Bi-Incomplete Tambara Functors As Coherent Monoids

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For Laura, Athena, Hermies, Artemis, Hera, and Nyx

Declaration

The work in this thesis is my own except where otherwise stated.

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Acknowledgements

While completing this thesis I have lived and worked upon the lands of the Ngunnawal and Ngambri peoples, lands which were never ceded. I pay my respects to their elders past and present.

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Abstract

By considering arbitrary finite G-sets for a finite group G rather than just the orbits G/H for subgroups H we realise the norm functor of Hoyer and the box product of Tambara functors as instances of the same construction. Using similar constructions and a new bicategorical framework we give characterisations of incomplete Mackey functors and bi-incomplete Tambara functors as coherent monoids. A recent conjecture of Blumberg and Hill is a corollary of our characterisation of bi-incomplete Tambara functors. In particular, our characterisation of $(\mathcal{O}'_a, \mathcal{O}'_m)$-Tambara functors as coherent $(\mathcal{O}'_a, \mathcal{O}'_m)$-monoids in $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functors allows both additive and multiplicative structure to be added simultaneously.

We also provide a notion of module category over a tight symmetric bimonoidal category and use this to relate our characterisation to work of Hill and Hopkins on equivariant symmetric monoidal structures.

Contents

Chapter 1

Introduction

In his 1967 paper [4] defining equivariant cohomology theories Bredon introduced coefficient systems to encode cohomological information in a way recognising the action of a discrete group G on a G-CW complex. Equivariant algebraic topology has developed significantly in the half-century since; a good survey is [21]. More recently, the solution of Hill, Hopkins, and Ravenel [11] to the Kervaire invariant one problem (in all dimensions except 126) has lead to increased research activity in equivariant stable homotopy theory (ESHT). They have recently published a primer [12] on the solution and the ESHT background required to understand it.

Coefficient systems come with restriction (and conjugation) maps. Mackey functors were defined by Green [9] and Dress [8] as axiomatisations of the representation theory of finite groups; they are coefficient systems equipped with transfer maps. The transfer and restriction maps interact according to a Mackey decomposition formula, similar to how induction and restriction interact in representation theory. Equivariant cohomology theories are represented by genuine G-spectra, stabilisation introduces transfer maps so equivariant cohomology is Mackey functor valued.

In [23] Tambara introduced TNR functors (now called Tambara functors) which are Mackey functors equipped with multiplicative transfer maps (now called norm maps). Norm maps generalise tensor induction. In particular, norm maps interact with restriction and transfer maps via relations analogous to the interactions of tensor induction with restriction and ordinary induction respectively in representation theory. Brun [5] showed that $\underline{\pi_0}$ of any commutative G-ring spectrum has the structure of a Tambara functor and conversely Ullman [24] showed that there is an Eilenberg-MacLane commutative G-ring spectrum for any Tambara functor.

A key construction in the solution to the Kervaire invariant one problem was that of the norm functor from H-spectra to G-spectra for H a subgroup of G. Ullman described this norm algebraically on Tambara functors in [24] in a way that still passed through a category of spectra. Mazur's thesis [22] contains an explicit description of the norm functor on Mackey functors for cyclic groups of prime power order: their construction does not involve topology; it is purely algebraic. A construction for the norm functor from H-Mackey functors to G-Mackey functors for H a subgroup of G was given in Hoyer's thesis [14] as a left Kan extension. This norm functor was key to Hoyer's characterisation of G-Tambara functors as a sort of coherent monoid in the category of G-Mackey functors.

In [19] and [20] Lewis considers incomplete Mackey functors, that is Mackey functors which are missing some transfer maps. Bousfield localisation can cause some norm maps to disappear and so Blumberg and Hill consider incomplete Tambara functors, which are missing some norm maps in [1]. They also consider incomplete Mackey functors in [3]. Considering the possibility of missing some transfers and some norm, Blumberg and Hill define bi-incomplete Tambara functors in [2].

The objects in equivariant algebra can be defined either with reference of the category of finite G-sets Set^G or, because finite G-sets are disjoint unions of orbits, the full subcategory Orb_G containing only the orbits G/H for H a subgroup of G. A common slogan in equivariant topology is that "orbits are points", for instance the 0-cells in G-CW complexes are orbits of points. It is also the case that the orbit approach lends itself to a natural representation on the page (at least for relatively small finite groups) in the form of Lewis diagrams. Hence there has been a bias toward the Orb_G definitions. In this thesis we will see that by considering slice categories of Set^G rather than Orb_H for subgroups of G we can make more general constructions and in so doing unify some concepts and simplify some proofs. One instance of this is our definition of the norm functor of incomplete Mackey functors where $\mathcal{O}$ is an appropriate wide subcategory of Set^G.

Definition 1.1. Given a map $\varphi : X \to Y$ of G-sets so that if $p_A : A \to X \in \mathcal{O}$ then $\pi_Y : \Pi_\varphi(A, p_A) \to Y, (y, \sigma) \mapsto y$ is also in $\mathcal{O}$, and $\underline{M} \in \mathrm{sMack}_X^{\mathcal{O}}$ we define $\mathcal{N}_\varphi \underline{M}$ to be the left Kan extension of $\underline{M}$ along Π_φ.

$$\begin{tikzcd}
\mathcal{A}_X^{\mathcal{O}} \arrow[r, "\underline{M}"] \arrow[d, "\Pi_\varphi"'] & \mathbf{Set} \\
\mathcal{A}_Y^{\mathcal{O}} \arrow[ur, "\mathrm{Lan}_{\Pi_\varphi} \underline{M} =: \mathcal{N}_\varphi \underline{M}"'] &
\end{tikzcd}$$

If $\mathcal{O} = \mathrm{Set}^G$ then when φ is the quotient map $G/H \to G/G$ our norm recovers the norm N_H^G of Hoyer, and when φ is the fold map $* \amalg * \to *$ our norm recovers the box product of Mackey functors. Hence our definition unifies these two important constructions.

The two main results of this thesis are characterisation results along the lines of Hoyer's characterisation on G-Tambara functors, the second one being a generalisation of it. Our first main theorem characterises incomplete Mackey functors.

Theorem 1.2. *For a finite group G and a pair $\mathcal{O}, \mathcal{O}'$ of indexing categories such that $\mathcal{O}$ is a subcategory of $\mathcal{O}'$, the $\mathcal{O}'$-Mackey functors are precisely the coherent $\mathcal{O}'$-monoids in* $\mathrm{Mack}_*^{\mathcal{O}}$*. That is, there is an equivalence between the category of coherent $\mathcal{O}'$-monoids in* $\mathrm{Mack}_*^{\mathcal{O}}$ *and the category* $\mathrm{Mack}_*^{\mathcal{O}'}$*.*

It is a corollary that G-Mackey functors are a type of coherent monoid in the category of G-coefficient systems. Our second main theorem is a characterisation of bi-incomplete Tambara functors.

Theorem 1.3. *Let be a finite group G, and let $(\mathcal{O}_a, \mathcal{O}_m)$ and $(\mathcal{O}'_a, \mathcal{O}'_m)$ be compatible pairs of indexing categories such that $(\mathcal{O}_a, \mathcal{O}'_m)$ is a compatible pair of indexing categories, $\mathcal{O}_a$ is a subcategory of $\mathcal{O}'_a$, and $\mathcal{O}_m$ is a subcategory of $\mathcal{O}'_m$. Then $(\mathcal{O}'_a, \mathcal{O}'_m)$-Tambara functors are precisely the coherent $(\mathcal{O}'_a, \mathcal{O}'_m)$-monoids in* $\mathrm{Tamb}_*^{(\mathcal{O}_a, \mathcal{O}_m)}$*. That is, there is an equivalence between the category of coherent $(\mathcal{O}'_a, \mathcal{O}'_m)$-monoids in* $\mathrm{Tamb}_*^{(\mathcal{O}_a, \mathcal{O}_m)}$ *and the category* $\mathrm{Tamb}_*^{(\mathcal{O}'_a, \mathcal{O}'_m)}$*.*

Hoyer's characterisation is a corollary of this theorem. Another corollary of this theorem is a conjecture of Blumberg and Hill [2, Conjecture 7.94] which is the case where $\mathcal{O}_a = \mathcal{O}'_a$. While this thesis was in preparation Chan [6] provided a proof of this conjecture by proving [2, Conjecture 7.90]. We note that our approach is different and our result more general.

In order to compare our results to previous work we develop a notion of module category over a tight symmetric bimonoidal category which is a categorification of the notion of a module over a ring. We see that the approach of previous work should also work to prove Theorem 1.3 when $\mathcal{O}_m = \mathcal{O}'_m$, but it is not clear how one would prove the general statement with that approach.

Chapter 2

Coefficient Systems and Module Categories

In this chapter we explain what coefficient systems and module categories are, showing that the category of the former is one of the latter.

In Section 2.1, we describe change of base functors between slice categories over finite G-sets and important relationships between them.

In Section 2.2, we define coefficient systems and some constructions on them.

In Section 2.3, we give a definition of module category over a tight symmetric bimonoidal category and use this to construct a new definition of a G-symmetric monoidal structure on a category. We show that there is such a structure on the category of coefficient systems.

2.1 G-set Preliminaries

Throughout this thesis G will always refer to a finite group and Set^G will always refer to the category of finite G-sets and G-equivariant functions between them. We write Set^G_X for the slice category of finite G-sets over a finite G-set X. In this first section we describe some important functors between such slice categories and their properties. These change of base functors will be the basis of important constructions on our objects of interest.

Notation 2.1. Objects in slice categories Set^G_X are pairs (A, p_A) with $p_A : A \to X$ a G-equivariant map. For the sake of readability we will often suppress p_A.

Definition 2.2. Given morphisms $f : A \to B$ and $g : B \to C$ of G-sets the **dependent product** of f along g is the set

$$\Pi_g(A, f) := \left\{ (c, \sigma) \;\middle|\; \begin{array}{c} c \in C, \sigma : g^{-1}(C) \to A \text{ is a function of sets such that} \\ f\sigma(b) = b \text{ for all } b \end{array} \right\}.$$

This is a G-set via the action $\gamma \cdot (c, \sigma) = (\gamma c, \gamma \cdot \sigma)$ where $(\gamma \cdot \sigma)(b) = \gamma(\sigma(\gamma^{-1}b))$ for all $b \in g^{-1}(\gamma c)$.

Definition 2.3. A morphism $\varphi : X \to Y$ of G-sets induces base change functors

$$\begin{tikzcd} \mathrm{Set}^G_Y \arrow[r, "\Delta_\varphi" description] & \mathrm{Set}^G_X \arrow[l, bend right, "\Sigma_\varphi"'] \arrow[l, bend left, "\Pi_\varphi"] \end{tikzcd}$$

defined as follows:

- On objects we have $\Sigma_\varphi(A, p_A) = (A, \varphi p_A)$ and on morphisms

$$\Sigma_\varphi : \left(\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p_A"'] & & B \arrow[dl, "p_B"] \\ & X & \end{tikzcd} \right) \mapsto \left(\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\varphi p_A"'] & & B \arrow[dl, "\varphi p_B"] \\ & Y & \end{tikzcd} \right).$$

- On objects we have $\Delta_\varphi(U, p_U) = (X \times_Y U, pr_X)$ where $pr_X : (x, u) \mapsto x$ is the pullback of p_U along φ (that is projection onto the X coordinate) and on morphisms

$$\Delta_\varphi : \left(\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "p_U"'] & & V \arrow[dl, "p_V"] \\ & Y & \end{tikzcd} \right) \mapsto \left(\begin{tikzcd} X \times_Y U \arrow[rr, "\mathrm{id}_X \times_Y f"] \arrow[dr, "pr^U_X"'] & & X \times_Y V \arrow[dl, "pr^V_X"] \\ & X & \end{tikzcd} \right).$$

- On objects we have that $\Pi_\varphi(A, p_A)$ is the dependent product of p_A along φ and on morphisms

$$\Pi_\varphi : \left(\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p_A"'] & & B \arrow[dl, "p_B"] \\ & X & \end{tikzcd} \right) \mapsto \left(\begin{tikzcd} \Pi_\varphi A \arrow[rr, "f_*"] \arrow[dr, "\pi_Y"'] & & \Pi_\varphi B \arrow[dl, "\pi_Y"] \\ & Y & \end{tikzcd} \right)$$

where $\pi_Y : (y, \sigma) \mapsto y$ is projection onto the Y coordinate, and $f_* : (y, \sigma) \mapsto (y, f\sigma)$.

Remark 2.4. For any subgroup H of G the categories $\mathrm{Set}^G_{G/H}$ and Set^H are equivalent via the functors

$$\mathrm{Set}^G_{G/H} \to \mathrm{Set}^H, A \mapsto p_A^{-1}(eH)$$
$$\mathrm{Set}^H \to \mathrm{Set}^G_{G/H}, A \mapsto (G \times_H A, [g, x] \mapsto gH).$$

Across this equivalence are functors

$$\begin{tikzcd} \mathrm{Set}^G \arrow[r, "i^G_H" description] & \mathrm{Set}^H \arrow[l, bend right, "G \times_H -"'] \arrow[l, bend left, "Map_H(G, -)"] \end{tikzcd}$$

where, for the collapse map $\varphi : G/H \to G/G$, the functors $\Sigma_\varphi, \Delta_\varphi$, and Π_φ correspond to $G \times_H -$, i^G_H, and $Map_H(G, -)$ respectively.

Proposition 2.5. *For any morphism of G-sets $\varphi : X \to Y$, the functor Δ_φ is right adjoint to Σ_φ and left adjoint to Π_φ.*

Proof. We construct an isomorphism

$$\Phi : \mathrm{Set}_Y^G(\Sigma_\varphi A, B) \xrightarrow{\cong} \mathrm{Set}_X^G(A, \Delta_\varphi B)$$

natural in (A, p_A) and (B, p_B) by

$$\Phi : \left(\begin{tikzcd} \Sigma_\varphi A \arrow[rr, "f"] \arrow[dr, "\varphi p_A"'] & & B \arrow[dl, "p_B"] \\ & Y & \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} A \arrow[rr, "\hat{f}"] \arrow[dr, "p_A"'] & & \Delta_\varphi B \arrow[dl, "pr_X"] \\ & X & \end{tikzcd}\right)$$

where $\hat{f} : a \mapsto (p_A(a), f(a))$. The inverse isomorphism is given by

$$\Phi^{-1} : \left(\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "p_A"'] & & \Delta_\varphi B \arrow[dl, "pr_X"] \\ & X & \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} \Sigma_\varphi A \arrow[rr, "\tilde{g}"] \arrow[dr, "\varphi p_A"'] & & B \arrow[dl, "p_B"] \\ & Y & \end{tikzcd}\right)$$

where $\tilde{g} : a \mapsto pr_B(g(a))$.

We construct an isomorphism

$$\Psi : \mathrm{Set}_X^G(\Delta_\varphi B, A) \xrightarrow{\cong} \mathrm{Set}_Y^G(B, \Pi_\varphi A)$$

natural in (A, p_A) and (B, p_B) by

$$\Psi : \left(\begin{tikzcd} \Delta_\varphi B \arrow[rr, "f"] \arrow[dr, "pr_X"'] & & A \arrow[dl, "p_A"] \\ & X & \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} B \arrow[rr, "\hat{f}"] \arrow[dr, "p_B"'] & & \Pi_\varphi A \arrow[dl, "\pi_Y"] \\ & Y & \end{tikzcd}\right)$$

where $\hat{f} : b \mapsto (p_B(b), x \mapsto f(x, b))$. The inverse isomorphism is given by

$$\Psi^{-1} : \left(\begin{tikzcd} B \arrow[rr, "g"] \arrow[dr, "p_B"'] & & \Pi_\varphi A \arrow[dl, "\pi_Y"] \\ & Y & \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} \Delta_\varphi B \arrow[rr, "\tilde{g}"] \arrow[dr, "pr_X"'] & & A \arrow[dl, "p_A"] \\ & X & \end{tikzcd}\right)$$

where $\tilde{g} : (x, b) \mapsto \pi_\sigma(g(b))(x)$ and $\pi_\sigma : (y, \sigma) \mapsto \sigma$. $\square$

The unit of the adjunction $\Delta_\varphi \dashv \Pi_\varphi$ has components

$$\eta_B : B \to \Pi_\varphi(X \times_Y B), b \mapsto (p_B(b), x \mapsto (x, b)).$$

Where necessary to distinguish between unit maps coming from different adjunctions we will write η_B^φ.

The following relationship between the change of base functors and pullback squares is the source of double coset formulas like the Mackey decomposition formula from representation theory. It will play a significant role in the later chapters.

Lemma 2.6. *Let*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "pr_Y"] \arrow[d, "pr_X"'] & Y \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"'] & Z \end{tikzcd}$$

be a pullback square in Set^G*. Then there are natural isomorphisms* $\Delta_\beta \Sigma_\alpha \cong \Sigma_{pr_Y} \Delta_{pr_X}$ *and* $\Delta_\beta \Pi_\alpha \cong \Pi_{pr_Y} \Delta_{pr_X}$.

Proof. We have

$$\Delta_\beta \Sigma_\alpha (A, p_A) = \Delta_\beta (A, \alpha p_A) = (A \times_Z Y, pr_Y)$$

and

$$\Sigma_{pr_Y} \Delta_{pr_X} (A, p_A) = \Sigma_{pr_Y} (A \times_X (X \times_Z Y), pr_{X \times_Z Y}) = (A \times_X (X \times_Z Y), pr_Y).$$

These objects are naturally isomorphic in Set^G_Y via the inverse maps

$$A \times_Z Y \to A \times_X (X \times_Z Y), (a, y) \mapsto (a, (p_A(a), y))$$

$$A \times_X (X \times_Z Y), (a, (x, y)) \mapsto (a, y).$$

We have

$$\Delta_\beta \Pi_\alpha (A, p_A) = Y \times_X \Pi_\alpha (A, p_A)$$

and

$$\Pi_{pr_Y} \Delta_{pr_X} (A, p_A) = \Pi_{pr_Y} (A \times_X (X \times_Z Y), pr_{X \times_Z Y}).$$

These objects are naturally isomorphic in Set^G_Y via the inverse maps

$$\begin{gathered} Y \times_X \Pi_\alpha (A, p_A) \to \Pi_{pr_Y} (A \times_X (X \times_Z Y), pr_{X \times_Z Y}), \\ (y, (z, \sigma)) \mapsto (y, (x, y) \mapsto (\sigma(x), (x, y))) \end{gathered}$$

and

$$\begin{aligned} \Pi_{pr_Y} (A \times_X (X \times_Z Y), pr_{X \times_Z Y}) &\to Y \times_Z \Pi_\alpha (A, p_A), \\ (y, \sigma) &\mapsto (y, (\beta(y), x \mapsto pr_A(\sigma(x, y)))). \end{aligned}$$ □

Remark 2.7. Typically one would replace $A \times_X (X \times_Z Y)$ with $A \times_Z Y$ because the universal properties they satisfy are equivalent so there is a unique isomorphism between them which preserves these universal properties. However, when we start considering things bicategorically in Section 3.3 it will be important that our machinery encodes these isomorphisms.

Definition 2.8. Given morphisms $f : X \to Y$ and $g : Y \to Z$ in Set^G the commutative diagram

$$\begin{tikzcd}
Y \times_Z \Pi_g(X,f) \arrow[rr, "pr_{\Pi_g(X,f)}"] \arrow[d, "ev"'] & & \Pi_g(X,f) \arrow[d, "\pi_Z"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

where ev is the evaluation map $(y, \sigma) \mapsto \sigma(y)$ is called the **canonical exponential diagram** generated by f and g. An **exponential diagram** is a diagram in Set^G which is isomorphic to a canonical exponential diagram. Exponential diagrams encode the distributivity of multiplicative structures over additive structures.

Proposition 2.9. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in Set^G. Then the functors $\Pi_g \Sigma_f$ and $\Sigma_{\pi_Z} \Pi_{pr_{\Pi_g X}} \Delta_{ev}$ are naturally isomorphic.*

Proof. We have

$$\Pi_g \Sigma_f(A, p_A) = (\Pi_g(A, gp_A), (z, \sigma) \mapsto z)$$

and

$$\begin{aligned}\Sigma_{\pi_Z} \Pi_{pr_{\Pi_g X}} \Delta_{ev}(A,p_A) &= \Sigma_{\pi_Z} \Pi_{pr_{\Pi_g X}}(A \times_X (Y \times_Z \Pi_g(X,f)), pr_{Y\times_Z \Pi_g(X,f)}) \\ &= (\Pi_{pr_{\Pi_g X}}(A \times_X (Y \times_Z \Pi_g(X,f)), pr_{Y\times_Z \Pi_g(X,f)}), ((z,\sigma),\omega) \mapsto z).\end{aligned}$$

These objects are naturally isomorphic in Set^G_Z via the inverse maps

$$\begin{aligned}&\Pi_g(A, gp_A) \to \Pi_{pr_{\Pi_g X}}(A \times_X (Y \times_Z \Pi_g(X,f)), pr_{Y\times_Z \Pi_g(X,f)}) \\ &\qquad (z,\sigma) \mapsto ((z, y \mapsto p_A\sigma(y)), (y, (g(y), \psi)) \mapsto (\sigma(y), (y, (g(y), \psi))))\end{aligned}$$

and

$$\begin{aligned}&\Pi_{pr_{\Pi_g X}}(A \times_X (Y \times_Z \Pi_g(X,f)), pr_{Y\times_Z \Pi_g(X,f)}) \to \Pi_g(A, fp_A) \\ &\qquad\qquad ((z,\sigma),\omega) \mapsto (z, y \mapsto pr_A(\omega(y,(z,\sigma)))).\end{aligned}$$

□

2.2 Coefficient Systems

We will develop the story of coefficient systems as a warm up to Mackey and Tambara functors.

Notation 2.10. As with objects in Set^G_X for objects $(A, p_A) \in (\mathrm{Set}^G_X)^{op}$ we will usually just write A suppressing the p_A to improve readability.

Definition 2.11. Let G be a finite group and let X be a G-set. A G**-semi-coefficient system** over X is a product-preserving functor from $(\mathrm{Set}^G_X)^{op}$ to the category of commutative monoids

$$\underline{C} : (\mathrm{Set}^G_X)^{op} \to \mathbf{CMon}.$$

The collection of G-semi-coefficient systems over X and natural transformations between them forms a category which we will denote sCoeff_X^G. The product in $(\mathrm{Set}_X^G)^{op}$ is disjoint union so, because $\underline{C}$ is product-preserving, $\underline{C}(\emptyset)$ is always a singleton. For $f : A \to B$ in Set_X^G we write res_f for the map $\underline{C}(f) : \underline{C}(B) \to \underline{C}(A)$ which we call the restriction map along f.

A ***G*-coefficient system** over X is a G-semi-coefficient system $\underline{C}$ over X such that $\underline{C}(A)$ is an abelian group for all $A \in (\mathrm{Set}_X^G)^{op}$. This is equivalent to requiring that $\underline{C}$ be a product-preserving functor $(\mathrm{Set}_X^G) \to \mathbf{Ab}$. We write Coeff_X^G for the full subcategory of G-coefficient systems over X in sCoeff_X^G. We will write sCoeff_*^G and Coeff_*^G for $\mathrm{sCoeff}_{G/G}^G$ and $\mathrm{Coeff}_{G/G}^G$, respectively, and drop "over G/G" when referring to their objects. We note that because $\mathrm{Set}_{G/H}^G$ and Set^H are equivalent the categories $\mathrm{sCoeff}_{G/H}^G$ and sCoeff_*^H are also equivalent, as are the categories $\mathrm{Coeff}_{G/H}^G$ and Coeff_*^H.

Coefficient systems first appeared in [4] as a replacement for abelian groups as the appropriate notion of coefficients for equivariant cohomology. It is common to define them as a functor out of the category Orb_G, the full subcategory of Set^G containing the orbits of G, that is, finite G-sets of the form G/H for H a subgroup of G. We can do this because every finite G-set is a disjoint union of orbits and any G-equivariant map between them can be written in terms of morphisms in Orb_G, so the product-preserving condition means that $\underline{C} \in \mathrm{sCoeff}_X^G$ is determined by what it does on Orb_G. This approach is natural from the view of equivariant topology where G-CW complexes are built from n-cells of the form $G/H \times D^n$, and so 0-cells are $G/H \times D^0 \cong G/H$ leading to the slogan "orbits are points". However we will see that for coefficient systems, and later Mackey and Tambara functors, that considering Set^G unifies some constructions that appear distinct from the Orb_G perspective.

The change of base functors from Definition 2.3 induce functors on the opposite categories, we will use the same notation for these functors. For purely formal reasons these functors still form adjoint pairs as in Proposition 2.5 but with the directions of the adjunctions reversed, so that Δ_φ is left adjoint to Σ_φ and right adjoint to Π_φ. For instance

$$\begin{aligned}(\mathrm{Set}_Y^G)^{op}(U, \Sigma_\varphi A) &= \mathrm{Set}_Y^G(\Sigma_\varphi A, U) \\ &\cong \mathrm{Set}_X^G(A, \Delta_\varphi U) = (\mathrm{Set}_X^G)^{op}(\Delta_\varphi U, A).\end{aligned}$$

A similar argument shows $\Pi_\varphi \dashv \Delta_\varphi$ between $(\mathrm{Set}_X^G)^{op}$ and $(\mathrm{Set}_Y^G)^{op}$.

Now we describe the restriction and transfer functor constructions on semi-coefficient systems, and prove some basic properties about them. These functors will be used to show that Coeff_X^G is a module category.

Definition 2.12. Let $\varphi : X \to Y$ be a morphism in Set^G and $\underline{C} \in \mathrm{sCoeff}_Y^G$. We define $\mathcal{R}_\varphi \underline{C}$ as the left Kan extension of $\underline{C}$ along Δ_φ.

$$\begin{tikzcd}
(\mathrm{Set}_Y^G)^{op} \arrow[r, "\underline{C}"] \arrow[d, "\Delta_\varphi"'] & \mathbf{CMon} \\
(\mathrm{Set}_X^G)^{op} \arrow[ur, "\mathrm{Lan}_{\Delta_\varphi}\underline{C} =: \mathcal{R}_\varphi \underline{C}"'] &
\end{tikzcd}$$

Write $\alpha_{\underline{C}} : \underline{C} \to \Delta_\varphi(\mathcal{R}_\varphi \underline{C})$ for the universal map of the left Kan extension. Given a morphism $\underline{C} \xrightarrow{f} \underline{D}$ in sCoeff_Y^G there is a morphism $\mathcal{R}_\varphi(f) : \mathcal{R}_\varphi \underline{C} \to \mathcal{R}_\varphi \underline{D}$ such that $\mathcal{R}_\varphi(f)\alpha_{\underline{C}} = \alpha_{\underline{D}} f$ by the universal property of $\alpha_{\underline{C}}$. This gives a functor $\mathcal{R}_\varphi : \mathrm{sCoeff}_Y^G \to \mathrm{sCoeff}_X^G$; we call this the **restriction functor** along φ.

The restriction functor admits a convenient explicit description.

Proposition 2.13. *Given $\underline{C} \in \mathrm{sCoeff}_Y^G$ and $\varphi : X \to Y$ in Set^G we have $\mathcal{R}_\varphi \underline{C} = \underline{C}\Sigma_\varphi$.*

Proof. This is a direct application of enriched Yoneda reduction [17, 4.28]. That is, the enriched version of Lemma A.2, setting $K = \Delta_\varphi$, $H = \Sigma_\varphi$, and $F = \underline{C}$. □

Lemma 2.14. *Let $\underline{C} \in \mathrm{sCoeff}_Y^G$ and $\varphi : X \to Y$ in Set^G, then $\mathcal{R}_\varphi \underline{C} \in \mathrm{sCoeff}_X^G$.*

Proof. We note that Σ_φ preserves products, so as a composition of product-preserving functors $\mathcal{R}_\varphi \underline{C}$ is product-preserving. □

The restriction functor takes coefficient systems to coefficient systems.

Lemma 2.15. *If $\underline{C} \in \mathrm{Coeff}_Y^G$ then $\mathcal{R}_\varphi \underline{C} \in \mathrm{Coeff}_X^G$.*

Proof. Given $A \in (\mathrm{Set}_X^G)^{op}$ we have $\mathcal{R}_\varphi \underline{C}(A) = \underline{C}(\Sigma_\varphi A)$ which is an abelian group since $\underline{C} \in \mathrm{Coeff}_Y^G$. □

Definition 2.16. Given a morphism $\varphi : X \to Y$ in Set^G and $\underline{C} \in \mathrm{sCoeff}_X^G$, we define $\mathcal{T}_\varphi \underline{C}$ as the right Kan extension of $\underline{C}$ along Σ_φ.

$$\begin{tikzcd}
(\mathrm{Set}_X^G)^{op} \arrow[r, "\underline{C}"] \arrow[d, "\Sigma_\varphi"'] & \mathbf{CMon} \\
(\mathrm{Set}_Y^G)^{op} \arrow[ur, "\mathrm{Ran}_{\Sigma_\varphi}\underline{C} =: \mathcal{T}_\varphi \underline{C}"'] &
\end{tikzcd}$$

Write $\alpha_{\underline{C}} : \Sigma_\varphi(\mathcal{R}_\varphi \underline{C}) \to \underline{C}$ for the universal map of the right Kan extension. Given a morphism $\underline{C} \xrightarrow{f} \underline{D}$ in sCoeff_X^G there is a morphism $\mathcal{T}_\varphi(f) : \mathcal{T}_\varphi \underline{C} \to \mathcal{T}_\varphi \underline{D}$ such that $\alpha_{\underline{D}} \mathcal{T}_\varphi(f) = f\alpha_{\underline{C}}$ by the universal property of $\alpha_{\underline{D}}$. This gives a functor $\mathcal{T}_\varphi : \mathrm{sCoeff}_X^G \to \mathrm{sCoeff}_Y^G$; we call this the **transfer functor** along φ.

The transfer functor admits a convenient explicit description.

Proposition 2.17. *Given $\underline{C} \in \mathrm{sCoeff}_X^G$ and $\varphi : X \to Y$ in Set^G we have $\mathcal{T}_\varphi \underline{C} = \underline{C}\Delta_\varphi$*

Proof. Using the objectwise formula for a right Kan extension the value on objects is given by

$$\begin{aligned}\mathcal{T}_\varphi\underline{C}(A, p_A) &\cong \int_{(\mathrm{Set}^G_X)^{op}} [(\mathrm{Set}^G_Y)^{op}(A, \Sigma_\varphi B), \underline{C}(B)]\, dB \\ &\cong \int_{(\mathrm{Set}^G_X)^{op}} [(\mathrm{Set}^G_X)^{op}(\Delta_\varphi A, B), \underline{C}(B)]\, dB \\ &\cong \underline{C}(\Delta_\varphi A) = \underline{C}(X \times_Y A, pr_X)\end{aligned}$$

where the second isomorphism uses the adjunction $\Delta_\varphi \dashv \Sigma_\varphi$ and the third uses the enriched (co)Yoneda lemma. So $\mathcal{T}_\varphi\underline{C}$ and $\underline{C}\Delta_\varphi$ agree on objects. As each of the isomorphisms are natural in (A, p_A) they also agree on morphisms. The above is an application of the dual statement to the enriched version of Lemma A.2. □

Lemma 2.18. *Let $\underline{C} \in \mathrm{sCoeff}^G_X$ and $\varphi : X \to Y$ be in $(\mathrm{Set}^G)^{op}$, then $\mathcal{T}_\varphi\underline{C} \in \mathrm{sCoeff}^G_Y$.*

Proof. As Δ_φ is a right adjoint it preserves limits and so in particular preserves products. So as a composition of product-preserving functors $\mathcal{T}_\varphi\underline{C}$ preserves products. □

The transfer functor takes coefficient systems to coefficient systems.

Lemma 2.19. *If $\underline{C} \in \mathrm{Coeff}^G_X$ then $\mathcal{T}_\varphi\underline{C} \in \mathrm{Coeff}^G_Y$.*

Proof. Given $(A, p_A) \in (\mathrm{Set}^G_Y)^{op}$ we have $\mathcal{T}_\varphi\underline{C}(A, p_A) = \underline{C}(\Delta_\varphi(A, p_A))$ which is an abelian group as $\underline{C} \in \mathrm{Coeff}^G_X$. □

Proposition 2.20. *Let*

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{pr_Y} & Y \\ {\scriptstyle pr_X}\downarrow & & \downarrow{\scriptstyle \beta} \\ X & \xrightarrow[\alpha]{} & Z \end{array}$$

be a pullback square in Set^G. Then the functors $\mathcal{R}_\beta\mathcal{T}_\alpha$ and $\mathcal{T}_{pr_Y}\mathcal{R}_{pr_X}$ are naturally isomorphic.

Proof. By applying the natural isomorphism $\Delta_\beta\Sigma_\varphi \cong \Sigma_{pr_Y}\Delta_{pr_X}$ from Lemma 2.6 we have

$$\mathcal{R}_\beta\mathcal{T}_\alpha\underline{C} = \underline{C}\Delta_\beta\Sigma_\alpha \cong \underline{C}\Sigma_{pr_Y}\Delta_{pr_X} = \mathcal{T}_{pr_Y}\mathcal{R}_{pr_X}\underline{C}.$$ □

Definition 2.21. For $(A, p_A) \in \mathrm{Set}^G_X$ and $\underline{C} \in \mathrm{sCoeff}^G_X$ define

$$A \odot \underline{C} := \mathcal{T}_{p_A}\mathcal{R}_{p_A}\underline{C}.$$

This gives a functor $- \odot - : (\mathrm{Set}^G_X)_{\cong} \times \mathrm{sCoeff}^G_X \to \mathrm{sCoeff}^G_X$.

Lemma 2.22. *For the fold map $\nabla : X \amalg X \to X$ we have*

$$(X \amalg X, \nabla) \odot \underline{C}(A, p_A) \cong \underline{C}(A, p_A) \oplus \underline{C}(A, p_A).$$

Proof. We have

$$\begin{aligned}(X \amalg X) \odot \underline{C}(A, p_A) &\cong \mathcal{T}_\nabla \mathcal{R}_\nabla \underline{C}(A, p_A) \cong \mathcal{R}_\nabla \underline{C}(\Delta_\nabla(A, p_A)) \\ &\cong \mathcal{R}_\nabla \underline{C}(A \amalg A, p_A \amalg p_A) \cong \underline{C}(\Sigma_\nabla(A \amalg A, p_A \amalg p_A)) \\ &\cong \underline{C}(A \amalg A, (p_A, p_A)) \cong \underline{C}(A, p_A) \oplus \underline{C}(A, p_A)\end{aligned}$$ □

Definition 2.23. The **sum** of two G-coefficient systems $\underline{C}$ and $\underline{D}$ is the G-coefficient system $\underline{C} \oplus \underline{D}$ with $(\underline{C} \oplus \underline{D})(A) = \underline{C}(A) \oplus \underline{D}(A)$.

We let $\underline{0}$ be the G-coefficient system with $\underline{0}(A) = 0$ for all $A \in (\mathrm{Set}^G)^{op}$. Then $\underline{0}$ acts as as identity object with respect to $\oplus$.

Lemma 2.24. *Both* $(\mathrm{sCoeff}^G_X, \oplus, \underline{0})$ *and* $(\mathrm{Coeff}^G_X, \oplus, \underline{0})$ *are symmetric monoidal categories.*

2.3 Module categories over bimonoidal categories

In [22] a G-symmetric monoidal structure on $\mathcal{C}$ is defined to be a functor

$$(-) \boxtimes (-) : \mathrm{Set}^G_{\cong} \times \mathcal{C} \to \mathcal{C},$$

where $\mathrm{Set}^G_{\cong}$ is the wide subcategory containing only isomorphisms, such that

- $(X \amalg Y) \boxtimes C$ is isomorphic to $(X \boxtimes C) \otimes (Y \boxtimes C)$, naturally in X, Y, and C,
- when X has a trivial G-action $X \boxtimes -$ is the canonical exponentiation functor with $X \boxtimes C = C^{\otimes |X|}$,
- and $X \boxtimes (Y \boxtimes C)$ is isomorphic to $(X \times Y) \boxtimes C$, naturally in X, Y, and C.

We have an alternate definition which is stricter but includes all cases the author is aware of. The following definition is [16, Volume 1, Definition 2.1.2] and is a categorification of the notion of a commutative ring. Our definition of G-symmetric monoidal (and generalisations) will be a categorification of the notion of a module over a ring for a specific tight symmetric bimonoidal category.

Definition 2.25. A **tight symmetric bimonoidal category** is a tuple

$$\big(\mathcal{C}, (\oplus, 0, \alpha^\oplus, \lambda^\oplus, \rho^\oplus, \beta^\oplus), (\otimes, 1, \alpha^\otimes, \lambda^\otimes, \rho^\otimes, \beta^\otimes), (\lambda^\bullet, \rho^\bullet), (\delta^L, \delta^R)\big)$$

consisting of the following data:

- **Additive Structure:** The tuple $(\mathcal{C}, \oplus, 0, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \beta^{\oplus})$ is a symmetric monoidal structure where $\alpha^{\oplus}$ is the associator, $\lambda^{\oplus}$ and $\rho^{\oplus}$ are the left and right unitors respectively, and $\beta^{\oplus}$ is the commutator.

- **Multiplicative Structure:** The tuple $(\mathcal{C}, \otimes, 1, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \beta^{\otimes})$ is a symmetric monoidal structure with $\alpha^{\otimes}$ is the associator, $\lambda^{\otimes}$ and $\rho^{\otimes}$ are the left and right unitors respectively, and $\beta^{\otimes}$ is the commutator.

- **Multiplicative Zero:** The natural isomorphisms $\lambda^{\bullet}$ and $\rho^{\bullet}$ with components $\lambda^{\bullet}_X : 0 \otimes X \xrightarrow{\cong} 0$ and $\rho^{\bullet}_X : X \otimes 0 \xrightarrow{\cong} 0$ for $X \in \mathcal{C}$ are the left and right multiplicative zero respectively.

- **Distributivity:** The natural isomorphisms δ^L and δ^R with components
$$\delta^L_{X,Y,Z} : X \otimes (Y \oplus Z) \xrightarrow{\cong} (X \otimes Y) \oplus (X \otimes Z),$$
$$\delta^R_{X,Y,Z} : (X \oplus Y) \otimes Z \xrightarrow{\cong} (X \otimes Z) \oplus (Y \otimes Z)$$
are left distributivity and right distributivity respectively.

This data is required to satisfy coherence axioms in the form of commutative diagrams. We do not list these here but the interested reader can find them under [16, Volume 1, Definition 2.1.2].

Our definitions of incomplete Mackey functors and bi-incomplete Tambara functors involve subcategories of Set^G satisfying the following properties.

Definition 2.26. Let $\mathcal{C}$ be a category. A subcategory $\mathcal{A}$ of $\mathcal{C}$ is

- **wide** if it contains all objects in $\mathcal{C}$;

- **pullback stable** if for any morphism f in $\mathcal{A}$ and any morphism g in $\mathcal{C}$, the pullback of f along g is contained in $\mathcal{A}$;

- **finite coproduct complete** if for any finite set of objects $\{X_1, \ldots, X_n\}$ in $\mathcal{A}$ their coproduct $\coprod_{i=1}^n X_i$ is contained in $\mathcal{A}$ and agrees with the coproduct in $\mathcal{C}$, this includes the empty coproduct.

Following [1], if $\mathcal{O}$ is a wide, pullback stable, finite coproduct complete subcategory of Set^G we call it an **indexing category**. Indexing categories will be used to define incomplete Mackey functors and bi-incomplete Tambara functors in later chapters. The following lemma is an immediate consequence of [1, Lemma 3.2] and [1, Proposition 3.1].

Lemma 2.27. *If $\mathcal{O}$ is an indexing category then $\mathcal{O}$ contains all monomorphisms.*

Definition 2.28. Let $\mathcal{O}$ be an indexing category and X be a finite G-set. We write $\mathcal{O}_X$ for the slice category of $\mathcal{O}$ over X.

We will see that these slice categories $\mathcal{O}_X$ are tight symmetric bimonoidal categories.

Definition 2.29. A symmetric monoidal category $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho, \beta)$ is **distributive** if $\mathcal{C}$ has all finite coproducts and the natural morphisms

$$(X \otimes Y) \amalg (X \otimes Z) \to X \otimes (Y \amalg Z) \tag{2.30}$$

$$(X \otimes Z) \amalg (Y \otimes Z) \to (X \amalg Y) \otimes Z \tag{2.31}$$

$$\emptyset \to X \otimes \emptyset \tag{2.32}$$

$$\emptyset \to \emptyset \otimes X \tag{2.33}$$

where $\emptyset$ is the empty coproduct are isomorphisms for all $X, Y, Z \in \mathcal{C}$.

The following proposition is [16, Volume 1, Proposition 2.3.2].

Proposition 2.34. *If $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho, \beta)$ is a distributive symmetric monoidal category then $\mathcal{C}$ has the structure of a tight symmetric bimonoidal category where we have:*

- ***Additive Structure:*** *Is given by the coproduct functor with the empty coproduct as the identity object, $(\mathcal{C}, \amalg, \emptyset, \alpha^{\amalg}, \lambda^{\amalg}, \rho^{\amalg}, \beta^{\amalg})$. The structure natural transformations are given by the universal properties of coproducts.*
- ***Multiplicative Structure:*** *Is the symmetric monoidal structure we started with, $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho, \beta)$.*
- ***Multiplicative Zero:*** *Are the inverse maps to 2.32 and 2.33.*
- ***Distributivity:*** *Are the inverse maps to 2.30 and 2.31.*

Proposition 2.35. *For $\mathcal{O}$ an indexing category and X a finite G-set, the category $\mathcal{O}_X$ is a tight symmetric bimonoidal category with respect to disjoint union and pullback over X.*

Proof. There is a symmetric monoidal structure $(\mathcal{O}_X, \times_X, \mathrm{id}_X, \alpha^{\times}, \lambda^{\times}, \rho^{\times}, \beta^{\times})$ on $\mathcal{O}_X$ where $\times_X$ is pullback over X, which is the categorical product in $\mathcal{O}_X$. The structure natural transformations are given by the universal properties of products. The categorical coproduct in $\mathcal{O}_X$ is disjoint union, and we note that there are obvious natural isomorphisms

$$(A \times_X B) \amalg (A \times_X C) \xrightarrow{\cong} A \times_X (B \amalg C)$$

$$(A \times_X C) \amalg (B \times_X C) \xrightarrow{\cong} (A \amalg B) \times_X C$$

$$\emptyset \xrightarrow{\cong} A \times_X \emptyset$$

$$\emptyset \xrightarrow{\cong} \emptyset \times_X A$$

so $(\mathcal{O}_X, \times_X, \mathrm{id}_X, \alpha^\times, \lambda^\times, \rho^\times, \beta^\times)$ is a distributive symmetric monoidal category, and hence is a tight symmetric bimonoidal category with the structure described in Proposition 2.34. $\square$

Note that the maximal groupoid (the wide subcategory containing precisely the isomorphisms) in $\mathcal{O}_X$ still contains all the structural morphisms and so is also a tight symmetric bimonoidal category, we will write $(\mathcal{O}_X)_\cong$ for this category.

Corollary 2.36. *The groupoid of finite G-sets over X,* $(\mathrm{Set}_X^G)_\cong$*, is a tight symmetric bimonoidal category under disjoint union and pullback over X.*

Now that we have defined our tight symmetric bimonoidal categories of interest we need to define what it means for a category to be a module over such a thing.

Definition 2.37. A **module category** over a tight symmetric bimonoidal category $(\mathcal{C}, (\oplus, 0, \alpha^\oplus, \lambda^\oplus, \rho^\oplus, \beta^\oplus), (\otimes, 1, \alpha^\otimes, \lambda^\otimes, \rho^\otimes, \beta^\otimes), (\lambda^\bullet, \rho^\bullet), (\delta^L, \delta^R))$ is a tuple

$$(\mathcal{A}, (\mathbb{I}, \square, \alpha^\square, \lambda^\square, \rho^\square, \beta^\square), (*, \lambda^*, \alpha^*), (d^L, d^R), (\xi^L, \xi^R))$$

consisting of the following data:

- **Symmetric Monoidal Structure:** The tuple $(\mathcal{A}, \mathbb{I}, \square, \alpha^\square, \lambda^\square, \rho^\square, \beta^\square)$ is a symmetric monoidal structure where $\alpha^\square$ is the associator, $\lambda^\square$ and $\rho^\square$ are the left and right unitors respectively, and $\beta^\square$ is the commutator.
- **Module Structure:** A functor $* : \mathcal{C} \times \mathcal{A} \to \mathcal{A}$ together with a (left) unit natural isomorphism $\lambda^* : 1 * - \xrightarrow{\cong} \mathrm{id}_\mathcal{A}$ and for each triple (X, Y, A) with $X, Y \in \mathcal{C}$ and $A \in \mathcal{A}$ an associativity isomorphism $\alpha^*_{X,Y,A} : X * (Y * A) \xrightarrow{\cong} (X \otimes Y) * A$ natural in X, Y, and A.
- **Distributivity Constraints:** For $X, Y \in \mathcal{C}$ and $A, B \in \mathcal{A}$ left and right distributivity isomorphisms $d^L_{X,Y,A} : (X \oplus Y) * A \xrightarrow{\cong} (X * A) \square (Y * A)$ and $d^R_{X,A,B} : X * (A \square B) \xrightarrow{\cong} (X * A) \square (X * B)$ natural in X, Y, A, and B.
- **Multiplicative Zero:** For $X \in \mathcal{C}$ and $A \in \mathcal{A}$ natural isomorphisms with components $\xi^L_A : 0 * A \xrightarrow{\cong} \mathbb{I}$ and $\xi^R_X : X * \mathbb{I} \xrightarrow{\cong} \mathbb{I}$.

This data is required to satisfy coherence axioms which take the form of diagrams which must commute for any $X, Y, Z \in \mathcal{C}$ and any $A, B \in \mathcal{A}$:

- **Braiding and Distributivity**

$$\begin{tikzcd}
(X \oplus Y) * A \arrow[r, "d^L"] \arrow[d, "\beta^\oplus * \mathrm{id}"'] & (X * A)\square(Y * A) \arrow[d, "\beta^\square"] \\
(Y \oplus X) * A \arrow[r, "d^L"'] & (Y * A)\square(X * A)
\end{tikzcd}
\qquad
\begin{tikzcd}
X * (A \square B) \arrow[r, "d^R"] \arrow[d, "\mathrm{id} * \beta^\square"'] & (X * A)\square(X * B) \arrow[d, "\beta^\square"] \\
X * (B \square A) \arrow[r, "d^R"'] & (X * B)\square(X * A)
\end{tikzcd}$$

- **Associativity and Distributivity:**

$$\begin{tikzcd}
((X \oplus Y) \oplus Z) * A \arrow[r, "d^L"] \arrow[d, "\alpha^{\oplus} * \mathrm{id}"'] & ((X \oplus Y) * A) \square (Z * A) \arrow[r, "d^L \square \mathrm{id}"] & ((X * A) \square (Y * A)) \square (Z * A) \arrow[d, "\alpha^{\square}"] \\
(X \oplus (Y \oplus Z)) * A \arrow[r, "d^L"'] & (X * A) \square ((Y \oplus Z) * A) \arrow[r, "\mathrm{id} \square d^L"'] & (X * A) \square ((Y * A) \square (Z * A))
\end{tikzcd}$$

$$\begin{tikzcd}
X * ((A \square B) \square C) \arrow[r, "d^R"] \arrow[d, "\mathrm{id} * \alpha^{\square}"'] & (X * (A \square B)) \square (X * C) \arrow[r, "d^R \square \mathrm{id}"] & ((X * A) \square (X * B)) \square (X * C) \arrow[d, "\alpha^{\square}"] \\
X * (A \square (B \square C)) \arrow[r, "d^R"'] & (X * A) \square (X * (B \square C)) \arrow[r, "\mathrm{id} \square d^R"'] & (X * A) \square ((X * B) \square (X * C))
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes Y) * (A \square B) \arrow[rr, "d^R"] \arrow[d, "\alpha^{*}"'] & & ((X \otimes Y) * A) \square ((X \otimes Y) * B) \arrow[d, "\alpha^{\square} \square \alpha^{\square}"] \\
X * (Y * (A \square B)) \arrow[r, "\mathrm{id} * d^R"'] & X * ((Y * A) \square (Y * B)) \arrow[r, "d^R"'] & (X * (Y * A)) \square (X * (Y * B))
\end{tikzcd}$$

- **2-by-2 Distributivity**

$$\begin{tikzcd}
(X \oplus Y) * (A \square B) \arrow[r, "d^R"] \arrow[d, "d^L"'] & ((X \oplus Y) * A) \square ((X \oplus Y) * B) \arrow[d, "d^L \square d^L"] \\
(X * (A \square B)) \square (Y * (A \square B)) \arrow[d, "d^R \square d^R"'] & ((X * A) \square (Y * A)) \square ((X * B) \square (Y * B)) \arrow[d, "\alpha^{\square}"] \\
((X * A) \square (X * B)) \square ((Y * A) \square (Y * B)) \arrow[d, "\alpha^{\square}"'] & (X * A) \square ((Y * A) \square ((X * B) \square (Y * B))) \arrow[d, "\mathrm{id} \square (\alpha^{\square})^{-1}"] \\
(X * A) \square ((X * B) \square ((Y * A) \square (Y * B))) \arrow[d, "\mathrm{id} \square (\alpha^{\square})^{-1}"'] & (X * A) \square (((Y * A) \square (X * B)) \square (Y * B)) \arrow[d, "\mathrm{id} \square (\beta^{\square} \square \mathrm{id})"] \\
(X * A) \square (((X * B) \square (Y * A)) \square (Y * B)) \arrow[r, equal] & (X * A) \square (((X * B) \square (Y * A)) \square (Y * B)
\end{tikzcd}$$

- **Multiplication of Additive Units:** $0 * \mathbb{I} \underset{\xi^R}{\overset{\xi^L}{\rightrightarrows}} \mathbb{I}$

- **Multiplication by Additive Unit and Distributivity:**

$$\begin{tikzcd}
0 * (A \square B) \arrow[r, "d^R"] \arrow[d, "\xi^L"'] & (0 * A) \square (0 * B) \arrow[d, "\xi^L \square \xi^L"] \\
\mathbb{I} & \mathbb{I} \square \mathbb{I} \arrow[l, "\lambda^{\square}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(X \oplus Y) * \mathbb{I} \arrow[r, "d^l"] \arrow[d, "\xi^R"'] & (X * \mathbb{I}) \square (Y * \mathbb{I}) \arrow[d, "\xi^R \square \xi^R"] \\
\mathbb{I} & \mathbb{I} \square \mathbb{I} \arrow[l, "\lambda^{\square}"]
\end{tikzcd}$$

- **Multiplication by Additive Unit and Associativity:**

$$\begin{tikzcd}
(X \otimes 0) * A \arrow[r, "\alpha^{*}"] \arrow[d, "\rho^{\bullet} * \mathrm{id}"'] & X * (0 * A) \arrow[r, "\mathrm{id} * \xi^L"] & X * \mathbb{I} \arrow[d, "\xi^R"] \\
0 * A \arrow[rr, "\xi^L"'] & & \mathbb{I}
\end{tikzcd}$$

$$\begin{tikzcd}
(0 \otimes X) * A \arrow[r, "\alpha^*"] \arrow[d, "\lambda^\bullet * \mathrm{id}"'] & 0 * (X * A) \arrow[d, "\xi^L"] \\
0 * A \arrow[r, "\xi^L"'] & \mathbb{I}
\end{tikzcd}
\qquad
\begin{tikzcd}
(X \otimes Y) * \mathbb{I} \arrow[r, "\alpha^*"] \arrow[d, "\xi^R"'] & X * (Y * \mathbb{I}) \arrow[d, "\mathrm{id} * \xi^R"] \\
\mathbb{I} & X * \mathbb{I} \arrow[l, "\xi^R"]
\end{tikzcd}$$

- **Multiplication by Additive Unit and Distributivity:**

$$\begin{tikzcd}
X * (\mathbb{I} \square A) \arrow[r, "d^R"] \arrow[d, "\mathrm{id} * \lambda^\square"'] & (X * \mathbb{I}) \square (X * A) \arrow[d, "\xi^R \square \mathrm{id}"] \\
X * A & \mathbb{I} \square (X * A) \arrow[l, "\lambda^\square"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X * (A \square \mathbb{I}) \arrow[r, "d^R"] \arrow[d, "\mathrm{id} * \rho^\square"'] & (X * A) \square (X * \mathbb{I}) \arrow[d, "\mathrm{id} \square \xi^R"] \\
X * A & (X * A) \square \mathbb{I} \arrow[l, "\rho^\square"]
\end{tikzcd}$$

$$\begin{tikzcd}
(0 \oplus X) * A \arrow[r, "d^L"] \arrow[d, "\lambda^\oplus * \mathrm{id}"'] & (0 * A) \square (X * A) \arrow[d, "\xi^L \square \mathrm{id}"] \\
X * A & \mathbb{I} \square (X * A) \arrow[l, "\lambda^\square"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(X \oplus 0) * A \arrow[r, "d^L"] \arrow[d, "\rho^\oplus"'] & (X * A) \square (0 * A) \arrow[d, "\mathrm{id} \square \xi^L"] \\
X * A & (X * A) \square \mathbb{I} \arrow[l, "\rho^\square"]
\end{tikzcd}$$

- **Multiplicative Unit and Distributivity:**

$$\begin{tikzcd}
1 * (A \square B) \arrow[rr, "d^R"] \arrow[dr, "\lambda^*"'] & & (1 * A) \square (\mathbb{I} * B) \arrow[dl, "\lambda^* \square \lambda^*"] \\
& A \square B &
\end{tikzcd}$$

- **Multiplicative Unit and Associativity:**

$$\begin{tikzcd}
(1 \otimes X) * A \arrow[rr, "\alpha^*"] \arrow[dr, "\lambda^\otimes * \mathrm{id}"'] & & 1 * (X * A) \arrow[dl, "\lambda^*"] \\
& X * A &
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes 1) * A \arrow[rr, "\alpha^*"] \arrow[dr, "\rho^\otimes * \mathrm{id}"'] & & X * (1 * A) \arrow[dl, "\mathrm{id} * \lambda^*"] \\
& X * A &
\end{tikzcd}$$

Definition 2.38. A **G-symmetric monoidal category over** X for X a finite G-set is a module category over $(\mathrm{Set}^G_X)_{\cong}$. We note that our definition is non-standard, there are a few different notions of G-symmetric monoidal category in the literature.

Theorem 2.39. *Let X be a finite G-set and $\mathcal{O}$ be an indexing category. The category of G-coefficient systems over X is a module category over $(\mathcal{O}_X)_{\cong}$ with the action functor $\odot : (\mathcal{O}_X)_{\cong} \times \mathrm{Coeff}^G_X \to \mathrm{Coeff}^G_X$ given by $(A, \varphi) \odot \underline{C} = \mathcal{T}_\varphi \mathcal{R}_\varphi \underline{C}$.*

Proof. By Lemma 2.24 $(\mathrm{Coeff}^G_X, \oplus, \underline{0})$ is a symmetric monoidal category. Noting that on objects we have $(\underline{C} \oplus \underline{D})(X) = \underline{C}(X) \oplus \underline{D}(X)$ the structure natural isomorphisms have the component maps $\alpha^\oplus_X : ((a,b),c) \mapsto (a,(b,c))$, $\lambda^\oplus_X : (0,a) \mapsto a$, $\rho^\oplus_X : (a,0) \mapsto a$, $\beta^\oplus_X : (a,b) \mapsto (b,a)$. For $(A,\varphi),(B,\psi) \in (\mathcal{O}_X)_{\cong}$ we declare the following structure natural isomorphisms:

- Let $\lambda^{\odot} : \mathrm{id}_X \odot \underline{C} = \mathcal{T}_{\mathrm{id}_X}\mathcal{R}_{\mathrm{id}_X}\underline{C} \to \underline{C}$ be the map specified on objects by
$$\mathcal{T}_{\mathrm{id}_X}\mathcal{R}_{\mathrm{id}_X}\underline{C}(V) = \underline{C}\Sigma_{\mathrm{id}_X}\Delta_{\mathrm{id}_X}(V) = \underline{C}(X \times_X V) \to \underline{C}(V)$$
which is $\underline{C}$ applied to (the opposite of) the map $V \xrightarrow{\cong} X \times_X V, v \mapsto (p_V(v), v)$ of G-sets.

- Let $\alpha^{\odot} : \psi \odot (\varphi \odot \underline{C}) = \mathcal{T}_\psi\mathcal{R}_\psi\mathcal{T}_\varphi\mathcal{R}_\varphi\underline{C} \to \mathcal{T}_{\psi pr_B}\mathcal{R}_{\varphi pr_A}\underline{C} = (\psi \times_X \varphi) \odot \underline{C}$ be the map
$$\mathcal{T}_\psi\mathcal{R}_\psi\mathcal{T}_\varphi\mathcal{R}_\varphi\underline{C} \xrightarrow{\cong} \mathcal{T}_\psi\mathcal{T}_{pr_B}\mathcal{R}_{pr_A}\mathcal{R}_\varphi\underline{C} \xrightarrow{\cong} \mathcal{T}_{\psi pr_B}\mathcal{R}_{\varphi pr_A}\underline{C}$$
where the first natural isomorphism comes from Proposition 2.20.

- Let $d^L_{\odot} : (\varphi \amalg \psi) \odot \underline{C} = \mathcal{T}_{\varphi\amalg\psi}\mathcal{R}_{\varphi\amalg\psi}\underline{C} \to \mathcal{T}_\varphi\mathcal{R}_\varphi\underline{C} \oplus \mathcal{T}_\psi\mathcal{R}_\psi\underline{C} = (\varphi \odot \underline{C}) \oplus (\psi \odot \underline{C})$ be the map coming from applying $\underline{C}$ to the isomorphism $(A \times_X V) \amalg (B \times_X V) \xrightarrow{\cong} (A \amalg B) \times_X V$, we have
$$\begin{aligned}\mathcal{T}_{\varphi\amalg\psi}\mathcal{R}_{\varphi\amalg\psi}\underline{C}(V) &= \underline{C}\Sigma_{(\varphi\amalg\psi)}\Delta_{(\varphi\amalg\psi)}(V) = \underline{C}((A \amalg B) \times_X V)\\ &\cong \underline{C}((A \times_X V) \amalg (B \times_X V) = \underline{C}(A \times_X V) \oplus \underline{C}(B \times_X V)\\ &= \underline{C}\Sigma_\varphi\Delta_\varphi(V) \oplus \underline{C}\Sigma_\psi\Delta_\psi(V) = \mathcal{T}_\varphi\mathcal{R}_\varphi\underline{C}(V) \oplus \mathcal{T}_\psi\mathcal{R}_\psi\underline{C}(V)\\ &= (\mathcal{T}_\varphi\mathcal{R}_\varphi\underline{C} \oplus \mathcal{T}_\psi\mathcal{R}_\psi\underline{C})(V).\end{aligned}$$

- Let $d^R_{\odot} : \varphi \odot (\underline{C} \oplus \underline{D}) = \mathcal{T}_\varphi\mathcal{R}_\varphi(\underline{C} \oplus \underline{D}) \to \mathcal{T}_\varphi\mathcal{R}_\varphi\underline{C} \oplus \mathcal{T}_\varphi\mathcal{R}_\varphi\underline{D} = (\varphi \odot \underline{C}) \oplus (\varphi \odot \underline{D})$ be the natural identity. We have
$$\begin{aligned}\mathcal{T}_\varphi\mathcal{R}_\varphi(\underline{C} \oplus \underline{D})(V) &= (\underline{C} \oplus \underline{D})\Sigma_\varphi\Delta_\varphi(V) = (\underline{C} \oplus \underline{D})(A \times_X V)\\ &= \underline{C}(A \times_X V) \oplus \underline{D}(A \times_X V) = \underline{C}\Sigma_\varphi\Delta_\varphi(V) \oplus \underline{D}\Sigma_\varphi\Delta_\varphi(V)\\ &= \mathcal{T}_\varphi\mathcal{R}_\varphi\underline{C}(V) \oplus \mathcal{T}_\varphi\mathcal{R}_\varphi\underline{D}(V) = (\mathcal{T}_\varphi\mathcal{R}_\varphi\underline{C} \oplus \mathcal{T}_\varphi\mathcal{R}_\varphi\underline{D})(V)\end{aligned}$$

- Note that if $\zeta : \emptyset \to X$ then $\mathcal{T}_\zeta\mathcal{R}_\zeta\underline{C} = \underline{0}$ so let $\xi^L_{\odot} : \zeta \odot \underline{C} = \mathcal{T}_\zeta\mathcal{R}_\zeta\underline{C} = \underline{0} \to \underline{0}$ be the identity natural transformation.

- Let $\xi^R_{\odot} : \varphi \odot \underline{0} = \mathcal{T}_\varphi\mathcal{R}_\varphi\underline{0} = \underline{0} \to \underline{0}$ be the identity natural transformation.

It remains to check that the coherence diagrams from Definition 2.37 are satisfied. This is a straightforward but lengthy exercise and is left to the interested reader. □

Finally, we define the notion of a monoid in a module category which we will refer back to in later chapters.

Definition 2.40. Let $\mathcal{A}$ be a module category over the maximal groupoid $\mathcal{C}_{\cong}$ of a tight symmetric bimonoidal category $\mathcal{C}$ with action functor $* : \mathcal{C}_{\cong} \times \mathcal{A} \to \mathcal{A}$, a **$\mathcal{C}$-monoid** is an object $X \in \mathcal{A}$ together with an extension $- * X$ to $\mathcal{C}$.

$$\begin{tikzcd} \mathcal{C}_{\cong} \arrow[r, "- * X"] \arrow[d, hook] & \mathcal{A} \\ \mathcal{C} \arrow[ur, dashed] & \end{tikzcd}$$

Notice that if $\mathcal{A}$ is a monoidal category, $\mathcal{C}$ is the category of finite sets, and we take $S * X = X^{\otimes |S|}$, then a $\mathcal{C}$-monoid is precisely a monoid object in $\mathcal{A}$.

Chapter 3

Incomplete Mackey Functors

Mackey functors, which are often considered an equivariant analogue for abelian groups, are coefficient systems equipped with transfer (or induction) maps corresponding to the restriction maps. Transfer maps are morphisms with the opposite variance to the restriction maps, and which satisfy a relation determined by pullback when composed with them. An incomplete Mackey functor is one which does not have all possible transfer maps. In this chapter we provide a characterisation of incomplete Mackey functors as less complete Mackey functors equipped with an appropriate pseudofunctor which encodes the information of the extra transfer maps.

In Section 3.1 we define Lindner categories.

In Section 3.2 we define (incomplete) Mackey functors and some constructions on them. In particular, we see that our approach to the norm functor realises previously defined norm functors of Mackey functors and the box product of Mackey functors as the same construction.

In Section 3.3 we define some bicategorical gadgets that are used for the main result of the chapter.

In Section 3.4 we prove the main result of the chapter: the category of coherent $\mathcal{O}'$-monoids in the category of $\mathcal{O}$-Mackey functors is equivalent to the category of $\mathcal{O}'$-Mackey functors. A corollary of this result is that G-Mackey functors are precisely the coherent Set^G-monoids in the category of G-coefficient systems.

In Section 3.5 we show that the category of $\mathcal{O}$-Mackey functors is a module category over a tight symmetric bimonoidal category determined by $\mathcal{O}$. We also provide a comparison to another characterisation of $\mathcal{O}$-Mackey functors by modifying some definitions appearing in [10].

3.1 $\mathcal{O}$-Lindner categories

Definition 3.1. Let X be a finite G-set, and $\mathcal{O}$ be a wide, pullback stable, finite coproduct complete subcategory of Set^G. Following [1], we call $\mathcal{O}$ an **indexing category**. An **$\mathcal{O}$-transfer span** over X from (A, p_A) to (C, p_C) is a diagram in Set^G of the form

$$\begin{tikzcd}
A \arrow[d, "p_A"] & B \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "p_B"] & C \arrow[d, "p_C"] \\
X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}$$

such that g is in $\mathcal{O}$. An isomorphism of $\mathcal{O}$-transfer spans over X from (A, p_A) to (C, p_C) is an isomorphism $\varphi : B \to B'$ of G-sets such that the following diagram commutes

$$\begin{tikzcd}
 & B \arrow[dl, bend right, "f"'] \arrow[d, "\varphi"] \arrow[dr, "g"] \arrow[dd, bend right, "p_B"'] & \\
A \arrow[dr, "p_A"'] & B' \arrow[l, "f'"] \arrow[r, "g'"'] \arrow[d, "p_{B'}"] & C \arrow[dl, "p_C"] \\
 & X &
\end{tikzcd}$$

Definition 3.2. The **$\mathcal{O}$-Lindner category** $\mathcal{A}_X^{\mathcal{O}}$ over X is the category whose objects are pairs (A, p_A) where A is a finite G-set and p_A is a G-equivariant map $A \to X$, and whose morphisms are isomorphism classes of $\mathcal{O}$-transfer spans over X. Composition in an $\mathcal{O}$-Lindner category is defined by pullback; given a pair of morphisms in $\mathcal{A}_X^{\mathcal{O}}$ represented by transfer spans over X, $(A \xleftarrow{f} B \xrightarrow{g} C)$ and $(C \xleftarrow{h} D \xrightarrow{i} E)$, their composition is represented by the transfer span over X, $(A \xleftarrow{fpr_B} B \times_C D \xrightarrow{ipr_D} E)$. We note that ipr_D is in $\mathcal{O}$ because pr_D is the pullback of g and $\mathcal{O}$ is pullback stable. Further, this composition is associative because we are considering isomorphism classes of transfer spans over X.

Notation 3.3. For objects (A, p_A) in $\mathcal{O}$-Lindner categories will usually just write A, suppressing p_A to improve readability.

Notation 3.4. When $\mathcal{O} = \mathrm{Set}^G$ we write $\mathcal{A}_X^G$ for $\mathcal{A}_X^{\mathcal{O}}$ and call it the **G-Lindner category** over X. More generally whenever $\mathcal{O} = \mathrm{Set}^G$ we will replace $\mathcal{O}$ with G.

Lemma 3.5. *Products and coproducts in $\mathcal{A}_X^{\mathcal{O}}$ are both given by disjoint union. The structure maps for the product $X_1 \amalg X_2$ are represented by $(X_1 \amalg X_2 \hookleftarrow X_i = X_i)$. The structure maps for the coproduct $X_1 \amalg X_2$ are represented by $(X_i = X_i \hookrightarrow X_1 \amalg X_2)$.*

The initial object in $\mathcal{A}_X^{\mathcal{O}}$ is $\emptyset$, which has the unique $\mathcal{O}$-transfer span to any other object, A, $(\emptyset = \emptyset \to A)$.

Remark 3.6. Let H be a subgroup of G, then given an indexing category $\mathcal{O}$ in Set^G we get an indexing category $i_H^*\mathcal{O}$ in Set^H which contains those morphisms $f \in \mathrm{Set}^H$ such that $G \times_H f \in \mathcal{O}$. We can apply the equivalence of categories $\mathrm{Set}^G_{G/H} \simeq \mathrm{Set}^H$ from Remark 2.4 to see that we also have an equivalence of categories $\mathcal{A}^{\mathcal{O}}_{G/H} \simeq \mathcal{A}^{i_H^*\mathcal{O}}_*$. Note that the objects in $\mathcal{A}^{i_H^*\mathcal{O}}_*$ are H-sets over a point and the morphisms are $i_H^*\mathcal{O}$-transfer spans, whose legs are H-equivariant maps.

There are a pair of functors

$$\begin{aligned} \mathcal{A}^{\mathcal{O}}_X \times \mathcal{A}^{\mathcal{O}}_Y &\to \mathcal{A}^{\mathcal{O}}_{X \amalg Y} \\ (A, B) &\mapsto (A \amalg B) \\ ((A \xleftarrow{\alpha} B \xrightarrow{\beta} C), (D \xleftarrow{\delta} E \xrightarrow{\gamma} F)) &\mapsto (A \amalg D \xleftarrow{\alpha \amalg \delta} B \amalg E \xrightarrow{\beta \amalg \gamma} C \amalg F) \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}^{\mathcal{O}}_{X \amalg Y} &\to \mathcal{A}^{\mathcal{O}}_X \times \mathcal{A}^{\mathcal{O}}_Y \\ (A) &\mapsto (p_A^{-1}(X), p_A^{-1}(Y)) \\ (A \xleftarrow{\alpha} B \xrightarrow{\beta} C) &\mapsto ((p_A^{-1}(X) \xleftarrow{\alpha} p_B^{-1}(X) \xrightarrow{\beta} p_C^{-1}(X)), (p_A^{-1}(Y) \xleftarrow{\alpha} p_B^{-1}(Y) \xrightarrow{\beta} p_C^{-1}(Y))). \end{aligned}$$

Lemma 3.7. *The two functors above are inverse to each other, defining an equivalence of categories $\mathcal{A}^{\mathcal{O}}_X \times \mathcal{A}^{\mathcal{O}}_Y \simeq \mathcal{A}^{\mathcal{O}}_{X \amalg Y}$.*

Under this equivalence a pair of $\mathcal{O}$-Mackey functors $(\underline{M}, \underline{N})$ in $\mathrm{Mack}^{\mathcal{O}}_X \times \mathrm{Mack}^{\mathcal{O}}_Y$ corresponds to an $\mathcal{O}$-Mackey functor which takes $S \amalg T \xrightarrow{\alpha \amalg \beta} X \amalg Y$ to $\underline{M}(S) \oplus \underline{N}(T) \cong \underline{M}(S) \times \underline{N}(T)$, we write $\underline{M \overline{\oplus} N}$ for this $\mathcal{O}$-Mackey functor over $X \amalg Y$.

Definition 3.8. A morphism $\varphi : X \to Y$ of G-sets the base change functors from Definition 2.3 can induce functors

$$\begin{tikzcd} \mathcal{A}^{\mathcal{O}}_Y \arrow[rr, "\Delta_\varphi"] & & \mathcal{A}^{\mathcal{O}}_X \arrow[ll, bend right, "\Sigma_\varphi"'] \arrow[ll, bend left, "\Pi_\varphi"] \end{tikzcd}$$

which agrees on objects with the original functors on slice categories of Set^G. On morphisms we apply the base of change functor to each leg of the $\mathcal{O}$-transfer span over X. The functors Σ_φ and Δ_φ are well-defined for any φ, however Π_φ is only well-defined if $f_* : \Pi_\varphi A \to \Pi_\varphi B, (\sigma, y) \mapsto (f\sigma, y)$ is in $\mathcal{O}$ for all $f : A \to B \in \mathcal{O}$. So for example we have

$$\Pi_\varphi : \left(\begin{tikzcd} A \arrow[d, "p_A"] & B \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "p_B"] & C \arrow[d, "p_C"] \\ X \arrow[r, equal] & X \arrow[r, equal] & X \end{tikzcd} \right) \mapsto \left(\begin{tikzcd} \Pi_\varphi(A, p_A) \arrow[d, "\pi_Y^A"] & \Pi_\varphi(B, p_B) \arrow[l, "f_*"'] \arrow[r, "g_*"] \arrow[d, "\pi_Y^B"] & \Pi_\varphi(C, p_C) \arrow[d, "\pi_Y^C"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd} \right).$$

It is immediate that Σ_φ is functorial as applying this functor to a span does not change the spanning maps, it changes the G-set the span is over. Given spans $(A \leftarrow B \rightarrow C)$ and $(C \leftarrow D \rightarrow E)$ we have $(B \times_Y X) \times_{(C \times_Y X)} (D \times_Y X) \cong (B \times_C D) \times_Y X$, along with the fact that pullbacks preserve identity morphisms we see that Σ_φ is functorial. Similarly, we have an isomorphism $\Pi_\varphi B \times_{\Pi_\varphi C} \Pi_\varphi D \to \Pi_\varphi (B \times_C D)$ given by

$$\big((\sigma, y), (\sigma', y')\big) \mapsto \big(x \mapsto (\sigma(x), \sigma'(x)), y\big)$$

(noting that the definition of the domain requires that $y = y'$) with inverse

$$(\omega, y) \mapsto ((pr_B\omega, y), (pr_D\omega, y)).$$

The following lemma is important because it allows us to define both the transfer functor and the restriction functor on categories of Mackey functors as left Kan extensions, recall that in the case of coefficient systems we defined the transfer functor as a right Kan extension. This greatly simplifies some proofs by allowing us to take advantage of Lemmas A.4 and A.5.

Lemma 3.9. *The functors Σ_φ and Δ_φ defined above form an ambidextrous adjunction. That is, they are both left and right adjoints to each other, $\Delta_\varphi \dashv \Sigma_\varphi \dashv \Delta_\varphi$.*

Proof. Given $A \in \mathcal{A}_Y^{\mathcal{O}}$ and $B \in \mathcal{A}_X^{\mathcal{O}}$ we have a natural isomorphism $\mathcal{A}_X^{\mathcal{O}}(\Delta_\varphi A, B) \cong \mathcal{A}_Y^{\mathcal{O}}(A, \Sigma_\varphi B)$ given by

$$\left(\begin{tikzcd} X \times_Y A \arrow[dr, "pr_X"'] & C \arrow[l, "f_1"'] \arrow[r, "f_2"] \arrow[d, "p_C"] & B \arrow[dl, "p_B"] \\ & X & \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} A \arrow[dr, "p_A"'] & C \arrow[l, "pr_A f_1"'] \arrow[r, "f_2"] \arrow[d, "\varphi p_C"] & B \arrow[dl, "\varphi p_B"] \\ & Y & \end{tikzcd}\right)$$

with inverse

$$\left(\begin{tikzcd} A \arrow[dr, "p_A"'] & C \arrow[l, "g_1"'] \arrow[r, "g_2"] \arrow[d, "p_C"] & B \arrow[dl, "\varphi p_B"] \\ & Y & \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} X \times_Y A \arrow[dr, "pr_X"'] & C \arrow[l, "(p_B g_2, g_1)"'] \arrow[r, "g_2"] \arrow[d, "p_B g_2"] & B \arrow[dl, "p_B"] \\ & X & \end{tikzcd}\right).$$

Similarly, there is a natural isomorphism $\mathcal{A}_Y^{\mathcal{O}}(\Sigma_\varphi B, A) \cong \mathcal{A}_X^{\mathcal{O}}(B, \Delta_\varphi A)$ given by

$$\left(\begin{tikzcd} B \arrow[dr, "\varphi p_B"'] & C \arrow[l, "f_1"'] \arrow[r, "f_2"] \arrow[d, "p_C"] & A \arrow[dl, "p_A"] \\ & Y & \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} B \arrow[dr, "p_B"'] & C \arrow[l, "f_1"'] \arrow[r, "(p_B f_1, f_2)"] \arrow[d, "p_B f_1"] & X \times_Y A \arrow[dl, "\varphi pr_X"] \\ & X & \end{tikzcd}\right)$$

with inverse

$$\left(\begin{tikzcd} B \arrow[dr, "p_B"'] & C \arrow[l, "g_1"'] \arrow[r, "g_2"] \arrow[d, "p_C"] & X \times_Y A \arrow[dl, "pr_X"] \\ & X & \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} B \arrow[dr, "\varphi p_B"'] & C \arrow[l, "g_1"'] \arrow[r, "pr_A g_2"] \arrow[d, "\varphi p_C"] & A \arrow[dl, "p_A"] \\ & Y & \end{tikzcd}\right).$$

$\square$

3.2 $\mathcal{O}$-Mackey functors

Definition 3.10. An **$\mathcal{O}$-semi-Mackey functor** $\underline{M}$ over X is a product-preserving functor $\mathcal{A}_X^{\mathcal{O}} \to \mathbf{Set}$. We write $\mathrm{sMack}_X^{\mathcal{O}}$ for the category of $\mathcal{O}$-semi-Mackey functors and natural transformations between them. As the product in $\mathcal{A}_X^{\mathcal{O}}$ is disjoint union and $\underline{M}$ is product-preserving, $\underline{M}(\emptyset)$ is always a singleton. For $A \in \mathcal{A}_X^{\mathcal{O}}$ the set $\underline{M}(A)$ is naturally a commutative monoid with unit map $\underline{M}(\emptyset = \emptyset \to A)$, and binary operation $\underline{M}(A \amalg A = A \amalg A \xrightarrow{\nabla} A)$. Here we are using that $\underline{M}$ is product-preserving so $\underline{M}(A \amalg A) \cong \underline{M}(A) \times \underline{M}(A)$. For any $\mathcal{O}$-transfer span $(A \leftarrow B \to C)$ over X the corresponding map $\underline{M}(A) \to \underline{M}(C)$ is a monoid homomorphism. Given a map $f : A \to B$ of finite G-sets we write tr_f for the map $\underline{M}(A = A \xrightarrow{f} B)$ from $\underline{M}(A)$ to $\underline{M}(B)$, and we write res_f for the map $\underline{M}(B \xleftarrow{f} A = A)$ from $\underline{M}(B)$ to $\underline{M}(A)$. We call tr_f the transfer along f, and res_f the restriction along f.

Definition 3.11. An **$\mathcal{O}$-Mackey functor**, $\underline{M}$, over X is an $\mathcal{O}$-semi-Mackey functor over X such that $\underline{M}(A)$ is an abelian group for all $A \in \mathcal{A}_X^{\mathcal{O}}$. We write $\mathrm{Mack}_X^{\mathcal{O}}$ for the category of $\mathcal{O}$-Mackey functors and natural transformations between them. We call an $\mathcal{O}$-Mackey functor an incomplete Mackey functor whenever $\mathcal{O} \neq \mathrm{Set}^G$. When G is the trivial group the category of G-Mackey functors is equivalent to the category of abelian groups, because such Mackey functors are determined up to isomorphism by the value they take on a singleton (and the spans which determine the structure maps).

Now we will describe the restriction, transfer, and norm functor constructions on (semi-)Mackey functors, and prove some basic properties about them. The transfer and restriction functors will play a key role in the main theorem of Chapter 3, and all three will play a similar role in the main result of Chapter 4.

Definition 3.12. Given a morphism $\varphi : X \to Y$ in Set^G and $\underline{M} \in \mathrm{sMack}_Y^{\mathcal{O}}$ we define the **restriction** $\mathcal{R}_\varphi \underline{M}$ of $\underline{M}$ along φ to be the left Kan extension of $\underline{M}$ along Δ_φ.

$$\begin{tikzcd} \mathcal{A}_Y^{\mathcal{O}} \arrow[r, "\underline{M}"] \arrow[d, "\Delta_\varphi"'] & Set \\ \mathcal{A}_X^{\mathcal{O}} \arrow[ur, "\mathrm{Lan}_{\Delta_\varphi} \underline{M} =: \mathcal{R}_\varphi \underline{M}"'] & \end{tikzcd}$$

Write $\alpha_{\underline{M}} : \underline{M} \to (\mathcal{R}_\varphi \underline{M})\Delta_\varphi$ for the universal map of the left Kan extension. Given a morphism $\underline{M} \xrightarrow{f} \underline{N}$ in $\mathrm{sMack}_Y^{\mathcal{O}}$ there is a morphism $\mathcal{R}_\varphi(f) : \mathcal{R}_\varphi \underline{M} \to \mathcal{R}_\varphi \underline{N}$ such that $\mathcal{R}_\varphi(f)\alpha_{\underline{M}} = \alpha_{\underline{N}} f$ by the universal property of $\alpha_{\underline{M}}$. This makes $\mathcal{R}_\varphi$ a functor $\mathrm{sMack}_Y^{\mathcal{O}} \to \mathrm{sMack}_X^{\mathcal{O}}$; we call this the **restriction functor** along φ.

When G is the trivial group a G-Mackey functor over X, $\underline{M}$, is an X-parameterised family of abelian groups. We write $\underline{M}(A)_x$ for the abelian group in $\underline{M}(A)$ corresponding to $x \in X$. In this case we have $\mathcal{R}_\varphi \underline{M}(A)_x = \underline{M}(A)_{\varphi(x)}$.

The restriction functor admits a convenient explicit description.

Proposition 3.13. *Given* $\underline{M} \in \mathrm{sMack}_Y^{\mathcal{O}}$ *and* $\varphi : X \to Y$ *in* Set^G *we have* $\mathcal{R}_\varphi \underline{M} = \underline{M}\Sigma_\varphi$.

Proof. This is a direct application of Yoneda reduction, setting $K = \Delta_\varphi$, $H = \Sigma_\varphi$ and $F = \underline{M}$ in Lemma A.2. □

Remark 3.14. The restriction functor could also have been defined using right Kan extensions because Σ_φ and Δ_φ form an ambidextrous adjunction. An alternative proof of the above proposition uses the objectwise formula for a right Kan extension which is described as an end rather than a coend.

Lemma 3.15. *Let* $\underline{M} \in \mathrm{sMack}_Y^{\mathcal{O}}$ *and* $\varphi : X \to Y$ *in* Set^G. *Then* $\mathcal{R}_\varphi \underline{M} \in \mathrm{sMack}_X^{\mathcal{O}}$.

Proof. As Σ_φ is a right adjoint it preserves limits and so in particular preserves products. So as a composition of product-preserving functors $\mathcal{R}_\varphi \underline{M}$ preserves products. □

The restriction functor takes Mackey functors to Mackey functors.

Lemma 3.16. *If* $\underline{M} \in \mathrm{Mack}_Y^{\mathcal{O}}$ *then* $\mathcal{R}_\varphi \underline{M} \in \mathrm{Mack}_X^{\mathcal{O}}$.

Proof. Given any $A \in \mathcal{A}_X^G$ we have $\mathcal{R}_\varphi \underline{M}(A) = \underline{M}(\Sigma_\varphi A) = \underline{M}(A, \varphi p_A)$ which is an abelian group since $\underline{M} \in \mathrm{Mack}_Y^{\mathcal{O}}$. □

Definition 3.17. Given a morphism $\varphi : X \to Y$ in Set^G and $\underline{M} \in \mathrm{sMack}_X^{\mathcal{O}}$ we define $\mathcal{T}_\varphi \underline{M}$ as the left Kan extension of $\underline{M}$ along Σ_φ.

$$\begin{tikzcd} \mathcal{A}_X^{\mathcal{O}} \arrow[r, "\underline{M}"] \arrow[d, "\Sigma_\varphi"'] & \mathbf{Set} \\ \mathcal{A}_Y^{\mathcal{O}} \arrow[ur, "\mathrm{Lan}_{\Sigma_\varphi} \underline{M} =: \mathcal{T}_\varphi \underline{M}"'] & \end{tikzcd}$$

Write $\alpha_{\underline{M}} : \underline{M} \to (\mathcal{T}_\varphi \underline{M})\Sigma_\varphi$ for the universal map of the left Kan extension. Given a morphism $\underline{M} \xrightarrow{f} \underline{N}$ in $\mathrm{sMack}_X^{\mathcal{O}}$ there is a unique morphism $\mathcal{T}_\varphi(f) : \mathcal{T}_\varphi \underline{M} \to \mathcal{T}_\varphi \underline{N}$ such that $\mathcal{T}_\varphi(f)\alpha_{\underline{M}} = \alpha_{\underline{N}} f$ by the universal property of $\alpha_{\underline{M}}$. This makes $\mathcal{T}_\varphi$ a functor $\mathrm{sMack}_X^{\mathcal{O}} \to \mathrm{sMack}_Y^{\mathcal{O}}$; we call this the **transfer functor** along φ.

When G is the trivial group and $\underline{M}$ is a G-Mackey functor over X we have $\mathcal{T}_\varphi \underline{M}(A)_y = \bigoplus_{x \in f^{-1}(y)} \underline{M}(A)_x$.

The transfer functor admits a convenient explicit description.

Proposition 3.18. *Given* $\underline{M} \in \mathrm{sMack}_X^{\mathcal{O}}$ *and* $\varphi : X \to Y$ *in* Set^G *we have* $\mathcal{T}_\varphi \underline{M} = \underline{M}\Delta_\varphi$.

Proof. This is a direct application of Yoneda reduction, setting $K = \Sigma_\varphi$, $H = \Delta_\varphi$ and $F = \underline{M}$ in Lemma A.2. $\square$

Lemma 3.19. *Let $\underline{M} \in \mathrm{sMack}_X^{\mathcal{O}}$ and $\varphi : X \to Y$ be in Set^G. Then $\mathcal{T}_\varphi \underline{M} \in \mathrm{sMack}_Y^{\mathcal{O}}$.*

Proof. As Δ_φ is a right adjoint it preserves limits and so in particular preserves products. So as a composition of product-preserving functors $\mathcal{T}_\varphi \underline{M}$ preserves products. $\square$

The transfer functor takes Mackey functors to Mackey functors.

Lemma 3.20. *If $\underline{M} \in \mathrm{Mack}_X^{\mathcal{O}}$ then $\mathcal{T}_\varphi \underline{M} \in \mathrm{Mack}_Y^{\mathcal{O}}$.*

Proof. Given $A \in \mathcal{A}_Y^{\mathcal{O}}$ we have $\mathcal{T}_\varphi \underline{M}(A) = \underline{M}(\Delta_\varphi A) = \underline{M}(X \times_Y A, pr_X)$ which is an abelian group as $\underline{M} \in \mathrm{Mack}_X^{\mathcal{O}}$. $\square$

Definition 3.21. Let $\varphi : X \to Y$ be a map of G-sets so that if $p_A : A \to X \in \mathcal{O}$ then $\pi_Y^A : \Pi_\varphi(A, p_A) \to Y, (y, \sigma) \mapsto y$ is also in $\mathcal{O}$, and let $\underline{M} \in \mathrm{sMack}_X^{\mathcal{O}}$. We define $\mathcal{N}_\varphi \underline{M}$ to be the left Kan extension of $\underline{M}$ along Π_φ.

$$\begin{tikzcd}
\mathcal{A}_X^{\mathcal{O}} \arrow[r, "\underline{M}"] \arrow[d, "\Pi_\varphi"'] & \mathbf{Set} \\
\mathcal{A}_Y^{\mathcal{O}} \arrow[ur, "\mathrm{Lan}_{\Pi_\varphi} \underline{M} =: \mathcal{N}_\varphi \underline{M}"']
\end{tikzcd}$$

This gives a functor $\mathcal{N}_\varphi : \mathrm{sMack}_X^{\mathcal{O}} \to \mathrm{sMack}_Y^{\mathcal{O}}$; we call this the **norm functor** along φ.

When G is the trivial group and $\underline{M}$ is a G-Mackey functor over X we have $\mathcal{N}_\varphi \underline{M}(A)_y = \bigotimes_{x \in f^{-1}(y)} \underline{M}(A)_x$.

Lemma 3.22. *Let $\underline{M} \in \mathrm{sMack}_X^{\mathcal{O}}$ and $\varphi : X \to Y$ in Set^G. Then $\mathcal{N}_\varphi \underline{M} \in \mathrm{sMack}_Y^{\mathcal{O}}$.*

Proof. By Lemma A.6 left Kan extensions of product-preserving functors are product-preserving. $\square$

The norm functor does not have a convenient explicit description along the lines of Propositions 3.13 and 3.18. However it does have an explicit description, by the objectwise formula for a left Kan extension we can write

$$\mathcal{N}_\varphi \underline{M}(A) \cong \int^{\mathcal{A}_X^{\mathcal{O}}} \mathcal{A}_Y^{\mathcal{O}}(\Pi_\varphi B, A) \times \underline{M}(B)\, dB.$$

Elements in this coend are represented by elements of the form $(\Pi_\varphi B \xleftarrow{k} E \xrightarrow{h} A, m)$ where the first coordinate is an $\mathcal{O}$-transfer span over Y and the second coordinate is an element of $\underline{M}(B)$. We write $[\Pi_\varphi B \xleftarrow{k} E \xrightarrow{h} A, m]$ for the element represented by $(\Pi_\varphi B \xleftarrow{k} E \xrightarrow{h} A, m)$.

Given a span $(S \xleftarrow{f} Q \xrightarrow{g} B) \in \mathcal{A}_X^G$ and $m \in \underline{M}(S)$ we say that

$$\begin{aligned}&((\Pi_\varphi B \xleftarrow{k} E \xrightarrow{h} A) \circ \Pi_\varphi(S \xleftarrow{f} Q \xrightarrow{g} B), m) = \\ &(\Pi_\varphi S \xleftarrow{(\Pi_\varphi(f))pr_{\Pi_\varphi Q}} \Pi_\varphi Q \times_{\Pi_\varphi B} E \xrightarrow{hpr_E} A, m) \in \mathcal{A}_Y^G(\Pi_\varphi S, A) \times \underline{M}(S)\end{aligned}$$

and

$$\begin{aligned}&(\Pi_\varphi B \xleftarrow{k} E \xrightarrow{h} A, \underline{M}(S \xleftarrow{f} Q \xrightarrow{g} B)(m)) = \\ &(\Pi_\varphi B \xleftarrow{k} E \xrightarrow{h} A, tr_g res_f(m)) \in \mathcal{A}_Y^G(\Pi_\varphi B, A) \times \underline{M}(B)\end{aligned}$$

are identified along

$$(\Pi_\varphi B \xleftarrow{k} E \xrightarrow{h} A, m) \in \mathcal{A}_Y^G(\Pi_\varphi B, A) \times \underline{M}(S)$$

via $(S \xleftarrow{f} Q \xrightarrow{g} B)$. So the two identified elements are representatives of the same element in the coend.

Recall that for $\underline{M} \in \mathrm{sMack}_X^{\mathcal{O}}$ and $A \in \mathcal{A}_X^{\mathcal{O}}$ the monoid structure is given by $tr_{\emptyset\to A} = \underline{M}(\emptyset = \emptyset \to A)$ for the unit map, and $tr_\nabla = \underline{M}(A \amalg A = A \amalg A \xrightarrow{\nabla} A)$ for the binary operation. The additive structure on $\mathcal{N}_\varphi \underline{M}(A)$ is given by

$$\mathcal{N}_\varphi \underline{M}(A) \times \mathcal{N}_\varphi \underline{M}(A) \xrightarrow{\cong} \mathcal{N}_\varphi \underline{M}(A \amalg A) \xrightarrow{tr_\nabla} \mathcal{N}_\varphi \underline{M}(A)$$

where the first map is the isomorphism from lemma A.6. On elements this map is given by

$$\begin{aligned}&([\Pi_\varphi B \xleftarrow{\alpha_1} C \xrightarrow{\alpha_2} A, b], [\Pi_\varphi D \xleftarrow{\beta_1} E \xrightarrow{\beta_2} A, d]) \\ &\mapsto [\Pi_\varphi(B \amalg D) \hookleftarrow (\Pi_\varphi B) \amalg (\Pi_\varphi D) \xleftarrow{\alpha_1 \amalg \beta_1} C \amalg E \xrightarrow{\alpha_2 \amalg \beta_2} A \amalg A, (b, d)] \\ &\mapsto [\Pi_\varphi(B \amalg D) \hookleftarrow (\Pi_\varphi B) \amalg (\Pi_\varphi D) \xleftarrow{\alpha_1 \amalg \beta_1} C \amalg E \xrightarrow{\nabla(\alpha_2 \amalg \beta_2)} A, (b, d)].\end{aligned}$$

Recalling that $\mathcal{N}_\varphi \underline{M}(\emptyset)$ is necessarily a singleton we know that $(\emptyset = \emptyset = \emptyset, *)$ is a representative of the only class. We can use this to pick out a representative of the additive identity in $\mathcal{N}_\varphi \underline{M}(A)$ using the unit map $\mathcal{N}_\varphi \underline{M}(\emptyset) \xrightarrow{tr_{\emptyset\to A}} \mathcal{N}_\varphi \underline{M}(A)$, under this map

$$[\emptyset = \emptyset = \emptyset, *] \mapsto [\emptyset = \emptyset \to A, *].$$

If 0_B is the identity element in $\underline{M}(B)$ then $[\Pi_\varphi B \leftarrow C \to A, 0_B] = [\emptyset = \emptyset \to A, *]$. We can identify these two representatives along $(\Pi_\varphi B \leftarrow C \to A, *)$ via the span $(\emptyset = \emptyset \to B)$. We also have $[\Pi_\varphi B \leftarrow \emptyset \to A, b] = [\emptyset \leftarrow \emptyset \to A, *]$, identified along $(\emptyset \leftarrow \emptyset \to A, b)$ via $(B \leftarrow \emptyset \to \emptyset)$.

The norm functor takes Mackey functors to Mackey functors.

Lemma 3.23. *If $\underline{M} \in \mathrm{Mack}_X^{\mathcal{O}}$ then $\mathcal{N}_\varphi \underline{M} \in \mathrm{Mack}_Y^{\mathcal{O}}$.*

Proof. It suffices to show that every element of $\mathcal{N}_\varphi \underline{M}(A)$ has an additive inverse. Let $[\Pi_\varphi B \leftarrow C \to A, b] \in \mathcal{N}_\varphi \underline{M}(A)$. Consider the pullback $P = \Pi_\varphi(B \amalg B) \times_{\Pi_\varphi B} C$ in the following diagram.

$$\begin{tikzcd} P \arrow[r] \arrow[d] & C \arrow[d] \\ \Pi_\varphi(B \amalg B) \arrow[r, "\nabla_*"'] & \Pi_\varphi(B) \end{tikzcd}$$

There are two subsets in P which are isomorphic to C as G-sets over Y. These correspond to elements $(y, \sigma) \in \Pi_\varphi(B \amalg B)$ where the image of σ in fully contained in one of the copies of $B \subset B \amalg B$. Let C' be the subset corresponding to those (y, σ) such that the image of σ is contained in the second copy of B. We have

$$\begin{aligned} &[\Pi_\varphi(B \amalg B) \leftarrow (P \setminus C') \to A, (-b, b)] + [\Pi_\varphi B \leftarrow C' \to A, b] \\ &= [\Pi_\varphi((B \amalg B) \amalg B) \hookleftarrow (\Pi_\varphi(B \amalg B)) \amalg (\Pi_\varphi B) \leftarrow (P \setminus C') \amalg C' \to A, ((-b, b), b)] \\ &= [\Pi_\varphi((B \amalg B) \amalg B) \leftarrow P \to A, ((-b, b), b)]. \end{aligned}$$

Now $[\Pi_\varphi((B \amalg B) \amalg B) \leftarrow P \to A, ((-b, b), b)] = [\Pi_\varphi(B \amalg B) \leftarrow P \to A, (-b, b)]$ by identification along

$$(\Pi_\varphi((B \amalg B) \amalg B) \leftarrow P \to A, (-b, b))$$

via

$$(B \amalg B \xleftarrow{id_B \amalg \nabla} B \amalg (B \amalg B) \cong (B \amalg B) \amalg B).$$

Finally we have $[\Pi_\varphi(B \amalg B) \leftarrow P \to A, (-b, b)] = [\Pi_\varphi B \leftarrow C \to A, (-b) + b = 0_B]$ by identification along $(\Pi_\varphi B \leftarrow C \to A, (-b, b))$. As shown above this is the identity element. Hence $[\Pi_\varphi(B \amalg B) \leftarrow (P \setminus C') \to A, (-b, b)]$ is the additive inverse for $[\Pi_\varphi B \leftarrow C' \to A, b]$. $\square$

If we consider the map $q : G/H \to *$ then the above description of additive inverses in $\mathcal{N}_q \underline{M}$ is an instance of in [13, Theorem 2.4]; Tambara reciprocity for sums. As we will see at the end of this chapter, $\mathcal{N}_q$ and $\mathcal{N}_H^G$ are the same functor across an equivalence of categories. The Tambara reciprocity formula in [13] is $N_H^G(a + b) = N_H^G(a) + N_H^G(b) + T(-)$ where $T(-)$ is a polynomial in transfers from proper subgroups, norms, and restrictions that depends only on G. In the case where $a = -b$ we have $N_H^G(b) = -N_H^G(-b) - T(-)$, we can think of $[\Pi_\varphi B \leftarrow C' \to A, b]$ as $N_H^G(b)$ and $[\Pi_\varphi(B \amalg B) \leftarrow (P \setminus C') \to A, (-b, b)]$ as $N_H^G(-b) + T(-)$ where $N_H^G(-b)$ corresponds to the subset of $\Pi_\varphi(B \amalg B)$ corresponding to the (y, σ) such that the image of σ is contained in the first copy of B and and $T(-)$ corresponds to the (y, σ) such that the image of σ is not contained in either copy of B.

Lemma 3.24. *Given $\varphi : X \to Y$ in Set^G the functors $\mathcal{T}_\varphi$ and $\mathcal{R}_\varphi$ form an ambidextrous adjunction. That is, they are both left and right adjoints to each other, $\mathcal{T}_\varphi \dashv \mathcal{R}_\varphi \dashv \mathcal{T}_\varphi$.*

Further the norm functor $\mathcal{N}_\varphi$ is left adjoint to precomposition with dependent product functor, Π_φ^*.

Proof. From Propositions 3.18 and 3.13 we know that the $\mathcal{T}_\varphi$ and $\mathcal{R}_\varphi$ are naturally isomorphic to precomposition by Δ_φ and precomposition by Σ_φ respectively. Noting that $\mathcal{T}_\varphi$ and $\mathcal{R}_\varphi$ are defined as $\mathrm{Lan}_{\Sigma_\varphi}$ and $\mathrm{Lan}_{\Delta_\varphi}$ respectively, the first claim is an application of Lemma A.7. The second claim is an immediate application of Lemma A.7 as $\mathcal{N}_\varphi$ is defined as $\mathrm{Lan}_{\Pi_\varphi}$. □

Notation 3.25. For any finite G-set X we will write $\mathcal{T}_X$, $\mathcal{R}_X$, and $\mathcal{N}_X$ for the transfer, restriction, and norm functors respectively along the map $X \to *$ to a point.

The following two propositions take advantage of being able to define the transfer, restriction, and norm functors as left Kan extensions. In particular, in the case of the transfer functor this is a consequence of Lemma 3.9.

Proposition 3.26. *Let*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "pr_Y"] \arrow[d, "pr_X"'] & Y \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"'] & Z \end{tikzcd}$$

be a pullback square in Set^G. *Then the functors $\mathcal{R}_\beta\mathcal{T}_\alpha$ and $\mathcal{T}_{pr_Y}\mathcal{R}_{pr_X}$ are naturally isomorphic. Similarly, the functors $\mathcal{R}_\beta\mathcal{N}_\alpha$ and $\mathcal{N}_{pr_Y}\mathcal{R}_{pr_X}$ are naturally isomorphic.*

Proof. Composing natural isomorphisms from Propositions 3.13 and 3.18, and Lemma 2.6 we have $\mathcal{R}_\beta\mathcal{T}_\alpha\underline{M} \cong \underline{M}\Delta_\beta\Sigma_\alpha \cong \underline{M}\Sigma_{pr_Y}\Delta_{pr_X} \cong \mathcal{T}_{pr_Y}\mathcal{R}_{pr_X}\underline{M}$.
By Lemma A.5 we have that

$$\mathcal{R}_\beta\mathcal{N}_\alpha\underline{M} = \mathrm{Lan}_{\Delta_\beta}(\mathrm{Lan}_{\Pi_\alpha}\underline{M}) \cong \mathrm{Lan}_{\Delta_\beta\,\Pi_\alpha}\underline{M}$$

and

$$\mathcal{N}_{pr_Y}\mathcal{R}_{pr_X}\underline{M} = \mathrm{Lan}_{\Pi_{pr_Y}}(\mathrm{Lan}_{\Delta_{pr_X}}\underline{M}) \cong \mathrm{Lan}_{\Pi_{pr_Y}\,\Delta_{pr_X}}\underline{M}.$$

We have $\Delta_\beta\,\Pi_\alpha \cong \Pi_{pr_Y}\,\Delta_{pr_X}$ by Lemma 2.6, hence by Lemmas A.4 and A.5,

$$\mathcal{R}_\beta\mathcal{N}_\alpha\underline{M} \cong \mathrm{Lan}_{\Delta_\beta\,\Pi_\alpha}\underline{M} \cong \mathrm{Lan}_{\Pi_{pr_Y}\,\Delta_{pr_X}}\underline{M} \cong \mathcal{N}_{pr_Y}\mathcal{R}_{pr_X}\underline{M}.$$ □

Proposition 3.27. *Let*

$$\begin{tikzcd} Y \times_Z \Pi_g X \arrow[rr, "pr_{\Pi_g X}"] \arrow[d, "ev"'] & & \Pi_g X \arrow[d, "\pi_Z"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

be an exponential diagram in Set^G*. Then the functors* $\mathcal{T}_{\pi_Z}\mathcal{N}_{pr_{\Pi_g X}}\mathcal{R}_{ev}$ *and* $\mathcal{N}_g\mathcal{T}_f$ *are naturally isomorphic.*

Proof. We have $\Sigma_{\pi_Z}\Pi_{pr_{\Pi_g X}}\Delta_{ev} \cong \Pi_g\Sigma_f$ by lemma 2.9, hence by lemmas A.4 and A.5,

$$\mathcal{T}_{\pi_Z}\mathcal{N}_{pr_{\Pi_g X}}\mathcal{R}_{ev}\underline{M} \cong \mathrm{Lan}_{\Sigma_{\pi_Z}\Pi_{pr_{\Pi_g X}}\Delta_{ev}}\underline{M} \cong \mathrm{Lan}_{\Pi_g\Sigma_f}\underline{M} \cong \mathcal{N}_g\mathcal{T}_f\underline{M}. \qquad \square$$

There are two symmetric monoidal structures on the category of semi-Mackey functors that come from Day convolution.

Definition 3.28. Given a pair $\underline{M}$ and $\underline{N}$ of $\mathcal{O}$-semi-Mackey functors over X we define their **sum** $\underline{M}\oplus\underline{N}$ to be the left Kan extension of the functor $\underline{M}\times\underline{N}$ along the functor $\amalg : \mathcal{A}_X^{\mathcal{O}}\times\mathcal{A}_X^{\mathcal{O}} \to \mathcal{A}_X^{\mathcal{O}}$.

$$\begin{tikzcd} \mathcal{A}_X^{\mathcal{O}}\times\mathcal{A}_X^{\mathcal{O}} \arrow[r, "(\underline{M},\underline{N})"] \arrow[d, "\amalg"'] & \mathbf{Set}\times\mathbf{Set} \arrow[r, "\times"] & \mathbf{Set} \\ \mathcal{A}_X^{\mathcal{O}} \arrow[urr, "\mathrm{Lan}_{\amalg}(\underline{M}\times\underline{N})=:\underline{M}\oplus\underline{N}"'] & & \end{tikzcd}$$

As $\underline{M}\times\underline{N}$ is product-preserving, by Lemma A.6 $\underline{M}\oplus\underline{N}$ is also product-preserving. Hence this defines a functor $\oplus : \mathrm{sMack}_X^{\mathcal{O}}\times\mathrm{sMack}_X^{\mathcal{O}} \to \mathrm{sMack}_X^{\mathcal{O}}$.
We define

$$\underline{M}\overline{\oplus}\underline{N} : \mathcal{A}_{X\amalg X}^{\mathcal{O}} \xrightarrow[\simeq]{\Omega} \mathcal{A}_X^{\mathcal{O}}\times\mathcal{A}_X^{\mathcal{O}} \xrightarrow{\underline{M}\times\underline{N}} \mathbf{Set}$$

where Ω is the equivalence from Lemma 3.7. Note that the functors involved are all product-preserving, so $\underline{M}\overline{\oplus}\underline{N} \in \mathrm{sMack}_{X\amalg X}^{\mathcal{O}}$. Moreover, $(\underline{M}\overline{\oplus}\underline{N})(A) = \underline{M}(p_A^{-1}(X_1)) \oplus \underline{N}(p_A^{-1}(X_2))$ where the index determines a copy of X in the coproduct. Hence if $\underline{M},\underline{N} \in \mathrm{Mack}_X^{\mathcal{O}}$ then $\underline{M}\overline{\oplus}\underline{N} \in \mathrm{Mack}_{X\amalg X}^{\mathcal{O}}$.

We can realise the sum operation on semi-Mackey functors as application of the transfer functor along the fold map.

Proposition 3.29. *The functors* $\underline{M}\oplus\underline{N}$ *and* $\mathcal{T}_\nabla(\underline{M}\overline{\oplus}\underline{N})$ *are naturally isomorphic.*

Proof. We have that $\amalg\Omega$ and Σ_∇ are naturally isomorphic functors $\mathcal{A}_{X\amalg X}^{\mathcal{O}} \to \mathcal{A}_X^{\mathcal{O}}$. The result follows from the fact that Ω is an equivalence and Lemmas A.4 and A.5. $\square$

We can explicitly describe the sum of two semi-Mackey functors.

Lemma 3.30. *For* $\underline{M},\underline{N} \in \mathrm{sMack}_X^{\mathcal{O}}$ *we have* $(\underline{M}\oplus\underline{N})(A) = \underline{M}(A)\oplus\underline{N}(A)$.

Proof. We have

$$\begin{aligned} \mathcal{T}_\nabla(\underline{M}\overline{\oplus}N)(A) &= (\underline{M}\overline{\oplus}\underline{N})(\Delta_\nabla A) = (\underline{M}\oplus\underline{N})(A\amalg A) \\ &= (\underline{M}\times\underline{N})(A,A) = \underline{M}(A)\times\underline{N}(A) \\ &= \underline{M}(A)\oplus\underline{N}(A). \end{aligned} \qquad \square$$

The category of Mackey functors is closed under the sum operation.

Lemma 3.31. *If* $\underline{M}, \underline{N} \in \mathrm{Mack}_X^{\mathcal{O}}$ *then* $\underline{M} \oplus \underline{N} \in \mathrm{Mack}_X^{\mathcal{O}}$.

Proof. As $\underline{M} \oplus \underline{N}$ and $\mathcal{T}_\nabla(\underline{M}\overline{\oplus}\underline{N})$ are naturally isomorphic and $\underline{M}\overline{\oplus}\underline{N} \in \mathrm{Mack}_{X\amalg X}^{\mathcal{O}}$ we have that $\underline{M} \oplus \underline{N}$ is an $\mathcal{O}$-Mackey functor over X by Lemma 3.20. □

Notice that if we let $\underline{0}$ be the $\mathcal{O}$-Mackey functor over X with $\underline{0}(A) = 0$ for all $A \in \mathcal{A}_X^{\mathcal{O}}$ then $\underline{0}$ acts as an identity with respect to $\oplus$. This is what we should expect from Day convolution as $\underline{0}$ is isomorphic to the functor $\mathcal{A}_X^{\mathcal{O}} \to \mathbf{Set}, A \mapsto \mathcal{A}_X^{\mathcal{O}}[\emptyset, A] \cong \{0\}$ corepresented by $\emptyset$ which is the identity in $\mathcal{A}_X^{\mathcal{O}}$ with respect to $\amalg$. As $\underline{M} \oplus \underline{N}$ was defined by Day convolution [7] of the symmetric monoidal category $(\mathcal{A}_X^{\mathcal{O}}, \amalg, \emptyset)$ we have the following.

Lemma 3.32. *Both* $(\mathrm{sMack}_X^{\mathcal{O}}, \oplus, \underline{0})$ *and* $(\mathrm{Mack}_X^{\mathcal{O}}, \oplus, \underline{0})$ *are symmetric monoidal categories.*

Definition 3.33. Given $\mathcal{O}$-semi-Mackey functors over X, $\underline{M}$ and $\underline{N}$, we define their **box product** $\underline{M}\square\underline{N}$ to be the left Kan extension of the functor $\underline{M} \times \underline{N}$ along the pullback over X functor $\times_X : \mathcal{A}_X^{\mathcal{O}} \times \mathcal{A}_X^{\mathcal{O}} \to \mathcal{A}_X^{\mathcal{O}}$.

$$\begin{tikzcd}
\mathcal{A}_X^{\mathcal{O}} \times \mathcal{A}_X^{\mathcal{O}} \arrow[r, "(\underline{M},\underline{N})"] \arrow[d, "\times_X"'] & \mathbf{Set} \times \mathbf{Set} \arrow[r, "\times"] & \mathbf{Set} \\
\mathcal{A}_X^{\mathcal{O}} \arrow[urr, "\mathrm{Lan}_{\times_X}(\underline{M}\times\underline{N}) =: \underline{M}\square\underline{N}"'] & &
\end{tikzcd}$$

As $\underline{M} \times \underline{N}$ is product-preserving, as a left Kan extension of a product-preserving functor $\underline{M} \oplus \underline{N}$ is also product-preserving by Lemma A.6. Hence this defines a functor $\square : \mathrm{sMack}_X^{\mathcal{O}} \times \mathrm{sMack}_X^{\mathcal{O}} \to \mathrm{sMack}_X^{\mathcal{O}}$.

The box product on semi-Mackey functors can be realised as an application of the norm functor along the fold map.

Proposition 3.34. *The functors* $\underline{M}\square\underline{N}$ *and* $\mathcal{N}_\nabla(\underline{M}\overline{\oplus}\underline{N})$ *are naturally isomorphic.*

Proof. We have that $\times_X \Omega$ and Π_∇ are naturally isomorphic functors $\mathcal{A}_{X\amalg X}^{\mathcal{O}} \to \mathcal{A}_X^{\mathcal{O}}$. The result follows from the fact that Ω is an equivalence and lemmas A.4 and A.5. □

The category of Mackey functors is closed under taking box products.

Lemma 3.35. *If* $\underline{M}, \underline{N} \in \mathrm{Mack}_X^{\mathcal{O}}$ *then* $\underline{M}\square\underline{N} \in \mathrm{Mack}_X^{\mathcal{O}}$.

Proof. As $\underline{M}\square\underline{N}$ and $\mathcal{N}_\nabla(\underline{M}\overline{\oplus}\underline{N})$ are naturally isomorphic and $\underline{M}\overline{\oplus}\underline{N} \in \mathrm{Mack}_{X\amalg X}^{\mathcal{O}}$ the result follows by Lemma 3.23. □

Remark 3.36. When G is the trivial group and $\underline{M}$ and $\underline{N}$ are G-Mackey functors the left Kan extension in Definition 3.33 is equivalent to the left Kan extension for the tensor product of abelian groups, so that $(\underline{M}\square\underline{N})(*) = \underline{M}(*) \otimes \underline{N}(*)$.

As with the sum operation there is an identity semi-Mackey functor with respect to the box product, however in the case of the box product this unit is not a Mackey functor.

Definition 3.37. The **group completion** $\mathrm{Gr}(\underline{M})$ of an $\mathcal{O}$-semi-Mackey functor $\underline{M}$ over X is the $\mathcal{O}$-Mackey functor over X with $\mathrm{Gr}(\underline{M})(C) = \mathrm{Gr}(\underline{M}(C))$, the group completion of the commutative monoid $\underline{M}(C)$.

Definition 3.38. We call the $\mathcal{O}$-semi-Mackey functor over X that is corepresented by X the **Burnside $\mathcal{O}$-semi-Mackey functor** over X,

$$\underline{B}_X : \mathcal{A}_X^{\mathcal{O}} \to \mathbf{Set}, Y \mapsto \mathcal{A}_X^{\mathcal{O}}(X, Y).$$

The **Burnside $\mathcal{O}$-Mackey functor** over X is the group completion of $\underline{B}_X$,

$$\underline{A}_X = \mathrm{Gr}(\underline{B}_X) : \mathcal{A}_X^{\mathcal{O}} \to \mathbf{Set}, Y \mapsto \mathrm{Gr}(\mathcal{A}^{\mathcal{O}}(X, Y))$$

As $\underline{M}\square\underline{N}$ was defined by Day convolution $(\mathrm{sMack}_X^{\mathcal{O}}, \square, \underline{B}_X)$ is a symmetric monoidal category. It can also be shown that $(\mathrm{Mack}_X^{\mathcal{O}}, \square, \underline{A}_X)$ is a symmetric monoidal category.

As with coefficient systems there is an equivalent definition for Mackey functors where one only considers the full subcategory of $\mathcal{A}_*^{\mathcal{O}}$ containing the finite G-sets G/H where H is a subgroup of G. These definitions are useful for explicit calculations but make proving general statements artificially difficult.

Let H be a subgroup of G. Following Remark 3.6 we have that $\mathcal{O}$-Mackey functors over G/H are equivalent to $i_H^*\mathcal{O}$-Mackey functors. In particular, G-Mackey functors over G/H are equivalent to H-Mackey functors. Across the equivalence $\mathcal{A}_{G/H}^{\mathcal{O}} \simeq \mathcal{A}_*^{i_H^*\mathcal{O}}$ with the quotient map $\varphi : G/H \to G/G$ we recover functors $N_H^G : \mathrm{Mack}^H \to \mathrm{Mack}^G$ from $\mathcal{N}_\varphi$, $T_H^G : \mathrm{Mack}^H \to \mathrm{Mack}^G$ from $\mathcal{T}_\varphi$, and $R_H^G : \mathrm{Mack}^G \to \mathrm{Mack}^H$ from $\mathcal{R}_\varphi$, as left Kan extensions along the functors $Map_H(G, -)$, $G \times_H -$, and i_H^G respectively. In particular, the norm defined by Hoyer on Mackey functors [14, Definition 2.3.2] and the box product on Mackey functors are both instances of our norm functor, along the quotient map and the fold map respectively. That the norms N_H^G are strong monoidal with respect to the box product is now an immediate consequence of Lemma A.5.

3.3 Bicategorical Setup

In the following bicategorical framework it is important to fix a specific choice of pullback object for every pullback diagram, so for any pair morphisms $\alpha : X \to Z$ and

$\beta : Y \to Z$ in Set^G when we write $X \times_Z Y$ we mean the set of ordered pairs (x, y) such that $x \in X, y \in Y$ and $\alpha(x) = \beta(y)$ together with projection maps $pr_X : (x, y) \mapsto x$ and $pr_Y : (x, y) \mapsto y$.

Definition 3.39. The **$\mathcal{O}$-Burnside bicategory** for a $\mathcal{O}$ a wide, pullback stable, finite-coproduct complete subcategory of Set^G is a bicategory $\underline{\mathcal{O}}$ with

- **Objects:** The 0-cells of the bicategory are finite G-sets
- **Hom Categories:** For X, Y a pair of finite G-sets $\underline{\mathcal{O}}(X, Y)$ is the category whose objects (1-cells of the bicategory) are $\mathcal{O}$-transfer spans $X \leftarrow A \to Y$ and whose morphisms (2-cells of the bicategory) are isomorphisms of $\mathcal{O}$-transfer spans. That is, a morphism $\psi : (X \xleftarrow{f} A \xrightarrow{g} Y) \to (X \xleftarrow{f'} A \xrightarrow{g'} Y)$ is an isomorphism $\psi : A \to B$ such that the following diagram commutes.

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, "\psi"] & \\ X & B \arrow[l, "f'"] \arrow[r, "g'"'] & Y \end{tikzcd}$$

  Composition in $\underline{\mathcal{O}}(X, Y)$ (called vertical composition) is composition of the underlying morphisms of G-sets. The identity 2-cell of an $\mathcal{O}$-transfer span $(X \xleftarrow{f} A \xrightarrow{g} Y)$ is id_A.

- **Identity 1-cells:** The identity 1-cell of a finite G-set X is the $\mathcal{O}$-transfer span $1_X(*) = (X = X = X)$.
- **Horizontal Composition:** For each triple of G-sets, X, Y, Z we have a horizontal composition functor $c_{X,Y,Z} : \underline{\mathcal{O}}(Y, Z) \times \underline{\mathcal{O}}(X, Y) \to \underline{\mathcal{O}}(X, Z)$. On 1-cells we have

$$c_{X,Y,Z} : \Big((Y \xleftarrow{\gamma} B \xrightarrow{\delta} Z), (X \xleftarrow{\alpha} A \xrightarrow{\beta} Y)\Big) \mapsto \Big(X \xleftarrow{\alpha pr_A} A \times_Y B \xrightarrow{\delta pr_B} Z\Big)$$

  and on 2-cells we have

$$c_{X,Y,Z} : \left(\begin{tikzcd} & B \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] \arrow[d, "\omega"] & \\ Y & B' \arrow[l, "\gamma'"] \arrow[r, "\delta'"'] & Z \end{tikzcd} , \begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[d, "\psi"] & \\ X & B \arrow[l, "\alpha'"] \arrow[r, "\beta'"'] & Y \end{tikzcd} \right) \mapsto \left(\begin{tikzcd} & A \times_Y B \arrow[dl, "\alpha pr_A"'] \arrow[dr, "\delta pr_B"] \arrow[d, "(\omega,\psi)"] & \\ X & A' \times_Y B' \arrow[l, "\alpha' pr_{A'}"] \arrow[r, "\delta' pr_{B'}"'] & Z \end{tikzcd} \right).$$

- **Associator:** For each quadruple of G-sets, W, X, Y, Z there is a natural isomorphism, called the associator,

$$a_{W,X,Y,Z} : c_{W,X,Z}(c_{X,Y,Z} \times \mathrm{id}_{\underline{\mathcal{O}}(W,X)}) \implies c_{W,Y,Z}(\mathrm{id}_{\underline{\mathcal{O}}(Y,Z)} \times c_{W,X,Y})$$

  of functors

$$\underline{\mathcal{O}}(Y, Z) \times \underline{\mathcal{O}}(X, Y) \times \underline{\mathcal{O}}(W, X) \to \underline{\mathcal{O}}(W, Z)$$

whose component on the triple

$$((Y \leftarrow C \rightarrow Z), (X \leftarrow B \rightarrow Y), (W \leftarrow A \rightarrow X))$$

is the invertible 2-cell is given by the isomorphism

$$A \times_X (B \times_Y C) \rightarrow (A \times_X B) \times_Y C, (a, (b, c)) \mapsto ((a, b), c).$$

- **Unitors:** For each pair of G-sets X, Y there are natural isomorphisms

$$\begin{aligned} l_{X,Y} : c_{X,Y,Y}(1_Y \times \mathrm{id}_{\underline{\mathcal{O}}(X,Y)}) &\implies \mathrm{id}_{\underline{\mathcal{O}}(X,Y)} \\ r_{X,Y} : c_{X,X,Y}(\mathrm{id}_{\underline{\mathcal{O}}(X,Y)} \times 1_X) &\implies \mathrm{id}_{\underline{\mathcal{O}}(X,Y)} \end{aligned}$$

called the left unitor and right unitor respectively, whose components on the $\mathcal{O}$-transfer span $(X \leftarrow A \rightarrow Y)$ are the invertible 2-cells given by the isomorphisms $A \times_Y Y \rightarrow A, (a, y) \mapsto a$ and $X \times_X A \rightarrow A, (x, a) \mapsto a$ respectively.

It is straightforward to check that the unity and pentagon axioms are satisfied.

Notation 3.40. Given an $\mathcal{O}$-transfer span $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ we write Λ_f for the functor $\mathcal{T}_{f_2}\mathcal{R}_{f_1} : \mathrm{Mack}_X^{\mathcal{O}} \rightarrow \mathrm{Mack}_Y^{\mathcal{O}}$.

Definition 3.41. Let CAT be the bicategory of categories, functors, and natural transformations and let $\mathcal{O}$ and $\mathcal{O}'$ be wide, pullback stable, finite-coproduct complete subcategories of Set^G such that $\mathcal{O}$ is a subcategory of $\mathcal{O}'$. We define the pseudofunctor $(\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}, (\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'})^2, (\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'})^0) : \underline{\mathcal{O}'} \rightarrow \mathrm{CAT}$ to be the triple consisting of the following data

- **Objects:** The assignment $\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}(X) = \mathrm{Mack}_X^{\mathcal{O}}$ for all finite G-sets X.
- **Hom Categories:** For each pair of G-sets X, Y the functor

$$\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'} : \underline{\mathcal{O}'}(X, Y) \rightarrow \mathrm{CAT}(\mathrm{Mack}_X^{\mathcal{O}}, \mathrm{Mack}_Y^{\mathcal{O}})$$

with

$$\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}(X \xleftarrow{f_1} A \xrightarrow{f_2} Y) = \Lambda_f : \mathrm{Mack}_X^{\mathcal{O}} \rightarrow \mathrm{Mack}_Y^{\mathcal{O}}$$

and

$$\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}\left(\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] \arrow[dd, "\psi"] & \\ X & & Y \\ & B \arrow[ul, "f_1'"] \arrow[ur, "f_2'"'] & \end{tikzcd}\right) = \left(\hat{\psi} : \Lambda_f \rightarrow \Lambda_{f'}\right)$$

where $\hat{\psi}$ has components $\hat{\psi}_{\underline{M}}(B) : \Lambda_f \underline{M}(B) \rightarrow \Lambda_{f'} \underline{M}(B)$ given by

$$[(C, f_2 p_C) \xleftarrow{g_1} D \xrightarrow{g_2} B, c] \mapsto [(C, f_2' \psi p_C) \xleftarrow{g_1} D \xrightarrow{g_2} B, c].$$

- **Pseudo functoriality constraint:** Write f and g to denote $\mathcal{O}'$-transfer spans $(X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ and $(Y \xleftarrow{g_1} B \xrightarrow{g_2} Z)$ respectively. The pseudo functoriality constraint is a natural isomorphism $(\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'})^2$ with component 2-cells given by the composite

$$\Lambda_g \Lambda_f = \mathcal{T}_{g_2}\mathcal{R}_{g_1}\mathcal{T}_{f_2}\mathcal{R}_{f_1} \to \mathcal{T}_{g_2}\mathcal{T}_{pr_B}\mathcal{R}_{pr_A}\mathcal{R}_{f_1} \to \mathcal{T}_{g_2 pr_B}\mathcal{R}_{f_1 pr_A} = \Lambda_{gf}$$

  where the first map is the natural isomorphism described in Proposition 3.26 and the second is two applications of the fact that left Kan extension respects composition, see Lemma A.5.

- **Pseudo unit constraint:** A natural isomorphism, $(\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'})^0$, with component 2-cells $\mathrm{id}_{[\mathrm{Mack}_X^{\mathcal{O}}, \mathrm{Mack}_X^{\mathcal{O}}]} \to \Lambda_{\mathrm{id}_X}$ which have components $\underline{M} \xrightarrow{\cong} \Lambda_{\mathrm{id}_X}\underline{M}$. On objects this is given by $\underline{M}(V \xleftarrow{\sigma} X \times_X V = X \times_X V)$ where σ is the isomorphism $X \times_X V \to V, (x, v) \mapsto v$.

- **Pseudo associativity and unity axioms:** For $f = (W \xleftarrow{f_1} A \xrightarrow{f_2} X)$, $g = (X \xleftarrow{g_1} B \xrightarrow{g_2} Y)$, and $h = (Y \xleftarrow{h_1} C \xrightarrow{h_2} Z)$, the pseudo associativity axiom is satisfied because the associator in $\underline{\mathcal{O}}$ is the natural isomorphism between the functors along which $\Lambda_{(hg)f}$ and $\Lambda_{h(gf)}$ are left Kan extensions. It is straightforward to check that the pseudo unity axioms are satisfied.

Definition 3.42. The Grothendieck construction on $\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}$ is the bicategory $\int \mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}$ with

- **Objects:** The 0-cells of the bicategory are pairs $(X, \underline{M})$ where X is a finite G-set and $\underline{M}$ is an $\mathcal{O}$-Mackey functor over X.

- **Hom Categories:** For a pair of 0-cells $(X, \underline{M})$ and $(Y, \underline{N})$ the category

$$\int \mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}((X, \underline{M}), (Y, \underline{N}))$$

  has objects (1-cells of the bicategory) pairs (f, α) where f is an $\mathcal{O}'$-transfer span $X \xleftarrow{f_1} A \xrightarrow{f_2} Y$ and $\alpha : \Lambda_f \underline{M} \to \underline{N}$ is a morphism in $\mathrm{Mack}_Y^{\mathcal{O}}$ and for morphisms (2-cells of the bicategory) $(f, \alpha) \to (g, \beta)$ has isomorphisms ψ of $\mathcal{O}'$-transfer spans such that the following diagram commutes.

$$\begin{tikzcd} \Lambda_f \underline{M} \arrow[rr, "\widehat{\psi}_{\underline{M}}"] \arrow[dr, "\alpha"'] & & \Lambda_g \underline{M} \arrow[dl, "\beta"] \\ & \underline{N} & \end{tikzcd} \tag{3.43}$$

- **Identity 1-cells:** The identity 1-cell of a pair $(X, \underline{M})$ is the pair

$$(1_X(*), ((\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'})^0)^{-1}_{\underline{M}})$$

where $1_X(*)$ is the span $X = X = X$ (that is, the identity 1-cell of X in $\underline{\mathcal{O}'}$) and $((\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'})^0)^{-1}_{\underline{M}}$ is the $\underline{M}$ component of the inverse of the pseudo unity constraint for $\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}$.

- **Horizontal Composition:** For each triple of 0-cells $(X, \underline{M})$, $(Y, \underline{N})$, and $(Z, \underline{P})$ we have a horizontal composition functor $c_{(X,\underline{M}),(Y,\underline{N}),(Z,\underline{P})}$:

$$\int \mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}((Y, \underline{N}), (Z, \underline{P})) \times \int \mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}((X, \underline{M}), (Y, \underline{N})) \to \int \mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}((X, \underline{M}), (Z, \underline{P})) .$$

For $(f, \alpha) \in \int \mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}((X, \underline{M}), (Y, \underline{N}))$ and $(g, \beta) \in \int \mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}((Y, \underline{N}), (Z, \underline{P}))$ a pair of 1-cells, the functor $c_{(X,\underline{M}),(Y,\underline{N}),(Z,\underline{P})}$ returns the pair

$$(c_{X,Y,Z}(g, f), \beta(\Lambda_g\alpha)((\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'})^2_{g,f})^{-1}_{\underline{M}})$$

where $c_{X,Y,Z}$ is the horizontal composition functor in $\underline{\mathcal{O}'}$ and $\beta(\Lambda_g\alpha)((\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'})^2_{g,f})^{-1}_{\underline{M}}$ is the composite map

$$\Lambda_{gf}\underline{M} \xrightarrow{((\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'})^2_{g,f})^{-1}_{\underline{M}}} \Lambda_g\Lambda_f\underline{M} \xrightarrow{\Lambda_g\alpha} \Lambda_g\underline{N} \xrightarrow{\beta} \underline{P}.$$

- **Associator:** The components of the associator $a_{(W,\underline{L}),(X,\underline{M}),(Y,\underline{N}),(Z,\underline{P})}$ are 2-cells which, on a given triple of composable 1-cells

$$((W \xleftarrow{f_1} A \xrightarrow{f_2} X), \alpha), \qquad ((X \xleftarrow{g_1} B \xrightarrow{g_2} Y), \beta), \qquad ((Y \xleftarrow{h_1} C \xrightarrow{h_2} Z), \gamma)$$

is the associator $a_{W,X,Y,Z}$ in $\underline{\mathcal{O}'}$. That is, the invertible 2-cell between $\mathcal{O}'$-transfer spans $\psi : A \times_X (B \times_Y C) \to (A \times_X B) \times_Y C, (a, (b, c)) \mapsto ((a, b), c)$. Our definition of 2-cells requires Diagram 3.43 to commute, for the associator this is the outside of the following diagram.

$$\begin{tikzcd}
\Lambda_{(hg)f}\underline{L} \arrow[rr, "\hat{\psi}_{\underline{L}}", "\cong"'] \arrow[d, "\cong"'] & & \Lambda_{h(gf)}\underline{L} \arrow[d, "\cong"] \\
\Lambda_{hg}\Lambda_f\underline{L} \arrow[r, "\cong"] \arrow[d, "\Lambda_{hg}\alpha"'] & \Lambda_h\Lambda_g\Lambda_f\underline{L} \arrow[d, "\Lambda_h\Lambda_g\alpha"] & \Lambda_h\Lambda_{gf}\underline{L} \arrow[l, "\cong"'] \arrow[ddl, "\Lambda_h(\beta\circ(\Lambda_g\alpha)\circ((\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'})^2_{g,f})^{-1}_{\underline{L}})"] \\
\Lambda_{hg}\underline{M} \arrow[r, "\cong"] & \Lambda_h\Lambda_g\underline{M} \arrow[d, "\Lambda_h\beta"'] & \\
& \Lambda_h\underline{N} \arrow[d, "\gamma"] & \\
& \underline{P} &
\end{tikzcd}$$

The pentagon is the pseudo associativity diagram for $\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}$ (backwards). The square is a naturality square for $(\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'})^2_{h,g}$. The triangle is functoriality of Λ_h.

- **Unitor:** For each pair of 0-cells, $(X, \underline{M})$ and $(Y, \underline{N})$ we have unitor natural isomorphisms

$$l_{(X,\underline{M}),(Y,\underline{N})} \qquad \text{and} \qquad r_{(X,\underline{M}),(Y,\underline{N})}$$

whose components on a 1-cell $(X \xleftarrow{f_1} A \xrightarrow{f_2} Y, \Lambda_f \underline{M} \xrightarrow{\alpha} \underline{N})$ are the isomorphisms $l : A \times_Y Y \to A, (a, y) \mapsto a$ and $r : X \times_X A \to A, (x, a) \mapsto a$ respectively. In order for l to be a 2-cell we require relevant version of Diagram 3.43 to commute, this is the outside of the following diagram.

$$\begin{tikzcd}
\Lambda_{\mathrm{id}_Y f}\underline{M} \arrow[r, "\hat{l}_M"] \arrow[d, "\cong"] & \Lambda_f \underline{M} \arrow[dd, "\alpha"] \\
\Lambda_{\mathrm{id}_X}\Lambda_f \underline{M} \arrow[ur, "\cong"'] \arrow[d, "\Lambda_{\mathrm{id}_Y}\alpha"] & \\
\Lambda_{\mathrm{id}_Y}\underline{N} \arrow[r, "\cong"] & \underline{N}
\end{tikzcd}$$

The triangle commutes by one of the pseudo unity axioms for $\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}$ and the square commutes because it is a naturality square for $((\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'})^0)^{-1}$. Showing that r is a 2-cell is similar.

It is straightforward to check that the unity and pentagon axioms are satisfied.

There is a projection pseudofunctor $\int \mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'} \to \underline{\mathcal{O}'}$ given by

$$\begin{aligned}
(X, \underline{M}) &\mapsto X \\
(f, \alpha) &\mapsto f \\
\psi &\mapsto \psi.
\end{aligned}$$

3.4 $\mathcal{O}'$-Mackey functors as $\mathcal{O}$-Mackey functors

We want to consider specific sections of the projection pseudofunctors. Given a section $F : \underline{\mathcal{O}'} \to \int \mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}$ for the projection pseudofunctor with $F(X) = (X, \underline{M})$ and $F(Y) = (Y, \underline{N})$ we will write

$$F(X \xleftarrow{f_1} A \xrightarrow{f_2} Y) = ((X \xleftarrow{f_1} A \xrightarrow{f_2} Y), \mu^F_{f_2,f_1} : \Lambda_f \underline{M} \to \underline{N}).$$

When it is clear what transfer span we are referring to we will write μ^F_f instead.

Definition 3.44. A **coherent $\mathcal{O}'$-monoid in** $\mathrm{Mack}_*^{\mathcal{O}}$ is an $\mathcal{O}$-Mackey functor $\underline{M}$ together with a strict functor (the pseudo functoriality and unity constraints are natural identities) $F : \underline{\mathcal{O}'} \to \int \mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}$ that is a section of the projection pseudofunctor such that:

- $F(X) = (X, \mathcal{R}_X \underline{M})$ for all G-sets X,
- for all $\mathcal{O}$-transfer spans $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ we have $\mu_f^F : \Lambda_f \mathcal{R}_X \underline{M} \to \mathcal{R}_Y \underline{M}$ is given by
$$\mu_f^F(B) : [C \xleftarrow{g_1} D \xrightarrow{g_2} B, c] \mapsto tr_{g_2} res_{g_1}(c).$$

A morphism $(\underline{M}, F) \to (\underline{N}, H)$ of coherent $\mathcal{O}'$-monoids in $\mathrm{Mack}_*^{\mathcal{O}}$ consists of a morphism $\varphi : \underline{M} \to \underline{N}$ in $\mathrm{Mack}_*^{\mathcal{O}}$ together with the strong transformation $\alpha : F \to H$ such that:

- **Component 1-cells:** For X a finite G-Set, the component 1-cell in $\int \mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'} \left((X, \mathcal{R}_X \underline{M}), (X, \mathcal{R}_X \underline{N}) \right)$ is
$$\alpha_X = \left(X = X = X, \overline{\alpha_X} : \Lambda_{\mathrm{id}_X} \mathcal{R}_X \underline{M} \to \mathcal{R}_X \underline{N} \right),$$
where $\overline{\alpha_X}$ is the composite
$$\Lambda_{\mathrm{id}_X} \mathcal{R}_X \underline{M} \xrightarrow{((\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'})^0_{\mathcal{R}_X \underline{M}})^{-1}} \mathcal{R}_X \underline{M} \xrightarrow{\mathcal{R}_X(\varphi)} \mathcal{R}_X \underline{N}.$$
- **Pseudo Naturality Constraints:** For an $\mathcal{O}'$-transfer span $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ the component 2-cell $\alpha_f : (Hf)\alpha_X \to \alpha_Y(Ff)$ in $\int \mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}((X, \mathcal{R}_X \underline{M}), (Y, \mathcal{R}_Y \underline{N}))$ is $\psi : X \times_X A \to A \times_Y Y, (x, a) \mapsto (a, f_2(a))$ which has inverse $(a, y) \mapsto (f_1(a), a)$. That is $\psi = l^{-1} r$. Note that being a 2-cell imposes the condition that the following diagram commutes.

$$\begin{tikzcd}
\Lambda_{f\,\mathrm{id}_X} R_{pr_X} \mathcal{R}_X \underline{M} \arrow[rr, "\hat{\psi}_M", "\cong"'] \arrow[d, "\cong"] & & \Lambda_{\mathrm{id}_Y\, f} \mathcal{R}_X \underline{M} \arrow[d, "\cong"'] \\
\Lambda_f \Lambda_{\mathrm{id}_X} \mathcal{R}_X \underline{M} \arrow[d, "\Lambda_f \overline{\alpha_X}"'] & & \Lambda_{\mathrm{id}_Y} \Lambda_f \mathcal{R}_X \underline{M} \arrow[d, "\Lambda_{\mathrm{id}_Y} \mu_f^F"] \\
\Lambda_f \mathcal{R}_X \underline{N} \arrow[dr, "\mu_f^H"'] & & \Lambda_{\mathrm{id}_Y} \mathcal{R}_Y \underline{M} \arrow[dl, "\overline{\alpha_Y}"] \\
& \mathcal{R}_Y \underline{N} &
\end{tikzcd} \tag{3.45}$$

For α to be a strong transformation there are axioms which must be satisfied. Checking the strong unity axiom boils down to noting that for any finite G-set X the map
$$X \times_X X \xrightarrow[(x,x) \mapsto x]{l} X \xrightarrow[x \mapsto (x,x)]{r^{-1}} X \times_X X \xrightarrow{id_{X \times_X X}} X \times_X X$$
is equal to the map
$$X \times_X X \xrightarrow{id_{X \times_X X}} X \times_X X \xrightarrow{id_{X \times_X X}} X \times_X X.$$

The strong naturality axiom is satisfied because, for any pair $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ and $g = (Y \xleftarrow{g_1} B \xrightarrow{g_2} Z)$ the map

$$\begin{tikzcd} X \times_X (A \times_Y B) \arrow[r, "\cong"] & (X \times_X A) \times_Y B \arrow[r, "((x,a),b)\mapsto((a,f_2(a)),b)"] & (A \times_Y Y) \times_Y B \arrow[r, "\cong"] & A \times_Y (Y \times_Y B) \arrow[dll, "(a,(b,g_2(b)))\leftarrow\!\shortmid(a,(y,b))"'] \\ & A \times_Y (B \times_Z Z) \arrow[r, "\cong"] & (A \times_Y B) \times_Z Z \arrow[r, "\mathrm{id}"] & (A \times_Y B) \times_Z Z \end{tikzcd}$$

is equal to the map

$$X \times_X (A \times_Y B) \xrightarrow{\mathrm{id}} X \times_X (A \times_Y B) \xrightarrow[(x,(a,b))\mapsto((a,b),g_2(n))]{} (A \times_Y B) \times_Z Z.$$

Given $(\underline{M}, F)$ the identity morphism $\mathrm{id}_{(\underline{M},F)}$ consists of $\mathrm{id}_{\underline{M}}$ together with the strong transformation it determines. Given two morphisms $(\varphi, \alpha) : (\underline{M}, F) \to (\underline{N}, H)$ and $(\omega, \beta) : (\underline{N}, H) \to (\underline{P}, I)$ of coherent $\mathcal{O}'$-monoids in $\mathrm{Mack}_*^{\mathcal{O}}$ we define their composite to be $(\omega\varphi, \beta\alpha) : (\underline{M}, F) \to (\underline{P}, I)$ where $\overline{(\beta\alpha)_X}$ is the composite

$$\Lambda_{\mathrm{id}_X} \mathcal{R}_X \underline{M} \xrightarrow{((\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'})^0_{\mathcal{R}_X \underline{M}})^{-1}} \mathcal{R}_X \underline{M} \xrightarrow{\mathcal{R}_X(\omega\varphi)} \mathcal{R}_X \underline{P}.$$

This is different from the usual composition of strong transformations, but is equivalent up to the invertible modification $X \times_X X \to X, (x, x) \mapsto x$.

Remark 3.46. Note that the information of α was entirely determined by φ, with the requirement that $\psi : (x, a) \mapsto (a, f_2(a))$ be a 2-cell encoding a compatibility condition involving F and H.

Definition 3.47. Let $(X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ be a $\mathcal{O}$-transfer span over X. There is a functor $\mathcal{E}_f : \mathrm{Mack}_X^{\mathcal{O}} \to \mathbf{Set}$ with $\mathcal{E}_f(\underline{M}) = \underline{M}(\mathrm{id}_X)$ and $\mathcal{E}_f(\varphi) = \varphi_X$. We also have a functor $\mathcal{F}_f : \mathrm{Mack}_X^{\mathcal{O}} \to \mathbf{Set}$ with $\mathcal{F}_f(\underline{M}) = \Lambda_f \underline{M}(\mathrm{id}_Y)$ with $\mathcal{F}_f(\varphi) = \Lambda_f(\varphi_{\mathrm{id}_Y})$. There is a natural transformation $\omega_f : \mathcal{E}_f \to \mathcal{F}_f$ with components $\underline{M}(\mathrm{id}_X) \to \Lambda_f \underline{M}(\mathrm{id}_Y)$ given by

$$x \mapsto [X \xleftarrow{f_1} A \xrightarrow{f_2} Y, x].$$

We now state and prove the main result of this chapter; our characterisation of incomplete Mackey functors.

Theorem 3.48. *For a finite group G and a pair $\mathcal{O}, \mathcal{O}'$ of indexing categories such that $\mathcal{O}$ is a subcategory of $\mathcal{O}'$, the $\mathcal{O}'$-Mackey functors are precisely the coherent $\mathcal{O}'$-monoids in $\mathrm{Mack}_*^{\mathcal{O}}$. That is, there is an equivalence between the category of coherent $\mathcal{O}'$-monoids in $\mathrm{Mack}_*^{\mathcal{O}}$ and the category $\mathrm{Mack}_*^{\mathcal{O}'}$.*

Proof. This proof proceeds as follows:

1. Define Φ (one direction of our equivalence) on objects.

2. Define Φ on morphisms.
3. Show Φ is functorial.
4. Define Ψ (the other direction of our equivalence) on objects.
5. Define Ψ on morphisms.
6. Show Ψ is functorial.
7. Finally, show that Φ and Ψ are inverse functors.

We will define a functor Φ from the category of $\mathcal{O}'$-Mackey functors to the category of coherent $\mathcal{O}'$-monoids in $\mathcal{O}$-Mackey functors.

1. Define Φ on objects

Let $\underline{M}$ be a an $\mathcal{O}'$-Mackey functor. We get an $\mathcal{O}$-Mackey functor, which we will call $\downarrow\underline{M}$ by precomposition with the inclusion functor

$$\downarrow\underline{M} : \mathcal{A}_*^{\mathcal{O}} \hookrightarrow \mathcal{A}_*^{\mathcal{O}'} \xrightarrow{\underline{M}} \mathrm{Set}\,.$$

We describe a strict functor $F : \underline{\mathcal{O}'} \to \int \mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}$ consisting of the following data:

- **Objects:** The assignment $F(X) = (X, \mathcal{R}_X{\downarrow}\underline{M})$.
- **Hom Categories:** For each pair of G-sets X, Y the functor

$$F : \underline{\mathcal{O}'}(X,Y) \to \int \mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}((X, \mathcal{R}_X{\downarrow}\underline{M}), (Y, \mathcal{R}_Y{\downarrow}\underline{M}))$$

  with

$$F(X \xleftarrow{f_1} A \xrightarrow{f_2} Y) = ((X \xleftarrow{f_1} A \xrightarrow{f_2} Y), \mu_f^F)$$

  on 1-cells, where $\mu_f^F(C)$ is given on elements by

$$[C \xleftarrow{g_1} D \xrightarrow{g_2} B, c] \mapsto tr_{g_2}(res_{g_1}(c)).$$

  1a. Show F is well-defined on 1-cells

  We note that if $\psi : D \to D'$ is an isomorphism between $\mathcal{O}'$-transfer spans $(C \xleftarrow{g_1} D \xrightarrow{g_2} B)$ and $(C \xleftarrow{g_1'} D' \xrightarrow{g_2'} B)$ then $tr_{g_2} res_{g_1} = tr_{g_2'} res_{g_1'}$ hence $\mu^F_{f_2,f_1}$ is well-defined.

  1b. Finish defining F

  On 2-cells we have

$$F\left(\begin{tikzcd}[column sep=small] & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] \arrow[dd, "\psi"] & \\ X & & Y \\ & A' \arrow[ul, "f_1'"] \arrow[ur, "f_2'"'] & \end{tikzcd}\right) = \left(\begin{tikzcd}[column sep=small] & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] \arrow[dd, "\psi"] & \\ X & & Y \\ & A' \arrow[ul, "f_1'"] \arrow[ur, "f_2'"'] & \end{tikzcd}\right).$$

We note that the condition of 2-cells in $\int \mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}$ is satisfied by the commutativity of the following diagram.

$$\begin{tikzcd}
\Lambda_f \mathcal{R}_X {\downarrow} \underline{M} \arrow[rr, "\hat{\psi}"] \arrow[dr, "\mu_f^F"'] & & \Lambda_{f'} \mathcal{R}_X {\downarrow} \underline{M} \arrow[dl, "\mu_{f'}^F"] \\
& \mathcal{R}_Y {\downarrow} \underline{M} &
\end{tikzcd}$$

We have

$$\begin{aligned}
\mu_{f'}^F \hat{\psi}([C \xleftarrow{g_1} D \xrightarrow{g_2} B, c]) &= \mu_{f'}^F([C \xleftarrow{g_1} D \xrightarrow{g_2} B, c]) \\
&= tr_{g_2}(res_{g_1}(c)) = \mu_f^F([C \xleftarrow{g_1} D \xrightarrow{g_2} B, c])
\end{aligned}$$

so the diagram commutes as required.

We note that, because F is a strict functor, commutativity of the required pseudo associativity and unity diagrams reduces to the requirement that the images under F of the associator and unitors in $\mathcal{O}'$ are precisely the associator and unitors in $\int \mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}$. This is immediate.

2. Define Φ on morphisms

We let $\Phi(\underline{M}) = (\Downarrow \underline{M}, F)$. Given a morphism $\varphi : \underline{M} \to \underline{N}$ of $\mathcal{O}'$-Mackey functors, we let $\Phi(\varphi) : (\Downarrow \underline{M}, F) \to (\Downarrow \underline{N}, H)$ be φ in the first coordinate and $\alpha : F \to H$ as determined by φ.

2a. Show Φ is well-defined on morphisms

In order for α to be a strong transformation we need to check for all $\mathcal{O}'$-transfer spans, $(X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$, that α_f does define a 2-cell. We consider following diagram.

$$\begin{tikzcd}
\Lambda_f \Lambda_{\mathrm{id}_X} \mathcal{R}_X \underline{M} \arrow[dd, "\Lambda_f \overline{\alpha_X}"'] & \Lambda_f \mathcal{R}_X \underline{M} \arrow[l, "\Lambda_f (\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'})^0_{\mathcal{R}_X \underline{M}}"', "\cong"] \arrow[r, "(\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'})^0_{\Lambda_f \mathcal{R}_X \underline{M}}", "\cong"'] \arrow[d, "\mu^F_{f_2, f_1}"] \arrow[ddl, "\Lambda_f \mathcal{R}_X \varphi"'] & \Lambda_{\mathrm{id}_Y} \Lambda_f \mathcal{R}_X \underline{M} \arrow[d, "\Lambda_{\mathrm{id}_Y} \mu_f^F"] \\
& \mathcal{R}_Y \underline{M} \arrow[r, "(\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'})^0_{\mathcal{R}_Y \underline{M}}", "\cong"'] \arrow[d, "\mathcal{R}_Y \varphi"] & \Lambda_{\mathrm{id}_Y} \mathcal{R}_Y \underline{M} \arrow[dl, "\overline{\alpha_Y}"] \\
\Lambda_f \mathcal{R}_X \underline{N} \arrow[r, "\mu_f^H"] & \mathcal{R}_Y \underline{N} &
\end{tikzcd} \tag{3.49}$$

Noting that the top row of the diagram is the same morphism as the top two rows of diagram 3.45 we see that the outside of this diagram is equivalent to Diagram 3.45, so to confirm that α_f is a 2-cell it suffices to show that this diagram commutes. The bottom right triangle commutes by the definition of $\overline{\alpha_Y}$. The top left triangle commutes because it is the image under Λ_f of the definition of $\overline{\alpha_X}$. The right square commutes because it is a naturality square for $\left(\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}\right)^0$. On objects the middle triangle is the following square.

$$\begin{tikzcd}
\Lambda_f \mathcal{R}_X \underline{M}(B) \arrow[r, "\mu_f^F(B)"] \arrow[d, "\Lambda_f \mathcal{R}_X \varphi(B)"'] & \mathcal{R}_Y \underline{M}(B) \arrow[d, "\mathcal{R}_Y \varphi(B)"] \\
\Lambda_f \mathcal{R}_X \underline{N}(B) \arrow[r, "\mu_f^H(B)"'] & \mathcal{R}_Y \underline{N}(B)
\end{tikzcd}$$

Going around the top of the square we have $[C \xleftarrow{g_1} D \xrightarrow{g_2} B, c] \mapsto \varphi(tr_{g_2}(res_{g_1}(c)))$ and going around the bottom we have $[C \xleftarrow{g_1} D \xrightarrow{g_2} B, c] \mapsto tr_{g_2}(res_{g_1}(\varphi(c))) = \varphi(tr_{g_2}(res_{g_1}(c)))$ because φ commutes with $\mathcal{O}$-transfers and with all restrictions. The square commutes, and as the diagram is natural with respect to B so does the middle triangle in 3.49. Hence 3.49 commutes and therefore α_f is a 2-cell as required.

<u>3. Show Φ is functorial</u>

Note that $\Phi(\mathrm{id}_{\underline{M}}) = \mathrm{id}_{\Phi(\underline{M})}$. Given $\varphi : \underline{M} \to \underline{N}$ and $\gamma : \underline{N} \to \underline{P}$ we have that $\Phi(\gamma)\Phi(\varphi)$ and $\Phi(\gamma\varphi)$ are both $\gamma\varphi$ in the first coordinate and both

$$\Lambda_{\mathrm{id}_X} \mathcal{R}_X \underline{M} \xrightarrow{((\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'})^0_{\mathcal{R}_X \underline{M}})^{-1}} \mathcal{R}_X \underline{M} \xrightarrow{\mathcal{R}_X(\gamma\varphi)} \mathcal{R}_X \underline{P}$$

in the second. Hence Φ is a functor.

Now we construct a functor Ψ from the category of coherent $\mathcal{O}'$-monoids in $\mathcal{O}$-Mackey functors to the category of $\mathcal{O}'$-Mackey functors.

<u>4. Define Ψ on objects</u>

Let $(\underline{M}, F)$ be a coherent $\mathcal{O}'$-monoid in $\mathrm{Mack}_*^{\mathcal{O}}$. We will extend $\underline{M}$ to an $\mathcal{O}'$-Mackey functor, $\uparrow_F \underline{M}$. On objects we let $\uparrow_F \underline{M}(X) = \underline{M}(X)$. Given a morphism in $\mathcal{A}_*^{\mathcal{O}'}$ represented by the $\mathcal{O}'$-transfer span $(X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ we let $\uparrow_F \underline{M}(X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ be the map $\uparrow_F \underline{M}(X) \to \uparrow_F \underline{M}(Y)$ given by the composition

$$\underline{M}(X) = \mathcal{R}_X(\mathrm{id}_X) \xrightarrow{\omega_f} \Lambda_f \mathcal{R}_X \underline{M}(\mathrm{id}_Y) \xrightarrow{\mu_f^F(\mathrm{id}_Y)} \mathcal{R}_Y \underline{M}(\mathrm{id}_Y) = \underline{M}(Y),$$

Because the morphisms in $\mathcal{A}_*^{\mathcal{O}'}$ are isomorphism classes of $\mathcal{O}'$-transfer spans we need to check that this composite is equal on isomorphic $\mathcal{O}'$-transfer spans. If we have an isomorphism $\psi : A \to A'$ of $\mathcal{O}'$-transfer spans $(X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ and $(X \xleftarrow{f_1'} A' \xrightarrow{f_2'} Y)$ then we consider the following diagram.

$$\begin{tikzcd}
 & & \Lambda_f \mathcal{R}_X \underline{M}(\mathrm{id}_Y) \arrow[dd, "\cong"', "\widehat{\psi}"] \arrow[dr, "\mu_f^F"] & & \\
\underline{M}(X) \arrow[r, equal] & \mathcal{R}_X \underline{M}(\mathrm{id}\, X) \arrow[ur, "\omega_f"] \arrow[dr, "\omega_{f'}"'] & & \mathcal{R}_Y \underline{M}(\mathrm{id}_Y) \arrow[r, equal] & \underline{M}(Y) \\
 & & \Lambda_{f'} \mathcal{R}_X \underline{M}(\mathrm{id}_Y) \arrow[ur, "\mu_{f'}^F"'] & &
\end{tikzcd}$$

Here $\widehat{\psi}$ is $\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}(\psi)$. The right triangle by the defining condition of 2-cells in $\int \mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}$. The bottom map in the left triangle is

$$x \mapsto [X \xleftarrow{f_1'} A' \xrightarrow{f_2'} Y, x]$$

and going around the top of the left triangle we have

$$\begin{aligned} x &\mapsto [X \xleftarrow{f_1} A \xrightarrow{f_2} Y, x] \\ &\mapsto [X \xleftarrow{f_1} A \xrightarrow{f_2} Y, x] \\ &= [X \xleftarrow{f_1'} A' \xrightarrow{f_2'} Y, x] \end{aligned}$$

where the equality comes from the isomorphism ψ. As this diagram commutes $\uparrow_F \underline{M}$ is well-defined on morphisms.
We need to check that $\uparrow_F \underline{M}$ is functorial. Given $\mathcal{O}'$-transfer spans $(X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ and $(X \xleftarrow{g_1} B \xrightarrow{g_2} Z)$ we consider the following diagram.

$$\begin{tikzcd}
\underline{M}(X) \arrow[d, equal] & & \underline{M} \arrow[d, equal] \\
\mathcal{R}_X\underline{M}(\mathrm{id}_X) \arrow[r, "\omega_f"] \arrow[ddd, "\omega_{gf}"'] & \Lambda_f\mathcal{R}_X\underline{M}(\mathrm{id}_Y) \arrow[r, "\mu_f^F(\mathrm{id}_Y)"] \arrow[d, "\omega_g"] & \mathcal{R}_Y\underline{M}(\mathrm{id}_Y) \arrow[d, "\omega_g"] \\
 & \Lambda_g\Lambda_f\mathcal{R}_X\underline{M}(\mathrm{id}_Z) \arrow[r, "\Lambda_g\mu_f^F(\mathrm{id}_Z)"] \arrow[ddl, "\cong"] & \Lambda_g\mathcal{R}_Y\underline{M}(\mathrm{id}_Z) \arrow[dd, "\mu_g^F"] \\
 & & \\
\Lambda_{gf}\mathcal{R}_X\underline{M}(\mathrm{id}_Z) \arrow[rr, "\mu_{gf}^F"] & & \mathcal{R}_Z\underline{M}(\mathrm{id}_Z) \arrow[d, equal] \\
 & & \underline{M}(Z)
\end{tikzcd}$$

The bottom region is the pseudo functoriality constraint for F and so commutes because F is a strict functor. the top right region commutes because it is a naturality square for ω_g. Going around the right side of the remaining region we have

$$\begin{aligned} x &\mapsto [X \xleftarrow{f_1} A \xrightarrow{f_2} Y, x] \\ &\mapsto [Y \xleftarrow{g_1} B \xrightarrow{g_2} Z, [X \xleftarrow{f_1} A \xrightarrow{f_2} Y, x]] \\ &\mapsto [X \xleftarrow{f_1 pr_A} A \times_Y B \xrightarrow{g_2 pr_B} Z, x] \\ &= \omega_{gf}(x) \end{aligned}$$

so the region commutes.
On an identity $\mathcal{O}'$-transfer span $(X = X = X)$ we have $\uparrow_F \underline{M}(X = X = X)$ given by the composition

$$x \mapsto [X = X = X, x] \mapsto tr_{\mathrm{id}_X}(res_{\mathrm{id}_X}(x)) = x.$$

Hence $\uparrow_F \underline{M}$ is a functor $\mathcal{A}^{\mathcal{O}'} \to \mathbf{Set}$. Note also that $\underline{M}$ takes values in abelian groups and is product-preserving because $\uparrow_F \underline{M}(X) = \underline{M}(X)$. Hence $\uparrow_F \underline{M}$ is an $\mathcal{O}'$-Mackey functor. We let $\Psi(\underline{M}, F) = \uparrow_F \underline{M}$.

5. Define Ψ on morphisms

Given a morphism $(\varphi, \alpha) : (\underline{M}, F) \to (\underline{N}, H)$ of coherent $\mathcal{O}'$-monoids in $\mathcal{O}$-Mackey functors we let $\Psi(\varphi, \alpha) = \varphi$ on objects, which is sufficient to describe a morphism $\uparrow_F \underline{M} \to \uparrow_H \underline{N}$. To see that this is sufficient we need to confirm that φ is natural in $\mathcal{A}^{\mathcal{O}'}$. That is, given $(X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ an $\mathcal{O}'$-transfer span we require that the following diagram commutes.

$$\begin{tikzcd}
\underline{M}(X) \arrow[r, equal] \arrow[d, "\varphi(X)"] & \mathcal{R}_X\underline{M}(\mathrm{id}_X) \arrow[r, "\omega_f"] \arrow[d, "\mathcal{R}\varphi(\mathrm{id}_X)"] & \Lambda_f\mathcal{R}_X\underline{M}(\mathrm{id}_Y) \arrow[r, "\mu_f^F(\mathrm{id}_Y)"] \arrow[d, "\Lambda_f\mathcal{R}_X\varphi(\mathrm{id}_Y)"] & \mathcal{R}_Y\underline{M}(\mathrm{id}_Y) \arrow[r, equal] \arrow[d, "\mathcal{R}_Y\varphi(\mathrm{id}_Y)"] & \underline{M}(Y) \arrow[d, "\varphi(Y)"] \\
\underline{N}(X) \arrow[r, equal] & \mathcal{R}_X\underline{N}(\mathrm{id}_X) \arrow[r, "\omega_f"] & \Lambda_f\mathcal{R}_X\underline{N}(\mathrm{id}_Y) \arrow[r, "\mu_f^H(\mathrm{id}_Y)"] & \mathcal{R}_Y\underline{N}(\mathrm{id}_Y) \arrow[r, equal] & \underline{N}(Y)
\end{tikzcd}$$

The second square commutes because it is a naturality square for ω_f. The third square is the middle triangle of Diagram 3.49 applied to id_Y, because we know that α_f is a 2-cell we know that the outside of this diagram commutes. The other regions of 3.49 commute for the same reasons as before and hence the middle triangle of 3.49 commutes. Hence the diagram above commutes, therefore φ is a morphism of $\mathcal{O}'$-Mackey functors, $\uparrow_F \underline{M} \to \uparrow_H \underline{N}$.

6. Show Ψ is functorial

The functoriality of Ψ is immediate.

7. Show that Φ and Ψ are inverse

7a. $\Phi\Psi$ is isomorphic to the identity

Given a coherent $\mathcal{O}'$-monoid $(\underline{M}, F)$ in $\mathrm{Mack}_*^{\mathcal{O}}$ we have $\Psi(\underline{M}, F) = \uparrow_F \underline{M}$ with $\uparrow_F \underline{M}(X) = \underline{M}(X)$ and for $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ in $\mathcal{A}_*^{\mathcal{O}'}(X, Y)$ the map $\uparrow_F \underline{M}(f)$ is given by

$$\underline{M}(X) \xrightarrow{\omega_f} \Lambda_f\mathcal{R}_X\underline{M}(\mathrm{id}_Y) \xrightarrow{\mu_f^F(\mathrm{id}_Y)} \mathcal{R}_Y\underline{M}(\mathrm{id}_Y) = \underline{M}(Y).$$

Then we have $\Phi(\Psi(\underline{M}, F)) = \Phi(\uparrow_F \underline{M}) = (\downarrow(\uparrow_F \underline{M}), F')$ with $\downarrow(\uparrow_F \underline{M})(X) = \uparrow_F \underline{M}(X) = \underline{M}(X)$ and for an $\mathcal{O}$-transfer span $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ the map $\downarrow(\uparrow_F \underline{M})(f)$ is given by

$$\underline{M}(X) \xrightarrow{\omega_f} \Lambda_f\mathcal{R}_X\underline{M}(\mathrm{id}_Y) \xrightarrow{\mu_f^F(\mathrm{id}_Y)} \mathcal{R}_Y\underline{M}(\mathrm{id}_Y) = \underline{M}(Y),$$

as $\mu_f^F(\mathrm{id}_Y)$ is the map

$$[C \xleftarrow{g_1} D \xrightarrow{g_2} Y, c] \mapsto tr_{g_2}(res_{g_1}(c))$$

we see that $\downarrow(\uparrow_F \underline{M})(f) = tr_{f_2}res_{f_1}$. Hence $\downarrow(\uparrow_F \underline{M}) = \underline{M}$. Now we have $F'(X) = (X, \mathcal{R}_X \downarrow(\uparrow_F \underline{M})) = (X, \mathcal{R}_X\underline{M})$. Given an $\mathcal{O}'$-transfer span $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ we have $F'(f) = (f, \mu_f^{F'})$ where $\mu_f^{F'}(B)$ is given by

$$\begin{aligned}[C \xleftarrow{g_2} D \xrightarrow{g_1} B, c] &\mapsto (\uparrow_F \underline{M})(C \xleftarrow{g_2} D \xrightarrow{g_1} B)(c) \mapsto \mu_f^F \omega_g(c) \\ &\mapsto \mu_f^F([C \xleftarrow{g_1} D \xrightarrow{g_2} B, c])\end{aligned}$$

which agrees with $\mu_f^F(B)$. Hence $\mu_f^{F'} = \mu_f^F$. Therefore $\Phi\Psi(\underline{M}, F) = (\underline{M}, F)$.
7b. $\Psi\Phi$ is isomorphic to the identity
Given on $\mathcal{O}'$-Mackey functor $\underline{M}$ we have $\Phi(\underline{M}) = (\downarrow \underline{M}, F)$ with $\downarrow \underline{M}(X) = \underline{M}(X)$ and for $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ in $\mathcal{A}_*^{\mathcal{O}}(X, Y)$ the map $\downarrow \underline{M}(f) = tr_{f_2} res_{f_1}$. Now we have $F(X) = (X, \mathcal{R}_X \downarrow \underline{M}) = (X, \mathcal{R}_X \underline{M})$ and for $\mathcal{O}'$-transfer spans $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ we have $\mu_f^F(B)$ given by

$$[C \xleftarrow{g_1} D \xrightarrow{g_2} B, c] \mapsto tr_{g_2}(res_{g_1}(c)).$$

Now we have $\uparrow_F (\downarrow \underline{M})(X) = \downarrow \underline{M}(X) = \underline{M}(X)$. For an $\mathcal{O}'$-transfer span $(X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ the map $\uparrow_F (\downarrow \underline{M})(f)$ is given by

$$\underline{M}(X) \xrightarrow{\omega_f} \Lambda_f \mathcal{R}_X \underline{M}(\mathrm{id}_Y) \xrightarrow{\mu_f^F(\mathrm{id}_Y)} \mathcal{R}_Y \underline{M}(\mathrm{id}_Y) = \underline{M}(Y)$$

which on elements is $x \mapsto [X \xleftarrow{f_1} A \xrightarrow{f_2} Y, x] \mapsto tr_{f_2} res_{f_1}(x)$. So $\uparrow_F (\downarrow \underline{M})(f) = \underline{M}(f)$. Hence $\uparrow_F (\downarrow \underline{M}) = \underline{M}$. Hence $\Psi\Phi(\underline{M}) = \underline{M}$. $\square$

Corollary 3.50. *G-Mackey functors are precisely the coherent* Set^G*-monoids in the category of G-coefficient systems.*

3.5 $\mathrm{Mack}_X^{\mathcal{O}}$ as a module category

We will now see that categories of Mackey functors are module categories over indexing categories. Recall Definition 2.37 of a module category over a tight symmetric bimonoidal category.

Notation 3.51. Given a map $\varphi : A \to X$ of G-sets we will write $\overline{\varphi}$ for the span $(X \xleftarrow{\varphi} A \xrightarrow{\varphi} X)$.

Theorem 3.52. *Let $\mathcal{O}$ and $\mathcal{O}'$ be indexing categories and X be a finite G-set. The category of $\mathcal{O}$-Mackey functors over X is a module category over $(\mathcal{O}'_X)_{\cong}$ with the action functor $\odot : (\mathcal{O}'_X)_{\cong} \times \mathrm{Mack}_X^{\mathcal{O}} \to \mathrm{Mack}_X^{\mathcal{O}}$ given by $(A, \varphi) \odot \underline{M} = \Lambda_{\overline{\varphi}} \underline{M}$.*

Proof. By Lemma 3.32 $(\mathrm{Mack}_X^{\mathcal{O}}, \oplus, \underline{0})$ is a symmetric monoidal category. Noting that on objects we have $(\underline{M} \oplus \underline{N})(X) = \underline{M}(X) \oplus \underline{N}(X)$ the structure natural isomorphisms have the component maps $\alpha_X^{\oplus} : ((a, b), c) \mapsto (a, (b, c))$, $\lambda_X^{\oplus} : (0, a) \mapsto a$, $\rho_X^{\oplus} : (a, 0) \mapsto a$, $\beta_X^{\oplus} : (a, b) \mapsto (b, a)$. For $(A, \varphi), (B, \psi) \in (\mathcal{O}'_X)_{\cong}$ we declare the following structure natural isomorphisms:

- Let $\lambda^{\odot} : \mathrm{id}_X \odot \underline{M} = \Lambda_{\overline{\mathrm{id}_X}} \underline{M} \to \underline{M}$ be the inverse of the $\underline{M}$ component of $(\mathrm{Mack}_{\mathcal{O}}^{\mathrm{Set}^G})^0$.

- Let $\alpha^{\odot} : \psi \odot (\varphi \odot \underline{M}) = \Lambda_{\overline{\psi}} \Lambda_{\overline{\varphi}} \underline{M} \to \Lambda_{\overline{\psi}\overline{\varphi}} \underline{M} = (\psi \times_X \varphi) \odot \underline{M}$ be the $\underline{M}$ component of the pseudo functoriality constraint $(\mathrm{Mack}_{\mathcal{O}}^{\mathrm{Set}^G})^2$.

- Let $d_{\odot}^L : (\varphi \amalg \psi) \odot \underline{M} = \Lambda_{\overline{\varphi \amalg \psi}} \underline{M} \to \Lambda_{\overline{\varphi}} \underline{M} \oplus \Lambda_{\overline{\psi}} \underline{M} = (\varphi \odot \underline{M}) \oplus (\psi \odot \underline{M})$ be the isomorphism from Lemma A.4 induced by the natural identity transformation between the functors
$$\mathcal{A}_X^{\mathcal{O}} \xrightarrow{\Delta_{(\varphi \amalg \psi)}} \mathcal{A}_{A \amalg B}^{\mathcal{O}} \xrightarrow{\Sigma_{(\varphi \amalg \psi)}} \mathcal{A}_X^{\mathcal{O}}$$
and
$$\mathcal{A}_X^{\mathcal{O}} \xrightarrow{\Delta} \mathcal{A}_X^{\mathcal{O}} \times \mathcal{A}_X^{\mathcal{O}} \xrightarrow{(\Delta_\varphi, \Delta_\psi)} \mathcal{A}_A^{\mathcal{O}} \times \mathcal{A}_B^{\mathcal{O}} \xrightarrow{\amalg} \mathcal{A}_X^{\mathcal{O}}.$$

- Let $d_{\odot}^R : \varphi \odot (\underline{M} \oplus \underline{N}) = \Lambda_{\overline{\varphi}}(\underline{M} \oplus \underline{N}) \to \Lambda_{\overline{\varphi}} \underline{M} \oplus \Lambda_{\overline{\varphi}} \underline{N} = (\varphi \odot \underline{M}) \oplus (\varphi \odot \underline{N})$ be the isomorphism from Lemma A.4 induced by the natural isomorphism between the functors
$$\mathcal{A}_X^{\mathcal{O}} \times \mathcal{A}_X^{\mathcal{O}} \xrightarrow{\amalg} \mathcal{A}_X^{\mathcal{O}} \xrightarrow{\Delta_\varphi} \mathcal{A}_A^{\mathcal{O}} \xrightarrow{\Sigma_\varphi} \mathcal{A}_X^{\mathcal{O}}$$
and
$$\mathcal{A}_X^{\mathcal{O}} \times \mathcal{A}_X^{\mathcal{O}} \xrightarrow{(\Delta_\varphi, \Delta_\varphi)} \mathcal{A}_A^{\mathcal{O}} \times \mathcal{A}_A^{\mathcal{O}} \xrightarrow{(\Sigma_\varphi, \Sigma_\varphi)} \mathcal{A}_X^{\mathcal{O}} \times \mathcal{A}_X^{\mathcal{O}} \xrightarrow{\amalg} \mathcal{A}_X^{\mathcal{O}}.$$

- Note that if $\zeta : \emptyset \to X$ then $\Lambda_{\overline{\zeta}} \underline{M} = \underline{0}$ so let $\xi_{\odot}^L : \zeta \odot \underline{M} = \Lambda_{\overline{\zeta}} \underline{M} = \underline{0} \to \underline{0}$ be the identity natural isomorphism.

- Let $\xi_{\odot}^R : \varphi \odot \underline{0} = \Lambda_{\overline{\varphi}} \underline{0} = \underline{0} \to \underline{0}$ be the identity natural isomorphism.

It remains to check that the coherence diagrams from Definition 2.37 are satisfied. This is a straightforward but lengthy exercise and is left to the interested reader. ☐

Recall Definition 2.40 of a $\mathcal{C}$-monoid in a module category $\mathcal{A}$ over $\mathcal{C}_{\cong}$ for $\mathcal{C}$ a tight symmetric bimonoidal category.

Proposition 3.53. *Let $\mathcal{O}$ and $\mathcal{O}'$ indexing categories such that $\mathcal{O}$ is a subcategory of $\mathcal{O}'$. If $(\underline{M}, F)$ is a coherent $\mathcal{O}'$-monoid in $\mathrm{Mack}_*^{\mathcal{O}}$ then $\mathcal{R}_X \underline{M}$ is an $\mathcal{O}'$-monoid in $\mathrm{Mack}_X^{\mathcal{O}}$ as a module category over $(\mathcal{O}'_X)_{\cong}$ for all finite G-sets X.*

Proof. We are asking for an extension of the functor $- \odot \mathcal{R}_X \underline{M} : (\mathcal{O}')_{\cong} \to \mathrm{Mack}_X^{\mathcal{O}}$ to $\mathcal{O}'$.

$$\begin{tikzcd} (\mathcal{O}'_X)_{\cong} \arrow[r, "- \odot \mathcal{R}_X \underline{M}"] \arrow[d, hook] & \mathrm{Mack}_X^{\mathcal{O}} \\ \mathcal{O}'_X \arrow[ur, dashed] & \end{tikzcd}$$

Because $(\mathcal{O}'_X)_{\cong}$ is a wide subcategory of $\mathcal{O}'_X$ we only need to describe what happens on the extra morphisms. Given $\sigma : (A, \varphi) \to (B, \psi)$ in $\mathcal{O}'_X$ we need a map $\Lambda_{\overline{\varphi}}\mathcal{R}_X\underline{M} \to \Lambda_{\overline{\psi}}\mathcal{R}_X\underline{M}$. We let this map be

$$\Lambda_{\overline{\varphi}}\mathcal{R}_X\underline{M}(C) = \underline{M}(A \times_X C) \xrightarrow{tr_{\sigma \times_X \mathrm{id}_C}} \underline{M}(B \times_X C) = \Lambda_{\overline{\psi}}\mathcal{R}_X\underline{M}$$

where $tr_{\sigma\times_X \mathrm{id}_C}$ is the transfer in $\uparrow_F \underline{M}$ from Theorem 3.48. □

It is possible to have an $\mathcal{O}$-Mackey functor $\underline{M}$ together with extensions of the functor $- \odot \mathcal{R}_X\underline{M}$ to $\mathcal{O}'_X$ for all finite G-sets X that do not assemble into a *coherent* $\mathcal{O}'$-monoid. We get objects equivalent to this (what we might call incoherent $\mathcal{O}'$-monoids) by replacing $\underline{\mathcal{O}}$ throughout Section 3.3 with the sub-bicategory containing all objects, with hom categories the full subcategories of $\underline{\mathcal{O}}(X,Y)$ containing only those 1-cells of the form $(X = X \to Y)$.

The following definition is equivalent to one appearing in [10, Definition 4.4] with the modification that we have replaced the orbit category Orb_G with Set^G.

Definition 3.54. Let $\mathcal{O}$ be an indexing category. A **symmetric monoidal $\mathcal{O}$-Mackey functor** is a pseudofunctor $F : \underline{\mathcal{O}} \to Sym$ landing in the bicategory of symmetric monoidal categories, strong monoidal functors, and monoidal natural transformations.

Definition 3.55. Let $\mathcal{O}$ be an indexing category. A **tight symmetric bimonoidal $\mathcal{O}$-Mackey functor** is a pseudofunctor $F : \underline{\mathcal{O}} \to \mathrm{Bi}^{sy}_{m,sg}$ such that $F(X)$ is a tight symmetric bimonoidal category for all finite G-sets X and $\mathrm{Bi}^{sy}_{m,sg}$ is the bicategory of symmetric bimonoidal categories, strong symmetric monoidal functors, and monoidal natural transformations. Let $\underline{\mathcal{O}}_{\cong}$ be the sub-bicategory of $\underline{\mathcal{O}}$ containing all objects, with the hom category $\underline{\mathcal{O}}_{\cong}(X,Y)$ being the full subcategory of $\underline{\mathcal{O}}(X,Y)$ containing only the invertible 1-cells. The **core** of a tight symmetric bimonoidal $\mathcal{O}$-Mackey functor F is the pseudofunctor $F_{\cong} : \underline{\mathcal{O}}_{\cong} \to \mathrm{Bi}^{sy}_{sg}$ such that $F_{\cong}(X) = F(X)_{\cong}$ and which on a 1-cell $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ is the restriction of $F(f)$ to $F(X)_{\cong}$.

Definition 3.56. Let F be a tight symmetric bimonoidal $\mathcal{O}$-Mackey functor. A **module over** F is a symmetric monoidal $\mathcal{O}$-Mackey functor H together with a strong transformation $\boxtimes : F_{\cong} \times H \to H$ (considered as pseudofunctors $\underline{\mathcal{O}}_{\cong} \to Sym$) such that the component 1-cell $\boxtimes_X : F(X)_{\cong} \times H(X) \to H(X)$ is an action functor which makes $H(X)$ a module category over $F(X)_{\cong}$ for all finite G-sets X.

Definition 3.57. Let $(H, \boxtimes)$ be a module over a tight symmetric bimonoidal $\mathcal{O}$-Mackey functor F. Given an object $M \in H(*)$ we have a pseudofunctor $- \boxtimes M : F_{\cong} \to H$ with components $- \boxtimes M_X : F(X)_{\cong} \to H(X)$ where M_X is the image of M under the functor $H(* \leftarrow X = X)$. An **F-monoid** in H is an object $M \in H(*)$ together with an extension of this pseudofunctor to F. When such an extension exists we will still refer to it by $-\boxtimes M$.

$$\begin{tikzcd} F_{\cong} \arrow[r, "-\boxtimes M"] \arrow[d, hook] & H \\ F \arrow[ur, dashed] & \end{tikzcd}$$

We write $\overline{\mathcal{O}}$ for the symmetric monoidal $\mathcal{O}$-Mackey functor with $\overline{\mathcal{O}}(X) = \mathcal{O}_X$ and $\overline{\mathcal{O}}(X \xleftarrow{f_1} A \xrightarrow{f_2} Y) = \Sigma_{f_2}\Delta_{f_1} : \mathcal{O}_X \to \mathcal{O}_Y$. On isomorphisms of $\mathcal{O}$-transfer spans (that is, 2-cells) $\overline{O}$ is the obvious natural isomorphism. If $\mathcal{O}$ and $\mathcal{O}'$ are indexing categories such that $\mathcal{O}$ is a subcategory of $\mathcal{O}'$ then $\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}$ (Definition 3.41) is a symmetric monoidal $\mathcal{O}'$-Mackey functor. By Theorem 3.52 we see that $(\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}, \odot)$ is a module over $\overline{\mathcal{O}'}$.

The following two propositions show that we now have two equivalent characterisations of $\mathcal{O}'$-Mackey functors.

Proposition 3.58. *If $(\underline{M}, F)$ is a coherent $\mathcal{O}'$-monoid in $\mathrm{Mack}_*^{\mathcal{O}}$ then $\underline{M}$ is an $\overline{\mathcal{O}'}$-module in $\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}$.*

Proof. By Proposition 3.53 we see that the required extension of $-\odot \underline{M}$ exists objectwise. We only need to check that these commute with the structure functors in the $\overline{\mathcal{O}'}$ and $\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}$. That is, given a $\mathcal{O}'$-transfer span $(X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ we want the following diagram to commute (up to natural isomorphism).

$$\begin{tikzcd} \mathcal{O}'_X \arrow[r, "-\odot\mathcal{R}_X\underline{M}"] \arrow[d, "\Sigma_{f_2}\Delta_{f_1}"'] & \mathrm{Mack}_X^{\mathcal{O}} \arrow[d, "\mathcal{T}_{f_2}\mathcal{R}_{f_1}"] \\ \mathcal{O}'_Y \arrow[r, "-\odot\mathcal{R}_Y\underline{M}"] & \mathrm{Mack}_Y^{\mathcal{O}} \end{tikzcd} \tag{3.59}$$

Given $(C, \varphi) \in \mathcal{O}'_X$ going around the bottom of the diagram we have

$$(C,\varphi) \mapsto (A \times_X C, f_2pr_A) \mapsto \mathcal{T}_{f_2pr_A}\mathcal{R}_{f_2pr_A}\mathcal{R}_Y\underline{M} \cong \mathcal{T}_{f_2pr_A}\mathcal{R}_{A\times_X C}\underline{M}$$

and going around the top we have

$$(C,\varphi) \mapsto \mathcal{T}_{\varphi}\mathcal{R}_{\varphi}\mathcal{R}_X\underline{M} \mapsto \mathcal{T}_{f_2}\mathcal{R}_{f_1}\mathcal{T}_{\varphi}\mathcal{R}_{\varphi}\mathcal{R}_X\underline{M} \cong \mathcal{T}_{f_2}\mathcal{T}_{pr_A}\mathcal{R}_{pr_C}\mathcal{R}_{\varphi}\mathcal{R}_X\underline{M} \cong \mathcal{T}_{f_2pr_A}\mathcal{R}_{A\times_X C}\underline{M}.$$

□

Commutativity of Diagram 3.59 is what we mean when we say that $(\underline{M}, F)$ is a *coherent* $\mathcal{O}'$-monoid functor.

Proposition 3.60. *If $\underline{M}$ is an $\overline{\mathcal{O}'}$-module in $\mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}$ then $\underline{M}$ is a coherent $\mathcal{O}'$-monoid in $\mathrm{Mack}_*^{\mathcal{O}}$.*

Proof. We need to describe a strict functor $F : \underline{\mathcal{O}'} \to \int \mathrm{Mack}_{\mathcal{O}}^{\mathcal{O}'}$ as per Definition 3.44. Because F is required to be a section we only need to describe μ_f^F from $F(f) = (f, \mu_f^F)$ for $\mathcal{O}'$-transfer spans $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$. From the Y component $\mathcal{O}'_Y \to \mathrm{Mack}_Y^{\mathcal{O}}$ of the extension of $- \odot \underline{M}$ and the map $(A, f_2) \xrightarrow{f_2} (Y, \mathrm{id}_Y)$ in $\mathcal{O}'_Y$ we get a map $\sigma : \mathcal{T}_{f_2}\mathcal{R}_{f_2}\mathcal{R}_Y\underline{M} \to \mathcal{R}_Y\underline{M}$ in $\mathrm{Mack}_Y^{\mathcal{O}}$. We let μ_f^F be the composite

$$\mathcal{T}_{f_2}\mathcal{R}_{f_1}\mathcal{R}_X\underline{M} \xrightarrow{\cong} \mathcal{T}_{f_2}\mathcal{R}_{f_2}\mathcal{R}_Y\underline{M} \xrightarrow{\sigma} \mathcal{R}_Y\underline{M}.$$

The commutativity of diagrams like Diagram 3.59 for any $\mathcal{O}'$-transfer span ensures that $\mu_{gf}^F = \mu_g^F(\Lambda_g \mu_f^F)$. □

Chapter 4

Bi-Incomplete Tambara Functors

Extending the analogy of Mackey functors as the equivariant analogue for abelian groups, Tambara functors are the equivariant version of commutative rings. These are Mackey functors with a multiplicative structure encoded by norm maps which correspond to the transfer maps. Compositions of restrictions and norms satisfy the same pullback-induced relation as compositions of restrictions and transfers. Compositions of transfers and norms satisfy a complicated relation encoded by exponential diagrams which describe distributivity. Similarly to incomplete Mackey functors, an incomplete Tambara functor is one which does not have all norm maps. A bi-incomplete Tambara functor is one which is missing some transfer maps and some norm maps. In this chapter we provide a characterisation of bi-incomplete Tambara functors as more bi-incomplete Tambara functors equipped with an appropriate pseudofunctor encoding the information of the extra transfer and norm maps. This chapter follows a very similar path the previous chapter, with more complicated but similar statements and proofs.

In Section 4.1 we define bispan categories of compatible pairs of indexing systems.

In Section 4.2 we define (bi-incomplete) Tambara functors and some constructions on them.

In Section 4.3 we define some bicategorical gadgets that are used for the main result of the chapter.

In Section 4.4 we prove the main result of the chapter: the category of coherent $(\mathcal{O}'_a, \mathcal{O}'_m)$-monoids in the category of $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functors is equivalent to the category of $(\mathcal{O}'_a, \mathcal{O}'_m)$-Tambara functors. A conjecture of Blumberg and Hill [2, Conjecture 7.94] is a corollary of our main result. It also immediately follows that G-Tambara functors are precisely the coherent $(\mathrm{Set}^G, \mathrm{Set}^G)$-monoids in the category of G-Mackey functors.

In Section 4.5 we show that the category of $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functors is a module category over a tight symmetric bimonoidal category determined by $(\mathcal{O}_a, \mathcal{O}_m)$. We also provide a comparison to another characterisation of $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functors but see that it is less general than the characterisation given for our main result.

4.1 $(\mathcal{O}_a, \mathcal{O}_m)$-bispan categories

Definition 4.1. We say a pair of indexing categories, $(\mathcal{O}_a, \mathcal{O}_m)$, is **compatible** if for any pair of morphisms $f : A \to X$ in $\mathcal{O}_a$ and $g : X \to Y$ in $\mathcal{O}_m$ the dependent product $\pi_Y : \Pi_g(A, f) \to Y, (y, \sigma) \mapsto y$ is in $\mathcal{O}_a$.

Let X be a finite G-set, and $(\mathcal{O}_a, \mathcal{O}_m)$ be a compatible pair of indexing categories. An $(\mathcal{O}_a, \mathcal{O}_m)$ **bispan** over X from (A, p_A) to (D, p_D) is a diagram in Set^G of the form

$$\begin{tikzcd} A \arrow[d, "p_A"] & B \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "p_B"] & C \arrow[r, "h"] \arrow[d, "p_C"] & D \arrow[d, "p_D"] \\ X \arrow[r, equal] & X \arrow[r, equal] & X \arrow[r, equal] & X \end{tikzcd}$$

such that g is in $\mathcal{O}_m$ and h is in $\mathcal{O}_a$. An isomorphism of $(\mathcal{O}_a, \mathcal{O}_m)$ bispans over X from (A, p_A) to (D, p_D) is a pair of isomorphisms $\varphi_B : B \to B'$, $\varphi_C : C \to C'$ of G-sets such that the following diagram commutes.

$$\begin{tikzcd} & B \arrow[r, "g"] \arrow[dl, bend right, "f"'] \arrow[d, "\varphi_B"] \arrow[dd, bend right, "p_B"'] & C \arrow[d, "\varphi_C"] \arrow[dr, "h"] \arrow[dd, bend right, "p_C"'] & \\ A \arrow[dr, "p_A"'] & B' \arrow[l, "f'"'] \arrow[r, "g'"] \arrow[d, "p_{B'}"] & C' \arrow[r, "h'"] \arrow[d, "p_{C'}"] & D \arrow[dl, "p_D"] \\ & X \arrow[r, equal] & X & \end{tikzcd}$$

The $(\mathcal{O}_a, \mathcal{O}_m)$**-bispan category** over X, $\mathcal{P}_X^{(\mathcal{O}_a, \mathcal{O}_m)}$, is the category whose objects are pairs (A, p_A) where A is a finite G-set and p_A is a G-equivariant map $A \to X$, and whose morphisms are isomorphism classes of $(\mathcal{O}_a, \mathcal{O}_m)$ bispans over X. When $(\mathcal{O}_a, \mathcal{O}_m) = (\mathrm{Set}^G, \mathrm{Set}^G)$ we write $\mathcal{P}_X^G$ for $\mathcal{P}_X^{(\mathcal{O}_a, \mathcal{O}_m)}$ and call it the G**-bispan category** over X. Given two $(\mathcal{O}_a, \mathcal{O}_m)$-bispans over X, $\gamma = (A \xleftarrow{f_1} B \xrightarrow{f_2} C \xrightarrow{f_3} D)$ and $\omega = (D \xleftarrow{g_1} E \xrightarrow{g_2} F \xrightarrow{g_2} H)$, their composition $\omega\gamma$ is extracted from the top of the following diagram, where '*pb*' indicates a pullback square and '*exp*' indicates an exponential diagram.

$$\begin{tikzcd}
 & & & B \times_C (E \times_F \Pi_{g_2}(C \times_D E)) \arrow[dl, "(b{,}(e{,}(f{,}\sigma)))\mapsto(b{,}e)"'] \arrow[dr, "pr_{E\times_F\Pi_{g_2}(C\times_D E)}"] & & & \\
 & & B \times_D E \arrow[ddl, "pr_B"'] \arrow[dr, "f_2\times_D id_E"] & pb & E \times_F \Pi_{g_2}(C \times_D E) \arrow[r, "pr_{\Pi_{g_2}(C\times_D E)}"] \arrow[dl, "ev"] & \Pi_{g_2}(C \times_D E) \arrow[dd, "\pi_F^{C\times_D E}"] & \\
 & & pb & C \times_D E \arrow[dl, "pr_C"] \arrow[dr, "pr_E"] & exp & & \\
 & B \arrow[r, "f_2"'] \arrow[dl, "f_1"] & C \arrow[dr, "f_3"'] & pb & E \arrow[r, "g_2"'] \arrow[dl, "g_1"] & F \arrow[dr, "g_3"] & \\
A & & & D & & & H
\end{tikzcd}$$

The composition $\omega\gamma$ is (the isomorphism class of) the $(\mathcal{O}_a, \mathcal{O}_m)$-bispan over X,

$$A \xleftarrow{f_1 pr_B} B \times_C (E \times_F \prod_{g_2}(C \times_D E)) \xrightarrow{pr_{\Pi_{g_2}(C\times_D E)}} \prod_{g_2}(C \times_D E) \xrightarrow{g_3\pi_F^{C\times_D E}} H.$$

Remark 4.2. Asking for $(\mathcal{O}_a, \mathcal{O}_m)$ to be compatible is precisely what is required to ensure that for any pair of morphisms $f : A \to X$ in $\mathcal{O}_a$ and $g : X \to Y$ in $\mathcal{O}_m$ the exponential diagram

$$\begin{tikzcd}
X \times_Y \Pi_g A \arrow[rr, "pr_{\Pi_g A}"] \arrow[d, "ev"'] & & \Pi_g A \arrow[d, "\pi_Y^A"] \\
A \arrow[r, "f"'] & X \arrow[r, "g"'] & Y
\end{tikzcd}$$

describes an $(\mathcal{O}_a, \mathcal{O}_m)$-bispan $(A \xleftarrow{ev} X \times_Y \Pi_g A \xrightarrow{pr_{\Pi_g A}} \Pi_g A \xrightarrow{\pi_Y^A} Y)$. Hence ensuring that the composition described above is well-defined.

Lemma 4.3. *Products in $\mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ is are given by disjoint union. The structure maps for the product $X_1 \amalg X_2$ are represented by $(X_1 \amalg X_2 \hookleftarrow X_i = X_i = X_i)$.*

Remark 4.4. Let H be a subgroup of G, then given compatible pair of indexing systems $(\mathcal{O}_a, \mathcal{O}_m)$ in Set^G we get a compatible pair of indexing systems $(i_H^*\mathcal{O}_a, i_H^*\mathcal{O}_m)$ in Set^H. Via the equivalence of categories $\mathrm{Set}^G_{G/H} \simeq \mathrm{Set}^H$ from Remark 2.4 we have an equivalence of categories $\mathcal{P}_{G/H}^{(\mathcal{O}_a,\mathcal{O}_m)} \simeq \mathcal{P}_*^{(i_H^*\mathcal{O}_a, i_H^*\mathcal{O}_m)}$.

There are a pair of functors

$$\mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \times \mathcal{P}_Y^{(\mathcal{O}_a,\mathcal{O}_m)} \to \mathcal{P}_{X\amalg Y}^{(\mathcal{O}_a,\mathcal{O}_m)}$$
$$((A, p_A), (B, p_B)) \mapsto (A \amalg B, p_A \amalg p_B)$$
$$((A \xleftarrow{f} B \xrightarrow{g} C \xrightarrow{h} D), (A' \xleftarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} D'))$$
$$\mapsto (A \amalg A' \xleftarrow{f\amalg f'} B \amalg B' \xrightarrow{g\amalg g'} C \amalg C' \xrightarrow{h\amalg h'} D \amalg D')$$

and

$$\begin{aligned}
\mathcal{P}^{(\mathcal{O}_a,\mathcal{O}_m)}_{X\amalg Y} &\to \mathcal{P}^{(\mathcal{O}_a,\mathcal{O}_m)}_{X} \times \mathcal{P}^{(\mathcal{O}_a,\mathcal{O}_m)}_{Y}\\
A &\mapsto (p_A^{-1}(X), p_A^{-1}(Y))\\
(A \xleftarrow{f} B \xrightarrow{g} C \xrightarrow{h} D)&\\
\mapsto ((p_A^{-1}(X) \xleftarrow{f} p_B^{-1}(X) \xrightarrow{g} p_C^{-1}(X) \xrightarrow{h} p_D^{-1}(X)), &(p_A^{-1}(Y) \xleftarrow{f} p_B^{-1}(Y) \xrightarrow{g} p_C^{-1}(Y) \xrightarrow{h} p_D^{-1}(Y))).
\end{aligned}$$

Lemma 4.5. *The two functors above are inverse to each other, defining an equivalence of categories* $\mathcal{P}^{(\mathcal{O}_a,\mathcal{O}_m)}_{X} \times \mathcal{P}^{(\mathcal{O}_a,\mathcal{O}_m)}_{Y} \cong \mathcal{P}^{(\mathcal{O}_a,\mathcal{O}_m)}_{X\amalg Y}$.

Definition 4.6. Given a morphism $\varphi : X \to Y$ of G-sets the base change functors from Definition 2.3 induce functors

$$\begin{tikzcd}
\mathcal{P}^{(\mathcal{O}_a,\mathcal{O}_m)}_{Y} \arrow[r, bend right, "\Delta_\varphi"'] & \mathcal{P}^{(\mathcal{O}_a,\mathcal{O}_m)}_{X} \arrow[l, bend right, "\Sigma_\varphi"']
\end{tikzcd}$$

which agrees on objects with the original functors on slice categories of Set^G and on morphisms we apply the base of change functor to each leg of the $(\mathcal{O}_a,\mathcal{O}_m)$-bispan over X. For Σ_φ we have

$$\Sigma_\varphi : \left(\begin{tikzcd}
A \arrow[d, "p_A"] & B \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "p_B"] & C \arrow[r, "h"] \arrow[d, "p_C"] & D \arrow[d, "p_D"]\\
X \arrow[r, equal] & X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}\right) \mapsto \left(\begin{tikzcd}
A \arrow[d, "\varphi p_A"] & B \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\varphi p_B"] & C \arrow[r, "h"] \arrow[d, "\varphi p_C"] & D \arrow[d, "\varphi p_D"]\\
Y \arrow[r, equal] & Y \arrow[r, equal] & Y \arrow[r, equal] & Y
\end{tikzcd}\right)$$

and it is immediate from this that Σ_φ is functorial. For Δ_φ we have

$$\Delta_\varphi : \left(\begin{tikzcd}
A \arrow[d, "p_A"] & B \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "p_B"] & C \arrow[r, "h"] \arrow[d, "p_C"] & D \arrow[d, "p_D"]\\
Y \arrow[r, equal] & Y \arrow[r, equal] & Y \arrow[r, equal] & Y
\end{tikzcd}\right)$$

$$\mapsto \left(\begin{tikzcd}
A \times_Y X \arrow[d, "pr_X"] & B \times_Y X \arrow[l, "f\times_Y \mathrm{id}_X"'] \arrow[r, "g\times_Y \mathrm{id}_X"] \arrow[d, "pr_X"] & C \times_Y X \arrow[r, "h\times_Y \mathrm{id}_X"] \arrow[d, "pr_X"] & D \times_Y X \arrow[d, "pr_X"]\\
X \arrow[r, equal] & X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}\right).$$

To see that this is functorial we must check that applying Δ_φ commutes with composition of bispans. This boils down to providing isomorphisms

$$\left(B \times_C \left(E \times_F \prod_{g_2} (C \times_D E)\right)\right) \times_Y X \xrightarrow{\cong}$$
$$\left((B \times_Y X) \times_{(C\times_Y X)} \left((E \times_Y X) \times_{(F\times_Y X)} \prod_{g_2 \times_Y \mathrm{id}_X} \left((C \times_Y X) \times_{(D\times_Y X)} (E \times_Y X)\right)\right)\right)$$

and

$$\prod_{g_2} (C \times_D E) \xrightarrow{\cong} \prod_{g_2 \times_Y \mathrm{id}_X} \big((C \times_Y X) \times_{(D \times_Y X)} (E \times_Y X)\big) .$$

The first isomorphism is given by

$$((b, (e, (f, \sigma))), x) \mapsto \big((b, x), \big((e, x), \big((f, x), (e', x) \mapsto \big((pr_C \sigma(e'), x), (e', x)\big)\big)\big)\big)$$

with inverse

$$((b, x), ((e, x), ((f, x), \sigma))) \mapsto \big(\big(b, \big(e, \big(f, e' \mapsto \big(pr_C \sigma(e'), e'\big)\big)\big)\big), x\big) .$$

The second isomorphism is given by

$$((f, \sigma), x) \mapsto \big((f, x), (e', x) \mapsto \big((pr_C \sigma(e'), x), (e', x)\big)\big)$$

with inverse

$$((f, x), \sigma) \mapsto \big(\big(f, e' \mapsto \big(pr_C \sigma(e'), e'\big)\big), x\big) .$$

Lemma 4.7. *The functor Δ_φ is left adjoint to Σ_φ.*

Proof. Given $A \in \mathcal{P}_Y^{(\mathcal{O}_a, \mathcal{O}_m)}$ and $D \in \mathcal{P}_X^{(\mathcal{O}_a, \mathcal{O}_m)}$ we have a natural isomorphism $\mathcal{P}_X^{(\mathcal{O}_a, \mathcal{O}_m)}(\Delta_\varphi A, D) \cong \mathcal{P}_Y^{(\mathcal{O}_a, \mathcal{O}_m)}(A, \Sigma_\varphi D)$ given by

$$\left(\begin{tikzcd} X \times_Y A \arrow[dr, "pr_A"'] & B \arrow[l, "f_1"'] \arrow[r, "f_2"] \arrow[d, "p_B"] & C \arrow[r, "f_3"] \arrow[d, "p_C"'] & D \arrow[dl, "p_D"] \\ & X \arrow[r, equal] & X & \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} A \arrow[dr, "p_A"'] & B \arrow[l, "pr_A f_1"'] \arrow[r, "f_2"] \arrow[d, "\varphi p_B"] & C \arrow[r, "f_3"] \arrow[d, "\varphi p_C"'] & D \arrow[dl, "\varphi p_D"] \\ & Y \arrow[r, equal] & Y & \end{tikzcd}\right)$$

with inverse

$$\left(\begin{tikzcd} A \arrow[dr, "p_A"'] & B \arrow[l, "g_1"'] \arrow[r, "g_2"] \arrow[d, "p_B"] & C \arrow[r, "g_3"] \arrow[d, "p_C"'] & D \arrow[dl, "\varphi p_D"] \\ & Y \arrow[r, equal] & Y & \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} X \times_Y A \arrow[dr, "pr_X"'] & B \arrow[l, "(p_D g_3 g_2, g_1)"'] \arrow[r, "g_2"] \arrow[d, "p_D g_3 g_2"] & C \arrow[r, "g_3"] \arrow[d, "p_D g_3"'] & D \arrow[dl, "p_D"] \\ & X \arrow[r, equal] & X & \end{tikzcd}\right).$$

□

4.2 $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functors

Definition 4.8. An $(\mathcal{O}_a, \mathcal{O}_m)$**-semi-Tambara functor** $\underline{S}$ over X is a product-preserving functor $\mathcal{P}_X^{(\mathcal{O}_a, \mathcal{O}_m)} \to \mathbf{Set}$. We write $\mathrm{sTamb}_X^{(\mathcal{O}_a, \mathcal{O}_m)}$ for the category of $(\mathcal{O}_a, \mathcal{O}_m)$-semi-Tambara functors and natural transformations between them. As the product in $\mathcal{P}_X^{(\mathcal{O}_a, \mathcal{O}_m)}$ is disjoint union and $\underline{S}$ is product-preserving, $\underline{S}(\emptyset)$ is always a singleton. For $A \in \mathcal{P}_X^{(\mathcal{O}_a, \mathcal{O}_m)}$ the set $\underline{S}(A)$ is naturally a commutative semiring with additive unit map $\underline{S}(\emptyset = \emptyset = \emptyset \to A)$, multiplicative unit map $\underline{S}(\emptyset = \emptyset \to A = A)$, additive binary operation $\underline{S}(A \amalg A = A \amalg A = A \amalg A \xrightarrow{\nabla} A)$, and multiplicative binary operation

$\underline{S}(A \amalg A = A \amalg A \xrightarrow{\nabla} A = A)$. For any map $f : X \to Y$ of G-sets, the corresponding map $res_f = \underline{S}(Y \xleftarrow{f} X = X = X) : \underline{S}(Y) \to \underline{S}(X)$ is a morphism of semirings. For any map $g : X \to Y$ in $\mathcal{O}_m$, the corresponding map $nm_g = \underline{S}(X = X \xrightarrow{g} Y = Y) : \underline{S}(X) \to \underline{S}(Y)$ is a monoid homomorphism with respect to the multiplicative operation. For any map $h : X \to Y$ in $\mathcal{O}_a$, the corresponding map $tr_h = \underline{S}(X = X = X \xrightarrow{h} X) : \underline{S}(X) \to \underline{S}(Y)$ is a monoid homomorphism with respect to the additive operation.

Definition 4.9. An $(\mathcal{O}_a, \mathcal{O}_m)$**-Tambara functor** over X is an $(\mathcal{O}_a, \mathcal{O}_m)$-semi-Tambara functor $\underline{S}$ such that $\underline{S}(A)$ is a ring for all $A \in \mathcal{P}_X^{(\mathcal{O}_a, \mathcal{O}_m)}$. We write $\mathrm{Tamb}_X^{(\mathcal{O}_a, \mathcal{O}_m)}$ for the category of $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functors over X and natural transformations between them. We call an $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functor incomplete when $\mathcal{O}_a = \mathrm{Set}^G$ and $\mathcal{O}_m \neq \mathrm{Set}^G$, and bi-incomplete when both $\mathcal{O}_a \neq \mathrm{Set}^G$ and $\mathcal{O}_m \neq \mathrm{Set}^G$. When G is the trivial group the category of G-Tambara functors is equivalent to the category of commutative rings, because such Tambara functors are determined up to isomorphism by the value they take on a singleton (and the bispans which determine the structure maps).

Definition 4.10. We can define a **forgetful functor**, $\mathcal{U}_X : \mathrm{sTamb}_X^{(\mathcal{O}_a, \mathcal{O}_m)} \to \mathrm{sMack}_X^{\mathcal{O}_a}$ via precomposition by the "inclusion" functor $\iota : \mathcal{A}_X^{\mathcal{O}_a} \to \mathcal{P}_X^{(\mathcal{O}_a, \mathcal{O}_m)}$ which is the identity on objects and on morphisms is

$$\iota(X \xleftarrow{f_1} A \xrightarrow{f_2} Y) = (X \xleftarrow{f_1} A = A \xrightarrow{f_2} Y).$$

That is, any $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functor contains an $\mathcal{O}_a$-Mackey functor.

Definition 4.11. Given a morphism $\varphi : X \to Y$ in Set^G and $\underline{S} \in \mathrm{sTamb}_Y^{(\mathcal{O}_a, \mathcal{O}_m)}$ we define the **restriction** $\mathcal{R}_\varphi \underline{S}$ of $\underline{S}$ along φ to be the left Kan extension of $\underline{S}$ along Δ_φ.

$$\begin{tikzcd}
\mathcal{P}_Y^{(\mathcal{O}_a, \mathcal{O}_m)} \arrow[r, "\underline{S}"] \arrow[d, "\Delta_\varphi"'] & Set \\
\mathcal{P}_X^{(\mathcal{O}_a, \mathcal{O}_m)} \arrow[ur, "\mathrm{Lan}_{\Delta_\varphi} \underline{S} =: \mathcal{R}_\varphi \underline{S}"'] &
\end{tikzcd}$$

Write $\alpha_{\underline{S}} : \underline{S} \to (\mathcal{R}_\varphi \underline{S})\Delta_\varphi$ for the universal map of the left Kan extension. Given a morphism $\underline{S} \xrightarrow{f} \underline{T}$ in $\mathrm{sMack}_Y^{\mathcal{O}}$ there is a morphism $\mathcal{R}_\varphi(f) : \mathcal{R}_\varphi \underline{S} \to \mathcal{R}_\varphi \underline{T}$ such that $\mathcal{R}_\varphi(f)\alpha_{\underline{S}} = \alpha_{\underline{T}} f$ by the universal property of $\alpha_{\underline{S}}$. This makes $\mathcal{R}_\varphi$ a functor $\mathrm{sTamb}_Y^{(\mathcal{O}_a, \mathcal{O}_m)} \to \mathrm{sTamb}_X^{(\mathcal{O}_a, \mathcal{O}_m)}$; we call this the **restriction functor** along φ.

When G is the trivial group a G-Tambara functor over X, $\underline{S}$, is an X-parameterised family of commutative rings. We write $\underline{S}(A)_x$ for the commutative ring in $\underline{S}(A)$ corresponding to $x \in X$. In this case we have $\mathcal{R}_\varphi \underline{S}(A)_x = \underline{S}(A)_{\varphi(x)}$.

The restriction functor admits a convenient explicit description.

Proposition 4.12. *Given $\underline{S} \in \mathrm{sTamb}_Y^{\mathcal{O}}$ and $\varphi : X \to Y$ in Set^G we have $\mathcal{R}_\varphi \underline{S} = \underline{S}\Sigma_\varphi$.*

Proof. This is a direct application of Yoneda reduction, setting $K = \Delta_\varphi$, $H = \Sigma_\varphi$ and $F = \underline{S}$ in Lemma A.2. $\square$

Lemma 4.13. *Let $\underline{S} \in \mathrm{sTamb}_Y^{(\mathcal{O}_a,\mathcal{O}_m)}$ and $\varphi : X \to Y$ then $\mathcal{R}_\varphi \underline{S} \in \mathrm{sTamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$.*

Proof. We note that Σ_φ preserves products in $\mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ (given by disjoint union) and so, as a composite of product-preserving functors, $\mathcal{R}_\varphi \underline{S}$ is product-preserving. $\square$

Definition 4.14. Given a morphism $\varphi : X \to Y$ in Set^G and $\underline{S} \in \mathrm{sTamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ we define the **transfer** $\mathcal{T}_\varphi \underline{S}$ of $\underline{S}$ along φ to be $\underline{S}\Delta_\varphi \in \mathrm{sTamb}_Y^{(\mathcal{O}_a,\mathcal{O}_m)}$. Precomposition with Δ_φ is natural in $\underline{S}$ so $\mathcal{T}_\varphi$ defines a functor $\mathrm{sTamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \to \mathrm{sTamb}_Y^{(\mathcal{O}_a,\mathcal{O}_m)}$ which we call the **transfer functor** along φ.

When G is the trivial group and $\underline{S}$ is a G-Tambara functor over X we have $\mathcal{T}_\varphi \underline{S}(A)_y = \prod_{x \in f^{-1}(y)} \underline{S}(A)_x$.

Lemma 4.15. *Let $\underline{S} \in \mathrm{sTamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ and $\varphi : X \to Y$ then $\mathcal{T}_\varphi \underline{S} \in \mathrm{sTamb}_Y^{(\mathcal{O}_a,\mathcal{O}_m)}$.*

Proof. We note that Δ_φ preserves products in $\mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ and so, as a composite of product-preserving functors, $\mathcal{T}_\varphi \underline{S}$ is product-preserving. $\square$

Lemma 4.16. *There are natural isomorphisms $\mathcal{R}_\varphi \mathcal{U}_Y \cong \mathcal{U}_X \mathcal{R}_\varphi$ and $\mathcal{T}_\varphi \mathcal{U}_X \cong \mathcal{U}_Y \mathcal{T}_\varphi$. That is, up to natural isomorphism the following diagrams commute.*

$$\begin{tikzcd}
\mathrm{sTamb}_Y^{(\mathcal{O}_a,\mathcal{O}_m)} \arrow[r, "\mathcal{R}_\varphi"] \arrow[d, "\mathcal{U}_Y"'] & \mathrm{sTamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \arrow[d, "\mathcal{U}_X"] \\
\mathrm{sMack}_Y^{\mathcal{O}_a} \arrow[r, "\mathcal{R}_\varphi"'] & \mathrm{sMack}_X^{\mathcal{O}_a}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{sTamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \arrow[r, "\mathcal{T}_\varphi"] \arrow[d, "\mathcal{U}_X"'] & \mathrm{sTamb}_Y^{(\mathcal{O}_a,\mathcal{O}_m)} \arrow[d, "\mathcal{U}_Y"] \\
\mathrm{sMack}_X^{\mathcal{O}_a} \arrow[r, "\mathcal{T}_\varphi"'] & \mathrm{sMack}_Y^{\mathcal{O}_a}
\end{tikzcd}$$

Proof. Commuting forgetful functors with restriction and transfer functors are immediate because they are naturally isomorphic to precomposition by Σ_φ and Δ_φ respectively, for both Mackey functors and Tambara functors. $\square$

Definition 4.17. Given a morphism $\varphi : X \to Y$ in Set^G and $\underline{S} \in \mathrm{sTamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ we define the **norm** $\mathcal{N}_\varphi \underline{S}$ of $\underline{S}$ along φ to be the left Kan extension of $\underline{S}$ along Σ_φ.

$$\begin{tikzcd}
\mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \arrow[r, "\underline{S}"] \arrow[d, "\Sigma_\varphi"'] & \mathrm{Set} \\
\mathcal{P}_Y^{(\mathcal{O}_a,\mathcal{O}_m)} \arrow[ur, "\mathrm{Lan}_{\Sigma_\varphi} \underline{S} =: \mathcal{N}_\varphi \underline{S}"']
\end{tikzcd}$$

Write $\alpha_{\underline{S}} : \underline{S} \to (\mathcal{N}_\varphi \underline{S})\Sigma_\varphi$ for the universal map of the left Kan extension. Given a morphism $\underline{S} \xrightarrow{f} \underline{T}$ in $\mathrm{sTamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ there is a morphism $\mathcal{N}_\varphi(f) : \mathcal{N}_\varphi \underline{S} \to \mathcal{N}_\varphi \underline{T}$ such that $\mathcal{N}_\varphi(f)\alpha_{\underline{S}} = \alpha_{\underline{T}} f$ by the universal property of $\alpha_{\underline{S}}$. This makes $\mathcal{N}_\varphi$ a functor $\mathrm{sTamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \to \mathrm{sTamb}_Y^{(\mathcal{O}_a,\mathcal{O}_m)}$; we call this the **norm functor** along φ.

When G is the trivial group and $\underline{S}$ is a G-Tambara functor over X we have $\mathcal{N}_\varphi \underline{S}(A)_y = \bigotimes_{x \in f^{-1}(y)} \underline{S}(A)_x$.

Notice that this norm functor is defined differently from the norm functor on Mackey functors. Unlike in the Mackey functor case there are no compatibility restrictions controlling which norm functors are well-defined for Tambara functors.

Lemma 4.18. *Let $\underline{S} \in \mathrm{sTamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ and $\varphi : X \to Y$. Then $\mathcal{N}_\varphi \underline{S} \in \mathrm{sTamb}_Y^{(\mathcal{O}_a,\mathcal{O}_m)}$.*

Proof. By Lemma A.6 left Kan extensions of product-preserving functors are product-preserving. □

The norm functor does not have a convenient explicit description along the lines of Proposition 4.12. However, similar to describing the norm on Mackey functors, there is an explicit description by the objectwise formula for a left Kan extension. We have

$$\mathcal{N}_\varphi \underline{S}(A) \cong \int^{\mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)}} \mathcal{P}_Y^{(\mathcal{O}_a,\mathcal{O}_m)}(\Sigma_\varphi B, A) \times \underline{S}(B)\, dB.$$

Elements in this coend are equivalence classes of elements of the form $(\Sigma_\varphi B \xleftarrow{f_1} C \xrightarrow{f_2} D \xrightarrow{f_3} A, b)$ where the first coordinate is an $(\mathcal{O}_a, \mathcal{O}_m)$-bispan over Y and the second coordinate is an element of $\underline{S}(B)$. We write $[\Sigma_\varphi B \xleftarrow{f_1} C \xrightarrow{f_2} D \xrightarrow{f_3} A, b]$ for the element represented by $(\Sigma_\varphi B \xleftarrow{f_1} C \xrightarrow{f_2} D \xrightarrow{f_3} A, b)$.

Recall that for $\underline{S} \in \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ and $D \in \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ the unit of the additive monoid structure is given by $tr_{\emptyset \to D} = \underline{S}(\emptyset = \emptyset = \emptyset \to D)$, and the binary operation by $tr_\nabla = \underline{S}(D \amalg D = D \amalg D = D \amalg D \xrightarrow{\nabla} D)$. The additive structure on $\mathcal{N}_\varphi \underline{S}(D)$ is given by

$$\mathcal{N}_\varphi \underline{S}(D) \times \mathcal{N}_\varphi \underline{S}(D) \cong \mathcal{N}_\varphi \underline{S}(D \amalg D) \xrightarrow{tr_\nabla} \mathcal{N}_\varphi \underline{S}(D)$$

where the first map is the isomorphism from Lemma A.6. On elements this map is given by

$$\begin{aligned}
&\left(\left[A \xleftarrow{\alpha_1} B \xrightarrow{\alpha_2} C \xrightarrow{\alpha_3} D, a\right], \left[A' \xleftarrow{\beta_1} B' \xrightarrow{\beta_2} C' \xrightarrow{\beta_3} D, a'\right]\right) \\
&\mapsto \left[A \amalg A' \xleftarrow{\alpha_1 \amalg \beta_1} B \amalg B' \xrightarrow{\alpha_2 \amalg \beta_2} C \amalg C' \xrightarrow{\alpha_3 \amalg \beta_3} D \amalg D, (a, a')\right] \\
&\mapsto \left[A \amalg A' \xleftarrow{\alpha_1 \amalg \beta_1} B \amalg B' \xrightarrow{\alpha_2 \amalg \beta_2} C \amalg C' \xrightarrow{\nabla(\alpha_3 \amalg \beta_3)} D, (a, a')\right].
\end{aligned}$$

Recalling that $\mathcal{N}_\varphi \underline{S}(\emptyset)$ is necessarily a singleton we know that $(\emptyset = \emptyset = \emptyset = \emptyset, *)$ is a representative of the only class. We can use this to pick out a representative of the additive identity in $\mathcal{N}_\varphi \underline{S}(D)$, under the unit map we have

$$[\emptyset = \emptyset = \emptyset = \emptyset, *] \mapsto [\emptyset = \emptyset = \emptyset \to D, *].$$

If 0_A is the additive identity in $\underline{S}(A)$ then $[A \leftarrow B \to C \to D, 0_A] = [\emptyset = \emptyset = \emptyset \to D, *]$. We can identify these two representatives along $(A \leftarrow B \to C \to D, *)$ via the bispan $(\emptyset = \emptyset = \emptyset \to A)$.

Lemma 4.19. *Let $\varphi : X \to Y$ be a map of finite G-sets. Given $\underline{S} \in \mathrm{Tamb}_X^{(\mathcal{O}_a, \mathcal{O}_m)}$ we have $\mathcal{T}_\varphi \underline{S} \in \mathrm{Tamb}_Y^{(\mathcal{O}_a, \mathcal{O}_m)}$ and $\mathcal{N}_\varphi \underline{S} \in \mathrm{Tamb}_Y^{(\mathcal{O}_a, \mathcal{O}_m)}$. Given $\underline{T} \in \mathrm{Tamb}_Y^{(\mathcal{O}_a, \mathcal{O}_m)}$ we have $\mathcal{R}_\varphi \underline{T} \in \mathrm{Tamb}_X^{(\mathcal{O}_a, \mathcal{O}_m)}$.*

Proof. By Lemma 4.16 restriction and transfer functors commute with the forgetful functor. So restriction and transfer functors take $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functors to $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functors, because they take $\mathcal{O}_a$-Mackey functors to $\mathcal{O}_a$-Mackey functors.
Given any element $[A \leftarrow B \to C \to D, a] \in \mathcal{N}_\varphi \underline{S}(D)$ we have

$$\begin{aligned}
[A \leftarrow B \to C \to D, a] &+ [A \leftarrow B \to C \to D, -a] \\
&= [A \amalg A \leftarrow B \amalg B \to C \amalg C \to D, (a, -a)] \\
&= [A \leftarrow B \to C \to D, a + (-a)] \\
&= [A \leftarrow B \to C \to D, 0_A]
\end{aligned}$$

where the representatives are identified across the second quality along $(A \leftarrow B \to C \to D, (a, -a))$ via the bispan $(A \amalg A = A \amalg A = A \amalg A \xrightarrow{\nabla} A)$. ☐

The norm functor on Tambara functors is compatible with the norm functor on Mackey functors in the following way.

Theorem 4.20. *Let $(\mathcal{O}_a, \mathcal{O}_m)$ be a compatible pair of indexing categories, and let $\varphi : X \to Y$ be a map in Set^G such that given any morphism $\gamma : A \to X$ in $\mathcal{O}_a$ the dependent product $\pi_Y : \Pi_\varphi A \to Y$ is in $\mathcal{O}_a$. Then there is a natural isomorphism $\theta : \mathcal{N}_\varphi \mathcal{U}_X \cong \mathcal{U}_Y \mathcal{N}_\varphi$. That is, up to a natural isomorphism the following diagram commutes.*

$$\begin{tikzcd}
\mathrm{sTamb}_X^{(\mathcal{O}_a, \mathcal{O}_m)} \arrow[r, "\mathcal{N}_\varphi"] \arrow[d, "\mathcal{U}_X"'] & \mathrm{sTamb}_Y^{(\mathcal{O}_a, \mathcal{O}_m)} \arrow[d, "\mathcal{U}_Y"] \\
\mathrm{sMack}_X^{\mathcal{O}_a} \arrow[r, "\mathcal{N}_\varphi"'] & \mathrm{sMack}_Y^{\mathcal{O}_a}
\end{tikzcd}$$

Proof. For $\underline{S} \in \mathrm{sTamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ the map $\theta_{\underline{S}} : \mathcal{N}_\varphi\mathcal{U}_X\underline{S} \to \mathcal{U}_Y\mathcal{N}_\varphi\underline{S}$ has components

$$\begin{aligned}&\mathcal{N}_\varphi\mathcal{U}_X\underline{S}(D) \to \mathcal{U}_Y\mathcal{N}_\varphi\underline{S}(D),\\ [\prod_\varphi B \xleftarrow{f_1} C \xrightarrow{f_2} D, x] &\mapsto [X \times_Y C = X \times_Y C \xrightarrow{pr_C} C \xrightarrow{f_2} D, res_{\tilde{f}_1}(x)]\end{aligned}$$

where $\tilde{f}_1 : X \times_Y C \to B, (x,c) \mapsto (\pi_\sigma^B(f_1(c))(x))$ is the adjoint map to f_1 across the adjunction $\Delta_\varphi \dashv \Pi_\varphi$. The inverse natural transformation, θ^{-1}, is given by

$$\begin{aligned}&\mathcal{U}_Y\mathcal{N}_\varphi\underline{S}(D) \to \mathcal{N}_\varphi\mathcal{U}_X\underline{S}(D),\\ [B \xleftarrow{g_1} C \xrightarrow{g_2} E \xrightarrow{g_3} D, x] &\mapsto [\prod_\varphi(X \times_Y E) \xleftarrow{\eta_E} E \xrightarrow{g_3} D, nm_{(p_Bg_1,g_2)}(res_{g_1}(x))]\end{aligned}$$

where η_E is the E component of the unit of the adjunction $\Delta_\varphi \dashv \Pi_\varphi$ and $(p_Bg_1, g_2) : C \to X \times_Y E, c \mapsto (p_Bg_1(c), g_2(c))$. The map $C \to X \times_Y C, c \mapsto (p_Bg_1(c), c)$ is a monomorphism so is in $\mathcal{O}_m$ by Lemma 2.27. The map $X \times_Y C \to X \times_Y E, (x,c) \mapsto (x, g_2(c))$ is in $\mathcal{O}_m$ as it is a pullback of g_2 which is in $\mathcal{O}_m$. As the map (p_Bg_1, g_2) is the composite of these two maps it is also in $\mathcal{O}_m$, so θ^{-1} is well-defined.
To check these are inverse to each other we note that $(pr_X\,\mathrm{id}_{X\times_Y C}, pr_C) = \mathrm{id}_{X\times_Y C}$ and so we can identify $(\Pi_\varphi B \xleftarrow{f_1} C \xrightarrow{f_2} D, x)$ with $(\Pi_\varphi(X \times_Y C) \xleftarrow{\eta_C} C \xrightarrow{f_2} D, res_{\tilde{f}_1}(x))$ along $(\Pi_\varphi(X \times_Y C) \xleftarrow{\eta_C} C \xrightarrow{f_2} D, x)$ via the span $(B \xleftarrow{\tilde{f}_1} X \times_Y C = X \times_Y C)$. In the other direction we note that $\tilde{\eta_E} = \mathrm{id}_{X\times_Y E}$ and so we can identify $(B \xleftarrow{g_1} C \xrightarrow{g_2} E \xrightarrow{g_3} D, x)$ with $(X \times_Y E = X \times_Y E \xrightarrow{pr_E} E \xrightarrow{g_3} D, nm_{(p_Bg_1,g_2)}(res_{g_1}(x)))$ along $(X \times_Y E = X \times_Y E \xrightarrow{pr_E} E \xrightarrow{g_3} D, x)$ via $(B \xleftarrow{g_1} C \xrightarrow{(p_Bg_1,g_2)} X \times_Y E = X \times_Y E)$. □

Remark 4.21. In the above proof we established the existence of useful representatives for elements in normed Mackey and Tambara functors. In particular, for $\underline{M}$ a Mackey functor every element of $\mathcal{N}_\varphi\underline{M}(A)$ can be written

$$[\prod_\varphi(X \times_Y B) \xleftarrow{\eta_B} B \xrightarrow{k} A, x].$$

Similarly, for $\underline{S}$ a Tambara functor every element of $\mathcal{N}_\varphi\underline{S}(A)$ can be written

$$[X \times_Y B = X \times_Y B \xrightarrow{pr_B} B \xrightarrow{k} A, x].$$

We note that the adjunction relationships in the following Lemma are different from those in the Mackey functor case where $\mathcal{T}_\varphi \dashv \mathcal{R}_\varphi \dashv \mathcal{T}_\varphi$ (Lemma 3.24).

Lemma 4.22. *Given $\varphi : X \to Y$ in Set^G the functor $\mathcal{T}_\varphi$ is left adjoint to the functor $\mathcal{R}_\varphi$ which is left adjoint to $\mathcal{N}_\varphi$. That is, $\mathcal{T}_\varphi \dashv \mathcal{R}_\varphi \dashv \mathcal{N}_\varphi$.*

Proof. The first claim is an immediate application of Lemma A.7 as $\mathcal{R}_\varphi$ is defined as $\mathrm{Lan}_{\Delta_\varphi}$ and $\mathcal{T}_\varphi$ is defined as precomposition by Δ_φ. By Proposition 4.12 we know that $\mathcal{R}_\varphi$ is naturally isomorphic to precomposition by Σ_φ. Noting that $\mathcal{N}_\varphi$ is defined as $\mathrm{Lan}_{\Sigma_\varphi}$ second claim is an application of Lemma A.7. □

As in the Mackey functor case, on Tambara functors the transfer and norm functors satisfy a pullback condition with respect to composition with the restriction functor.

Proposition 4.23. *Let*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "pr_Y"] \arrow[d, "pr_X"'] & Y \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"'] & Z \end{tikzcd}$$

be a pullback square in Set^G*. Then the functors* $\mathcal{R}_\beta \mathcal{T}_\alpha$ *and* $\mathcal{T}_{pr_Y}\mathcal{R}_{pr_X}$ *are naturally isomorphic. Similarly, the functors* $\mathcal{R}_\beta\mathcal{N}_\alpha$ *and* $\mathcal{N}_{pr_Y}\mathcal{R}_{pr_X}$ *are naturally isomorphic.*

Proof. By the natural isomorphism $\Delta_\beta\Sigma_\alpha \cong \Sigma_{pr_Y}\Delta_{pr_X}$ in Lemma 2.6 we have $\mathcal{R}_\beta\mathcal{T}_\alpha\underline{S} \cong \underline{S}\Delta_\beta\Sigma_\alpha \cong \underline{S}\Sigma_{pr_Y}\Delta_{pr_X} \cong \mathcal{T}_{pr_Y}\mathcal{R}_{pr_X}\underline{S}$. By Lemma A.5 we have that

$$\mathcal{R}_\beta\mathcal{N}_\alpha\underline{S} = \mathrm{Lan}_{\Delta_\beta}(\mathrm{Lan}_{\Sigma_\alpha}\underline{S}) \cong \mathrm{Lan}_{\Delta_\beta\Sigma_\alpha}\underline{S}$$

and

$$\mathcal{N}_{pr_Y}\mathcal{R}_{pr_X}\underline{S} = \mathrm{Lan}_{\Sigma_{pr_Y}}(\mathrm{Lan}_{\Delta_{pr_X}}\underline{S}) \cong \mathrm{Lan}_{\Sigma_{pr_Y}\Delta_{pr_X}}\underline{S}.$$

We have $\Delta_\beta\Sigma_\alpha \cong \Sigma_{pr_Y}\Delta_{pr_X}$ by Lemma 2.6, hence by Lemmas A.4 and A.5,

$$\mathcal{R}_\beta\mathcal{N}_\alpha\underline{S} \cong \mathrm{Lan}_{\Delta_\beta\Sigma_\alpha}\underline{S} \cong \mathrm{Lan}_{\Sigma_{pr_Y}\Delta_{pr_X}}\underline{S} \cong \mathcal{N}_{pr_Y}\mathcal{R}_{pr_X}\underline{S}. \qquad \square$$

Again, as in the Mackey functor case, the transfer, norm, and restriction functors on Tambara functors satisfy an exponential diagram relation. The proof is less straightforward in this case as the transfer functor is no longer a left Kan extension. Instead we take advantage of the norm, transfer, and restrictions functors being compatible with the forgetful functors.

Lemma 4.24. *Let* $(\mathcal{O}_a, \mathcal{O}_m)$ *and* $(\mathcal{O}'_a, \mathcal{O}'_m)$ *be compatible pairs of indexing categories such* $(\mathcal{O}_a, \mathcal{O}'_m)$ *is also a compatible pair of indexing categories, and such that* $\mathcal{O}_a$ *and* $\mathcal{O}_m$ *are subcategories of* $\mathcal{O}'_a$ *and* $\mathcal{O}'_m$*. If*

$$\begin{tikzcd} Y \times_Z \Pi_g X \arrow[rr, "pr_{\Pi_g X}"] \arrow[d, "ev"'] & & \Pi_g X \arrow[d, "\pi_Z"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

is an exponential diagram in with $f \in \mathcal{O}'_a$ *and* $g \in \mathcal{O}'_m$ *then* $\mathcal{T}_{\pi_Z}\mathcal{N}_q\mathcal{R}_{ev}\underline{S}(A) \cong \mathcal{N}_g\mathcal{T}_f\underline{S}(A)$ *for any* $A \in \mathrm{Set}^G_Z$*. We will write* q *for the projection map* $Y \times_Z \Pi_g X \to \Pi_g X$ *for readability.*

Proof. Because the transfer, norm, and restriction functors commute with the forgetful functors we can check this at the level of Mackey functors. We have

$$\begin{aligned}
\mathcal{U}_Z\mathcal{T}_{\pi_Z}\mathcal{N}_q\mathcal{R}_{ev}\underline{S}(A) &= \mathcal{U}_{\Pi_g X}\mathcal{N}_q\mathcal{R}_{ev}\underline{S}(A\times_Z \prod_g X)\\
&\cong \mathcal{N}_q\mathcal{U}_{Y\times_Z\Pi_g X}\mathcal{R}_{ev}\underline{S}(A\times_Z \prod_g X)\cong \mathcal{T}_{\pi_Z}\mathcal{N}_q\mathcal{R}_{ev}\mathcal{U}_X\underline{S}(A)\\
&\cong \mathcal{N}_g\mathcal{T}_f\mathcal{U}_X\underline{S}(A)\cong \mathcal{U}_Z\mathcal{N}_g\mathcal{T}_f\underline{S}(A)
\end{aligned}$$

where the isomorphisms are acting on elements by

$$\begin{aligned}
&[C \xleftarrow{f_1} D \xrightarrow{f_2} E \xrightarrow{f_3} A\times_Z \prod_g X, c]\\
&\mapsto \left[\prod_q((Y\times_Z \prod_g X)\times_{\Pi_g X} E) \xleftarrow{\eta_E^q} E \xrightarrow{f_3} A\times_Z \prod_g X, \hat{c}\right]\\
&\mapsto \left[\prod_q((Y\times \prod_g X)\times_X ((Y\times_Z \prod_g X)\times_{\Pi_g X} E)) \xleftarrow{(pr_{Y\times_Z\Pi_g X},\mathrm{id})_*\eta_E^q} E \xrightarrow{pr_A f_3} A, \hat{c}\right]\\
&\mapsto \left[\prod_g((Y\times_Z \prod_g X)\times_{\Pi_g X} E) \xleftarrow{\omega} E \xrightarrow{pr_A f_3} A, \hat{c}\right]\\
&\mapsto \left[Y\times_Z E = Y\times_Z E \xrightarrow{pr_E} E \xrightarrow{pr_A f_3} A, nm_{(pr_Y p_C f_1, f_2)}(res_{f_1}(c))\right].
\end{aligned}$$

Here we have $\hat{c} = nm_{(p_C f_1, f_2)}(res_{f_1}(c))$ and we note that ω is η_E^g when postcomposed with the isomorphism $\Pi_g((Y\times_Z \Pi_g X)\times_{\Pi_g X} E)\cong \Pi_g(Y\times_Z E)$. □

Remark 4.25. The compatibility conditions in the previous lemma and the following proposition are there to ensure that the norm functors in question exist at the level of Mackey functors.

Proposition 4.26. *Let $(\mathcal{O}_a,\mathcal{O}_m)$ and $(\mathcal{O}'_a,\mathcal{O}'_m)$ be compatible pairs of indexing categories such $(\mathcal{O}_a,\mathcal{O}'_m)$ is also a compatible pair of indexing categories, and such that $\mathcal{O}_a$ and $\mathcal{O}_m$ are subcategories of $\mathcal{O}'_a$ and $\mathcal{O}'_m$. If*

$$\begin{tikzcd}
Y\times_Z \Pi_g X \arrow[rr, "pr_{\Pi_g X}"] \arrow[d, "ev"'] & & \Pi_g X \arrow[d, "\pi_Z"]\\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

is an exponential diagram in with $f\in\mathcal{O}'_a$ and $g\in\mathcal{O}'_m$ then $\mathcal{T}_{\pi_Z}\mathcal{N}_{pr_{\Pi_g X}}\mathcal{R}_{ev}$ and $\mathcal{N}_g\mathcal{T}_f$ are naturally isomorphic functors $\mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}\to\mathrm{Tamb}_Z^{(\mathcal{O}_a,\mathcal{O}_m)}$.

Proof. As the transfer, norm, and restriction functors commute with the forgetful functors we have that $\mathcal{U}_Z\mathcal{T}_{\pi_Z}\mathcal{N}_{pr_{\Pi_g X}}\mathcal{R}_{ev}\cong\mathcal{T}_{\pi_Z}\mathcal{N}_{pr_{\Pi_g X}}\mathcal{R}_{ev}\mathcal{U}_X\cong\mathcal{N}_g\mathcal{T}_f\mathcal{U}_X\cong\mathcal{U}_Z\mathcal{N}_g\mathcal{T}_f$ and so $\mathcal{T}_{\pi_Z}\mathcal{N}_{pr_{\Pi_g X}}\mathcal{R}_{ev}$ and $\mathcal{N}_g\mathcal{T}_f$ agree on objects, and transfer and restrictions. So it remains to check that they agree on norm maps. That is, we need to show that the following diagram commutes. Note the top isomorphism is the one described in the preceding lemma.

$$\begin{tikzcd}
\mathcal{U}_Z\mathcal{T}_{\pi_Z}\mathcal{N}_{pr_{\Pi_g X}}\mathcal{R}_{ev}\underline{S}(A) \arrow[r, "\cong"] \arrow[d, "nm_\varphi"] & \mathcal{U}_Z\mathcal{N}_g\mathcal{T}_f\underline{S}(A) \arrow[d, "nm_\varphi"] \\
\mathcal{U}_Z\mathcal{T}_{\pi_Z}\mathcal{N}_{pr_{\Pi_g X}}\mathcal{R}_{ev}\underline{S}(B) \arrow[r, "\cong"] & \mathcal{U}_Z\mathcal{N}_g\mathcal{T}_f\underline{S}(B)
\end{tikzcd}$$

We go on to show explicitly that this diagram commutes, however it suffices to note that the horizontal isomorphisms act on element (representatives) on both coordinates, in particular by precomposition in the first and the vertical norm maps act on the first coordinate by postcomposition and do not act on the second coordinate.

The norm maps act by postcomposition of bispans with $(A = A \xrightarrow{\varphi} B = B)$. Going around the top of the diagram we have

$$\begin{aligned}
&[C \xleftarrow{f_1} D \xrightarrow{f_2} E \xrightarrow{f_2} A \times_Z \prod_g X, c] \\
&\quad\mapsto [Y \times_Z E \xleftarrow{\mathrm{id}_Y \times_Z ev} Y \times_Z (A \times_B \prod_\varphi E) \xrightarrow{pr_{\Pi_\varphi E}} \prod_\varphi E \xrightarrow{\pi_B} B, nm_{(pr_Y p_C f_1, f_2)}(res_{f_1}(c))] \\
&\quad= [C \xleftarrow{f_1 pr_D} D \times_{Y \times_Z E} (Y \times_Z (A \times_B \prod_\varphi E)) \xrightarrow{pr_{\Pi_\varphi E}} \prod_\varphi E \xrightarrow{\pi_B} B, c]
\end{aligned}$$

where the equality comes from identifying along $(Y \times_Z E \xleftarrow{\mathrm{id}_Y \times_Z ev} Y \times_Z (A \times_B \Pi_\varphi E) \xrightarrow{pr_{\Pi_\varphi E}} \Pi_\varphi E \xrightarrow{\pi_B} B, c)$ via the bispan $(C \xleftarrow{f_1} D \xrightarrow{(pr_Y p_C f_1, f_2)} Y \times_Z E = Y \times_Z E)$. Going around he bottom of the diagram we have

$$\begin{aligned}
&[C \xleftarrow{f_1} D \xrightarrow{f_2} E \xrightarrow{f_2} A \times_Z \prod_g X, c] \\
&\mapsto [Y \times_Z \prod_{\varphi \times_Z \mathrm{id}_{\Pi_g X}} E = Y \times_Z \prod_{\varphi \times_Z \mathrm{id}_{\Pi_g X}} E \xrightarrow{pr_{\Pi E}} \prod_{\varphi \times_Z \mathrm{id}_{\Pi_g X}} E \xrightarrow{pr_B \pi_{(B \times_Z \Pi_g X)}} B, nm_{(pr_Y p_C f_1 pr_D, pr_{\Pi E})}(res_{f_1 pr_D}(c))] \\
&= [C \xleftarrow{f_1 pr_D} D \times_E ((A \times_Z \prod_g X) \times_{B \times_Z \Pi_g X} \prod_{\varphi \times_Z \mathrm{id}_{\Pi_g X}} E) \xrightarrow{pr_{\Pi E}} \prod_{\varphi \times_Z \mathrm{id}_{\Pi_g X}} E \xrightarrow{pr_B \pi_{(B \times_Z \Pi_g X)}} B, c] \\
&= [C \xleftarrow{f_1 pr_D} D \times_{Y \times_Z E} (Y \times_Z (A \times_B \prod_\varphi E)) \xrightarrow{pr_{\Pi_\varphi E}} \prod_\varphi E \xrightarrow{\pi_B} B, c].
\end{aligned}$$

where the fist equality comes from identifying along

$$(Y \times_Z \prod_{\varphi \times_Z \mathrm{id}_{\Pi_g X}} E = Y \times_Z \prod_{\varphi \times_Z \mathrm{id}_{\Pi_g X}} E \xrightarrow{pr_{\Pi E}} \prod_{\varphi \times_Z \mathrm{id}_{\Pi_g X}} E \xrightarrow{pr_B \pi_{(B \times_Z \Pi_g X)}} B, c)$$

via the bispan

$$(C \xleftarrow{f_1 pr_D} D \times_E ((A \times_Z \prod_g X) \times_{(B \times_Z \Pi_g X)} \prod_{\varphi \times_Z \mathrm{id}_{\Pi_g X}} E) \xrightarrow{pr_{\Pi E}} \prod_{\varphi \times_Z \mathrm{id}_{\Pi_g X}} E \xrightarrow{pr_B \pi_{(B \times_Z \Pi_g X)}} B).$$

The second equality comes from the obvious isomorphism of bispans. $\square$

The forgetful functors commute with the important natural transformations between restriction, transfer, and norm functors.

Lemma 4.27. *Let*

$$\begin{tikzcd} X \times_Y Z \arrow[r, "pr_Z"] \arrow[d, "pr_X"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

be a pullback square in Set^G. *Then the following square commutes for any Tambara functor* $\underline{S}$.

$$\begin{tikzcd} \mathcal{U}_Z \mathcal{R}_g \mathcal{T}_f \underline{S} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathcal{U}_Z \mathcal{T}_{pr_Z} \mathcal{R}_{pr_X} \underline{S} \arrow[d, "\cong"] \\ \mathcal{R}_g \mathcal{T}_f \mathcal{U}_X \underline{S} \arrow[r, "\cong"'] & \mathcal{T}_{pr_Z} \mathcal{R}_{pr_X} \mathcal{U}_X \underline{S} \end{tikzcd}$$

Proof. This follows immediately from the fact that restriction functors, $\mathcal{R}_\varphi$, and transfer functors, $\mathcal{T}_\varphi$ are naturally isomorphic to precomposition be Σ_φ and Δ_φ respectively for both Mackey and Tambara functors. □

Lemma 4.28. *Let* $(\mathcal{O}_a, \mathcal{O}_m)$ *and* $(\mathcal{O}'_a, \mathcal{O}'_m)$ *be compatible pairs of indexing categories such* $(\mathcal{O}_a, \mathcal{O}'_m)$ *is also a compatible pair of indexing categories, and such that* $\mathcal{O}_a$ *and* $\mathcal{O}_m$ *are subcategories of* $\mathcal{O}'_a$ *and* $\mathcal{O}'_m$. *If*

$$\begin{tikzcd} X \times_Y Z \arrow[r, "pr_Z"] \arrow[d, "pr_X"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback square in Set^G *with* $f \in \mathcal{O}'_m$. *Then the following square commutes for any* $(\mathcal{O}_a, \mathcal{O}_m)$*-Tambara functor* $\underline{S}$.

$$\begin{tikzcd} \mathcal{U}_Z \mathcal{R}_g \mathcal{N}_f \underline{S} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathcal{U}_Z \mathcal{N}_{pr_Z} \mathcal{R}_{pr_X} \underline{S} \arrow[d, "\cong"] \\ \mathcal{R}_g \mathcal{N}_f \mathcal{U}_X \underline{S} \arrow[r, "\cong"'] & \mathcal{N}_{pr_Z} \mathcal{R}_{pr_X} \mathcal{U}_X \underline{S} \end{tikzcd}$$

Proof. Recall from Remark 4.21 that we have nice representatives for elements of normed Mackey and Tambara functors. On objects the top natural isomorphism,

$\mathcal{U}_Z \mathcal{R}_g \mathcal{N}_f \underline{S}(A,h) \to \mathcal{U}_Z \mathcal{N}_{pr_Z} \mathcal{R}_{pr_X} \underline{S}(A,h)$, is given by

$$
\begin{aligned}
&\left[X \times_Y B = X \times_Y B \xrightarrow{pr_B} B \xrightarrow{j} A, \alpha\right] \\
\mapsto &\left[Z \times_Y A \xleftarrow{(h,\mathrm{id}_A)} A = A = A, \left[X \times_Y B = X \times_Y B \xrightarrow{pr_B} B \xrightarrow{j} A, \alpha\right]\right] \\
\mapsto &\left[(X \times_Y B) \times_Y Z \xleftarrow{(\mathrm{id}_{X \times_Y B}, hjpr_B)} X \times_Y B \xrightarrow{pr_B} B \xrightarrow{j} A, \alpha\right] \\
\mapsto &\left[(X \times_Y B) \times_X (X \times_Y Z) \xleftarrow{(\mathrm{id}_{X \times_Y B}, (pr_X, hjpr_B))} X \times_Y B \xrightarrow{j} A, \alpha\right] \\
\mapsto &\left[(X \times_Y B) \times_X (X \times_Y Z) \xleftarrow{(\mathrm{id}_{X \times_Y B}, (pr_X, hjpr_B))} X \times_Y B \xrightarrow{j} A, \left[\mathrm{id}_{\Delta_{pr_X}(X \times_Y B)}, \alpha\right]\right] \\
\mapsto &\left[(X \times_Y B) \times_X (X \times_Y Z) \xleftarrow{(\mathrm{id}_{X \times_Y B}, (pr_X, hjpr_B))} X \times_Y B \xrightarrow{j} A, res_{pr_{X \times_Y B}}(\alpha)\right].
\end{aligned}
$$

In order, we are using natural isomorphisms from Lemmas A.2, A.5, A.4, A.5, A.2. On objects the bottom natural isomorphism, $\mathcal{R}_g \mathcal{N}_f \mathcal{U}_X \underline{S}(A,h) \to \mathcal{N}_{pr_Z} \mathcal{R}_{pr_X} \mathcal{U}_X \underline{S}(A,h)$, is given by

$$
\begin{aligned}
&\left[\prod_f (X \times_Y B) \xleftarrow{\eta^f_B} B \xrightarrow{j} A, \alpha\right] \\
\mapsto &\left[Z \times_Y A \xleftarrow{(h,\mathrm{id}_A)} A = A, \left[\prod_f (X \times_Y B) \xleftarrow{\eta^f_B} B \xrightarrow{j} A, \alpha\right]\right] \\
\mapsto &\left[Z \times_Y \prod_f (X \times_Y B) \xleftarrow{(hj, \eta^f_B)} B \xrightarrow{j} A, \alpha\right] \\
\mapsto &\left[\prod_{pr_Z} ((X \times_Y B) \times_X (X \times_Y Z)) \xleftarrow{\varphi} B \xrightarrow{j} A, \alpha\right] \\
\mapsto &\left[\prod_{pr_Z} ((X \times_Y B) \times_X (X \times_Y Z)) \xleftarrow{\varphi} B \xrightarrow{j} A, \left[\mathrm{id}_{\Delta_{pr_X}(X \times_Y B)}, \alpha\right]\right] \\
\mapsto &\left[\prod_{pr_Z} ((X \times_Y B) \times_X (X \times_Y Z)) \xleftarrow{\varphi} B \xrightarrow{j} A, res_{pr_{X \times_Y B}}(\alpha)\right]
\end{aligned}
$$

where $\varphi : b \mapsto (hj(b), (x,z) \mapsto ((x,b),(x,z)))$. In order, we are using natural isomorphisms from Lemmas A.2, A.5, A.4, A.5, A.2. Around the bottom of the square we have

$$
\begin{aligned}
&\left[X \times_Y B = X \times_Y B \xrightarrow{pr_B} B \xrightarrow{j} A, \alpha\right] \\
\mapsto &\left[\prod_f (X \times_Y B) \xleftarrow{\eta^f_B} B \xrightarrow{j} A, \alpha\right] \\
\mapsto &\left[\prod_{pr_Z} ((X \times_Y B) \times_X (X \times_Y Z)) \xleftarrow{\varphi} B \xrightarrow{j} A, res_{pr_{X \times_Y B}}(\alpha)\right]
\end{aligned}
$$

where the first map is from Theorem 4.20. Around the top of the square we have

$$\begin{aligned}
&\left[X \times_Y B = X \times_Y B \xrightarrow{pr_B} B \xrightarrow{j} A, \alpha\right] \\
\mapsto &\left[(X \times_Y B) \times_X (X \times_Y Z) \xleftarrow{(\mathrm{id}_{X \times_Y B},(pr_X, hjpr_B))} X \times_Y B \xrightarrow{j} A, res_{pr_{X \times_Y B}}(\alpha)\right] \\
= &\left[(X \times_Y Z) \times_Z B = (X \times_Y Z) \times_Z B \xrightarrow{pr_B} B \xrightarrow{j} A, nm_{((pr_X, hjpr_B),pr_B)}(\alpha)\right] \\
= &\left[X \times_Y B = X \times_Y B \xrightarrow{pr_B} B \xrightarrow{j} A, \alpha\right] \\
\mapsto &\left[\prod_{pr_Z} ((X \times_Y Z)) \times_Z B \xleftarrow{\eta_B^{pr_Z}} B \xrightarrow{j} A, nm_{((pr_X, hjpr_B),pr_B)}(\alpha)\right] \\
= &\left[\prod_{pr_Z} ((X \times_Y Z)) \times_Z B \xleftarrow{\eta_B^{pr_Z}} B \xrightarrow{j} A, res_{(pr_X, pr_B)}(\alpha)\right].
\end{aligned}$$

The first equality is changing to the nice representative from Remark 4.21. The second equality is an identification using the bispan

$$\left(X \times_Y B = X \times_Y B \xrightarrow{\cong} (X \times_Y Z) \times_Z B = (X \times_Y Z) \times_Z B\right)$$

note that this is not putting it into normal form. The second map is from Theorem 4.20. The third equality holds because we have the following isomorphism of bispans.

$$\begin{tikzcd}
 & X \times_Y B \arrow[r, "\cong"] \arrow[d, "\cong"] \arrow[dl, equal] & (X \times_Y Z) \times_Z B \arrow[d, equal] \arrow[dr, equal] & \\
X \times_Y B & (X \times_Y Z) \times_Z B \arrow[l, "\cong"] \arrow[r, equal] & (X \times_Y Z) \times_Z B \arrow[r, equal] & (X \times_Y Z) \times_Z B
\end{tikzcd}$$

Finally the elements

$$\left[\prod_{pr_Z} ((X \times_Y B) \times_X (X \times_Y Z)) \xleftarrow{\varphi} B \xrightarrow{j} A, res_{pr_{X \times_Y B}}(\alpha)\right]$$

and

$$\left[\prod_{pr_Z} ((X \times_Y Z)) \times_Z B \xleftarrow{\eta_B^{pr_Z}} B \xrightarrow{j} A, res_{(pr_X, pr_B)}(\alpha)\right]$$

are identified along $(\Pi_{pr_Z} ((X \times_Y Z)) \times_Z B \xleftarrow{\eta_B^{pr_Z}} B \xrightarrow{j} A, res_{pr_{X \times_Y B}}(\alpha))$ via the span $(((X \times_Y B) \times_X (X \times_Y Z)) \xleftarrow{\psi} (X \times_Y Z) \times_Z B = (X \times_Y Z) \times_Z B)$ where ψ is the composite

$$(X \times_Y Z) \times_Z B \xrightarrow{\cong} X \times_Y B \xrightarrow{((\mathrm{id}_{X \times_Y B},(pr_X, hjpr_B)))} ((X \times_Y B) \times_X (X \times_Y Z)). \quad \square$$

Lemma 4.29. *Let $(\mathcal{O}_a, \mathcal{O}_m)$ and $(\mathcal{O}'_a, \mathcal{O}'_m)$ be compatible pairs of indexing categories such $(\mathcal{O}_a, \mathcal{O}'_m)$ is also a compatible pair of indexing categories, and such that $\mathcal{O}_a$ and*

$\mathcal{O}_m$ are subcategories of $\mathcal{O}'_a$ and $\mathcal{O}'_m$. If

$$\begin{tikzcd} Y \times_Z \Pi_g X \arrow[rr, "pr_{\Pi_g X}"] \arrow[d, "ev"'] & & \Pi_g X \arrow[d, "\pi_Z"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

is an exponential diagram with $f \in \mathcal{O}'_a$ and $g \in \mathcal{O}'_m$ then the following square commutes for any $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functor $\underline{S}$.

$$\begin{tikzcd} \mathcal{U}_Z \mathcal{T}_{\pi_Z} \mathcal{N}_{pr_{\Pi_g X}} \mathcal{R}_{ev} \underline{S} \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathcal{U}_Z \mathcal{N}_g \mathcal{T}_f \underline{S} \arrow[d, "\cong"] \\ \mathcal{T}_{\pi_Z} \mathcal{N}_{pr_{\Pi_g X}} \mathcal{R}_{ev} \mathcal{U}_X \underline{S} \arrow[r, "\cong"] & \mathcal{N}_g \mathcal{T}_f \mathcal{U}_Z \underline{S} \end{tikzcd}$$

Proof. This is immediate because the exponential diagram isomorphism was defined at the level of Tambara functors in Lemma 4.24 by passing over the forgetful functor and using the exponential diagram isomorphism at the Mackey functor level. □

Lemma 4.30. *Let $f : X \to Y$ and $g : Y \to Z$ be maps in Set^G. Then for $\underline{S}$ and $\underline{T}$ Tambara functors the following diagrams commute.*

$$\begin{tikzcd} \mathcal{U}_X \mathcal{R}_f \mathcal{R}_g \underline{T} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathcal{U}_X \mathcal{R}_{gf} \underline{T} \arrow[d, "\cong"] \\ \mathcal{R}_f \mathcal{R}_g \mathcal{U}_Z \underline{T} \arrow[r, "\cong"'] & \mathcal{R}_{gf} \mathcal{U}_Z \underline{T} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{U}_Z \mathcal{T}_g \mathcal{T}_f \underline{S} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathcal{U}_Z \mathcal{T}_{gf} \underline{S} \arrow[d, "\cong"] \\ \mathcal{T}_g \mathcal{T}_f \mathcal{U}_X \underline{S} \arrow[r, "\cong"'] & \mathcal{T}_{gf} \mathcal{U}_X \underline{S} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{U}_Z \mathcal{N}_g \mathcal{N}_f \underline{S} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathcal{U}_Z \mathcal{N}_{gf} \underline{S} \arrow[d, "\cong"] \\ \mathcal{N}_g \mathcal{N}_f \mathcal{U}_X \underline{S} \arrow[r, "\cong"'] & \mathcal{N}_{gf} \mathcal{U}_X \underline{S} \end{tikzcd}$$

The commutativity of the third diagram is contingent on $\mathcal{N}_f$ and $\mathcal{N}_g$ existing at the level of Mackey functors.

Proof. The first diagram commutes because $\Sigma_g \Sigma_f = \Sigma gf$ and the second diagram commutes because there is a natural isomorphism $\Delta_f \Delta_g \cong \Delta_{gf}$.

Recall from Remark 4.21 that we have nice representatives for elements of normed Mackey and Tambara functors. Around the top of the diagram we have

$$\left[Y \times_Z B = Y \times_Z B \xrightarrow{pr_B} B \xrightarrow{k} A, \left[X \times_Y C = X \times_Y C \xrightarrow{pr_C} C \xrightarrow{j} Y \times_Z B, \alpha \right] \right]$$

$$\mapsto \left[X \times_Y C \xleftarrow{\mathrm{id}_X \times_Y ev} X \times_Y ((Y \times_Z B) \times_B \prod_{pr_B} C) \xrightarrow{pr_{\Pi_{pr_B} C}} \prod_{pr_B} C \xrightarrow{k\pi_B} A, \alpha \right]$$

$$\mapsto \left[\prod_{gf} (X \times_Z \prod_{pr_B} C) \xleftarrow{\eta^{gf}_{\Pi_{pr_B} C}} \prod_{pr_B} C \xrightarrow{k\pi_B} A, nm_{(pr_X, pr_{\Pi_{pr_B} C})}(res_{\mathrm{id}_X \times_Y ev}(\alpha)) \right]$$

$$= \left[\prod_{gf} (X \times_Z \prod_{pr_B} C) \xleftarrow{\eta^{gf}_{\Pi_{pr_B} C}} \prod_{pr_B} C \xrightarrow{k\pi_B} A, res_{(pr_X, ((fpr_X, \pi_B), (\pi_B, \pi_\sigma)))}(res_{\mathrm{id}_X \times_Y ev}(\alpha)) \right]$$

$$= \left[\prod_{gf} (X \times_Z \prod_{pr_B} C) \xleftarrow{\eta^{gf}_{\Pi_{pr_B} C}} \prod_{pr_B} C \xrightarrow{k\pi_B} A, res_\varphi(\alpha)) \right]$$

where the first map is an application of Lemma A.5, the second is from the proof of Theorem 4.20, the first equality holds because we have the following isomorphism of bispans, and the second equality is composition of restriction maps with $\varphi : (x, (b, \sigma)) \mapsto (x, \sigma(f(x), b))$.

$$\begin{tikzcd}
& X \times_Y ((Y \times_Z B) \times_B \Pi_{pr_B} C) \arrow[r, "\cong"] \arrow[dl, equal] \arrow[dd, "\cong"] & X \times_Z \Pi_{pr_B} C \arrow[dd, equal] \arrow[dr, equal] & \\
X \times_Y ((Y \times_Z B) \times_B \Pi_{pr_B} C) & & & X \times_Z \Pi_{pr_B} C \arrow[dl, equal] \\
& X \times_Z \Pi_{pr_B} C \arrow[ul, "\cong"] \arrow[r, equal] & X \times_Z \Pi_{pr_B} C &
\end{tikzcd}$$

Around the bottom of the diagram we have

$$\begin{aligned}
&\left[Y \times_Z B = Y \times_Z B \xrightarrow{pr_B} B \xrightarrow{k} A, \left[X \times_Y C = X \times_Y C \xrightarrow{pr_C} C \xrightarrow{j} Y \times_Z B, \alpha\right]\right] \\
\mapsto &\left[\prod_g (Y \times_Z B) \xleftarrow{\eta^g_B} B \xrightarrow{k} A, \left[\prod_f (X \times_Y C) \xleftarrow{\eta^f_C} C \xrightarrow{j} Y \times_Z B, \alpha\right]\right] \\
\mapsto &\left[\prod_{gf} (X \times_Y C) \xleftarrow{\psi} \prod_{pr_B} C \xrightarrow{k} A, \alpha\right]
\end{aligned}$$

where $\psi : (b, \sigma) \mapsto (hk(b), x \mapsto (x, \sigma(f(x), b)))$ is the composition

$$\prod_{pr_B} C \xrightarrow{\cong} \prod_g C \times_{\Pi_g(Y \times_Z B)} B \xrightarrow{pr_{\Pi_g C}} \prod_g C \xrightarrow{\Pi_g \eta^f_C} \prod_g \prod_f (X \times_Y C) \xrightarrow{\cong} \prod_{gf} (X \times_Y C).$$

Finally, the elements

$$\left[\prod_{gf} (X \times_Z \prod_{pr_B} C) \xleftarrow{\eta^{gf}_{\Pi_{pr_B} C}} \prod_{pr_B} C \xrightarrow{k\pi_B} A, res_\varphi(\alpha))\right]$$

and

$$\left[\prod_{gf} (X \times_Y C) \xleftarrow{\psi} \prod_{pr_B} C \xrightarrow{k} A, \alpha\right]$$

are identified along $(\Pi_{gf}(X \times_Z \Pi_{pr_B} C) \xleftarrow{\eta^{gf}_{\Pi_{pr_B} C}} \Pi_{pr_B} C \xrightarrow{k\pi_B} A, \alpha)$ via the span $(X \times_Y C \xleftarrow{\varphi} X \times_Z \Pi_{pr_B} C = X \times_Z \Pi_{pr_B} C)$. □

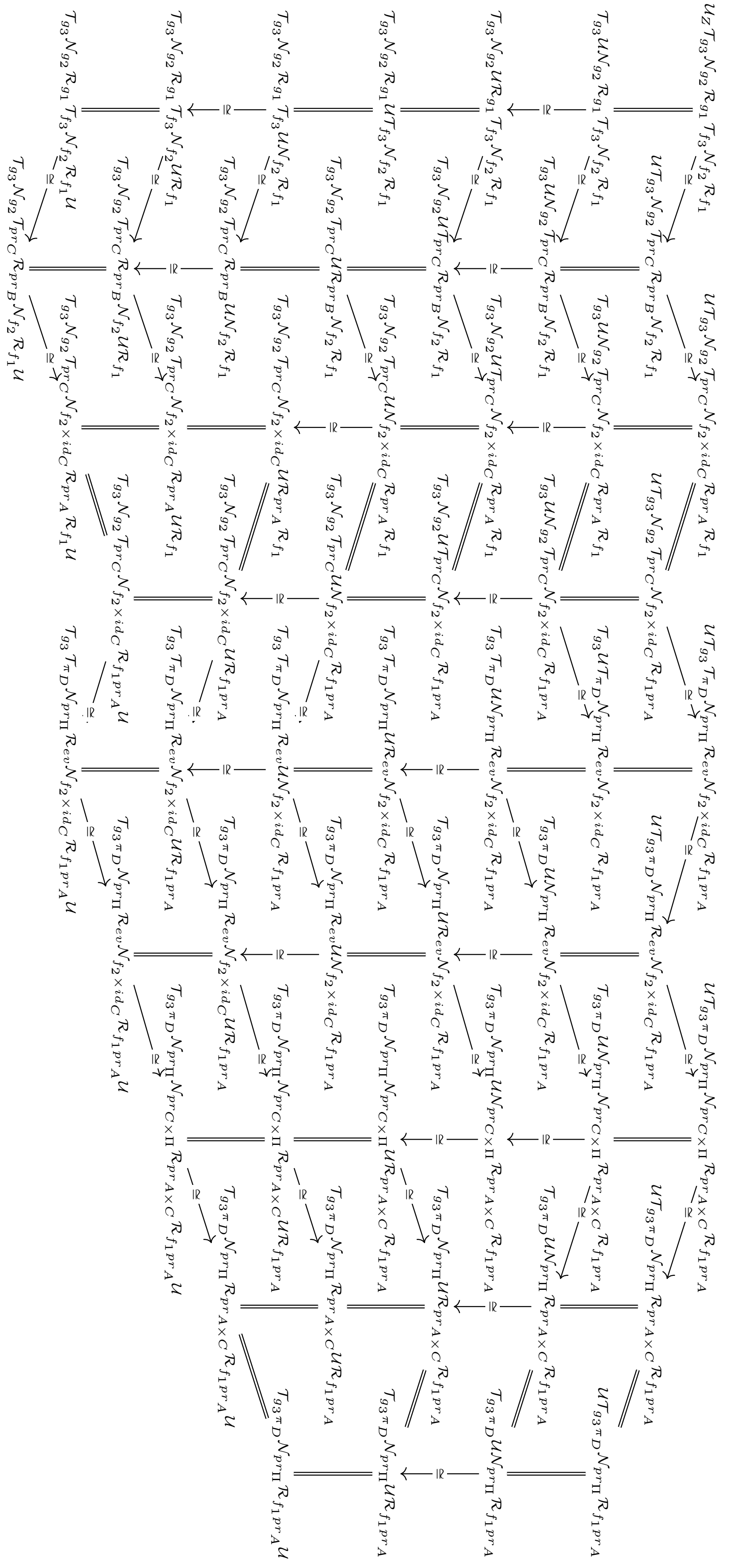
Notation 4.31. Given a bispan $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ we write Γ_{f_3,f_2,f_1} or Γ_f for the functor $\mathcal{T}_{f_3}\mathcal{N}_{f_2}\mathcal{R}_{f_1} : \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \to \mathrm{Tamb}_Y^{(\mathcal{O}_a,\mathcal{O}_m)}$ and $\mathcal{T}_{f_3}\mathcal{N}_{f_2}\mathcal{R}_{f_1} : \mathrm{Mack}_X^{\mathcal{O}} \to \mathrm{Mack}_Y^{\mathcal{O}}$. The later requires a compatibility condition is satisfied so that $\mathcal{N}_{f_2}$ exists.

Lemma 4.32. *Let $(\mathcal{O}_a, \mathcal{O}_m)$ and $(\mathcal{O}'_a, \mathcal{O}'_m)$ be compatible pairs of indexing categories such $(\mathcal{O}_a, \mathcal{O}'_m)$ is also a compatible pair of indexing categories, and such that $\mathcal{O}_a$ and $\mathcal{O}_m$ are subcategories of $\mathcal{O}'_a$ and $\mathcal{O}'_m$. Given $(\mathcal{O}'_a, \mathcal{O}'_m)$-bispans $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$*

and $g = (Y \xleftarrow{g_1} C \xrightarrow{g_2} D \xrightarrow{g_3} Z)$ *and an* $(\mathcal{O}_a, \mathcal{O}_m)$*-Tambara functor* $\underline{S}$ *the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{U}_Z\Gamma_g\Gamma_f\underline{S} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathcal{U}\Gamma_{gf}\underline{S} \arrow[dd, "\cong"] \\
\Gamma_g\mathcal{U}_Y\Gamma_f\underline{S} \arrow[d, "\cong"'] & \\
\Gamma_g\Gamma_f\mathcal{U}_X\underline{S} \arrow[r, "\cong"'] & \Gamma_{gf}\mathcal{U}_X\underline{S}
\end{tikzcd}$$

Proof. This follows from applications of the results for forgetful functors commuting with restriction, transfer, and norm functors, and the natural transformations involving combinations of them. That is, Theorem 4.20 and Lemmas 4.16, 4.27, 4.28, 4.29, 4.30. On the following page we have included the relevant commutative diagram. The diagram has been rotated and its rows zigzagged to make it fit on the page. The columns of the diagram consist of isomorphisms moving a forgetful functors across another functor. The rows consist of isomorphisms applying composition, pullback, and exponential diagram lemmas to combinations of restriction, norm, and transfer functors. □



We get a symmetric monoidal structure on the category of semi-Tambara functors by taking the pointwise product.

Definition 4.33. Given $(\mathcal{O}_a, \mathcal{O}_m)$-semi-Tambara functors $\underline{S}$ and $\underline{T}$ over X we define their **box sum** $\underline{S} \boxplus \underline{T}$ to be the functor $\mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \to \mathbf{Set}$ with $(\underline{S} \boxplus \underline{T})(A) = \underline{S}(A) \times \underline{T}(A)$ and for an $(\mathcal{O}_a, \mathcal{O}_m)$-bispan $(A \xleftarrow{f_1} B \xrightarrow{f_2} C \xrightarrow{f_3} D)$ over X we have $(\underline{S} \boxplus \underline{T})(f) = \underline{S}(f) \times \underline{T}(f)$. We have

$$\begin{aligned}(\underline{S} \boxplus \underline{T})(A \amalg B) &= \underline{S}(A \amalg B) \times \underline{T}(A \amalg B) = (\underline{S}(A) \times \underline{S}(B)) \times (\underline{T}(A) \times \underline{T}(B)) \\ &\cong (\underline{S}(A) \times \underline{T}(A)) \times (\underline{S}(B) \times \underline{T}(B)) = (\underline{S} \boxplus \underline{T})(A) \times (\underline{S} \boxplus \underline{T})(B).\end{aligned}$$

So $\underline{S} \boxplus \underline{T}$ is product-preserving and hence is an $(\mathcal{O}_a, \mathcal{O}_m)$-semi-Tambara functor.

Categories of Tambara functors are closed under taking box sums.

Lemma 4.34. *If* $\underline{S}, \underline{T} \in \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ *then* $\underline{S} \boxplus \underline{T} \in \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$.

Proof. For any $(x, y) \in \underline{S}(A) \times \underline{T}(A) = (\underline{S} \boxplus \underline{T})(A)$ we have $(x, y) + (-x, -y) = (0, 0) \in (\underline{S} \boxplus \underline{T})(A)$, so $\underline{S} \boxplus \underline{T}$ has additive inverses. $\square$

Notice that if we let $\underline{0}$ be the $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functor over X with $\underline{0}(A) = 0$ for all $A \in \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ then $\underline{0}$ acts as an identity with respect to $\boxplus$.

Lemma 4.35. *Both* $(\mathrm{sTamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}, \boxplus, \underline{0})$ *and* $(\mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}, \boxplus, \underline{0})$ *are symmetric monoidal categories.*

There is a symmetric monoidal structure on the category of semi-Tambara functors that comes from Day convolution.

Definition 4.36. Given $(\mathcal{O}_a, \mathcal{O}_m)$-semi-Tambara functors over X, $\underline{S}$ and $\underline{T}$, we define their **box product** $\underline{S} \Box \underline{T}$ to be the left Kan extension of the functor $\underline{S} \times \underline{T}$ along the functor $\mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \times \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \to \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$.

$$\begin{tikzcd} \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \times \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \arrow[r, "(\underline{S},\underline{T})"] \arrow[d, "\amalg"'] & \mathbf{Set} \times \mathbf{Set} \arrow[r, "\times"] & \mathbf{Set} \\ \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \arrow[urr, "\mathrm{Lan}_{\amalg}(\underline{S} \times \underline{T}) =: \underline{S} \Box \underline{T}"'] & & \end{tikzcd}$$

By Lemma A.6 the box product $\underline{S} \Box \underline{T}$ is product-preserving, as it is the left Kan extension of $\underline{S} \times \underline{T}$ which is product-preserving. Hence we have a functor

$$\Box : \mathrm{sTamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \times \mathrm{sTamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \to \mathrm{sTamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}.$$

We define

$$\underline{S} \overline{\otimes} \underline{T} : \mathcal{P}_{X \amalg X}^{(\mathcal{O}_a,\mathcal{O}_m)} \underset{\simeq}{\xrightarrow{\Omega}} \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \times \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \xrightarrow{\underline{S} \times \underline{T}} \mathbf{Set}$$

where the Ω is the equivalence from Lemma 4.5. Note that all the functors involved are product-preserving, so $\underline{S}\overline{\otimes}\underline{T} \in \mathrm{sTamb}_{X\amalg X}^{(\mathcal{O}_a,\mathcal{O}_m)}$. Moreover, $\underline{S}\overline{\otimes}\underline{T}(A) = \underline{S}(p_A^{-1}(X_1)) \times \underline{T}(p_A^{-1}(X_2))$ where the index determines a copy of X in the summand. Hence if $\underline{S}, \underline{T} \in \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ then $\underline{S}\overline{\otimes}\underline{T} \in \mathrm{Tamb}_{X\amalg X}^{(\mathcal{O}_a,\mathcal{O}_m)}$.

We can realise the box product on semi-Tambara functors as an application of the norm functor along the fold map.

Proposition 4.37. *The functors $\underline{S}\square\underline{T}$ and $\mathcal{N}_\nabla(\underline{S}\overline{\otimes}\underline{T})$ are naturally isomorphic.*

Proof. We have that $\amalg\Omega$ and Σ_∇ are naturally isomorphic functors $\mathcal{P}_{X\amalg X}^{(\mathcal{O}_a,\mathcal{O}_m)} \to \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$. The result follows from the fact that Ω is an equivalence and Lemmas A.4 and A.5. □

Categories of Tambara functors are closed under taking box products.

Lemma 4.38. *If $\underline{S}, \underline{T} \in \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ then $\underline{S}\square\underline{T} \in \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$.*

Proof. As $\underline{S}\square\underline{T}$ and $\mathcal{N}_\nabla(\underline{S}\overline{\otimes}\underline{T})$ are naturally isomorphic and $\underline{S}\overline{\otimes}\underline{T} \in \mathrm{Tamb}_{X\amalg X}^{(\mathcal{O}_a,\mathcal{O}_m)}$ we have that $\underline{S}\square\underline{T}$ is an $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functor over X by Corollary 4.19. □

Remark 4.39. When G is the trivial group and $\underline{S}$ and $\underline{T}$ are G-Tambara functors the left Kan extension in Definition 4.36 is equivalent to the left Kan extension describing the tensor product of commutative rings as $\mathbb{Z}$-algebras, so that $(\underline{S}\square\underline{T})(*) = \underline{S}(*)\otimes\underline{T}(*)$. The tensor product is the coproduct in the category of commutative rings, similarly the box product is the coproduct in the category of Tambara functors.

Definition 4.40. The **group completion** $\mathrm{Gr}(\underline{S})$ of an $(\mathcal{O}_a, \mathcal{O}_m)$-semi-Tambara functor $\underline{S}$ over X is the $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functor over X with $\mathrm{Gr}(\underline{S})(C) = \mathrm{Gr}(\underline{S}(C))$, the group completion of $\underline{S}(C)$ as a commutative monoid so that it is a commutative ring.

It is immediate that group completion commutes with restriction and transfer functors.

Lemma 4.41. *Group completion commutes with norms. That is, there is a natural isomorphism $\mathrm{Gr}\,\mathcal{N}_\varphi \cong \mathcal{N}_\varphi\,\mathrm{Gr}$.*

Proof. For $\underline{S} \in \mathrm{sTamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ the isomorphism is given on objects by

$$\begin{aligned} \mathrm{Gr}\,\mathcal{N}_\varphi\underline{S}(D) &\to \mathcal{N}_\varphi\,\mathrm{Gr}\,\underline{S}(D) \\ [A \xleftarrow{f_1} B \xrightarrow{f_2} C \xrightarrow{f_3} D, a] &\mapsto [A \xleftarrow{f_1} B \xrightarrow{f_2} C \xrightarrow{f_3} D, a] \\ -[A \xleftarrow{f_1} B \xrightarrow{f_2} C \xrightarrow{f_3} D, a] &\mapsto [A \xleftarrow{f_1} B \xrightarrow{f_2} C \xrightarrow{f_3} D, -a]. \end{aligned}$$ □

Definition 4.42. We call the $(\mathcal{O}_a, \mathcal{O}_m)$-semi-Tambara functor over X that is corepresented by $\emptyset$ the **Burnside** $(\mathcal{O}_a, \mathcal{O}_m)$**-semi-Tambara functor** over X,

$$\underline{B}_X : \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \to \mathbf{Set}, Y \mapsto \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)}(\emptyset, Y).$$

The **Burnside** $(\mathcal{O}_a, \mathcal{O}_m)$**-Tambara functor** over X is the group completion of $\underline{B}_X$,

$$\underline{A}_X = \mathrm{Gr}(\underline{B}_X) : \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \to \mathbf{Set}, Y \mapsto \mathrm{Gr}(\mathcal{P}^{(\mathcal{O}_a,\mathcal{O}_m)}(\emptyset, Y)).$$

Remark 4.43. When $(\mathcal{O}_a, \mathcal{O}_m) = (\mathrm{Set}^G, \mathrm{Set}^G)$ the elements of $\underline{B}_*(*)$ are isomorphisms of bispans of the form $(\emptyset \leftarrow \emptyset \to A \to *)$. That is, the elements of $\underline{B}_*(*)$ are in bijection with the isomorphism classes of finite G-sets, so we will write A for the element $[\emptyset \leftarrow \emptyset \to A \to *]$. Addition in $B_*(*)$ is given by postcomposition with $(* \amalg * = * \amalg * = * \amalg * \xrightarrow{\nabla} *)$ so $A + A = A \coprod A$. Multiplication in $B_*(*)$ is given by postcomposition with $(* \amalg * = * \amalg * \xrightarrow{\nabla} * = *)$ so $A \cdot A = A \times A$ because $\Pi_\nabla(A \amalg A) = A \times A$. We see from this that by taking the group completion $\underline{A}_*(*)$ is isomorphic to the free $\mathbb{Z}$-module on isomorphism classes of orbits G/H. That is, $\underline{A}_*(*)$ is the Burnside ring of G.

As $\underline{S} \Box \underline{T}$ was defined by Day convolution $(\mathrm{sTamb}^{(\mathcal{O}_a,\mathcal{O}_m)}, \Box, \underline{B}_X)$ is a symmetric monoidal category.

Lemma 4.44. $(\mathrm{Tamb}^{(\mathcal{O}_a,\mathcal{O}_m)}, \Box, \underline{A}_X)$ *is also a symmetric monoidal category.*

Proof. We only need to see that $\underline{A}_X$ acts as an identity with respect to the box product. Let $\underline{S} \in \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ then $\underline{A}_X \Box \underline{S} = \mathrm{Gr}\, \underline{B}_X \Box \mathrm{Gr}\, \underline{S} \cong \mathrm{Gr}(\underline{B}_X \Box \underline{S}) \cong \mathrm{Gr}\, \underline{S} = \underline{S}$ where we are using Lemma 4.41 to commute Gr with $\Box$ (which by Proposition 4.37 can be realised as a norm). $\Box$

Following Remark 4.4 if H is a subgroup of G, then $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functors over G/H are equivalent to $(i_H^*\mathcal{O}_a, i_H^*\mathcal{O}_m)$-Tambara functors. In particular, G-Tambara functors over G/H are equivalent to H-Tambara functors. Across the equivalence $\mathcal{P}_{G/H}^{(\mathcal{O}_a,\mathcal{O}_m)} \simeq \mathcal{P}_*^{(i_H^*\mathcal{O}_a, i_H^*\mathcal{O}_m)}$ with the quotient map $\varphi : G/H \to G/G$ we recover functors $N_H^G : \mathrm{Tamb}^H \to \mathrm{Tamb}^G$ from $\mathcal{N}_\varphi$, and $R_H^G : \mathrm{Tamb}^G \to \mathrm{Tamb}^H$ from $\mathcal{R}_\varphi$, as left Kan extensions along the functors $G \times_H -$, and i_H^G respectively. In particular, the norm functor on Tambara functors defined by Hoyer in [14] and the box product on Tambara functors are both instances of our norm functor, along the quotient map and the fold map respectively. That the norms N_H^G are strong monoidal with respect to the box product is now an immediate consequence of Lemma A.5.

4.3 Bicategorical Setup

Definition 4.45. The $(\mathcal{O}_a, \mathcal{O}_m)$**-bispan bicategory** for $(\mathcal{O}_a, \mathcal{O}_m)$ a compatible pair of transfer systems is a bicategory, $\underline{(\mathcal{O}_a, \mathcal{O}_m)}$ with

- **Objects:** The 0-cells of the bicategory are finite G-sets
- **Hom Categories:** For X, Y a pair of finite G-sets $\underline{(\mathcal{O}_a, \mathcal{O}_m)}(X, Y)$ is the category whose objects (1-cells of the bicategory) are $(\mathcal{O}_a, \mathcal{O}_m)$-bispans $X \leftarrow A \to B \to Y$ and

whose morphisms (2-cells of the bicategory) are isomorphisms of $(\mathcal{O}_a, \mathcal{O}_m)$-bispans. That is, a pair of morphisms $\psi : (X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y) \to (X \xleftarrow{f_1'} A' \xrightarrow{f_2'} B' \xrightarrow{f_3'} Y)$ is a pair of isomorphisms $\psi_1 : A \to A'$ and $\psi_2 : B \to B'$ such that the following diagram commutes.

$$\begin{tikzcd}
& A \arrow[r, "f_2"] \arrow[dl, "f_1"'] \arrow[dd, "\cong"', "\psi_1"] & B \arrow[dr, "f_3"] \arrow[dd, "\cong"', "\psi_2"] & \\
X & & & Y \\
& A' \arrow[r, "f_2'"'] \arrow[ul, "f_1'"] & B' \arrow[ur, "f_3'"'] &
\end{tikzcd}$$

Composition in $(\mathcal{O}_a, \mathcal{O}_m)(X, Y)$ (called vertical composition) is composition of the underlying morphisms of G-sets. The identity 2-cell of an $(\mathcal{O}_a, \mathcal{O}_m)$-bispan $(X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ is $(\mathrm{id}_A, \mathrm{id}_B)$.

- **Identity 1-cells:** The identity 1-cell of a finite G-set X is the $(\mathcal{O}_a, \mathcal{O}_m)$-bispan $1_X(*) = (X = X = X = X)$.

- **Horizontal Composition:** For each triple of G-sets, X, Y, Z we have a horizontal composition functor $c_{X,Y,Z} : \underline{\mathcal{O}}(Y, Z) \times \underline{\mathcal{O}}(X, Y) \to \underline{\mathcal{O}}(X, Z)$. On 1-cells we have

$$\begin{aligned}
c_{X,Y,Z} :& ((Y \xleftarrow{g_1} C \xrightarrow{g_2} D \xrightarrow{g_3} Z), (X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)) \\
&\mapsto (X \xleftarrow{f_1 pr_A} A \times_B (C \times_D \prod_{g_2}(B \times_Y C)) \xrightarrow{pr_{\Pi_{g_2}(B \times_Y C)}} \prod_{g_2}(B \times_Y C) \xrightarrow{g_3 \pi_D^{B \times_Y C}} Z)
\end{aligned}$$

and on 2-cells we have

$$c_{X,Y,Z} : \left(\begin{tikzcd}
& C \arrow[r, "g_2"] \arrow[dl, "g_1"'] \arrow[dd, "\cong"', "\beta_1"] & D \arrow[dr, "g_3"] \arrow[dd, "\cong"', "\beta_2"] & \\
Y & & & Z \\
& C' \arrow[r, "g_2'"'] \arrow[ul, "g_1'"] & D' \arrow[ur, "g_3'"'] &
\end{tikzcd}, \begin{tikzcd}
& A \arrow[r, "f_2"] \arrow[dl, "f_1"'] \arrow[dd, "\cong"', "\alpha_1"] & B \arrow[dr, "f_3"] \arrow[dd, "\cong"', "\alpha_2"] & \\
X & & & Y \\
& A' \arrow[r, "f_2'"'] \arrow[ul, "f_1'"] & B' \arrow[ur, "f_3'"'] &
\end{tikzcd} \right)$$

$$\mapsto \left(\begin{tikzcd}
& A \times_B (C \times_D \Pi_{g_2}(B \times_Y C)) \arrow[r, "pr_{\Pi_{g_2}(B \times_Y C)}"] \arrow[dl, "f_1 pr_A"'] \arrow[dd, "\cong"] & \Pi_{g_2}(B \times_Y C) \arrow[dr, "g_3 \pi_D"] \arrow[dd, "\cong"] & \\
X & & & Z \\
& A' \times_{B'} (C' \times_{D'} \Pi_{g_2'}(B' \times_Y C')) \arrow[r, "pr_{\Pi_{g_2'}(B' \times_Y C')}"'] \arrow[ul, "f_1' pr_{A'}"] & \Pi_{g_2'}(B' \times_Y C') \arrow[ur, "g_3' \pi_{D'}"'] &
\end{tikzcd} \right)$$

where the map $\Pi_{g_2}(B \times_Y C) \to \Pi_{g_2'}(B' \times_Y C')$ is

$$(d, \sigma) \mapsto (\beta_2(d), c' \mapsto (\alpha_2 pr_B \sigma(\beta_1^{-1}(c')), c'))$$

and the other isomorphism is

$$(a,(c,(d,\sigma))) \mapsto (\alpha_1(a),(\beta_1(c),(\beta_2(d),c' \mapsto (\alpha_2 pr_B \sigma(\beta_1^{-1}(c')),c')))).$$

- **Associator:** For each quadruple of G-sets, W,X,Y,Z there is a natural isomorphism, called the associator,

$$a_{W,X,Y,Z} : c_{W,X,Z}(c_{X,Y,Z} \times \mathrm{id}_{\underline{(\mathcal{O}_a,\mathcal{O}_m)}(W,X)}) \implies c_{W,Y,Z}(\mathrm{id}_{\underline{(\mathcal{O}_a,\mathcal{O}_m)}(Y,Z)} \times c_{W,X,Y})$$

of functors

$$\underline{(\mathcal{O}_a,\mathcal{O}_m)}(Y,Z) \times \underline{(\mathcal{O}_a,\mathcal{O}_m)}(X,Y) \times \underline{(\mathcal{O}_a,\mathcal{O}_m)}(W,X) \to \underline{(\mathcal{O}_a,\mathcal{O}_m)}(W,Z)$$

whose component on the triple

$$((Y \xleftarrow{h_1} E \xrightarrow{h_2} F \xrightarrow{h_3} Z),(X \xleftarrow{g_1} C \xrightarrow{g_2} D \xrightarrow{g_3} Y),(W \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} X))$$

is an invertible 2-cell between $(\mathcal{O}_a,\mathcal{O}_m)$-bispans

$X \xleftarrow{f_1 pr_A}$
$\leftarrow A \times_B ((C \times_D (E \times_F \Pi_{h_2}(D \times_Y E))) \times_{\Pi_{h_2}} \Pi_{pr_{\Pi_{h_2}}}(B \times_X (C \times_D (E \times_F \Pi_{h_2}(D \times_Y E)))))$
$\xrightarrow{pr} \Pi_{pr_{\Pi_{h_2}}}(B \times_X (C \times_D (E \times_F \Pi_{h_2}(D \times_Y E)))) \xrightarrow{h_3 \pi_F \pi_{\Pi_{h_2}(D \times_Y E)}} Z$

and

$$\begin{aligned} X \xleftarrow{f_1 pr_A} &(A \times_B (C \times_D \Pi_{g_2}(B \times_X C))) \times_{\Pi_{g_2}} (E \times_F \Pi_{h_2}((\Pi_{g_2}(B \times_X C)) \times_Y E)) \\ &\xrightarrow{pr} \Pi_{h_2}((\Pi_{g_2}(B \times_X C)) \times_Y E) \xrightarrow{h_3 \pi_F} Z. \end{aligned}$$

Because the middle map for each of these is a projection, it is sufficient to describe the isomorphism just on the second term. The isomorphism is given by

$$\begin{aligned} &(a,((c,(e,(f,\sigma))),((f,\sigma),\psi))) \\ &\mapsto ((a,(c,(g_2(c),\hat{c} \mapsto (f_2(a),\hat{c})))),(e,(f,\hat{e} \mapsto ((g_2(c),\hat{c} \mapsto (f_2(a),\hat{c})))))) \end{aligned}$$

- **Unitors:** For each pair of G-sets X,Y there are natural isomorphisms

$$\begin{aligned} l_{X,Y} &: c_{X,Y,Y}(1_Y \times \mathrm{id}_{(\mathcal{O}_a,\mathcal{O}_m)(X,Y)}) \to \mathrm{id}_{(\mathcal{O}_a,\mathcal{O}_m)(X,Y)} \\ r_{X,Y} &: c_{X,X,Y}(\mathrm{id}_{(\mathcal{O}_a,\mathcal{O}_m)(X,Y)} \times 1_X) \to \mathrm{id}_{(\mathcal{O}_a,\mathcal{O}_m)(X,Y)} \end{aligned}$$

called the left unitor and right unitor respectively. The left unitor has component on the $(\mathcal{O}_a,\mathcal{O}_m)$-bispan $(X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ given by the invertible 2-cell $(pr_A,(y,\sigma) \mapsto \sigma(y))$. The right unitor has component on the $(\mathcal{O}_a,\mathcal{O}_m)$-bispan $(X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ given by the invertible 2-cell (pr_A,π_B).

It is straightforward, if painful, to check that the unity and pentagon axioms are satisfied.

Definition 4.46. Let CAT be the bicategory of categories, functors, and natural transformations. Let $(\mathcal{O}_a, \mathcal{O}_m)$ and $(\mathcal{O}'_a, \mathcal{O}'_m)$ compatible pairs of indexing categories such that $(\mathcal{O}_a, \mathcal{O}'_m)$ is a compatible pair of indexing categories, $\mathcal{O}_a$ is a subcategory of $\mathcal{O}'_a$, and $\mathcal{O}_m$ is a subcategory of $\mathcal{O}'_m$. We define a pseudofunctor $\left(\mathrm{Tamb}^{(\mathcal{O}'_a,\mathcal{O}'_m)}_{(\mathcal{O}_a,\mathcal{O}_m)}, \left(\mathrm{Tamb}^{(\mathcal{O}'_a,\mathcal{O}'_m)}_{(\mathcal{O}_a,\mathcal{O}_m)}\right)^2, \left(\mathrm{Tamb}^{(\mathcal{O}'_a,\mathcal{O}'_m)}_{(\mathcal{O}_a,\mathcal{O}_m)}\right)^0\right) : \underline{(\mathcal{O}'_a, \mathcal{O}'_m)} \to CAT$ to be the triple consisting of the following data

- **Objects:** The assignment $\mathrm{Tamb}^{(\mathcal{O}'_a,\mathcal{O}'_m)}_{(\mathcal{O}_a,\mathcal{O}_m)}(X) = \mathrm{Tamb}^{(\mathcal{O}_a,\mathcal{O}_m)}_X$ for all finite G-sets X.

- **Hom Categories:** For each pair of finite G-sets X, Y the functor

$$\mathrm{Tamb}^{(\mathcal{O}'_a,\mathcal{O}'_m)}_{(\mathcal{O}_a,\mathcal{O}_m)} : \underline{(\mathcal{O}'_a, \mathcal{O}'_m)} \to CAT\left(\mathrm{Tamb}^{(\mathcal{O}_a,\mathcal{O}_m)}_X, \mathrm{Tamb}^{(\mathcal{O}_a,\mathcal{O}_m)}_Y\right)$$

with

$$\mathrm{Tamb}^{(\mathcal{O}'_a,\mathcal{O}'_m)}_{(\mathcal{O}_a,\mathcal{O}_m)}(X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y) = \Gamma_f : \mathrm{Tamb}^{(\mathcal{O}_a,\mathcal{O}_m)}_X \to \mathrm{Tamb}^{(\mathcal{O}_a,\mathcal{O}_m)}_Y$$

and

$$\mathrm{Tamb}^{(\mathcal{O}'_a,\mathcal{O}'_m)}_{(\mathcal{O}_a,\mathcal{O}_m)}\left(\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[r, "f_2"] \arrow[dd, "\cong"', "\psi_1"] & B \arrow[dr, "f_3"] \arrow[dd, "\cong"', "\psi_2"] & \\ X & & & Y \\ & A' \arrow[ul, "f'_1"] \arrow[r, "f'_2"'] & B' \arrow[ur, "f_3"'] & \end{tikzcd}\right) = \left(\hat{\psi} : \Gamma_f \to \Gamma_{f'}\right)$$

where $\hat{\psi}$ has components $\hat{\psi}_{\underline{S}}$ given by

$$\begin{aligned}&\Gamma_f \underline{S}(C) \to \Gamma_{f'} \underline{S}(C) : \left((D, f_2 p_D) \xleftarrow{g_1} E \xrightarrow{g_2} F \xrightarrow{g_3} (C \times_Y B, pr_B), d\right) \\ &\mapsto \left((D, f'_2\psi_1 p_D) \xleftarrow{g_1} (E, \psi_2 p_E) \xrightarrow{g_2} (F, \psi_2 p_F) \xrightarrow{(pr_C g_3, \psi_2 pr_B g_3)} (C \times_Y B', pr_{B'}), d\right).\end{aligned}$$

Note that we are identifying elements in the second coordinate across the isomorphisms $\mathcal{R}_{f_1}\underline{S}(D, p_D) \cong \underline{S}(D, f_1 p_D) \cong \mathcal{R}_{f'_1}\underline{S}(D, f'_1\psi_1 p_D)$.

- **Pseudo functoriality constraint:** Write f and g to denote $(\mathcal{O}'_a, \mathcal{O}'_m)$-bispans $(X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ and $(Y \xleftarrow{g_1} C \xrightarrow{g_2} D \xrightarrow{g_3} Z)$ respectively. The pseudo functoriality constraint is the natural isomorphism $\left(\mathrm{Tamb}^{(\mathcal{O}'_a,\mathcal{O}'_m)}_{(\mathcal{O}_a,\mathcal{O}_m)}\right)^2$ with component

2-cells given by the composite

$$\begin{aligned}\Gamma_g\Gamma_f &= \mathcal{T}_{g_3}\mathcal{N}_{g_2}\mathcal{R}_{g_1}\mathcal{T}_{f_3}\mathcal{N}_{f_2}\mathcal{R}_{f_1} \xrightarrow{\cong} \mathcal{T}_{g_3}\mathcal{N}_{g_2}\mathcal{T}_{pr_C}\mathcal{R}_{pr_B}\mathcal{N}_{f_2}\mathcal{R}_{f_1}\\
&\xrightarrow{\cong} \mathcal{T}_{g_3}\mathcal{N}_{g_2}\mathcal{T}_{pr_C}\mathcal{N}_{f_2\times_Y \mathrm{id}_C}\mathcal{R}_{pr_A}\mathcal{R}_{f_1} \xrightarrow{\cong} \mathcal{T}_{g_3}\mathcal{N}_{g_2}\mathcal{T}_{pr_C}\mathcal{N}_{f_2\times_Y \mathrm{id}_C}\mathcal{R}_{f_1 pr_A}\\
&\xrightarrow{\cong} \mathcal{T}_{g_3}\mathcal{T}_{\pi_D}\mathcal{N}_{pr_{\Pi_{g_2}(B\times_Y D)}}\mathcal{R}_{ev}\mathcal{N}_{f_2\times_Y \mathrm{id}_C}\mathcal{R}_{f_1 pr_A}\\
&\xrightarrow{\cong} \mathcal{T}_{g_3\pi_D}\mathcal{N}_{pr_{\Pi_{g_2}(B\times_Y D)}}\mathcal{R}_{ev}\mathcal{N}_{f_2\times_Y \mathrm{id}_C}\mathcal{R}_{f_1 pr_A}\\
&\xrightarrow{\cong} \mathcal{T}_{g_3\pi_D}\mathcal{N}_{pr_{\Pi_{g_2}(B\times_Y D)}}\mathcal{N}_{pr_{C\times_D \Pi_{g_2}(B\times_Y D)}}\mathcal{R}_{pr_{A\times_Y C}}\mathcal{R}_{f_1 pr_A}\\
&\xrightarrow{\cong} \mathcal{T}_{g_3\pi_D}\mathcal{N}_{pr_{\Pi_{g_2}(B\times_Y D)}}\mathcal{R}_{f_1 pr_B} = \Gamma_{gf}\end{aligned}$$

where the isomorphisms are from Propositions 4.23 and 4.26.

- **Pseudo unit constraint:** A natural isomorphism $\left(\mathrm{Tamb}^{(\mathcal{O}'_a,\mathcal{O}'_m)}_{(\mathcal{O}_a,\mathcal{O}_m)}\right)^0$ with component 2-cells $\mathrm{id}_{[\mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)},\mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}]} \to \Gamma_{\mathrm{id}_X}$ which have components $\underline{S}(A) \xrightarrow{\cong} \Gamma_{\mathrm{id}_X}\underline{S}(A)$. On objects this is given by $a \mapsto [A = A = A = A, a]$. The inverse natural isomorphism has components given by $[B \xleftarrow{f_1} C \xrightarrow{f_2} D \xrightarrow{f_3} A, b] \mapsto tr_{f_3}nm_{f_2}res_{f_1}(b)$. Note that $[A = A = A = A, tr_{f_3}nm_{f_2}res_{f_1}(b)] = [B \xleftarrow{f_1} C \xrightarrow{f_2} D \xrightarrow{f_3} A, b]$ because we can identity the two representatives along $(A = A = A = A, b)$ via $(B \xleftarrow{f_1} C \xrightarrow{f_2} D \xrightarrow{f_3} A)$.

We now define a bicategory $\int \mathrm{Tamb}^{(\mathcal{O}'_a,\mathcal{O}'_m)}_{(\mathcal{O}_a,\mathcal{O}_m)}$. Unlike in Chapter 3, where $\int \mathrm{Mack}^{\mathcal{O}'}_{\mathcal{O}}$ was the Grothendieck construction on $\mathrm{Mack}^{\mathcal{O}'}_{\mathcal{O}}$, the bicategory $\int \mathrm{Tamb}^{(\mathcal{O}'_a,\mathcal{O}'_m)}_{(\mathcal{O}_a,\mathcal{O}_m)}$ is not the Grothendieck construction on $\mathrm{Tamb}^{(\mathcal{O}'_a,\mathcal{O}'_m)}_{(\mathcal{O}_a,\mathcal{O}_m)}$ due to the inclusion of forgetful functors in the description of its 1-cells.

Definition 4.47. We have a bicategory $\int \mathrm{Tamb}^{(\mathcal{O}'_a,\mathcal{O}'_m)}_{(\mathcal{O}_a,\mathcal{O}_m)}$ with

- **Objects:** The 0-cells of the bicategory are pairs $(X, \underline{S})$ where X is a finite G-set and $\underline{S}$ is an $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functor over X.

- **Hom Categories:** For a pair of 0-cells $(X, \underline{S})$ and $(Y, \underline{T})$ the category $\int \mathrm{Tamb}^{(\mathcal{O}'_a,\mathcal{O}'_m)}_{(\mathcal{O}_a,\mathcal{O}_m)}((X, \underline{S}), (Y, \underline{T}))$ has objects pairs (f, α) where f is an $(\mathcal{O}'_a, \mathcal{O}'_m)$-bispan $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ and $\alpha : \mathcal{U}\Gamma_f\underline{S} \to \mathcal{U}\underline{T}$ is a morphism in $\mathrm{Mack}^{\mathcal{O}_a}_Y$, and for morphisms $(f, \alpha) \to (g, \beta)$ has isomorphisms ψ of $(\mathcal{O}'_a, \mathcal{O}'_m)$-bispans such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{U}\Gamma_f\underline{S} \arrow[rr, "\mathcal{U}\hat{\psi}_{\underline{S}}"] \arrow[dr, "\alpha"'] & & \mathcal{U}\Gamma_g\underline{S} \arrow[dl, "\beta"] \\ & \mathcal{U}\underline{T} & \end{tikzcd} \tag{4.48}$$

Defining α to be a morphism of underlying Mackey functors rather than the Tambara functors themselves is the reason that $\int \mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}$ is not the Grothendieck construction on $\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}$.

- **Identity 1-cells:** The identity 1-cell of a pair $(X, \underline{S})$ is the pair

$$\left(1_X(*), \mathcal{U}\left(\left(\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}\right)^0\right)_{\underline{S}}^{-1}\right)$$

where $1_X(*)$ is $(X = X = X = X)$, the identity 1-cell of X in $\underline{(\mathcal{O}'_a, \mathcal{O}'_m)}$ and $\left(\left(\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}\right)^0\right)_{\underline{S}}^{-1}$ is the $\underline{S}$ component of the inverse of the pseudo unity constraint for $\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}$.

- **Horizontal Composition:** For each triple of 0-cells $(X, \underline{S}), (Y, \underline{T}), (Z, \underline{V})$ we have a horizontal composition functor $c_{(X,\underline{S}),(Y,\underline{T}),(Z,\underline{V})}$:

$$\int \mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}((Y,\underline{T}),(Z,\underline{V})) \times \int \mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}((X,\underline{S}),(Y,\underline{T})) \\ \to \int \mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}((X,\underline{S}),(Z,\underline{V})).$$

On a pair of 1-cells

$$(f,\alpha) \in \int \mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}((X,\underline{S}),(Y,\underline{T})) \text{ and } (g,\beta) \in \int \mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}((Y,\underline{T}),(Z,\underline{V}))$$

the horizontal composition returns the pair

$$\left(c_{X,Y,Z}(g,f), \beta\,(\Gamma_g\alpha)\,\mathcal{U}\left(\left(\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}\right)^2_{g,f}\right)_{\underline{S}}^{-1}\right)$$

where $c_{X,Y,Z}$ is the horizontal composition functor in $\underline{(\mathcal{O}'_a, \mathcal{O}'_m)}$ in the first coordinate, and in the second coordinate is the composite map

$$\mathcal{U}\Gamma_{gf}\underline{S} \xrightarrow{\mathcal{U}\left(\left(\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}\right)^2_{g,f}\right)_{\underline{S}}^{-1}} \mathcal{U}\Gamma_g\Gamma_f\underline{S} \xrightarrow{\cong} \Gamma_g\mathcal{U}\Gamma_f\underline{S} \xrightarrow{\Gamma_g\alpha} \Gamma_g\mathcal{U}\underline{T} \xrightarrow{\cong} \mathcal{U}\Gamma_g\underline{T} \xrightarrow{\beta} \underline{V}.$$

- **Associator:** The components of the associator $a_{(W,\underline{P}),(X,\underline{Q}),(Y,\underline{S}),(Z,\underline{T})}$ are 2-cells which, on a given triple of composable 1-cells

$$\left((W \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} X), \alpha\right), \left((X \xleftarrow{g_1} C \xrightarrow{g_2} D \xrightarrow{g_3} Y), \beta\right), \left((Y \xleftarrow{h_1} E \xrightarrow{h_2} F \xrightarrow{h_3} Z), \gamma\right)$$

is the associator $a_{W,X,Y,Z}$ in $\underline{(\mathcal{O}'_a, \mathcal{O}'_m)}$. That is, the invertible 2-cell between $(\mathcal{O}'_a, \mathcal{O}'_m)$-bispans. Our definition of 2-cells requires Diagram 4.48 to commute, for the associator this is the outside of the following diagram.

$$\begin{tikzcd}
\mathcal{U}\Gamma_{(hg)f}\underline{P} \arrow[rrr, "\mathcal{U}\psi_{\underline{P}}", "\cong"'] \arrow[d, "\cong"'] & & & \mathcal{U}\Gamma_{h(gf)}\underline{P} \arrow[d, "\cong"] \\
\mathcal{U}\Gamma_{hg}\Gamma_f\underline{P} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathcal{U}\Gamma_h\Gamma_g\Gamma_f\underline{P} \arrow[r, equal] \arrow[d, "\cong"] & \mathcal{U}\Gamma_h\Gamma_g\Gamma_f\underline{P} \arrow[d, "\cong"] & \mathcal{U}\Gamma_h\Gamma_{gf}\underline{P} \arrow[l, "\cong"'] \arrow[d, "\cong"] \\
\Gamma_{hg}\mathcal{U}\Gamma_f\underline{P} \arrow[r, "\cong"] \arrow[d, "\Gamma_{hg}\alpha"'] & \Gamma_h\Gamma_g\mathcal{U}\Gamma_f\underline{P} \arrow[d, "\Gamma_h\Gamma_g\alpha"] & \Gamma_h\mathcal{U}\Gamma_g\Gamma_f\underline{P} \arrow[l, "\cong"'] & \Gamma_h\mathcal{U}\Gamma_{gf}\underline{P} \arrow[l, "\cong"] \arrow[dddl, bend left] \\
\Gamma_{hg}\mathcal{U}\underline{Q} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \Gamma_h\Gamma_g\mathcal{U}\underline{Q} \arrow[r, equal] & \Gamma_h\Gamma_g\mathcal{U}\underline{Q} \arrow[d, "\cong"] & \\
\mathcal{U}\Gamma_{hg}\underline{Q} \arrow[r, "\cong"] & \mathcal{U}\Gamma_h\Gamma_g\underline{Q} \arrow[r, "\cong"] & \Gamma_h\mathcal{U}\Gamma_g\underline{Q} \arrow[d, "\Gamma_h\beta"'] & \\
& & \Gamma_h\mathcal{U}\underline{S} \arrow[d, "\cong"] & \\
& & \mathcal{U}\Gamma_h\underline{S} \arrow[d, "\gamma"] & \\
& & \mathcal{U}\underline{T} &
\end{tikzcd}$$

The top region is the pseudo associativity diagram for $\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}$ (backwards). The remaining regions commute for one of the following reasons: naturality of $\left(\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}\right)^2_{h,g}$, functoriality of Γ_h, applications of Lemma 4.32.

- **Unitor:** For each pair of 0-cells, $(X,\underline{S})$ and $(Y,\underline{T})$ we have unitor natural isomorphisms

$$l_{(X,\underline{S}),(Y,\underline{T})} \qquad \text{and} \qquad r_{(X,\underline{S}),(Y,\underline{T})}$$

whose components on a 1-cell $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y, \mathcal{U}\Gamma_f\underline{S} \xrightarrow{\alpha} \mathcal{U}\underline{T})$ are the pairs of isomorphisms

$$(l_1 = pr_A : A\times_B(Y\times_Y\Pi_{\mathrm{id}_Y} B\times_Y Y) \to A, l_2 : \Pi_{\mathrm{id}_Y}(B\times_Y Y) \to B, (y,\sigma) \mapsto pr_B\sigma(y))$$

and

$$(r_1 = pr_A : X \times_X (A \times_B \Pi_{f_2}(X \times_X A)) \to A, r_2 = \pi_B : \Pi_{f_2}(X \times_X A)) \to B$$

respectively. In order for l to be a 2-cell we require Diagram 4.48 to commute, this is the outside of the following diagram.

$$\begin{tikzcd}
\mathcal{U}\Gamma_{\mathrm{id}_Y f}\underline{S} \arrow[rr, "\hat{\imath}_{\underline{S}}"] \arrow[d, "\cong"'] & & \mathcal{U}\Gamma_f\underline{S} \arrow[dd, equal] \\
\mathcal{U}\Gamma_{\mathrm{id}_Y}\Gamma_f\underline{S} \arrow[urr, "\cong"'] \arrow[d, "\cong"'] & & \\
\Gamma_{\mathrm{id}_Y}\mathcal{U}\Gamma_f\underline{S} \arrow[rr, "\cong"] \arrow[d, "\Gamma_{\mathrm{id}_Y}\alpha"'] & & \mathcal{U}\Gamma_f\underline{S} \arrow[d, "\alpha"] \\
\Gamma_{\mathrm{id}_Y}\mathcal{U}\underline{T} \arrow[rr, "\cong"] & & \mathcal{U}\underline{T}
\end{tikzcd}$$

The triangle commutes by one of the pseudo unity axioms for $\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}$, the middle region commutes by Lemma 4.32, and the bottom region commutes because it is a naturality square for $\left(\left(\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}\right)^0\right)^{-1}$.

It is straightforward to check that the unity and pentagon axioms are satisfied. There is a projection pseudofunctor $\int \mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)} \to \underline{(\mathcal{O}'_a,\mathcal{O}'_m)}$ given by

$$\begin{aligned}
(X,\underline{S}) &\mapsto X\\
(f,\alpha) &\mapsto f\\
\psi &\mapsto \psi.
\end{aligned}$$

4.4 $(\mathcal{O}'_a,\mathcal{O}'_m)$-Tambara functors as coherent $(\mathcal{O}_a,\mathcal{O}_m)$-monoids

We want to consider specific sections of the projection pseudofunctor. Given a section $F:\underline{(\mathcal{O}_a,\mathcal{O}_m)} \to \int \mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}$ for the projection pseudofunctor with $F(X)=(X,\underline{S})$ and $F(Y)=(Y,\underline{T})$ we will write

$$F\left(X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y\right) = \left(\left(X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y\right), \mu^F_{f_3,f_2,f_1} : \mathcal{U}_Y\Gamma_{f_3,f_2,f_1}\underline{S} \to \mathcal{U}_Y\underline{T}\right),$$

when it is clear what bispan we are referring to we will write μ^F_f instead. Note that $\Gamma_f\underline{S}(C) = \mathcal{T}_{f_3}\mathcal{N}_{f_2}\mathcal{R}_{f_1}\underline{S}(C) = \mathcal{N}_{f_2}\mathcal{R}_{f_1}\underline{S}\Delta_{f_3}(C) = \mathcal{N}_{f_2}\mathcal{R}_{f_1}\underline{S}(C\times_Y B)$ so $\Gamma_f\underline{S}(C)$ has elements of the form $[D \xleftarrow{g_1} E \xrightarrow{g_2} F \xrightarrow{g_3} C\times_Y B, d]$ where the first coordinate is an $(\mathcal{O}_a,\mathcal{O}_m)$-bispan over B and $d\in\mathcal{R}_{f_1}\underline{S}(D)\cong\underline{S}(D)$.

Definition 4.49. A **coherent** $(\mathcal{O}'_a,\mathcal{O}'_m)$**-monoid in** $\mathrm{Tamb}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$ is an $(\mathcal{O}_a,\mathcal{O}_m)$-Tambara functor $\underline{S}$ together with a functor $F:\underline{(\mathcal{O}'_a,\mathcal{O}'_m)} \to \int \mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}$ that is a section of the projection pseudofunctor such that:

- $F(X)=(X,\mathcal{R}_X\underline{S})$ for all G-sets X,

- for all $(\mathcal{O}_a, \mathcal{O}_m)$-bispans $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ we have that on objects μ_f^F is $\mathcal{U}\Gamma_f \mathcal{R}_X \underline{S}(C) \to \mathcal{U}\mathcal{R}_Y \underline{S}(C), [D \xleftarrow{g_1} E \xrightarrow{g_2} F \xrightarrow{g_3} C \times_Y B, d] \mapsto tr_{\tilde{g_3}}(nm_{g_2}(res_{g_1}(d)))$. Recall that $\tilde{g_3} = pr_C g_3$. We note that μ_f^F is a map of Mackey functors, this claim is demonstrated in part 1a of the proof of Theorem 4.53.

A morphism $(\underline{S}, F) \to (\underline{T}, H)$ of coherent $(\mathcal{O}'_a, \mathcal{O}'_m)$-monoids in $\mathrm{Tamb}_*^{(\mathcal{O}_a, \mathcal{O}_m)}$ consists of a morphism $\varphi : \underline{S} \to \underline{T}$ in $\mathrm{Tamb}_*^{(\mathcal{O}_a, \mathcal{O}_m)}$ together with the strong transformation $\alpha : F \to H$ such that:

- **Component 1-cells:** For X a finite G-Set, the component 1-cell in $\int \mathrm{Tamb}_{(\mathcal{O}_a, \mathcal{O}_m)}^{(\mathcal{O}'_a, \mathcal{O}'_m)} \left((X, \mathcal{R}_X \underline{S}), (X, \mathcal{R}_X \underline{T})\right)$ is

$$\alpha_X = (X = X = X = X, \overline{\alpha_X} : \mathcal{U}\Gamma_{\mathrm{id}_X} \mathcal{R}_X \underline{S} \to \mathcal{U}\mathcal{R}_X \underline{T})$$

  where $\overline{\alpha_X}$ is the composite

$$\mathcal{U}\Gamma_{\mathrm{id}_X} \mathcal{R}_X \underline{S} \xrightarrow{\mathcal{U}\left(\left(\mathrm{Tamb}_{(\mathcal{O}_a, \mathcal{O}_m)}^{(\mathcal{O}'_a, \mathcal{O}'_m)}\right)^0_{\mathcal{R}_X \underline{S}}\right)^{-1}} \mathcal{U}\mathcal{R}_X \underline{S} \xrightarrow{\mathcal{U}\mathcal{R}_X(\varphi)} \mathcal{U}\mathcal{R}_X \underline{T}.$$

- **Pseudo Naturality Constraints** For an $(\mathcal{O}'_a, \mathcal{O}'_m)$-bispan

$$f = (X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$$

  the component 2-cell $\alpha_f : (Hf)\alpha_X \to \alpha_Y(Ff)$ in $\int \mathrm{Tamb}_{(\mathcal{O}_a, \mathcal{O}_m)}^{(\mathcal{O}'_a, \mathcal{O}'_m)}((X, \mathcal{R}_X \underline{S}), (Y, \mathcal{R}_Y \underline{T}))$ is given by $l^{-1}r$, the right unitor followed by the inverse of the left unitor. Note that being a 2-cell imposes the condition that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{U}\Gamma_{f\,\mathrm{id}_X} \mathcal{R}_X \underline{S} \arrow[rr, "\mathcal{U}\hat{\psi}_{\underline{S}}"] \arrow[d, "\cong"'] & & \mathcal{U}\Gamma_{\mathrm{id}_Y f} \mathcal{R}_X \underline{S} \arrow[d, "\cong"] \\
\mathcal{U}\Gamma_f \Gamma_{\mathrm{id}_X} \mathcal{R}_X \underline{S} \arrow[d, "\cong"'] & & \mathcal{U}\Gamma_{\mathrm{id}_Y} \Gamma_f \mathcal{R}_X \underline{S} \arrow[d, "\cong"] \\
\Gamma_f \mathcal{U}\Gamma_{\mathrm{id}_X} \mathcal{R}_X \underline{S} \arrow[d, "\Gamma_f \overline{\alpha_X}"'] & & \Gamma_{\mathrm{id}_Y} \mathcal{U}\Gamma_f \mathcal{R}_X \underline{S} \arrow[d, "\Gamma_{\mathrm{id}_Y} \mu_f^F"] \\
\Gamma_f \mathcal{U}\mathcal{R}_X \underline{T} \arrow[d, "\cong"'] & & \Gamma_{\mathrm{id}_Y} \mathcal{U}\mathcal{R}_Y \underline{S} \arrow[d, "\cong"'] \\
\mathcal{U}\Gamma_f \mathcal{R}_X \underline{T} \arrow[dr, "\mu_f^H"'] & & \mathcal{U}\Gamma_{\mathrm{id}_Y} \mathcal{R}_Y \underline{S} \arrow[dl, "\overline{\alpha_Y}"] \\
& \mathcal{U}\mathcal{R}_Y \underline{T} &
\end{tikzcd} \tag{4.50}$$

There are strong unity and strong naturality axioms which α must satisfy to be a strong transformation. We omit them here because they are similar to case for coherent $\mathcal{O}'$-monoids in $\mathcal{O}$-Mackey functors but using isomorphisms of different representatives of a fixed bispan isomorphism class which are difficult to parse and ultimately unenlightening.

Given $(\underline{S}, F)$ the identity morphism $\mathrm{id}_{(\underline{S},F)}$ consists of $\mathrm{id}_{\underline{S}}$ together with the strong transformation it determines. Given two morphisms $(\varphi, \alpha) : (\underline{S}, F) \to (\underline{T}, H)$ and $(\omega, \beta) : (\underline{T}, H) \to (\underline{V}, I)$ of coherent $(\mathcal{O}'_a, \mathcal{O}'_m)$-monoids in $\mathrm{Tamb}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$ we define their composite to be $(\omega\varphi, \beta\alpha) : (\underline{S}, F) \to (\underline{V}, I)$ where $\overline{(\beta\alpha)_X}$ is the composite

$$\mathcal{U}\Gamma_{\mathrm{id}_X}\mathcal{R}_X\underline{S} \xrightarrow{\mathcal{U}\left(\left(\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}\right)^0_{\mathcal{R}_X\underline{S}}\right)^{-1}} \mathcal{U}\mathcal{R}_X\underline{S} \xrightarrow{\mathcal{U}\mathcal{R}_X(\omega\varphi)} \mathcal{U}\mathcal{R}_X\underline{V}.$$

This is different from the usual composition of strong transformations, but is equivalent up to an invertible modification.

Definition 4.51. Let $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ be an $(\mathcal{O}_a, \mathcal{O}_m)$-bispan. There is a functor $\mathcal{E}_f : \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \to \mathbf{Set}$ with $\mathcal{E}_f(\underline{S}) = \underline{S}(\mathrm{id}_X)$ and $\mathcal{E}_f(\varphi) = \varphi_X$. We also have a functor $\mathcal{F}_f : \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \to \mathbf{Set}$ with $\mathcal{F}_f(\underline{S}) = \Gamma_f\underline{S}(\mathrm{id}_Y)$ with $\mathcal{F}_f(\varphi) = \Gamma_f(\varphi_{\mathrm{id}_Y})$. There is a natural transformation $\omega_f : \mathcal{E}_f \to \mathcal{F}_f$ with components $\underline{S}(\mathrm{id}_X) \to \Gamma_f\underline{S}(\mathrm{id}_Y)$ given by

$$x \mapsto [A = A \xrightarrow{f_2} B \xrightarrow{(f_3,\mathrm{id}_B)} Y \times_Y B, res_{f_1}(x)].$$

Remark 4.52. The proof of the following theorem is structurally similar to the proof of Theorem 3.48, the main result of Chapter 3. Rather than being the Grothendieck construction on the nose $\int \mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}$ is complicated by including forgetful functors in the description of its 1-cells, see Definition 4.47. This necessitates cluttering the diagrams in the following proof with isomorphisms passing forgetful functors across Γ functors. By Lemma 4.32 the regions these isomorphisms add the diagrams all commute. To ensure readability we will suppress these extra regions, so that some morphisms will have the correct domain or codomain only up to commuting forgetful functors.

We now state and prove the main theorem of this thesis; our characterisation of bi-incomplete Tambara functors.

Theorem 4.53. *Let be a finite group G, and let $(\mathcal{O}_a, \mathcal{O}_m)$ and $(\mathcal{O}'_a, \mathcal{O}'_m)$ be compatible pairs of indexing categories such that $(\mathcal{O}_a, \mathcal{O}'_m)$ is a compatible pair of indexing categories, $\mathcal{O}_a$ is a subcategory of $\mathcal{O}'_a$, and $\mathcal{O}_m$ is a subcategory of $\mathcal{O}'_m$. Then $(\mathcal{O}'_a, \mathcal{O}'_m)$-Tambara functors are precisely the coherent $(\mathcal{O}'_a, \mathcal{O}'_m)$-monoids in $\mathrm{Tamb}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$. That is, there is an equivalence between the category of coherent $(\mathcal{O}'_a, \mathcal{O}'_m)$-monoids in $\mathrm{Tamb}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$ and the category $\mathrm{Tamb}_*^{(\mathcal{O}'_a,\mathcal{O}'_m)}$.*

Proof. This proof proceeds as follows:

1. Define Φ (one direction of our equivalence) on objects.

2. Define Φ on morphisms.

3. Show Φ is functorial.

4. Define Ψ (the other direction of our equivalence) on objects.

5. Define Ψ on morphisms.

6. Show Ψ is functorial.

7. Finally, show that Φ and Ψ are inverse functors.

We will define a functor Φ from the category of $(\mathcal{O}'_a, \mathcal{O}'_m)$-Tambara functors to the category of coherent $(\mathcal{O}'_a, \mathcal{O}'_m)$-monoids in the category of $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functors.

1. Define Φ on objects

Let $\underline{S}$ be a an $(\mathcal{O}'_a, \mathcal{O}'_m)$-Tambara functor. We get an $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functor, which we will call $\downarrow\underline{S}$ by precomposition with the inclusion functor

$$\downarrow\underline{S} : \mathcal{P}_*^{(\mathcal{O}_a, \mathcal{O}_m)} \hookrightarrow \mathcal{P}_*^{(\mathcal{O}'_a, \mathcal{O}'_m)} \xrightarrow{\underline{S}} \mathrm{Set}\,.$$

Given an $\underline{(\mathcal{O}'_a, \mathcal{O}'_m)}$-Tambara functor $\underline{S}$ we describe a strict functor $F : \underline{(\mathcal{O}'_a, \mathcal{O}'_m)} \to \int \mathrm{Tamb}^{(\mathcal{O}'_a, \mathcal{O}'_m)}_{(\mathcal{O}_a, \mathcal{O}_m)}$ consisting of the following data:

- **Objects:** The assignment $F(X) = (X, \mathcal{R}_X \downarrow \underline{S})$.

- **Hom Categories:** For each pair of G-sets X, Y the functor

$$F : \underline{(\mathcal{O}'_a, \mathcal{O}'_m)}(X, Y) \to \int \mathrm{Tamb}^{(\mathcal{O}'_a, \mathcal{O}'_m)}_{(\mathcal{O}_a, \mathcal{O}_m)}((X, \mathcal{R}_X \downarrow \underline{S}), (Y, \mathcal{R}_Y \downarrow \underline{S}))$$

  with

$$F(X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y) = \Big((X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y), \mu_f^F : \mathcal{U}\Gamma_f \mathcal{R}_X \downarrow \underline{S} \to \mathcal{U}\mathcal{R}_Y \downarrow \underline{S}\Big)$$

  on 1-cells, where $\mu_f^F(C)$ is given on element representatives by

$$[D \xleftarrow{g_1} E \xrightarrow{g_2} F \xrightarrow{g_3} C \times_Y B, d] \mapsto tr_{\tilde{g_3}}(nm_{g_2}(res_{g_1}(d)))$$

  where $d \in \mathcal{R}_A \downarrow \underline{S}(D)$, and $\tilde{g_3} = pr_C g_3 : F \to C$ is the adjoint to g_3 across the adjunction $\Sigma_{f_3} \dashv \Delta_{f_3}$ at the level of G-sets.

  1a. Show F is well-defined on 1-cells

  We note that if (ψ_1, ψ_2) is an isomorphism between $(\mathcal{O}'_a, \mathcal{O}'_m)$-bispans g and g' then (ψ_1, ψ_2) also describes an isomorphism between $\overline{g} = (D \xleftarrow{g_1} E \xrightarrow{g_2} F \xrightarrow{\tilde{g_3}} C)$ and $\overline{g'} = (D \xleftarrow{g'_1} E' \xrightarrow{g'_2} F' \xrightarrow{\tilde{g_3}'} C)$ so $tr_{\tilde{g_3}} nm_{g_2} res_{g_1} = tr_{\tilde{g_3}'} nm_{g'_2} res_{g'_1}$ hence μ_f^F is well defined. We need to check that μ_f^F is a map of $\mathcal{O}_a$-Mackey functors. Let $h = (C \xleftarrow{h_1} J \xrightarrow{h_2} K)$ be an $\mathcal{O}_a$-transfer span then we have

$$\begin{aligned}
&\mu_f^F\left(\mathcal{U}_Y\Gamma_f\mathcal{R}_X \downarrow \underline{S}(h)\left(\left[D \xleftarrow{g_1} E \xrightarrow{g_2} F \xrightarrow{g_3} C \times_Y B, d\right]\right)\right) \\
&= \mu_f^F\Big(\Big[D \xleftarrow{g_1 pr_E} E \times_{C\times_Y B} (J \times_Y B) \\
&\qquad \xrightarrow{(g_2pr_E,(pr_J,pr_B))} F \times_{C\times_Y B} (J \times_Y B) \xrightarrow{(h_2pr_J,pr_B)} K \times_Y B, d\Big]\Big) \\
&= tr_{h_2pr_J}(nm_{(g_2pr_E,(pr_J,pr_B))}(res_{g_1pr_E}(d))).
\end{aligned}$$

Going the other way, we have

$$\begin{aligned}
&\mathcal{U}_Y\mathcal{R}_Y \downarrow \underline{S}(h)\left(\mu_f^F\left(\left[D \xleftarrow{g_1} E \xrightarrow{g_2} F \xrightarrow{g_3} C \times_Y B, d\right]\right)\right) \\
&= \mathcal{U}_Y\mathcal{R}_Y \downarrow \underline{S}(h)(tr_{\tilde{g_3}}(nm_{g_2}(res_{g_1}(d)))) \\
&= tr_{h_2}(res_{h_1}(tr_{\tilde{g_3}}(nm_{g_2}(res_{g_1}(d))))) \\
&= tr_{h_2pr_J}(nm_{(g_2pr_E,pr_J)}(res_{g_1pr_E}(d))).
\end{aligned}$$

These are equal by the following isomorphism of bispans.

$$\begin{tikzcd}
& E \times_{C\times_Y B} (J \times_Y B) \arrow[r, "{(g_2pr_E,(pr_J,pr_B))}"] \arrow[dl, "g_1pr_E"'] \arrow[dd, "{(pr_E,pr_J)}", "\cong"'] & F \times_{C\times_Y B} (J \times_Y B) \arrow[dr, "h_2pr_J"] \arrow[dd, "{(pr_F,pr_J)}", "\cong"'] & \\
D & & & K \\
& E \times_C J \arrow[ul, "g_1pr_E"] \arrow[r, "{(g_2pr_E,pr_J)}"'] & F \times_C J \arrow[ur, "h_2pr_J"'] &
\end{tikzcd}$$

Hence μ_f^F is a map of $\mathcal{O}_a$-Mackey functors as required.

<u>1b. Finish defining F</u>

On 2-cells we have

$$F\left(\begin{tikzcd}
& A \arrow[r, "f_2"] \arrow[dl, "f_1"'] \arrow[dd, "\psi_1", "\cong"'] & B \arrow[dr, "f_3"] \arrow[dd, "\psi_2", "\cong"'] & \\
X & & & Y \\
& A' \arrow[ul, "f_1'"] \arrow[r, "f_2'"'] & B' \arrow[ur, "f_3'"'] &
\end{tikzcd}\right) = \left(\begin{tikzcd}
& A \arrow[r, "f_2"] \arrow[dl, "f_1"'] \arrow[dd, "\psi_1", "\cong"'] & B \arrow[dr, "f_3"] \arrow[dd, "\psi_2", "\cong"'] & \\
X & & & Y \\
& A' \arrow[ul, "f_1'"] \arrow[r, "f_2'"'] & B' \arrow[ur, "f_3'"'] &
\end{tikzcd}\right).$$

The condition on 2-cells in $\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}_a',\mathcal{O}_m')}$ requires that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{U}\Gamma_f\mathcal{R}_X{\downarrow}\underline{S} \arrow[rr, "\hat{\psi}"] \arrow[dr, "\mathcal{U}\mu_f^F"'] & & \mathcal{U}\Gamma_{f'}\mathcal{R}_X{\downarrow}\underline{S} \arrow[dl, "\mu_{f'}^F"] \\
& \mathcal{U}{\downarrow}\mathcal{R}_Y\underline{S} &
\end{tikzcd}$$

We have

$$\begin{aligned}\mu^F_{f'}\hat{\psi}(D \xleftarrow{g_1} E \xrightarrow{g_2} F \xrightarrow{g_3} C \times_Y B, d) &= \mu^F_{f'}(D \xleftarrow{g_1} E \xrightarrow{g_2} F \xrightarrow{(pr_C g_3, \psi_2 pr_B g_3)} C \times_Y B', d) \\ &= tr_{\tilde{g}_3}(nm_{g_2}(res_{g_1}(s))) = \mu^F_f(D \xleftarrow{g_1} E \xrightarrow{g_2} F \xrightarrow{g_3} C \times_Y B, d)\end{aligned}$$

so the diagram commutes as required.

We let $\Phi(\underline{S}) = (\Downarrow \underline{S}, F)$.

2. Define Φ on morphisms

Given a morphism $\varphi : \underline{S} \to \underline{T}$ of $(\mathcal{O}'_a, \mathcal{O}'_m)$-Tambara functors, we let $\Phi(\varphi) : (\Downarrow \underline{S}, F) \to (\Downarrow \underline{T}, H)$ be φ in the first coordinate and $\alpha : F \to H$ as determined by φ.

2a. Show Φ is well-defined on morphisms

We need to check that α_f defines a 2-cell for all $(\mathcal{O}'_a, \mathcal{O}'_m)$-bispans, $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$. Consider the following diagram.

$$\begin{tikzcd}
\Gamma_f\mathcal{U}\Gamma_{\mathrm{id}_X}\mathcal{R}_X{\downarrow}\underline{S} \arrow[dd, "\Gamma_f\overline{\alpha_X}"'] & \mathcal{U}\Gamma_f\mathcal{R}_X{\downarrow}\underline{S} \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "\mu_f^F"] \arrow[ddl, "\Gamma_f\mathcal{U}\mathcal{R}_X\varphi"'] & \Gamma_{\mathrm{id}_Y}\mathcal{U}\Gamma_f\mathcal{R}_X{\downarrow}\underline{S} \arrow[d, "\Gamma_{\mathrm{id}_Y}\mu_f^F"] \\
 & \mathcal{U}\mathcal{R}_Y{\downarrow}\underline{S} \arrow[r, "\cong"] \arrow[d, "\mathcal{U}\mathcal{R}_Y\varphi"'] & \Gamma_{\mathrm{id}_Y}\mathcal{U}\mathcal{R}_Y{\downarrow}\underline{S} \arrow[dl, "\overline{\alpha_Y}"] \\
\Gamma_f\mathcal{U}\mathcal{R}_X{\downarrow}\underline{T} \arrow[r, "\mu_f^H"] & \mathcal{U}\mathcal{R}_Y{\downarrow}\underline{T} &
\end{tikzcd} \tag{4.54}$$

Noting that the top row of the diagram is the same morphism as the top two rows of 4.50 we see that the outside of this diagram is equivalent to 4.50. Showing that this diagram commutes we confirm that α_f is a 2-cell. The bottom right triangle commutes by the definition of $\overline{\alpha_Y}$. The top left triangle commutes because it is the image under Γ_f of the defining diagram of $\overline{\alpha_X}$. The right square commutes because it is a naturality square for $\left(\mathrm{Tamb}^{(\mathcal{O}'_a, \mathcal{O}'_m)}_{(\mathcal{O}_a, \mathcal{O}_m)}\right)^0$. For $C \in \mathcal{P}^{(\mathcal{O}_a, \mathcal{O}_m)}_Y$ the map $\Gamma_f\mathcal{R}_X\varphi(C)$ is given on element representatives by

$$\left(D \xleftarrow{g_1} E \xrightarrow{g_2} F \xrightarrow{g_3} C \times_Y B, d\right) \mapsto \left(D \xleftarrow{g_1} E \xrightarrow{g_2} F \xrightarrow{g_3} C \times_Y B, \varphi(d)\right).$$

For any such element representative we have

$$tr_{\tilde{g}_3}(nm_{g_2}(res_{g_1}(\varphi(d)))) = \varphi(tr_{\tilde{g}_3}(nm_{g_2}(res_{g_1}(d))))$$

because φ is a map of $(\mathcal{O}'_a, \mathcal{O}'_m)$-Tambara functors and so commutes with transfers, norms, and restrictions. Hence the middle triangle commutes. So the diagram commutes and therefore α_f is a 2-cell as required.

3. Show Φ is functorial

Note that $\Phi(\mathrm{id}_{\underline{S}}) = \mathrm{id}_{\Phi(\underline{S})}$. Given $\varphi : \underline{S} \to \underline{T}$ and $\omega : \underline{T} \to \underline{V}$ we have that $\Phi(\omega)\Phi(\varphi)$

and $\Phi(\omega\varphi)$ are both $\omega\varphi$ in the first coordinate and both

$$\Gamma_{\mathrm{id}_X}\mathcal{R}{\downarrow}\underline{S} \xrightarrow{\left(\left(\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}\right)^0_{\mathcal{R}_X{\downarrow}\underline{S}}\right)^{-1}} \mathcal{R}_X\underline{S} \xrightarrow{\mathcal{R}_X(\omega\varphi)} \mathcal{R}_X\underline{V}$$

in the second. Hence Φ is a functor.

Now we construct a functor Ψ from the category of coherent $(\mathcal{O}'_a,\mathcal{O}'_m)$-monoids in $(\mathcal{O}_a,\mathcal{O}_m)$-Tambara functors to the category of $(\mathcal{O}'_a,\mathcal{O}'_m)$-Tambara functors.

4. Define Ψ on objects

Let $(\underline{S},F)$ be a coherent $(\mathcal{O}'_a,\mathcal{O}'_m)$-monoid in $\mathrm{Tamb}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$. We will extend $\underline{S}$ to an $(\mathcal{O}'_a,\mathcal{O}'_m)$-Tambara functor, $\uparrow_F \underline{S}$. On objects we let $\uparrow_F \underline{S}(X) = \underline{S}(X)$. Given a morphism $f \in \mathcal{P}^{(\mathcal{O}'_a,\mathcal{O}'_m)}$ represented by the $(\mathcal{O}'_a,\mathcal{O}'_m)$-bispan $(X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ we let $\uparrow_F \underline{S}([X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y])$ be the function $\uparrow_F \underline{S}(X) \to\uparrow_F \underline{S}(Y)$ given by the composition

$$\underline{S}(X) = \mathcal{R}_X\underline{S}(\mathrm{id}_X) \xrightarrow{\omega_f} \Gamma_f\mathcal{R}_X\underline{S}(\mathrm{id}_Y) = \mathcal{U}\Gamma_f\mathcal{R}_X\underline{S}(\mathrm{id}_Y) \xrightarrow{\mu_f^F} \mathcal{U}\mathcal{R}_Y\underline{S}(\mathrm{id}_Y) = \underline{S}(Y).$$

Because the morphisms in $\mathcal{P}^{(\mathcal{O}'_a,\mathcal{O}'_m)}$ are isomorphism classes of $(\mathcal{O}'_a,\mathcal{O}'_m)$-bispans we need to check that this composite is equal on isomorphic $(\mathcal{O}'_a,\mathcal{O}'_m)$-bispans. Given an isomorphism ψ of $(\mathcal{O}'_a,\mathcal{O}'_m)$-bispans $(X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ and $(X \xleftarrow{f'_1} A' \xrightarrow{f'_2} B' \xrightarrow{f'_3} Y)$ then we consider the following diagram

$$\begin{tikzcd}
 & & \mathcal{U}\Gamma_f\mathcal{R}_X\underline{S}(\mathrm{id}_Y) \arrow[d, "\mathcal{U}\hat{\psi}"] \arrow[dr, "\mu_f^F"] & & \\
\underline{S}(X) \arrow[r, equal] & \mathcal{R}_X\underline{S}(\mathrm{id}_X) \arrow[ur, "\omega_f"] \arrow[dr, "\omega_{f'}"'] & & \mathcal{U}\mathcal{R}_Y\underline{S}(\mathrm{id}_Y) \arrow[r, equal] & \underline{S}(Y) \\
 & & \mathcal{U}\Gamma_{f'}\mathcal{R}_X\underline{S}(\mathrm{id}_Y) \arrow[ur, "\mu_{f'}^F"'] & &
\end{tikzcd}$$

Here $\hat{\psi} = \mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}(\psi)$. The right triangle by the defining condition on 2-cells in $\int \mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}'_m)}$. The bottom map in the left triangle is

$$x \mapsto [A' = A' \xrightarrow{f'_2} B' \xrightarrow{(f'_3,\mathrm{id}_{B'})} Y \times_Y B', res_{f'_1}(x)]$$

and going around the top of the left triangle we have

$$\begin{aligned}
x \mapsto &[A = A \xrightarrow{f_2} B \xrightarrow{(f_3,\mathrm{id}_B)} Y \times_Y B, res_{f_1}(x)] \\
&\mapsto [A = A \xrightarrow{f_2} B \xrightarrow{(f_3,\psi_2)} Y \times_Y B', res_{f_1}(x)] \\
&= [A' \xleftarrow{\psi_1} A \xrightarrow{f_2} B \xrightarrow{(f_3,\psi_2)} Y \times_Y B', res_{\psi_1^{-1}} res_{f_1}(x)] \\
&= [A' \xleftarrow{\psi_1} A \xrightarrow{f_2} B \xrightarrow{(f_3,\psi_2)} Y \times_Y B', res_{f'_1}(x)] \\
&= [A' = A' \xrightarrow{f'_2} B' \xrightarrow{(f'_3,\mathrm{id}_{B'})} Y \times_Y B', res_{f'_1}(x)].
\end{aligned}$$

As this diagram commutes $\uparrow_F \underline{S}$ is well-defined on morphisms. We need to check that $\uparrow_F \underline{S}$ is functorial. Given $(\mathcal{O}'_a, \mathcal{O}'_m)$-bispans $(X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ and $(Y \xleftarrow{g_1} C \xrightarrow{g_2} D \xrightarrow{g_3} Z)$ we consider the following diagram.

$$\begin{tikzcd}
\underline{S}(X) \arrow[d, equal] & & \underline{S}(Y) \arrow[d, equal] \\
\mathcal{R}_X\underline{S}(\mathrm{id}_X) \arrow[r, "\omega_f"] \arrow[dd, "\omega_{gf}"'] & \mathcal{U}_Y\Gamma_f\mathcal{R}_X\underline{S}(\mathrm{id}_Y) \arrow[r, "\mu_f^F(\mathrm{id}_Y)"] \arrow[d, "\omega_g"] & \mathcal{U}_Y\mathcal{R}_Y\underline{S}(\mathrm{id}_Y) \arrow[d, "\omega_g"] \\
 & \Gamma_g\mathcal{U}_Y\Gamma_f\mathcal{R}_X\underline{S}(\mathrm{id}_Z) \arrow[r, "\Gamma_g\mu_f^F(\mathrm{id}_Z)"] \arrow[dl, "\cong"] & \Gamma_g\mathcal{U}_Y\mathcal{R}_Y\underline{S}(\mathrm{id}_Z) \arrow[d, "\mu_g^F"] \\
\mathcal{U}_Z\Gamma_{gf}\mathcal{R}_X\underline{S}(\mathrm{id}_Z) \arrow[rr, "\mu_{gf}^F"] & & \mathcal{U}_Z\mathcal{R}_Z\underline{S}(\mathrm{id}_Z) \arrow[d, equal] \\
 & & \underline{S}(Z)
\end{tikzcd}$$

The bottom region is the pseudo functoriality constraint for F and so commutes because F is a strict functor. The top right region commutes because it is a naturality square for ω_g. Going around the right side of the remaining region we have

$$\begin{aligned}
x \mapsto & [X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{(f_3, \mathrm{id}_B)} Y \times_Y B, x] \\
\mapsto & [Y \xleftarrow{g_1} C \xrightarrow{g_2} D \xrightarrow{(g_3, \mathrm{id}_D)} Z \times_Z D, [X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{(f_3, \mathrm{id}_B)} Y \times_Y B, x]] \\
\mapsto & [X \xleftarrow{f_1 pr_A} A \times_B (C \times_D \prod_{g_2}(B \times_Y C)) \xrightarrow{pr_{\Pi_{g_2}(B \times_Y C)}} \prod_{g_2}(B \times_Y C) \xrightarrow{(g_3\pi_D, \mathrm{id})} Z \times_Z \prod_{g_2}(B \times_Y C), x] \\
= & \, \omega_{gf}(x)
\end{aligned}$$

so the region commutes.
On an identity bispan, id_X, we have $\uparrow_F \underline{S}(\mathrm{id}_X)$ given by

$$x \mapsto [X = X = X = X \times_X X, x] \mapsto tr_{\mathrm{id}_X} nm_{\mathrm{id}_X} res_{\mathrm{id}_X}(x) = x.$$

Hence $\uparrow_F \underline{S}$ is a functor $\mathcal{P}_*^{(\mathcal{O}'_a, \mathcal{O}'_m)} \to \mathbf{Set}$. Note also that $\uparrow_F \underline{S}$ takes values in commutative rings and is product-preserving because $\uparrow_F \underline{S}(X) = \underline{S}(X)$. Hence $\uparrow_F \underline{S}$ is an $(\mathcal{O}'_a, \mathcal{O}'_m)$-Tambara functor. We let $\Psi(\underline{S}, F) = \uparrow_F \underline{S}$.

<u>5. Define Ψ on morphisms</u>

Given a morphism $(\varphi, \alpha) : (\underline{S}, F) \to (\underline{T}, H)$ of coherent $(\mathcal{O}'_a, \mathcal{O}'_m)$-monoids in $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functors we let $\Psi(\varphi, \alpha) = \varphi$ on objects, which is sufficient to describe a morphism $\uparrow_F \underline{S} \to \uparrow_H \underline{T}$. To see that this is sufficient we need to check that φ is natural in $\mathcal{P}_{(\mathcal{O}_a, \mathcal{O}_m)}^{(\mathcal{O}'_a, \mathcal{O}'_m)}$. That is, given $(X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ an $(\mathcal{O}'_a, \mathcal{O}'_m)$-bispan we require the following diagram to commute.

$$\begin{tikzcd}
\underline{S}(X) \arrow[r, equal] \arrow[d, "\varphi(X)"] & \mathcal{R}_X\underline{S}(\mathrm{id}_X) \arrow[r, "\omega_f"] \arrow[d, "\mathcal{R}_X\varphi(\mathrm{id}_X)"] & \mathcal{U}_Y\Gamma_f\mathcal{R}_X\underline{S}(\mathrm{id}_Y) \arrow[r, "\mu_f^F(\mathrm{id}_Y)"] \arrow[d, "\mathcal{U}_Y\Gamma_f\mathcal{R}_X\varphi(\mathrm{id}_Y)"] & \mathcal{U}_Y\mathcal{R}_Y\underline{S}(\mathrm{id}_Y) \arrow[r, equal] \arrow[d, "\mathcal{R}_Y\varphi(\mathrm{id}_Y)"] & \underline{S}(Y) \arrow[d, "\varphi(Y)"] \\
\underline{T}(X) \arrow[r, equal] & \mathcal{R}_X\underline{T}(\mathrm{id}_X) \arrow[r, "\omega_f"'] & \mathcal{U}_Y\Gamma_f\mathcal{R}_X\underline{T}(\mathrm{id}_Y) \arrow[r, "\mu_f^H(\mathrm{id}_Y)"'] & \mathcal{U}_Y\mathcal{R}_Y\underline{T}(\mathrm{id}_Y) \arrow[r, equal] & \underline{T}(Y)
\end{tikzcd}$$

The second square is a naturality square for ω_f and so commutes. Note that the outside of Diagram 4.54 commutes because α is a strong transformation $F \to H$, hence the middle triangle of Diagram 4.54 commutes. The third square in the above diagram is the middle triangle of Diagram 4.54 applied to id_Y and so commutes. Therefore $\varphi :\uparrow_F \underline{S} \to\uparrow_H \underline{T}$ is a morphism of $(\mathcal{O}'_a, \mathcal{O}'_m)$-Tambara functors.

<u>6. Show Ψ is functorial</u>

The functoriality of Ψ is immediate.

<u>7. Show that Φ and Ψ are inverse</u>

<u>7a. $\Phi\Psi$ is isomorphic to the identity</u>

Let $(\underline{S}, F)$ be a coherent $(\mathcal{O}'_a, \mathcal{O}'_m)$-monoid in $\mathrm{Tamb}_*^{(\mathcal{O}'_a,\mathcal{O}'_m)}$. We have $\Psi(\underline{S}, F) =\uparrow_F \underline{S}$ with $\uparrow_F \underline{S}(X) = \underline{S}(X)$ and for $(X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ in $\mathcal{P}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$ the map $\uparrow_F \underline{S}(f)$ is given by

$$\underline{S}(X) \xrightarrow{\omega_f} \mathcal{U}_Y\Gamma_f\mathcal{R}_X\underline{S}(\mathrm{id}_Y) \xrightarrow{\mu_f^F(\mathrm{id}_Y)} \mathcal{U}_Y\mathcal{R}_Y\underline{S}(\mathrm{id}_Y) = \underline{S}(Y).$$

Then we have $\Phi(\Psi(\underline{S}, F)) = \Phi(\uparrow_F \underline{S}) =\downarrow (\uparrow_F \underline{S}, F')$ with $\downarrow (\uparrow_F \underline{S})(X) =\uparrow_F \underline{S}(X) = \underline{S}(X)$ and for $(X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ in $\mathcal{P}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$ the map $\uparrow_F \underline{S}(f)$ is given by

$$\underline{S}(X) \xrightarrow{\omega_f} \mathcal{U}\Gamma_f\mathcal{R}_X\underline{S}(\mathrm{id}_Y) \xrightarrow{\mu_f^F(\mathrm{id}_Y)} \mathcal{U}_Y\mathcal{R}_Y\underline{S}(\mathrm{id}_Y) = \underline{S}(Y),$$

as f is an $(\mathcal{O}_a, \mathcal{O}_m)$-bispan and $(\underline{S}, F)$ is a coherent $((\mathcal{O}'_a, \mathcal{O}'_m))$-monoid this map is

$$x \mapsto [A = A \xrightarrow{f_2} B \xrightarrow{(f_3,\mathrm{id}_B)} Y \times_Y B, res_{f_1}(x)] \mapsto tr_{f_3}(nm_{f_2}(res_{f_1}(x)))$$

so $\downarrow (\uparrow_F \underline{S})(f) = tr_{f_3}nm_{f_2}res_{f_1}$. Hence $\downarrow (\uparrow_F \underline{S}) = \underline{S}$. Now we have $F'(X) = (X, \mathcal{R}_X \downarrow (\uparrow_F \underline{S}) = (X, \mathcal{R}_X\underline{S})$. Given an $(\mathcal{O}'_a, \mathcal{O}'_m)$-bispan $(X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ we have $F'(f) = (f, \mu_f^{F'})$ where $\mu_f^{F'}(C)$ is given by

$$\begin{aligned}[D \xleftarrow{g_1} E \xrightarrow{g_2} F \xrightarrow{g_3} C \times_Y B, d] &\mapsto (\uparrow_F \underline{S})(D \xleftarrow{g_1} E \xrightarrow{g_2} F \xrightarrow{\tilde{g_3}} C)(d) \mapsto \mu_f^F\omega_g(d) \\ &\mapsto \mu_f^F([D \xleftarrow{g_1} E \xrightarrow{g_2} F \xrightarrow{g_3} C \times_Y B, d]).\end{aligned}$$

Hence $\mu_f^{F'} = \mu_f^F$ and so $F' = F$. Therefore $\Phi\Psi(\underline{S}, F) = (\underline{S}, F)$.

<u>7b. $\Psi\Phi$ is isomorphic to the identity</u>

Let $\underline{S} \in \mathrm{Tamb}_*^{(\mathcal{O}'_a,\mathcal{O}'_m)}$. We have $\Phi(\underline{S}) = (\downarrow \underline{S}, F)$ with $\downarrow \underline{S}(X) = \underline{S}(X)$ and for $(X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ an $(\mathcal{O}_a, \mathcal{O}_m)$-bispan the map $\downarrow \underline{S}(f)$ is the map $\underline{S}(f)$. The strict functor F has $F(X) = (X, \mathcal{R}_X \downarrow \underline{S})$, and on an $(\mathcal{O}'_a, \mathcal{O}'_m)$-bispan $(X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ we have $F(f) = (f, \mu_f^F)$ where $\mu_f : \Gamma_f\mathcal{R}_X \downarrow \underline{S} \to \mathcal{R}_Y \downarrow \underline{S}$ with

$$\mu_f^F(C) : [D \xleftarrow{g_1} E \xrightarrow{g_2} F \xrightarrow{g_3} C \times_Y B, d] \mapsto tr_{\tilde{g_3}}(nm_{g_2}(res_{g_1}(d))).$$

Then we have $\Psi(\Phi(\underline{S})) = \Psi(\Downarrow \underline{S}, F) = \uparrow_F (\Downarrow \underline{S})$ with $\uparrow_F (\Downarrow \underline{S})(X) = \Downarrow \underline{S}(X) = \underline{S}(X)$, and for an $(\mathcal{O}'_a, \mathcal{O}'_m)$-bispan $(X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ the map $\uparrow_F (\Downarrow \underline{S})(f)$ is

$$\underline{S}(X) \xrightarrow{\omega_f} \mathcal{U}_Y \Gamma_f \mathcal{R}_X \underline{S}(\mathrm{id}_Y) \xrightarrow{\mu_f^F(\mathrm{id}_Y)} \mathcal{U}_Y \mathcal{R}_Y \underline{S}(\mathrm{id}_Y) = \underline{S}(Y)$$

which, on elements, is

$$x \mapsto [A = A \xrightarrow{f_2} B \xrightarrow{(f_3, \mathrm{id}_B)} Y \times_Y B, res_{f_1}(x)] \mapsto tr_{f_3}(nm_{f_2}(res_{f_1}(x))).$$

Hence $\uparrow_F (\Downarrow \underline{S})(f) = \underline{S}(f)$, so $\uparrow_F \Downarrow \underline{S} = \underline{S}$. Therefore $\Psi\Phi(\underline{S}) = \underline{S}$. $\square$

As a corollary of this theorem we have an equivalent statement to [2, Conjecture 7.94]

Corollary 4.55. *The $(\mathcal{O}_a, \mathcal{O}'_m)$-Tambara functors are precisely the $(\mathcal{O}_a, \mathcal{O}'_m)$-monoids in the category of $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functors.*

We note that an equivalent statement to [14, Theorem 2.7.4] is also a corollary.

Corollary 4.56. *The G-Tambara functors are precisely the coherent $(\mathrm{Set}^G, \mathrm{Set}^G)$-monoids in the category of G-Mackey functors.*

Remark 4.57. When G is the trivial group this is equivalent to the statement that the category of commutative rings is equivalent to the category of monoid objects in the category of abelian groups. In this case, given a G-Tambara functor $\underline{S}$ we find that given a bispan $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$ the map $\mu_f^F : \mathcal{U}\Gamma_f \mathcal{R}_X \underline{S} \to \mathcal{R}_Y \underline{S}$ in the construction of the coherent monoid structure from Theorem 4.53 is determined by a map ϕ_y of abelian groups for each $y \in Y$. This map is

$$\bigoplus_{f_3(b)=y} \left(\bigotimes_{f_2(a)=b} \underline{S}(*) \right) \to \underline{S}(*)$$
$$\left(\otimes_{f_2(a)=b} \gamma_a\right)_{f_3(b)=y} \mapsto \sum_{f_3(b)=y} \left(\prod_{f_2(a)} \gamma_a \right).$$

We end this section with two conjectures:

Conjecture A. *The compatibility conditions for Proposition 4.26, and Lemmas 4.28 and 4.29 are unnecessary. That is, the claimed isomorphisms hold for the relevant diagrams without requiring any morphisms to belong to a specific indexing category.*

If this is true then the compatibility condition that $(\mathcal{O}_a, \mathcal{O}'_m)$ is a compatible pair of indexing categories in Definition 4.46 can be removed. In this case the proof, without any modification, of Theorem 4.53 is a proof of the following.

Conjecture B. *Let be a finite group G, and let $(\mathcal{O}_a, \mathcal{O}_m)$ and $(\mathcal{O}'_a, \mathcal{O}'_m)$ be compatible pairs of indexing categories such that $\mathcal{O}_a$ is a subcategory of $\mathcal{O}'_a$, and $\mathcal{O}_m$ is a subcategory of $\mathcal{O}'_m$. Then $(\mathcal{O}'_a, \mathcal{O}'_m)$-Tambara functors are precisely the coherent $(\mathcal{O}'_a, \mathcal{O}'_m)$-monoids in $\mathrm{Tamb}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$. That is, there is an equivalence between the category of coherent $(\mathcal{O}'_a, \mathcal{O}'_m)$-monoids in $\mathrm{Tamb}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$ and the category $\mathrm{Tamb}_*^{(\mathcal{O}'_a,\mathcal{O}'_m)}$.*

4.5 $\mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ as a module category

We will now see that categories of Tambara functors are module categories over indexing categories. Recall Definition 2.37 of a module category over a tight symmetric bimonoidal category.

Notation 4.58. Given an Set^G-transfer span $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ we write Θ_{f_1,f_2} or Θ_f for the functor $\mathcal{N}_{f_2}\mathcal{R}_{f_1} : \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \to \mathrm{Tamb}_Y^{(\mathcal{O}_a,\mathcal{O}_m)}$. As we did for Mackey functors, we write Λ_f for the functor $\mathcal{T}_{f_2}\mathcal{R}_{f_1} : \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \to \mathrm{Tamb}_Y^{(\mathcal{O}_a,\mathcal{O}_m)}$. Given a map $\varphi : A \to X$ of G-sets we will write $\overline{\varphi}$ for the span $(X \xleftarrow{\varphi} A \xrightarrow{\varphi} X)$.

Theorem 4.59. *Let $\mathcal{O}'_a$ be an indexing category, $(\mathcal{O}_a, \mathcal{O}_m)$ be a compatible pair of indexing categories such that $\mathcal{O}_a$ is a subcategory of $\mathcal{O}'_a$, and X be a finite G-set. The category of $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functors over X is a module category over $((\mathcal{O}'_a)_X)_{\cong}$ with the action functor $\odot : ((\mathcal{O}'_a)_X)_{\cong} \times \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \to \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ given by $(A, \varphi) \odot \underline{S} = \Lambda_{\overline{\varphi}}\underline{S}$.*

Proof. By Lemma 4.35 $(\mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}, \boxplus, \underline{0})$ is a symmetric monoidal category. For $(A, \varphi), (B, \psi) \in ((\mathcal{O}'_a)_X)_{\cong}$ we declare the following structure natural isomorphisms:

- Let $\lambda^{\odot} : \mathrm{id}_X \odot \underline{S} = \Lambda_{\overline{\mathrm{id}_X}}\underline{S} \to \underline{S}$ be the isomorphism
$$\Lambda_{\overline{\mathrm{id}_X}}\underline{S}(C) = \underline{S}(X \times_X C) \xrightarrow{tr_{pr_C}} \underline{S}(C)$$
for $pr_C : X \times_X C \to C, (x, c) \mapsto c$.

- Let $\alpha^{\odot} : \psi \odot (\varphi \odot \underline{S}) = \Lambda_{\overline{\psi}}\Lambda_{\overline{\varphi}}\underline{S} \to \Lambda_{\overline{\psi}\overline{\varphi}}\underline{S} = (\psi \times_X \varphi) \odot \underline{S}$ be the isomorphism
$$\mathcal{T}_\psi\mathcal{R}_\psi\mathcal{T}_\varphi\mathcal{R}_\varphi \to \mathcal{T}_\psi\mathcal{T}_{pr_B}\mathcal{R}_{pr_A}\mathcal{R}_\varphi \to \mathcal{T}_{\psi pr_B}\mathcal{R}_{\varphi pr_A}$$
where the first map is the natural isomorphism described in Proposition 4.23 and the second isomorphism follows from $\Sigma_\varphi\Sigma_{pr_A} = \Sigma_{\varphi pr_A}$ and $\Delta_{pr_B}\Delta_\psi \cong \Delta_{\psi pr_B}$.

- Let $d^L_{\odot} : (\varphi \amalg \psi) \odot \underline{S} = \Lambda_{\overline{\varphi\amalg\psi}}\underline{S} \to \Lambda_{\overline{\varphi}}\underline{S} \boxplus \Lambda_{\overline{\psi}}\underline{S} = (\varphi \odot \underline{S}) \boxplus (\psi \odot \underline{S})$ be
$$\begin{aligned}\mathcal{T}_{\varphi\amalg\psi}\mathcal{R}_{\varphi\amalg\psi}\underline{S}(C) &= \underline{S}((A \amalg B) \times_X C) \cong \underline{S}((A \times_X C) \amalg (B \times_X C)) \\ &= \underline{S}(A \times_X C) \times \underline{S}(B \times_X C) = \mathcal{T}_\varphi\mathcal{R}_\varphi\underline{S}(C) \times \mathcal{T}_\psi\mathcal{R}_\psi\underline{S}(C)\end{aligned}$$
where the isomorphism is tr_γ for the obvious isomorphism
$$\gamma : (A \amalg B) \times_X C \xrightarrow{\cong} (A \times_X C) \amalg (B \times_X C).$$

- Let $d^R_{\odot} : \varphi \odot (\underline{S} \boxplus \underline{T}) = \Lambda_{\overline{\varphi}}(\underline{S} \boxplus \underline{T}) \to \Lambda_{\overline{\varphi}}\underline{S} \boxplus \Lambda_{\overline{\varphi}}\underline{T} = (\varphi \odot \underline{S}) \boxplus (\varphi \odot \underline{T})$ be the identity because

$$\begin{aligned}\mathcal{T}_\varphi\mathcal{R}_\varphi(\underline{S} \boxplus \underline{T})(C) &= (\underline{S} \boxplus \underline{T})(A \times_X C) = \underline{S}(A \times_X C) \times \underline{T}(A \times_X C)\\ &= \mathcal{T}_\varphi\mathcal{R}_\varphi\underline{S}(C) \times \mathcal{T}_\varphi\mathcal{R}_\varphi\underline{T}(C) = (\mathcal{T}_\varphi\mathcal{R}_\varphi\underline{S} \boxplus \mathcal{T}_\varphi\mathcal{R}_\varphi\underline{T})(C).\end{aligned}$$

- For $\zeta : \emptyset \to X$ we have $\Lambda_{\overline{\zeta}}\underline{S} = \underline{0}$ so let $\xi^L_{\odot} : \zeta \odot \underline{S} = \Lambda_{\overline{\zeta}}\underline{S} \to \underline{0}$ be the identity.

- For any $\varphi : X \to Y$ we have $\Lambda_{\overline{\varphi}}\underline{0} = \underline{0}$ so let $\xi^R_{\odot} : \varphi \odot \underline{0} = \Lambda_{\overline{\varphi}}\underline{0} \to \underline{0}$ be the identity.

It remains to check that the coherence diagrams from Definition 2.37 are satisfied. This is a straightforward but lengthy exercise and is left to the interested reader. $\square$

Theorem 4.60. *Let $\mathcal{O}'_m$ be an indexing category, $(\mathcal{O}_a, \mathcal{O}_m)$ be a compatible pair of indexing categories such that $\mathcal{O}_m$ is a subcategory of $\mathcal{O}'_m$, and X be a finite G-set. The category of $(\mathcal{O}_a, \mathcal{O}_m)$-Tambara functors over X is a module category over $((\mathcal{O}'_m)_X)_{\cong}$ with the action functor $\boxtimes : ((\mathcal{O}'_m)_X)_{\cong} \times \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \to \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ given by $(A,\varphi) \boxtimes \underline{S} = \Theta_{\overline{\varphi}}\underline{S}$.*

Proof. By Lemma 4.44 $(\mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}, \square, \underline{A}_X)$ is a symmetric monoidal category. For $(A,\varphi),(B,\psi) \in ((\mathcal{O}'_m)_X)_{\cong}$ we declare the following structure natural isomorphisms:

- Let $\lambda^{\boxtimes} : \mathrm{id}_X \boxtimes \underline{S} = \Theta_{\overline{\mathrm{id}_X}}\underline{S} \to \underline{S}$ be the isomorphism

$$\Theta_{\overline{\mathrm{id}_X}}\underline{S}(C) \to \underline{S}(C), [D \xleftarrow{f_1} E \xrightarrow{f_2} F \xrightarrow{f_3} C, d] \mapsto tr_{f_3}(nm_{f_2}(res_{f_1}(d)))$$

  which has inverse $c \mapsto [C = C = C = C, c]$.

- Let $\alpha^{\boxtimes} : \psi \boxtimes (\varphi \boxtimes \underline{S}) = \Theta_{\overline{\psi}}\Theta_{\overline{\varphi}}\underline{S} \to \Theta_{\overline{\psi}\overline{\varphi}}\underline{S} = (\psi \times_X \varphi) \boxtimes \underline{S}$ be the natural isomorphism

$$\mathcal{N}_\psi\mathcal{R}_\psi\mathcal{N}_\varphi\mathcal{R}_\varphi \to \mathcal{N}_\psi\mathcal{N}_{pr_B}\mathcal{R}_{pr_A}\mathcal{R}_\varphi \to \mathcal{N}_{\psi pr_B}\mathcal{R}_{\varphi pr_A}$$

  where the first map is the natural isomorphism described in Proposition 3.26 and the second is two applications of Lemma A.5.

- Let $d^L_{\boxtimes} : (\varphi \amalg \psi) \boxtimes \underline{S} = \Theta_{\overline{\varphi\amalg\psi}}\underline{S} \to \Theta_{\overline{\varphi}}\underline{S} \square \Theta_{\overline{\psi}}\underline{S} = (\varphi \boxtimes \underline{S}) \square (\psi \boxtimes \underline{S})$ be the isomorphism from Lemma A.4 induced by the natural identity transformation between the functors

$$\mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \xrightarrow{\Delta_{(\varphi\amalg\psi)}} \mathcal{P}_{A\amalg B}^{(\mathcal{O}_a,\mathcal{O}_m)} \xrightarrow{\Sigma_{(\varphi\amalg\psi)}} \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$$

  and

$$\mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \xrightarrow{\Delta=(\mathrm{id},\mathrm{id})} \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \times \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \xrightarrow{(\Delta_\varphi,\Delta_\psi)} \mathcal{P}_A^{(\mathcal{O}_a,\mathcal{O}_m)} \times \mathcal{P}_B^{(\mathcal{O}_a,\mathcal{O}_m)} \xrightarrow{\amalg} \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)}.$$

- Let $d^R_{\boxtimes} : \varphi \boxtimes (\underline{S}\square\underline{T}) = \Theta_{\overline{\varphi}}(\underline{S}\square\underline{T}) \to \Theta_{\overline{\varphi}}\underline{S}\square\Theta_{\overline{\varphi}}\underline{T} = (\varphi \boxtimes \underline{S})\square(\varphi \boxtimes \underline{T})$ be the isomorphism from Lemma A.4 induced by the natural isomorphism between the functors

$$\mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \times \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \xrightarrow{\amalg} \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \xrightarrow{\Delta_\varphi} \mathcal{P}_A^{(\mathcal{O}_a,\mathcal{O}_m)} \xrightarrow{\Sigma_\varphi} \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$$

and

$$\mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \times \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \xrightarrow{(\Delta_\varphi,\Delta_\varphi)} \mathcal{P}_A^{(\mathcal{O}_a,\mathcal{O}_m)} \times \mathcal{P}_A^{(\mathcal{O}_a,\mathcal{O}_m)} \xrightarrow{(\Sigma_\varphi,\Sigma_\varphi)} \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \times \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \xrightarrow{\amalg} \mathcal{P}_X^{(\mathcal{O}_a,\mathcal{O}_m)}.$$

- For $\zeta : \emptyset \to X$ we have $\Theta_{\overline{\zeta}}\underline{S} \cong \underline{A}_X$ so let $\xi^L_{\boxtimes} : \zeta \boxtimes \underline{S} = \Theta_{\overline{\zeta}}\underline{S} \to \underline{A}_X$ be this natural isomorphism.

- For any $\varphi : X \to Y$ we have $\Theta_{\overline{\varphi}}\underline{A}_X \cong \underline{A}_X$ so let $\xi^R_{\boxtimes} : \varphi \boxtimes \underline{A}_X = \Theta_{\overline{\varphi}}\underline{A}_X \to \underline{A}_X$ be this natural isomorphism.

It remains to check that the coherence diagrams from Definition 2.37 are satisfied. This is a straightforward but lengthy exercise and is left to the interested reader. $\square$

Recall Definition 2.40 of a $\mathcal{C}$-monoid in a module category $\mathcal{A}$ over $\mathcal{C}_{\cong}$ for C a tight symmetric bimonoidal category.

Proposition 4.61. *Let $(\mathcal{O}_a, \mathcal{O}_m)$ and $(\mathcal{O}'_a, \mathcal{O}_m)$ be pairs of compatible indexing categories such that $\mathcal{O}_a$ is a subcategory $\mathcal{O}'_a$. If $(\underline{S}, F)$ is a coherent $(\mathcal{O}'_a, \mathcal{O}_m)$-monoid in $\mathrm{Tamb}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$ then $\mathcal{U}\mathcal{R}_X\underline{S}$ is an $\mathcal{O}'_a$-monoid in $\mathrm{Mack}_X^{\mathcal{O}_a}$ as a module category over $((\mathcal{O}'_a)_X)_{\cong}$ for all finite G-sets X.*

Proof. We are asking for an extension of the functor $-\odot\mathcal{U}\mathcal{R}_X\underline{S} : ((\mathcal{O}'_a)_X)_{\cong} \to \mathrm{Mack}_X^{\mathcal{O}_a}$ to $(\mathcal{O}'_a)_X$.

$$\begin{tikzcd} ((\mathcal{O}'_a)_X)_{\cong} \arrow[r, "-\odot\mathcal{U}\mathcal{R}_X\underline{S}"] \arrow[d, hook] & \mathrm{Mack}_X^{\mathcal{O}_a} \\ (\mathcal{O}'_a)_X \arrow[ur, dashed] & \end{tikzcd}$$

Because $((\mathcal{O}'_a)_X)_{\cong}$ is a wide subcategory of $(\mathcal{O}'_a)_X$ we only need to describe what happens on the extra morphisms. Given $\sigma : (A, \varphi) \to (B, \psi)$ in $(\mathcal{O}'_a)_X$ we have a map $\Lambda_{\overline{\varphi}}\mathcal{U}\mathcal{R}_X\underline{S} \to \Lambda_{\overline{\psi}}\mathcal{U}\mathcal{R}_X\underline{S}$ given on objects by

$$\Lambda_{\overline{\varphi}}\mathcal{U}\mathcal{R}_X\underline{S}(C) = \underline{S}(A \times_X C) \xrightarrow{tr_{\sigma\times_X \mathrm{id}_C}} \underline{S}(B \times_X C) = \Lambda_{\overline{\psi}}\mathcal{U}\mathcal{R}_X\underline{S}(C)$$

where the transfer map is the one in $\uparrow_F \underline{S}$. $\square$

Proposition 4.62. *Let $(\mathcal{O}_a, \mathcal{O}_m)$ and $(\mathcal{O}_a, \mathcal{O}'_m)$ be pairs of compatible indexing categories such that $\mathcal{O}_m$ is a subcategory $\mathcal{O}'_m$. If $(\underline{S}, F)$ is a coherent $(\mathcal{O}_a, \mathcal{O}'_m)$-monoid in $\mathrm{Tamb}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$ then $\mathcal{R}_X\underline{S}$ is an $\mathcal{O}'_m$-monoid in $\mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ as a module category over $((\mathcal{O}'_m)_X)_{\cong}$ for all finite G-sets X.*

Proof. We are asking for an extension of the functor $-\boxtimes\mathcal{R}_X\underline{S} : ((\mathcal{O}'_m)_X)_{\cong} \to \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)}$ to $(\mathcal{O}'_m)_X$.

$$\begin{tikzcd}
((\mathcal{O}'_m)_X)_{\cong} \arrow[r, "-\boxtimes\mathcal{R}_X\underline{S}"] \arrow[d, hook] & \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \\
(\mathcal{O}'_m)_X \arrow[ur, dashed] &
\end{tikzcd}$$

Because $((\mathcal{O}'_m)_X)_{\cong}$ is a wide subcategory of $(\mathcal{O}'_m)_X$ we only need to describe what happens on the extra morphisms. Given $\sigma : (A,\varphi) \to (B,\psi)$ in $(\mathcal{O}'_m)_X$ we have a map $\Gamma_{\overline{\varphi}}\mathcal{R}_X\underline{S} \to \Gamma_{\overline{\psi}}\mathcal{R}_X\underline{S}$ given on objects by

$$\Gamma_{\overline{\varphi}}\mathcal{R}_X\underline{S}(H) \to \Gamma_{\overline{\psi}}\mathcal{R}_X\underline{S}(H),$$
$$[(C,p_C) \xleftarrow{g_1} D \xrightarrow{g_2} E \xrightarrow{g_3} H, c] \mapsto [(C,\sigma p_C) \xleftarrow{g_1} D \xrightarrow{g_2} E \xrightarrow{g_3} H, c]. \qquad \square$$

We write $\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{\mathcal{O}'_a}$ for the symmetric monoidal $\mathcal{O}'$-Mackey functor

$$\underline{\mathcal{O}'_a} \hookrightarrow \underline{(\mathcal{O}'_a,\mathcal{O}_m)} \xrightarrow{\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}_m)}} \mathrm{CAT}$$

where $\underline{\mathcal{O}'_a}$ includes into $\underline{(\mathcal{O}'_a,\mathcal{O}_m)}$ by taking 1-cells $(X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ to $(X \xleftarrow{f_1} A = A \xrightarrow{f_2} Y)$ and 2-cells to the obvious corresponding 2-cells. By Theorem 4.59 we see that $(\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{\mathcal{O}'_a}, \boxplus)$ is a module over $\overline{\mathcal{O}'_a}$.

Similarly, we write $\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{\mathcal{O}'_m}$ for the symmetric monoidal $\mathcal{O}'$-Mackey functor

$$\underline{\mathcal{O}'_m} \hookrightarrow \underline{(\mathcal{O}_a,\mathcal{O}'_m)} \xrightarrow{\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}_a,\mathcal{O}'_m)}} \mathrm{CAT}$$

where $\underline{\mathcal{O}'_m}$ includes into $\underline{(\mathcal{O}_a,\mathcal{O}'_m)}$ by taking 1-cells $(X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ to $(X \xleftarrow{f_1} A \xrightarrow{f_2} Y = Y)$ and 2-cells to the obvious corresponding 2-cells. By Theorem 4.60 we see that $(\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{\mathcal{O}'_m}, \boxtimes)$ is a module over $\overline{\mathcal{O}'_m}$.

Proposition 4.63. *If $(\underline{S},F)$ is a coherent $(\mathcal{O}'_a,\mathcal{O}_m)$-monoid in $\mathrm{Tamb}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$ then $\mathcal{U}\underline{S}$ is an $\overline{\mathcal{O}'_a}$-module in $\mathrm{Mack}_{\mathcal{O}_a}^{\mathcal{O}'_a}$.*

Proof. By Proposition 4.61 we see that the required extension of $-\odot\mathcal{U}\underline{S}$ exists objectwise. We only need to check that these commute with the structure functors in $\overline{\mathcal{O}'_a}$ and $\mathrm{Mack}_{\mathcal{O}_a}^{\mathcal{O}'_a}$. That is, given a $\mathcal{O}'_a$-transfer span $(X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ we want the following diagram to commutes (up to natural isomorphism).

$$\begin{tikzcd}
(\mathcal{O}'_a)_X \arrow[r, "-\odot\mathcal{U}\mathcal{R}_X\underline{S}"] \arrow[d, "\Sigma_{f_2}\Delta_{f_1}"'] & \mathrm{Mack}_X^{\mathcal{O}_a} \arrow[d, "\mathcal{T}_{f_2}\mathcal{R}_{f_1}"] \\
(\mathcal{O}'_a)_Y \arrow[r, "-\odot\mathcal{U}\mathcal{R}_Y\underline{S}"] & \mathrm{Mack}_Y^{\mathcal{O}_a}
\end{tikzcd} \tag{4.64}$$

Given $(C,\varphi) \in (\mathcal{O}'_m)_X$ going around the bottom of the diagram we have

$$(C,\varphi) \mapsto (A \times_X C, f_2pr_A) \mapsto \mathcal{T}_{f_2pr_A}\mathcal{R}_{f_2pr_A}\mathcal{U}\mathcal{R}_Y\underline{S} \cong \mathcal{U}\mathcal{T}_{f_2pr_A}\mathcal{R}_{A\times_X C}\underline{S}$$

and going around the top we have

$$(C,\varphi) \mapsto \mathcal{T}_\varphi\mathcal{R}_\varphi\mathcal{R}_X\underline{S} \mapsto \mathcal{T}_{f_2}\mathcal{R}_{f_1}\mathcal{T}_\varphi\mathcal{R}_\varphi\mathcal{U}\mathcal{R}_X\underline{S} \cong \mathcal{U}\mathcal{T}_{f_2}\mathcal{T}_{pr_A}\mathcal{R}_{pr_C}\mathcal{R}_\varphi\mathcal{R}_X\underline{S} \cong \mathcal{U}\mathcal{T}_{f_2pr_A}\mathcal{R}_{A\times_X C}\underline{S}.$$

□

Proposition 4.65. *If* $\underline{S} \in \mathrm{Tamb}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$ *and* $\mathcal{U}\underline{S}$ *is an* $\overline{\mathcal{O}'_a}$*-module in* $\mathrm{Mack}_{\mathcal{O}_a}^{\mathcal{O}'_a}$ *then* $\underline{S}$ *is a coherent* $(\mathcal{O}'_a,\mathcal{O}_m)$*-monoid in* $\mathrm{Tamb}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$.

Proof. We need to describe a strict functor $F : \underline{(\mathcal{O}'_a,\mathcal{O}_m)} \to \int \mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{(\mathcal{O}'_a,\mathcal{O}_m)}$ as per Definition 4.49. Because F is required to be a section we only need to describe μ_f^F from $F(f) = (f,\mu_f^F)$ for $(\mathcal{O}'_a,\mathcal{O}_m)$-bispans $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} Y)$. From the Y component $(\mathcal{O}'_a)_Y \to \mathrm{Mack}_Y^{\mathcal{O}_a}$ of the extension of $- \odot \mathcal{U}\underline{S}$ and the map $(B,f_3) \xrightarrow{f_3} (Y,\mathrm{id}_Y)$ in $\mathcal{O}'_Y$ we get a map $\beta_f : \mathcal{U}\mathcal{T}_{f_3}\mathcal{R}_{f_3}\mathcal{R}_Y\underline{S} = \mathcal{U}\mathcal{T}_{f_3}\mathcal{R}_B\underline{S} \to \mathcal{U}\mathcal{R}_Y\underline{S}$ in $\mathrm{Mack}_B^{\mathcal{O}_a}$. Let $\alpha_f : \mathcal{U}\mathcal{N}_{f_2}\mathcal{R}_A\underline{S} \to \mathcal{U}\mathcal{R}_B\underline{S}$ given on an object K by

$$[C \xleftarrow{g_1} D \xrightarrow{g_2} E \xrightarrow{g_3} K, c] \mapsto tr_{g_3}nm_{g_2}res_{g_1}(c).$$

We let μ_f^F be the composite

$$\mathcal{U}\mathcal{T}_{f_3}\mathcal{N}_{f_2}\mathcal{R}_{f_1}\mathcal{R}_X\underline{S} \xrightarrow{\cong} \mathcal{U}\mathcal{T}_{f_3}\mathcal{N}_{f_2}\mathcal{R}_A\underline{S} \xrightarrow{\mathcal{U}\mathcal{T}_{f_3}\alpha_f} \mathcal{U}\mathcal{T}_{f_3}\mathcal{R}_B\underline{S} \xrightarrow{\beta_f} \mathcal{U}\mathcal{R}_Y\underline{S}.$$

The commutativity of diagrams like Diagram 4.64 for any $\mathcal{O}'_a$-transfer span ensures that $\mu_{gf}^F = \mu_g^F(\Gamma_g\mu_f^F)$. □

Proposition 4.66. *If* $(\underline{S},F)$ *is a coherent* $(\mathcal{O}_a,\mathcal{O}'_m)$*-monoid in* $\mathrm{Tamb}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$ *then* $\underline{S}$ *is an* $\overline{\mathcal{O}'_m}$*-module in* $\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{\mathcal{O}'_m}$.

Proof. By Proposition 4.62 we see that the required extension of $- \boxtimes \underline{S}$ exists objectwise. We only need to check that these commute with the structure functors in $\overline{\mathcal{O}'_m}$ and $\mathrm{Tamb}_{(\mathcal{O}_a,\mathcal{O}_m)}^{\mathcal{O}'_m}$. That is, given a $\mathcal{O}'_m$-transfer span $(X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$ we want the following diagram to commutes (up to natural isomorphism).

$$\begin{tikzcd}
(\mathcal{O}'_m)_X \arrow[r, "-\boxtimes\mathcal{R}_X\underline{S}"] \arrow[d, "\Sigma_{f_2}\Delta_{f_1}"'] & \mathrm{Tamb}_X^{(\mathcal{O}_a,\mathcal{O}_m)} \arrow[d, "\mathcal{N}_{f_2}\mathcal{R}_{f_1}"] \\
(\mathcal{O}'_m)_Y \arrow[r, "-\boxtimes\mathcal{R}_Y\underline{S}"] & \mathrm{Tamb}_Y^{(\mathcal{O}_a,\mathcal{O}_m)}
\end{tikzcd} \tag{4.67}$$

Given $(C,\varphi) \in (\mathcal{O}'_m)_X$ going around the bottom of the diagram we have

$$(C,\varphi) \mapsto (A \times_X C, f_2pr_A) \mapsto \mathcal{N}_{f_2pr_A}\mathcal{R}_{f_2pr_A}\mathcal{R}_Y\underline{S} \cong \mathcal{N}_{f_2pr_A}\mathcal{R}_{A\times_X C}\underline{S}$$

and going around the top we have

$$(C,\varphi)\mapsto\mathcal{N}_\varphi\mathcal{R}_\varphi\mathcal{R}_X\underline{S}\mapsto\mathcal{N}_{f_2}\mathcal{R}_{f_1}\mathcal{N}_\varphi\mathcal{R}_\varphi\mathcal{R}_X\underline{S}\cong\mathcal{N}_{f_2}\mathcal{N}_{pr_A}\mathcal{R}_{pr_C}\mathcal{R}_\varphi\mathcal{R}_X\underline{S}\cong\mathcal{N}_{f_2pr_A}\mathcal{R}_{A\times_X C}\underline{S}.$$

□

Proposition 4.68. *If $\underline{S}$ is an $\overline{\mathcal{O}'_m}$-module in $\mathrm{Tamb}^{\mathcal{O}'_m}_{(\mathcal{O}_a,\mathcal{O}_m)}$ then $\underline{S}$ is a coherent $(\mathcal{O}_a,\mathcal{O}'_m)$-monoid in $\mathrm{Tamb}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$.*

Proof. We need to describe a strict functor $F:\underline{(\mathcal{O}_a,\mathcal{O}'_m)}\to\int\mathrm{Tamb}^{(\mathcal{O}_a,\mathcal{O}'_m)}_{(\mathcal{O}_a,\mathcal{O}_m)}$ as per Definition 4.49. Because F is required to be a section we only need to describe μ^F_f from $F(f)=(f,\mu^F_f)$ for $(\mathcal{O}_a,\mathcal{O}'_m)$-bispans $f=(X\xleftarrow{f_1}A\xrightarrow{f_2}B\xrightarrow{f_3}Y)$. From the B component $\mathcal{O}'_B\to\mathrm{Tamb}_B^{(\mathcal{O}_a,\mathcal{O}_m)}$ of the extension of $-\boxtimes\underline{S}$ and the map $(A,f_2)\xrightarrow{f_2}(B,\mathrm{id}_B)$ in $\mathcal{O}'_B$ we get a map $\sigma_f:\mathcal{N}_{f_2}\mathcal{R}_{f_2}\mathcal{R}_B\underline{S}\to\mathcal{R}_B\underline{S}$ in $\mathrm{Tamb}_B^{(\mathcal{O}_a,\mathcal{O}_m)}$. Let $\gamma_f:\mathcal{U}\mathcal{T}_{f_3}\mathcal{R}_B\underline{S}\to\mathcal{U}\mathcal{R}_Y\underline{S}$ given by

$$\mathcal{U}\mathcal{T}_{f_3}\mathcal{R}_B\underline{S}(C)=\mathcal{U}\underline{S}(B\times_Y C)\xrightarrow{tr_{pr_C}}\mathcal{U}\underline{S}(C)=\mathcal{U}\mathcal{R}_Y\underline{S}(C).$$

We let μ^F_f be the composite

$$\mathcal{U}\mathcal{T}_{f_3}\mathcal{N}_{f_2}\mathcal{R}_{f_1}\mathcal{R}_X\underline{S}\xrightarrow{\cong}\mathcal{U}\mathcal{T}_{f_3}\mathcal{N}_{f_2}\mathcal{R}_{f_2}\mathcal{R}_B\underline{S}\xrightarrow{\mathcal{U}\mathcal{T}_{f_3}\sigma_f}\mathcal{U}\mathcal{T}_{f_3}\mathcal{R}_B\underline{S}\xrightarrow{\gamma_f}\mathcal{U}\mathcal{R}_Y\underline{S}.$$

The commutativity of diagrams like Diagram 4.67 for any $\mathcal{O}'_m$-transfer span ensures that $\mu^F_{gf}=\mu^F_g(\Gamma_g\mu^F_f)$. □

Corollary 4.69. *If $(\underline{S},F)$ is a coherent $(\mathcal{O}'_a,\mathcal{O}'_m)$-monoid in $\mathrm{Tamb}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$ then $\underline{S}$ is an $\overline{\mathcal{O}'_a}$-module in $\mathrm{Mack}^{\mathcal{O}'_a}_{\mathcal{O}_a}$ and an $\overline{\mathcal{O}'_m}$-module in $\mathrm{Tamb}^{\mathcal{O}'_m}_{(\mathcal{O}_a,\mathcal{O}_m)}$.*

Proof. Restricting F to the appropriate sub-bicategories makes $(\underline{S},F)$ both a coherent $\mathcal{O}'_a$-monoid in $\mathrm{Mack}_*^{\mathcal{O}_a}$ and a coherent $(\mathcal{O}_a,\mathcal{O}'_m)$-monoid in $\mathrm{Tamb}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$. The result follows by Propositions 4.63 and 4.66. □

Corollary 4.70. *Let $(\mathcal{O}_a,\mathcal{O}_m)$ and $(\mathcal{O}'_a,\mathcal{O}'_m)$ be compatible pairs of indexing categories such that $(\mathcal{O}_a,\mathcal{O}'_m)$ is a compatible pair of indexing categories, $\mathcal{O}_a$ is a subcategory of $\mathcal{O}'_a$, and $\mathcal{O}_m$ is a subcategory of $\mathcal{O}'_m$. If $\underline{S}$ is an $\overline{\mathcal{O}'_m}$-module in $\mathrm{Tamb}^{\mathcal{O}'_m}_{(\mathcal{O}_a,\mathcal{O}_m)}$ and $\mathcal{U}\underline{S}$ is an $\overline{\mathcal{O}'_a}$-module in $\mathrm{Mack}^{\mathcal{O}'_a}_{\mathcal{O}_a}$ then $\underline{S}$ is a coherent $(\mathcal{O}'_a,\mathcal{O}'_m)$-monoid in $\mathrm{Tamb}_*^{(\mathcal{O}_a,\mathcal{O}_m)}$.*

Proof. We need to describe a strict functor $F:\underline{(\mathcal{O}'_a,\mathcal{O}'_m)}\to\int\mathrm{Tamb}^{(\mathcal{O}'_a,\mathcal{O}'_m)}_{(\mathcal{O}_a,\mathcal{O}_m)}$ as in Definition 4.49. Because F is required to be a section we only need to describe μ^F_f from $F(f)=(f,\mu^F_f)$ for $(\mathcal{O}'_a,\mathcal{O}'_m)$-bispans $f=(X\xleftarrow{f_1}A\xrightarrow{f_2}B\xrightarrow{f_3}Y)$. We let μ^F_f be the composite

$$\mathcal{U}\mathcal{T}_{f_3}\mathcal{N}_{f_2}\mathcal{R}_{f_1}\mathcal{R}_X\underline{S}\xrightarrow{\cong}\mathcal{U}\mathcal{T}_{f_3}\mathcal{N}_{f_2}\mathcal{R}_A\underline{S}\xrightarrow{\mathcal{U}\mathcal{T}_{f_3}\sigma_f}\mathcal{U}\mathcal{T}_{f_3}\mathcal{R}_B\underline{S}\xrightarrow{\beta_f}\mathcal{U}\mathcal{R}_Y\underline{S}$$

where σ_f is the map from Proposition 4.68 and β_f is the map from Proposition 4.65. The commutativity of Diagrams 4.64 and 4.67 for any $\mathcal{O}'_a$-transfer spans and $\mathcal{O}'_m$-transfer spans respectively ensures that $\mu^F_{gf} = \mu^F_g(\Gamma_g \mu^F_f)$. $\square$

We see that under the formalism of this section when can either extend the additive structure or the multiplicative structure, but not both simultaneously. The formalism from Sections 4.3 and 4.4 allows us to extend both structures at the same time.

Appendix A

Left Kan Extensions

We collect some basic results about left Kan extensions. We will restrict our attention to left Kan extensions of functors valued in **Set** as all the left Kan extensions appearing in this thesis of this form, and it allows us to describe the relevant maps explicitly in terms of element representatives.

Lemma A.1. *Let $\mathcal{C}$ be an essentially small category, $\mathcal{D}$ be a locally small category. Given $K : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{C} \to$ **Set** the left Kan extension, $\mathrm{Lan}_K F$ of F along K is given on an object $X \in \mathcal{D}$ by the coend formula*

$$\mathrm{Lan}_K F(X) = \int^{\mathcal{C}} \mathcal{D}(KA, X) \times FA \, dA.$$

Elements of $\mathrm{Lan}_K F(X)$ are represented by pairs (φ, a) where $\varphi : KA \to X$ in $\mathcal{D}$ and $a \in FA$. Two pairs (φ, a) and (φ', a') represent the same element in $\mathrm{Lan}_K F(X)$ if there exists a morphism $\sigma : A \to A'$ in $\mathcal{C}$ such that $\varphi = \varphi' K\sigma$ and $F(\sigma)(a) = a'$, we say that (φ, a) and (φ', a') are identified along (φ', a) via σ.
Given a morphism $\omega : X \to Y$ in $\mathcal{D}$ the map $\mathrm{Lan}_K F(\omega)$ is given by $[(\varphi, a)] \mapsto [(\omega\varphi, a)]$.

Lemma A.2 (Yoneda Reduction)**.** *Let $\mathcal{C}$ be an essentially small category, $\mathcal{D}$ be a locally small category. If $K : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{C} \to$ **Set** such that K is left adjoint to some functor $H : \mathcal{D} \to \mathcal{C}$. Then $\mathrm{Lan}_K F \cong FH$.*

We have

$$\begin{aligned} \mathrm{Lan}_K F(X) &= \int^{\mathcal{C}} \mathcal{D}(KA, X) \times FA \, dA \\ &\cong \int^{\mathcal{C}} \mathcal{D}(A, HX) \times FA \, dA \\ &\cong FH(X) \end{aligned}$$

where the first isomorphism acts on elements by $[(KA \xrightarrow{\varphi} X, a)] \mapsto [(A \xrightarrow{\tilde{\varphi}} HX, a)]$ where $\tilde{\varphi}$ is the adjoint map to φ across the adjunction $K \dashv H$, and the second isomorphism is given by $[(A \xrightarrow{\psi} HX, a)] \mapsto F(\psi)(a)$. The inverse to the second isomorphism is given by $x \mapsto [(\mathrm{id}_{HX}, x)]$. So the isomorphism $FH(X) \cong \mathrm{Lan}_K F(X)$ is given by $x \mapsto [(KH(X) \xrightarrow{\varepsilon_X} X, x)]$ where ε is the counit of the adjunction $K \dashv H$.

Corollary A.3. *Let $\mathcal{C}$ be an essentially small category, $\mathcal{D}$ be a locally small category. If $K : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{C} \to$ **Set** such that K is left adjoint to some functor $H : \mathcal{D} \to \mathcal{C}$ then every element of* $\mathrm{Lan}_K F(X)$ *has a representative of the form* $(KHX \xrightarrow{\varepsilon_X} X, x)$.

Specifically, the maps $\mathrm{Lan}_K F(X) \to FH(X) \to \mathrm{Lan}_K F(X)$ in the previous lemma send $[(KA \xrightarrow{\varphi} X, a)] \mapsto [(KHX \xrightarrow{\varepsilon_X} X, F(\tilde{\varphi})(a))]$. The two representatives are identified along $(KHX \xrightarrow{\varepsilon_X} X, a)$ by the map $\tilde{\varphi}$, noting that $\varepsilon_X K(\tilde{\varphi}) = \varphi$.
Next we see that left Kan extensions along isomorphic functors are isomorphic.

Lemma A.4. *Let $\mathcal{C}$ be an essentially small category, $\mathcal{D}$ be a locally small category. If $K, K' : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{C} \to$ **Set** such that $\theta : K \to K'$ is a natural isomorphism then* $\mathrm{Lan}_K F$ *is naturally isomoprhic to* $\mathrm{Lan}_{K'} F$.

In particular, the natural isomorphism between left Kan extensions has components given by

$$\mathrm{Lan}_K F(X) \to \mathrm{Lan}_{K'} F, [(KA \xrightarrow{\varphi} X, a)] \mapsto [(K'A \xrightarrow{\varphi\theta_A^{-1}} X, a)].$$

We will also need that taking left Kan extensions respects composition.

Lemma A.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be essentially small categories, $\mathcal{D}$ and $\mathcal{E}$ be locally small categories. If $K : \mathcal{C} \to \mathcal{D}$, $J : \mathcal{D} \to \mathcal{E}$ and $F : \mathcal{C} \to$ **Set** then* $\mathrm{Lan}_J(\mathrm{Lan}_K F)$ *is naturally isomorphic to* $\mathrm{Lan}_{JK} F$.

The natural isomorphism between $\mathrm{Lan}_J(\mathrm{Lan}_K F)$ and $\mathrm{Lan}_{JK} F$ has components given by

$$[(JB \xrightarrow{\varphi} X, (KA \xrightarrow{\psi} B, a))] \mapsto [(JKA \xrightarrow{J\psi} JB \xrightarrow{\varphi} X, a)]$$

with inverse

$$[(JKA \xrightarrow{\varphi} X, a)] \mapsto [(JKA \xrightarrow{\varphi} X, (KA \xrightarrow{\mathrm{id}_{KA}} KA, a))].$$

Another important property is that left Kan extensions of product-preserving functors are product-preserving. For a proof of the following lemma see proposition 2.5 in [18].

Lemma A.6. *Let $\mathcal{C}$ be an essentially small category, $\mathcal{D}$ be a locally small category. If $K : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{C} \to$ **Set** such that F is product-preserving then* $\mathrm{Lan}_K F$ *is product-preserving. That is, there is a natural isomorphism* $\mathrm{Lan}_K F(X \times Y) \cong (\mathrm{Lan}_K F(X)) \times (\mathrm{Lan}_K F(Y))$.

This natural isomorphism is given on elements by

$$[(KA \xrightarrow{\varphi} X \times Y, a)] \mapsto ([(KA \xrightarrow{pr_X\varphi} X, a)], [(KA \xrightarrow{pr_Y\varphi} Y, a)])$$

with inverse

$$([(KA \xrightarrow{\alpha} X, a)], [(KB \xrightarrow{\beta} Y, b)]) \mapsto [(K(A \times B) \xrightarrow{(\alpha K(pr_A), \beta K(pr_B))} X \times Y, (a, b))]$$

noting that we are abusing notation by writing (a, b) to represent the image of $(a, b) \in F(A) \times F(B)$ under the isomorphism $F(A) \times F(B) \cong F(A \times B)$. In particular the map $\mathrm{Lan}_K F(X \times Y) \to \mathrm{Lan}_K F(X) \times \mathrm{Lan}_K F(Y) \to \mathrm{Lan}_K F(X \times Y)$ acts on elements by

$$[(KA \xrightarrow{\varphi} X \times Y, a)] \mapsto [(K(A \times A) \xrightarrow{(pr_X \varphi K(pr_1), pr_Y \varphi K(pr_2))} X \times Y, (a, a))]$$

which is the identity, the two representatives are identified along

$$(K(A \times A) \xrightarrow{(pr_X \varphi K(pr_1), pr_Y \varphi K(pr_2))} X \times Y, a)$$

via the diagonal map $\Delta : A \to A \times A$. In the other direction the map $\mathrm{Lan}_K F(X) \times \mathrm{Lan}_K F(Y) \to \mathrm{Lan}_K F(X \times Y) \to \mathrm{Lan}_K F(X) \times \mathrm{Lan}_K F(Y)$ acts on elements by

$$\begin{aligned} &([(KA \xrightarrow{\alpha} X, a)], [(KB \xrightarrow{\beta} Y, b)]) \\ &\mapsto ([(K(A \times B) \xrightarrow{\alpha K(pr_A)} X, (a, b))], [(K(A \times B) \xrightarrow{\beta K(pr_B)} Y, (a, b))]) \end{aligned}$$

which is the identity we identity the first coordinate along $(KA \xrightarrow{\alpha} X, (a, b))$ and the second along $(KB \xrightarrow{\beta} Y, b)$ via the projection maps pr_A and pr_B out of $A \times B$ respectively.

Lemma A.7. *Let $\mathcal{C}$ be an essentially small category, $\mathcal{D}$ be a locally small category. Given $K : \mathcal{C} \to \mathcal{D}$ the functor $\mathrm{Lan}_K(-) : [\mathcal{C}, \boldsymbol{Set}] \to [\mathcal{D}, \boldsymbol{Set}]$ is left adjoint to the precomposition by K functor $(-) \circ K : [\mathcal{D}, \boldsymbol{Set}] \to [\mathcal{C}, \boldsymbol{Set}]$.*

Appendix B

Bicategorical Background

We collect some basic bicategoical definitions following [15].

Definition B.1. A **bicategory** is a tuple $(\mathcal{B}, 1, c, a, l, r)$ consisting of the following data:

- **Objects:** $\mathcal{B}$ is equipped with a class $\mathcal{B}_0$ of **objects** or **0-cells**.
- **Hom Categories:** For each pair of objects X, Y the hom category $\mathcal{B}(X, Y)$ is a category whose objects are called **1-cells** and whose morphisms are called **2-cells**. The composition in a hom category is called **vertical composition** and the identity morphisms are called **identity 2-cells**.
- **Identity 1-cells:** Each object X in $\mathcal{B}$ has a functor $1_X : \mathbf{1} \to \mathcal{B}(X, X)$ from the category with one object and one morphism. The 1-cell that 1_X picks out is called the **identity 1-cell of** X.
- **Horizontal Composition:** For each triple of objects $X, Y, Z \in \mathcal{B}$ there is a **horizontal composition** functor

  $$c_{X,Y,Z} : \mathcal{B}(Y, Z) \times \mathcal{B}(X, Y) \to \mathcal{B}(X, Z).$$

  On 1-cells we write $c_{X,Y,Z}(g, f) = gf$ and on 2-cells we write $c_{X,Y,Z}(\beta, \alpha) = \beta * \alpha$.
- **Associator:** For each quadruple of objects $W, X, Y, Z \in \mathcal{B}$ there is a natural isomorphism

  $$a_{W,X,Y,Z} : c_{W,X,Z}\left(c_{X,Y,Z} \times \mathrm{id}_{\mathcal{B}(W,X)}\right) \to c_{W,Y,Z}(\mathrm{id}_{\mathcal{B}_{Y,Z}} \times c_{W,X,Y})$$

  which are functors $\mathcal{B}(Y, Z) \times \mathcal{B}(X, Y) \times \mathcal{B}(W, X) \to \mathcal{B}(W, Z)$.
- **Unitors:** For each pair off objects $X, Y \in \mathcal{B}$ there two natural isomorphisms, the **left unitor**

  $$l_{X,Y} : c_{X,Y,Y}(1_Y \times \mathrm{id}_{\mathcal{B}(X,Y)}) \to \mathrm{id}_{\mathcal{B}(X,Y)}$$

and the **right unitor**

$$r_{X,Y} : c_{X,X,Y}(\mathrm{id}_{\mathcal{B}(X,Y)} \times 1_X)$$

which are both between endofunctors of $\mathcal{B}(X,Y)$.

We will use the subscripts for the components of the natural transformations in the following axioms. The data above is required to satisfy the following axioms for any quadruple of horizontally composable 1-cells $f \in \mathcal{B}(V,W), g \in \mathcal{B}(W,X), h \in \mathcal{B}(X,Y), k \in \mathcal{B}(Y,Z)$:

- **Unity Axiom:** The following diagram in $\mathcal{B}(V,X)$ commutes.

$$\begin{tikzcd}
(g1_W)f \arrow[rr, "a_{g,1_W,f}"] \arrow[dr, "r_g * 1_f"'] & & g(1_W f) \arrow[dl, "1_g * l_f"] \\
& gf &
\end{tikzcd}$$

- **Pentagon Axiom:** The following diagram in $\mathcal{B}(V,Z)$ commutes.

$$\begin{tikzcd}
& (kh)(gf) \arrow[dr, "a_{k,h,gf}"] & \\
((kh)g)f \arrow[ur, "a_{kh,g,f}"] \arrow[d, "a_{k,h,g} * 1_f"] & & k(h(gf)) \\
(k(hg))f \arrow[rr, "a_{k,hg,f}"] & & k((hg)f) \arrow[u, "1_k * a_{h,g,f}"']
\end{tikzcd}$$

When it is clear from context a bicategory $(\mathcal{B}, 1, c, a, l, r)$ may be referred to by $\mathcal{B}$ alone.

Definition B.2. Let $(\mathcal{B}, 1, c, a, l, r)$ and $(\mathcal{B}', 1', c', a', l', r')$ be bicategories. A **pseudofunctor** from $\mathcal{B}$ to $\mathcal{B}'$ is a triple $(F, F^2, F^0) : \mathcal{B} \to \mathcal{B}'$ consisting of the following data:

- **Objects:** A function $F : \mathcal{B}_0 \to \mathcal{B}'_0$ on objects.

- **Hom Categories:** For each pair of objects $X, Y \in \mathcal{B}$ a functor

$$F : \mathcal{B}(X,Y) \to \mathcal{B}'(FX, FY).$$

- **Pseudo Functoriality Constraint:** For each triple of objects $X, Y, Z \in \mathcal{B}$ a natural isomorphism

$$F^2 : c'(F \times F) \to Fc$$

between functors $\mathcal{B}(Y,Z) \times \mathcal{B}(X,Y) \to \mathcal{B}'(FX, FZ)$ with component 2-cells $F^2_{g,f} : Fg \circ Ff \to F(gf)$.

- **Pseudo Unity Constraint:** For each object $X \in \mathcal{B}$ a natural isomorphism
$$F^0 : 1'_{FX} \to F1_X$$
between functors $\mathbf{1} \to \mathcal{B}'(FX, FX)$ with component 2-cells $F^0_X : 1'_{FX} \to F1_X$.

The above data is required to satisfy the following axioms for any triple of horizontally composable 1-cells $f \in \mathcal{B}(W, X), g \in \mathcal{B}(X, Y), h \in \mathcal{B}(Y, Z)$:

- **Pseudo Associativity Axiom:** The following diagram in $\mathcal{B}'(FW, FZ)$ commutes.

$$\begin{tikzcd}
(Fh \circ Fg) \circ Ff \arrow[r, "a'"] \arrow[d, "F^2_{h,g} * 1_{Ff}"] & Fh \circ (Fg \circ Ff) \arrow[d, "1_{Fh} * F^2_{g,f}"] \\
F(hg) \circ Ff \arrow[d, "F^2_{hg,f}"] & Fh \circ F(gf) \arrow[d, "F^2_{h,gf}"] \\
F((hg)f) \arrow[r, "Fa"] & F(h(gf))
\end{tikzcd}$$

- **Pseudo Left and Right Unity Axioms:** The following diagrams in $\mathcal{B}'(FW, FX)$ commute.

$$\begin{tikzcd}
1'_{FX} \circ Ff \arrow[r, "l'"] \arrow[d, "F^0_X * 1_{Ff}"'] & Ff \\
F1_X \circ Ff \arrow[r, "F^2_{1_X, f}"'] & F(1_X \circ f) \arrow[u, "Fl"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Ff \circ 1'_{FW} \arrow[r, "r'"] \arrow[d, "1_{Ff} * F^0_W"'] & Ff \\
Ff \circ F1_W \arrow[r, "F^2_{f, 1_W}"'] & F(f \circ 1_W) \arrow[u, "Fr"']
\end{tikzcd}$$

A **strict functor** is a pseudofunctor in which F^2 and F^0 are identity natural transformations.

Definition B.3. Let (F, F^2, F^0) and (G, G^2, G^0) be pseudofunctors $\mathcal{B} \to \mathcal{B}'$. A **strong transformation** $\alpha : F \to G$ consists of the following data:

- **Component 1-cells:** For each object $X \in \mathcal{B}$ a 1-cell $\alpha_X \in \mathcal{B}'(FX, GX)$.
- **Pseudo Naturality Constraints:** For each pair of objects $X, Y \in \mathcal{B}$ a natural isomorphism $\alpha : \alpha_X^* G \to (\alpha_Y)_* F$, with component 2-cells
$$\alpha_f : (Gf)\alpha_X \to \alpha_Y(Ff).$$

The above data is required to satisfy the following axioms for any pair of horizontally composable 1-cells $f \in \mathcal{B}(X, Y), g \in \mathcal{B}(Y, Z)$:

- **Strong Unity Axiom:** The following diagram in $\mathcal{B}'(FX, GX)$ commutes.

$$\begin{tikzcd}
1_{GX}\alpha_X \arrow[r, "l"] \arrow[d, "G^0 * 1_{\alpha_X}"] & \alpha_X \arrow[r, "r^{-1}"] & \alpha_X 1_{FX} \arrow[d, "1_{\alpha_X} * F^0"] \\
(G1_X \alpha_X) \arrow[rr, "\alpha_{1_X}"] & & \alpha_X(F1_X)
\end{tikzcd}$$

- **Strong Naturality Axiom:** The following diagram in $\mathcal{B}'(FX, GX)$ commutes.

$$\begin{tikzcd}
(Gg)\,(\alpha_Y Ff) \arrow[r, "a^{-1}"] & ((Gg)\,\alpha_Y)\,Ff \arrow[r, "\alpha_g * 1_{Ff}"] & (\alpha_Z Fg)\,Ff \arrow[d, "a"] \\
(Gg)\,((Gf)\,\alpha_X) \arrow[u, "1_{Gg} * \alpha_f"] & & \alpha_Z\,((Fg)\,(Ff)) \arrow[d, "1_{\alpha_Z} * F^2"] \\
((Gg)\,(Gf))\,\alpha_X \arrow[u, "a"] \arrow[r, "G^2 * 1_{\alpha_X}"] & G(gf)\alpha_X \arrow[r, "\alpha_{gf}"] & \alpha_Z F(gf)
\end{tikzcd}$$

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