ELSEVIER

24 November 1997

PHYSICS LETTERS A

Physics Letters A 235 (1997) 591–596

Equilibrium configurations for a system of N particles in the plane

Kathryn Glass

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge CB2 1ST, UK

Received 25 July 1997; revised manuscript received 1 September 1997; accepted for publication 2 September 1997
Communicated by A.P. Fordy

Abstract

We bring together work on central configurations of the classical N-body problem and on asymptotic properties of a system of repelling particles. By introducing a parameter β, we consider configurations that occur in these problems to be special cases of a more general system. We describe an asymmetric configuration obtained in the eight-particle system and investigate a bifurcation occurring in the four-particle system. © 1997 Elsevier Science B.V.

Keywords: Repulsive interactions; Particles; Central configurations; Bifurcation; N-body problem

1. Introduction

A *central configuration* of a system of masses attracting one another according to a potential function $U(r) = 1/r$ is an initial configuration of the masses that gives rise to periodic behaviour under a suitable choice of initial velocities. The set of central configurations of this system can be shown to be equivalent to the solutions of a certain system of nonlinear equations (see Ref. [1]), and there has been much work done to investigate the central configurations that occur for various numbers of masses. In particular, Albouy has recently identified all central configurations with four equal masses (see Refs. [2,3]).

In studying repelling particles, Fusco and Oliva define the *asymptotic shape* of a system of particles (see Ref. [4]). They consider the system induced by the potential function $U(r) = -\ln r$, and prove that only a restricted set of shapes are possible for this potential. These asymptotic shapes have been shown to be the solutions of a system of nonlinear equations, but there has been no real investigation of the shapes that can occur. This Letter will outline some results on asymptotic shapes for this system, and in particular will present an asymptotic shape of the eight-particle system in the plane that has no axis of symmetry.

In order to compare the attracting and repelling problems, we will introduce a general system of differential equations such that the potential function $U(r)$ satisfies $U'(r) = U'(1)/r^{1+\beta}$. From this, we will derive a system of nonlinear equations in terms of the parameter β, and define the solutions of these equations to be *equilibrium configurations*. For $\beta = 1$, the equilibrium configurations are the central configurations of the attracting problem, and for $\beta = 0$, the equilibrium configurations are asymptotic shapes of the repelling problem. We will then compare the results for these two problems. In particular, we will look at the case of four particles, where we briefly discuss a bifurcation also mentioned by Albouy [3].

PII S0375-9601(97)00720-2

2. Framework of the problem

Although much of the following is applicable to particles in higher-dimensional systems, we will consider the motion of N point-particles in the plane. We denote the position of the ith particle at time t by $x_i(t) \in \mathbb{R}^2$, and make the following assumptions about the system. Firstly, we assume that all forces arise from pairwise interactions between the particles, and may be described by a potential function $U(r)$. Secondly, we assume that the masses of the particles are equal, so that the equations may be expressed in the form

$$\ddot{x}_i = -\sum_{\substack{j=1 \\ j\neq i}}^{N} U'(|x_i - x_j|)\frac{(x_i - x_j)}{|x_i - x_j|}, \quad i = 1, \ldots, N.$$

Finally, we restrict our attention to a class of these potentials, $U(r)$, such that $U'(r)$ is a homogeneous function of degree $-1-\beta$. The equations of motion then become

$$\ddot{x}_i = -U'(1)\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{(x_i - x_j)}{|x_i - x_j|^{2+\beta}}, \quad i = 1, \ldots, N. \qquad (1)$$

The sign of $U'(1)$ determines whether the system is attracting or repelling. Varying the parameter β changes the strength of the forces between particles. At $\beta = -1$, the force between two particles is independent of the distance between them. We will assume $\beta \geqslant -1$ to avoid forces which grow as the distance between particles increases.

We may assume, without loss of generality, that the centre of mass of this system is at the origin. In order to avoid singularities in the potential, we also assume that no two particles can be in the same position at the same time, and thus we let

$$X = \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^{2N} : \right.$$
$$\left. x_i \neq x_j, i \neq j; \sum_{i=1}^{N} x_i = 0 \right\}$$

be the configuration space of the system.

We would like to observe the changes that occur over time in the shape that is made by the system of particles. To do so, we introduce new variables in which the size of the system remains constant.

Definition 2.1. We define the *shape*, $u(t)$, of a system of particles at time t by

$$u_i(t) = \frac{x_i(t)}{|x(t)|}, \quad i = 1, \ldots, N.$$

It follows from this and the definition of X that the space of possible shapes is given by

$$S = \left\{ (u_1, \ldots, u_N) \in \mathbb{R}^{2N} : u_i \neq u_j, i \neq j; \right.$$
$$\left. \sum_{i=1}^{N} u_i = 0; \sum_{i=1}^{N} |u_i|^2 = 1 \right\}.$$

We consider two shapes, u and $\bar{u}$ to be *equivalent*, and write $u \sim \hat{u}$ if u can be transformed into $\hat{u}$ by a rotation about the origin.

Definition 2.2. An equivalence class of the relation $\sim$ is a *configuration* of particles.

Thus $S/\sim$ is the set of possible configurations of the system. We will look at two special types of solution to Eq. (1) called *homothetic* solutions and *homographic* solutions. They are defined as follows.

Definition 2.3. A solution $(x_i(t), \ldots, x_N(t))$ of Eq. (1) is a *homothetic solution* if the shape formed by the system of particles remains constant for all time.

Definition 2.4. A solution $(x_i(t), \ldots, x_N(t))$ of Eq. (1) is a *homographic solution* if the configuration formed by the system of particles remains constant for all time.

A homothetic solution is a special type of homographic solution in which there is no rotation of the system. To find homothetic solutions, we set $x_i(t) = \rho(t)u_i$ for each i, where ρ is a real function, and $u \in S$ is an arbitrary shape that is kept fixed. Substituting this change of variable into Eq. (1), we obtain the following system,

$$u_i = \frac{1}{\sum_{j<k} |u_j - u_k|^{-\beta}} \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{(u_i - u_j)}{|u_i - u_j|^{2+\beta}},$$
$$i = 1, \ldots, N$$

$$\rho''(t) = -\frac{U'(1)\rho(t)}{|\rho(t)|^{2+\beta}} \sum_{j<k} |u_j - u_k|^{-\beta}. \qquad (2)$$

We are interested in the shapes, u, that satisfy the above equations.

Definition 2.5. A shape, $u \in S$, is an *equilibrium shape* if there exists a function ρ over the reals such that if $x_i(t) = \rho(t)u_i$ for all i, then $x(t)$ is a homothetic solution.

Clearly, if a shape is an equilibrium shape, then all other shapes in its equivalence class are also equilibrium shapes. In general, we will want to group these equilibrium shapes together and consider only the distinct configurations satisfying Eqs. (2). We thus define an *equilibrium configuration* as follows,

Definition 2.6. A configuration is an *equilibrium configuration* if the shapes in this equivalence class are equilibrium shapes.

The classical N-body problem concerns N masses interacting under gravitational forces with equations

$$\ddot{x}_i = -\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j(x_i - x_j)}{|x_i - x_j|^3}, \quad i = 1, \ldots, N,$$

where m_j is the mass of the jth particle. All homothetic solutions of the problem consist of the particles collapsing to the origin, while the homographic solutions include periodic orbits. Configurations of particles that give rise to homographic solutions of the N-body problem are called *central configurations*. If the masses of the N particles are set to one, these central configurations are the equilibrium configurations of the general system (1) with $\beta = 1$.

3. Asymptotic shapes

Consider a system of repelling particles with equations of motion defined by Eq. (1) with $U'(1) = -1$. We are interested in the behaviour of this system in the limit as time tends to infinity. In order to describe this behaviour, Fusco and Oliva (see Ref. [4]) define *asymptotic velocities* and the *asymptotic shape* of the system as follows.

Definition 3.1. If the velocity of every particle approaches a limit as time tends to infinity, we set

$$v_i^+ = \lim_{t\to\infty} \dot{x}_i(t), \quad i = 1, \ldots, N$$

and refer to the limits, v_i^+ as *asymptotic velocities*.

Definition 3.2. We define the shape, u, of the system by $u_i(t) = x_i(t)/|x(t)|$ for all i. If the limit

$$u_i^+ = \lim_{t\to\infty} u_i(t), \quad i = 1, \ldots, N$$

exists for all i, then the limiting shape, u^+ is the *asymptotic shape* of the system.

It follows from these definitions that any system for which asymptotic velocities exist will also have an asymptotic shape. If we release a repelling system from an equilibrium shape with zero velocity, the particles will diverge to infinity such that the shape made by the particles remains the same for all time. Thus, the set of equilibrium shapes is a subset of the set of asymptotic shapes for the values of β for which asymptotic shapes exist.

Let us now consider the values of β for which our system will possess these asymptotic properties. Galperin has proved that asymptotic velocities will exist for systems with $\beta > 0$ (see Ref. [5] or Ref. [6] for an alternative proof). Fusco and Oliva have then shown in Refs. [7,6] that for systems with $\beta > 0$, the set of asymptotic shapes coincides with the set of all shapes, S. That is, any shape that does not have two particles at one location is an asymptotic shape for some initial condition if $\beta > 0$.

In Ref. [4], Fusco and Oliva consider the system with $\beta = 0$, which is induced by the potential $U(r) = -\ln r$. They prove that under this potential, the velocities of the particles do not approach an asymptotic limit, but the system always approaches an asymptotic shape. Moreover, they show that all asymptotic shapes, u, are solutions of the equations

$$u_i = \frac{2}{N(N-1)} \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{(u_i - u_j)}{|u_i - u_j|^2}, \quad i = 1, \ldots, N.$$

Comparing these equations to Eqs. (2), we see that the set of asymptotic shapes is exactly equal to the set of equilibrium shapes for this problem. Therefore, for any initial position of the particles, the shape made by them will approach an equilibrium shape in the limit as time tends to infinity. In Ref. [4], Fusco and Oliva conjecture that asymptotic shapes will exist for $\beta > -1$, and it seems possible that if they do exist, the asymptotic shapes will coincide with equilibrium shapes for $-1 < \beta \leqslant 0$; however, no results have been proved as yet for $\beta < 0$.

4. Equilibrium configurations

Let us return to the general system with equations of motion (1). Recall that an equilibrium configuration is an equivalence class of equilibrium shapes, where we say that two shapes are equivalent if they may be transformed into one another under a rotation. In order to find equilibrium shapes, we must solve Eqs. (2). We thus look for zeros of the following system of equations,

$$f_i(u) = u_i \sum_{j<k} |u_j - u_k|^{-\beta} - \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{(u_i - u_j)}{|u_i - u_j|^{2+\beta}}, \quad i = 1, \ldots, N. \tag{3}$$

Since we are assuming that all masses in the problem are equal, we will not distinguish between two configurations that may be transformed into one another by permuting the particles.

As discussed in Ref. [4], the symmetries in Eqs. (3) allow us to predict some configurations. For example, an equilibrium configuration in the form of an N-gon of radius $1/\sqrt{N}$ exists for all values of N, as does a configuration in the form of an $(N-1)$-gon of radius $1/\sqrt{N-1}$ with a particle at the origin. While N is small, we may solve Eq. (3) fully to find all possible equilibrium configurations. For the two-particle system, there is one distinct equilibrium configuration, which corresponds to setting $u_2 = -u_1$. We have also solved the three-particle case algebraically (see Ref. [8]) for $\beta \geqslant -1$ to find two configurations (modulo permutations of particles) only: an equilateral triangle configuration and a linear configuration with one particle at the origin.

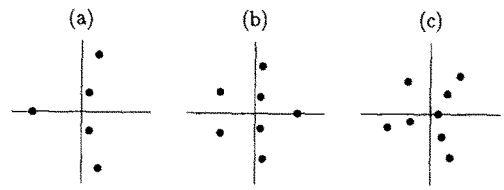

Fig. 1. Some examples of equilibrium configurations for the system with $\beta = 0$. Diagram (c) is an example of a configuration with no symmetries.

In order to find further equilibrium configurations, we use numerical methods. The NAG Fortran library contains routines that will search for zeros of a system of equations, given an initial condition. A very simplistic method of finding equilibrium configurations is then to choose random initial conditions for the system and then apply a numerical routine to search for a nearby zero. By iterating this procedure many times, we hope to find a number of equilibrium configurations. Clearly this method is ad hoc, and we certainly cannot claim that it will find all the zeros of the equations. However, for small values of N, it does appear to find all the configurations we can predict algebraically, and in addition, finds some more unusual and interesting configurations. Some examples of these are given in Fig. 1. All configurations pictured in Fig. 1 are equilibrium configurations for the system with $\beta = 0$. Fig. 1a has five particles and Fig. 1b has seven particles. The early numerical results suggested that all configurations might have (at least) one axis of symmetry, and indeed, we have not found any asymmetric configurations for $N \leqslant 7$. However, the conjecture that all equilibrium configurations should have an axis of symmetry is contradicted by the configura-

Table 1
Location of the particles in the eight-particle asymmetric equilibrium configuration with $\beta = 0$. Positions are given to four significant figures

x-coordinate	y-coordinate
0.07806	0.0000
0.1760	0.1944
0.2996	0.3700
−0.2306	0.3142
−0.4317	−0.1342
−0.2048	−0.07813
0.1157	−0.2335
0.1977	−0.4327

tion pictured in Fig. 1c, which has eight particles in an equilibrium configuration with no symmetries. Table 1 gives the (numerically computed) positions of the particles in this configuration to four significant figures. We have traced this configuration using the AUTO 97 package (see Ref. [9]) to verify that it exists for β in the range $-1 \leqslant \beta \leqslant 1$. Some unpublished diagrams resulting from a search by Moeckel for central configurations with eight particles or fewer were brought to our attention by a referee. For the eight-particle system, Moeckel has found two asymmetric configurations, which agree with our bifurcation diagram at $\beta = 1$.

5. An example: $N = 4$

The four-particle system is the simplest case that has not been solved algebraically. It is an interesting case to consider, because the results for $\beta = 0$ and $\beta = 1$ suggest that a bifurcation takes place in the system between these values. Let us first look at these two cases, and then attempt to explain the changes that take place.

5.1. The system with $\beta = 1$

Equilibrium configurations with $\beta = 1$ are central configurations of the four-body problem with equal masses. Much research has been devoted to this problem in the general case where the masses of the particles are varied (see, for example, Refs. [2,1,10]). One of the most significant results is that of Albouy (see Ref. [2]), which states that *every* central configuration of four particles with equal masses has an axis of symmetry. In Ref. [3], he then extends this result to prove that the only central configurations of this problem areas follows.

(a) A configuration with four particles lying on a line through the origin.

(b) A configuration with particles at the vertices of a square.

(c) A configuration with particles at the vertices of an equilateral triangle and one particle at the origin.

(d) A configuration with particles at the vertices of an isosceles (but not equilateral) triangle and one particle within the triangle and on its symmetry axis.

These configurations are pictured in Fig. 2.

5.2. The system with $\beta = 0$

The form of Eqs. (2) with $\beta = 0$ is simpler than with $\beta = 1$. Albouy considers this system in Ref. [3], and states that the symmetry result in Ref. [2] is still valid. With this additional result, these equations may, as noted by Albouy, be solved algebraically to give three solutions only: the linear configuration, the configuration in the form of a square and the equilateral configuration with a particle at the origin. This proves that the isosceles triangle configuration does not exist for $\beta = 0$.

5.3. Varying the value of β

The results in Sections 5.1 and 5.2 suggest that a bifurcation occurs in the system, leading to the annihilation of the isosceles triangle configuration somewhere in the region $0 \leqslant \beta < 1$. To investigate the bifurcations of the isosceles solution in the region $-1 \leqslant \beta \leqslant 1$ numerically, we use the AUTO 97 package (see Ref. [9]), which allows us to follow the zeros of our system of nonlinear functions as we vary the parameter β. Applying this to our equations we produce the bifurcation diagram given in Fig. 3. This figure confirms some remarks of Albouy [3] that the isosceles solution approaches the equilateral solution at $\beta = -1$ and $\beta = 0$ and gives some numerical support to his conjecture that there exists only one isosceles solution. Fig. 3 shows two bifurcations (marked by squares) occurring between the isosceles triangle configuration and the equilateral triangle configuration. The x-axis of this diagram represents β, while the y-axis gives the distance, d, from the origin of the particle of the triangle lying on the axis of symmetry. For the equilateral triangle, the distance remains fixed. For the isosceles triangle, however, this distance varies, coinciding with the equilateral triangle at -1 and 0, where bifurcations take place. These bifurcations are examples of a generic D_3 transcritical steady-state bifurcation [11]. Some discussion of bifurcations with D_3 symmetry is given in Ref. [12].

6. Conclusions

In looking at equilibrium configurations of a system of repelling particles, we bring together work on

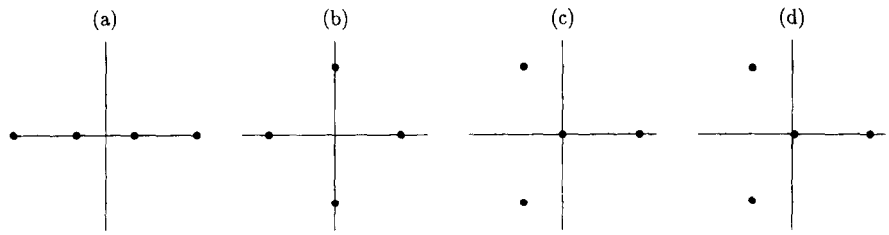

Fig. 2. The four equilibrium configurations for the four-particle system with $\beta = 1$. See Section 5.1 for a description of the configurations

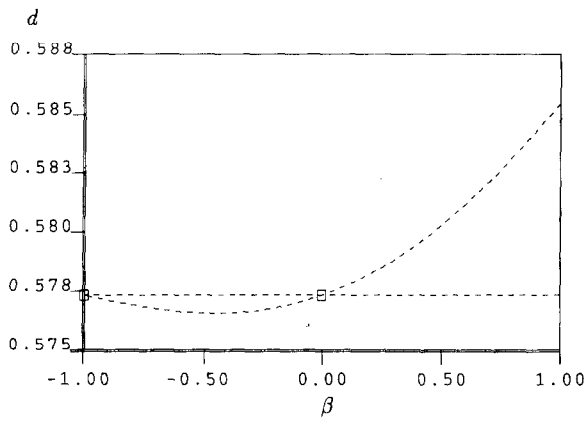

Fig. 3. A bifurcation diagram showing the equilateral triangle and isosceles triangle configurations for $-1 \leqslant \beta \leqslant 1$ The squares mark bifurcations detected by the AUTO software, the x-axis represents β and the y-axis gives the distance, d, of the particle of the triangle lying on the axis of symmetry from the origin.

central configurations of the N-body problem and on asymptotic properties of a system of repelling particles. By introducing a parameter, β, which allows us to vary the underlying potential function, we may consider the configurations that occur in these two problems to be special cases of equilibrium configurations of a more general system of particles. Numerical investigation of this system has yielded many varied configurations, some of which have been presented in this Letter. In particular, we have discovered an asymmetric configuration with eight particles, and this is the smallest number of particles for which we have found a configuration with no axis of symmetry. Bifurcations occur in the general system of particles as the parameter, β, is varied and we have discussed an example with four particles. We have made a preliminary study of bifurcations in the system with $N > 4$, and we have found that at $N = 8$, the system has a very complicated bifurcation structure over the region $-1 \leqslant \beta \leqslant 1$. The eight-particle system is also interesting for its asymmetric configurations, and the number of configurations we have found without symmetry varies between one and three in the range $-1 \leqslant \beta \leqslant 1$. Investigating the general system in this way, we will obtain a better understanding of central configurations of the N-body problem and asymptotic shapes of repelling particles.

References

[1] R. Moeckel, Math. Zeit. 205 (1990) 499.
[2] A. Albouy, C.R. Acad. Sci. Serie I 320 (1995) 217.
[3] A. Albouy, Contemp. Math. 198 (1996) 131.
[4] G. Fusco, W.M. Oliva, Formation of symmetric structures in the dynamics of repelling particles (1996), in preparation.
[5] G.A. Galperin, Comm. Math. Phys. 84 (1982) 547.
[6] G. Fusco, W.M. Oliva, J. Diff. Eqs. 135 (1907) 16.
[7] G. Fusco, W.M. Oliva, Asymptotic behaviour of a system of repelling particles: asymptotic velocities, phases, scattering operators and integrability, to appear in the Proc. of Equadiff 95.
[8] K. Glass, Properties of a system of N particles (1997), unpublished.
[9] E.J. Doedel, H.B. Keller, J.P. Kerneévez, Int. J. Bif. Chaos 1 (1991) 493.
[10] Z. Yan, Y. Sun, Sci. Sin. Series A 31 (1988) 724.
[11] P. Ashwin, private communication.
[12] M. Golubitsky, I. Stewart, D. Schaeffer, Singularities and Groups in Bifurcation Theory, Vol. 2 (Springer, Berlin, 1988).