Some Local Multipliers Associated with the Ornstein-Uhlenbeck Operator

Zanbing Dai

May 2017

A thesis submitted for the degree of Master of Mathematical Science
of the Australian National University

Australian
National
University

Declaration

The work in this thesis is my own except where otherwise stated.

Zanbing Dai

Acknowledgements

First, I would like to thank my supervisor Pierre Portal, for developing my interests in math (a supervisor), providing general guidance (a mind mentor), and being always patient and kind (a friend). Without his helps, this work cannot be accomplished. I should appreciate to Andrew Hassell, John Urbas, Tim Trudgian, Joan Licata and Ben Andrew. Thanks to every one of them for their encouragement.

I am grateful to Sean Harris for always helping me in mathematics during the last three semesters.

Finally, I would like to thank my parents. Without their supports, I won't be able to keep studying in mathematics.

Abstract

This thesis combines ideas from [19] and [21] to introduce two main spectral multipliers results for the Ornstein Uhlenbeck operator in Section 4. Chapter 1 introduces basic definitions and properties of Markov semigroups and shows how they relate to the Ornstein Uhlenbeck operator. Lots of researchers study the hypercontractivity result via Markov segmigroups. Chapter 2 describes the notions of weighted Nash inequalities and show how to deduce a hypercontrativity result (Theorem 2.7) from these inequalities based on paper [13]. A brief introduction to the Spectral theory and the main theorem of paper [19] is given in Section 3.

Introduction

Hypercontractivity

The term "Hypercontractivity" was first presented and discussed in developing the constructive quantum filed theory. A hypercontractivity usually refers to a map being contractive from L^q to L^2 for $q < 2$. In 1973, E. Nelson proved a hypercontractivity theorem for Ornstein Uhlenbeck operator L [1, Theorem 2] that $\exp(-tL)$ is bounded from L^q to L^p if $\exp(-t) \leq \sqrt{(q-1)/(p-1)}$. His Ph.D student Jay B. Epperson then established a result extending Nelson's theorem from real number t to complex numbers z in [2].
On the other hand, combining probability theory and semigroup theory, people realized that one can define a semigroup from a Markov process, which is called the Markov semigroup. Each Markov semigroup is associated to an infinitesimal generator, and the Ornstein Uhlenbeck operator is one of them. In 1975 [3], L. Gross introduced the notion of log-Sobolev inequalities:

$$\int_{\mathbb{R}^n} |f|^2 \log |f| d\gamma \leq \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma + \|f\|_2^2 \log \|f\|_2$$

where γ is the Gaussian measure on $\mathbb{R}^n$ and deduced a hypercontractivity result for from these inequalities [4, Theorem 1]. In recent works, F.-Y. Wang obtained a hypercontractivity result for Markov semigroup through weighted Super-Poincare inequalities in [7, Theorem 3.3]. D. Bakry, F. Bolley, I. Gentil and P. Maheux then introduced the notion of weighted Nash inequalities and confirmed his result by providing a simpler proof in [13, Theorem 2.5].
Recently, Jan Van Neervan and Pierre Portal presented a restricted boundedness result for the Ornstein-Uhlenbeck Semigroup in [21].

Spectral Multiplier theory

In classical ways, by the Hormander-Mikhlin theorem [5],[6] and the spectral theory, the spectral multipliers $F(\Delta)$ of the Laplacian are bounded $L^p - L^p, 1 < p < \infty$ for all functions F defined on $(-\infty, 0]$ satisfying $|\partial_\xi^\alpha F(-4\pi|\xi|^2)| \leq C|\xi|^{-|\alpha|}$ for all $|\alpha| \leq \frac{2}{n}+1$. In 2016, P. Chen, A. Sikora and L. Yan introduced the notions of resolvent type estimates to the non-doubling space and showed how this relates to the spectral multipliers [19].

Results

Two spectrum multiplier boundedness results Theorem 4.6 and Theorem 4.8 follow from [19, P. Chen, A. Sikora, L. Yang] and [21, J. Van Neerven, P. Portal].

Theorem. *Let L be the Ornstein-Uhlenbeck operator. Suppose a function $F \in L^\infty(\mathbb{R})$ satisfies that there exists an $\alpha > 1$ such that $\xi \mapsto (1+|\xi|^2)^{\frac{\alpha}{2}}\hat{F}(\xi) \in L^1(\mathbb{R})$. For $p \in (\frac{2d}{d+4}, 2]$, then*

$$F(L) \in \mathcal{B}(L^p(\mathbb{R}^d, \gamma_{2/p}), L^2(\mathbb{R}^d, \gamma))$$

Theorem. *Let L be the Ornstein-Uhlenbeck operator. For a bounded Borel function F such that $\operatorname{supp} F \subset [-1, 1]$ and $F \in H^s(\mathbb{R})$ for some $s > \frac{d}{2} + 1$, the operator $F(L)$ is bounded on $L^p(\mathbb{R}^d, \gamma_2)$ for all $1 \leq p \leq 2$, and there exists a constant $C_{d,s} > 0$ such that:*

$$\|F(tL)\|_{\mathscr{L}(L^1(\mathbb{R}^d,\gamma_2),L^1(\mathbb{R}^d,\gamma_2))} \leq C_{d,s}\|F\|_{H^s}$$

for all $t > 0$.

Remark. As recently noticed by Sean Harris, the results in Section 4 could also be obtained by transferring known results regarding the quantum harmonic oscillator through the natural isometry U between $L^2(\gamma)$ and L^2, noting that U is also an isomorphism between $L^p(\gamma_{2/p})$ and L^p. The proofs given here, however, interesting in their own right as a way to combine [19, P. Chen, A. Sikora, L. Yang] and [21, J. Van Neerven, P. Portal]. In particular, our proofs could most likely be used to prove results on $L^p(\gamma_\alpha)$ for $\alpha < 2/p$, such that $L \log L$ estimates that would exploit a log Sobolev rather that a full Sobolev embedding.

Contents

Chapter 1

Introduction

1.1 Markov Semigroups and Invariant Measures

Let G be a set. Consider a σ-algebra $\mathscr{G}$ on G. A pair $(G, \mathscr{G})$ is called a measurable space. We usually write a measure space as $(G, \mathscr{G}, \mu)$, where $(G, \mathscr{G})$ is a measurable space and μ is a measure on it. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and T be a index set. A probability space is a measurable space with $\mathbb{P}(\Omega) = 1$. Let $(G_i, \mathscr{G}_i)_{i \in I}$ be a collection of measurable spaces and f_i be a map from Ω to E_i. Define $\sigma(f_i, i \in I)$ as the σ-algebra on Ω generated by the sets $\{f_i^{-1}(E_i) : E_i \in \mathscr{G}_i, i \in I\}$. It is the smallest σ-algebra such that that all f_i are measurable. A stochastic process with values in $(G, \mathscr{G})$ and a index set T over probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is a collection $X = \{X_t, t \in T\}$ of maps X_t from Ω to G. Each of these maps is measurable relative to $\mathscr{F}$ and $\mathscr{G}$, which means that for every $E \in \mathscr{G}$, $X_t^{-1}(E) \in \mathscr{F}$.

Definition 1.1. In the above setting, let $T = \mathbb{R}^+ \cup \{0\}$ and let $\{\mathscr{F}_t, t \in T\}$ be an increasing family of sub-σ-algebra of $\mathscr{F}$ such that $\mathscr{F}_s \subset \mathscr{F}_t$ if $s < t$. Then, X is called a Markov process with respect to $\{\mathscr{F}_t, t \in T\}$ if

1. X is adapted to $\{\mathscr{F}_t\}$. We say the process X is adapted to $\{\mathscr{F}_t\}$ if X_t is measurable relative to $\mathscr{F}_t$ and $\mathscr{G}$ for each $t \in T$.

2. For each $t \in T$ the σ-algebra $\mathscr{F}_t$ and $\sigma(X_s, s \geq t)$ are conditionally independent given X_t, that is:

$$\mathbb{P}(A \cap B | X_t) = \mathbb{P}(A | X_t)\mathbb{P}(B | X_t)$$

   where A is in $\mathscr{F}_t$ and B is in $\sigma(X_s; s \geq t)$.

In general, we say X is a Markov process if it is a Markov process with respect to the family of σ-algebra $\mathscr{A}_t := \sigma(X_s; s \leq t)$.

There are several equivalent statements with Definition 1.1.

Theorem 1.2. *[9, Theorem 1.3 on p.12] Let $X = \{X_t; t \in T\}$ be a stochastic process adapted to the family $\{\mathscr{F}_t\}$. The following statements are equivalent:*

(i) X is a Markov process with respect to $\{\mathscr{F}_t\}$;

(ii) For each $t \in T$ and Y is in $\sigma(X_s : s \geq t)$, then we have

$$\mathbb{E}(Y|\mathscr{F}_t) = \mathbb{E}(Y|X_t)$$

(iii) If $t, s \in T$ and $t \leq s$, then we have:

$$\mathbb{E}(f \circ X_s|\mathscr{F}_t) = \mathbb{E}(f \circ X_s|X_t)$$

for all $f \in \mathscr{G}$.

Definition 1.3. Let $X = \{X_t, t \in T\}$ be a Markov process with respect to $\{\mathscr{F}_t, t \in T\}$. The associated Markov semigroup $(P_t)_{t\geq 0}$ on bounded measurable function $f : G \to \mathbb{R}$, is defined as:

$$P_t f(x) = \mathbb{E}(f(X_t)|X_0 = x)$$

We can easily check that it does indeed define a semigroup:

$$P_0 f(x) = \mathbb{E}\Big(f(X_0)|X_0 = x\Big) = f(x)$$

In other words, $P_0 = Id$. Next, for all $t, s \geq 0$, by law of iterated expectations and (iii) of Theorem (1.2)

$$\begin{aligned} P_{t+s} f(x) = \mathbb{E}\Big(f(X_{t+s})|X_0 = x\Big) &= \mathbb{E}\Big(\mathbb{E}\Big(f(X_{t+s})\big|\mathscr{F}_t\Big)\Big|X_0 = x\Big) \\ &= \mathbb{E}\Big(\mathbb{E}\Big(f(X_{t+s})\big|X_t\Big)\Big|X_0 = x\Big) \\ &= \mathbb{E}\Big(P_s f(X_t)\big|X_0 = x\Big) = P_t(P_s f)(x) \end{aligned}$$

It implies that for every $t, s \geq 0$, $P_{t+s} = P_t \circ P_s$. Besides these two fundamental properties, there are several more.

Proposition 1.4. *Let $X = \{X_t, t \in T\}$ be a Markov process on $(G, \mathscr{G})$ and $(P_t)_{t\geq 0}$ be the associated Markov semigroup. Then:*

1. *$P_t(\chi) = \chi$, when χ is the constant function equal to 1 (mass conservation)*

2. *If $f \geq 0$, then $P_t f \geq 0$ (positivity preserving)*

3. *For every $t \geq 0$, P_t is a linear operator sending bounded measurable functions on $(G, \mathscr{G})$ to bounded measurable functions.*

Proof. 1 and 2 follow immediately from the properties of expectation.
For 3, by the linearity of the expectation,

$$|P_t(\psi)(x)| \leq \mathbb{E}(|\psi(X_t^x)|) \leq \|\psi\|_\infty, \ \forall x \in G \implies \|P_t(\psi)(x)\|_\infty \leq \|\psi\|_\infty, \ \forall t \geq 0$$

□

Note that, since 1 and 2 of Proposition 1.4 hold, by Jensen's inequality, for every convex function $\phi : \mathbb{R} \to \mathbb{R}$, for every bounded measurable function $f : G \to \mathbb{R}$:

$$\begin{aligned} P_t\big(\phi(f)\big)(x) &= \mathbb{E}\Big(\phi\big(f(X_t)\big|X_0 = x\big)\Big) \\ &\geq \phi\Big(\mathbb{E}\big(f(X_t)\big|X_0 = x\big)\Big) = \phi\big(P_t f\big)(x), \ \forall t \geq 0, \forall x \in G \end{aligned} \tag{1.1}$$

where $\phi(f)(x) := \phi(f(x))$.

Definition 1.5. (Invariant measure) Given a Markov semigroup $(P_t)_{t\geq 0}$ acting on $C_b(G)$, where $(G, \mathscr{G})$ is a measurable space. A (positive) $\sigma-$finite measure μ on $(G, \mathscr{G})$ is said to be invariant for $(P_t)_{t\geq 0}$ if for every bounded positive measurable function $f : G \to \mathbb{R}$ and every $t \geq 0$

$$\int_E P_t f d\mu = \int_E f d\mu$$

In other words, μ is invariant if $P_t^* \mu = \mu$ for every $t \geq 0$.

If μ is an invariant measure, then P_t is a bounded operator from $L^p(\mu)$ to $L^p(\mu)$ by (1.1) such that for $1 \leq p < \infty$:

$$\|P_t f\|_{L^p(\mu)}^p = \int_E |P_t f|^p d\mu \leq \int_E P_t(|f|^p) d\mu = \int_E |f|^p d\mu = \|f\|_{L^p(\mu)}^p$$

Moreover, the semigroup P_t is strongly continuous in $L^p(\mu)$.

Proposition 1.6. *Suppose that μ is a $\sigma-$finite invariance measure for $(P_t)_{t\geq 0}$. For every $f \in L^p(\mu)$, $P_t f$ converges to f in $L^p(\mu)$ as $t \to 0$ (continuity property)*

1.2 Infinitesimal Generators

To a strongly-continuous Markov semigroup $(P_t)_{t\geq 0}$, one can associate a generator L in the following way.

Definition 1.7. The infinitesimal generator L of a Markov semigroup $(P_t)_{t\geq 0}$ is defined as:

$$Lf := \lim_{s\to 0^+} \frac{P_s f - f}{s}, \quad \mathscr{D}(L) = \left\{ f \in L^2(G) : \lim_{s\to 0^+} \frac{P_s f - f}{s} \text{ exists} \right\}$$

where $\mathscr{D}(L)$ is called the domain of L.

Proposition 1.8. *Let $(P_t)_{t\geq 0}$ be a Markov semigroup and L be its associated infinitesimal generator, then:*

1. If $f \in \mathscr{D}(L)$, then $P_t f \in \mathscr{D}(L)$ and:

$$\frac{d}{dt} P_t f = L P_t f = P_t L f$$

   for all $t \geq 0$.

2. If $f \in \mathscr{D}(L)$, then:

$$P_t f - f = \int_0^t L P_s f ds = \int_0^t P_s L f ds$$

   for all $t \geq 0$.

Proof. 1. For all $f \in \mathcal{D}(L)$,

$$\frac{1}{s}[P_{t+s} f - P_t f] = P_t\Big(\frac{1}{s}[P_s f - f]\Big) = \frac{1}{s}\Big[P_s(P_t f) - P_t f\Big]$$

Letting $s \to 0$, we get $P_t f \in \mathcal{D}(L)$ and:

$$\frac{d}{dt} P_t f = P_t L f = L P_t f \tag{1.2}$$

2. By (1.2), integrating over $s > 0$:

$$P_t f - f = \int_0^t \frac{d}{ds} P_s f ds = \int_0^t L P_s f ds = \int_0^t P_s L f ds$$

□

We now turn to a question: can we obtain semigroup from a generator L? Considering there are two different types of operators, bounded or unbounded, does this affect the continuity condition of the semigroup constructed? We start with a case where L is bounded linear operator.

Proposition 1.9. *Consider that L is a bounded operator on $L^p(G)$. We define:*

$$P_t f := e^{tL} f = \sum_{n=0}^{\infty} \frac{(tL)^n}{n!} f$$

for all $t \geq 0$ and $f \in L^p(G)$. Then, $(P_t)_{t\geq 0}$ is a uniformly continuous semigroup.

Proof. Since L is bounded, then $\sum_{n=0}^{\infty} \frac{(tL)^n}{n!}$ converges for each $t \geq 0$ to a linear operator e^{tL}. We can easily check that $P_0 f = f$ and $P_t \circ P_s = P_{t+s}$, because

$$\sum_{n=0}^{\infty} \frac{(t)^n}{n!} \sum_{n=0}^{\infty} \frac{(s)^n}{n!} = \sum_{n=0}^{\infty} \frac{(t+s)^n}{n!}$$

To check P_t that is uniformly continuous, since $\|L\| < \infty$, then:

$$\|P_t - I\| = \|\sum_{n=1}^{\infty} \frac{(tL)^n}{n!}\| \leq \sum_{n=1}^{\infty} \frac{(t\|L\|)^n}{n!} = e^{t\|L\|} - 1$$

Hence, $\|P_t - I\| \to 0$ as $t \to 0$. □

When L is unbounded, we need a more powerful tool to connect it to its semigroup. Let X be a Banach space over $\mathbb{C}$, and let A be an unbounded operator acting on X.

Definition 1.10. The resolvent set $\rho(A)$ of A is given by:

$$\rho(A) := \{\lambda \in \mathbb{C}; \lambda I - A : \mathscr{D}(A) \to X \text{ bijective}, (\lambda I - A)^{-1} \in \mathcal{B}(X)\}$$

where $\mathcal{B}(X)$ denotes the space of bounded linear operators in X, with domain all of X.

Definition 1.11. The operator $R(\lambda, L) = (\lambda I - L)^{-1}$ is called the resolvent of L at λ, and

$$R(\cdot\,, L) : \rho(A) \to \mathcal{B}(X)$$

Remark 1.12. If $\rho(A) \neq \emptyset$ and $\lambda \in \rho(A)$, then $(\lambda I - A)^{-1} \in \mathcal{L}(X)$ is closed. It follows that $\lambda I - A$ is closed and A is closed. If A is a closed operator and pick $\lambda \in \mathbb{C}$ such that $\lambda I - A : \mathscr{D}(A) \to X$ is bijective. Then the inverse $(\lambda I - A)^{-1}$ is closed. By the closed graph theorem, $(\lambda I - A)^{-1} \in \mathcal{B}(X)$.

Theorem 1.13. *[10, Theorem 2.7 on p.16] Consider a strongly continuous semigroup T on X and its generator A. Let $M \geq 1, \omega \in \mathbb{R}$ be such that:*

$$\|T(t)\| \leq Me^{\omega t} \qquad (t \geq 0)$$

Then $\{\lambda \in \mathbb{C}; \Re\lambda > \omega\} \subseteq \rho(A)$. Moreover, for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$, we have:

$$R(\lambda, L) = \int_0^\infty e^{-\lambda t} P_t dt$$
$$\|R(\lambda, L)^n\| \leq \frac{M}{(\Re\lambda - \omega)^n}, \quad \forall n \in \mathbb{N}.$$

This theorem guarantees that the operator $(I - \frac{t}{n}A)^{-n}$ is defined for n large enough. Assume that $\omega > 0$ and let $\alpha > 0$. If $0 \leq t \leq \alpha$, then $\frac{n}{t} \geq \frac{n}{\alpha}$. Thus, $\frac{n}{t} \in \rho(L)$ if $n \geq \alpha\omega$. Therefore, for n large enough, operator $(I - \frac{t}{n}A)^{-1} = \frac{n}{t}(\frac{n}{t}I - A)^{-1}$ is defined.

Definition 1.14. Let X be a Banach space and A be a closed unbounded operator on X. Define $T(t)$ as

$$T(t)x = \lim_{n\to\infty} \big(I - \frac{t}{n}A\big)^{-n} x \tag{1.3}$$

for all $x \in X$ and $t \geq 0$.

Proposition 1.15. *In the setting above, $(T(t))_{t\geq 0}$ is a strongly continuous semigroup.*

Proof. For n large enough,

$$T(t)x = e^{tA}x$$

It satisfies two properties of strongly continuous semigroup that:

(i) $\lim_{t\to 0^+} T(t)x = x$ for all $x \in X$

(ii) $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$.

□

1.3 Kernels and Chapman-Kolmogorov Equations

Another representation for a Markov semigroup $(P_t)_{t\geq 0}$ is through its kernel. One often has the following representation: for every bounded measurable function $f : G \to \mathbb{R}$,

$$P_t f(x) = \int_E f(y) p_t(x, dy), \quad x \in G, \ t \geq 0$$

where $p_t(x, dy)$ is a probability measure, but usually it has density with respect to a reference measure.

Definition 1.16. (Density kernel) A Markov semigroup $(P_t)_{t\geq 0}$ acting on bounded measurable functions $f : G \to \mathbb{R}$, where $(G, \mathscr{G})$ is a measurable space, is said to admit density kernels with respect to a reference $\sigma-$finite measure μ on $\mathcal{F}$ if there exists, for every $t > 0$, a positive measurable function $p_t(x, y)$ defined on $G \times G$ such that for every bounded measurable function $f : G \to \mathbb{R}$ and (μ-almost) every $x \in G$,

$$P_t f(x) = \int_E f(y) p_t(x, y) d\mu(y)$$

In this case, $\int_G p(x, y) d\mu(y) = 1$ for ($\mu-$almost) every $x \in E$. (Reflecting 1. of Proposition 1.4)

Before discussing the existence of density kernel, we need the following definition first. The construction of density kernels has to be under good measure spaces such as polish spaces.

Definition 1.17. A Polish space is a separable topological space X for which exists a compatible metric d such that (X, d) is a complete metric space.

The following proposition explains what kind of operators have a density kernel.

Proposition 1.18. *(Existence of density kernel) Let $(G, \mathscr{G}, \mu)$ be a a Polish space with probability measure μ. Let P be a linear operator mapping $L^1(\mu)$ into $L^\infty(\mu)$ with operator norm $\|P\|_{\mathscr{L}(L^1(\mu), L^\infty(\mu))} \leq M$. Then, there exists a measurable function (density kernel) $p(x, y)$ on $G \times G$ with $|p(x, y)| \leq M$ for $m \otimes m$-almost every $(x, y) \in G \times G$ such that, for any $f \in L^1(G, d\mu)$ and (μ-almost) every $x \in G$,*

$$Pf(x) = \int_E f(y) p(x, y) d\mu(y),$$

Proof. Referring to [11, Theorem 1.3 on p.8], in a Polish space, the collection M of probability measures on $(G, \mathscr{G})$ is tight. The collection M is called tight if $i \in \mathbb{N}$, there exists a sequence $(K_i)_{i\in\mathbb{N}}$ of compact measurable sets such that $\mu(K_i) > 1 - 1/i$. Define $S_i = G/K_i$, then $\mu(S_i) \to 0$ as $i \to \infty$. For each $i \in \mathbb{N}$, let $\mathscr{G}_i$ be the $\sigma-$field generated by $S_0, S_1...S_i$. Let $(B_1^i, ...B_{J_i}^i)$ be a partition of G from which the sets removing the measure 0, which also generates $\mathscr{G}_i$. Define the operator P_i as $P_i f = \mathbb{E}(Pf|\mathscr{G}_i)$. We represent it as:

$$P_i f(x) = \sum_{j=1}^{J_i} Q_j^i f \chi_{B_j^i}(x), \ \forall x \in G$$

where

$$Q_j^i f = \frac{1}{m(B_j^i)} \int_{B_j^i} Pf d\mu$$

Since $\|P\|_{\mathscr{L}(L^1(\mu), L^\infty(\mu))}$ is bounded, we have $|Q_j^i f| \leq \|Pf\|_{L^\infty(\mu)} \leq M\|f\|_{L^1(\mu)}$. Therefore, $Q_j^i \in \left(L^1(\mu)\right)^*$, and thus there exists some bounded (by M) measurable function q_j^i such that

$$Q_j^i f = \int_E f(y) q_j^i(y) d\mu(y), \quad \forall i \in \mathbb{N}, \forall j = 1, 2...J_i, \forall f \in L^1(\mu)$$

It follows that there exists a kernel $p_i = p_i(x,y) := \sum_{j=1}^{J_i} q_j^i(y)\chi_{B_j^i}(x)$ bounded by M, which is $\mathscr{G}_i \otimes \mathscr{G}_i-$measurable such that for any function $f \in L^1(\mu)$:

$$P_i f(x) = \int_E f(y) p_i(x,y) d\mu(y), \ \ \forall i \in \mathbb{N}$$

Given a σ-algebra $\mathscr{G}_i \otimes \mathscr{G}_i$, for $k > i$, we have:

$$\begin{aligned}\mathbb{E}(p_k|\mathscr{G}_i \otimes \mathscr{G}_i) &= \mathbb{E}\Big(\sum_{j=1}^{J_k} q_j^k(y)\chi_{B_j^k}(x)|\mathscr{G}_i \otimes \mathscr{G}_i\Big) \\ &= \mathbb{E}\Big(\sum_{j=1}^{J_i} q_j^i(y)\chi_{B_j^i}(x)|\mathscr{G}_i \otimes \mathscr{G}_i\Big) = p_i\end{aligned}$$

Thus, $\big(p_i(x,y)\big)_{i\in\mathbb{N}}$ is a martingale. Furthermore, $\mathbb{E}(|p_i(x,y)|) < M$ for all $i \in \mathbb{N}$. By martingale convergence theorem, it converges to a measurable function $p(x,y)$ bounded by M in expectation. Using Levy's zero-one law, we have convergence almost surely, and thus

$$|p(x,y)| \leq M, \quad \text{a.e. } x, y \in G$$

$\square$

Using the semigroup property that $P_{t+s} = P_t \circ P_s$, through the kernel representation, we have the Chapman-Kolmogorov Equation such that:

$$p_{t+s}(x, dy) = \int_{z \in E} p_t(z, dy) p_s(x, dz)$$

for all $t, s \geq 0$ and $x \in G$. Let $(X_t)_{t \geq 0}$ be a Markov process starting at any $x \in G$. By Chapman-Kolmogorov equation, we have

$$\mathbb{E}(f(X_{t_1}, X_{t_2} ... X_{t_k})) = \int_{E \times E ... \times E} f(y_1, y_2 ... y_k) p_{t_k - t_{k-1}}(y_{k-1}, dy_k) \times p_{t_{k-1} - t_{k-2}}(y_{k-2}, dy_{k-1}) ... p_{t_1}(x, dy_1) \tag{1.4}$$

for any partition $0 \leq t_1 \leq t_2 \leq ... \leq t_k$ and any bounded measurable $f \in L^1(G \times G .... \times G)$. If the initial distribution of X_0 is given by a measure ν, then by (1.4) we have:

$$\mathbb{E}(f(X_0, X_{t_1}, X_{t_2} ... X_{t_k})) = \int_{E \times E ... \times E} f(y_0, y_1, y_2 ... y_k) p_{t_k - t_{k-1}}(y_{k-1}, dy_k) \times p_{t_{k-1} - t_{k-2}}(y_{k-2}, dy_{k-1}) ... p_{t_1}(y_0, dy_1) \nu(dy_0) \tag{1.5}$$

1.4 Symmetric Markov Semigroups

One important subclass of Markov semigroup, is when the density kernel p_t is a symmetric function on $G \times G$.

Definition 1.19. (Symmetric Markov Semigroup) The Markov semigroup $(P_t)_{t \geq 0}$ is said to be symmetric with respect to an invariant measure μ (or μ is reversible for $\mathbf{P}$), if for all functions $f, g \in L^2(G, d\mu)$ and all $t \geq 0$,

$$\int_G f P_t g d\mu = \int_G g P_t f d\mu \tag{1.6}$$

Differentiating equation (1.6) at $t = 0$, we have:

$$\int_G f L g d\mu = \int_G g L f d\mu \quad \forall f, g \in \mathscr{D}(L) \tag{1.7}$$

Reversibility means that if the process starts with an invariant distribution μ, then it has the same distribution for any time. Furthermore, if the measure is reversible and a process $(X_t)_{t \geq 0}$ starting with a initial distribution μ, then

$(X_0, X_{t_1}...X_{t_k}, X_t)$ and $(X_t, X_{t_k}...X_{t_1}, X_0)$ have the same law. We can understand this through its kernel representation. Suppose that the initial measure ν is μ in (1.5). Since the measure $p_t(x, dy)\mu(dx)$ is symmetric in (x, y) for all $t \geq 0$, then $p_{t_k - t_{k-1}}(y_{k-1}, dy_k)p_{t_{k-1}-t_{k-2}}(y_{k-2}, dy_{k-1})...p_{t_1}(y_0, dy_1)\mu(dy_0)$ is invariant when $(y_0....y_k)$ changes to $(y_k.....y_0)$. It implies that the law of process $(X_t)_{0\leq t\leq T}$ and $(X_{T-t})_{0\leq t\leq T}$ are the same. This is called "reversible in time". Furthermore, applying martingale theory, we have the following result.

Lemma 1.20. *(Rota's Lemma) Let $(P_t)_{t\geq 0}$ be a Markov semigroup symmetric with respect to an invariant measure μ. Then for any $1 < p < \infty$, there exists a $C_p > 0$ such that for any measurable function $f : G \to \mathbb{R}$*

$$\Big\| \sup_{s\geq 0} P_s f \Big\|_{L^p(G,d\mu)} \leq C_p \|f\|_{L^p(G,d\mu)}$$

Proof. This proof only works in the case where μ is a probability measure. We may assume that f is positive and bounded. Let $\{X_t\}_{t\geq 0}$ be the Markov process with initial distribution μ associated with $(P_t)_{t\geq 0}$. Define $M_t = P_{T-t}(f)(X_t), t \in [0, T]$. By Markov property and law of total expectation:

$$\mathbb{E}(M_t|\mathcal{F}_s) = \mathbb{E}(\mathbb{E}(f(X_T)|\mathcal{F}_t)|\mathcal{F}_s) = \mathbb{E}(f(X_T)|\mathcal{F}_s) = M_s, \ \forall T \geq t \geq s$$

Thus, it is a martingale. By Doob's maximal inequality, there exists a constant C_p such that:

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} M_t^p\Big) \leq C_p^p \mathbb{E}(M_T^p)$$

Note that $M_T = f(X_T)$ and that, by the homogeneity of X_T, $\mathbb{E}(M_T^p) = \|f\|^p_{L^p(G,d\mu)}$. By Jensen's inequality and law of total expectation:

$$\begin{aligned}\mathbb{E}\Big(\mathbb{E}\Big(\sup_{0\leq t\leq T} M_t|X_T\Big)^p\Big) &\leq \mathbb{E}\Big(\mathbb{E}\Big((\sup_{0\leq t\leq T} M_t)^p|X_T\Big)\Big) \\ &= \mathbb{E}\Big(\sup_{0\leq t\leq T} M_t^p\Big) \\ &\leq C_p^p \|f\|^p_{L^p(\mu)}\end{aligned}$$

Since invariance and symmetry implies that the Markov process is "reversible in time", then the law of (X_t, X_T) is the same as (X_{2T-t}, X_T). It follows that

$$\begin{aligned}\mathbb{E}(M_t|X_T) &= \mathbb{E}\Big(P_{T-t}(f)(X_t)|X_T\Big) \\ &= \mathbb{E}\Big(\mathbb{E}(f(X_T)|X_t)\Big|X_T\Big) \\ &= \mathbb{E}\Big(\mathbb{E}(f(X_{2T-t})|X_T)\Big|X_T\Big) \\ &= \mathbb{E}\Big(f(X_{2T-t})\Big|X_T\Big) = P_{2T-t}f(X_T)\end{aligned}$$

$$\sup_{0\le t\le 2T} P_{2T-t}f(X_T) = \sup_{0\le t\le 2T} \mathbb{E}(M_t|X_T) \le \mathbb{E}\Big(\sup_{0\le t\le 2T} M_t|X_T\Big)$$

It follows that:

$$\Big\|\sup_{0\le t\le 2T} P_t f\Big\|_{L^p(G,d\mu)} \le C_p\|f\|_{L^p(G,d\mu)}$$

Let $T\to\infty$, then we have the result by monotone convergence theorem. □

1.5 Carre du Champ Operators and Dirichlet Forms

To each Markov semigroup $(P_t)_{t\ge 0}$, we can associate a bilinear map Γ and a bilinear form $\mathcal{E}$.

Definition 1.21. (Carre du champ operator) The bilinear map Γ:

$$\Gamma(f,g) = \frac{1}{2}[L(fg) - fLg - gLf]$$

defined for every $(f,g)\in\mathcal{A}\times\mathcal{A}$, where $\mathcal{A}$ is a subspace of the domain $\mathscr{D}(L)$ such that for every pair (f,g), the product fg is in the domain of L, is called the carre du champ operator of the Markov generator L.

For a symmetric Markov semigroup $(P_t)_{t\ge 0}$ with infinitesimal generator L, invariant measure μ and carre du champ operator Γ on a class $\mathcal{A}$ of functions on G, define the bilinear operator:

$$\mathcal{E}(f,g) = \int_G \Gamma(f,g)d\mu, \qquad (f,g)\in\mathcal{A}\times\mathcal{A}$$

By (1.7) and $\int_G L(fg)d\mu = 0$,

$$\mathcal{E}(f,g) = -\int_G fLgd\mu, \qquad (f,g)\in\mathcal{A}\times\mathcal{A}$$

Remark 1.22. If $f,g\in\mathscr{D}(L)$, then $\mathcal{E}(f,g) = \int_G f(-Lg)d\mu$ implies that $\mathscr{D}(L)\subset\mathscr{D}(\mathcal{E})\subset L^2(\mu)$. For any $f\in L^2(\mathbb{R},d\mu)$, the function P_tf is always in the domain of L and

$$\int_{\mathbb{R}} (P_tf)^2 d\mu \le e^{-2t}\int_{\mathbb{R}} f^2 d\mu \tag{1.8}$$

Moreover,

$$\|LP_tf\|^2_{L^2(\mathbb{R},d\mu)} \le \frac{C}{t^2}\|f\|^2_{L^2(\mathbb{R},d\mu)} \tag{1.9}$$

We will prove (1.8) and (1.9) by using spectral decomposition argument in the next section.

Definition 1.23. (Dirichlet form) For a symmetric Markov semigroup $(P_t)_{t\geq 0}$ with invariance measure μ, the energy $\mathcal{E}(f)$ is defined as the limit as $t \to 0$ of:

$$\frac{1}{t}\int_G f(f - P_t f) d\mu$$

for all functions $f \in L^2(G, d\mu)$ for which this limit exists.(Defining in this way the domain $\mathscr{D}(\mathcal{E})$.) The Dirichlet form $\mathcal{E}(f,g)$ is defined by polarization for f and g in the Dirichlet domain $\mathscr{D}(\mathcal{E})$, and $\mathcal{E}(f) = \mathcal{E}(f,f)$.

1.6 The Ornstein-Uhlenbeck Semigroup and Spectral Decompositions

Consider that a stochastic differential equation:

$$dX_t = \sqrt{2}dB_t - X_t dt, \qquad X_0 = x$$

with a solution:

$$X_t = e^{-t}\Big(x + \sqrt{2}\int_0^t e^s dB_s\Big)$$

Indeed, $(X_t)_{t\geq 0}$ is a Markov semigroup. Let γ be the invariant measure such that:

$$d\gamma(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{|x|^2}{2}} dx$$

Then, X_t has the following integral form:

$$\begin{aligned} X_t f(x) &= \mathbb{E}(f(e^{-t}x + \sqrt{1-e^{-2t}}G)) \\ &= \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1-e^{-2t}}y) d\gamma(y) \quad t \geq 0, x \in \mathbb{R}^n \end{aligned}$$

for precise $f : \mathbb{R}^n \to \mathbb{R}$, where G is a standard Gaussian variable on $\mathbb{R}^n$. Futhermore, the associated infinitesimal generator L on $\mathbb{R}^n$ is defined as:

$$Lf = \Delta f - x \cdot \nabla f$$

We usually call it the Ornstein-Uhlenbeck operator.
The eigenvectors of the Ornstein-Uhlenbeck operator L are the Hermite polynomials which are orthogonal to each others with respect to the invariant measure γ. We discuss the case when dimension equals to one.

Definition 1.24. The Hermite polynomial of degree n and parameter $\lambda > 0$ is defined as:

$$H_0(x, \lambda) = 1,$$
$$H_n(x, \lambda) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2\lambda}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2\lambda}}), \quad x \in \mathbb{R}$$

We usually use the notation $H_n(x) = H_n(x, 1)$.

Hermite polynomials have the following properties [12]:

$$\frac{\partial}{\partial x} H_n(x, \lambda) = H_{n-1}(x, \lambda), \quad n \geq 1 \tag{1.10}$$
$$(n+1)H_{n+1}(x, \lambda) = xH_n(x, \lambda) - \lambda H_{n-1}(x, \lambda), \quad n \geq 1 \tag{1.11}$$
$$H_n(-x, \lambda) = (-1)^n H_n(x, \lambda) \quad n \geq 1 \tag{1.12}$$
$$\frac{\partial}{\partial \lambda} H_n(x, \lambda) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} H_n(x, \lambda), \quad n \geq 1 \tag{1.13}$$

Proposition 1.25. *For each $n \in \mathbb{N}$, H_n is of degree n and is an eigenvector of $-L$ such that:*

$$-LH_n = nH_n, \quad \lambda_n > 0$$

Proof. Consider that:

$$-LH_n = -\frac{\partial^2}{\partial x^2} H_n + x \frac{\partial}{\partial x} H_n$$

By (1.10) and (1.11), picking $\lambda = 1$, we have that:

$$-LH_n = -H_{n-2} + xH_{n-1} = nH_n$$

□

Hermite polynomials form an orthogonal basis in the Hilbert space $L^2(\mathbb{R}, \gamma)$. For any function $f \in L^2(\mathbb{R}, d\gamma)$ on the real line can be decomposed into the sum of Hermite orthonormal polynomials $(H_n)_{n\in\mathbb{N}}$ such that:

$$f = \sum_{n=0}^{\infty} a_n H_n$$

where $a_n = \int_{\mathbb{R}} f H_n d\gamma$ and thus:

$$Lf = \sum_{n=0}^{\infty} a_n LH_n = -\sum_{n=0}^{\infty} n a_n H_n$$

Therefore,

$$P_t f = \sum_{n=0}^{\infty} e^{-nt} a_n H_n, \quad t \geq 0$$

Observe that $a_0 = \int_{\mathbb{R}} f H_0 d\gamma = \int_{\mathbb{R}} f d\gamma$, then

$$\Big\| P_t\Big(f - \int_{\mathbb{R}} f d\gamma\Big)\Big\|^2_{L^2(\mathbb{R},d\gamma)} = \sum_{n\geq 1} e^{-2nt} a_n^2 \leq e^{-2t} \sum_{n\geq 1} a_n^2 = e^{-2t} \Big\| f - \int_{\mathbb{R}} f d\gamma\Big\|^2_{L^2(\mathbb{R},d\gamma)}$$

Assuming that $\int_{\mathbb{R}} f d\mu = 0$, we have the inequality (1.8):

$$\int_{\mathbb{R}} (P_t f)^2 d\gamma \leq e^{-2t} \int_{\mathbb{R}} f^2 \gamma$$

Furthermore, $LP_t H_n = \sum_{n=0}^{\infty} n e^{-nt} a_n H_n$ and there exists a constant C such that for all $t > 0$ and every $n \in \mathbb{N}$, $n^2 t^2 e^{-2nt} \leq C$. The inequality (1.9) thus follows:

$$\|LP_t H_n\|^2_{L^2(\mathbb{R},d\gamma)} = \sum_{n=1}^{\infty} n^2 e^{-2nt} a_n^2 \leq \frac{C}{t^2} \|f\|^2_{L^2(\mathbb{R},d\gamma)}$$

1.7 Poincare Inequality and Variance Decay

In this subsection, there are some general results for a probability measure. Let $(G, \mathscr{G}, \mu)$ be a probability space. The variance of a function $f \in L^2(G, d\mu)$ is defined as:

$$Var_\mu(f) = \int_G f^2 d\mu - \Big(\int_G f d\mu\Big)^2$$

Once we define the variance, there is a important class of operators which satisfies the following inequality.

Definition 1.26. (Poincare inequality) A measurable space $(G, \mathscr{G}, \mu)$ endowed with an energy form $\mathcal{E}$ (with corresponding L), is said to satisfy a Poincare inequality $P(C)$ with constant $C > 0$, if for all function $f : G \to \mathbb{R}$ in $\mathscr{D}(\mathcal{E})$,

$$Var_\mu(f) \leq C\mathcal{E}(f)$$

Note that this inequality is also called a spectrum gap inequality. Suppose that f is an eigenfunction of $-L$ with eigenvalue λ, by Poincare inequality with $\int_G f d\nu = 0$

$$\int_G f^2 d\mu = Var_\mu(f) \leq C\mathcal{E}(f) = C \int_G f(-Lf) d\mu = C\lambda \int_G f^2 d\mu$$

Therefore, for a symmetric positive operator $-L$, this inequality implies the existence of a spectral gap such that that $\lambda \in \{0\} \cup [\frac{1}{C}, \infty)$.
One essential property for Poincare inequality is stability under tensorization.

Proposition 1.27. *If (G_1, μ_1, Γ_1) and (G_2, μ_2, Γ_2) satisfy Poincare inequalities with respective constants C_1 and C_2, then the product $(G_1 \times G_2, \mu_1 \otimes \mu_2, \Gamma_1 \oplus \Gamma_2)$ satisfies a Poincare inequality with constant $C = \max(C_1, C_2)$.*

Proof. Begin with a simple case where the corresponding generators $-L_1$ and $-L_2$ have discrete spectra $(\lambda_k^1)_{k\in\mathbb{N}}$ and $(\lambda_l^2)_{l\in\mathbb{N}}$. It follows that $-(L_1 \oplus L_2)$ has spectra $(\lambda_k^1 + \lambda_l^2)_{k,l\in\mathbb{N}}$. Thus, $C_1 = \frac{1}{\lambda_1^1}$ and $C_2 = \frac{1}{\lambda_1^2}$.
Let $\mathcal{E}_1$ be the Dirichlet form acting on the first variable of f and $\mathcal{E}_2$ be the Dirichlet form acting on the second. Define the Dirichlet form $\mathcal{E}$ associated to product $(G_1 \times G_2, \mu_1 \otimes \mu_2, \Gamma_1 \oplus \Gamma_2)$ on functions $f : G_1 \times G_2 \to \mathbb{R}$ as:

$$\mathcal{E}(f) = \int_{G_2} \mathcal{E}_1(f) d\mu_2 + \int_{G_1} \mathcal{E}_2(f) d\mu_1,$$

where for every $x_2 \in G_2$, $\mathcal{E}_1(f(., x_2))$ is finite and for every $x_1 \in G_1, \mathcal{E}_2(f(x_1, .))$ is finite. Since the Dirichlet forms are bilinear and positive, then such quadratic forms are convex. So applying Jensen' inequality, for such functions $f : G_1 \times G_2 \to \mathbb{R}$ satisfying above conditions, we have:

$$\mathcal{E}_1\Big(\int_{G_2} f(x_1, x_2) d\mu_2\Big) \leq \int_{G_2} \mathcal{E}_1\Big(f(x_1, x_2)\Big) d\mu_2$$

Let $h(x_1) = \int_{G_2} f(x_1, x_2) d\mu_2$ be a function of x_1. It follows that the variance decomposes as:

$$Var_{\mu_1 \otimes \mu_2}(f) = Var_{\mu_1}(h) + \int_{G_1} \Big(\int_{G_2} [f(x_1, x_2) - h(x_1)]^2 d\mu_2(x_2)\Big) d\mu_1(x_1)$$

Since (G_1, μ, Γ_1) and (G_2, μ_2, Γ_2) satisfy Poincare inequalities with respective constants C_1 and C_2,

$$Var_{\mu_1}(h) \leq C_1 \mathcal{E}_1(h) \leq C_1 \int_{G_2} \mathcal{E}_1(f) d\mu_2$$

and the second term:

$$\begin{aligned}\int_{G_1} \Big(\int_{G_2} [f(x_1, x_2) - h(x_1)]^2 d\mu_2(x_2)\Big) d\mu_1(x_1) &= \int_{G_1} \Big(Var_{\mu_2}(f)\Big) d\mu_1 \\ &\leq C_2 \int_{G_1} \mathcal{E}_2(f) d\mu_1\end{aligned}$$

In conclusion,

$$Var_{\mu_1\otimes\mu_2}(f) \leq C_1 \int_{G_2} \mathcal{E}_1(f)d\mu_2 + C_2 \int_{G_1} \mathcal{E}_2(f)d\mu_1 \leq \max(C_1, C_2)\mathcal{E}(f)$$

□

Applying this proposition inductively, we construct the same type of inequality for the Gaussian measure on $\mathbb{R}^n$.

Proposition 1.28. *(Poincare inequality for the Gaussian measure) Let γ be the standard Gaussian measure on the Borel sets of $\mathbb{R}^n$. For every function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f \in \mathcal{D}(\mathcal{E})$:*

$$\int_{\mathbb{R}^n} f^2 d\gamma - \Big(\int_{\mathbb{R}^n} f d\gamma\Big)^2 \leq \mathcal{E}(f) = \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma$$

Moreover, the variance of Markov semigroup performs an exponential decay in respect of t.

Theorem 1.29. *(Exponential decay in variance) Given a measure space (G, ν) associated with a bilinear form Γ and a Markov semigroup $(P_t)_{t\geq 0}$, the following statements are equivalent:*

1. *(G, ν, Γ) satisfies a Poincare inequality $P(C)$.*
2. *For every function $f : E \to \mathbb{R}$ in $L^2(\nu)$, and every $t \geq 0$,*

$$Var_\nu(P_t f) \leq \frac{2}{C} e^{-2t/C} Var_\nu(f)$$

3. *For every function $f \in L^2(\nu)$, there exists a constant $c(f) > 0$ such that for every $t \geq 0$:*

$$Var_\nu(P_t f) \leq c(f) e^{-2t/C}$$

Proof. $(1 \Rightarrow 2)$ For simplicity, assume that $\int_E f d\nu = 0$. Note that by invariance, $\int_G P_t f d\nu = \int_G f d\nu = 0$, and

$$\begin{aligned} \frac{d}{dt} Var_\nu(P_t f) &= \frac{d}{dt} \int_G (P_t f)^2 d\nu \\ &= \big\langle \frac{d}{dt} P_t f, P_t f \big\rangle + \big\langle P_t f, \frac{d}{dt} P_t f \big\rangle \quad \text{(By integration by parts)} \quad (1.14) \\ &= 2 \int_G P_t f L P_t f d\nu = -2\mathcal{E}(P_t f) \end{aligned}$$

By inequality $P(C)$,

$$Var_\nu(P_tf) \le -\frac{C}{2}\frac{d}{dt}Var_\nu(P_tf)$$

It implies that function $\frac{C}{2}e^{2t/C}Var_\nu(P_tf)$ is non-increasing in terms of t.
$(2 \Rightarrow 1)$ Conversely, 2 implies that $\frac{2}{C}e^{2t/C}Var_\nu(P_tf) \le Var_\mu(f)$ for all $t \ge 0$. Thus,

$$\frac{d}{dt}\Big(\frac{2}{C}e^{2t/C}Var_\nu(P_tf)\Big) \le 0$$

It implies that:

$$\frac{4}{C^2}e^{\frac{C}{2}}Var_\nu(P_tf) + \frac{2}{C}e^{\frac{C}{2}}\frac{d}{dt}Var_\nu(P_tf) \le 0$$

By (1.14), 1 holds
$(2 \Rightarrow 3)$ Pick $c(f) \ge \frac{2}{C}Var_\nu(f)$. Finally, $(3 \Rightarrow 2)$. Denote $f(t) = \log\big(\int_G (P_tf)^2 d\nu\big)$, then

$$\begin{aligned}\frac{d^2}{dt^2}f(t) &= -\frac{d}{dt}\Big(\frac{2\mathcal{E}(P_tf)}{\int_G [P_tf]^2 d\nu}\Big)\\ &= -\frac{2\mathcal{E}'(P_tf)\big(\int_G (P_tf)^2 d\nu\big) - 4[\mathcal{E}(P_tf)]^2}{(\int_G (P_tf)^2 d\nu)^2}\\ &= \frac{4\int_E L(P_tf)L(P_tf)d\mu}{\int_G (P_tf)^2 d\nu} + \frac{4[\mathcal{E}(P_tf)]^2}{(\int_G (P_tf)^2 d\nu)^2} \ge 0\end{aligned}$$

Therefore, $f(t)$ is convex on $\mathbb{R}^+$. Since $x \mapsto \log x$ is an increasing function and $\int_G f d\nu = 0$, inequality in 3 implies $f(t)$ is bounded above by $\log[c(f)] - \frac{2t}{C}$. It follows that function $f(t) + \frac{2t}{C}$ is convex and bounded from above for $t \ge 0$. Hence, $f(t) \le f(0) - \frac{2t}{C}$. □

1.8 Curvature and dimension

In this section, we define functional notions of curvature and dimension using the carre du champ operators. Suppose that $(P_t)_{t\ge 0}$ is a Markov semigroup with generator L on E. Let $\mathcal{A}$ be a subclass of all C^∞ compactly support function such that $\mathcal{A} \in \mathscr{D}(L)$ and $\forall f, g \in \mathcal{A}, (Lf, g), (Lf, g), (f, g) \in \mathscr{D}(\Gamma)$ and $\Gamma(f, g) \in \mathscr{D}(L)$. The iterated carre du champ is defined as a bilinear symmetric operator on $\mathcal{A}\times\mathcal{A}$:

$$2\Gamma_2(f, g) = L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf), \quad f, g \in \mathcal{A}$$

We start with the following definition, which was introduced by Bakry and Emery in 1985.

Definition 1.30. A generator L satisfies a curvature-dimension condition $CD(\lambda, n)$ of curvature $\lambda \in \mathbb{R}$ and dimension $n \geq 1$ if:

$$\Gamma_2(f) \geq \lambda\Gamma(f) + \frac{1}{n}(Lf)^2$$

for all functions $f \in \mathcal{A}$, given $\Gamma(f) = \Gamma(f, f)$.
In the case when $n = \infty$, the generator L satisfies the curvature condition $CD(\lambda, \infty)$ if

$$\Gamma_2(f) \geq \lambda\Gamma(f)$$

for every $f \in \mathcal{A}$.

Example 1.31. Consider the classical Laplacian operator $L = \Delta$ on $\mathbb{R}^n$ and then $\Gamma(f, g) = \nabla f \cdot \nabla g$. Hence, the iterated carre du champ operator is given by:

$$\Gamma_2(f) = \|\text{Hess}f\|_2^2$$

where:

$$\|\text{Hess}f\|_2^2 = \Bigg(\sum_{i,j}^{n} \Big(\frac{\partial^2 f}{\partial x_i \partial x_j}\Big)^2\Bigg)^{\frac{1}{2}}$$

By Cauchy Schwartz inequality,

$$\|\text{Hess}f\|_2^2 = \sum_{i,j}^{n} \Big(\frac{\partial^2 f}{\partial x_i \partial x_j}\Big)^2 \geq \sum_{i=1}^{n} \Big(\frac{\partial^2 f}{\partial x_i^2}\Big)^2 \geq \frac{1}{n}(\Delta f)^2$$

Therefore, the heat semigroup satisfies $CD(0, n)$.

Example 1.32. Suppose that $L = \Delta - \nabla U(x) \cdot \nabla$ and then $\Gamma(f, g) = \nabla f \cdot \nabla g$. Thus,

$$\Gamma_2(f) = \|\text{Hess}f\|_2^2 + \nabla f \cdot \text{Hess}f \nabla f$$

If $\text{Hess}f \geq \lambda Id$ for some $\lambda \in \mathbb{R}$, then

$$\nabla f \cdot \text{Hess}f \nabla f \geq \lambda |\nabla f|^2$$

Since $\|\text{Hess}f\|_2^2$ is always positive, then:

$$\Gamma_2(f) \geq \lambda |\nabla f|^2 = \lambda\Gamma(f) \tag{1.15}$$

It satisfies $CD(\lambda, \infty)$.

In [16], Bakery lists equivalent statements that relate the $CD(\lambda, n)$ condition for a diffusion semigroup to the functional inequalities.

Definition 1.33. An operator L on with carre du champ operator Γ, is said to be a diffusion operator if

$$L\Psi(f) = \Psi'(f)Lf + \Psi''\Gamma(f), \quad \forall f \in \mathcal{A}$$

for every $\Psi : \mathbb{R} \to \mathbb{R}$ in $C^\infty(\mathbb{R})$. A semigroup associated with a diffusion operator is called the diffusion semigroup.

Example 1.34. Consider a smooth function $U : \mathbb{R}^n \to \mathbb{R}$. Let $d\mu = e^{-U(x)}dx$. It is a invariant measure associated to the generator :

$$L = \Delta - \nabla U \cdot \nabla$$

with $\Gamma(f) = |\nabla f|^2$. For every $\Psi : \mathbb{R} \to \mathbb{R}$ in $C^\infty(\mathbb{R})$ and $\forall f \in \mathcal{A}$, we have

$$\begin{aligned} L\Psi(f) &= \Delta\Psi(f) - \nabla U \cdot \nabla\Psi(f) \\ &= \Psi''(f)|\nabla f|^2 + \Psi'(f)\Delta f - \Psi'(f)\nabla U \cdot \nabla f \\ &= \Psi'(f)Lf + \Psi''\Gamma(f) \end{aligned}$$

Indeed, it is a diffusion operator. The Ornstein Uhlenbeck operator is a special case when $U = |x|^2/2$.

Proposition 1.35. *Suppose that $(P_t)_{t\geq 0}$ is a diffusion semigroup with generator L. Let λ be any real number, then the following statements are equivalent.*

1. *$CD(\lambda, \infty)$ holds*
2. *For any function $f \in \mathcal{A}$, $\Gamma(P_t f) \leq \exp(-2\lambda t)P_t(\Gamma(f))$*
3. *For any function $f \in \mathcal{A}$, $\Gamma(P_t f)^{\frac{1}{2}} \leq \exp(-\lambda t)P_t(\Gamma(f)^{\frac{1}{2}})$*
4. *For any function $f \in \mathcal{A}$, $P_t(f^2) - (P_t f)^2 \leq \frac{1-\exp(-2\lambda t)}{\lambda}P_t(\Gamma(f))$*
5. *For any function $f \in \mathcal{A}$, $P_t(f^2) - (P_t f)^2 \geq \frac{\exp(-2\lambda t)-1}{\lambda}\Gamma(P_t f)$*
6. *For some $\alpha \in (1, 2)$, any function $f \in \mathcal{A}$*

$$P_t(f^\alpha) - (P_t f)^\alpha \leq \alpha(\alpha - 1)\frac{1 - \exp(-2\lambda t)}{2\lambda}P_t(f^{\alpha-2}\Gamma(f))$$

7. *For some* $\alpha \in (1,2)$*, any function* $f \in \mathcal{A}$

$$P_t(f^\alpha) - (P_t f)^\alpha \geq \alpha(\alpha-1)\frac{\exp(-2\lambda t) - 1}{2\lambda}(P_t f)^{\alpha-2}\Gamma(P_t f)$$

8. *For any function* $f \in \mathcal{A}$*,*

$$P_t(f\log f) - (P_t f)\log P_t f \leq \frac{1-\exp(-2\lambda t)}{2\lambda}P_t(\frac{\Gamma(f)}{f})$$

9. *For any function* $f \in \mathcal{A}$*,*

$$P_t(f\log f) - (P_t f)\log P_t f \geq \frac{\exp(-2\lambda t) - 1}{2\lambda}\frac{\Gamma(P_t f)}{P_t f})$$

Chapter 2

Weighted Nash inequalities

In this section, I present a family of weighted Nash inequalities and several density estimates introduced by Dominique Bakry, Francois Bolley, Ivan Gentil and Patrick Maheux in [13]. The classical Nash inequality in $\mathbb{R}^n$ is given by:

$$\|f\|_{L^2(\mathbb{R}^n)}^{1+n/2} \leq C_n \|f\|_{L^1(\mathbb{R}^n)} \|\nabla f\|_{L^2(\mathbb{R}^n)}^{n/2} \tag{2.1}$$

for all smooth function f. This inequality was introduced by John Nash in 1958 and used to study the boundedness of the solution of linear parabolic equations. It follows from properties of the Fourier transform. Firstly, for all $\rho > 0$ and $u \in W^{1,2}(\mathbb{R}^n)$.

$$\int_{|\xi|\geq\rho} |\hat{u}(\xi)| d\xi \leq \int_{|\xi|\geq\rho} \frac{|\xi|^2}{\rho^2} |\hat{u}(\xi)|^2 d\xi \leq \rho^{-2} \int_{\mathbb{R}^n} |\nabla u|^2 dx \tag{2.2}$$

On the other hand:

$$|\hat{u}(\xi)| \leq \|u\|_{L^1(\mathbb{R}^n)}, \quad \forall \xi \in \mathbb{R}$$

and thus,

$$\int_{|\xi|\leq\rho} |\hat{u}(\xi)|^2 d\xi \leq C\rho^n \|u\|_{L^1(\mathbb{R}^n)}^2 \tag{2.3}$$

Summing inequalities (2.2) and (2.3) and picking ρ to minimise the right-hand side gives the classical Nash inequality.

Applying this inequality to the semigroup, we obtain the several following results. Let $(P_t)_{t\geq 0}$ be a symmetric Markov semigroup, which is strongly continuous. Let $h(t) = \|P_t f\|_{L^2(\mathbb{R}^n)}^2$, then it has derivative $h'(t) = -2\mathcal{E}(P_t f, P_t f)$. Replacing f by $P_t f$ in the classical Nash inequality:

$$h(t)^{1+n/2} \leq C_n^2 \|P_t f\|_{L^1(\mathbb{R}^n)}^2 (-h'(t)/2)^{n/2} \leq C_n^2 \|f\|_{L^1(\mathbb{R}^n)}^2 (-h'(t)/2)^{n/2}, \quad \forall t > 0$$

It follows that $1 \leq -\frac{1}{2}C_n^{4/n}\|f\|_{L^1(\mathbb{R}^n)}^{4/n}\frac{h'}{h^{2/n+1}}$. Integrating both sides in t, we have that $\|P_t f\|_{L^2(\mathbb{R}^n)} \leq C' t^{-n/4}\|f\|_{L^1(\mathbb{R}^n)}$ for $t > 0$. By the symmetry of the semigroup and duality, $\|P_t f\|_{L^\infty(\mathbb{R}^n)} \leq C' t^{-n/4}\|f\|_{L^2(\mathbb{R}^n)}$. The ultracontractive bound thus follows

$$\|P_{2t} f\|_{L^\infty(\mathbb{R}^n)} \leq C' t^{-n/4}\|P_t f\|_{L^2(\mathbb{R}^n)} \leq C'^2 t^{-n/2}\|f\|_{L^1(\mathbb{R}^n)}$$

Use the kernel representation, this inequality implies uniform bounds on the kernel density such that:

$$|p_t(x,y)| \leq C'^2 t^{-n/2}$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$. This is, of course, true for $(P_t)_{t\geq 0} = (e^{t\nabla})_{t\geq 0}$, but now these kernel bounds extend to every strongly continuous Markov semigroup. More generally, Nash inequalities are of the form:

$$\|f\|_{L^2(\mathbb{R}^n)}^{1+n/2} \leq \|f\|_{L^1(\mathbb{R}^n)}\left[a\mathcal{E}(f,f) + b\|f\|_{L^2(\mathbb{R}^n)}^2\right]^{n/4}, \quad \forall f \in \mathscr{D}(\mathcal{E})$$

F.-Y Wang introduces an even more general family of inequalities in [14], called super-Poincare inequalities, as follows. He proves the following theorem in a d-dimensional connected complete Riemannian manifold. Suppose that E is a d-dimensional connected complete Riemanian manifold with boundary ∂M. Let dx be the Riemannian volume element. Let $L = \nabla + \Delta V$ for some $V \in W_{loc}^{1,2}(M, dx)$ such that $Z := \int \exp[V]dx < \infty$. Moreover, assume that $(L, C_0^\infty(M))$ is essentially self-adjoint on $L^2(\mu)$, where $d\mu = Z^{-1}\exp[V]dx$.

Theorem 2.1. *[14] Let $\epsilon > 0$, we have that $\sigma_{ess}(-L) \subset [1/\epsilon, \infty)$ if and only of there exists some decreasing function $b \in C[\epsilon, \infty)$ such that:*

$$\mu(f^2) \leq a\mu(|\nabla f|^2) + b(a)\mu(|f|)^2, \qquad a > \epsilon \tag{2.4}$$

Consequently, $\sigma_{ess}(-L) = \emptyset$ if and only if $\epsilon = 0$.

Corollary 2.2. *Let $f \in L^1(\mu)$. If $\sigma_{ess}(-L) = \emptyset$, then:*

$$\frac{\|f\|_2^2}{\|f\|_1^2} \leq \psi\Big(\frac{\mathcal{E}(f)}{\|f\|_1^2}\Big) \tag{2.5}$$

for some positive $\psi(x) \in C[0,\infty)$ with $x^{-1}\psi(x) \to 0$ as $x \to \infty$.

Proof. It is sufficient to show that (2.4) with $\epsilon = 0$ implies (2.5). Picking $\psi(x) := \inf\{ax + b(a) : a > 0\}$, then (2.4) implies (2.5). □

Remark 2.3. If $\|f\|_1 = 1$, then (2.4) and (2.5) are equivalent by picking $b(x) = \sup_{s\geq 0}\{xs - \psi(s)\}$.

Equivalently,

$$\phi(\frac{\|f\|_2^2}{\|f\|_1^2}) \leq \frac{\mathcal{E}(f)}{\|f\|_1^2} \tag{2.6}$$

where ϕ is an increasing and convex function. The following proposition implies that (2.6) leads to ultracontractive boundedness.

Proposition 2.4. *[15] Let $(P_t)_{t\geq 0}$ be a strongly continuous semigroup on $L^p, 1 \leq p \leq \infty$, with infinitesimal generator L. Suppose that $(P_t)_{t\geq 0}$ is equicontinuous on L^1 and L^∞,i.e.,*

$$\sup_t \|T_t\|_{1\to 1}, \sup_t \|T_t\|_{\infty\to\infty} \leq M < \infty$$

and that:

$$\phi(\|f\|_2^2) \leq \Re(Lf, f), \qquad \forall f \in \mathscr{D}(L), \ \ \|f\|_1 \leq M, \tag{2.7}$$

for some continuous functions $\phi : (0,\infty) \to (0,\infty)$ sarisfying $U(t) = \int_t^\infty \frac{1}{\phi(x)}dx < \infty$. Then $(P_t)_{t\geq 0}$ is ultracontractive and:

$$\|P_t\|_{1\to\infty} \leq U^{-1}(t), \qquad \forall t > 0$$

where U^{-1} means the inverse function.

Proof. Let $f \in \mathscr{D}(L)$ and $t > 0$. By our assumption that $\|P_t f\|_1 \leq M\|f\|_1$, substituting f by $P_t f$ in (2.7), then for some continuous function $\phi : (0,\infty) \to (0,\infty)$.

$$\phi(\|P_t f\|_2^2) \leq -\frac{1}{2}\frac{d}{dt}\|P_t f\|_2^2, \qquad \forall f \in \mathcal{D}(L) \ \ \|f\|_1 = 1$$

following the definition of the infinitesimal generator. Let $S(t) = \|P_t f\|_2^2$, then:

$$S'(t) \leq -2\phi(S(t)), \qquad t \geq 0$$

Since ϕ is positive, moving $\phi(S(t))$ to the left-hand side without changing the inequality and integrating on both sides, by changing of variables, we obtain

$$\int_0^t -\frac{S'(s)}{\phi(S(s))}ds \geq 2t \implies \int_{S(t)}^{S(0)} \frac{1}{\phi(x)}dx \geq 2t$$

Note that this changing of variables is valid, because $S(t)$ is decreasing and differentiable. Thus, it is a diffeomorphism. Moreover, by the definition of $U(t)$

$$\int_{U^{-1}(2t)}^{\infty} \frac{1}{\phi(x)} dx = 2t$$

Recalling U is decreasing, then $\|P_t f\|_2^2 = S(t) \leq U^{-1}(2t)\|f\|_1^2$, with $\|f\|_1 = 1$.

$$\|P_t\|_{1\to 2}^2 \leq U^{-1}(2t)$$

Apply the same argument to P_t^*,

$$\|P_t\|_{2\to\infty}^2 \leq U^{-1}(2t)$$

The results follows that $\|P_t\|_{1\to\infty} \leq \|P_{t/2}\|_{1\to 2}\|P_{t/2}\|_{2\to\infty} \leq U^{-1}(t)$. □

In general, the classical Nash inequality is not adapted to the case where the kernel density is not uniformly bounded, e.g. the Ornstein-Uhlenbeck semigroup on $\mathbb{R}^n$. In [13], D. Bakry, F. Bolley, I. Gentil and P. Maheux introduce the idea of adding a special kind of weighted function to obtain a new type inequality of the form:

$$\phi\left(\frac{\|f\|_2^2}{\|fV\|_1^2}\right) \leq \frac{\mathcal{E}(f)}{\|fV\|_1^2}$$

where V is a positive function. They called this a weighted Nash inequality.

Definition 2.5. Let V be a positive function on E, let M be a non-negative real number and let ϕ be a positive function defined on (M, ∞) with $\phi(x)/x$ non decreasing. The Dirichlet form $\mathcal{E}_\mu$ satisfies a weighted Nash inequality with weight V and rate function ϕ if:

$$\phi\left(\frac{\|f\|_2^2}{\|fV\|_1^2}\right) \leq \frac{\mathcal{E}_\mu(f)}{\|fV\|_1^2}$$

for all functions in the domain of the Dirichlet form such that:

$$\|f\|_2^2 > M\|fV\|_1^2$$

Definition 2.6. A Lyapunov function is a positive function V on E in the domain of the generator L such that:

$$LV \leq cV$$

for a real number constant c, called the Lyapunov constant.

By adding a Lyapunov constant, a bound $\|P_t f\|_2 \leq K(t)\|fV\|$ can be obtained.

Theorem 2.7. *[13] Let $(P_t)_{t\geq 0}$ be a Markov semigroup on E with generator symmetric in $L^2(\mu)$. Assume that there exists a Lyapunov function V in $L^2(\mu)$ with Lyapunov constant $c \geq 0$, and that the Dirichlet form $\mathcal{E}_\mu$ satisfies a weighted Nash inequality with weight V and rate function ϕ on $(M, +\infty)$ such that:*

$$\int_0^\infty \frac{1}{\phi(x)} dx < \infty$$

Then:

$$\|P_t f\|_2 \leq K(2t)\exp(ct)\|fV\|_1$$

for all $t > 0$ and all $f \in L^2(\mu)$. Thus, the function K is defined by:

$$K(x) = \begin{cases} \sqrt{U^{-1}(x)} & \text{if } 0 < x < U(M) \\ \sqrt{M} & \text{if } x \geq U(M) \end{cases}$$

with U defined as:

$$U(x) = \int_x^\infty \frac{1}{\phi(u)} du$$

Proof. Suppose $f \in L^2(\mu)$. Let $H(t) = \int VP_t f d\mu$, where $VP_t f \in \mathscr{D}(L)$. This guarantees the integration by parts formula holds, as follows:

$$H'(t) = \int VLP_t f d\mu = \int LVP_t f d\mu \leq cH(t), \forall t \geq 0$$

where c is the Lyapunov constant. Move $H(t)$ to the left and then integrate on both sides over t. It follows that:

$$\int VP_t f d\mu \leq \exp(ct) \int V f d\mu, \forall t \geq 0 \tag{2.8}$$

Next, fix $0 \leq t \leq T$ and define a new function of s such that

$$I(s) = \frac{\|P_s f\|_2^2}{\left[\exp(ct) \int fV d\mu\right]^2}$$

for $s \in [0, t]$. Since $\partial_s \|P_s f\|_2^2 = -2\mathcal{E}(P_s f)$,

$$-\frac{I'(s)}{2} = \frac{\mathcal{E}_\mu(P_s f)}{\left(e^{ct} \int fV d\mu\right)^2} = \frac{\mathcal{E}_\mu(P_s f)}{\left(\int P_s fV d\mu\right)^2} \left(\frac{\int P_s fV d\mu}{e^{ct} \int fV d\mu}\right)^2 \tag{2.9}$$

Note that $I(s)$ is decreasing in terms of s. There are two cases we need to discuss. If there exists $s \in [0,t]$ such that $I(s) \leq M$, then $I(t) \leq I(s) \leq M$. The result follows. Assume now, $I(s) \geq M$ on $[0,t]$. By (2.8),

$$\begin{aligned}\frac{\|P_s f\|_2^2}{\left(\int fVd\mu\right)^2} &= \frac{\|P_s f\|_2^2}{\left[\exp(ct)\int fVd\mu\right]^2}\frac{\left[\exp(ct)\int fVd\mu\right]^2}{\left(\int VP_s fd\mu\right)^2}\\ &= I(s)\exp(2c(t-s))\frac{\left[\exp(cs)\int fVd\mu\right]^2}{\left(\int VP_s fd\mu\right)^2} \geq M\exp(2c(t-s))\end{aligned}$$

with the Lyapunov constant $c \geq 0$. Applying the weighted Nash inequality to P_s,

$$\phi\Big(\frac{\|P_s f\|_2^2}{(\int VP_s fd\mu)^2}\Big) \leq \frac{\mathcal{E}_\mu(P_s f)}{\left(\int VP_s fd\mu\right)}$$

By (2.9),

$$\phi\Big(\frac{\|P_s f\|_2^2}{(\int VP_s fd\mu)^2}\Big)\Big(\frac{\int VP_s fd\mu}{\exp(ct)\int fVd\mu}\Big)^2 \leq -\frac{I'(s)}{2}$$

Since the rate function $\phi(x)/x$ is non decreasing, with $\frac{\|P_s f\|_2^2}{(\int VP_s fd\mu)^2} \geq \frac{\|P_s f\|_2^2}{[\exp(ct)\int Vfd\mu]^2}$

$$\phi\Big(\frac{\|P_s f\|_2^2}{(\int VP_s fd\mu)^2}\Big)/\Big(\frac{\|P_s f\|_2^2}{(\int VP_s fd\mu)^2}\Big) \geq \phi\Big(\frac{\|P_s f\|_2^2}{[\exp(ct)\int fVd\mu]^2}\Big)/\Big(\frac{\|P_s f\|_2^2}{[\exp(ct)\int fVd\mu]^2}\Big)$$

It follows that:

$$-\frac{I'(s)}{2} \geq \phi(I(s))$$

We thus have:

$$U(I(s))' \geq 2$$

Integrating over t on both sides:

$$U(I(t)) \geq 2t$$

Since U is decreasing, then U^{-1} is decreasing and defined in $(0, U(M)]$. Therefore,

$$I(t) \leq U^{-1}(2t)$$

for all $t \leq U(M)/2$. If $t \geq U(M)/2$, then $I(t) \leq I(U(M)/2) \leq M$. □

Example 2.8. Consider a differential operator L on $\mathbb{R}^n$

$$L = \nabla \cdot (A\nabla \cdot)$$

where $A = (a_{i,j})$ is symmetric, smooth with bounded derivatives of all orders and uniformly elliptic. The operator A is said to be uniformly elliptic on $\mathbb{R}^n$ if there exists a constant $\theta > 0$ such that:

$$\sum_{i,j}^{n} a_{i,j}(x)\xi_i\xi_j \geq \theta|\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and almost every x in $\mathbb{R}^d$. Take $f \in C_c^\infty(\mathbb{R}^n)^+$ such that $\int f dx = 1$, where dx is the Lebesgue measure. Thus, $\|P_t f\|_1 = \|f\|_1$. Let $h(t) = \int (P_t f)^2 dx$, then using classical Nash inequality we have:

$$\begin{aligned}
h(t)' &= 2\int P_t L P_t f dx \\
&= -2\int \nabla(P_t f) \cdot (A\nabla(P_t f))dx \quad \text{(By integration by parts)} \\
&\leq -2\theta \int |\nabla(P_t f)|^2 dx \quad \text{(By the definition of uniform ellipticity)} \\
&\leq -C\|P_t f\|_2^{2+4/n} \|P_t f\|_1^{-4/n} \\
&= -Ch(t)^{1+2/n}
\end{aligned}$$

It is equivalent to an inequality:

$$[h(t)^{-2/n}]' \geq C_{n,\theta}$$

Integrating on both sides from 0 to t,

$$h(t)^{-2/n} - h(0)^{-2/n} \geq C_{n,\theta}$$

Since $h(0)$ is a constant independent of t, then:

$$h(t) \leq \left(\frac{1}{C_{n,\theta}}\right)^{\frac{n}{2}} t^{-\frac{n}{2}}, \quad \forall t \geq 0$$

Equivalently,

$$\|P_t f\|_2 \leq C_{n,\theta} t^{-\frac{n}{4}} \|f\|_1 \tag{2.10}$$

for all $f \in (C_c^\infty(\mathbb{R}^n))^+$ with $\|f\|_1 = 1$. We can extend to any function $f \in C_c^\infty(\mathbb{R}^n)$ by writing it as $f = f^+ - f^-$.

We can understand (2.10) as a Nash weighted inequality with weight $V = 1$, the rate function $\phi(x) = Cx^{1+2/n}$ and $c = 0$ in Theorem 2.7.

Note that the measure μ need not be a probability measure. A converse to Theorem (2.7) is the following.

Theorem 2.9. *[13] Let $(P_t)_{t\geq 0}$ be a Markov semigroup on E with generator L symmetric in $L^2(\mu)$. Assume that there exists a positive function V and a positive function K defined on $(0,\infty)$ such that:*

$$\|P_t f\|_2 \leq K(t)\|fV\|_1$$

for all $t > 0$, then the weighted Nash inequality holds with the same function $V, M = 0$ and the function:

$$\phi(x) = \sup_{t>0} \frac{x}{2t} \log \frac{x}{K(t)^2}, \qquad \forall x \geq 0$$

Proof. Let $h(t) = \|P_t f\|_2^2$, then $h'(t) = 2\int_E P_t f L(P_t f) d\mu = -2\mathcal{E}(P_t f)$ and $h''(t) = 4\int_E L(P_t f)L(P_t f)d\mu$ by integration by parts. By Cauchy Schwartz inequality:

$$h'^2(t) = \Big(2\int_E P_t f L(P_t f) d\mu\Big)^2 \leq \Big(\int_E (P_t f)^2 d\mu\Big)\Big(4\int_E L(P_t f)L(P_t f)d\mu\Big) = hh''$$

It follows that:

$$(\log h)'' = \frac{h''h - h'^2}{h^2} \geq 0$$

Equivalently,

$$\frac{1}{u}[\log h(u) - \log h(0)] \leq \frac{1}{t}[\log h(t) - \log h(0)]$$

for $0 < u < t$. Let $u \to 0$,

$$\frac{h'(0)}{h(0)} \leq \frac{1}{t}\log\frac{h(t)}{h(0)}$$

Since $h'(0) = -2\mathcal{E}(f)$ and $\|fV\|_1^2 > 0$, by our assumption that $h(t) \leq K^2(t)\|fV\|_1^2$ we have,

$$-2\frac{\mathcal{E}(f)}{\|fV\|_1^2} \leq \frac{\|f\|_2^2}{\|fV\|_1^2}\frac{1}{t}\log\Big(\frac{h(t)}{\|f\|_2^2}\Big) \leq \frac{\|f\|_2^2}{\|fV\|_1^2}\frac{1}{t}\log\Big(\frac{K^2(t)\|fV\|_1^2}{\|f\|_2^2}\Big), \quad \forall t > 0$$

After rearranging, for all $t > 0$

$$\frac{\mathcal{E}(f)}{\|fV\|_1^2} \geq -\frac{\|f\|_2^2}{\|fV\|_1^2}\frac{1}{2t}\log\Big(\frac{K^2(t)\|fV\|_1^2}{\|f\|_2^2}\Big) = \frac{\|f\|_2^2}{\|fV\|_1^2}\frac{1}{2t}\log\Big(\frac{\|f\|_2^2}{\|fV\|_1^2}\cdot\frac{1}{K^2(t)}\Big)$$

Therefore, the weighted Nash inequality holds with

$$\phi(x) = \sup_{t>0}\frac{x}{2t}\log\frac{x}{K(t)^2}, \quad \forall x > 0$$

□

Remark 2.10. Theorem (2.7) and Theorem (2.9) build up a quantitative equivalence between the weighted Nash inequality and the bound on $\|P_t f\|_2$.

Bakry, Bolley, Gentil and Maheux in [13] develop a universal weighted Nash inequality on $\mathbb{R}^n$ for the operator $Lf = \Delta f + \nabla \log \rho \cdot \nabla f$ with a specific weighted function $V = \rho^{-1/2}$ and the measure $d\nu = \rho(x)dx$, given that $\rho : \mathbb{R}^n \to \mathbb{R}$ is a positive smooth function. In this case, V is not necessarily a L^2 function.

Theorem 2.11. *[13] In the context above, the classical Nash inequality is:*

$$\|f\|_2^{2+\frac{4}{n}} \leq C_n^{\frac{4}{n}} \|fV\|_1^{\frac{4}{n}} \Big(\mathcal{E}(f,f) + \int_{\mathbb{R}^n} \frac{LV}{V} f^2 d\nu\Big)$$

for all smooth functions f on $\mathbb{R}^n$, with compact support. Moreover, if $LV \leq cV$ for $c \in \mathbb{R}$, then:

$$\|f\|_2^{2+\frac{4}{n}} \leq C_n^{\frac{4}{n}} \|fV\|_1^{\frac{4}{n}} \Big(\mathcal{E}(f,f) + c\int_{\mathbb{R}^n} f^2 d\nu\Big)$$

where $\|.\|_p$ represents the $L^p(d\nu)$ norm.

Proof. Let g be a smooth function with compact support and let $f = g\sqrt{\rho}$. Then:

$$\int_{\mathbb{R}^n} |f|^2 dx = \|g\|^2_{L^2(\nu)}, \quad \int_{\mathbb{R}^n} |f| dx = \|gV\|_{L^1(\nu)}.$$

Moreover,

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla f|^2 dx &= \int_{\mathbb{R}^n} |\nabla(g\sqrt{\rho})|^2 dx = \int_{\mathbb{R}^n} |\frac{1}{V}\nabla g + \nabla(\frac{1}{V})g|^2 dx \\ &= \int_{\mathbb{R}^n} |\nabla g|^2 d\nu + \int_{\mathbb{R}^n} 2\frac{g}{V}\nabla g \cdot \nabla\frac{1}{V} dx + \int_{\mathbb{R}^n} g^2 \big|\nabla\big(\frac{1}{V}\big)\big|^2 dx\end{aligned}$$

By observation, using integration by parts:

$$\int_{\mathbb{R}^n} 2\frac{g}{V}\nabla g \cdot \nabla\frac{1}{V} dx = \int_{\mathbb{R}^n} \nabla(g^2) \cdot (\frac{1}{V}\nabla\frac{1}{V}) dx = \int_{\mathbb{R}^n} g^2 (\frac{\Delta V}{V} - 3\frac{|\nabla V|^2}{V^2}) d\nu$$

Therefore:

$$\int_{\mathbb{R}^n} |\nabla f|^2 dx = \int_{\mathbb{R}^n} |\nabla g|^2 d\nu + \int_{\mathbb{R}^n} g^2 \Big(\frac{\Delta V}{V} - 2\frac{|\nabla V|^2}{V^2}\Big) d\nu$$

Further,

$$\frac{LV}{V} = \frac{1}{V}(\Delta V - 2\nabla(\log V) \cdot \nabla V) = \frac{\Delta V}{V} - 2\frac{|\nabla V|^2}{V^2}$$

Finally, apply (2.1) to f,

$$\|g\|^{2+\frac{4}{n}}_{L^2(\nu)} = \|f\|^{2+\frac{4}{n}}_{L^2(x)} \leq C_n^{\frac{4}{n}} \|f\|^{\frac{4}{n}}_{L^1(x)} \|\nabla f\|^2_{L^2(x)} = C_n^{\frac{4}{n}} \|gV\|^{\frac{4}{n}}_{L^1(\nu)} \Big(\mathcal{E}(g,g) + c\int_{\mathbb{R}^n} g^2 d\nu\Big)$$

□

The following corollary estimates the kernel density.

Corollary 2.12. *[13] With same notation, assume that* ν *is a probability measure and that* $V \in L^1(\nu)$ *satisfies* $LV \leq cV$ *with* $c \geq 0$*. If the Hessian of* $\log \rho$ *is uniformly bounded from above on* $\mathbb{R}^n$ *and if:*

$$\sup_{|x|=r} \rho(x)^{\frac{1}{2}} r^{n-1} \to 0 \quad \textit{and} \quad \sup_{|x|=r} |\nabla \rho(x)| \rho^{-\frac{1}{2}} r^{n-1} \to 0$$

as r *tends to infinity, then* P_t *has a density* p_t *which satisfies:*

$$p_{2t}(x, y) \leq \frac{d}{t^{\frac{n}{2}}} e^{2ct} V(x) V(y) \tag{2.11}$$

for some $d > 0$ *and for all* $x, y \in \mathbb{R}^n, t > 0$*.*

Proof. The proof is very similar to Theorem 2.7. However, $V \notin L^2(\nu)$. The key step we need to show is:

$$\int_{\mathbb{R}^n} V L P_t f d\nu = \int_{\mathbb{R}^n} (LV) P_t f d\nu$$

Let $r > 0, B_r$ be the centred ball of $\mathbb{R}^n$ with radius r and $\vec{v}$ be its outward unit normal vector. The idea used here is to use integration by parts on ball and let $r \to \infty$. Using integration by parts twice, we have:

$$\begin{aligned}\int_{B_r} V L P_t f d\nu = \int_{B_r} (LV) P_t f d\nu - \int_{S^{n-1}} P_t f(rw) \nabla V(rw) \cdot \vec{v} \rho(rw) r^{n-1} dw \\ + \int_{S^{n-1}} V(rw) \nabla P_t f(rw) \cdot \vec{v} \rho(rw) r^{n-1} dw\end{aligned}$$

Since $\|P_t f\|_\infty \leq \|f\|_\infty$, for fixed f,

$$\begin{aligned}&\int_{S^{n-1}} P_t f(rw) \nabla V(rw) \cdot \vec{v} \rho(rw) r^{n-1} dw \\ &= \int_{S^{n-1}} P_t f(rw) \rho^{-\frac{3}{2}}(rw) \nabla \rho(rw) \cdot \vec{v} \rho(rw) r^{n-1} dw \\ &= \int_{S^{n-1}} P_t f(rw) \rho^{-\frac{1}{2}}(rw) \nabla \rho(rw) \cdot \vec{v} r^{n-1} dw\end{aligned}$$

The second assumption of ρ ensures this terms goes to 0 as $r \to \infty$. To finish our proof, it requires that $\nabla P_t f$ unifrom bound. Referring to Example 1.32, $U(x) = -\log \rho$ in this case. Since the Hessian of $\log \rho$ is uniformly bounded from above by λ on $\mathbb{R}^n$, then by (1.15):

$$\Gamma_2(f) \geq -\lambda \Gamma(f)$$

It follows that L satisfies curvature-dimension criterion $CD(-\lambda, \infty)$. Moreover, L is a special case of Example 1.34 with $U = -\log\rho$ and then it is a diffusion operator. By (2) in Proposition 1.35,

$$|\nabla P_t f| \leq e^{\lambda t} P_t |\nabla f| \leq e^{\lambda t} \|\nabla f\|_\infty$$

□

Chapter 3

Spectral Multiplier Theory

3.1 Continuous Functional Calculus for Bounded Operators

Let $L \in \mathcal{B}(H)$ be a self-adjoint, bounded operator acting on a Hilbert space H.

Definition 3.1. The spectrum $\sigma(L)$ of the operator L is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - L$ is not invertible, which means that it does not have an inverse that is a bounded linear operator.

Given a polynomial of the form:

$$p(z) = \sum_{i=0}^{d} a_i z^i, \forall z \in \sigma(L)$$

We can define $p(L)$ as:

$$p(L) = \sum_{i=0}^{d} a_i L^i \in \mathcal{B}(H)$$

Can we extend this definition to any continuous function? We can approach any $F \in C(\sigma(L))$ uniformly by polynomials. Building up a general definition requires the following theorem.

Theorem 3.2. *(Weierstrass Approximation Theorem) Let $X \subset \mathbb{R}$ be a compact set of real numbers with the induced topology. Then the space of polynomials on X is dense in $C(X)$ for the L^∞ norm.*

We want to define for any $F \in C(\sigma(L))$,

$$F(L) = \lim_{n \to \infty} p_n(L) \tag{3.1}$$

where (p_n) is a sequence of polynomials in $\mathbb{C}(X)$ such that $\|F - p_n\|_{C(X)} \to 0$. The following theorem provides basic properties of such a continuous functional calculus.

Theorem 3.3. *Let H be a Hilbert space and $L \in \mathcal{B}(H)$ be a self-adjoint bounded operator. There exists a unique linear map:*

$$\phi = \phi_L : C(\sigma(L)) \to \mathcal{B}(H)$$

with the following properties:

1. *This map is a Banach-algebra isometric homomorphism*

$$\phi(F_1 F_2) = \phi(F_1)\phi(F_2) \tag{3.2}$$

for all $F_1, F_2 \in C(\sigma(L))$, $\phi(Id) = Id$ and

$$\|\phi(F)\| = \|F\|_{C(\sigma(L))}, \forall F \in C(\sigma(L))$$

2. *For any $F \in C(\sigma(L))$, we have $\phi(f)^* = \phi(\bar{f})$ and $F(L)$ is normal for all $F \in C(\sigma(L))$. Moreover,*

$$F \geq 0 \implies \phi(F) \geq 0$$

3. *If $\lambda \in \sigma(L)$ is in the point spectrum and $h \in Ker(\lambda - L)$, then $h \in Ker(F(\lambda) - F(L))$. The point spectrum is defined as the set of eigenvalues.*

4. *More generally, we have the spectral mapping theorem*

$$\sigma(F(L)) = F(\sigma(L)) = \sigma(F) \tag{3.3}$$

where $\sigma(F)$ is computed for $F \in C(\sigma(L))$.

In particular, the following lemma corresponds to the case when F is a polynomial in $\mathbb{C}[X]$.

Lemma 3.4. *Let H be a Hilbert space.*
(1) For any $L \in \mathcal{B}(H)$ and any polynomial $p \in \mathbb{C}[X]$, define $\phi(p) = p(L) \in \mathcal{B}(H)$. Then we have:

$$\sigma(\phi(p)) = p(\sigma(L))$$

(2) Let $L \in \mathcal{B}(H)$ be normal and let $p \in \mathbb{C}[X]$ be polynomial. Then we have:

$$\|\phi(p)\|_{\mathcal{B}(H)} = \|p\|_{C(\sigma(L))}$$

Proof. For any $\lambda \in \mathbb{C}$, the polynomial $p(X) - \lambda$ in $\mathbb{C}[X]$ with degree n can be factorised as:

$$p(X) - \lambda = \alpha \prod_{1 \leq i \leq n} (X - \lambda_i)$$

for some $\alpha \in \mathbb{C}^{\times}$ and complex number $\lambda_i \in \mathbb{C}$. Since the map $p \to p(T)$ is an algebra homomorphism, then

$$p(L) - \lambda \cdot Id = \alpha \prod_{1 \leq i \leq n} (L - \lambda_i) \tag{3.4}$$

Assume that $\lambda \notin p(\sigma(L))$, then $\lambda \neq p(\lambda_i)$ for all $1 \leq i \leq n$ and thus λ_i are not in $\sigma(L)$. Thus, $(T - \lambda_i)$ is invertible for each i and so is $p(L) - \lambda$. By contrapositive, we have:

$$\sigma(p(L)) \subset p(\sigma(L))$$

Conversely, if $\lambda \in p(\sigma(L))$, then one of $\lambda_i \in \sigma(L)$. Hence, (3.4) implies $p(L) - \lambda$ is not invertible. It follows that $\lambda \in \sigma(p(L))$, which finishes the proof of (1).
For (2), let L be a normal operator. Since the map ϕ is an algebra homomorphism, then

$$p(L)^* p(L) = \phi(\bar{p})\phi(p) = \phi(\bar{p}p) = \phi(p\bar{p}) = p(L)p(L)^*$$

It follows that $p(L)$ is a normal operator. By the spectral radius formula, we have

$$\|p(L)\| = r(p(L)) = \max_{\lambda \in \sigma(p(L))} |\lambda|$$

Applying the result from part (1) such that $\sigma(p(L)) = p(\sigma(L))$, we have:

$$\|p(L)\| = \max_{\lambda \in \sigma(p(L))} |\lambda| = \max_{\lambda \in p(\sigma(L))} |\lambda| = \max_{\mu \in \sigma(L)} |p(\mu)| = \|p\|_{C(\sigma(L))}$$

□

Remark 3.5. The above lemma shows that the map

$$\Phi : (\mathbb{C}[X], \|.\|_{C(\sigma(L))}) \to \mathcal{B}(H)$$

on polynomials is linear, continuous and isometric. It allows us pass to the limit to any continuous function in the norm of $\|.\|_{C(\sigma(L))}$.

Now we start to prove the four properties in Theorem 3.3.

Proof. For (1), equality (3.2) holds for all polynomials. Let $(p_n)_{n=1}^{\infty}$ be a sequence of polynomials satisfying

$$\|F - p_n\|_{C(X)} \to 0$$

By the continuity of ϕ , passing the limit, so equality (3.2) holds for all continuous function F.
For (2), using the same argument for polynomials before, we can show $F(L)$ is always normal. If F is real valued, then $F(L)$ is self-adjoint. Moreover, if $F \geq 0$, then there exists a continuous function $f \geq 0$ on $\sigma(L)$ such that:

$$F = f^2$$

It follows that $f(L)$ is well defined. Since f is real valued, then $f(L)$ is self-adjoint. Hence, for all $h \in H$

$$\begin{aligned} \big\langle F(L)h, h \big\rangle &= \big\langle f^2(L)h, h \big\rangle \\ &= \big\langle f(L)h, f(L)^* h \big\rangle \\ &= \|f(L)h\|^2 \geq 0 \end{aligned}$$

It finishes the proof of (2).
For (3), assume that $h \in \mathrm{Ker}(\lambda - L)$, then $Lh = \lambda h$. By induction, $L^n(h) = \lambda^n h$. It follows that $p_n(L)h = p_n(\lambda)h$ for any polynomial p_n. Writing F as the uniform limit of a sequence of polynomials again, $F(L)h = F(\lambda)h$ by continuity. In other words, $h \in \mathrm{Ker}(F(\lambda) - F(L))$.
For (4), assume that $\lambda \notin F(\sigma(L))$, then the function:

$$f(x) = \frac{1}{F(x) - \lambda}$$

is continuous. Therefore, $f(F - \lambda) = (F - \lambda)f = 1$ are valid in $C(\sigma(L))$. By the result of property 1,

$$f(L)(F(L) - \lambda) = (F(L) - \lambda)f(L) = Id$$

Hence, $f(L) = (F(L) - \lambda)^{-1}$ is well-defined. So, λ is not in the spectrum of $F(L)$.
Conversely, pick $\lambda \in \sigma(L)$. There are three cases to discuss. Since L is self-adjoint, then the residual spectrum is empty. If λ is in the point spectrum, then (3) implies $F(\lambda)$ is in the point spectrum of $F(L)$. Note that if λ is in the continuous spectrum, then there exists a vector $v \in H$ with $\|v\|_H = 1$ such that

$\|(L-\lambda)v\|$ is arbitrary small, say is less than ϵ. Fix this ϵ, finding a polynomial p such that $\|f-p\|_{C(\sigma(L))} \leq \epsilon/3$ and $|p(\lambda)-F(\lambda)| \leq \epsilon/3$. Since λ is root of $p(X)-p(\lambda)$, then

$$p(X)-p(\lambda) = q(X)(X-\lambda)$$

for some fixed $q(X) \in \mathbb{C}[X]$. Therefore, $q(L)$ is bounded and

$$\begin{aligned}
\|(F(L)-F(\lambda))v\| &\leq \|(F(L)-p(L))v\| + \|(p(L)-p(\lambda))v\| + \|(p(\lambda)-F(\lambda))v\| \\
&\leq \|F(L)-p(L)\|_{\mathcal{B}(H)}\|v\| + \|q(L)(L-\lambda)v\| + |p(\lambda)-F(\lambda)|\|v\| \\
&\leq \|F-p\|_{C(\sigma(L))} + \|q(L)\|\|(L-\lambda)v\| + |p(\lambda)-F(\lambda)| \\
&\leq \frac{2}{3}\epsilon + \epsilon\|q\|_\infty
\end{aligned}$$

It implies that $F(\lambda)$ is in the continuous spectrum of $F(L)$. So, $F(\sigma(L)) \subset \sigma(F(L))$. Note that spectrum is always compact [18, Theorem 10.13 on p.253]. Given $C(\sigma(L))$ with the supremum norm, this Banach space is a Banach algebra. Thus, for $F \in C(\sigma(L))$,

$$\sigma(F) = F(\sigma(L))$$

□

3.2 Spectral Measures for Bounded Operators

Using the properties of the continuous functional calculus, we can construct a measure which relates an operator to its spectrum. The following theorem is needed to build such a measure.

Theorem 3.6. *[18](Riesz-Markov-Kakutani Theorem) Let X be a locally compact topological space, let $C_b(X)$ be the Banach space of bounded functions on X with L^∞ norm. Let $l : C_b(X) \to \mathbb{C}$ be a linear map such that $F \geq 0$ implies $l(F) \geq 0$. Then there exists a unique Randon measure μ such that*

$$\int_X F(x)d\mu(x) = l(f)$$

for all $f \in C_b(X)$.

The Riesz-Markov-Kakutani theorem represents positive linear functionals on spaces of continuous functions on a locally compact space.

Proposition 3.7. *Let H be Hilbert space. Let $L \in \mathcal{B}(H)$ be a self-adjoint operator and $h \in H$. There exists a unique positive Radon measure μ on $\sigma(L)$, depending on L and h such that:*

$$\int_{\sigma(L)} F(x)d\mu(x) = \langle F(L)h, h \rangle$$

for all $F \in C(\sigma(L))$. In particular,

$$\mu(\sigma(T)) = \|h\|^2$$

so μ is a finite measure. This measure is called the spectral measure associated with h and L.

Proof. Define a linear functional $l : C(\sigma(L)) \to \mathbb{C}$:

$$l : F \mapsto \langle F(L)h, h \rangle$$

By 2 of Theorem 3.3, $F \geq 0$ implies $l(F) \geq 0$. Hence, by Theorem 3.6, there exists a unique Radon measure μ on $\sigma(L)$ such that

$$l(f) = \langle F(L)h, h \rangle = \int_{\sigma(L)} F(x)d\mu(x)$$

for all $F \in C(\sigma(L))$. □

Example 3.8. Let L be a compact self-adjoint operator acting on a separable Hilbert space$H$. For any $h \in H$, we write:

$$L(h) = \sum_{n\geq 1} \lambda_n \langle h, e_n \rangle$$

where λ_n are eigenvalues and e_n are corresponding eigenvectors. Since L is a compact operator, then for all continuous function F, $F(L)$ is defined on $\sigma(L) = \{0\} \cup \{\lambda_n\}$ by:

$$F(T)h = F(0)P_0(h) + \sum_{n\geq 1} F(\lambda_n)\langle h, e_n \rangle e_n$$

where P_0 is the orthogonal projection on $\mathrm{Ker}(L)$. By the definition of the spectral measure,

$$\int_{\sigma(L)} F(x)d\mu(x) = F(0)\|P_0(h)\|_2^2 + \sum_{n\geq 1} F(\lambda_n)\left|\langle h, e_n \rangle\right|^2$$

This expression implies that the unique Radon measure μ is a sequence of Dirac measures at each eigenvalue.

3.3 The Spectral Theorem for Bounded Self-adjoint Operators

Once the spectral measure is defined, we can describe a bounded self-adjoint operator by using its spectrum and the functional calculus. Let H be a Hilbert space. Let $L \in \mathcal{B}(H)$ be a self-adjoint operator and $h \in H$. Applying Proposition 3.7 to continuous function $|F|^2 = F\bar{F}$, then

$$\|F(L)h\|^2 = \int_{\sigma(L)} |F(x)|^2 d\mu_h(x) = \|F\|^2_{L^2(\sigma(L),\mu_h)}$$

It implies that the map $(C(\sigma(L)), \|.\|_{L^2}) \to H$ defined as

$$F \mapsto F(L)h$$

is isometric. Note that continuous functions are dense in $L^2(\sigma(L), \mu_h)$. There exists an isometric extension U such that

$$U : L^2(\sigma(L), \mu_h) \to H$$

Let $H_h = \mathrm{Im}(U)$. It is closed because U is an isometry. Suppose p is a polynomial, then

$$L(U(p)) = L(p(L)h) = L(\sum_i \alpha_i L^i)(h) = \sum_i \alpha_i L^{i+1}(h) = (xp)(L)(h) \in H_h \tag{3.5}$$

It follows that $L(U(F)) \in H_h$ for all continuous functions F. So, $L(H_h) \subset H_h$. Denote L_h be the restriction of L to H_h. Hence,

$$U : L^2(\sigma(L_h), \mu_h) \to H_h$$

is an isomorphism. Therefore,

$$L_h(U(F)) = (xF)(L)(h) = U(xF)$$

There exists a operator $S = U^{-1}L_hU$ on $L^2(\sigma(L), \mu_h)$. Further, we extend polynomials to functions in $L^2(\sigma(L), \mu_h)$ in (3.5), then

$$S(F)(x) = xF(x) \in L^2$$

It is a simple case of spectral theorem if H_h is the whole space for some vector h, which shows a bounded self-adjoint operator can be represented by a multiplication operator via spectrum measures. Now we consider a general case.

Definition 3.9. Let H be a Hilbert space and let $L \in \mathcal{B}(H)$. A vector $h \in H$ is called a cyclic vector for L if the vector $\{L^n(h)\}_{n=0}^{\infty}$ span a dense subset of H for all $n \geq 0$.

Note that a vector h is cyclic for a self-adjoint operator if and only if $H_h = H$, where H_h is defined as above.

Lemma 3.10. *[17, Lemma 2 on p.226] Let H be a Hilbert space and let $L \in \mathcal{B}(H)$ be a self-adjoint operator. There exists a family $(H_i)_{i \in I}$ of non-zero, pairwise orthogonal, closed subspaces of H such that $H = \bigoplus_{i=1}^{N} H_i$ with $N = 1, 2, ...\infty$. Moreover, $L(H_i) \subset H_i$ and L restrict to H_i is a bounded self-adjoint operator for all $i \in I$, which has a cyclic vector.*

Note that if H is separable, then the index set I is either finite or countable.

Theorem 3.11. *Let H be a separable Hilbert space and L be a bounded self-adjoint operator. There exists a finite measure space (X, μ), a unitary operator:*

$$U : H \to L^2(X, \mu)$$

and a bounded function $g \in L^{\infty}(X, \mu)$, such that:

$$M_g \circ U = U \circ L$$

where M_g is the multiplication operator on $L^2(X, \mu)$ defined by $M_g f = gf$ for all $f \in L^2(X, \mu)$.

Proof. By the previous lemma, there exists a finite or countable family of $(H_n)_{n \geq 1}$ such that H is the orthogonal direct sum of H_n and the operator L has a cyclic vector $\overline{h}_n$ on H_n. Let $h_n = 2^{-n/2} \frac{\overline{h}_n}{\|\overline{h}_n\|}$. Let $\mu_n = \mu_{h_n}$ be the spectral measure associated with h_n such that:

$$\mu_n(\sigma(L)) = \|h_n\|^2 = 2^{-n}$$

Define a product space with product topology,

$$X = \{1, 2..., n...\} \times \sigma(L)$$

with

$$\mu_n(S) = \mu(\{n\} \times S)$$

for all $n \geq 0$ and a measurable subset S of $\sigma(L)$. In addition,

$$\mu(X) = \sum_{n\geq 1} \mu_n(\sigma(L)) = \sum_{n\geq 1} 2^{-n} < \infty$$

So, (X, μ) is a finite measure space. As we shown in the first part of this section, we have unitary maps

$$U_n; L^2(\sigma(L), \mu_n) \to H_n$$

for each H_n such that $U_n^{-1} L U_n = M_x$. Consider a surjective map $V : L^2(X, \mu) \to \bigoplus_{n\geq 1} L^2(\sigma(L), \mu_n)$ mapping f to $f(n, x)$ and:

$$\int_X f(x) d\mu(x) = \sum_{n\geq 1} \int_{\sigma(L)} f(n, x) d\mu_n(x)$$

We construct U by defining:

$$U(\sum_{n\geq 1} v_n) = V^{-1}(\sum_{n\geq 1} U_n^{-1}(v_n))$$

for all $v_n \in H_n$. Since all H_n span H, thus it is a unitary map defined on H with inverse:

$$U^{-1}(f) = \sum_{n\geq 1} U_n(f(n, x))$$

Finally, note that the n-th component of $U(\sum_{n\geq 1} U_n(f(n, x)))$ is $f(n, x)$. Therefore, the n-th component of $U(L(\sum_{n\geq 1} U_n(f(n, x))))$ is

$$U_n^{-1}(L U_n f(n, x)) = x f(n, x)$$

It implies that $U \circ T = M_g \circ U$ where g is a bounded and measurable function such that

$$g = \begin{cases} X \to \mathbb{C} \\ (n, x) \mapsto x \end{cases}$$

□

3.4 Spectral Theorem for Bounded Normal Operators

Another way to interpret the spectral theorem is using resolutions of the identity (or projection-valued measures).

Definition 3.12. Let H be a Hilbert space and $\mathscr{M}$ be a σ-algebra over a set Ω. A resolution of the identity on $\mathscr{M}$ is a mapping:

$$E : \mathscr{M} \to \mathcal{B}(H)$$

with the following properties. For all $\omega, \omega', \omega'' \in \mathscr{M}$:

(a) $E(\emptyset) = 0, E(\Omega) = I$.

(b) Each $E(\omega)$ is a self-adjoint projection

(c) $E(\omega' \cap \omega'') = E(\omega')E(\omega'')$

(d) If $\omega' \cap \omega'' = \emptyset$, then $E(\omega' \cup \omega'') = E(\omega') + E(\omega'')$.

(e) For every $x, y \in H$, the function $E : \mathscr{M} \to \mathbb{C}$ defined by:

$$E_{x,y}(\omega) = \big\langle E(\omega)x, y \big\rangle$$

is a complex measure.

Remark 3.13. Property (b) implies $E_{x,x}(\omega) = \big\langle E(\omega)x, x \big\rangle = \|E(\omega)x\|^2$ for all $x \in H$. Each $E_{x,x}$ is a positive measure with total variation $\|E_{x,x}\| = E_{x.x}(\Omega) = \|x\|^2$. Property (c) shows that any two of the projections commute. Properties (a) and (c) imply that $E(\omega')$ and $E(\omega'')$ are orthogonal to each others if $\omega' \cap \omega'' = \emptyset$.

There are some further important consequences.

Proposition 3.14. *Consider a sequence of the disjoint sets* $(\omega_n) \subset \mathscr{M}$ *such that* $\omega = \bigcup_{n \geq 1} \omega_n$. *For fixed* $x \in H$,

$$\sum_{n=1}^{\infty} E(\omega_n)x = E(\omega)x$$

Proof. Properties (a) and (c) imply that $\{E(\omega_n)x\}_{n \geq 1}$ is a sequence of pairwise orthogonal vectors in H. By property (e),

$$\sum_{n=1}^{\infty} \big\langle E(\omega_n)x, y \big\rangle = \big\langle E(\omega)x, y \big\rangle$$

for all $y \in H$. Since x is fixed, then $\sum_{n=1}^{\infty} E(\omega_n)$ converges to $E(\omega)$ in the strong operator topology on $\mathcal{B}(H)$. $\square$

For some functions, we can construct a bounded operator via resolutions of identity. Let f be a complex $\mathscr{M}$-measurable function on Ω. Suppose that there exists a countable sequence of open discs $\{D_i\}$ such that $E(f^{-1}(D_i)) = 0$. Let V be the union of D_i. Thus, $E_{x,x}(f^{-1}(D_i)) = 0$ for all $x \in H$. By countably additivity of the norm topology of H, $E_{x,x}(f^{-1}(V)) = 0$ and then $E(f^{-1}(V)) = 0$. The complement of V is called the essential range of f. We can give a topology to the space of such f by defining the norm as the largest value through the essential range of f, which is denoted by $\|.\|_\infty$. Let B be the algebra of all bounded complex $\mathscr{M}$-measurable functions on Ω with norm:

$$\|f\| = \sup\{|f(p)| : p \in \Omega\}$$

and a subset $N \subset B$ such that

$$N = \{f \in B : \|f\|_\infty = 0\}$$

Hence, we denote B/N by $L^\infty(E)$. Indeed, it is a Banach algebra. Note that the Banach algebra is an associative algebra over the complex numbers and also a Banach space with a norm satisfying multiplicative inequality. The norm of any coset $[f]$ is equal to $\|f\|_\infty$.
Now, we can construct a operator from resolutions of the identity.

Theorem 3.15. *[18, Theorem 12.21 on p.319] Let H be a Hilbert space and let E be a resolution of the identity. For any bounded Borel function $f \in L^\infty(E)$, there exists an isometric-isomorphism $\Phi : L^\infty(E) \to A$, given A is a closed normal subalgebra of $\mathcal{B}(H)$ such that:*

$$\langle \Phi(f)x, y \rangle = \int_\Omega f dE_{x,y}$$

for $x, y \in H$.

Remark 3.16. A normal subalgebra A of $\mathcal{B}(H)$ is a commutative subalgebra containing L^* for every $L \in A$. A map Φ is said to be isometric-isomorphism if Φ is linear, one to one and

$$\Phi(\overline{f}) = \Phi(f)^*$$

for all $f \in L^\infty(E)$.

Now, we start to develop a spectral theorem for bounded normal operators in the resolutions of the identity form. We need to construct an approximate resolution of the identity, To do so, we use Gelfand transforms. (see [18, p.280])

Definition 3.17. Let Δ be the set of all complex homomorphisms of a commutative Banach algebra A. Let $x \in A$ $\hat{x} : \Delta \to \mathbb{C}$ defined by

$$\hat{x}(h) = h(x) \qquad (h \in \Delta)$$

is called the Gelfand transform of x .

The Gelfand topology of Δ is the weakest topology that makes every $\hat{x}$ continuous. In other words, it is the weak topology induced by $\hat{A}$, where $\hat{A}$ is the set of all $\hat{x}$ for $x \in A$. The set Δ equipped with the Gelfand topology is called the maximal maximal ideal space of A.

Theorem 3.18. *[18, Theorem 12.22 on p.321] If A is a closed normal subalgebra of $\mathcal{B}(H)$ which contains the identity operator I and if Δ is the maximal ideal space of A, then the following assertions are true:*

(a) There exits a unique resolution E of the identity on the Borel subsets of Δ which satisfies:

$$L = \int_{\Delta} \hat{L} dE$$

for every $L \in A$, where $\hat{L}$ is the Gelfand transform of L.

(b) The inverse of the Gelfand transform extends to an isometric-isomorphism Φ of the algebra $L^\infty(E)$ onto a closed subalgebra B of $\mathcal{B}(H)$, $A \subset B$, given by

$$\Phi f = \int_{\Delta} f dE$$

Moreover, Φ is linear and multiplicative and satisfies:

$$\Phi \overline{f} = (\Phi f)^*, \quad \|\Phi f\| = \|f\|_\infty$$

for all $f \in L^\infty(E)$.

(c) The subalgebra B is the closure of the set of all finite linear combinations of the projections $E(\omega)$ in the topology norm of $\mathcal{B}(H)$.

(d) If $\omega \subset \Delta$ is open and nonempty, then $E(\omega) \neq 0$.

(e) *An operator $S \in \mathcal{B}(H)$ commutes with every $L \in A$ if and only if S commutes with every projection $E(\omega)$.*

Now, we have a spectral theorem found bounded normal operators.

Theorem 3.19. *[18, Theorem 12.23 on p.324] If $L \in \mathcal{B}(H)$ and L is normal, there exists a unique resolution of the identity E on the Borel subsets of $\sigma(L)$ which satisfies:*

$$L = \int_{\sigma(L)} \lambda dE(\lambda)$$

Furthermore, every projection $E(\omega)$ commutes with every $S \in \mathcal{B}(H)$, which commutes with L. This E is also called the spectral decomposition of L.

Remark 3.20. If f is a bounded Borel function on $\sigma(L)$, given E is the spectrum decomposition of L, then the operator:

$$\Phi(f) = \int_{\sigma(L)} f dE$$

is well-defined. In particular, if $f \in C(\sigma(L))$, then $f \to f(L)$ is an isomorphism on $C(\sigma(L))$ satisfying:

$$\|f(T)x\|^2 = \int_{\sigma(L)} |f|^2 dE_{x,x}$$

3.5 Spectrum Theorem for Unbounded Normal Operators

In most cases, operators are not bounded. Let H be a Hilbert space. An unbounded operator is a pair $\big(\mathscr{D}(L), L\big)$, where $\mathscr{D}(L) \subset H$ and $L : \mathscr{D}(L) \to H$ is linear. The graph $\mathcal{G}(L)$ of L is defined as the set

$$\{(x, y) \subset H \times H | Lx = y, \forall x \in \mathcal{D}(L)\}$$

We say that L is a closed operator if its graph is a closed subspace of $H \times H$. A operator S is said to be an extension of L if $\mathcal{G}(L) \subset \mathcal{G}(S)$ (That is $\mathscr{D}(L) \subset \mathscr{D}(S)$ and $Lx = Sx$ for $x \in \mathscr{D}(L)$).
In this situation, $\mathscr{D}(L^*) \neq \mathscr{D}(L)$ in general. Consider a continuous linear functional

$$x \to \big\langle Lx, y \big\rangle, \quad x \in \mathscr{D}(L), \quad y \in H$$

By Hahn-Banach theorem, it can be extended to any $x \in H$. If $y \in \mathscr{D}(L^*)$, there exists a $L^*y \in H$ satisfying:

$$\big\langle Lx, y \big\rangle = \big\langle x, L^*y \big\rangle, \quad x \in \mathscr{D}(L)$$

Therefore, L^*y is uniquely determined if and only L is densely defined (That is $\mathscr{D}(L)$ is dense in H).

Definition 3.21. An operator L in H is said to be symmetric if:

$$(Lx, y) = (x, Ly)$$

where $x, y \in \mathscr{D}(L)$.

It follows that $L \subset L^*$. Further, L is said to be self-adjoint if $L = L^*$. A symmetric operator L is said to be maximally symmetric if L has no proper symmetric extension. (That is $L \subset S$ and S is symmetric, then $L = S$)

Theorem 3.22. *Let L be symmetric operator on H (not necessarily densely defined), the following statements hold:*

(a) $\|Lx + ix\|^2 = \|x\|^2 + \|Lx\|^2, \quad \forall x \in \mathscr{D}(L)$

(b) L is a closed operator if and only of $\mathscr{R}(L + iI)$ is closed.

(c) $L + iI$ is one-to-one.

(d) If $\mathscr{R}(L + iI) = H$, then L is maximally symmetric

(e) Statements 1-4 are still true if we replace i by $-i$.

Proof. Since L is a symmetric operator, then:

$$\begin{aligned} \|Lx + ix\|^2 &= \|x\|^2 + \|Lx\|^2 + (ix, Lx) + (Lx, ix) \\ &= \|x\|^2 + \|Lx\|^2 - (Lx, ix) + (Lx, ix) \quad \text{(By symmetricity)} \\ &= \|x\|^2 + \|Lx\|^2 \end{aligned}$$

It proves (a). For (b), the result in (a) shows there is an one-to-one correspondence between the range of $L+iI$ and $\mathscr{D}(L)$. Hence, (c) is also followed. For (d), assume that there exists a proper extension S. Since the range of $L + iI$ is the whole space H, then $S+iI$ is not one-to-one. By the statement (c), S is not symmetric. It finished the proof of (d). Finally, all proofs above work when we replace i by $-i$. □

Since the above theorem shows that

$$\|Lx + ix\|^2 = \|x\|^2 + \|Lx\|^2 = \|Lx - ix\|^2$$

then there exists an isometry U such that:

$$U(Lx + ix) = Lx - ix$$

for $x \in \mathscr{D}(L)$ with

$$\mathscr{D}(U) = \mathscr{R}(L + iI), \quad \mathscr{R}(U) = \mathscr{R}(L - iI)$$

Since $(L + iI)^{-1}$ is a map from $\mathscr{D}(U)$ to $\mathscr{D}(L)$, thus U can be written as:

$$U = (L + iI)(L - iI)^{-1}$$

The map $L \to U$ is called the Cayley transform of the symmetric operator L. Several important statements about Cayley transform are stated below.

Theorem 3.23. *Suppose U is the Cayley transform of a symmetric operator L in H. Then, the following statements are true:*

(a) U is closed if and only if L is closed.

(b) $\mathscr{R}(I - U) = \mathscr{D}(L)$, $I - U$ is one-to one and T can be reconstructed from U by

$$L = i(I + U)(I - U)^{-1}$$

(c) U is unitary if and only if T is self-adjoint.

Proof. I will mainly prove statement (b), which will be used in constructing the spectral theorem for unbounded operators. Others can be found in [18, Theorem 13.19 on p.358]. By Theorem 3.22, $L + iI$ is one-to-one and thus there exists one to one correspondence between $\mathscr{D}(L)$ and $\mathscr{R}(L + iI)$. Denote $z = Lx + ix \in \mathscr{R}(L + iI)$ and $Uz = Lx - ix$ for $x \in \mathscr{D}(L)$. Subtracting and summing them lead to $(I - U)z = 2ix$ and $(I + U)z = 2Lx$. Thus, $I - U$ is one-to-one and $\mathscr{R}(I - U) = \mathscr{D}(L)$. It follows that $(I - U)^{-1}$ maps from $\mathscr{D}(L)$ onto H and so,

$$2Lx = (I + U)z = (I + U)(I - U)^{-1}(2ix), \quad x \in \mathscr{D}(L)$$

□

Theorem 3.18 holds for every $f \in L^\infty(E)$ and $\Phi(f) \in \mathcal{B}(H)$. Given the same resolution of the identity $E : \mathscr{M} \to \mathcal{B}(H)$, it can extend to any unbounded measurable functions. The following lemma insures that such a construction is valid.

Lemma 3.24. *[18, Lemma 13.23 on p.361] Let $f : \Omega \to \mathbb{C}$ be measurable. Let*

$$\mathscr{D}_f = \{x \in H : \int_\Omega |f|^2 dE_{x,x} < \infty\}$$

Then $\mathscr{D}_f$ is a dense subset of H. Further, if $x, y \in H$, then

$$\int_\Omega |f| d|E_{x,y}| \leq \|y\| \Big\{ \int_\Omega |f|^2 dE_{x,x} \Big\}^{\frac{1}{2}}$$

Theorem 3.25. *[18, Theorem 13.24 on p.362] Let E be a resolution of the identity on a set Ω.*

(a) To every measurable $f : \Omega \to \mathbb{C}$ corresponds a densely defined closed operator $\Phi(f)$ in H, with domain $\mathscr{D}(\Phi(f)) = \mathscr{D}(f)$, which is characterised by

$$\big\langle \Phi(f)x, y \big\rangle = \int_\Omega f dE_{x,y}$$

and satisfies

$$\|\Phi(f)x\|^2 = \int_\Omega |f|^2 dE_{x,x}$$

(b) If and g are measurable, then

$$\Phi(f)\Phi(g) = \Phi(fg) \quad \text{and} \quad \mathscr{D}(\Phi(f)\Phi(g)) = \mathscr{D}_g \cap \mathscr{D}_{fg}$$

Hence, $\Phi(f)\Phi(g) = \Phi(fg)$ if and only if $\mathscr{D}_{fg} \subset \mathscr{D}_g$

(c) For every measurable $f : \Omega \to \mathbb{C}$

$$\Phi(f^*) = \Phi(\overline{f})$$

and

$$\Phi(f)\Phi(f)^* = \Phi(|f|^2) = \Phi(f)^*\Phi(f)$$

The following theorem relates the essential range of f to the spectrum of $\Phi(f)$.

Theorem 3.26. *[18, Theorem 13.27 on p.366] Suppose E is a resolution of the identity on Ω. Let $f : \Omega \to \mathbb{C}$ be a measurable function, and:*

$$\omega_\alpha = \{p \in \Omega : f(p) = \alpha\} \quad \alpha \in \mathbb{C}$$

(a) If α is in the essential range of f and $E(\omega_\alpha) \neq 0$, then α is in the point spectrum of $\Phi(f)$.

(b) If α is in the essential range of f but $E(\omega_\alpha) = 0$, then α is in the continuous spectrum of $\Phi(f)$.

(c) $\sigma(\Phi(f))$ is the essential range of f.

Finally, we reach our main theorem.

Theorem 3.27. *Let H be a Hilbert space. Let L be a self-adjoint operator in H. There exists a unique resolution E of the identity such that:*

$$\langle Lx, y\rangle = \int_{\sigma_L} t dE_{x,y}(t) \quad (\forall x \in \mathscr{D}(L).\forall y \in H)$$

In this case, we also call E the spectral decomposition.

Proof. Let U be the Cayley transform of a self-adjoint operator L. Let $\bar{E}$ be the spectral decomposition of U. By Theorem 3.23, U is unitary (so it is normal) and $I - U$ is one-to-one. Therefore, 1 is not an eigenvalue of L. Referring to the page 328 Rudin's book [18], then $\bar{E}(\{1\}) = 0$. Since the unitary operator U is bounded and normal, then applying Theorem 3.19

$$\langle Ux, y\rangle = \int_\Omega \lambda \bar{E}_{x,y}(\lambda)$$

where Ω is a unit ball without point 1 and $E_{x,y}$ in unque. Define a measurable function

$$f(\lambda) = \frac{i(1+\lambda)}{1-\lambda} \quad (\lambda \in \Omega)$$

and thus by Theorem 3.25, there exists a densely defined closed operator:

$$\langle \Phi(f)x, y\rangle = \int_\Omega f d\overline{E}_{x,y}$$

Since $f(\lambda)(1-\lambda) = i(1+\lambda)$, then by (b) of Theorem 3.25

$$\Phi(f)(I - U) = i(I + U)$$

It implies $\mathscr{R}(I-U) \subset \mathscr{D}(\Phi(f))$. In addition, by Theorem 3.23, we can reconstruct L from U such that:

$$L(I-U) = i(I+U)$$

Therefore, $\mathscr{D}(L) = \mathscr{R}(I-U) \subset \mathscr{D}(\Phi(f))$. Since f is a one-to-one map from a unit circle to the real line, then $\Phi(f)$ is a self-adjoint extension of L. Every self-adjoint operator is maximally symmetric, then $\Phi(f) = L$. By (c) of Theorem 3.26, $\sigma(L)$ is the essential range of f. Hence, define $E(f(\omega)) := \bar{E}(\omega)$ for every Borel set $\omega \subset \Omega$, we have

$$\langle Lx, y \rangle = \int_\Omega f d\bar{E}_{x,y} = \int_{\sigma(L)} t dE_{x,y}(t), \quad (\forall x \in \mathscr{D}(L), \forall y \in H)$$

The uniqueness of E comes from the uniqueness of $\bar{E}$. □

3.6 Chen, Sikora and Yan's Paper

This section is mainly based on P. Chen, A. Sikora and L.Yan's paper [19]. They introduce a main theorem about resolvent type estimates. Let L be a self-adjoint operator acting on a Hilbert space. By Theorem 3.15 and Theorem 3.27, for any bounded Borel function F, we can define the operator:

$$F(L) = \int_0^\infty F(\lambda) dE_L(\lambda)$$

where dE_L is a spectral resolution of the operator L. Spectral multiplier theory deals with the question whether $F(L)$ can extend to a bounded operator on L^p for some $p \neq 2$. Before stating the main theorem, several important definitions are needed:

Definition 3.28. A metric measure space (X, d, μ) is said to satisfy the doubling condition if there exists constant C such that:

$$V(x, 2r) \leq CV(x, r)$$

for all $x \in X$ and $r > 0$, where $V(x, r) = \mu(B(x, r))$. Consequently, there exists constants $C, n > 0$ such that:

$$V(x, sr) \leq Cs^n V(x, r) \quad \text{for all } s \geq 1, r > 0$$

such a space is called a homogeneous space.

Definition 3.29. (Finite propagation speed) Let L be a self-adjoint non-negative operator acting on $L^2(X)$. We say that L satisfies the finite propagation speed property for the corresponding wave equation if, for all $r > 0$ and all balls B_1, B_2 where

$$\langle \cos(r\sqrt{L})f_1, f_2 \rangle = 0$$

for all $f_i \in L^2(B_i, \mu), i = 1, 2$.

Conventionally, we write:

$$\mathbf{supp}\ T \subseteq D_r, \qquad D_r = \{(x, y) \in X \times X : d(x, y) \leq r\}$$

if $\langle Tf_1, f_2 \rangle = 0$ when $f_1 \in L^2(B_1, \mu), f_2 \in L^2(B_2, \mu)$, and B_1, B_2 are balls such that $d(B_1, B_2) > r$. So we can restate the finite propagation speed property as:

$$\mathbf{supp}\ \cos(r\sqrt{L}) \subseteq D_r, \quad \forall r > 0$$

Lemma 3.30. *[19] Assume that L satisfies the finite propagation speed property for the corresponding wave equation. Let $\Phi \in L^1(\mathbb{R})$ be an even function such that* **supp** $\hat{\Phi} \subset [-1, 1]$*. Then:*

$$\mathbf{supp}\ \Phi(r\sqrt{L}) \subseteq D_r$$

for all $r > 0$.

Proof. Since Φ is an even function, then

$$\begin{aligned} \hat{\Phi}(-\xi) &= \int_{\mathbb{R}} \Phi(-x) \exp(-ix\xi)dx \\ &= \int_{\mathbb{R}} \Phi(x) \exp(-ix\xi)dx = \hat{\Phi}(\xi) \end{aligned}$$

Hence, $\hat{\Phi}$ is even. By the Fourier inversion formula and the spectral theorem, for even function $\hat{\Phi}$:

$$\begin{aligned} \Phi(r\sqrt{L}) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\Phi}(s) \exp(irs\sqrt{L})ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\Phi}(s) \big[\cos(rs\sqrt{L} + i \sin(rs\sqrt{L})\big] ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\Phi}(s) \cos(rs\sqrt{L})ds \end{aligned}$$

The last step is because the product of an even function and an odd function is odd. Since supp $\hat{\Phi} \subseteq [-1, 1]$, then the result holds. □

Main theorem is following. Define a volume function V of a ball $B(x,r)$ by

$$V_r(x) = V(x,r) = \mu(B(x,r))$$

Theorem 3.31. *[19] Suppose that L is self-adjoint, non negative operator which satisfies the finite propagation speed property for the corresponding wave equation. Assume that for some $\sigma > 0, \kappa \geq 0$ and all $t \geq 0$:*

$$\|V_t^{1/2}(I + t^2 L)^{-\sigma}\|_{\mathscr{L}(L^2,L^\infty)} \leq C(1+t^2)^\kappa \tag{3.6}$$

then

(1) There exists a constant $C > 0$ such that

$$\|e^{i\xi tL} e^{-tL}\|_{\mathscr{L}(L^1,L^\infty)} \leq C(1+\xi^2)^{\sigma+\kappa+1/4}(1+t)^\kappa \tag{3.7}$$

for all $t > 0$ and $\xi \in \mathbb{R}$.

(2) For a bounded Borel function F such that $\operatorname{supp} F \subset [-1,1]$ and $F \in H^s(\mathbb{R})$ for some $s > 2\sigma + 2\kappa + 1$, the operator $F(L)$ is bounded on L^p for all $1 \leq p < \infty$, and there exists constant $C(s) > 0$ such that:

$$\|F(tL)\|_{\mathscr{L}(L^p,L^p)} \leq C(1+t)^\kappa \|F\|_{H^s}$$

for all $t > 0$.

Proof. Define:

$$(t)_+ = t \quad \text{if} \quad t \geq 0 \quad \text{and} \quad (t)_+ = 0 \quad \text{if} \quad t < 0$$

Consider that for all $a \geq 0, \xi \in \mathbb{R}$:

$$\frac{1}{\Gamma(a+1)} \int_0^\infty (s-\xi^2)_+^a e^{-s} ds = e^{-\xi^2} \frac{1}{\Gamma(a+1)} \int_{\xi^2}^\infty (s-\xi^2)^a e^{-(s-\xi^2)} ds$$

Replacing $s - \xi^2$ by z in the last term,

$$\frac{1}{\Gamma(a+1)} \int_0^\infty (s-\xi^2)_+^a e^{-s} ds = e^{-\xi^2} \frac{1}{\Gamma(a+1)} \int_0^\infty t^a e^{-z} ds$$

Realizing that the last integral is a gamma function, then we have

$$e^{-\xi^2} = C_a \int_0^\infty (s-\xi^2)_+^a e^{-s} ds$$

where $C_a = \frac{1}{\Gamma(a+1)}$. Thus,

$$e^{-\frac{\xi^2}{4}} = C_a' \int_0^\infty (1 - \frac{\xi^2}{s})_+^a e^{-\frac{s}{4}} s^a ds$$

Hence, letting $u = \frac{\xi}{\sqrt{s}}$ and taking inverse Fourier transform on both sides

$$\begin{aligned} e^{-x^2} &= \int_0^\infty \Big[\frac{1}{2\pi}\int_{\mathbb{R}} (1 - \frac{\xi^2}{s})_+^a \exp(ix\xi)dx\Big] s^a e^{-\frac{s}{4}} ds \\ &= \int_0^\infty \Big[\int_{\mathbb{R}} (1-u^2)_+^a \exp(iu\sqrt{s}x)du\Big] s^{a+\frac{1}{2}} e^{(-\frac{s}{4})} ds \\ &= \int_0^\infty F_a(\sqrt{s}x) s^{a+\frac{1}{2}} \exp(-\frac{s}{4}) ds \end{aligned}$$

where F_a is the inverse Fourier transform of the function $u \mapsto (1-u^2)_+^a$. Therefore, by spectral theory, for $v > 0$

$$\int_0^\infty F_a(\sqrt{svL}) s^{a+\frac{1}{2}} \exp(-\frac{s}{4}) ds = \exp(-vL)$$

Thus, we can write:

$$e^{i\xi tL} e^{-tL} = \int_0^\infty F_a(\sqrt{sL}) s^{a+\frac{1}{2}} (t - i\xi t)^{-a-\frac{3}{2}} \exp\Big(-\frac{s}{4(t-i\xi t)}\Big) ds \tag{3.8}$$

Since supp $\hat{F}_a \subset [-1,1]$, then by Lemma 3.30

$$\text{supp } F_a(\sqrt{sL}) \subseteq D_{\sqrt{s}}, \forall s > 0$$

For an integral operator $F_a(\sqrt{sL})$ with kernel $K_{F_a(\sqrt{sL})}$, it implies that $K_{F_a(\sqrt{sL})} = 0$ for $(x,y) \notin D_{\sqrt{s}}$, where $D_{\sqrt{s}} := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : d(x,y) \le \sqrt{s}\}$.

$$\begin{aligned} \|F_a(\sqrt{sL})\|_{\mathscr{L}(L^1,L^1)} &= \sup_y \int_X |K_{F_a(\sqrt{sL})}(x,y)| d\mu(x) \\ &= \sup_y \int_{B(y,\sqrt{s})} |K_{F_a(\sqrt{sL})}(x,y)| d\mu(x) \\ &\le \sup_y \mu(B(y,\sqrt{s}))^{1/2} \Big(\int_X |K_{F_a(\sqrt{sL})}(x,y)|^2 d\mu(x)\Big)^{\frac{1}{2}} \\ &= \|V_{\sqrt{s}}^{1/2} F_a(\sqrt{sL})\|_{\mathscr{L}(L^2,L^\infty)} \\ &\le \|V_{\sqrt{s}}^{1/2} (I+sL)^{-\sigma}\|_{\mathscr{L}(L^2,L^\infty)} \|(1+sL)^\sigma F_a(\sqrt{sL})\|_{\mathscr{L}(L^2,L^2)} \end{aligned}$$

Recalling the Bocher-Riesz summability in [20, p.388-394], consider a multiplier operator S^a on $L^p(\mathbb{R}^n)$ defined by:

$$f(x) \mapsto (S^a f)(x) = \frac{1}{2\pi} \int_{|\xi| \le 1} \hat{f}(\xi)(1 - |\xi|^2)^a \exp(ix\xi) d\xi$$

It can be written as a convolution [20, Lemma 2.2.1 on page 390-391]:

$$S^a(f) = f * K^a$$

where K^a is bounded, and that:

$$K^a(x) \sim |x|^{-(n+1)/2-a}\left[e^{2\pi i|x|}\sum_{j=0}^{\infty}\alpha_j|x|^{-j} + e^{-2\pi i|x|}\sum_{j=0}^{\infty}\beta_j|x|^{-j}\right]$$

For any function f in Schwartz class $\mathscr{S}$:

$$f * F_a(\lambda) = \frac{1}{2\pi}\int_{|u|\leq 1}\hat{f}(u)(1-u^2)_+^a \exp{(iu\lambda)}du$$

Therefore, F_a behaves in the following patten:

$$F_a(\lambda) \sim |\lambda|^{-1-a}\left[e^{2\pi i|\lambda|}\sum_{j=0}^{\infty}\alpha_j|\lambda|^{-j} + e^{-2\pi i|\lambda|}\sum_{j=0}^{\infty}\beta_j|\lambda|^{-j}\right]$$

as $|\lambda| \to \infty$ for suitable constants α_j and β_j.
As long as $a + 1 \geq 2\sigma$, then function $\lambda \mapsto (1+\lambda^2)^\sigma F_a(\lambda) \in L^\infty(\mathbb{R})$. Hence, by spectral theory

$$(1+sL)^\sigma F_a(\sqrt{sL}) \in \mathcal{B}(L^2)$$

Combing this with our assumption (3.6),

$$\|F_a(\sqrt{sL})\|_{\mathscr{L}(L^1,L^1)} \leq C(1+s)^\kappa$$

It follows that:

$$\begin{aligned}\|e^{i\xi tL}e^{-tL}\|_{\mathscr{L}(L^1,L^1)} &\leq \int_0^\infty (1+s)^\kappa s^{a+\frac{1}{2}}(t^2+(\xi t)^2)^{-\frac{a}{2}-\frac{3}{4}}\exp\Big(-\frac{s}{4(t+\xi^2 t)}\Big)ds \\ &\leq C(1+t)^\kappa(1+\xi^2)^{\frac{a}{2}+\kappa+\frac{3}{4}}\end{aligned}$$

Thus, (3.7) holds.
For (2), writing $G(\lambda) = F(\lambda)e^\lambda$, then by inverse Fourier transform and spectral theory

$$F(tL) = G(tL)e^{-tL} = \int_{\mathbb{R}}\hat{G}(\xi)e^{it\xi L}e^{-tL}d\xi$$

Hence,

$$\begin{aligned}\|F(tL)\|_{\mathscr{L}(L^1,L^1)} &\leq \int_{\mathbb{R}}|\hat{G}(\xi)|\|e^{it\xi L}e^{-tL}\|_{\mathscr{L}(L^1,L^1)}d\xi \\ &\leq C\int_{\mathbb{R}}|\hat{G}(\xi)|(1+\xi^2)^{\sigma+\kappa+\frac{1}{4}}(1+t)^\kappa d\xi \\ &\leq C(1+t)^\kappa\|G\|_{H^s}\Big(\int_{\mathbb{R}}(1+\xi^2)^{2\sigma+2\kappa+\frac{1}{2}-s}d\xi\Big)^{\frac{1}{2}}\end{aligned}$$

Recalling the assumption in (2), as long as $s > 2\sigma + 2\kappa + 1$

$$\|F(tL)\|_{\mathscr{L}(L^1,L^1)} \leq C(1+t)^\kappa \|G\|_{H^s}$$

Since supp $F \subset [-1,1]$, then $\|G\|_{H^s} \leq C\|F\|_{H^s}$ and so

$$\|F(tL)\|_{\mathscr{L}(L^1,L^1)} \leq C_s(1+t)^\kappa \|G\|_{H^s} \leq C_s(1+t)^\kappa \|F\|_{H^s}$$

By duality, we can obtain $L^\infty - L^\infty$ boundedness. In addition, for any Borel function $F(L)$ is bounded on L^2. By interpolation, our result can be extend to any $1 \leq p \leq \infty$.

□

Chapter 4

Ornstein-Uhlenbeck Operator

4.1 Restricted Embedding Result

In Chapter 2, the weighted Nash inequality provides an idea that we possibly can develop a bounded spectral multiplier theorems by adding a weight function. This function associated with the original measure can be treated as a new measure. Jan Van Neerven and Pierre Portal in [21] introduce a Gaussian version of Weyl calculus and prove a restricted $L^p - L^q$ boundedness property of the Ornstein-Uhlenbeck semigroup by changing measures. For $\rho > 0$, setting

$$d\gamma_\rho = (2\pi\rho)^{-d/2} \exp{(-\frac{|x|^2}{2\rho})} dx$$

and

$$r_\pm(s) := \frac{1}{2}\Re(\frac{1}{s} \pm s), \forall s \in \mathbb{C}$$

where γ_1 is the standard Gaussian measure. We write γ instead of γ_1 in what follows.

Theorem 4.1. *[21] Let $p, q \in [1, \infty)$ and $\alpha, \beta > 0$. Let $s \in \mathbb{C}$ with $\Re s > 0$ satisfying $1 - \frac{2}{\alpha p} + r_+(s) > 0$, $\frac{2}{\beta q} - 1 + r_+(s) > 0$ and*

$$(r_-(s))^2 \leq (1 - \frac{2}{\alpha p} + r_+(s))(\frac{2}{\beta q} - 1 + r_+(s))$$

Define $z \in \mathbb{C}$ by $s = \frac{1-e^{-z}}{1+e^{-z}}$, then

$$\begin{aligned}&\|\exp{(-zL)}\|_{\mathscr{L}(L^p(\mathbb{R}^d,\gamma_\alpha),L^q(\mathbb{R}^d,\gamma_\beta))}\\ \leq&\frac{2^d C}{|1-e^{-2z}|^{\frac{d}{2}}}\frac{1}{(1-\frac{2}{\alpha p}+\Re\frac{1+e^{-2z}}{1-e^{-2z}})^{\frac{d}{2}(1-\frac{1}{p})}(\frac{2}{\beta q}-1+\Re\frac{1+e^{-2z}}{1-e^{-2z}})^{\frac{d}{2}\frac{1}{q}}}\end{aligned}$$

where $C = (\frac{1}{2r})^{d/2}(\frac{\alpha r}{2})^{d/2p}(\frac{\beta r}{2})^{-d/2q}$.

Remark 4.2. For small values of $t > 0$, $|1-e^{-2t}| \sim t$ and $\Re(\frac{1+e^{-2t}}{1-e^{-2t}}) \sim \frac{1}{t}$, therefore

$$\| \exp(-tL)\|_{\mathscr{L}(L^p(\mathbb{R}^d,\gamma_{2/p}),L^q(\mathbb{R}^d,\gamma_{2/q}))} \preceq C_{d,p,q} t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}$$

One important consequence of Theorem 4.1 is the Sobolev embedding result.

Corollary 4.3. *[21] Let L be the Ornstein-Uhlenbeck operator. Let $p \in (\frac{2d}{d+4}, 2]$. The resolvent $(I+L)^{-1}$ maps $L^p(\mathbb{R}^d, \gamma_{2/p})$ into $L^2(\mathbb{R}^d, \gamma)$ such that:*

$$\|(I+L)^{-1}u\|_{L^2(\mathbb{R}^d,\gamma)} \preceq \|u\|_{L^p(\mathbb{R}^d,\gamma_{2/p})}$$

Proof. Picking $q = 2$, then

$$\| \exp(-tL)\|_{\mathscr{L}(L^p(\mathbb{R}^d,\gamma_{2/p}),L^2(\mathbb{R}^d,\gamma))} \preceq C_{d,p,q} t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{2})}$$

Let $p \in (\frac{2d}{d+4}, 2]$, then

$$\| \exp(-t(I+L))\|_{\mathscr{L}(L^p(\mathbb{R}^d,\gamma_{2/p}),L^2(\mathbb{R}^d,\gamma))} \preceq C_{d,p,q} t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{2})} \exp(-t)$$

For $u \in L^p(\mathbb{R}^d, \gamma_{2/p}) \cap L^2(\mathbb{R}^d, \gamma)$, we have

$$\|(I+L)^{-1}u\|_{L^2(\mathbb{R}^d,\gamma)} \preceq \|u\|_{L^p(\mathbb{R}^d,\gamma_{2/p})} \int_0^\infty \exp(-t) t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{2})} dt$$

Since $p \in (\frac{2d}{d+4}, 2]$, then $\frac{d}{2}(\frac{1}{p} - \frac{1}{2}) < 1$. Using integration by parts, the second term is finite. □

Remark 4.4. If $\frac{d}{2}(\frac{1}{p} - \frac{1}{q}) < 1$, then the above proof works for a $L^p - L^q$ boundedness result.

In addition, running an iteration argument, we can obtain a $L^1 - L^2$ boundedness result for $(I+L)^{-\beta}$, for some $\beta \in \mathbb{N}$.

Theorem 4.5. *Let $\beta > \frac{d-1}{3}$, then $(I+L)^{-\beta} \in \mathcal{B}(L^1(\mathbb{R}^d, \gamma_2), L^2(\mathbb{R}^d, \gamma))$.*

Proof. For $p \in (\frac{dq}{d+2q}, q]$, we have $\frac{d}{2}(\frac{1}{p} + \frac{1}{q}) < 1$. Let $\epsilon \in (0, 1]$ and $a = 4 - \epsilon$. For $\frac{2d}{d+a} \in (\frac{2d}{d+4}, 2]$, then

$$\frac{d(\frac{2d}{d+a})}{d + 2(\frac{2d}{d+a})} = \frac{2d}{d+a+4}$$

Thus, $(I+L)^{-2} \in \mathcal{B}(L^p(\mathbb{R}^d, \gamma_{2/p}), L^2(\mathbb{R}^d, \gamma))$ for $p \in (\frac{2d}{d+a+4}, 2]$. In the same way, $(I+L)^{-3} \in \mathcal{B}(L^p(\mathbb{R}^d, \gamma_{2/p}), L^2(\mathbb{R}^d, \gamma))$ for $p \in (\frac{2d}{d+2a+4}, 2]$. Repeating this process, when $\beta = k+1$ for $k \in \mathbb{N} \cup \{0\}$, we have

$$(I+L)^{-\beta} \in \mathcal{B}(L^p(\mathbb{R}^d, \gamma_{2/p}), L^2(\mathbb{R}^d, \gamma))$$

for $p \in (\frac{2d}{d+ka+4}, 2]$. Since $\beta - 1 > \frac{d-4}{3}$, then

$$\frac{2d}{d+ka+4} \leq \frac{2d}{d+3k+4} < \frac{2d}{2d} = 1$$

□

For the Ornstein Uhlenbeck operator, we only have $L^p(\mathbb{R}^d, \gamma) - L^2(\mathbb{R}^d, \gamma)$ boundedness of $(I+L)^{-1}$ if $p = 2$. Fortunately, the restricted Sobolev embedding result provides a clue that we can obtain a $L^p(\mathbb{R}^d, \gamma_{2/p}) - L^2(\mathbb{R}^d, \gamma)$ spectral multiplier theorem.

4.2 Applications

Combing the idea of studying the multiplier problem via inverse Fourier transform (as in Chen, Sikora and Yan's paper), we obtain the following theorem.

Theorem 4.6. *Let L be the Ornstein-Uhlenbeck operator. Suppose a function $F \in L^\infty(\mathbb{R})$ satisfies that there exists an $\alpha > 1$ such that $\xi \mapsto (1+|\xi|^2)^{\frac{\alpha}{2}} \hat{F}(\xi) \in L^1(\mathbb{R})$. For $p \in (\frac{2d}{d+4}, 2]$, then*

$$F(L) \in \mathcal{B}(L^p(\mathbb{R}^d, \gamma_{2/p}), L^2(\mathbb{R}^d, \gamma))$$

Proof. Let $F \in \{f \in L^\infty(\mathbb{R}); \xi \mapsto (1+|\xi|^2)^{\frac{\alpha}{2}} \hat{f}(\xi) \in L^1\}$ and $p \in (\frac{2d}{d+4}, 2]$, then by Fourier inverse transform and the spectral theorem; for $f \in L^2(\mathbb{R}^d, \gamma)$:

$$\begin{aligned} F(L)f &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{F}(\xi) e^{i\xi L} f d\xi \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{F}(\xi) e^{i\xi L} (I+L^2)^{\frac{\alpha}{2}} (I+L^2)^{-\frac{\alpha}{2}} f d\xi \end{aligned}$$

Let $\hat{F}_\alpha(\xi) = \hat{F}(\xi)(1+|\xi|^2)^{\frac{\alpha}{2}}$, thus

$$\begin{aligned} \|F(L)f\|_{L^2(\mathbb{R}^d,\gamma)} =& \Big\| \int_{\mathbb{R}} \hat{F}_a(\xi) e^{i\xi L} (I+L^2)^{-\frac{\alpha}{2}} (I+L)(I+L)^{-1} f d\xi \Big\|_{L^2(\mathbb{R}^d,\gamma)} \\ \leq& \int_{\mathbb{R}} |\hat{F}_a(\xi)| \Big\| e^{i\xi L} (I+L^2)^{-\frac{\alpha}{2}} (I+L) \Big\|_{\mathscr{L}(L^2(\mathbb{R}^d,\gamma), L^2(\mathbb{R}^d,\gamma))} \\ & \Big\| (I+L)^{-1} f \Big\|_{L^2(\mathbb{R}^d,\gamma)} d\xi \end{aligned}$$

Since the function $x \mapsto e^{i\xi x} \frac{1+x}{(1+x^2)^{\frac{\alpha}{2}}} \in L^\infty(\mathbb{R})$ for $\alpha > 1$, by spectral theorem

$$e^{i\xi L}(I+L^2)^{-\frac{\alpha}{2}}(I+L) \in \mathcal{B}(L^2(\mathbb{R}^d, \gamma), L^2(\mathbb{R}^d, \gamma))$$

Since $\hat{F}_a \in L^1(\mathbb{R})$, by Theorem 4.3,

$$\|F(L)f\|_{L^2(\mathbb{R}^d,\gamma)} \leq C\|\hat{F}_a\|_{L^1(\mathbb{R})}\|f\|_{L^p(\mathbb{R}^d,\gamma_{2/p})}$$

□

Remark 4.7. Note that the Sobolev space $W^{k,p}(\mathbb{R}^n)$ for $1 \leq p \leq \infty$ is defined as:

$$W^{k,p}(\mathbb{R}^n) := \left\{ f \in L^p(\mathbb{R}^n) : \mathcal{F}^{-1}\left[(1+|\xi|^2)^{\frac{k}{2}}\mathcal{F}f\right] \in L^p(\mathbb{R}^d) \right\}$$

The space, which was studied in Theorem 4.6, is not exactly a Sobolev space $W^{\alpha,\infty}$, because the inverse Fourier transform of a function in $L^1(\mathbb{R})$ is not necessary a $L^\infty(\mathbb{R})$ function. We don't know what exactly this space is, but at least it contains the Schwartz class $\mathscr{S}(\mathbb{R})$.

Ideally, if we prove the result $(I+L)^{-\beta} \in \mathcal{B}(L^p(\mathbb{R}^d,\gamma_{2/p}), L^2(\mathbb{R}^d,\gamma))$ for a smaller positive β, then the function $e^{i\xi x}\frac{(1+x)^\beta}{(1+x^2)^{\frac{\alpha}{2}}}$ is in $L^\infty(\mathbb{R})$ for $\alpha > \beta$. Our spectral multiplier result will hold for a larger function space. On the other hand, if we obtain a $L^p - L^2$ boundedness result for $(I+L)^{-\beta}$, similar spectral multiplier results will hold.

Conjecture: Suppose a function F in $L^\infty(\mathbb{R})$ such that $\xi \mapsto (1+|\xi|^2)^{\frac{\alpha}{2}}\hat{F}(\xi) \in L^1(\mathbb{R})$ and $\alpha > \frac{1}{2}(\frac{1}{n} - \frac{1}{p})$, then $F(L) \in \mathcal{B}(L^p(\mathbb{R}^d,\gamma_{2/p}))$.

Combining Theorem 4.1 and the argument from Theorem 3.31, there is a $L^p - L^p, 1 \leq p \leq 2$ boundedness consequence with respect to the measure γ_2 for any compactly supported Borel function $F \in H^s(\mathbb{R})$

Theorem 4.8. *Let L be the Ornstein-Uhlenbeck operator. For a bounded Borel function F such that* supp $F \subset [-1,1]$ *and $F \in H^s(\mathbb{R})$ for some $s > \frac{d}{2}+1$, the operator $F(L)$ is bounded on $L^p(\mathbb{R}^d,\gamma_2)$ for all $1 \leq p \leq 2$, and there exists a constant $C_{d,s} > 0$ such that:*

$$\|F(tL)\|_{\mathscr{L}(L^1(\mathbb{R}^d,\gamma_2),L^1(\mathbb{R}^d,\gamma_2))} \leq C_{d,s}\|F\|_{H^s}$$

for all $t > 0$.

Proof. Recall the settings of Theorem 4.1. Let $z = t + it\xi$ for $t > 0$. Define $s = \frac{1-e^{-z}}{1+e^{-z}}$, then

$$\begin{aligned} s &= \frac{1 - e^{-t}\cos(t\xi) + e^{-t}\sin(t\xi)i}{1 + e^{-t}\cos(t\xi) - e^{-t}\sin(t\xi)i} \\ &= \frac{[1 - e^{-t}\cos(t\xi) + e^{-t}\sin(t\xi)i][1 + e^{-t}\cos(t\xi) + e^{-t}\sin(t\xi)i]}{[1 + e^{-t}\cos(t\xi)]^2 + [e^{-t}\sin(t\xi)]^2} \end{aligned}$$

Thus,

$$\Re s = \frac{1 - e^{-2t}}{[1 + e^{-t}\cos{(t\xi)}]^2 + [e^{-t}\sin{(t\xi)}]^2} > 0$$

Further

$$r_+(s) = \frac{1}{2}\Re(s + \frac{1}{s}) = \Re(\frac{1 + e^{-2z}}{1 - e^{-2z}}) = \frac{1 - e^{-4t}}{[1 - e^{-2t}\cos(2t\xi)]^2 + e^{-4t}\sin^2{(2t\xi)}}$$

and

$$r_-(s) = \frac{1}{2}\Re(s - \frac{1}{s}) = \Re(-\frac{e^{-z}}{1 - e^{-2z}}) = \frac{(-e^{-t} + e^{-3t})\cos{(t\xi)}}{[1 - e^{-2t}\cos(2t\xi)]^2 + e^{-4t}\sin^2{(2t\xi)}}$$

Since $e^{-t} - e^{-3t} \leq 1 - e^{-4t}$ for $t > 0$, then $|-e^{-t} + e^{-3t}\cos{(t\xi)}|^2 \leq |1 - e^{-4t}|^2$. Let $p = q = 1$ and $\alpha = \beta = 2$, then $(r_-(s))^2 \leq (r_+(s))^2$, which satisfies the condition in Theorem 4.1. Thus,

$$\begin{aligned}
&\|\exp{(-tL - it\xi L)}\|_{\mathscr{L}(L^1(\mathbb{R}^d,\gamma_2),L^1(\mathbb{R}^d,\gamma_2))}\\
\leq&\Big(\frac{4C}{|1 - e^{-2z}||\Re(\frac{1+e^{-2z}}{1-e^{-2z}})|}\Big)^{\frac{d}{2}}\\
=&(4C)^{d/2}\Big(\sqrt{[1 - e^{-2t}\cos{(2t\xi)}]^2 + [e^{-2t}\sin{(2t\xi)}]^2}/1 - e^{-4t}\Big)^{d/2}\\
=&(4C)^{d/2}\Big(\frac{1 - 2e^{-2t}\cos{(2t\xi)} + e^{-4t}}{1 - 2e^{-4t} + e^{-8t}}\Big)^{d/4}
\end{aligned}$$

Using L'Hopital's rule, differentiating in terms of t twice on denominator and numerator, we have:

$$\begin{aligned}
&\lim_{t\to 0}\frac{1 - 2e^{-2t}\cos{(2t\xi)} + e^{-4t}}{1 - 2e^{-4t} + e^{-8t}}\\
=&\lim_{t\to 0}\frac{4e^{-2t}\cos{(2t\xi)} + 4\xi e^{-2t}\sin{(2t\xi)} - 4e^{-4t}}{8e^{-4t} - 8e^{-8t}}\\
=&\lim_{t\to 0}\frac{-8e^{-2t}\cos{(2t\xi)} - 8\xi e^{-2t}\sin{(2t\xi)} - 8\xi e^{-2t}\sin{(2t\xi)} + 8\xi^2 e^{-2t}\cos{(2t\xi)} + 16e^{-4t}}{64e^{-8t} - 32e^{-4t}}\\
=&\frac{1}{4}(\xi^2 + 1)
\end{aligned}$$

When t is away from 0, the function $t \mapsto 1 - 2e^{-2t}\cos{(2t\xi)} + e^{-4t}$ is positive and bounded above by $3 + e^{-4t}$ uniformly over ξ, which approaches to 3 as $t \to \infty$. It is sufficient to conclude that $\forall t > T$ for some $T > 0$:

$$\begin{aligned}
\|\exp{(-tL - it\xi L)}\|_{\mathscr{L}(L^1(\mathbb{R}^d,\gamma_2),L^1(\mathbb{R}^d,\gamma_2))} &\preceq C_d(1 + \xi^2)^{\frac{d}{4}}\Big(\frac{\sqrt{3 + e^{-4t}}}{1 - e^{-4t}}\Big)^{d/2}\\
&\preceq C_d(1 + \xi^2)^{\frac{d}{4}}\Big(\frac{2}{1 - e^{-4T}}\Big)^{d/2}
\end{aligned}$$

Therefore, for $t > T$

$$\|\exp(-tL - it\xi L)\|_{\mathscr{L}(L^1(\mathbb{R}^d,\gamma_2),L^1(\mathbb{R}^d,\gamma_2))} \preceq C_d(1+\xi^2)^{\frac{d}{4}} \tag{4.1}$$

Now combining Chen, Sikora and Yan's argument, writing $G(\lambda) = F(\lambda)e^{\lambda}$ and then using inverse Fourier transform and spectral theory

$$F(tL) = G(tL)e^{-tL} = \int_{\mathbb{R}} \hat{G}(\xi)e^{it\xi L}e^{-tL}d\xi$$

Hence, by (4.1):

$$\begin{aligned}
\|F(tL)\|_{\mathscr{L}(L^1(\mathbb{R}^d,\gamma_2),L^1(\mathbb{R}^d,\gamma_2))} &\leq \int_{\mathbb{R}} |\hat{G}(\xi)| \|e^{it\xi L}e^{-tL}\|_{\mathscr{L}(L^1(\mathbb{R}^d,\gamma_2),L^1(\mathbb{R}^d,\gamma_2))}d\xi \\
&\leq C_d \int_{\mathbb{R}} |\hat{G}(\xi)|(1+\xi^2)^{\frac{d}{4}}d\xi \\
&\leq C_d\Big(\int_{\mathbb{R}} |\hat{G}(\xi)|^2(1+\xi^2)^s d\xi\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}}(1+\xi^2)^{\frac{d}{2}-s}d\xi\Big)^{\frac{1}{2}} \\
&\leq C_d\|G\|_{H^s}\Big(\int_{\mathbb{R}}(1+\xi^2)^{\frac{d}{2}-s}d\xi\Big)^{\frac{1}{2}}
\end{aligned}$$

The last step is because Fourier transform is an isometry between L^2 spaces. As long as $\frac{d}{2} - s < -1$, then we can conclude that:

$$\|F(tL)\|_{\mathscr{L}(L^1(\mathbb{R}^d,\gamma_2),L^1(\mathbb{R}^d,\gamma_2))} \leq C_{d,s}\|G\|_{H^s}$$

Since supp $F \subset [-1, 1]$, then $\|G\|_{H^s} \leq C\|F\|_{H^s}$ and so

$$\|F(tL)\|_{\mathscr{L}(L^1(\mathbb{R}^d,\gamma_2),L^1(\mathbb{R}^d,\gamma_2))} \leq C_{d,s}\|G\|_{H^s} \leq C_{d,s}\|F\|_{H^s}$$

For any Borel function F, $F(L)$ is bounded on $L^2(\mathbb{R}^d, \gamma_2)$. By interpolation theorem, our result can be extended to any $1 \leq p \leq 2$. □

Bibliography

[1] Edward, Nelson *The free Markov field* Journal of Functional Analysis, Volume 12, Issue 2, Feb. 1973, p.211-227

[2] Jay B. Epperson *The Hypercontractive Approach to Exactly Bounding an Operator with Complex Gaussian Kernel* Journal of Functional Analysis, Volume 87, Issue 1, Nov. 1989, pp.1-30

[3] Leonard Gross *Logarithmic Sobolev inequalities* American Journal of Mathematics, Volume 97, No.4, Winter, 1975, pp. 1061-1083

[4] Leonard Gross *Hypercontractivity and Logarithmic Sobolev inequalities for the Clifford-Dirichelet Form,* Duke Mathematical Journal, Vol. 42 No.3, September 1971, pp. 383-396

[5] Las Hormander *Estimates for translation invariant operators in L^p spaces,* Acta Math. (1) 104 (1960), pp.93-140.

[6] Solomon Mikhlin *Multidimensional singular integrals and integral equations,* Translated from the Russian by W. J. A. Whyte. Pergamon Press, Oxford-New York-Paris 1965.

[7] F.-Y. *Functional Inequalities and Spectrum Estimates: the Infinite Measure Case* Journal of Functional Analysis, Volume 194, Issue 2, Oct 2002, p.288-310

[8] Dominique Bakry, Ivan Gentil, Michel Ledoux, *Analysis and Geometry of Markov Diffusion Operators.* Springer International Publishing Switzerland 2014

[9] Robert M. Blumenthal and Ronald K. Getoor, *Markov Process and Potential Theory.* Dover Publications, 2017

[10] Wolfgang Arendt, Ralph Ghill, Christian Seifert, Hendrik Vogt and Jurgen Voigt, *Form Methods for Evolution Equations, and Applications, Lecture Notes*
https://www.mat.tuhh.de/veranstaltungen/isem18/images/5/59/LectureNotes.pdf

[11] Patrick Billingsley, *Convergence of Probability Measures* Volume 493 of Wiley Series in Probability and Statistics, Wiley, 2009

[12] Eulalia Nualart, *Lectures on Malliavin calculus and its applications to finance*
http://www.math.wisc.edu/ kurtz/NualartLectureNotes.pdf

[13] Dominique Bakry, Francois Bolley, Ivan Gentil and Patrick Maheux, *Weighted Nash Inequalities* Revista Matematica Iberoamericana 28, 3, pp. 879-906, 2012

[14] Feng-Yu Wang, *Functional Inequalities for Empty Essential Spectrum* Journal of Functional Analysis, Volume 170, issue 1, Jan. 2000, pp. 219-245

[15] T. Coulhon, *Ultracontractivity and Nash type inequalities* Journal of Functional Analysis, Volume 141, issue 2, Nov. 1996, pp. 510-539

[16] Dominique Bakry, *Functional inequalities for Markov semigroups* Probability measures on groups, 2004, Mumbai, India. Tata Institute of Fundamental Research, Mubai, pp.91-147, 2006.

[17] Michael Reed and Barry Simon, *I: Functional Analysis, Volume 1 of Methods of Modern Mathematical Physics, revised* Academic Press, 1981

[18] Walter Rudin, *Functional Analysis, Second Addition* 1991

[19] Peng Chen, Adam Sikora and Lixin Yan, *Spectral Multipliers via Resolvent Type Estimates on Non-Homogeneous Metric Measure Spaces,* Arxiv,
https://arxiv.org/pdf/1609.01871.pdf

[20] Elias M.Stein, *Harmonic Analysis, second printing* 1995

[21] Jan Van Neervan and Pierre Portal, *Weyl Calculus with Respect to the Gaussian Measure and Restricted $L^p - L^q$ Boundedness of the Ornstein-Uhlenbeck Semigroup in Complex Time,* Arxiv,
https://arxiv.org/pdf/1702.03602.pdf