Reach Almost Sure Consensus with Only Group Information ☆

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Abstract

This brief presents a new distributed scheme to solve the consensus problem for a group of agents if neither their absolute states nor inter-agent relative states are available. The new scheme considers a random partition of agents into two subgroups at each step and then uses the relative group representative state as feedback information for the consensus purpose. It is then shown that almost sure consensus can be achieved under the proposed scheme in both discrete time and continuous time. For the discrete time case, almost sure consensus is achieved if and only if the weighting parameter for state update is greater than one. For the continuous time case, almost sure consensus is realized when the weighting parameter is positive. Moreover, it is shown that if a uniform probability is considered for group selection, then the group of agents can reach average consensus in mean.

Key words: Consensus; multi-agent systems; random networks

1. Introduction

Recent years have witnessed an increasing attention on the study of consensus and synchronization for multi-agent systems due to broad applications, for example, distributed synchronization of coupled chaotic oscillators and laser arrays, distributed beamforming of cooperative communicating antennas, wireless sensor networks, and robotic swarms. A fundamental concern of this topic is the design of local interaction protocols using the shared information to ensure that all the agents asymptotically agree on some variable of interest in an appropriate sense.

Most of existing works assume that either the absolute state of each agent or inter-agent relative states are available through local communications so that a typical consensus control law uses the relative states of neighbors. With this type of consensus control laws, early works on this topic mainly focus on discovering the topological properties of underlying deterministic networks related to consensus by considering simple agent dynamics such as single or double integrators (Jadbabaie et al., 2003; Lin et al., 2004; Olfati-Saber and Murray, 2004; Ren and Beard, 2005; Lin et al., 2007). It has been obtained that to achieve consensus, the underlying networks must contain a spanning tree for fixed topologies while for switching topologies, the union of the networks over every bounded time interval should contain a spanning tree. More recent works take into account general linear dynamics for consensus (Ma and Zhang, 2010; Li et al., 2010; You and Xie, 2011), i.e., the dynamics of each agent is described by an identical linear time-invariant system. It has been shown that for consensus of linear multi-agent systems, the coupling gain should be strong enough to dominate the instability of agent dynamics in addition to the topological connectivity condition. On the other hand, motivated by the stochastic nature of real wireless communication, stochastic consensus of multi-agent systems is investigated by assuming that the interaction topologies are randomly switching or Markovian switching (Boyd et al., 2006; Fagnani and Zampieri, 2008; Tahbaz-Salehi and Jadbabaie, 2010; Huang and Manton, 2010; Zhang and Tian, 2012). It has been derived that for randomly switching or Markovian switching topologies, the mean graph with respect to a stationary distribution or the union of graphs associated with the positive recurrent states of the Markov processes should meet certain connectivity properties. In addition, the consensus problem has also been addressed with quantized data or with delayed data. See for example Kashyap et al. (2007); Cai and Ishii (2011); Tian and Liu (2008) and the references therein.

However, in certain practical applications, neither the absolute states of the agents nor the inter-agent relative states are available, for which the above consensus schemes are not applicable. For example, in collaborative communication of multiple transmitters, the state (i.e., the phase of each transmission signal) and the inter-agent relative state (i.e., the phase difference between any two transmission signals) can not be obtained. But by some techniques, the phase difference between two groups

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of transmitters can be measured by a cooperative receiver (Hou et al., 2012) and sent back to all the transmitters to help achieve synchronization of signal phases at the receiver end. This paper proposes a new distributed scheme to solve the consensus problem when neither the absolute states nor the inter-agent relative states are available. The new scheme considers a random partition of agents into two subgroups at each step based on each agent's self-nominated probability for subgroup selection. Thus, the partition evolves randomly as time goes on. Moreover, the relative group representative state of two subgroups is used to update the states of all the agents, where the group representative state is a random state lying inside the convex hull spanned by the states of the agents in each subgroup. Thus, the control input for the agents in the same subgroup takes the same value. It is then shown that almost sure consensus can be achieved under the proposed scheme in both discrete time and continuous time. For the discrete time case, a necessary and sufficient condition to achieve almost sure consensus is that the weighting parameter for state update is greater than one. For the continuous time case, a group of agents reaches consensus almost surely as long as the weighting parameter is positive. Moreover, if every agent takes equal probability to nominate oneself in one subgroup or the other, then the agents reach average consensus in mean.

Notation : Throughout the paper, we use $\mathbb{P}\{\cdot\}$ and $\mathbb{E}[\cdot]$ to represent the probability measure and the expectation measurement respectively. Moreover, we denote by $|\cdot|$ the cardinality of a set.

2. Problem Formulation

2.1. Discrete-time Case

Consider n agents and suppose each agent is modeled by a discrete-time system

$$x_i(k+1) = x_i(k) + u_i(k), \quad i = 1, \cdots, n \tag{1}$$

where $x_i(k) \in \mathbb{R}$ and $u_i(k) \in \mathbb{R}$ represent the state and control input of agent i, respectively. The initial state of the n agents is denoted by $\mathbf{x}(0) = [x_1(0), \cdots, x_n(0)]^T$.

In this brief, we assume that no information exchange occurs among neighboring agents and thus no inter-agent relative state $(x_j - x_i)$ is available. This happens in many applications such as distributed beamforming (Hou et al., 2012) and page ranking (Ishii et al., 2012). Instead, a piece of alternative information is available to each agent. For example, as in distributed beamforming (Hou et al., 2012), a group of agents is divided into two subgroups, for which the *relative group representative state* is available to each agent via feedback information from a cooperative destination.

More formally, we consider a partition at step k such that a group of agents is divided into two subgroups, denoted by $G(\sigma(k))$ and $\bar{G}(\sigma(k))$, where σ represents one partition for the set $\{1, \ldots, n\}$, taking values from $\Upsilon := \{1, 2, \ldots, s = 2^n\}$. The total number of agents in each group is represented by $|G(\sigma(k))|$ and $|\bar{G}(\sigma(k))|$ respectively. Thus, $|G(\sigma(k))| + |\bar{G}(\sigma(k))| = n$.

Now we are ready to introduce the *representative state* for a subgroup, which is the average of the states of agents in the subgroup. The formal definition is given below.

Definition 2.1. *Let $r_G : \mathbb{R}^{|G|} \to \mathbb{R}$ and $r_{\bar{G}} : \mathbb{R}^{|\bar{G}|} \to \mathbb{R}$ be defined as follows:*

$$r_G(\mathbf{x}) = \frac{1}{|G|} \sum_{i \in G} x_i \tag{2}$$

and

$$r_{\bar{G}}(\mathbf{x}) = \frac{1}{|\bar{G}|} \sum_{i \in \bar{G}} x_i. \tag{3}$$

We call $r_G(\mathbf{x})$ and $r_{\bar{G}}(\mathbf{x})$ the representative state *of subgroup G and $\bar{G}$ respectively.*

In this brief, we assume that at each step k,

$$\phi_{\sigma(k)}(\mathbf{x}(k)) := r_{G(\sigma(k))}(\mathbf{x}(k)) - r_{\bar{G}(\sigma(k))}(\mathbf{x}(k)) \tag{4}$$

is available to each agent though the agents do not know what the partition $\sigma(k)$ is. The variable $\phi_{\sigma(k)}(\mathbf{x}(k))$ is called the *relative group representative state* at step k.

Remark 2.1. *The relative group representative state is a piece of global information to some extent. However, in some applications such as distributed beamforming for spatially distributed multiple transmitters, the state (i.e., the phase of each transmission signal) and the inter-agent relative state (i.e., the phase difference between any two transmission signals) can not be obtained as the agents themselves do not know their clock offsets with respect to a common clock and also they do not know how much the phase shifts when the transmission signal reaches the destination due to different distances from it. But the phase difference between two groups of transmitters can be measured by a cooperative receiver and sent back to the transmitters to help achieve synchronization of the transmission signal phases at the destination. Though the group representative state is a common variable available to all the agents, it can not be used to achieve consensus trivially since each agent can only update its state according to* (1) *using u_i for adjustment without knowing its own absolute state.*

2.2. Continuous-time Case

In continuous time, the n agents are governed by the following continuous dynamics

$$\dot{x}_i(t) = u_i(t), \qquad i = 1, \cdots, n \tag{5}$$

where $x_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ are the state and control input of agent i.

We consider a switching signal $\sigma(t) : \mathbb{R} \to \Upsilon$ indicating the group partition sequence. Assume without loss of generality that a random group partitioning occurs at $t_1, t_2, \ldots$. In other words, during the time interval $(t_i, t_{i+1}]$, the partition remains unchanged.

For any switching signal $\sigma(t)$, the two subgroups at time t are represented by $G(\sigma(t))$ and $\bar{G}(\sigma(t))$. The available relative representative state at time t is

$$\phi_{\sigma(t)}(\mathbf{x}(t)) = r_{G(\sigma(t))}(\mathbf{x}(t)) - r_{\bar{G}(\sigma(t))}(\mathbf{x}(t)) \tag{6}$$

where $r_G(\mathbf{x})$ and $r_{\bar{G}}(\mathbf{x})$ are defined in Definition 2.1.

2.3. Our Objective

The objective of this brief is to achieve state consensus using the *relative group representative state* rather than inter-agent relative states as investigated in most consensus literature. To this goal, we suppose each agent knows which subgroup it belongs to and access only the relative group representative state $\phi_{\sigma(k)}(\mathbf{x}(k))$ in discrete time and $\phi_{\sigma(t)}(\mathbf{x}(t))$ in continuous time.

In this brief, we consider a random partitioning scheme. That is, $\{\sigma(k) : k = 1, 2, \dots\}$ is assumed to be an i.i.d. random sequence with the common probability distribution $\mathbb{P}\{\sigma(k) = m\} = p_m$ for $m = 1, 2, \dots, s$. The following notions of consensus will be used.

Definition 2.2. *(***Almost sure consensus***) The system (1) reaches consensus almost surely under an appropriate control law if for any initial state* $\mathbf{x}(0)$*, it holds that*

$$\mathbb{P}\left\{\lim_{k\to\infty}(x_i(k) - x_j(k)) = 0\right\} = 1 \tag{7}$$

for all $i, j = 1, \dots, n$*.*

Definition 2.3. *(***Consensus in mean***) The system (1) reaches* consensus in mean *under an appropriate control law if for any initial state* $\mathbf{x}(0)$*, it holds that*

$$\lim_{k\to\infty} \mathbb{E}[x_i(k) - x_j(k)] = 0 \tag{8}$$

for all $i, j = 1, \dots, n$*. If in addition for every* i*,*

$$\lim_{k\to\infty} \mathbb{E}[x_i(k)] = \frac{x_1(0) + \cdots + x_n(0)}{n}, \tag{9}$$

then the system (1) is said to reach average consensus in mean*.*

The notions of almost sure consensus and consensus in mean are similar in continuous time and thus are omitted here.

3. Consensus in Discrete-time

In this section, we investigate the consensus problem in discrete time. Two partitioning schemes will be considered. The first case assumes that each agent i takes its own probability q_i to classify itself into either subgroup G or $\bar{G}$ at each step k, for which q_i may not equal to q_j for different agents i and j. The second case assumes that all the agents take the same probability $q_1 = \cdots = q_n = 0.5$ for themselves to be partitioned into subgroup G and $\bar{G}$. For both cases, one knows that $\{\sigma(k) : k = 1, 2, \dots\}$ is an i.i.d. random sequence with the common probability distribution $\mathbb{P}\{\sigma(k) = m\} = p_m \neq 0$ for $m = 1, 2, \dots, s$. Finally, we define

$$N(\sigma(k)) = |G(\sigma(k))||\bar{G}(\sigma(k))|.$$

3.1. Non-uniform probability for group selection

In this subsection, we assume that at step k, each agent i has probability q_i in a subgroup $G(\sigma(k))$ and has probability $1 - q_i$ in the other subgroup $\bar{G}(\sigma(k))$.

Each agent takes the following control law based on the relative group representative state feedback $\phi_{\sigma(k)}$ for the consensus purpose. If $N(\sigma(k)) = 0$, then $u_i(k) = 0$; Otherwise,

$$u_i(k) = \begin{cases} -\frac{1}{\gamma}\phi_{\sigma(k)}(\mathbf{x}(k)) & \text{when } i \in G(\sigma(k)) \\ \frac{1}{\gamma}\phi_{\sigma(k)}(\mathbf{x}(k)) & \text{when } i \in \bar{G}(\sigma(k)) \end{cases} \tag{10}$$

where $\gamma > 0$ is a control parameter.

With the above control law, the overall system can be written as

$$\mathbf{x}(k+1) = \mathbf{x}(k) + \mathbf{W}(\sigma(k))\mathbf{x}(k) \tag{11}$$

where $\sigma(k)$ is the switching signal indicating the partition at step k. If $N(\sigma(k)) = 0$, then $\mathbf{W}(\sigma(k)) = \mathbf{0}$; Otherwise, the ijth entry of $\mathbf{W}(\sigma(k))$ is

$$w_{ij}(\sigma(k)) = \begin{cases} -\frac{1}{\gamma|G(\sigma(k))|} & \text{if } i \in G(\sigma(k)), j \in G(\sigma(k)) \\ \frac{1}{\gamma|\bar{G}(\sigma(k))|} & \text{if } i \in G(\sigma(k)), j \in \bar{G}(\sigma(k)) \\ \frac{1}{\gamma|G(\sigma(k))|} & \text{if } i \in \bar{G}(\sigma(k)), j \in G(\sigma(k)) \\ -\frac{1}{\gamma|\bar{G}(\sigma(k))|} & \text{if } i \in \bar{G}(\sigma(k)), j \in \bar{G}(\sigma(k)). \end{cases}$$

Now we present our first main result for the discrete-time case.

Theorem 3.1. *A group of agents reaches consensus almost surely under the control law (10) if and only if* $\gamma > 1$*.*

Proof: Consider the following positive definite function

$$V(\mathbf{x}) = \frac{1}{2}\sum_{i\neq j}(x_i - x_j)^2.$$

The function $V(\mathbf{x})$ equals to 0 if $x_i = x_j$ for all i and j, and is positive otherwise. Along the solution of (11), the following is obtained.

(1) If $N(\sigma(k)) = 0$, then

$$V(\mathbf{x}(k+1)) = V(\mathbf{x}(k)).$$

(2) If $N(\sigma(k)) \neq 0$, then

$$\begin{aligned} & V(\mathbf{x}(k+1)) \\ = \;& \sum_{\substack{i \in G(\sigma(k)) \\ j \in \bar{G}(\sigma(k))}} \left[x_i(k) - x_j(k) - \tfrac{2}{\gamma}\phi_{\sigma(k)}(\mathbf{x}(k))\right]^2 \\ & + \tfrac{1}{2}\textstyle\sum_{i_1, i_2 \in G(k)} \left[x_{i_1}(k) - x_{i_2}(k)\right]^2 \\ & + \tfrac{1}{2}\textstyle\sum_{j_1, j_2 \in \bar{G}(k)} \left[x_{j_1}(k) - x_{j_2}(k)\right]^2 \\ = \;& V(\mathbf{x}(k)) + \tfrac{4N(\sigma(k))}{\gamma^2}\phi^2_{\sigma(k)}(\mathbf{x}(k)) \\ & - \tfrac{4}{\gamma}\phi_{\sigma(k)}(\mathbf{x}(k)) \textstyle\sum_{i\in G(\sigma(k)), j\in\bar{G}(\sigma(k))} \left[x_i(k) - x_j(k)\right] \\ = \;& V(\mathbf{x}(k)) + (1-\gamma)\tfrac{4N(\sigma(k))}{\gamma^2}\phi^2_{\sigma(k)}(\mathbf{x}(k)). \end{aligned} \tag{12}$$

Notice that for an i.i.d. random sequence $\{\sigma(k) : k = 1, 2, \ldots\}$, $\{\mathbf{x}(k) : k = 1, 2, \ldots\}$ is also a random sequence. Thus, we obtain

$$\begin{aligned}\mathbb{E}[V(\mathbf{x}(k+1))] &= \mathbb{E}[V(\mathbf{x}(k))] \\ &\quad + \tfrac{4(1-\gamma)}{\gamma^2} \textstyle\sum_{m=1}^{s} p_m N(m) \mathbb{E}[\phi_m^2(\mathbf{x}(k))].\end{aligned} \tag{13}$$

(**Sufficiency**) Suppose $\gamma > 1$. Then it follows from (13) that

$$\mathbb{E}[V(\mathbf{x}(k+1))] \leq \mathbb{E}[V(\mathbf{x}(k))] \leq \mathbb{E}[V(\mathbf{x}(0))] < \infty.$$

Therefore, $\mathbb{E}[V(\mathbf{x}(k))]$ has a limit as k tends to ∞, which in turn implies that

$$\lim_{k\to\infty} \mathbb{E}[\phi_m^2(\mathbf{x}(k))] = 0 \ \ \forall m = 1, \ldots, s \tag{14}$$

due to (13) and $\gamma > 1$.

Moreover, from (13), it is known that $V(\mathbf{x}(k))$ is a nonnegative supermartingale. So by Doob's first martingale convergence theorem (Ash and Doleans-Dade, 2000), $V(\mathbf{x}(k))$ has a limit as $k \to \infty$ in a pointwise sense. Considering (13), this also implies that the nonnegative sequence $\phi_m^2(\mathbf{x}(k))$, $m = 1, \ldots, s$, has a limit as $k \to \infty$ in a pointwise sense as well, denoted as $\phi_m^2(\mathbf{x}(\infty))$. Thus, by Fatou's Lemma (Ash and Doleans-Dade, 2000), it follows that

$$\mathbb{E}[\phi_m^2(\mathbf{x}(\infty))] \leq \lim_{k\to\infty} \mathbb{E}[\phi_m^2(\mathbf{x}(k))] = 0, \ \ \forall m = 1, \ldots, s.$$

Then it is clear that $\phi_m^2(\mathbf{x}(\infty)) = 0$ with probability one for every $m = 1, \ldots, s$.

Without loss of generality, we consider a partition $G(m_1) = \{i\}$ and $\bar{G}(m_1) = \{1, \ldots, n\} - \{i\}$ and another partition $G(m_2) = \{j\}$ and $\bar{G}(m_2) = \{1, \ldots, n\} - \{j\}$, for which $i \neq j$. Thus,

$$\phi_{m_1}^2(\mathbf{x}(\infty)) = \left(x_i(\infty) - \frac{\sum_{l\neq i} x_l(\infty)}{n-1} \right)^2 = 0$$

with probability one implies

$$x_i(\infty) = \frac{\sum_{l\neq i} x_l(\infty)}{n-1} \tag{15}$$

with probability one. Similarly, we get

$$x_j(\infty) = \frac{\sum_{l\neq j} x_l(\infty)}{n-1} \tag{16}$$

with probability one. Subtracting (16) from (15) leads to the conclusion that

$$x_i(\infty) - x_j(\infty) = 0$$

with probability one. Since i and j are arbitrarily chosen, we conclude that the group of agents reach consensus almost surely.

(**Necessity**): Suppose $\gamma \leq 1$. Then from (12), it follows that for any initial state satisfying $x_i(0) \neq x_j(0)$ for $i \neq j$, $V(\mathbf{x}(k)) \geq \mathbb{E}[V(\mathbf{x}(0))] > 0$ for any sequence $\{\sigma(k) : k = 1, 2, \ldots\}$. It is therefore not able to reach consensus almost surely. ■

3.2. *Uniform probability for group selection*

In this subsection, we assume that all the agents take the same probability $q_1 = \cdots = q_n = 0.5$ for themselves to be partitioned into subgroup G and $\bar{G}$ at each step k. For this case, we show that not only almost sure consensus can be achieved, but also average consensus in mean is achieved.

Theorem 3.2. *Suppose $q_1 = \cdots = q_n = 0.5$ and $\gamma > 1$. Then a group of agents reaches average consensus in mean under the control law* (10).

Proof: Let

$$S(\mathbf{x}(k)) = \sum_{i=1}^{n} x_i(k).$$

Considering the control law (10), we obtain that

$$\begin{aligned}&\mathbb{E}[S(\mathbf{x}(k+1))] \\ &= \mathbb{E}\left[\textstyle\sum_{i\in G(\sigma(k))} x_i(k+1) + \sum_{j\in\bar{G}(\sigma(k))} x_j(k+1)\right] \\ &= \mathbb{E}[S(\mathbf{x}(k))] - \mathbb{E}\left[\tfrac{|G(\sigma(k))| - |\bar{G}(\sigma(k))|}{\gamma} \phi_{\sigma(k)}(\mathbf{x}(k))\right].\end{aligned}$$

Since $q_1 = \cdots = q_n = 0.5$, it is known that

$$\mathbb{P}\{\sigma(k) = 1\} = \cdots = \mathbb{P}\{\sigma(k) = s\} = \frac{1}{2^n},$$

from which it is obtained that

$$\mathbb{E}\left[\frac{|G(\sigma(k))| - |\bar{G}(\sigma(k))|}{\gamma} \phi_{\sigma(k)}(\mathbf{x}(k))\right] = 0.$$

Therefore,

$$\mathbb{E}[S(\mathbf{x}(k+1))] = \mathbb{E}[S(\mathbf{x}(k))].$$

That is, the mean of the sum of all the agents' states is constant, i.e.

$$\lim_{k\to\infty} \mathbb{E}[S(\mathbf{x}(k))] = S(\mathbf{x}(0)).$$

Moreover, by Theorem 3.1, we know that the group of agents reaches consensus almost surely when $\gamma > 1$. Hence, it follows that

$$\lim_{k\to\infty} \mathbb{E}[x_i(k)] = \frac{x_1(0) + \cdots + x_n(0)}{n}, \ i = 1, \ldots, n. \qquad ■$$

4. Consensus in Continuous-time

In this section, we investigate the continuous-time case, for which each agent i takes probability q_i to partition itself in a subgroup G and $1 - q_i$ in the other subgroup $\bar{G}$, and keeps this choice until next re-selection time. Thus, it leads to a switching signal, denoted by $\sigma(t)$, representing the evolution of partitions. Suppose the switching occurs at the time instants $t_0 = 0, t_1, t_2, \ldots$. Without loss of generality, assume that the time intervals between any consecutive switching instants are uniformly upper bounded. Then $\{\sigma(t_k) : k = 0, 1, 2, \ldots\}$ is an i.i.d. random sequence with the common probability distribution $\mathbb{P}\{\sigma(t_k) = m\} = p_m \neq 0$ for $m = 1, 2, \ldots, s$. For

$t \in [t_k, t_{k+1})$, we define

$$N(\sigma(t)) = |G(\sigma(t_k))||\bar{G}(\sigma(t_k))|.$$

In continuous time, our control law at time t is given by

$$u_i(t) = \begin{cases} -\frac{1}{\gamma}\phi_{\sigma(t)}(\mathbf{x}(t)) & \text{if } i \in G(\sigma(t)) \\ \frac{1}{\gamma}\phi_{\sigma(t)}(\mathbf{x}(t)) & \text{if } i \in \bar{G}(\sigma(t)) \end{cases} \tag{17}$$

when $N(\sigma(t)) \neq 0$, and $u_i(t) = 0$ otherwise, where γ is a positive constant. With the above control law, the overall system becomes

$$\dot{\mathbf{x}}(t) = \mathbf{W}(\sigma(t))\mathbf{x}(t), \tag{18}$$

for which the ijth entry of $\mathbf{W}(\sigma(t))$ is defined as

$$w_{ij}(\sigma(t)) = \begin{cases} -\frac{1}{\gamma|G(\sigma(t))|} & \text{if } i \in G(\sigma(t)), j \in G(\sigma(t)) \\ \frac{1}{\gamma|\bar{G}(\sigma(t))|} & \text{if } i \in G(\sigma(t)), j \in \bar{G}(\sigma(t)) \\ \frac{1}{\gamma|G(\sigma(t))|} & \text{if } i \in \bar{G}(\sigma(t)), j \in G(\sigma(t)) \\ -\frac{1}{\gamma|\bar{G}(\sigma(t))|} & \text{if } i \in \bar{G}(\sigma(t)), j \in \bar{G}(\sigma(t)). \end{cases}$$

Next we give our main result in continuous time with a randomly switching group partition scheme.

Theorem 4.1. *Under the control law (17), a group of agents reaches consensus almost surely.*

Proof: We consider

$$V(\mathbf{x}(t)) = \frac{1}{2}\sum_{i \neq j}(x_i(t) - x_j(t))^2,$$

which is positive when the states are not in consensus and 0 otherwise. During any time interval $(t_k, t_{k+1}]$, $k = 0, 1, \ldots$, we have

$$\begin{aligned} \dot{V} &= \frac{dV}{dx}\frac{dx}{dt} = \sum_{i \neq j}(x_i - x_j)(\dot{x}_i - \dot{x}_j) \\ &= \sum_{i \neq j}(x_i - x_j)\left(\sum_l w_{il}(\sigma(t))x_l - \sum_l w_{jl}(\sigma(t))x_l\right). \end{aligned}$$

Note that $w_{il}(\sigma(t)) - w_{jl}(\sigma(t))$ equals to 0 if i and j are in the same group, and equals to $2w_{il}(\sigma(t))$ otherwise. Thus, we obtain that

$$\begin{aligned} \dot{V} = &\sum_{i \in G(\sigma(t)), j \in \bar{G}(\sigma(t))}(x_i(t) - x_j(t))\left[2\sum_l w_{il}(\sigma(t))x_l(t)\right] + \\ &\sum_{i \in \bar{G}(\sigma(t)), j \in G(\sigma(t))}(x_i(t) - x_j(t))\left[2\sum_l w_{il}(\sigma(t))x_l(t)\right] \\ &= -\frac{4}{\gamma}\phi_{\sigma(t)}(\mathbf{x}(t))\sum_{i \in G(\sigma(t)), j \in \bar{G}(\sigma(t))}\left[x_i(t) - x_j(t)\right] \\ &= -\frac{4N(\sigma(t))}{\gamma}\phi^2_{\sigma(t)}(\mathbf{x}(t)). \end{aligned}$$

Considering an i.i.d. random sequence $\{\sigma(t_k) : k = 0, 1, \ldots\}$, we take the expectation of $\dot{V}$ and obtain

$$\mathbb{E}[\dot{V}] = -\frac{4}{\gamma}\sum_{m=1}^{s} p_m N(m)\mathbb{E}[\phi^2_{\sigma(t)}(\mathbf{x}(t))] \leq 0.$$

Thus, it is known that $\mathbb{E}[V(\mathbf{x}(t))]$ is non-increasing in any time interval $(t_k, t_{k+1}]$ for $k = 0, 1, \ldots$. Since $V(\mathbf{x}(t))$ is a continuous function, it follows that $\mathbb{E}[V(t_{k+1})] \leq \mathbb{E}[V(t_k)]$ for all k. Thus, the remaining proof is the same as the one for Theorem 3.1. ■

5. Simulation results

In this section, we will give several numerical simulations to verify our theoretical results. In the simulations, we consider totally $n = 30$ agents. The initial state of each agent is a random value between 1 and 100.

5.1. Discrete-time Case

In this subsection, we first consider the discrete-time control law (10), for which all the agents take probability $q_1 = \cdots = q_n = 0.7$ in G and probability 0.3 in $\bar{G}$ at each step k. The simulation results with $\gamma = 1.2$ and $\gamma = 0.9$ are presented in Fig. 1 and Fig. 2 respectively. As shown in Fig. 1(a) where $\gamma = 1.2 > 1$, a group of agents reaches consensus, while in Fig. 2(a) where $\gamma = 0.9 < 1$, they diverge. These phenomena can also be observed in Fig. 1(b) and Fig. 2(b), where $V(\mathbf{x}(k)) = \frac{1}{2}\sum_{i \neq j}(x_i(k) - x_j(k))^2$ is used as a metric. The simulation results validate our theoretic result that a group of agents reach consensus almost surely if and only if $\gamma > 1$.

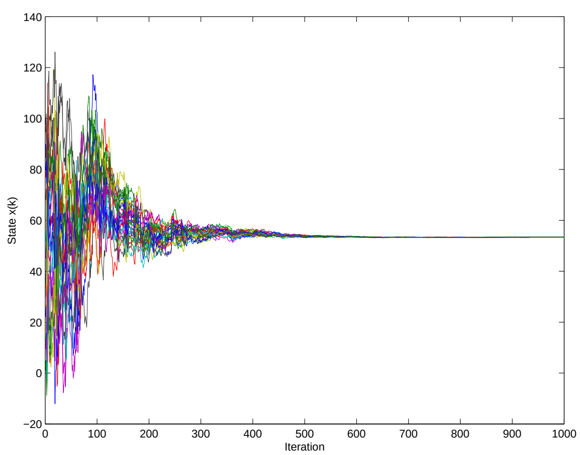
Second, we consider the uniform partition probability $q_1 = \cdots = q_n = 0.5$ and run the simulations 100 times with the same initial states. The final consensus values versus the 100 simulation runs are given in Fig. 3. It is known that the average of the consensus values approximately equals to the average of the initial states for the 100 runs.

5.2. Continuous-time Case

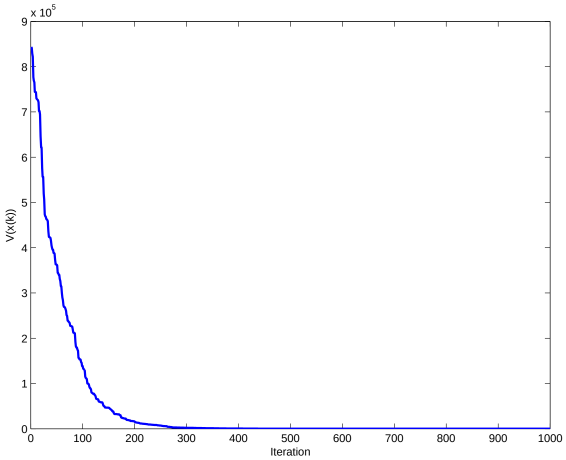
In the following, we give a simulation result for the continuous-time case. The partition is made about every second with probability $q_1 = \cdots = q_n = 0.6$. Two parameters, namely, $\gamma = 0.2$ and $\gamma = 1.2$, are used in the simulation. The simulated trajectories of $\mathbf{x}(t)$ are plotted in Fig. 4. It shows that a group of agents reaches consensus for both $\gamma = 0.2$ and $\gamma = 1.2$, but the convergence using $\gamma = 0.2$ is faster than the one using $\gamma = 1.2$.

6. Conclusions

In this brief, we present a new distributed scheme to reach consensus for a group of agents if their inter-agent relative states are not available. The new scheme considers a random partition of agents into two subgroups at each step and then uses the relative group representative state as feedback information for the consensus purpose. It is shown in this brief that almost sure consensus can be achieved under the proposed scheme in both discrete time and continuous time. For the discrete time case, a

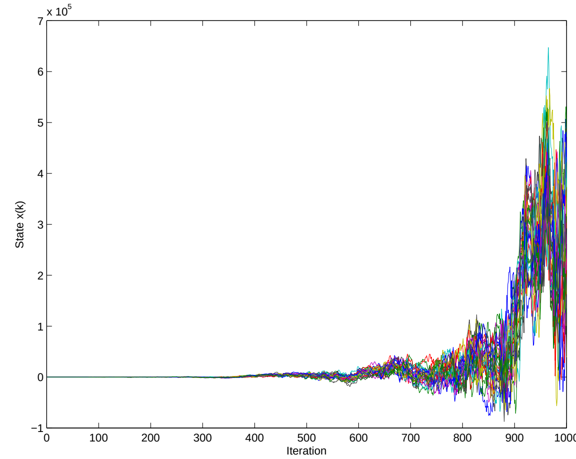


(a)

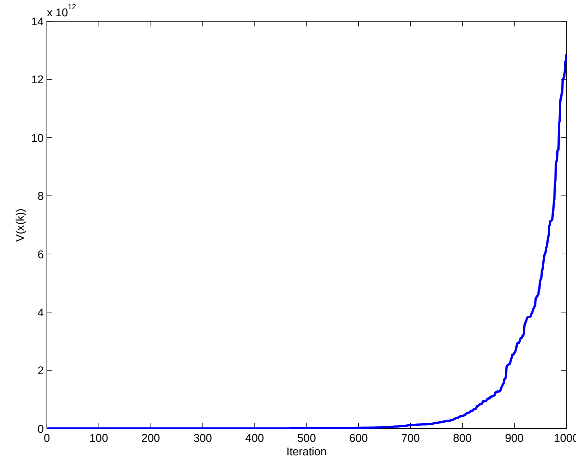


(b)

Figure 1: Reach consensus in discrete time under (11) with $\gamma = 1.2 > 1$. Each of totally 30 agent has probability 0.7 in G and probability 0.3 in $\bar{G}$ at each step. (a) The state evolution of the thirty agents; (b) The evolution of $V(\mathbf{x}(k)$ that converges to 0.



(a)



(b)

Figure 2: Do not reach consensus in discrete time under (11) with $\gamma = 0.9 < 1$. Each of totally 30 agent has probability 0.7 in G and probability 0.3 in $\bar{G}$ at each step. (a) The state evolution of the thirty agents; (b) The evolution of $V(\mathbf{x}(k)$ that diverges to ∞.

necessary and sufficient condition to achieve almost sure consensus is that the weighting parameter for state update is greater than one. For the continuous time case, almost sure consensus can be realized if the weighting parameter is positive. Moreover, it is shown that if a uniform probability is considered for group partitioning, then the group of agents reaches average consensus in mean. Numerical simulations also demonstrate the effectiveness of the proposed scheme.

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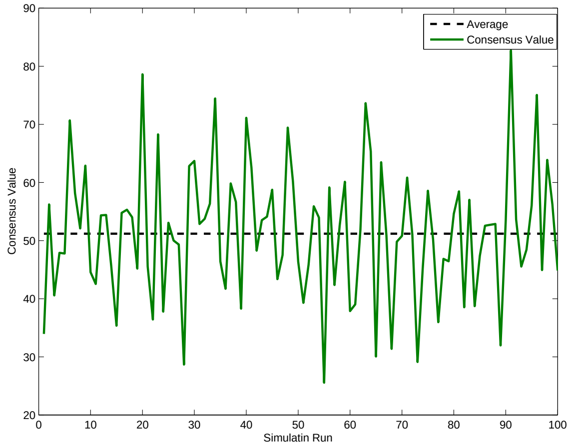

Figure 3: The consensus values with respect to different simulation runs under (11) with $\gamma = 1.2$. In each simulation, the initial state takes the same one and $q_1 = \cdots = q_{30} = 0.5$. The average of the consensus value in totally 100 simulation runs approximately equals to the average of the initial states.

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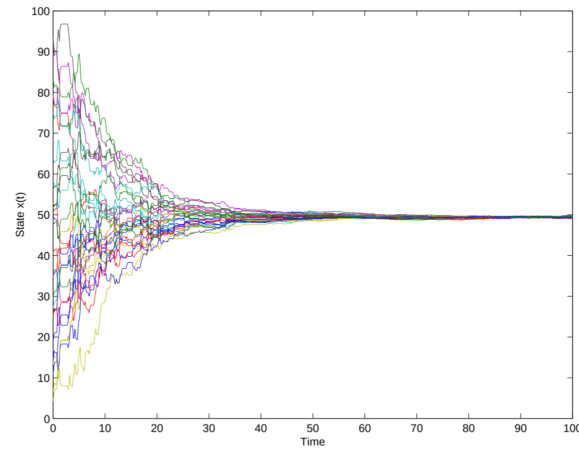
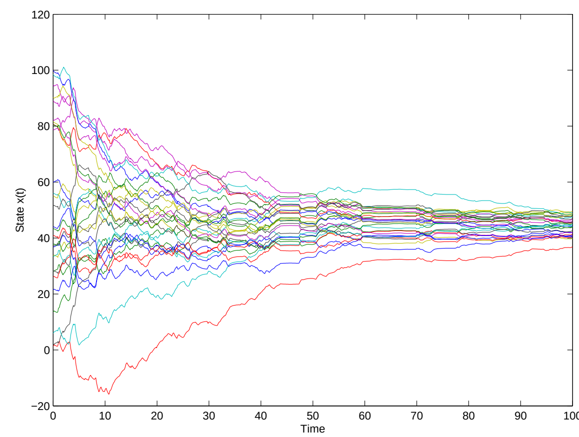


(a) $\gamma = 0.2$



(b) $\gamma = 1.2$

Figure 4: Reach consensus in continuous-time under (18) with $\gamma = 0.2$ and $\gamma = 1.2$. Each of totally 30 agents has probability 0.6 in G and probability 0.4 in $\bar{G}$.