^B \otimes_{\mathbb{F}_j\, j} P^B(1)$, where $\mathbb{F}_1 = \mathbb{R}$ and $\mathbb{F}_j = \mathbb{C}$ when $j \geq 2$. It is easy to check that:

Proposition 2.6.7. *The following are isomorphic as (path length graded) $(\mathscr{B}_n, \mathscr{B}_n)$-bimodules:*

$$\begin{aligned}
\mathcal{U}_j^2 &\cong U_j(1) \oplus \mathcal{U}_j(-1) \\
\mathcal{U}_j\mathcal{U}_k &\cong 0 \quad \text{if } |j-k| > 1, \\
\mathcal{U}_j\mathcal{U}_k\mathcal{U}_j &\cong \mathcal{U}_k \quad \text{if } |j-k| = 1 \text{ and } j,k > 1, \\
\mathcal{U}_j\mathcal{U}_k\mathcal{U}_j\mathcal{U}_k &\cong \mathcal{U}_j\mathcal{U}_k \oplus \mathcal{U}_j\mathcal{U}_k \quad \text{if } \{j,k\} = \{1,2\}.
\end{aligned}$$

Comparing this with the relations of $TL_v(B_n)$ above, it follows that U_j provides a (degenerate) functor realisation of the type B Temperley-Lieb algebra.

Theorem 2.6.8 (=Theorem 1.4.6)**.** *The (weak) action of $\mathcal{A}(B_n)$ on the category $Kom(\mathscr{B}_n\text{-}p_r g_r mod)$ given in 1.3.13 is faithful.*

Proof. Please refer to Theorem 1.4.6 and Remark 1.7.4. Q.E.D.

2.7 Faithfulness of the 2-Braid Group in Type B

The aim of this section is to construct a faithful 2-representation of the type B 2-braid group using the action of $\mathcal{A}(B_n)$ on $K^b(\mathscr{B}_n\text{-}g_r\text{mod})$.

Fix $n \geq 2$. Denote by $\mathscr{B}$ the additive monoidal subcategory of $(\mathscr{B}_n, \mathscr{B}_n)$-bimod generated by the U_i for $1 \leq i \leq n$ with morphisms being their grading shifts with bimodule homomorphisms of all degrees. Let $\overline{\mathscr{B}}$ be the graded version of $\mathscr{B}$. In this way, $\overline{\mathscr{B}}$ is a full additive graded monoidal subcategory of $(\mathscr{B}_n, \mathscr{B}_n)$-bimod generated by the U_i for $1 \leq i \leq n$ where objects are direct sums of tenor porducts of copies of the U_i's and morphisms are grading-preserving bimodule maps.

Fix the realization in Example 2.2.4 and consider its diagrammatic category $\mathcal{D}$. We want to construct a functor $G : \mathcal{D} \to \mathscr{B}$ which induces a grading-preserving functor $\overline{G} : K^b(Kar(\overline{\mathcal{D}})) \to K^b(Kar(\overline{\mathscr{B}}))$ macthing the Rouquier and the Khovanov Type B complexes associated to a braid word up to a perverse shift. Since $\mathcal{D}$ is a strict monoidal category with generators and relations, it suffices to define the functor G on the generating objects and morphisms as well as to verify that all relations in $\mathcal{D}$ hold in $\mathscr{B}$.

In the following section, we use red or violet for a simple reflection s_j, blue for an adjacent simple reflection $s_{j\pm1}$, and green for a distant simple reflection s_k with $|j-k| > 1$. We are going to define G on generating objects by sending s_j in S to $\mathcal{U}_j$ and on generating morphisms as in the proof of Theorem 2.7.2 where the following $(\mathscr{B}_n, \mathscr{B}_n)$-bimodule homomorphisms are defined as follows:

$$\begin{array}{lccccl}
\alpha_j : & \mathcal{U}_j & \to & \mathcal{U}_j(-1) \oplus \mathcal{U}_j(1) & \xrightarrow{\cong} & \mathcal{U}_j \otimes_{\mathscr{B}_n} \mathcal{U}_j \quad \text{of degree -1,} \\
& e_j \otimes e_j & \mapsto & (0, e_j \otimes e_j) & \mapsto & e_j \otimes e_j \otimes e_j \otimes e_j
\end{array}$$

$$\begin{array}{lccccl}
\delta_j : & \mathcal{U}_j \otimes_{\mathscr{B}_n} \mathcal{U}_j & \xrightarrow{\cong} & \mathcal{U}_j(-1) \oplus \mathcal{U}_j(1) & \to & \mathcal{U}_j \quad \text{of degree -1,} \\
& e_j \otimes X_j \otimes e_j \otimes e_j & \mapsto & (e_j \otimes e_j, 0) & \mapsto & e_j \otimes e_j \\
& e_j \otimes e_j \otimes e_j \otimes e_j & \mapsto & (0, e_j \otimes e_j) & \mapsto & 0
\end{array}$$

$$\epsilon_j : \mathscr{B}_n \to \mathscr{B}_n \quad \text{of degree 2,}$$

$$1 \mapsto \begin{cases} (-1)^{i+1}(2X_j + 2X_{j+1}), & \text{for } j = 1; \\ (-1)^{i+1}(2X_j + X_{j-1} + X_{j+1}), & \text{for } 1 < j < n; \\ (-1)^{i+1}(2X_j + X_{j-1}), & \text{for } j = n, \end{cases}$$

and, recall from Lemma 2.6.3,

$$\begin{aligned}&\beta_j : \mathcal{U}_j \to \mathscr{B}_n \text{ of degree } 1,\\ &e_j \otimes e_j \mapsto e_j\end{aligned}$$

$$\begin{aligned}&\gamma_j : \mathscr{B}_n \to \mathcal{U}_j \text{ of degree } 1,\\ &(1) \mapsto \begin{cases} X_j \otimes e_j + e_j \otimes X_j + (j+1|j) \otimes (j|j+1) & \\ \qquad +(-ie_{j+1})(j+1|j) \otimes (j|j+1)(ie_{j+1}), & \text{for } j = 1;\\ X_j \otimes e_j + e_j \otimes X_j + (j-1|j) \otimes (j|j-1) + (j+1|j) \otimes (j|j+1), & \text{for } 1 < j < n;\\ X_j \otimes e_j + e_j \otimes X_j + (j-1|j) \otimes (j|j-1), & \text{for } j = n, \end{cases}\end{aligned}$$

Lemma 2.7.1. *Denote $k_j := \mathbb{R}$ when $j = 1$ and $k_j := \mathbb{C}$ when $j \geq 2$. The space of $(\mathscr{B}_n, \mathscr{B}_n)$-bimodule homomorphism:*

1. *$\mathcal{U}_j \to \mathscr{B}_n$ of degree 1 is isomorphic to $k_j\beta_j$;*
2. *$\mathscr{B}_n \to \mathcal{U}_j$ of degree 1 is isomorphic to $k_j\gamma_j$;*
3. *$\mathcal{U}_j \to \mathcal{U}_j \otimes_{\mathscr{B}_n} \mathcal{U}_j$ of degree -1 is isomorphic to $k_j\alpha_j$;*
4. *$\mathcal{U}_j \otimes_{\mathscr{B}_n} \mathcal{U}_j \to \mathcal{U}_j$ of degree -1 is isomorphic to $k_j\delta_j$;*
5. *$\mathscr{B}_n \to \mathscr{B}_n$ of degree 2 is isomorphic to $\oplus_{1\leq j\leq n} k_j X_j$, where X_j is interpreted with left multiplication or right multiplication with X_j giving an $\mathscr{B}_n$-bimodule homomorphism as all paths of lengths $\geq$ are killed in $\mathscr{B}_n$.*

Proof. This is an easy verification. Q.E.D.

Recall that, in previous section (Definition 2.6.4), we define the following complexes of graded $(\mathscr{B}_n, \mathscr{B}_n)$-bimodules: for each $j \in \{1, 2, \cdots, n\}$,

$$\begin{aligned}R_j &:= (0 \to \mathcal{U}_j(-1) \xrightarrow{\beta_j} \mathscr{B}_n \to 0), \text{and}\\ R'_j &:= (0 \to \mathscr{B}_n \xrightarrow{\gamma_j} \mathcal{U}_j(1) \to 0).\end{aligned}$$

with both $\mathscr{B}_n$ in cohomological degree 0, $\mathbb{F}_1 = \mathbb{R}$ and $\mathbb{F}_j = \mathbb{C}$ for $j \geq 2$.

Theorem 2.7.2. *$G : \mathcal{D} \to \mathscr{B}$ is a well-defined monoidal functor sending*

(i) the empty sequence in S to $\mathscr{B}_n$, the monoidal identity in $\mathscr{B}$, s_i in $\mathscr{B}$;

(ii) s_i in S to U_i;

(iii) a sequence $s_{i_1}s_{i_2}\cdots s_{i_k}$ in S of length $k \geq 2$ to $G_0(s_{i_1}s_{i_2}\cdots s_{i_{k-1}}) \otimes_{\mathscr{B}_n} G_0(s_{i_k})$.

For two sequences $\underline{w}$ and $\underline{w}'$ in S, the coherence morphism $G_0(\underline{w}) \otimes_{A_n} G_0(w') \xrightarrow{\cong} G_0(\underline{ww'})$ is given by either a chosen sequence of associator morphisms moving all parentheses of the right block corresponding to the left to match the configuration of parenthese on the right hand side or a left or right unitor depending on whether $\underline{w}$ or $\underline{w}'$ is the empty sequence in S.

Proof. The assignment on objects is done in the theorem statement whereas on generating morphisms is carried out as follows:


$\longmapsto \quad b_j\beta_j : \mathcal{U}_j \to \mathscr{B}_n$ of degree 1


$\longmapsto \quad c_j\gamma_j : \mathscr{B}_n \to \mathcal{U}_j$ of degree 1


$\longmapsto \quad a_j\alpha_j : \mathcal{U}_j \to \mathcal{U}_j \otimes_{\mathscr{B}_n} \mathcal{U}_j$ of degree -1


$\longmapsto \quad d_j\delta_j : \mathcal{U}_j \otimes_{\mathscr{B}_n} \mathcal{U}_j \to \mathcal{U}_j$ of degree -1


$\longmapsto \quad \epsilon_j : \mathscr{B}_n \to \mathscr{B}_n, 1 \mapsto \sum_{k=1}^{n} f_k^j X_k$ of degree 2

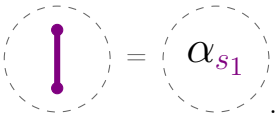

$\longmapsto \quad 0$ as $\mathcal{U}_j \otimes_{\mathscr{B}_n} \mathcal{U}_k = 0$, for $|j-k| > 1$


$\longmapsto \quad 0$


$\longmapsto \quad 0$

where a_j, b_j, c_j, d_j, $f_k^j \in \mathbb{R}$ for $j = 1$ and $\in \mathbb{C}$ for $j > 1$. Next, we want to find a set of scalars such that the restrictions imposed by the relations in $\mathcal{D}$ are satisfied.

To illustrate, let us consider the barbell relation (2.4.1) for s_1:

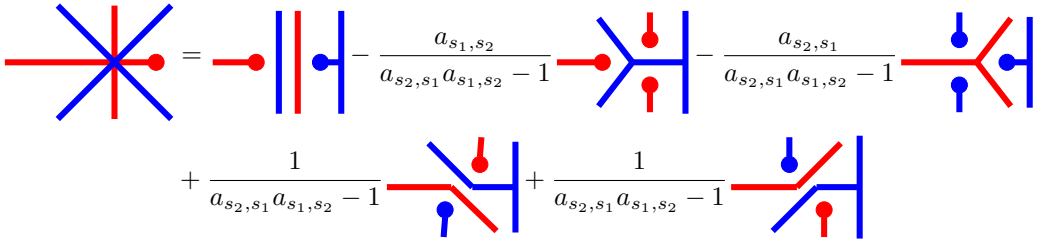


By definition,

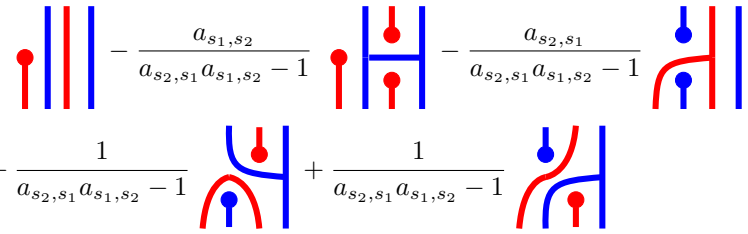
$$\begin{aligned} 1 &\mapsto c_1[X_1 \otimes e_1 + e_1 \otimes X_1 + (2|1) \otimes (1|2) + (-ie_2)(2|1) \otimes (1|2)(ie_2)] \\ &\mapsto c_1b_1[X_1 + X_1 + X_2 + X_2] \\ &= c_1b_1[2X_1 + 2X_2] \end{aligned}$$

Equating with the right hand side, we get $c_1b_1[2X_1 + 2X_2] = \sum_{k=1}^{n} f_k^1 X_k$ which, in turn, implies $2c_1b_1 = f_1^1$ and $2c_1b_1 = f_2^1$.

By the same token, let us look at the type B 8-valences relation (2.4.10) again: suppose red encodes s_1 and blue encodes s_2,



which, by construction, is equivalent to

$0 =$  $- \frac{a_{s_1,s_2}}{a_{s_2,s_1}a_{s_1,s_2}-1}$ $- \frac{a_{s_2,s_1}}{a_{s_2,s_1}a_{s_1,s_2}-1}$

$+ \frac{1}{a_{s_2,s_1}a_{s_1,s_2}-1}$ $+ \frac{1}{a_{s_2,s_1}a_{s_1,s_2}-1}$

Observe that there are five terms of Soergel graphs on the right hand side. One thing to note here is that $\mathcal{U}_1 \otimes \mathcal{U}_2 \otimes \mathcal{U}_1 \otimes \mathcal{U}_2$ is spanned by $e_1 \otimes_{\mathbb{R}} (1|2) \otimes_{\mathscr{B}_n} e_2 \otimes_{\mathbb{C}} e_2 \otimes_{\mathscr{B}_n} (2|1) \otimes_{\mathbb{R}} (1|2) \otimes_{\mathscr{B}_n} e_2 \otimes_{\mathbb{C}} e_2$ and $e_1 \otimes_{\mathbb{R}} (1|2) \otimes_{\mathscr{B}_n} e_2 \otimes_{\mathbb{C}} e_2 \otimes_{\mathscr{B}_n} (-ie_2)(2|1) \otimes_{\mathbb{R}} (1|2) \otimes_{\mathscr{B}_n} e_2 \otimes_{\mathbb{C}} e_2$. Without the coefficients, looking at the second term in the above equation and applying appropriate, we get

$$\begin{aligned} e_1 \otimes (1|2) \otimes e_2 \otimes e_2 \otimes (2|1) \otimes (1|2) \otimes e_2 \otimes e_2 &\xmapsto{\beta_1} b_1 (1|2) \otimes e_2 \otimes (2|1) \otimes (1|2) \otimes e_2 \otimes e_2 \\ &\xmapsto{\beta_1} b_1^2 (1|2) \otimes X_2 \otimes e_2 \otimes e_2 \\ &\xmapsto{\delta_2} d_2 b_1^2 (1|2) \otimes e_2 \\ &\xmapsto{\alpha_2} a_2 d_2 b_1^2 (1|2) \otimes e_2 \otimes e_2 \otimes e_2 \\ &\xmapsto{\gamma_1} c_1 a_2 d_2 b_1^2 (1|2) \otimes e_2 \otimes (-ie_2)(2|1) \otimes (1|2)(ie_2) \otimes e_2 \otimes e_2 \\ &\quad + c_1 a_2 d_2 b_1^2 (1|2) \otimes e_2 \otimes (2|1) \otimes (1|2) \otimes e_2 \otimes e_2. \end{aligned}$$

Similarly,

$$\begin{aligned} &e_1 \otimes (1|2) \otimes e_2 \otimes e_2 \otimes (-ie_2)(2|1) \otimes (1|2) \otimes e_2 \otimes e_2 \\ &\quad \mapsto c_1 a_2 d_2 b_1^2 \left[(1|2) \otimes e_2 \otimes (2|1) \otimes (1|2)(-ie_2) \otimes e_2 \otimes e_2 + (1|2) \otimes e_2 \otimes (-ie_2)(2|1) \otimes (1|2) \otimes e_2 \otimes e_2 \right]. \end{aligned}$$

Once the calculations for all the five Soergel graphs, comparing coefficients coming from four basis elements in the codomain will yield four defining equations.

Before giving you all the relations, we will deal with the unnecessary or overlapped relations. For (2.4.3), it says $b_j d_j a_j \beta_j \delta_j \alpha_j = 0$ which is true as $\delta_j \alpha_j = 0$. On the other hand, (2.4.4) (equiv. (2.4.16) and (2.4.17)) do not impose any restrictions on the coefficients whereas (2.4.5) is replaced by (2.4.18) and (2.4.19). In addition, (2.4.6), (2.4.7), (2.4.11) (2.4.12), (2.4.13), (2.4.14), and (2.4.15) are all trivially satisfied as the $2m_{st}$-valents vertices are killed for every $s, t \in S$. Moreover, (2.4.8) has both side equal as a zero object in $\mathbb{B}_a$.

Now, we summarise all the relations: (Mostly, the same from the type A case, except the two colour relation.)

(2.4.1) $\Rightarrow$ $f_1^1 = 2b_1c_1, f_2^1 = 2b_1c_1, f_j^j = 2b_jc_j, f_{j\pm1}^j = b_jc_j, f_k^j = 0$ for $j, k \geq 2$ and $|j-k| > 1$,

(2.4.2) $\Rightarrow$ $f_1^1 = 2b_1c_1, f_2^1 = -2b_2c_2, f_j^j = 2b_jc_j, f_j^{j\pm1} = -b_jc_j, f_k^j = 0$ for $j, k \geq 2$ and $|j-k| > 1$,

(2.4.9) (Type A 6-valences relation) $\Rightarrow$ $d_{j\pm1}b_jc_j = -b_{j\pm1}$ for suitable j,

(2.4.10) (Type B 8-valences relation) $\Rightarrow$ Since $a_{s_2,s_1}a_{s_1,s_2} - 1 = (-2)(-1) - 1 = 1$, we will simplify the denominator first.

For U_1 left-aligned, we get

$$b_1 - a_{s_1,s_2}b_1^2d_2a_2c_1 - a_{s_2,s_1}b_2d_1c_2 + b_2d_1b_1a_2c_1 + b_1d_2c_2 = 0,$$

$$b_1 - a_{s_1,s_2}b_1^2d_2a_2c_1 = 0, \quad -a_{s_1,s_2}b_1^2d_2a_2c_1 + b_2d_1b_1a_2c_1 = 0, \quad -a_{s_1,s_2}b_1^2d_2a_2c_1 + b_1d_2c_2 = 0,$$

while for U_2 left-aligned, we get

$$b_2 - a_{s_2,s_1}b_2^2d_1a_1c_2 - a_{s_1,s_2}b_1d_2c_1 + b_1d_2a_1b_2c_2 + b_2d_1c_1 = 0,$$

$$b_2 - a_{s_1,s_2}b_1d_2c_1 = 0, \quad -a_{s_1,s_2}b_1d_2c_1 + b_1d_2a_1b_2c_2 = 0, \quad -a_{s_1,s_2}b_1d_2c_1 + b_2d_1c_1 = 0,$$

(2.4.18) $\Rightarrow$ $a_jb_j = 1$ for all $1 \leq j \leq n$,

(2.4.19) $\Rightarrow$ $c_jd_j = 1$ for all $1 \leq j \leq n$,

(2.4.20) $\Rightarrow$ $a_jb_jc_jd_j = 1$ for all $1 \leq j \leq n$,

(2.4.21) $\Rightarrow$ $a_jb_jd_j = d_j$ for all $1 \leq j \leq n$,

(2.4.22) $\Rightarrow$ $a_jc_jd_j = a_j$ for all $1 \leq j \leq n$,

(2.4.23) $\Rightarrow$ $b_jc_jd_j = b_j$ for all $1 \leq j \leq n$,

(2.4.24) $\Rightarrow$ $a_jb_jc_j = c_j$ for all $1 \leq j \leq n$.

The solution $a_j = b_j = (-1)^{j+1}, c_j = d_j = 1, f_2^1 = 2, f_j^j = (-1)^{j+1}2, f_{j\pm1}^j = (-1)^{j+1}$ and $f_k^j = 0$ for $|j-k| > 1$ gives our desired functor. Q.E.D.

Corollary 2.7.3. *There is a well-defined additive, monoidal functor $\overline{G} : K^b(Kar(\overline{\mathcal{D}})) \to K^b(Kar(\overline{\mathscr{B}}))$ which sends F_{s_j} to $R_j[-1](1)$ and E_{s_1} to $R'_j[1](-1)$ and thus matches Rouquier and the Khovanov Type B complex associated to a braid word in Br_{n+1} up to inner grading shift and cohomological shift.*

Corollary 2.7.4 (Faithfulness of the 2-braid group in type B)**.** *Let (W, S) be a Coxeter group of type B_n, for $n \geq 2$. For any two distinct braids $\sigma \neq \beta \in Br_{(W,S)}$ the corresponding Rouquier complexes F_σ and F_β in the 2-braid group are non-isomorphic.*

Proof. This is due to the faithfulness result of the type B zigzag algebra in Theorem 2.6.8 and the type B 2-braid group action factors through the faithful map. Q.E.D.

Chapter 3

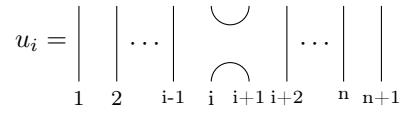
A Graded Fock Vector for a Crossingless Matching.

3.1 Background and Outline of the Chapter

In [Jon85], Vaughan Jones introduced a polynomial invariant for knots, thanks to the uniqueness of the trace on a class of von Neumann algebras called the type II_1 factors. This is the well-known Jones polynomial. He introduced the Jones polynomial via a representation of braid groups through the representation of Temperley-Lieb algebra. More on the the relation between Jones polynomial and Temperley-Lieb algebra can be found in [Jon87; Jon90; Jon16].

The goal of this chapter is to seek a new presentation of Jones polynomial via the Heisenberg algebra. This comes from a conjectural braid group action on the Fock space, which is a representation of the Heisenberg algebra. The braid group action has its origin in Lusztig's braid group action constructed from a quantum group [Lus10] in the form of vertex operators [CL14], which is categorical in flavour. Though it still remains a basic conjectural statement that the newly-defined braid operators in this chapter indeed braid, we start off the program by identifying a vector subspace of Fock space which is conjectured to be a Temperley-Lieb representation. Recall that the Temperley-Lieb algebra $\mathcal{TL}_{n+1}$ has a graphical interpretation with the following assignment to its generator u_i:

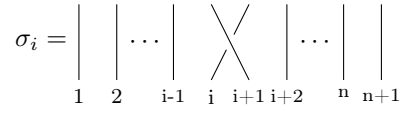
$u_i =$



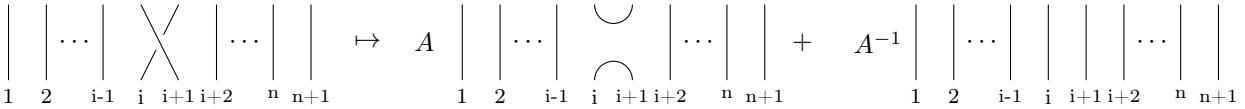
and this generator acts on the set of crossingless matchings. We devise a way to assign each crossingless matching a graded vector in the Fock space.

On the other hand, the $(n+1)$-strand braid group or the type A_n braid group, $\mathcal{A}(A_n)$ has a graphical interpretation with the following assignment to its generators s_i :

$\sigma_i =$



Following [Kau87], note that there is a representation π of the $(n+1)$-strand braid group into the Temperley-Lieb algebra which is given by



The choice of scalar here is due to the requirement of the invariant under type II Reidemeister move. As a side note, through state models, Kauffman's series of papers [Kau88; Kau98a; Kau98b; Kau97; Kau91] draft an outline of the importance of graphical calculus to Jones polynomial.

Hence, the vector subspace spanned by the crossingless matchings is conjectured to be preserved by the braid operators via the Temperley-Lieb representation.

Outline of the chapter

In Section 3.2, we introduce various definitions of combinatorics related to tableaux which are essential to the theory in the chapter.

Section 3.3 is where we recall and reconcile the definition of type A quantum lattice Heisenberg algebra from various literature.

In Section 3.4, we present the main battlefield of the theory – the Fock space representation of the Heisenberg algebra.

Section 3.5 is where we define the braid operator in the Fock space representation coming from private communication with my supervisor, Licata [Lic19].

Section 3.6 is the key section where we introduce a graded Fock vector for a crossingless matching. We also make two conjectures on these graded Fock vectors.

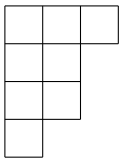
3.2 Combinatorics of Partitions of a Number

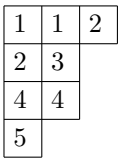
For the general study of Young tableaux, the readers are encouraged to read [Ful97; Sag01; Sta97]. In this section, we will also recall a specific combinatoric tool from [CL16].

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r, \dots)$ with an infinite number of non-zero terms satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \cdots$ be a *partition* of a natural number $n \in \mathbb{N}_{>0}$. We identify a finite sequence $(\lambda_1, \lambda_2, \dots, \lambda_r)$ with the infinite sequence obtained by setting $\lambda_i = 0$ for $i > r$. These λ_i are called the *parts* of λ. We then have the sum $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. We also denote $\lambda \vdash n$. We could also write $\lambda = (1^{m_1} 2^{m_2} \cdots)$ where m_i is the number of parts of λ equals to i. For example, $\lambda = (3, 2, 2, 1) = (1 \cdot 2^2 \cdot 3)$.

To each λ, we can associate a *Young diagram* (in English notation) which is a collection of boxes arranged in left-justified rows with a weakly decreasing number of boxes in each row. We define the *conjugate partition* or the *transpose* of a partition λ^t to be the Young diagram obtained by reflection in the main diagonal, in other words, interchanging rows and columns. It follows that if $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$, then the number of parts of λ^t that equals to i is $\lambda_{i+1} - \lambda_i$. For example, if $\lambda = (3, 2, 2, 1)$, then $\lambda^t = (4, 3, 1)$. We write $\mu \subset \lambda$ if the Young diagram of μ is contained in that of λ, or equivalently $\mu_i \leq \lambda_i$ for all i. If $\mu \subset \lambda$, we can then form a skew diagram $\lambda - \mu$ by removing μ from λ in such a way that the first μ_i boxes are removed from the part λ_i for all i.

Next, we want to introduce the lattice permutation. A *Young tableau* or a *semistandard Young tableau* is a filling of a Young diagram by positive integers that is weakly increasing across each row and strictly increasing down each column and denote a Young tableau by $SSYT(\lambda)$. If the Young diagram is filled with the natural numbers $1, 2, \dots, k$, then it is called the standard Young tableau and denoted by $SYT(\lambda)$. The *word* $w(\lambda_T)$ of a tableau T is the set of numbers in the diagram read from right to left and top to bottom. On the other hand, the *content* $w(\lambda)$ of a partition is the word of the tableau of shape λ with i-th row filled with the number i. The word $a_1 a_2 \cdots a_r$ is a *lattice permutation*[1] if for any i and $k \leq r$, the number of i's in $a_1 a_2 \cdots a_r$ is at least as great as the number of $i+1$'s. For example,

$\lambda =$ 

$\lambda_T =$ 

$w(\lambda) = 11122334$ $w(\lambda_T) = 21132445$

It is a lattice permutation. It is not a lattice permutation because of the first 2.

A *skew partition* is a pair of partitions (λ, ν) such that the Young diagram of λ contains the Young diagram of ν, and is denoted by $(\lambda - \nu)$. A *skew Young diagram* of a skew partition $(\lambda - \nu)$ are the set of boxes that belong to the Young diagram of λ but not to that of ν. The definition of a *skew (semi)standard Young tableau* follows similarly from its non-skew counterpart. We denote the set of skew Young tableaux $(\lambda - \nu)_T$ with content $w(\mu - \kappa)$ whose concatenation of words $w(\kappa)w((\lambda - \nu)_T)$ are lattice permutations as $\mathfrak{lp}(\lambda - \nu, \mu - \kappa)$. We also define *$t$-Young tableau* $\lambda_{\mathfrak{T}}$ as a filling of monomials of the form at^b for $a \geq 1$ and $b \in \{0, 1\}$ with a total order given by

$$at^b \leq ct^d \iff b < d \text{ or } b = d \text{ and } a \leq c. \tag{3.2.1}$$

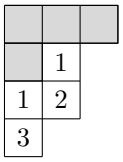
The word of t-Young tableau reads the coefficients of the monomials from right to left and top to bottom with the monomials of the form at^0 read before the monomials of the form at^1. We denote the set of skew t-Young tableaux$(\lambda - \nu)_{\mathfrak{T}}$ with monomials having coefficient in content $w(\mu - \kappa)$ and whose concatenation of words $w(\kappa)w((\lambda - \nu)_{\mathfrak{T}})$ are lattice permutation as $t\text{-}\mathfrak{lp}(\lambda - \nu, \mu - \kappa)$. We also define a statistic on t-Young

[1] Also known as *Yamanouchi word* or *ballot sequence* in different literature.

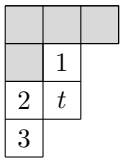
tableau by

$$c(\lambda_{\mathfrak{T}}) = \prod_{at^b \in \lambda_{\mathfrak{T}}} t^b. \tag{3.2.2}$$

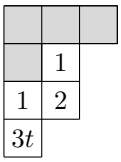
For example, let $\lambda = (3,2,2,1), \nu = (3,1), \mu = (3,1,1), \kappa = (1)$. Then, $w(\mu - \kappa) = 1123$. Below is the list of elements in t-$\mathfrak{lp}(\lambda - \nu, \mu - \kappa)$ with the its first element in $\mathfrak{lp}(\lambda - \nu, \mu - \kappa)$.



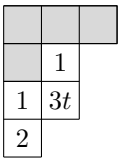
$w = 11213$
$c = 1$



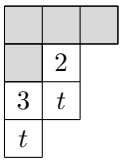
$w = 11231$
$c = t$



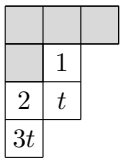
$w = 11213$
$c = t$



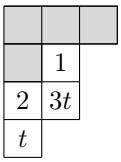
$w = 11123$
$c = t$



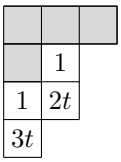
$w = 12311$
$c = t^2$



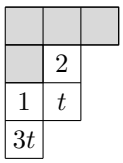
$w = 11213$
$c = t^2$



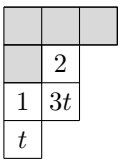
$w = 11231$
$c = t^2$



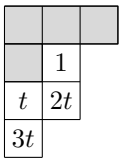
$w = 11123$
$c = t^2$



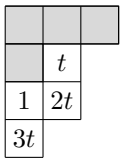
$w = 12113$
$c = t^2$



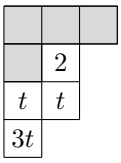
$w = 12131$
$c = t^2$



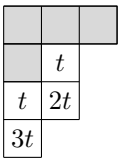
$w = 11213$
$c = t^3$



$w = 11123$
$c = t^3$



$w = 12113$
$c = t^3$



$w = 11213$
$c = t^4$

3.3 Type A Quantum Lattice Heisenberg Algebra, $\mathfrak{h}_A$

We now give an explicit description of $\mathfrak{h}_A$. Fix an orientation ϵ of the type A Dynkin diagram. Denote I_A to be the set of vertices in the type A Dynkin diagram. Given two vertices i, j, we set their inner product to be

$$\langle i,j\rangle = \begin{cases} 0 & \text{if } i,j \text{ are not connected by an edge;} \\ 1 & \text{if } i,j \text{ are connected by an edge with tail } i \text{ and head } j \text{ agree with the orientation } \epsilon; \\ -1 & \text{if } i,j \text{ are connected by an edge with tail } i \text{ and head } j \text{ disagree with the orientation } \epsilon. \end{cases}$$

It is thus apparent that $\langle i,j\rangle = -\langle j,i\rangle$.

Let $[r+1] = t^{-r} + t^{-r+2} + \cdots + t^{r-2} + t^r = \frac{t^{-r}-t^r}{t^{-1}-t}$ be the shifted quantum integer. We define the *type A quantum Heisenberg algebra* $\mathfrak{h}_A$ to be the unital $\mathbb{C}[t,t^{-1}]$-algebra with generators $p_i^{(n)}, q_i^{(n)}$ for $i \in I_\Gamma, n \in \mathbb{N}_{\geq 0}$

and relations

$$\begin{cases} q_i^{(n)}p_i^{(m)} = \sum_{r\geq 0}[r+1]p_i^{(m-r)}q_i^{(n-r)}, & \\ q_i^{(n)}p_j^{(m)} = p_j^{(m)}q_i^{(n)} + p_j^{(m-1)}q_i^{(n-1)}, & \text{if } j = i \pm 1, \\ q_i^{(n)}p_j^{(m)} = p_j^{(m)}q_i^{(n)}, & \text{if } |i-j| > 1, \\ p_i^{(n)}p_j^{(m)} = p_j^{(m)}p_i^{(n)}, & \\ q_i^{(n)}q_j^{(m)} = q_j^{(m)}q_i^{(n)}. & \end{cases}$$

By convention, $p_i^{(0)} = q_i^{(0)} = 1$ is the identity element in $\mathfrak{h}_A$.

Alternatively, $\mathfrak{h}_\Gamma$ can be generated by $a_i(r)$, for $i \in I_\Gamma$ and $r, s \in \mathbb{Z} - 0$, satisfying

$$[a_i(r), a_j(s)] = \delta_{r,-s}[s\langle i,j\rangle]\frac{[s]}{s}.$$

For any partition $\lambda \vdash n$, we can define p_i^λ using generating functions involving $a_i(r)$. For example,

$$exp\left(\sum_{r\geq 1}\frac{a_i(-r)}{[r]}z^r\right) = \sum_{s\geq 0}p_i^{(s)}z^s \text{ and } exp\left(\sum_{r\geq 1}\frac{a_i(r)}{[r]}z^r\right) = \sum_{s\geq 0}q_i^{(s)}z^s,$$

$$exp\left(-\sum_{r\geq 1}\frac{a_i(-r)}{[r]}z^r\right) = \sum_{s\geq 0}(-1)^s p_i^{(1^s)}z^s \text{ and } exp\left(-\sum_{r\geq 1}\frac{a_i(r)}{[r]}z^r\right) = \sum_{s\geq 0}(-1)^s q_i^{(1^s)}z^s.$$

In general, for partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t) = (1^{m_1}2^{m_2}\cdots)$, we first define $a_{i,+}^\lambda = a_i(\lambda_1)a_i(\lambda_2)\cdots a_i(\lambda_s)$, and similarly $a_{i,-}^\lambda = a_i(-\lambda_1)a_i(-\lambda_2)\cdots a_i(-\lambda_s)$. Let $[z]_\lambda = [1]^{m_1}m_1![2]^{m_2}m_2!\cdots$, and $h_{i,\pm}^n = \sum_{\lambda\vdash n}[z]_\lambda^{-1}a_{i,\pm}^\lambda$. We define

$$p_i^\lambda = det(h_{i,-}^{\lambda_j-k+l})_{k,l=1}^t, \text{ and } q_i^\lambda = det(h_{i,+}^{\lambda_j-k+l})_{k,l=1}^t.$$

Note that this resembles the Jacobi-Trudi identity for expressing Schur functions in terms of complete homogeneous symmetric function. In fact, as pointed out in [CL16], there is an isometric isomorphism between the Heisenberg algebra generators $a_i(r)$ and the power sum symmetric functions. Combining with the transition matrix between complete homogeneous symmetric function, we obtain the formula for the Schur functions. We could have defined p_i^λ, q_i^λ using the character table of the symmetric group which serves as the transition matrix between the Schur functions and the power symmetric functions as explained in [Mac15]. However, we found that the definitions given above made the required coefficients apparent and therefore, are practical for computations.

Consequently, deriving from [CL16], we will then have the commuting relation between q_i^λ and p_j^μ :

$$\begin{cases} q_i^{(\mu)}p_i^{(\lambda)} = \sum_{\nu\subseteq\lambda,\kappa\subseteq\mu,|\lambda|-|\nu|=|\mu|-|\kappa|}\mathfrak{C}_{\nu\kappa}^{\mu\lambda}(t)p_i^{(\nu)}q_i^{(\kappa)}, & \\ q_i^{(\mu)}p_j^{(\lambda)} = \sum_{\nu\subseteq\lambda^t,\kappa\subseteq\mu,|\lambda|-|\nu|=|\mu|-|\kappa|}C_{\nu\kappa}^{\mu\lambda^t}p_i^{(\nu)}q_i^{(\kappa)}, & \text{if } j = i \pm 1, \\ q_i^{(\mu)}p_j^{(\lambda)} = p_j^{(\lambda)}q_i^{(\mu)}, & \text{if } |i-j| > 1, \\ p_i^{(\mu)}p_j^{(\lambda)} = p_j^{(\lambda)}p_i^{(\mu)}, & \\ q_i^{(\mu)}q_j^{(\lambda)} = q_j^{(\lambda)}q_i^{(\mu)}, & \end{cases}$$

where

$$C_{\nu\kappa}^{\mu\lambda} = \sum_{\lambda_T\in \mathfrak{lp}(\lambda-\nu,\mu-\kappa)} 1 \text{ , and} \tag{3.3.1}$$

$$\mathfrak{C}_{\nu\kappa}^{\mu\lambda}(t) = \sum_{\lambda_{\bar{T}}\in t\text{-}\mathfrak{lp}(\lambda-\nu,\mu-\kappa)} t^{-(|\lambda|-|\nu|)}c(\lambda_{\mathfrak{p}})(t^2). \tag{3.3.2}$$

3.4 Fock Space Representation of $\mathfrak{h}_A$

In this section, we recall the main construction of Fock space representation in [LS12].

Consider the subalgebra $\mathfrak{h}_A^- \subset \mathfrak{h}_A$ generated by the $q_i^{(n)}$ for all $i \in I_A, n \geq 0$. If $\mathfrak{h}_A^+$ is the subalgebra of $\mathfrak{h}_A$ generated by the $p_i^n, n \geq 0$, then $\mathfrak{h}_A \cong \mathfrak{h}_A^+ \otimes \mathfrak{h}_A^-$ as vector spaces. Let $triv_0$ be the trivial representation of $\mathfrak{h}_A^-$, where 1 acts as the identity and $q_i^{(n)}$ acts by 0 for $n > 0$. Then, the $\mathfrak{h}_A$-module

$$\mathcal{F}_A := \mathrm{Ind}_{\mathfrak{h}_A^-}^{\mathfrak{h}_A}(triv_0) = \mathfrak{h}_A \otimes_{\mathfrak{h}_A^-} triv_0$$

is the *Fock space representation* of $\mathfrak{h}_A$. Let v_0 be the vacuum vector in the Fock space where $q_i^{(n)} \cdot v_0 = 0$ for $n > 0$. This will help to distinguish between the elements of $\mathfrak{h}_A$ and its representation $\mathcal{F}_A$.

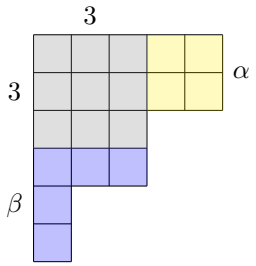
Note that the only relation that holds in $\mathfrak{h}_A^-$ and $\mathfrak{h}_A^+$ respectively is the abelian multiplication of generators. One can readily see that the Fock space $\mathcal{F}_\Gamma$ is naturally isomorphic to the space of polynomials in the commuting variables $\{p_i^{(n)} v_0\}_{i \in I_\Gamma}$.

3.5 Braid Operator in the Fock Space Representation

We define an integer statistic d_λ associated to a partition λ as follows: if $\lambda' \subset \lambda$ is the largest $(r \times r)$ square, also known as the *Durfee square*, whose Young diagram fits properly inside of λ, then the complement $\lambda \setminus \lambda'$ is a skew partition with the top horizontal component α and the bottom vertical component β and we set

$$d_\lambda := |\alpha| - |\beta| + r.$$

For example, let $\lambda = (5, 5, 3, 3, 1, 1)$ be the following Young diagram.



$$\begin{aligned} & d_{(5,5,3,3,1,1)} \\ &= |(2,2)| - |(3,1,1)| + 3 \\ &= 4 - 5 + 3 \\ &= 2 \end{aligned}$$

We will concentrate on the Heisenberg algebra $\mathfrak{h}_{A_{2n-1}}$ associated to the type A_{2n-1} Dynkin diagram. The type A_{2n-1} Dynkin diagram has its vertices labelled by $\{1, 2, \ldots, 2n-1\}$ and only i and $i+1$ vertices are connected by one edge. Then, we consider the Fock space representation $F_{A_{2n-1}}$ of $\mathfrak{h}_{A_{2n-1}}$. It is known that the Fock space $F_{A_{2n-1}}$ is a graded vector space which is isomorphic to $\oplus_{m \in \mathbb{N}_{\geq 0}} F_{A_{2n-1}}^m$, with

$$F_{A_{2n-1}}^m = \text{ span } \left\{ p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_{2n-1}^{\lambda_{2n-1}} v_0 \,\middle|\, \sum_i |\lambda_i| = m \right\}.$$

For each $m \in \mathbb{N}_{\geq 0}$ $i \in I_\Gamma$, we define the distinguished element $\varsigma_{i,m} \in \mathfrak{h}_{A_{2n-1}}$ to be

$$\varsigma_{i,m} := \sum_{\{\lambda : 0 \leq |\lambda| \leq m\}} (-1)^{d_\lambda} t^{-d_\lambda} p_i^\lambda q_i^{\lambda^T}$$

where λ^T denotes the transpose of λ and the other distinguished element $(\varsigma_i^m)' \in \mathfrak{h}_{A_{2n-1}}$ to be

$$\varsigma'_{i,m} := \sum_{\{\lambda : 0 \leq |\lambda| \leq m, \lambda = \lambda^T\}} (-1)^{d_\lambda} t^{d_\lambda} p_i^\lambda q_i^{\lambda^T} + \sum_{\{\lambda : 0 \leq |\lambda| \leq m, \lambda \neq \lambda^T\}} (-1)^{d_\lambda + 2} t^{d_\lambda + 2} p_i^\lambda q_i^{\lambda^T}$$

Note that $\varsigma_{i,m} \in \mathfrak{h}_{A_{2n-1}}$ can be expressed as the truncation of the generating function

$$P(x,y,t) = 1 + \sum_{a=1}^{\infty} \frac{(xyt)^a}{\prod_{b=1}^{a}(1-(xyt)^b)(1-(xyt^{-1})^b)}$$

where the P is a variant of Euler partition function enumerated based on Durfee square of partitions.

Next, we introduce the braid operators $\Sigma_{i,m}$, $\Sigma'_{i,m} : F^m_{A_{2n-1}} \to F^m_{A_{2n-1}}$ of multiplying with $\varsigma_{i,m}, \varsigma'_{i,m}$ respectively:

$$\Sigma_{i,m}(x) = \varsigma_{i,m} \cdot x, \qquad \Sigma'_{i,m}(x) = \varsigma'_{i,m} \cdot x, \qquad x \in F^m_{A_{2n-1}}.$$

Conjecture 3.5.1. *These braid operators define a braid group action representation on the Fock space.*

Strategy of Proof. Prove the following identity

$$\left(\sum_{\{\lambda:0\geq|\lambda|\leq n\}} (-1)^{d_\lambda} t^{-d_\lambda} p_i^\lambda q_i^{\lambda^T}\right)\left(\sum_{\{\lambda:0\leq|\lambda|\leq n,\lambda=\lambda^T\}} (-1)^{d_\lambda} t^{d_\lambda} p_i^\lambda q_i^{\lambda^T} + \sum_{\{\lambda:0\leq|\lambda|\leq n,\lambda\neq\lambda^T\}} (-1)^{d_\lambda+2} t^{d_\lambda+2} p_i^\lambda q_i^{\lambda^T}\right) = id$$

$$\left(\sum_{\{\lambda:0\geq|\lambda|\leq n\}} (-1)^{d_\lambda} t^{-d_\lambda} p_i^\lambda q_i^{\lambda^T}\right)\left(\sum_{\{\lambda:0\geq|\lambda|\leq n\}} (-1)^{d_\lambda} t^{-d_\lambda} p_{i+1}^\lambda q_i^{\lambda^T}\right)\left(\sum_{\{\lambda:0\geq|\lambda|\leq n\}} (-1)^{d_\lambda} t^{-d_\lambda} p_i^\lambda q_i^{\lambda^T}\right)$$

$$= \left(\sum_{\{\lambda:0\geq|\lambda|\leq n\}} (-1)^{d_\lambda} t^{-d_\lambda} p_{i+1}^\lambda q_i^{\lambda^T}\right)\left(\sum_{\{\lambda:0\geq|\lambda|\leq n\}} (-1)^{d_\lambda} t^{-d_\lambda} p_i^\lambda q_i^{\lambda^T}\right)\left(\sum_{\{\lambda:0\geq|\lambda|\leq n\}} (-1)^{d_\lambda} t^{-d_\lambda} p_{i+1}^\lambda q_i^{\lambda^T}\right)$$

We first realize the braid operators as generating functions. Then, we require generating functions for operation on (generalised) Littlewood-Richardson rule. Next, we probably need appropriate basic hypergeometric series identity, partition identity, or q-series identity (might be from Ramanujan) to help us do cancellation. Do the maths and find out the unnecessary terms cancelled nicely. The author has checked the braid relations to $n = 3$ case by brute force.

3.6 Crossingless Matching to Algebra

Recall that a *crossingless matching* c on a three-sphere $\mathbb{S}_3$ consist of multiple curves c_r with endpoints up to ambient isotopy. Given a crossingless matching c of $2n$ points, we construct a graded Fock vector $v_c \in F^n_{A_{2n-1}}$ as follows.

3.6.1 Crossingless Matching to Cross Diagram

In this section, we introduce some essential terminology and describe steps obtaining cross diagrams from crossingless matchings.

A crossingless matching is called *decomposable* if there exists a vertical line v_i that doesn't intersect any curves. Subsequently, a decomposable crossingless matching is partitioned into disjoint parts. A *subcrossingless matching* is a crossingless matching obtained by successively removing the curves with the largest difference of endpoints (it is possible to have more than one curve of such property) or curves in the other disjoint parts of a decomposable crossingless matching. A crossingless matching is called *nested* if it contains a decomposable subcrossingless matching, but it is not decomposable.

1. Enumerate the crossingless matching c starting with curve possessing the least difference of endpoints from left to right by $c_1, \ldots, c_n$.

2. Draw $2n-1$ vertical lines v_i that bisect perpendicularly the horizontal line connecting i-th and $(i+1)$-th vertices from left to right, for $i \in \{1, \ldots, 2n-1\}$. Note that every curve in crossingless matching intersects the vertical lines in odd number $2m+1$. Then, we define $n-m$ to be the *level* of a curve.

3. We shall label each intersection point $c_r \cap v_i$ with a triple $(r,i,\ell_{r,i})$ where $\ell_{r,i} \in \mathbb{N}_{\geq 0}$ enumerates intersections on each v_i from top to bottom. This intersection points are called the *distinguished crosses*.

4. A *cross diagram* is a set of distinguished crosses sitting in $\mathbb{Z} \times \mathbb{Z}$ lattice. Replace the multicurves diagram (e.g. Figure 3.6.1.1) with a cross diagram (e.g. Figure 3.6.1.2) labelled by the triples $(r,i,\ell_{r,i})$; a distinguished cross (x, y) is picked for each intersection point in a way that x is the level of the curve where it lies and y is subscript i of the vertical line it intersects. Note that the crosses are arranged in a floating pyramid-like arrangement reflecting their relative positions in the original crossingless matching.

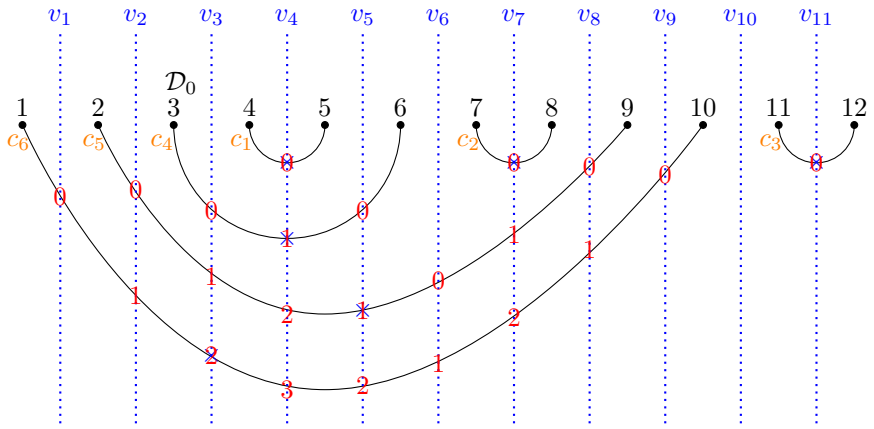


Figure 3.6.1.1: An example of crossingless matching c in a $2n$-punctured 3-sphere.

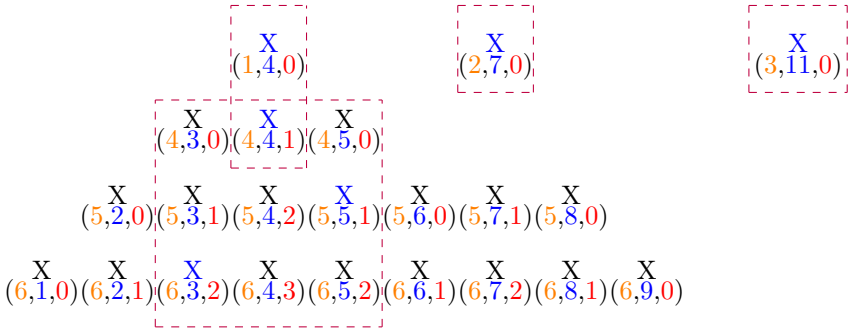


Figure 3.6.1.2: A cross diagram for the example with counting fence in purple dashed line.

A *counting rectangle* is a $\mathbb{Z} \times \mathbb{Z}$ sublattice containing either complete rows of distinguished crosses or empty rows of distinguished crosses. The rows of distinguished crosses contained in a counting rectangle are called the *lattice segments*. A marked distinguished cross, or marked cross for simplicity is a choice of distinguished cross. Given a cross diagram and a choice of b marked distinguished crosses, we can form the (possibly empty) smallest counting rectangle containing the b marked distinguished crosses. A lattice segment is *effective* if it contains marked crosses and *potential* if it doesn't. Observe that a counting rectangle could have more potential lattice segments than effective lattice segments. When $b = 1$, the smallest counting rectangle is the one-by-one grid by definition. Now, for $1 \leq a \leq b$, we consider the set of all counting rectangles coming from possible a-wise marked crosses but remove those rectangles that are contained in a larger one. This particular set of counting rectangles is called a counting fence.

In particular, we are interested in the set of sets of marked crosses where each of them comes from different curves in a crossingless matching c, denoted by $\mathcal{X}_c$. To give an overview of the remaining chapter,

we provide our end goal here without going into the details. Details will be laid out in the future sections. Every elements in $\mathcal{X}_c$ will be realised as a vector in the Fock space via a map ψ. Finally, we will associate c to a vector v_c defined as

$$v_c = \sum_{p_{i_1}^{\lambda_1} p_{i_2}^{\lambda_2} \dots p_{i_{2n-1}}^{\lambda_{2n-1}} v_0 \in \psi(\mathcal{X}_c)} \mathfrak{sp}(p_{i_1}^{\lambda_1} p_{i_2}^{\lambda_2} \dots p_{i_{2n-1}}^{\lambda_{2n-1}} v_0) \cdot p_{i_1}^{\lambda_1} p_{i_2}^{\lambda_2} \dots p_{i_{2n-1}}^{\lambda_{2n-1}} v_0 \tag{3.6.1}$$

where $\mathfrak{sp}$ is called the *spreading polynomial* of the vector in a given crossingless matching. The spreading polynomial $\mathfrak{sp}$ lives in the Laurent ring $\mathbb{Z}[t, t^{-1}]$ and will be defined based on several combinatoric invariants.

3.6.2 Cross Diagram to Fock Vector

In this section, we will lay out the details of constructing the the spreading polynomial $\mathfrak{sp}$, and subsequently, introducing a graded Fock vector for a crossingless matching. At the end, we present some conjectures relating to the vectors we established.

5. Given a crossingless matching c with curves $c_1, c_2, \cdots, c_n$, pick exactly one intersection point from each curve c_r in c. These are n marked crosses each lying on distinct curves with the label $(r,i,\ell_{r,i})$. Consider the set $\mathcal{I}_{i_1,\dots,i_n}$ containing them:

   $$\mathcal{I}_{i_1,i_2\dots,i_n} := \{\ (1,i_1,\ell_{1,i})\ (2,i_2,\ell_{2,i}),\ \cdots\ (n,i_n,\ell_{n,i}) \mid\ 1 \le i_r \le 2n-1,\ 0 \le \ell \le n-1\},$$

   which we call a marked cross configuration. We associate a Fock vector $p_{i_1} p_{i_2} \cdots p_{i_n} v_0$ to each set $\mathcal{I}_{i_1,i_2,\cdots,i_n}$.

   For example, the two set $I_{4,7,11,4,5,3} = \{(1,4,0),(2,7,0),(3,11,0),(4,4,1),\ (5,5,1),\ (6,3,2)\} \neq \{(1,4,0),(2,7,0),(3,11,0),(4,3,0),(5,4,2),(6,5,2)\ \} = I_{4,7,11,3,4,5}$ give $p_4p_7p_{11}p_4p_5p_3v_0$ and $p_4p_7p_{11}p_3p_4p_5v_0$ respectively. But, we want to remark that the two vectors are equal in the Fock space by the Heisenberg algebra relations.

6. Next, we define the sets

   $$\mathcal{X}_{i_1,i_2,\dots,i_n} = \bigsqcup_{s \in \mathrm{Perm}(i_1,i_2,\dots,i_n)} \mathcal{I}_s, \quad \text{and} \quad \mathcal{X}_c = \bigsqcup_{p \in \{(i_1,i_2,\dots,i_n)\}} \mathcal{X}_p$$

   where the first set is taking the disjoint union over all permutations of $(i_1, i_2, \dots, i_n)$ and the second set is taking the disjoint union of all combinations of $(i_1, i_2, \dots, i_n)$ in $\mathcal{X}_{i_1,i_2,\dots,i_n}$. Then, $\mathcal{X}_c$ is the set of all marked cross configurations of c. By previous step, there exists a a map $\psi : \mathcal{X}_c \to F^n_{A_{2n-1}}$ sending $\mathcal{I}_{i_1,i_2,\cdots,i_n} \mapsto p_{i_1}p_{i_2}\cdots p_{i_n}v_0$. By abuse of notation, we set $\mathcal{X}_{p_{i_1}p_{i_2}\cdots p_{i_n}} := \mathcal{X}_{i_1,i_2,\dots,i_n}$.

7. After that, we will associate a spreading polynomial $\mathfrak{sp}$ to each Fock vector $p_1^{k_1}p_2^{k_2}\dots p_{2n-1}^{k_{2n-1}}v_0$ which are defined based on its preimage

   $$\psi^{-1}\left(p_1^{k_1}p_2^{k_2}\dots p_{2n-1}^{k_{2n-1}}v_0\right)$$

   following the steps below:

   (a) First, we would like to compute the cardinality $\left|\psi^{-1}\left(p_1^{k_1}p_2^{k_2}\dots p_{2n-1}^{k_{2n-1}}v_0\right)\right|$ of the preimage of a Fock vector $p_1^{k_1}p_2^{k_2}\dots p_{2n-1}^{k_{2n-1}}$ under the map ψ . We outline a series of steps to express the cardinality as a **specific product** of binomial coefficients, that is the total number of a Fock vector $p_1^{k_1}p_2^{k_2}\dots p_{2n-1}^{k_{2n-1}}v_0$ appearing in c will be in the form of $\prod_j \binom{y_j}{z_j}$ for some y_j and z_j. It amounts to compute the number of ways getting each p_j or the corresponding j-crosses. We start by choosing one element in the preimage $\psi^{-1}\left(p_1^{k_1}p_2^{k_2}\dots p_{2n-1}^{k_{2n-1}}v_0\right)$.

   Starting with a cross diagram of c, for curves that has only one intersection with v_i, it is clear that the sole choice we can make is p_i, in other words, they contribute $\binom{1}{1}$. We can then delete those marked crosses/rows of crosses with the same r-label from the diagram.

After that, we form a counting fence for the remaining marked crosses. If the counting fence consists of one and only one counting rectangle, it is obvious that the number of effective lattice segments, in this case, the number of rows of the unique rectangle counts the possible combination of marked cross configuration in the form of product of binomial coefficients.

When p_j is present in an intersection of more than one counting rectangles, this implies that there exists other marked crosses not in the lattice segment of the same counting rectangle containing p_j. The counting of possible p_j combination will then be dependant on multiple rectangles. In fact, each p_j can only be contained in one and only one counting triangle or in the intersection of two counting rectangles. Since if it is in the intersection of more than two counting rectangles, there must exist a marked cross configuration such that it is only contained in the intersection of two counting rectangles by the nature of crossingless matching.

Hence, we can always find a i and at least one p_i of p_i^k such that p_i lies in curve of the highest or lowest level among the other p_j's lie and stays in one and only one counting rectangle. We would call it a *single* p_i. In particular, p_i's coming from curves that has only one intersection with v_i are single. We should start by counting the $p_i^{k_i}$ which contain single p_i. Let y_i be the number of effective lattice segments in the unique counting rectangle. Then, we have $\binom{y_i}{k_i}$ way to pick $p_i^{k_i}$ from that unique counting rectangle containing p_i. We will then delete all the single p_i's and the rows containing it. We continue to do the same for the other single p_j's in the unique counting rectangle containing p_i. Collectively, the remaining p_j's including p_i will all contain in this unique counting rectangle containing p_i and we will regard them as coming from the same column when forming the next counting fence. In principle, by counting and deleting the single p_j's, we have fixed a configuration that they could appear in another counting rectangle that intersects with the original unique rectangle and subsequently, the column p_j present become irrelevant.

We will keep picking out the single p_j's until we reach a counting fence consisting of one and only one counting rectangle. Taking the product of the possible $\binom{y_j}{z_j}$ for some y_j and z_j, we had computed the desired cardinality.

To justify this, this is the least upper bound for the cardinality since to turn any potential lattice segments into effective lattice segments, it must be the case that there are other marked crosses in the lattice segments below or above the potential segments. These are counted in the maximality of a counting fence. On the other hand, it is also the greatest lower bound because it realises the number of distinct combinations of set of marked crosses.

For example, in Figure 3.6.1.1, there are $\binom{3}{1}\binom{2}{1}$ of way realising $p_3p_4^2p_5p_7p_{11}v_0$. Depending on whether we count p_3 or p_4 first, we have $\binom{3}{1} = \binom{3}{2}$ of way realising $p_3p_4^3p_7p_{11}v_0$ in c.

(b) Then, we associate a Fock vector $p_1^{k_1}p_2^{k_2}\dots p_{2n-1}^{k_{2n-1}}v_0$ a polynomial ϕ in $\mathbb{Z}[t,t^{-1}]$ based on an expression of the cardinality $\left|\mathcal{X}_{p_1^{k_1}p_2^{k_2}\dots p_{2n-1}^{k_{2n-1}}}\right| = \prod_{j=1}^{k}\binom{y_j}{z_j}$ of its preimage obtained from item 7a. To that expression, we can associate the product ϕ of the graded dimension of the cohomology of Grassmannian $Gr(z_j,y_j)$, or the Gaussian polynomial, that is,

$$\phi(p_1^{k_1}p_2^{k_2}\dots p_{2n-1}^{k_{2n-1}}v_0) = \prod_{j=1}^{k} g_r\dim Gr(z_j,y_j) = \prod_{j=1}^{k}\begin{bmatrix} y_j \\ z_j \end{bmatrix}$$

where $[n] := \frac{1-t^n}{1-t} = 1+t+t^2+\dots+t^{n-1}$, $[n]! := \prod_{j=1}^{n}[j] = \sum_{w\in S_n} t^{\mathrm{inv}(w)}$ $\begin{bmatrix} y \\ z \end{bmatrix} := \frac{[y]!}{[y]![y-z]!} = \frac{(1-t^y)(1-t^{y-1})\cdots(1-t^{y-z+1})}{(1-t)(1-t^2)\cdots(1-t^z)}$. Note that the quantum integer defined here is not the shifted one as in Section 2.2.2.

At first sight, it might be lured that different expressions could give different ϕ. For example, $\phi(p_3p_4^3p_7p_{11}v_0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 1+t+t^2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. To explain it, we need the foloowing lemmas.

Lemma 3.6.1. *Suppose $\omega \in S_n$. Then,*

$$\begin{bmatrix} j_1+j_2+\dots+j_n \\ j_n \end{bmatrix}\cdots\begin{bmatrix} j_1+j_2 \\ j_2 \end{bmatrix}\begin{bmatrix} j_1 \\ j_1 \end{bmatrix} = \begin{bmatrix} j_{\omega(1)}+j_{\omega(2)}+\dots+j_{\omega(n)} \\ j_{\omega(n)} \end{bmatrix}\cdots\begin{bmatrix} j_{\omega(1)}+j_{\omega(2)} \\ j_{\omega(2)} \end{bmatrix}\begin{bmatrix} j_{\omega(1)} \\ j_{\omega(1)} \end{bmatrix}.$$

Proof. This is trivial as $\frac{[j_1+\cdots+j_n]}{[j_1]!\cdots[j_n]!} = \frac{[j_1+\cdots+j_n]}{[j_{w(1)}]!\cdots[j_{w(n)}]!}$.

Q.E.D.

Lemma 3.6.2. $\begin{bmatrix} r+s \\ s \end{bmatrix} = \begin{bmatrix} r+s \\ r \end{bmatrix}$.

Proof. This is left as an exercise. Q.E.D.

By the Lemma 3.6.1, the Gaussian polynomial of a Fock vector associated to a crossingless matching is an invariant of the cardinality obtained in item 7a. We need to argue why the order of "t-counting" doesn't matter. If there are only two choices to make in the counting rectangle, then it is equivalent to *Lemma* 3.6.2. If there are more than two choices exist in the counting rectangle, then it is covered by *Lemma* 3.6.1. It is not hard to extend the argument to p_j's containing in the intersection of two rectangles.

Lemma 3.6.3. *The spreading polynomial* $\mathfrak{sp}(p_{i_1}^{\lambda_1} p_{i_2}^{\lambda_2} \dots p_{i_{2n-1}}^{\lambda_{2n-1}} v_0)$ *of a Fock vector doesn't depend on the order you taking the* p_{i_j}*, or in another word, it does not depend on the preimage* $\psi^{-1}(p_{i_1}^{k_1} p_{i_2}^{k_2} \dots p_{i_{2n-1}}^{k_{2n-1}} v_0)$.

(c) As mentioned before, a single Fock vector can be further decomposed into the basis of Fock space in terms of multipartitioned Fock vector $p_1^{\lambda_1} p_2^{\lambda_2} \dots p_{2n-1}^{\lambda_{2n-1}} v_0$, $\lambda_i \vdash k_i$.

Each basis element will also be associated to a polynomial $\theta \in \mathbb{Z}[t, t^{-1}]$ based on the major statistic with standard Young tableaux fillings on multipartition.

For each partition $\lambda \vdash n$, the total number of standard Young tableaux associated can be interpreted as the dimension of irreducible representation of S_n or more directly the number of Gelfand-Tsetlin basis. We now define a major statistic to these standard Young tableaux. Given a standard Young tableau λ_S of λ, a descent of T is defined to be a value of i, $1 \le i \le |\lambda|$ for which $i+1$ occurs in a row below i in λ_S. A *major statistic* m_{λ_S} associated to a standard Young tableau λ_S is the sum

$$m_{\lambda_S} := \sum_{i \in \text{descent of } \lambda_S} i.$$

Let q be the dimension of irreducible representation of λ. Then

$$\theta_\lambda := \sum_{i=1}^{q} t^{m_{\lambda_{S_i}}}$$

and, for multipartitioned case,

$$\theta(p_1^{\lambda_1} p_2^{\lambda_2} \dots p_{2n-1}^{\lambda_{2n-1}} v_0) := \theta_{\lambda_1} \theta_{\lambda_2} \cdots \theta_{\lambda_{2n-1}}.$$

For example, in the Fock space representation, $p_3 p_4^2 p_5 p_7 p_{11} v_0 = p_3 p_4^{(2)} p_5 p_7 p_{11} v_0 + p_3 p_4^{(1^2)} p_5 p_7 p_{11} v_0$. Then, $\theta(p_3 p_4^{(2)} p_5 p_7 p_{11} v_0) = 1$ and $\theta(p_3 p_4^{(1^2)} p_5 p_7 p_{11} v_0) = t$. In the case of $p_4^{(2,1)} p_7 p_{11} p_9 v_0$, we get $\theta(p_4^{(2,1)} p_7 p_{11} p_9 v_0) = t + t^2$. Another example would be $p_4^{(2)} p_5^{(1^2)} p_7 p_{11} v_0$, we get $\theta(p_4^{(2)} p_5^{(1^2)} p_7 p_{11} v_0) = 1 \cdot t = t$.

(d) It can be shown(Lemma 3.6.4) that, all the elements in a single set of intersection points has the same total sum ℓ of $\ell_{r,s}$. Then, we will assign the spreading polynomial $\mathfrak{sp}(p_{i_1}^{\lambda_1} p_{i_2}^{\lambda_1} \dots p_{i_{2n-1}}^{\lambda_{2n-1}} v_0) \in \mathbb{Z}[t, t^{-1}]$ of an existing basis element $p_{s_1}^{\lambda_1} p_{i2}^{\lambda_1} \dots p_{i_{2n-1}}^{\lambda_{2n-1}} v_0$ in c by

$$\mathfrak{sp}(p_{i_1}^{\lambda_1} p_{i_2}^{\lambda_2} \dots p_{i_{2n-1}}^{\lambda_{2n-1}} v_0) = \theta\left(p_{i_1}^{\lambda_1} p_{i_2}^{\lambda_1} \dots p_{i_{2n-1}}^{\lambda_{2n-1}} v_0\right) \phi\left(p_{i_1}^{|\lambda_1|} p_{i_2}^{|\lambda_2|} \dots p_{i_{2n-1}}^{|\lambda_{2n-1}|} v_0\right)\Big|_{t \mapsto t^2} \cdot t^{-\ell}.$$

Note that the definition of $\mathfrak{sp}$ can be extended linearly, so that $\mathfrak{sp}(p_{i_1}^{k_1} p_{i_2}^{k_2} \dots p_{i_{2n-1}}^{k_{2n-1}} v_0)$, for $k_i \in \mathbb{Z}$ is defined to be the sum of the spreading polynomial of the basis elements that it decomposes into.

For example, {(1,4,0),(2,7,0),(3,11,0),(4,4,1),(5,5,1),(6,3,2)}
$\neq$ {(1,4,0),(2,7,0),(3,11,0),(4,3,0),(5,4,2), (6,5,2)}, but $0+0+0+1+1+2 = 4 = 0+0+0+0+2+2$. So, $\ell_{p_3p_4^2p_5p_7p_{11}v_0} = 4$. Then,

$$\mathfrak{sp}\left(p_3p_4^{(2)}p_5p_7p_{11}v_0\right) = 1\cdot(1+2t+2t^2+t^3)\Big|_{t\mapsto t^2}\cdot t^{-4} = t^{-4}+2t^{-2}+2+t^2,$$

and

$$\mathfrak{sp}\left(p_3p_4^{(1^2)}p_5p_7p_{11}v_0\right) = t\cdot(1+2t+2t^2+t^3)\Big|_{t\mapsto t^2}\cdot t^{-4} = t^{-2}+2+2t^2+t^4.$$

We can extend the definition as a homomorphism to a general Fock vector $p_1^{k_1}p_2^{k_2}\dots p_{2n-1}^{k_{2n-1}}$. For example,

$$\begin{aligned}\mathfrak{sp}(p_3p_4^2p_5p_7p_{11}v_0) &= \mathfrak{sp}\left(p_3p_4^{(2)}p_5p_7p_{11}v_0\right)+\mathfrak{sp}\left(p_3p_4^{(1^2)}p_5p_7p_{11}v_0\right)\\ &= t^{-4}+3t^{-2}+4+3t^2+t^4\end{aligned}$$

which has $5 = \ell+1$ terms. This is a fact that we will prove in Lemma 3.6.4.

8. Consider all such sets of intersection points. We will associate a crossingless matching c a finite sum $F(c)$ of Fock vectors based on the set

$$\bigcup_{r,i}\{(c_r,v_i,\ell_{r,i})|\text{ no repeated } r\}.$$

The set above will give us a list of Fock vectors contained in c. We then have

$$v_c = \sum_{p_{i_1}^{\lambda_1}p_{i_2}^{\lambda_2}\dots p_{i_{2n-1}}^{\lambda_{2n-1}}v_0\in\psi(X_c)}\mathfrak{sp}(p_{i_1}^{\lambda_1}p_{i_2}^{\lambda_2}\dots p_{i_{2n-1}}^{\lambda_{2n-1}}v_0)\cdot p_{i_1}^{\lambda_1}p_{i_2}^{\lambda_2}\dots p_{i_{2n-1}}^{\lambda_{2n-1}}v_0.$$

Lemma 3.6.4. *The spreading polynomial* $\sum_{\lambda_1\vdash k_1,\lambda_2\vdash k_2,\cdots\lambda_{2n-1}\vdash k_{2n-1}}\mathfrak{sp}(p_{i_1}^{\lambda_1}p_{i_2}^{\lambda_2}\dots p_{i_{2n-1}}^{\lambda_{2n-1}}v_0)$ *has* $\ell+1$ *terms. Moreover, the spreading polynomial* $\sum_{\lambda_1\vdash k_1,\lambda_2\vdash k_2,\cdots\lambda_{2n-1}\vdash k_{2n-1}}\mathfrak{sp}(p_{i_1}^{\lambda_1}p_{i_2}^{\lambda_2}\dots p_{i_{2n-1}}^{\lambda_{2n-1}}v_0)\big|_{t=1}$ *counts the total number of basis elements required to express* $p_{i_1}^{|\lambda_1|}p_{i_2}^{|\lambda_2|}\dots p_{i_{2n-1}}^{|\lambda_{2n-1}|}v_0$.

Proof. Suppose $\ell = 0$. That means there is only one possible choice of $p_{i_1}^{\lambda_1}p_{i_2}^{\lambda_2}\dots p_{i_{2n-1}}^{\lambda_{2n-1}}v_0$. Note that in this case $\lambda_i\in\{0,1\}$. Then, by construction, it is a single graded vector over $\mathbb{Z}$. Suppose now, we only look at a Fock vector $p_i^{k_i}$ consisting of a repeated indices i. Then, we have $p_i^{k_i} = \sum_{\lambda_i\vdash k_i}p_i^{\lambda_i}$. Then, $\ell+1 = 0+1+2+\dots+k_i-2+k_i-1+1$, but $\mathfrak{sp}(p_i^{k_i}) = \sum_{\lambda_i\vdash k_i}\mathfrak{sp}(p_i^{\lambda_i}) = [k_i]!$ which has $k_i-1+k_i-2+k_i-3+\dots+1$, as desired.

For the rest of the case, note that we can reduce the case of a single non-decomposable crossingless matching since the function $\mathfrak{sp}$ is multiplicative on the decomposable crossingless matchings. First, we discuss the case when the crossingless matching is non-nested. Consider its associated lattice diagram where it is noticed as a pyramid with the tip, say p_k. By discussion before, we can pick a specific representative of the vector by marking the crosses from top to the bottom in the following sequence: $p_k^{\lambda_k}, p_{k-1}^{\lambda_{k-1}}, p_{k+1}^{\lambda_{k+1}}, p_{k-2}^{\lambda_{k-2}}, p_{k+2}^{\lambda_{k+2}},\dots$. Forming the counting fence, we see that the counting rectangles are those with top vertices having ℓ-label as 0. Note that, by precious paragraph, ℓ-label on $p_{i_j}^{\lambda_{i_j}}$ carries the contribution of $\lambda_{i_j}-1+\lambda_{i_j}-2+\dots+1$. Then, from the desingated sequence, it is easy to see that the ℓ-label on $p_{i_j}^{\lambda_{i_j}}$ is the total contribution from the way to choose p_{i_j} in the lattice diagram and the spreading polynomial function. To see it, we could start counting the way to choose the last term p_{i_j} in the sequence since it is not in the intersection with other counting rectangles or, in other words, choosing it is independent of choosing the other. The ℓ-label of the first appearing p_{i_j} indicates there are $\binom{\ell+\lambda_{i_j}}{\lambda_{i_j}}$ choices of it which its Gaussian polynomial

$$\begin{bmatrix}\ell+\lambda_{i_j}\\ \lambda_{i_j}\end{bmatrix}$$

has degree $\ell \cdot \lambda_{i_j}$. Here, we recall that a degree of a polynomial with one indeterminate is the highest power attaining by the indeterminate appearing in the summand. Similarly, we apply the method to all the other p_{i_j}. We then obtained the desired result.

For the case of nested crossingless matching, it might be the case that all the counting rectangles has nontrivial intersections on marked crosses. However, it is not hard to extend the argument to get the desired result.

Statement 2 is obvious. Q.E.D.

Conjecture 3.6.5. *The span of the graded vectors v_c in $F^n_{A_{2n-1}}$ form a Temperley-Lieb representation.*

Conjecture 3.6.6. *By realising a link in 3-sphere as a plat closure of a braid [Bir74], through Temperley-Lieb representation [Kau87] , it consists of a combination of gluing crossingless matchings c and upside down crossingless matchings c^t. This topological gluing can be interpreted as multiplying the vector v_{c^t} (replacing the generator p with q) with the vector v_c. Then, one can recover the Jones polynomial of the link by counting the graded vacuum vectors in the Fock space.*

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