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Option-based portfolio and consumption insurance (OBPCI)

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ABSTRACT

This paper examines portfolio insurance (PI) problems for investors who require a minimum level of consumption. Its key contribution is introducing the Option-Based Portfolio and Consumption Insurance (OBPCI) strategy, which extends the popular OBPI framework. As the optimal solution to a constrained utility maximization problem under a general local covariance model, OBPCI is derived using the martingale approach and can be interpreted as a portfolio of options. Given the lack of valid benchmarks involving consumption in the literature, we also introduce and formalize two competitive strategies, optimal on their own, albeit suboptimal to our main problem: the Synthetic Constant Proportion Portfolio and Consumption Insurance (SCPPCI) and the Synthetic Option-Based Portfolio and Consumption Insurance (SOBPCI). Our numerical analysis shows that, under realistic parameters, SCPPCI and SOBPCI can lead to equivalent welfare losses of up to 9% for a short investment horizon and 5% for low risk aversion, respectively, relative to OBPCI.

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1. Introduction

Portfolio Insurance (PI) is a dynamic asset allocation strategy aimed at protecting against downside risk while preserving the potential for gains in rising markets (see, for example, Balder & Mahayni, 2010; Basak, 2002; Zagst & Kraus, 2011). It is designed to meet specified insurance constraints over the investment horizon, ensuring that the portfolio exceeds a benchmark. Consequently, PI has been widely adopted by both institutional and retail investors. The benchmark can be another portfolio comprising market assets, including the risk-free asset. For instance, a pension fund may require a pre-specified minimum terminal fund value due to regulatory capital constraints.

However, in practice, investors often seek not only a minimum terminal wealth but also a minimum level of consumption during the investment period. For example, the Guaranteed Minimum Withdrawal Benefit (GMWB), a type of variable annuity in the real world, provides income security for policyholders, with the income primarily intended to support minimum retirement consumption needs. Similarly, in the context of pension funds, a minimum guaranteed withdrawal can be reflected by a lower bound on consumption, regardless of investment performance. Such a guarantee ensures that retirees can maintain a minimum standard of living throughout retirement. It also provides individuals with greater certainty when planning for long-term consumption. In the literature, such a minimum consumption level (also known as a consumption subsistence constraint) is considered as a form of consumption habit in portfolio optimization problems (see Cheng & Wei, 2005; Gong & Li, 2006; Lim et al., 2018; Shim & Shin, 2014).

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Against this background, this paper aims to explore PI when the investor also requires a minimum consumption. In the literature, the consumption constraint is sparsely discussed compared to terminal wealth constraints. With terminal wealth constraints solely, the two prominent portfolio insurance strategies are Option-Based Portfolio Insurance (OBPI) and Constant Proportion Portfolio Insurance (CPPI). OBPI is introduced by Leland and Rubinstein (1976), involving a portfolio invested in a risky asset and a synthetic put option written on it. At the same time, CPPI is developed by Perold (1986) for fixed-income assets and by Black and Jones (1987) for equities. The CPPI strategy sets a floor, typically invested in cash, equal to the required minimum portfolio value. In contrast, the cushion – the excess of the portfolio value over the floor – is invested in risky assets. In the standard expected utility maximization framework proposed by Neumann and Morgenstern (1944), the OBPI strategy is optimal for investors with constant relative risk aversion (CRRA) preferences (see, for example, El Karoui et al., 2005), while the CPPI strategy is optimal for investors with hyperbolic absolute risk aversion (HARA) preferences (Black & Perold, 1992; Brennan & Solanki, 1981; Leland, 1980). As identified by Basak (2002), this distinction can be attributed to whether the marginal utility of wealth jumps discontinuously or goes smoothly to infinity.

This paper focuses on extending the OBPI strategy to account for the consumption constraint.[1] In the literature, extensions of the OBPI strategy have been explored under various terminal wealth constraints, often using the martingale method developed by Cox and Huang (1989) and Karatzas et al. (1987). For example, Basak (1995), Grossman and Zhou (1996), Basak (2002), and Korn (2005) consider a constant lower bound on terminal wealth, while Teplá (2001) introduces a stochastic lower bound based on the performance of a benchmark portfolio, where consumption is not allowed. El Karoui et al. (2005) investigates an American-style guarantee on the portfolio over the entire horizon, with the European-style guarantee (a terminal constraint) treated as a special case. Kronborg (2014) extends El Karoui et al. (2005) by incorporating consumption and labor income, showing that the optimal portfolio corresponds to an adjusted OBPI strategy. A typical result from these papers is that the optimal portfolio for the constrained problem can be decomposed into the optimal portfolio for an associated unconstrained problem and a synthetic put option. Terminal constraints can also take the form of Value-at-Risk (VaR) or expected shortfall (see, for example, Basak & Shapiro, 2001; Boyle & Tian, 2007; Chen et al., 2018; Kraft & Steffensen, 2013; Mi & Xu, 2023), where solutions imply that the investor can protect the portfolio by purchasing options with partial coverage against downside risk. However, both strategies will lose their optimality and show limitations when the inter-temporal consumption is incorporated.

Although Basak (1995), Basak (2002), and Kronborg (2014) consider consumption in their models, they do not impose constraints on the consumption rate. Bardhan (1994) pioneered existence results for utility maximization with lower bounds on consumption and terminal wealth; paving the way for explicit solutions and applications as per the work of Teplá (2001) and Lakner and Ma Nygren (2006). In Lakner and Ma Nygren (2006), these lower bounds are constants, and the optimal investment strategy is explicitly derived using the Malliavin calculus. Yuan and Hu (2009) considers a time-varying lower bound on consumption and a constant lower bound on terminal wealth, with the investment strategy derived using Ito's lemma. However, the strategies in these papers are challenging to implement in actual practice due to the need for continuous portfolio reallocation. In contrast, the OBPI strategy is static and does not require rebalancing after the initial purchase of put options, making it more practical and thus appealing. Drawing inspiration from the OBPI strategy, our objective is to develop a strategy that is straightforward to implement and optimal when consumption constraints are considered.

In this paper, we address a CRRA utility maximization problem with predetermined lower bounds on intermediate consumption rates and terminal wealth, extending the classical OBPI strategy to a general family of local covariance structures; like for instance, the multivariate geometric

[1] Extending the CPPI strategy under the same constraint is relatively straightforward: Minimum consumption can be secured through a cash investment. We name this the SCPPCI strategy and discuss it in the main text of this paper.

Brownian motion (GBM) model and a multivariate version of the LVO-CEV model proposed by Anel and Fan (2025). The latter model is a specific type of constant elasticity of variance (CEV) model (see Beckers, 1980; Cox & Huang, 1989; Cox & Ross, 1976; MacBeth & Merville, 1980), with the excess return considered linear in the volatility.

We derive the optimal solution by applying the martingale approach and Lagrangian methods, similarly to Teplá (2001) and Lakner and Ma Nygren (2006), but assuming a broader general covariance model and targeting closed-form representations. By the OBPI literature and expressing the optimal wealth process in terms of options, we find that a time-integrated put option (see Aytekin & Birge, 2004, for an application of such a product) is required to ensure the minimum consumption guarantee. Nielsen and Steffensen (2008) also find that a continuum of options should serve as a component of the total assets in the optimal life insurance and investment strategy to hedge against the event of death, where the sum insured is a control variable subject to maximum and/or minimum constraints, with no consumption incorporated. Similarly to El Karoui et al. (2005) and Kronborg (2014), we identify a put-based separation of the initial wealth: a proportion λ is allocated to unconstrained consumption and investment. At the same time, the remainder $(1 - \lambda)$ is used to purchase a time-integrated European put option and a classic European put option. This strategy is referred to as the Option-Based Portfolio and Consumption Insurance (OBPCI) strategy due to its similarity to OBPI. We provide an expression for λ in terms of Lagrangian multipliers, illuminating the relationship between the constrained and unconstrained problems.

To perform a comparative analysis, we propose two sub-optimal strategies: the Synthetic Constant Proportion Portfolio and Consumption Insurance (SCPPCI) and the Synthetic Option-Based Portfolio and Consumption Insurance (SOBPCI). These strategies substitute one or both put options in the OBPCI strategy with a cash investment. We also show that these two strategies correspond to optimal solutions to constrained problems with different utility functions. Our numerical analysis shows that the SOBPCI outperforms the SCPPCI strategy. However, when the time horizon is long and the risk aversion parameter is low, the OBPCI strategy performs significantly better.

In summary, our main contribution is the proposal of the Option-Based Portfolio and Consumption Insurance (OBPCI) strategy, developed under a general family of covariance structures to accommodate investors with minimum consumption needs. The OBPCI strategy is straightforward to implement due to its static nature – no rebalancing is required after the initial purchase of put options, making it practically applicable, for example, in pension funds. In addition, we investigate two sub-optimal strategies, SCPPCI and SOBPCI, and highlight their differences from the OBPCI strategy through theoretical analysis and numerical performance comparisons. We also analyze the impact of using different market models on the optimal strategy and show the welfare loss arising from the mismatch between the GBM-based optimal strategy and the LVO-CEV market.

The remainder of the paper is structured as follows. Section 2 describes the market environment, while Section 3 presents the optimization problem and solves it using the Martingale method. Then, we propose the OBPCI strategy based on put options written on the Merton portfolio, both with and without consumption. The sub-optimal strategies, SCPPCI and SOBPCI, are also proposed and investigated. In Section 4, we discuss the implied investment strategy needed to replicate OBPCI, SCPPCI, and SOBPCI using market-available assets in the GBM model as the more straightforward application. Section 5 provides a numerical analysis of the OBPCI strategy. We also offer a comparative study of the OBPCI with the two proposed sub-optimal strategies. Furthermore, we show the welfare loss from applying a GBM-based optimal strategy in an LVO-CEV market. Section 6 concludes this paper.

2. Problem formulation

Let us consider a complete continuous-time financial market with a finite horizon $[0, T]$, consisting of a risk-free bond and N risky stocks. Uncertainty in the financial market is represented by a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where $\mathbb{P}$ is the real-world probability measure and

$\mathcal{F}_t$ is the natural filtration generated by an N-dimensional independent standard Brownian motion $\boldsymbol{Z}_t = (Z_{1,t}, Z_{2,t}, \ldots, Z_{N,t})^\top$.

The bond price B_t is given by:

$$\mathrm{d}B_t = rB_t\,\mathrm{d}t, \quad B_0 = 1;$$

while the stock prices $\boldsymbol{S}_t = (S_{1,t}, S_{2,t}, \ldots, S_{N,t})^\top$ are assumed to be described by:

$$\mathrm{d}S_{j,t} = S_{j,t}\left[\mu_{j,t}\,\mathrm{d}t + \boldsymbol{\sigma}_{j,t}\,\mathrm{d}\boldsymbol{Z}_t\right], \quad j = 1, \ldots, N,$$

where r is the risk-free interest rate, while the drift vector $\boldsymbol{\mu}_t = (\mu_{1,t}, \mu_{2,t}, \ldots, \mu_{N,t})^\top$ and the volatility vector $\boldsymbol{\sigma}_{j,t} = (\sigma_{j1,t}, \sigma_{j2,t}, \ldots, \sigma_{jN,t})$ are assumed to be progressively measurable with respect to $\mathcal{F}_t$. The volatility matrix is defined by $\boldsymbol{\sigma}_t = (\sigma_{jk,t})_{j,k=1,\ldots,N}$, with the condition that the covariance matrix of the log-returns of the stocks, $\boldsymbol{\sigma}_t\boldsymbol{\sigma}_t^\top$, is strictly positive definite.

In this paper, we will consider a drift structure that satisfies $\boldsymbol{\mu}_t = r\bar{\mathbf{1}} + \boldsymbol{\sigma}_t\boldsymbol{\kappa}$ where $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \ldots, \kappa_N)^\top$ is a constant vector and $\bar{\mathbf{1}} \equiv (1, 1, \ldots, 1)^\top$. It should be noted that special cases of our proposal are the multivariate Brownian motion (GBM) and the LVO-CEV model (with the linear excess return in the volatility proposed by Anel & Fan, 2025). For these models, some details will be presented in Section 4.

Due to completeness and no arbitrage in the market, there exists a unique change of measure defined by the Radon-Nikodym derivative η_t, i.e.

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} \equiv \eta_t = \exp\left\{-\int_0^t \boldsymbol{\kappa}^\top \mathrm{d}\boldsymbol{Z}_s - \frac{1}{2}\int_0^t \|\boldsymbol{\kappa}\|^2\,\mathrm{d}s\right\}, \quad t \in [0, T].$$

Under the risk-neutral measure $\mathbb{Q}$, we can define a N-dimensional standard Brownian motion:

$$\mathrm{d}\boldsymbol{Z}_t^{\mathbb{Q}} = \mathrm{d}\boldsymbol{Z}_t + \boldsymbol{\kappa}\,\mathrm{d}t.$$

Consequently, the state price density process ζ_t is defined as follows:

$$\zeta_t = \exp\left\{-rt - \int_0^t \boldsymbol{\kappa}^\top\,\mathrm{d}\boldsymbol{Z}_s - \frac{1}{2}\int_0^t \|\boldsymbol{\kappa}\|^2\,\mathrm{d}s\right\},$$

with dynamics

$$\mathrm{d}\zeta_t = -\zeta_t\left[r\mathrm{d}t + \boldsymbol{\kappa}^\top\,\mathrm{d}\boldsymbol{Z}_t\right]. \tag{1}$$

It is assumed that an investor can continuously consume his wealth and invest in this financial market over the period $[0, T]$ with an initial wealth W_0. This investor requires his wealth at the terminal date T to be at least greater than a non-negative floor $D(T)$, while maintaining a non-negative minimum consumption rate $a(t)$ through $[0, T]$. In this study, we let $D(T)$ and $a(t), t \in [0, T]$ be deterministic functions of time to ensure an analytical solution to this optimal control problem.

The dynamics of the investor's wealth are

$$\mathrm{d}W_t = \left(W_t\left[r + \boldsymbol{\pi}_t^\top(\boldsymbol{\mu}_t - r\bar{\mathbf{1}})\right] - c_t\right)\,\mathrm{d}t + W_t\boldsymbol{\pi}_t^\top\boldsymbol{\sigma}_t\,\mathrm{d}\boldsymbol{Z}_t, \tag{2}$$

where $\boldsymbol{\pi}_t \equiv (\pi_{1,t}, \ldots, \pi_{N,t})^\top$ represents the fraction of the wealth W_t invested in each stock; c_t is the consumption rate, which should be no less than $a(t)$ as indicated above.

From the dynamics Equations (2) and (1), it could be shown by Ito's formula that

$$\mathrm{d}\left(\zeta_t W_t\right) = -\zeta_t c_t \,\mathrm{d}t + \zeta_t W_t \left(\boldsymbol{\pi}_t^\top \boldsymbol{\sigma}_t - \boldsymbol{\kappa}^\top\right) \,\mathrm{d}\boldsymbol{Z}_t,$$

which is equivalent to

$$\zeta_T W_T + \int_t^T \zeta_s c_s \,\mathrm{d}s = \zeta_t W_t + \int_t^T \zeta_s W_s \left(\boldsymbol{\pi}_s^\top \boldsymbol{\sigma}_s - \boldsymbol{\kappa}^\top\right) \,\mathrm{d}\boldsymbol{Z}_s \quad \forall\, t \in [0, T].$$

Since the last term in the above equation is a martingale under measure $\mathbb{P}$, we have

$$W_t = \frac{1}{\zeta_t} \mathbb{E}\left[\int_t^T \zeta_s c_s \,\mathrm{d}s + \zeta_T W_T \middle| \mathcal{F}_t\right] \quad \forall\, t \in [0, T]. \tag{3}$$

It should be noted that Equation (3) could also be written as the expectation under measure $\mathbb{Q}$:

$$W_t = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)} c_s \,\mathrm{d}s + e^{-r(T-t)} W_T \middle| \mathcal{F}_t\right] \quad \forall\, t \in [0, T]. \tag{4}$$

Specifically, if we let $t = 0$ in Equation (3), then

$$\begin{aligned} W_0 &= \mathbb{E}\left[\int_0^T \zeta_s c_s \,\mathrm{d}s + \zeta_T W_T\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rs} c_s \,\mathrm{d}s + e^{-rT} W_T\right], \end{aligned} \tag{5}$$

which is the budget constraint at time 0.

Definition 2.1: A consumption and investment strategy $(c_t, \boldsymbol{\pi}_t)_{t\in[0,T]}$ is said to be admissible if the following conditions are satisfied:

(1) c_t and $\boldsymbol{\pi}_t$ are progressively measurable with respect to $\mathcal{F}_t$ such that

$$\int_0^T \left(\|\boldsymbol{\pi}_t\|^2 + c_t\right) \,\mathrm{d}t < \infty, \quad \text{a.s.}$$

(2) The strategy is self-financing, i.e. the budget constraint (5) is met.

Let us define

$$D(t) = \int_t^T e^{-r(s-t)} a(s) \,\mathrm{d}s + e^{-r(T-t)} D(T), \tag{6}$$

representing the outstanding liability of the investor at time t. We note that $D(T)$ represents an exogenous lower bound that depends on both the choice of $a(t)$ and the time horizon T, and is more dynamic than the usual (constant) OBPI liability. To ensure at least one admissible strategy, according to the constraints $W_T \geq D(T)$ and $c_t \geq a(t), \forall\, t \in [0, T]$, we should have

$$\begin{aligned} W_0 &\geq \mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rs} a(s) \,\mathrm{d}s + e^{-rT} D(T)\right] \\ &= \int_0^T e^{-rs} a(s) \,\mathrm{d}s + e^{-rT} D(T) \\ &= D(0), \end{aligned}$$

which provides a necessary condition for the optimization problem.

3. The optimization problem and the OBPCI strategy

In this section, we will present and solve a constrained optimization problem. Then, we will give an optimal solution to the corresponding unconstrained optimization problem. Finally, we will propose the Option-Based Portfolio and Consumption Insurance (OBPCI) strategy by developing the relationship between these two optimal solutions.

It is assumed that the investor gains utility from continuous consumption c_t and terminal wealth W_T through a constant relative risk aversion (CRRA) utility function $u : [0,\infty) \to (-\infty,\infty)$ in the form:

$$u(x) = \begin{cases} \dfrac{x^{1-\gamma}}{1-\gamma} & \text{if } x > 0 \\ \lim\limits_{x \searrow 0} \dfrac{x^{1-\gamma}}{1-\gamma} & \text{if } x = 0 \end{cases},$$

for some $\gamma \in (0,\infty)\backslash\{1\}$.

The investor seeks a consumption and investment strategy to maximize the utility over a finite time horizon, while maintaining the minimum consumption and terminal wealth requirements throughout the period $[0,T]$. For given non-negative deterministic $a(t), t \in [0,T]$ and $D(T)$, the dynamic optimal control problem, which is named as **Constrained Problem**, could be written using the objective function J_C,

$$\begin{cases} J_C\left(t=0; W_0, \{a(t)\}_{t\in[0,T]}, D(T)\right) = \sup_{c_t,\boldsymbol{\pi}_t} \mathbb{E}\left[\int_0^T u\left(c_t\right)\,\mathrm{d}t + u\left(W_T\right)\right] \\ \text{s.t.} \quad c_t \geq a(t), \quad \text{a.e.} \quad \forall\, t \in [0,T], \\ W_T \geq D(T), \quad \text{a.e.} \end{cases} \tag{7}$$

By the change of measure, it can be seen that the objective function J_C is equivalent to the static one:

$$\begin{cases} J_C\left(t=0; W_0, \{a(t)\}_{t\in[0,T]}, D(T)\right) = \sup_{c_t,W_T} \mathbb{E}\left[\int_0^T u\left(c_t\right)\,\mathrm{d}t + u\left(W_T\right)\right] \\ \text{s.t.} \quad W_0 = \mathbb{E}\left[\int_0^T \zeta_t c_t\,\mathrm{d}t + \zeta_T W_T\right], \\ c_t \geq a(t), \quad \text{a.e.} \quad \forall\, t \in [0,T], \\ W_T \geq D(T), \quad \text{a.e.} \end{cases} \tag{8}$$

where the admissible strategy introduces the three constraints.

Proposition 3.1: *The optimal consumption and terminal wealth to* ***Constrained Problem*** (8) *are presented next.*

The optimal consumption rate is

$$c_t^* = \max\left(\left(\psi_1\zeta_t\right)^{-\frac{1}{\gamma}}, a(t)\right), \tag{9}$$

and the optimal terminal wealth is

$$W_T^* = \max\left(\left(\psi_1\zeta_T\right)^{-\frac{1}{\gamma}}, D(T)\right), \tag{10}$$

where ψ_1 is chosen to satisfy the budget constraint, i.e.

$$W_0 = \mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rt}c_t^*\,\mathrm{d}t + e^{-rT}W_T^*\right]. \tag{11}$$

Proof: See Appendix A.1. ■

Before we explore the implications of the optimal solutions for the investor, let us first give the optimal solutions to consumption, terminal wealth, and the investment strategy in the **Unconstrained Problem**, defined next for convenience.

$$\begin{cases} J_U\left(t=0; W_0, \{a(t)\}_{t\in[0,T]}, D(T)\right) = \sup_{c_t,\boldsymbol{\pi}_t} \mathbb{E}\left[\int_0^T u\left(c_t\right)\,\mathrm{d}t + u\left(W_T\right)\right] \\ \text{s.t.} \quad W_0 = \mathbb{E}\left[\int_0^T \zeta_t c_t\,\mathrm{d}t + \zeta_T W_T\right], \end{cases} \tag{12}$$

According to Proposition 3.1, it can be seen that when there are no constraints, i.e. $a(t) = 0, \forall\ t \in [0,T]$ and $D(T) = 0$, we have the optimal solutions:

$$c_t^{U,*} = (\phi_1\zeta_t)^{-\frac{1}{\gamma}},$$

and

$$W_T^{U,*} = (\phi_1\zeta_T)^{-\frac{1}{\gamma}},$$

where ϕ_1 satisfies the budget constraint

$$W_t^{U,*} = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)} c_s^{U,*}\,\mathrm{d}s + e^{-r(T-t)} W_T^{U,*}\right] \quad \forall\ t \in [0,T]. \tag{13}$$

Let us define

$$Y_t = (\phi_1\zeta_t)^{-\frac{1}{\gamma}} e^{\theta t},$$

with $\theta = \left(\frac{\gamma-1}{\gamma}\right)\left(r + \frac{\|\boldsymbol{\kappa}\|^2}{2\gamma}\right)$. Then Y_t is a discounted martingale under the risk neutral measure $\mathbb{Q}$, as identified by its dynamics:

$$\frac{\mathrm{d}Y_t}{Y_t} = r\,\mathrm{d}t + \frac{1}{\gamma}\boldsymbol{\kappa}^\top\,\mathrm{d}\mathbf{Z}_t^{\mathbb{Q}}, \tag{14}$$

Thus, Equation (13) could be written as

$$\begin{aligned} W_t^{U,*} &= \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)} e^{-\theta s} Y_s\,\mathrm{d}s + e^{-r(T-t)} e^{-\theta T} Y_T\right] \\ &= \int_t^T e^{-\theta s}\mathbb{E}^{\mathbb{Q}}\left[e^{-r(s-t)} Y_s\right]\mathrm{d}s + e^{-\theta T}\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} Y_T\right] \\ &= Y_t\left[\int_t^T e^{-\theta s}\,\mathrm{d}s + e^{-\theta T}\right] \\ &= (\phi_1\zeta_t)^{-\frac{1}{\gamma}}\left[\left(1 - \frac{1}{\theta}\right) e^{-\theta(T-t)} + \frac{1}{\theta}\right]. \end{aligned} \tag{15}$$

Let $t = 0$ in Equation (15), then ϕ_1 can be found explicitly:

$$\phi_1 = \left(\frac{W_0}{\left(1 - \frac{1}{\theta}\right) e^{-\theta T} + \frac{1}{\theta}}\right)^{-\gamma}. \tag{16}$$

With $f(t)$ defined by

$$f(t) = \left(1 - \frac{1}{\theta}\right) e^{-\theta(T-t)} + \frac{1}{\theta}, \tag{17}$$

we can rewrite $c_t^{U,*}$ as

$$c_t^{U,*} = (\phi_1 \zeta_t)^{-\frac{1}{\gamma}} = \frac{W_t^{U,*}}{f(t)}. \tag{18}$$

According to Equation (15), the dynamics of $W_t^{U,*}$ becomes

$$\mathrm{d}W_t^{U,*} = \left[\left(r + \frac{1}{\gamma}\boldsymbol{\kappa}^\top \boldsymbol{\kappa}\right) W_t^{U,*} - c_t^{U,*}\right] \mathrm{d}t + \frac{1}{\gamma}\boldsymbol{\kappa}^\top W_t^{U,*} \mathrm{d}\boldsymbol{Z}_t. \tag{19}$$

By comparing the diffusion term of Equations (19) and (2), it can be seen that the optimal investment strategy $\boldsymbol{\pi}_t^{U,*}$ to the **Unconstrained Problem**, (12), where $D_T = 0$ and $a(t) = 0, t \in [0, T]$, is given by

$$\boldsymbol{\pi}_t^{U,*} = \frac{1}{\gamma}\left(\boldsymbol{\sigma}_t^{-1}\right)^\top \boldsymbol{\kappa}. \tag{20}$$

It should be noted that if $\boldsymbol{\sigma}_t$ is a constant matrix (as in the GBM model), then $\boldsymbol{\pi}_t^{U,*}$ implies a constant proportion in each stock, which is well known as Merton's constant mix strategy. In the remainder of this paper, $\boldsymbol{\pi}_t^{U,*}$ is the so-called Merton-Analogous strategy.

Although optimal consumption and terminal wealth for the **Constrained Problem** have been solved, the investor may care about how to implement those solutions in practice. Proposition 3.2 provides such a method of implementation to the investor based on European put options, called the OBPCI strategy.

Proposition 3.2: *With $\lambda = (\psi_1/\phi_1)^{-\frac{1}{\gamma}}$, consider the strategy $\{\lambda c_t^{U,*}, \boldsymbol{\pi}_t^{U,*}\}_{t\in[0,T]}$ to manage $\lambda W_U^*(t)$ throughout the period $[0, T]$, combined with a time-integrated European put option with strike prices $a(t)$ and maturity t, written on $\lambda c_t^{U,*}$, $\forall\, t \in [0, T]$, whose value at time 0 is given by*

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rt} \max\left(a(t) - \lambda c_t^{U,*}, 0\right) \mathrm{d}t\right],$$

and a European put option with strike price $D(T)$ and maturity T, written on the portfolio $(\lambda W_t^{U,*})_{t\in[0,T]}$, with time-0 value to be

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-rT} \max\left(D(T) - \lambda W_t^{U,*}, 0\right)\right].$$

This strategy is named as ***Option-Based Portfolio and Consumption Insurance (OBPCI) strategy****, which lead to the optimal solution $(c_t^*, W_t^*)_{t\in[0,T]}$ to the* ***Constrained Problem****.*

Proof: Let us substitute the optimal solution $c^*(s)$ and W_T^* given in Proposition 3.1 to Equation (4), i.e. the budget constraint at time t:

$$\begin{aligned} W_t^* &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} W_T^* \middle| \mathcal{F}_t\right] + \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)} c_s^* \,\mathrm{d}s \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \max\left((\psi_1 \zeta_T)^{-\frac{1}{\gamma}}, D(T)\right) \middle| \mathcal{F}_t\right] \end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)}\max\left((\psi_1\zeta_s)^{-\frac{1}{\gamma}},a(s)\right)\,\mathrm{d}s\middle|\mathcal{F}_t\right]\\
&=\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(\psi_1\zeta_T)^{-\frac{1}{\gamma}}+\int_t^T e^{-r(s-t)}(\psi_1\zeta_s)^{-\frac{1}{\gamma}}\,\mathrm{d}s\middle|\mathcal{F}_t\right]\\
&+\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\max\left(D(T)-(\psi_1\zeta_T)^{-\frac{1}{\gamma}},0\right)\middle|\mathcal{F}_t\right]\\
&+\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)}\max\left(a(s)-(\psi_1\zeta_s)^{-\frac{1}{\gamma}},0\right)\,\mathrm{d}s\middle|\mathcal{F}_t\right].
\end{aligned}
$$

Given $\lambda=(\psi_1/\phi_1)^{-\frac{1}{\gamma}}$, $W_T^{U,*}=(\phi_1\zeta_T)^{-\frac{1}{\gamma}}$, and $c_s^{U,*}=(\phi_1\zeta_s)^{-\frac{1}{\gamma}}$, the above equation could be written as

$$
\begin{aligned}
W_t^*=\lambda\mathbb{E}^{\mathbb{Q}}&\left[e^{-r(T-t)}W_T^{U,*}+\int_t^T e^{-r(s-t)}c_s^{U,*}\,\mathrm{d}s\middle|\mathcal{F}_t\right]\\
&+\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\max\left(D(T)-\lambda W_T^{U,*},0\right)\middle|\mathcal{F}_t\right]\\
&+\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)}\max\left(a(s)-\lambda c_s^{U,*},0\right)\,\mathrm{d}s\middle|\mathcal{F}_t\right].
\end{aligned}\tag{21}
$$

According to the budget constraint (13) at time t in the **Unconstrained Problem**, the first term in Equation (21) is equal to $\lambda W_t^{U,*}$, and thus we have

$$
\begin{aligned}
W_t^*=\lambda W_t^{U,*}&+\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\max\left(D(T)-\lambda W_T^{U,*},0\right)\middle|\mathcal{F}_t\right]\\
&+\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)}\max\left(a(s)-\lambda c_s^{U,*},0\right)\,\mathrm{d}s\middle|\mathcal{F}_t\right].
\end{aligned}\tag{22}
$$

■

Remark 3.1: According to Equation (22) with $W_0^*=W_0=W_0^{U,*}$, the OBPCI strategy given in Proposition 3.2 means that the investor should reserve the initial amount of money λW_0 to follow the optimal unconstrained consumption and investment strategy $(\lambda c_t^{U,*},\boldsymbol{\pi}_t^{U,*})_{t\in[0,T]}$ while using the remaining $(1-\lambda)W_0$ to finance the purchase of the time-integrated and classic European options. Specifically, we have

$$
c_t^*=\lambda c_t^{U,*}+\max\left(a(t)-\lambda c_t^{U,*},0\right)\quad\forall\, t\in[0,T],
$$

and

$$
W_T^*=\lambda W_t^{U,*}+\max\left(D(T)-\lambda W_t^{U,*},0\right).
$$

The instantaneous payoff from the time-integrated European option, $\max(a(t)-\lambda c_t^{U,*},0)$, will be combined with $\lambda c_t^{U,*}$ for consumption, which guarantees a minimum consumption rate of $a(t)$ $\forall\, t\in[0,T]$. At the terminal time T, the initial reserve has a value $\lambda W_T^{U,*}$, thus combined with the payoff from the classic European option, $\max(D(T)-\lambda W_T^{U,*},0)$, will guarantee a minimum terminal wealth of $D(T)$.

Since $W_t^{U,*}$ is the portfolio that follows the optimal consumption rate $c_t^{U,*}$ and the Merton-Analogous strategy $\boldsymbol{\pi}_t^{U,*}$, we call the process $W_t^{U,*}$ a **Merton portfolio with consumption**. Consequently, $\lambda W_t^{U,*}$, as a multiple of $W_t^{U,*}$, is also a **Merton portfolio with consumption** with initial wealth λW_0, which can also be verified by the dynamics of $\lambda W_t^{U,*}$.

Remark 3.2: According to Equation (22), the OBPCI strategy in Proposition 3.2 is composed by a **Merton portfolio with consumption** $\lambda W_t^{U,*}$, a classic European option on that portfolio, and a time-integrated European option on $\lambda c_t^{U,*}$. Alternatively, compared with the expression $\lambda W_T^{U,*} = (\psi_1 \zeta_T)^{-\frac{1}{\gamma}}$, the consumption rate $\lambda c_t^{U,*} = (\psi_1 \zeta_t)^{-\frac{1}{\gamma}}$ can be regarded as the optimal terminal wealth in the **Constrained Problem** where the initial wealth is λW_0 and the time horizon is $[0, t]$. Thus, the time-integrated European option required in the OBPCI strategy (for consumption) can be interpreted as a series of European put options on the terminal values of the **Merton portfolio with consumption**, each having the same initial value λW_0 but different maturities, $t \in [0, T]$.

Although the synthetic options in Remark 3.2 are likely not available in the exchange market due to the underlying consumption, options with the **Merton portfolio without consumption** as underlying asset are likely to be available, as these are investments in the stock market. For this reason, we show next how the underlying assets of the options required in the OBPCI strategy can be transformed to **Merton portfolio without consumption**. As implied by Equations (20) and (14), Y_t is the portfolio value at time t by adopting $\boldsymbol{\pi}_t^{U,*}$ without consumption. Since λ is a scalar, the process λY_t is a **Merton Portfolio without consumption**, while the initial value is

$$\lambda Y_0 = \psi_1^{-\frac{1}{\gamma}}.$$

Noting that $\lambda W_T^{U,*} = \lambda(\phi_1 \zeta_T)^{-\frac{1}{\gamma}} = \lambda Y_T e^{-\theta T}$ and $\lambda c_t^{U,*} = \lambda(\phi_1 \zeta_t)^{-\frac{1}{\gamma}} = \lambda Y_t e^{-\theta t}$, the two options in Proposition 3.2 can be transformed to options on the portfolio λY_t:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\max\left(D(T) - \lambda W_T^{U,*}, 0\right)\right] = e^{-\theta T}\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\max\left(D(T)e^{\theta T} - \lambda Y_T, 0\right)\right], \tag{23}$$

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rt}\max\left(a(t) - \lambda c_t^{U,*}, 0\right)\,\mathrm{d}t\right] = \int_0^T e^{-\theta t}\mathbb{E}^{\mathbb{Q}}\left[e^{-rt}\max\left(a(t)e^{\theta t} - \lambda Y_t, 0\right)\,\mathrm{d}t\right]. \tag{24}$$

By the mean-value-theorem, there exists a $\xi \in (e^{-\theta T}, 1)$ that satisfies

$$\int_0^T e^{-\theta t}\mathbb{E}^{\mathbb{Q}}\left[e^{-rt}\max\left(a(t)e^{\theta t} - \lambda Y_t, 0\right)\,\mathrm{d}t\right] = \xi \int_0^T \mathbb{E}^{\mathbb{Q}}\left[e^{-rt}\max\left(a(t)e^{\theta t} - \lambda Y_t, 0\right)\right]\mathrm{d}t. \tag{25}$$

This equation implies that the time-integrated put option required in the OBPCI strategy is also equivalent to ξ share of the so-called standard time-integrated put option, which can be designed over-the-counter. The value of ξ can be found by (25) either analytically for some choices of $a(t)$ or through simulations.

Remark 3.3: The classic European put option given in the OBPCI strategy is equivalent to $e^{-\theta T}$ share of the European put option on λY_T with strike price $D(T)e^{\theta T}$. Similarly, the time-integrated European put option required by the OBPCI strategy is equivalent to an integrated put option on λY_t with strike price $a(t)e^{\theta t}$ and weight of $e^{-\theta t}$, $\forall\, t \in [0, T]$. These new options track the value of λY_t, a **Merton portfolio without consumption** and thus a possible tradable asset on the market, which makes the investor more likely to implement the OBPCI strategy presented by Proposition 3.2.

3.1. Sub-optimal strategy analysis

In this section, we propose two sub-optimal strategies to the **Constrained Problem**, named the Synthetic Constant Proportion Portfolio and Consumption Insurance (SCPPCI) and the Synthetic Option-Based Portfolio and Consumption Insurance (SOBPCI). The performance of these two sub-optimal strategies will be compared with that of the OBPCI strategy in Section 5.2.

In these two sub-optimal strategies, the put options in the OBPCI strategy are substituted with the investments in the risk-free bond to meet the consumption and terminal wealth constraints. Investors will likely adopt this substitution in practice due to its ease. On the other hand, the SCPPCI strategy represents the optimal solution to the constrained problem with HARA preference. In contrast, the SOBPCI strategy represents the optimal solution to the constrained problem of HARA's preference for consumption and CRRA's preference for terminal wealth. It should be noted that SCPPCI and SOBPCI each include both an investment and a consumption strategy.

Let us first define the HARA utilities for consumption and terminal wealth:

$$v_1\left(c_t\right) = \begin{cases} \dfrac{\left(c_t - a(t)\right)^{1-\gamma}}{1-\gamma} & \text{if } c_t > a(t) \\ \lim\limits_{c_t \searrow a(t)} \dfrac{\left(c_t - a(t)\right)^{1-\gamma}}{1-\gamma} & \text{if } c_t = a(t) \end{cases},$$

and

$$v_2\left(W_T\right) = \begin{cases} \dfrac{\left(W_T - D(T)\right)^{1-\gamma}}{1-\gamma} & \text{if } W_T > D(T) \\ \lim\limits_{W_T \searrow D(T)} \dfrac{\left(W_T - D(T)\right)^{1-\gamma}}{1-\gamma} & \text{if } W_T = D(T) \end{cases}.$$

3.1.1. SCPPCI strategy

In the constrained problem with HARA preference, the objective function is denoted by J_H, while $a(t)$ and $D(T)$ serve as the subsistence levels:

$$\begin{cases} J_H\left(t=0; W_0, \{a(t)\}_{t\in[0,T]}, D(T)\right) = \sup_{c_t,\boldsymbol{\pi}_t} \mathbb{E}\left[\int_0^T v_1\left(c_t\right)\,\mathrm{d}t + v_2\left(W_T\right)\right] \\ \text{s.t.} \quad W_0 = \mathbb{E}\left[\int_0^T \zeta_t c_t\,\mathrm{d}t + \zeta_T W_T\right], \\ c_t \geq a(t), \quad \text{a.e.} \quad \forall\, t \in [0,T], \\ W_T \geq D(T), \quad \text{a.e.} \end{cases} \tag{26}$$

Corollary 3.3: *The optimal solutions to the constrained problem with HARA preferences are as follows: The optimal consumption rate is*

$$c_t^{H,*} = \left(\psi^H \zeta_t\right)^{-\frac{1}{\gamma}} + a(t), \tag{27}$$

and the optimal terminal wealth is

$$W_T^{H,*} = \left(\psi^H \zeta_T\right)^{-\frac{1}{\gamma}} + D(T), \tag{28}$$

where ψ^H is chosen to satisfy the budget constraint, i.e.

$$W_0 = \mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rt} c_t^{H,*}\,\mathrm{d}t + e^{-rT} W_T^{H,*}\right]. \tag{29}$$

Proof: The proof can be done using the same method in Appendix A.1. ■

By substituting Equations (27) and (28) into (29), we have

$$\psi^H = \left(\frac{W_0 - D(0)}{\left(1 - \frac{1}{\theta}\right) e^{-\theta T} + \frac{1}{\theta}} \right)^{-\gamma}.$$

and the optimal consumption strategy is

$$c_t^{H,*} = \frac{W_t^{H,*} - D(t)}{f(t)} + a(t), \tag{30}$$

where $f(t)$ is defined by Equation (17).

The optimal investment strategy $\boldsymbol{\pi}_t^{H,*}$ to the constrained problem with HARA preference, also known as a CPPI investment strategy (see Balder & Mahayni, 2010), is given by

$$\boldsymbol{\pi}_t^{H,*} = \frac{1}{\gamma} \frac{W_t^{H,*} - D(t)}{W_t^{H,*}} \left(\boldsymbol{\sigma}_t^{-1}\right)^\top \boldsymbol{\kappa}. \tag{31}$$

This solution $(c_t^{H,*}, \boldsymbol{\pi}_t^{H,*})$ can be achieved by dividing the total portfolio into two parts:

$$W_t^H := D(t) + \left[W_t^H - D(t)\right], \tag{32}$$

and then adopting the strategy $(a(t), \mathbf{0})$[2] on segment $D(t)$, and adopting the strategy $(c_t^{U,*}, \boldsymbol{\pi}_t^{U,*})$ for the portion $W_t^H - D(t)$. This combination defines the SCPPCI strategy for managing the entire portfolio and consumption activities. The so-called SCPPCI strategy mimics the CPPI strategy by splitting the initial wealth W_0 into two segments at time 0 and managing each separately. However, it is essential to emphasize that the CPPI strategy is strictly an investment approach, while the SCPPCI strategy integrates both consumption and investment strategies.

3.1.2. SOBPCI strategy

In the constrained problem with HARA preference on consumption and CRRA preference on terminal wealth (i.e. mixed preference), the objective function is denoted by J_M:

$$\begin{cases} J_M\left(t = 0; W_0, \{a(t)\}_{t \in [0,T]}, D(T)\right) = \sup_{\boldsymbol{\pi}_t, c_t} \mathbb{E}\left[\int_0^T v_1(c_t)\, \mathrm{d}t + u(W_T)\right] \\ \text{s.t.} \quad W_0 = \mathbb{E}\left[\int_0^T \zeta_t c_t \,\mathrm{d}t + \zeta_T W_T\right], \\ c_t \geq a(t), \quad \text{a.e.} \quad \forall\, t \in [0,T], \\ W_T \geq D(T), \quad \text{a.e.} \end{cases} \tag{33}$$

Corollary 3.4: *The optimal solutions to the constrained problem with HARA preference in consumption and CRRA preference in terminal wealth are presented below.*

The optimal consumption rate is

$$c_t^{M,*} = \left(\psi^M \zeta_t\right)^{-\frac{1}{\gamma}} + a(t), \tag{34}$$

and the optimal terminal wealth is

$$W_T^{M,*} = \max\left(\left(\psi^M \zeta_T\right)^{-\frac{1}{\gamma}}, D(T)\right), \tag{35}$$

[2] This implies that the consumption rate is $a(t)$, while the allocation to risky investments is 0.

where ψ^M is chosen to satisfy the budget constraint, i.e.

$$W_0 = \mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rt} c_t^{M,*}\,\mathrm{d}t + e^{-rT} W_T^{M,*}\right]. \tag{36}$$

Proof: The proof can be done using the same method in Appendix A.1. ■

By substituting Equations (34) and (35) into (36), ψ^M should satisfy the following condition:

$$\begin{aligned} W_0 - \int_0^T e^{-r(s-t)} a(s)\,\mathrm{d}s &= \mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rt}\left(\psi^M \zeta_t\right)^{-\frac{1}{\gamma}}\,\mathrm{d}t + e^{-rT} W_T^{M,*}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rt}\left(\psi^M \zeta_t\right)^{-\frac{1}{\gamma}}\,\mathrm{d}t + e^{-rT}\max\left(\left(\psi^M \zeta_T\right)^{-\frac{1}{\gamma}}, D(T)\right)\right]. \end{aligned} \tag{37}$$

Let us define

$$X_t^M := W_t^M - \int_t^T e^{-r(s-t)} a(s)\,\mathrm{d}s.$$

According to Proposition 3.1, Equation (37) is equivalent to the budget constraint in the **Constrained Problem**, which includes an initial wealth of X_0^M, a minimum consumption level of 0, and a terminal wealth floor of $D(T)$. Notably, ψ^M matches ψ_1 under these conditions.

Following the result of Proposition 3.2 and Equation (22), the value of the portfolio X_t^M is given by

$$X_t^M = \lambda_M X_t^M + \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\max\left(D(T) - \lambda_M X_t^M, 0\right)\middle|\,\mathcal{F}_t\right].$$

Thus, we have

$$\begin{aligned} W_t^M &= \int_t^T e^{-r(s-t)} a(s)\,\mathrm{d}s + X_t^M \\ &= \int_t^T e^{-r(s-t)} a(s)\,\mathrm{d}s + \lambda_M X_t^M + \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\max\left(D(T) - \lambda_M X_t^M, 0\right)\middle|\,\mathcal{F}_t\right]. \end{aligned} \tag{38}$$

As Equation (38) indicates, the SOBPCI strategy involves splitting the entire portfolio into three segments. For the segment $\int_t^T e^{-r(s-t)} a(s)\,\mathrm{d}s$, the strategy $(a(t), \mathbf{0})$ is utilized, while $(c_t^{U,*}, \boldsymbol{\pi}_t^{U,*})$ is implemented for $\lambda_M X_t^M$. Additionally, a European put option is held, with $\lambda_M X_t^M$ as the underlying asset, expiring at T and with a strike price of $D(T)$.

The SOBPCI strategy is inspired by Kronborg (2014) using the European put option to hedge against the minimum terminal wealth constraint. In contrast, the consumption minimum constraint is secured by investing its present value in the risk-free bond. However, we note that Kronborg (2014) does not incorporate the consumption constraint.

4. The OBPCI's implied strategy on assets

We have identified the optimality of the OBPCI strategy in the general volatility model. As an alternative, the investor can allocate his wealth to the assets available in the market to replicate the OBPCI investment strategy. However, the replicating strategy depends on the option pricing formula and the specific volatility structure. In this section, we will discuss the replicating strategy under the GBM model in detail.

But first, to emphasize the approach's flexibility, we remind the reader that our methodology can be implemented on the larger family of local covariance models with a pre-specified structure on the drift. These are models where the covariance is measurable concerning the vector of Brownian Motion, for instance, a functional of the underlying asset prices. An example of such models is the LVO-CEV model. This is a specific type of constant elasticity of variance (CEV) model, with the excess return considered linear in the volatility, and the volatility with the following structure

$$\boldsymbol{\sigma}_t = \boldsymbol{S}(\boldsymbol{\beta}, t) \cdot \boldsymbol{\sigma},$$

where $\boldsymbol{\beta} \equiv (\beta_1, \beta_2, \ldots, \beta_N)^\top$ be the vector of those constant elasticity factors; $\boldsymbol{\sigma} = (\sigma_{jk})_{j,k=1,\ldots,N}$ is a constant matrix and

$$\boldsymbol{S}(\boldsymbol{\beta}, t) = \begin{pmatrix} S_{1,t}^{\beta_1} & 0 & \cdots & 0 \\ 0 & S_{2,t}^{\beta_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{N,t}^{\beta_N} \end{pmatrix}.$$

Since the LVO-CEV model admits a constant $\boldsymbol{\kappa}$, all the derivations in Section 3 are still valid.

Corollary 4.1: *Consider two continuous-time, frictionless, complete markets defined on the same filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, with the same risk-free rate r. Suppose that both markets admit the same constant $\boldsymbol{\kappa}$. Then, according to Proposition* 3.1, *the optimal consumption process c_t^* and the optimal terminal wealth W_T^* in the* ***Constrained Problem****, obtained via the martingale method, are identical in distribution across the two markets. Furthermore, the SCPPCI and SOBPCI strategies, as defined in Corollaries* 3.3 *and* 3.4*, respectively, lead to the same consumption process and terminal wealth across the two markets.*

Proof: The proof is straightforward. The optimal solution to the **Constrained Problem** depends solely on the process ζ_t. Since identical values of r and $\boldsymbol{\kappa}$ imply the same process ζ_t in distribution, the resulting optimal consumption process and terminal wealth coincide in distribution across the two markets. Similarly, the SCPPCI and SOBPCI strategies are constructed based on the optimal solutions associated with J_H and J_M, which also depend only on the process ζ_t. This completes the proof. ■

Corollary 4.1 implies that given the same r and $\boldsymbol{\kappa}$, each of the OBPCI, SCPPCI and SOBPCI strategies, across the two markets, will yield identical consumption processes and terminal wealth distributions. As a consequence, the lifetime utility, defined by $\mathbb{E}[\int_0^T u(c_t)\,\mathrm{d}t + u(W_T)]$ in the **Constrained Problem**, as well as the associated welfare loss, are identical across the two markets. Hence, the sub-optimal analysis presented in Section 5.2 under the LVO-CEV model (given the same r and $\boldsymbol{\kappa}$) leads to the same results as those under the GBM model.

Although identical ζ_t imply identical consumption and terminal wealth distributions, the trading strategies required to replicate these payoffs (in the OBPCI, SCPPCI and SOBPCI strategies), depend on the local volatility structure of the underlying assets. Consequently, even under matched process of ζ_t, GBM and LVO-CEV markets generally induce distinct portfolio strategies.

From Equation (22), the option pricing formula in the CEV model (see Delbaen & Shirakawa, 2002, for example), and the concept of delta-hedging, we can derive an explicit form for the investment strategy. However, due to the complexity of the option pricing formula in the CEV model, we will not expand it here; instead, we will focus on the special case of GBM.

Needless to say, the reader should be aware that two distinct models, such as GBM and LVO-CEV, when estimated on the same dataset, would produce different values of $\boldsymbol{\kappa}$, namely $\boldsymbol{\kappa}^{\mathrm{GBM}}$ and $\boldsymbol{\kappa}^{\mathrm{CEV}}$; see Table 2 in Anel and Fan (2025), where the notation $\bar{\lambda}$ is used for $\boldsymbol{\kappa}$. The cited study shows that $\boldsymbol{\kappa}^{\mathrm{CEV}}$ is approximately 25% larger than $\boldsymbol{\kappa}^{\mathrm{GBM}}$ across a range of assets. This observation motivates a new type of welfare-equivalent-loss (WEL) analysis, in which we examine the under-performance of an investor who follows an optimal strategy derived under the GBM specification while the true market dynamics are governed by the LVO-CEV model. Equivalently, this analysis quantifies the magnitude of the error incurred by using $\boldsymbol{\kappa}^{\mathrm{GBM}}$ when the true parameter is $\boldsymbol{\kappa}^{\mathrm{CEV}}$. The proposed analysis is carried out in Section 5.3 for our main setting, namely the OBPCI objective. Importantly, this approach is based entirely on the martingale method and reveals a novel and alternative way to compute WEL without requiring the explicit, and often difficult-to-compute, portfolio strategies.

4.1. GBM model

As a special case of the LVO-CEV model (and also the CEV model) when $\boldsymbol{\beta} = (0, 0, \dots, 0)^\top$, the GBM model admits $\boldsymbol{\sigma}_t = \boldsymbol{\sigma}$, i.e. the volatility matrix is time-invariant. Consequently, options can be priced using the straightforward and widely recognized Black-Scholes formula. Building on the approach previously suggested for the LVO-CEV model, we will derive the optimal investment strategy $\boldsymbol{\pi}_t^*$ implied by the **Constrained Problem**, (8), i.e. the proportion of wealth invested in each asset, as an alternative to the allocation to options on the OBPCI strategy, and show its limiting behaviors.

Proposition 4.2: *The optimal wealth at time t can be expressed by*[3]

$$W_t^* = D(t) + (\psi_1\zeta_t)^{-\frac{1}{\gamma}} \left(\int_t^T e^{-\theta(s-t)} \Phi\left(x_1(s)\right)\, \mathrm{d}s + e^{-\theta(T-t)} \Phi(d_1) \right) - \int_t^T e^{-r(s-t)} a(s) \Phi\left(x_2(s)\right)\, \mathrm{d}s - e^{-r(T-t)} D(T) \Phi\left(d_2\right), \tag{39}$$

where

$$\begin{aligned}
x_1(s) &= \frac{-\frac{1}{\gamma}\ln\left(\psi_1\zeta_t\right) - \ln\left(a(s)\right) + \left(r - \theta + \frac{1}{2}\nu^2\right)(s-t)}{\nu\sqrt{s-t}}, \\
x_2(s) &= x_1(s) - \nu\sqrt{s-t}, \\
d_1 &= \frac{-\frac{1}{\gamma}\ln\left(\psi_1\zeta_t\right) - \ln\left(D(T)\right) + \left(r - \theta + \frac{1}{2}\nu^2\right)(T-t)}{\nu\sqrt{T-t}}, \\
d_2 &= d_1 - \nu\sqrt{T-t}, \\
\theta &= \left(\frac{\gamma - 1}{\gamma}\right)\left(r + \frac{\|\boldsymbol{\kappa}\|^2}{2\gamma}\right), \\
\nu &= \frac{\|\boldsymbol{\kappa}\|}{\gamma}
\end{aligned}$$

and $\Phi(\cdot)$ is the cumulative density function of the standard normal distribution.

Proof: According to the Black-Scholes formula and put-call parity, the price of the European put options in Equation (22) can be easily derived with the convenience provided by Equation (23) and (24) because the process λY_t is a martingale under measure $\mathbb{Q}$. Teplá (2001) also gives a similar application of Margrabe (1978) on pricing exchange options in a utility maximization problem with terminal constraints solely. ■

Proposition 4.3: *The optimal investment strategy for the* ***Constrained Problem*** *is*

$$\boldsymbol{\pi}_t^* = \frac{1}{\gamma}\frac{A_t}{W_t^*}\left(\boldsymbol{\sigma}^{-1}\right)^\top \boldsymbol{\kappa}, \tag{40}$$

[3] According to Subramanian (2004), this expression can be made more explicit when $a(s)$ takes some exponential form, e.g. $a(s) = e^{\alpha s}$.

where A_t is defined as

$$A_t = W_t^* - \left[\int_t^T e^{-r(s-t)} a(s) \left(1 - \Phi\left(x_2(s)\right)\right) \, \mathrm{d}s + e^{-r(T-t)} D(T) \left(1 - \Phi\left(d_2\right)\right) \right]. \tag{41}$$

Proof: See Appendix A.2. ■

Let us define the constraint-adjusted liability at time t by

$$\bar{D}(t) = \int_t^T e^{-r(s-t)} a(s) \left(1 - \Phi\left(x_2(s)\right)\right) \, \mathrm{d}s + e^{-r(T-t)} D(T) \left(1 - \Phi\left(d_2\right)\right). \tag{42}$$

According to Equations (40) and (41), the investment strategy $\boldsymbol{\pi}_t^*$ can also be expressed by

$$\boldsymbol{\pi}_t^* = \frac{1}{\gamma} \frac{W_t^* - \bar{D}(t)}{W_t^*} \left(\boldsymbol{\sigma}^{-1}\right)^\top \boldsymbol{\kappa}, \tag{43}$$

In the SOBPCI strategy under the GBM model, applying Proposition 4.2 to its inner **Constrained Problem** with initial wealth $X_M(0)$, consumption lower bound 0, and terminal wealth lower bound $D(T)$, we have

$$\begin{aligned} A_t^M &= X_t^M - e^{-r(T-t)} D(T) \left(1 - \Phi\left(d_2\right)\right) \\ &= W_t^M - \left[\int_t^T e^{-r(s-t)} a(s) \, \mathrm{d}s + e^{-r(T-t)} D(T) \left(1 - \Phi\left(d_2\right)\right) \right]. \end{aligned}$$

Thus, the constraint-adjusted liability at time t for the SOBPCI strategy is then given by

$$\bar{D}^M(t) = \int_t^T e^{-r(s-t)} a(s) \, \mathrm{d}s + e^{r(T-t)} D(T) \left(1 - \Phi\left(d_2\right)\right), \tag{44}$$

and the equivalent investment strategy of the SOBPCI strategy is

$$\boldsymbol{\pi}_t^M = \frac{1}{\gamma} \frac{W_t^M - \bar{D}^M(t)}{W_t^M} \left(\boldsymbol{\sigma}^{-1}\right)^\top \boldsymbol{\kappa}. \tag{45}$$

In the SCPPCI strategy under the GBM model, according to Equation (31), the equivalent investment strategy becomes

$$\boldsymbol{\pi}_t^H = \frac{1}{\gamma} \frac{W_t^H - D(t)}{W_t^H} \left(\boldsymbol{\sigma}^{-1}\right)^\top \boldsymbol{\kappa}. \tag{46}$$

According to Equations (43), (45) and (46), it can be seen that the similarity between $\boldsymbol{\pi}_t^*$, $\boldsymbol{\pi}_t^M$ and $\boldsymbol{\pi}_t^H$ depends on how close $\bar{D}(t)$, $\bar{D}^M(t)$ and $D(t)$ are.

5. Numerical results

This section illustrates our optimal OBPCI solution and the two sub-optimal benchmarks, the SCPPCI and SOBPCI, for a realistic parametric setting. The first subsection deals with the actual optimal allocation and consumption for the OBPCI, while the second subsection explores the losses investors may incur if they fail to follow our optimal solution.

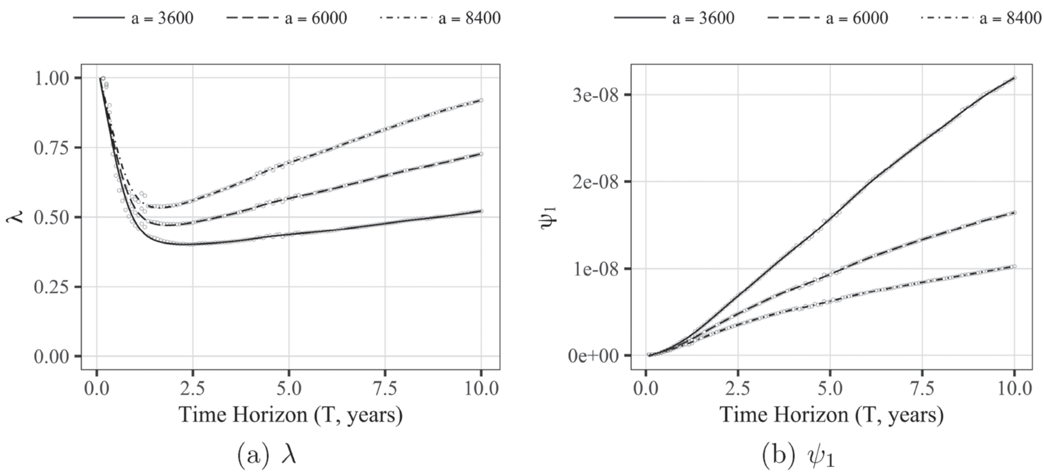

Figure 1. Selected parameters in the OBPCI strategy for different time horizon T. (a) λ and (b) ψ_1. *Note:* The gray points represent the simulated empirical data points, while the smooth curves are generated using a cubic spline interpolation method with a smoothing factor of 0.6, providing a continuous representation of the underlying trend.

5.1. Optimal strategy

In this subsection, we investigate the OBPCI strategy as the optimal solution to the **Constrained Problem** (8) in the GBM model (see Section 4.1) as a function of the time horizon T.

Specifically, let us consider an investor with initial wealth $W_0 = \$100{,}000$. It is assumed that there is one risky stock on the market, with $\boldsymbol{\mu} \equiv 0.1$ and $\boldsymbol{\sigma} \equiv 0.2$, while the risk-free interest rate is set to $r = 0.04$. The risk aversion parameter γ is set to 2.

The time horizon T can take any value in the interval $[0, 10]$, and this 10-year period is evenly discretized into 120 monthly time points in the numerical tests. When T increases from 0 to 10, the **Constrained Problem** (8) and its optimal solution will change dramatically if we keep the same $a(t)$ and $D(T)$. Thus, to maintain the consistency of the **Constrained Problem** in terms of T, we let $a(t)$ and $D(T)$ satisfy Equation (6) and fix $D(0)$, which represents the overall effect of the constraints. In this way, the **Constrained Problem** remains comparable as T changes. Specifically, $D(0) = \$80{,}000$, which is selected to make the constraints in **Constrained Problem** likely to bind.

We will explore the OBPCI strategy versus the time horizon T in three different cases of $a(t)$, namely $a(t) \equiv 3600$, $a(t) \equiv 6000$, and $a(t) \equiv 8400$, corresponding to low, medium and high monthly consumption floors of \$300, \$500, and \$700, respectively. In each case, as mentioned above, given $D(0) = \$80{,}000$, the value $D(T)$ is calculated by Equation (6) for different horizons T.

Figure 1 illustrates how the key parameters in the OBPCI strategy evolve with time horizon T. The parameter $\lambda = (\psi_1/\phi_1)^{-\frac{1}{\gamma}}$ represents the proportion of wealth invested in the **Merton portfolio with consumption**. A higher λ indicates that less wealth is allocated to purchasing put options, implying a lower solvency risk to be hedged.

As shown in Figure 1(a), the solvency risk is lowest at $T = 0$ because, when $T = 0$, the constraint problem is deterministic and free of uncertainty. The solvency risk rises sharply over short horizons and gradually declines for longer horizons. This pattern may arise because sequencing risk is higher over short horizons, as investors have less time to recover from poor investment outcomes. Furthermore, Figure 1(a) also suggests that a higher value of $a(t)$ reduces solvency risk, primarily due to a lower value of $D(T)$ given that $D(0)$ is fixed.

Although with lesser practical implications and interpretations, Figure 1(b) demonstrates a consistent monotonic increase in ψ_1 across all three $a(t)$ scenarios as the time horizon extends. This behavior aligns with the wealth allocation constraint under fixed initial wealth W_0: when the planning

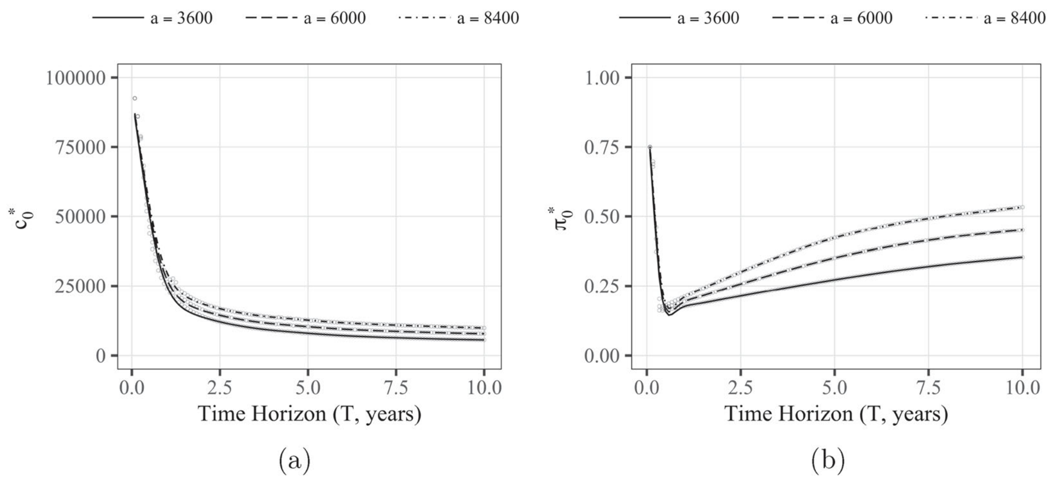

Figure 2. The consumption and investment strategy for different time horizons T. (a) The initial consumption c_0^*. (b) The investment strategy $\boldsymbol{\pi}_0^*$. *Note:* The gray points represent the simulated empirical data points, while the smooth curves are generated using a cubic spline interpolation method with smoothing factor to be 0.6 in Figure 2(a) and 0.4 in Figure 2(b) to illustrate local trends.

horizon T increases, the system must simultaneously reduce both the consumption rate c_t^* and terminal wealth W_T^* to maintain feasibility. The observed growth in ψ_1 directly reflects this dual reduction mechanism, as formally characterized by the closed-form solutions in Equations (9) and (10).

Figure 2 illustrates how the consumption and the equivalent investment strategy $\boldsymbol{\pi}_0^*$ change across different time horizons T. Due to the stochastic nature of consumption c_t when $t > 0$, we only show the optimal consumption c_0^*, which is deterministic, in Figure 2(a). In each of the three cases, c_0^* shows a sharp initial decrease followed by a gradual decline towards its lower bound $a(0)$ as T extends. This behavior is attributable to the rising value of ψ_1 for the same reason previously mentioned.

Figure 2(b) displays $\boldsymbol{\pi}_0^*$ (see Equations (40) and (43)), the initial equivalent investment in the risky asset. The corresponding equivalent portfolio comprising the risky and risk-free assets can replicate the options required in the OBPCI strategy. A comparison of Figures 2(b) and 1(a) reveals their similar patterns, caused by the same sequencing risk, where $\boldsymbol{\pi}_0^*$ first drops significantly, then gradually rises. Given our parameter choice, where $\frac{1}{\gamma}(\boldsymbol{\sigma}^{-1})^\top\boldsymbol{\kappa} = 0.75$, the ratio 0.75λ determines the proportion of wealth invested in the risky asset as in the OBPCI strategy. It should be noted that $\boldsymbol{\pi}_0^*$ should always be lower than 0.75λ because put options in the OBPCI strategy represent a long position in the risk-free bond and a short position in the risky asset, equivalently.

5.2. Comparison between alternative strategies

In this subsection, we examine how the SCPPCI and SOBPCI strategies perform relative to the OBPCI strategy across varying time horizons T and levels of risk aversion γ. This comparison allows us to highlight the enhancement achieved by the OBPCI strategy, an essential contribution of our paper.

To measure the performance of the sub-optimal strategies, we utilize the certainty equivalent wealth alongside the associated welfare loss. Specifically, the certainty equivalent wealth W_0^i for a sub-optimal strategy $i \in \{\text{SCPPCI, SOBPCI}\}$ is characterized as the wealth amount under the OBPCI strategy that provides identical utility to what is achieved by W_0 under strategy i, i.e.

$$J_C\left(t=0, \text{OBPCI}; W_0^i, \{a(t)\}_{t\in[0,T]}, D(T)\right) = J_C\left(t=0, \text{strategy } i; W_0, \{a(t)\}_{t\in[0,T]}, D(T)\right).$$

Table 1. The welfare loss across different parameters for the SCPPCI and SOBPCI strategies, compared with the OBPCI strategy.

Parameter	SCPPCI	SOBPCI
Panel A: Time Horizon Variation ($\gamma = 2$)		
Time Horizon T		
1/52	−3.31%	−0.01%
1	−9.49%	−0.04%
10	−4.64%	−4.28%
Panel B: Risk Aversion Variation ($T = 10$)		
Risk aversion γ		
1.5	−5.07%	−5.02%
4	−3.56%	−2.60%

Since OBPCI represents the optimal solution to the **Constrained Problem**, we must have $W_0^i \leq W_0$. Accordingly, the welfare loss under strategy i is defined as a percentage:

$$W_0^i / W_0 - 1.$$

The certainty equivalent for any feasible strategy to the **Constrained Problem** should be greater than $D(0)$ because $c_t \geq a(t)$ and $W_T \geq D(T)$ are satisfied. At the same time, an amount of $D(0)$, as the value of initial wealth, can only lead to $c_t = a(t)$ and $W_T = D(T)$. Thus, we have a lower bound for the welfare loss, which is $D(0)/W_0 - 1$.

Under the same parameter setting as in Section 5.1, i.e. $W_0 = \$100{,}000$, $D(0) = \$80{,}000$ and the same market-available assets, we take $a(t) \equiv 8400$ for an illustration. The lower bound of the welfare loss in this specific case is -20%.

The performance of each sub-optimal strategy over the time horizon T when $\gamma = 2$ is shown in Figure 3(a) and *Panel A* in Table 1. We evaluate the certainty equivalent and welfare loss at 10 discrete time points corresponding to $T = 1, 2, \ldots, 10$ for computational ease. The welfare loss of the SOBPCI strategy decreases smoothly from 0 as T increases, indicating a growing deviation from the performance of the OBPCI strategy. This trend arises because the SOBPCI strategy is inherently similar to the OBPCI strategy when the impact of $a(t)$ is minor, as both strategies hedge $D(T)$ through options. Conversely, the SCPPCI strategy prohibits option use. The maximum disparity in performance between the SCPPCI and OBPCI (or SOBPCI) strategies occurs when T is near zero.[4] As T lengthens, the SCPPCI strategy's performance improves, due to the heightened influence of $a(t)$ bringing $D(t)$ closer to $\bar{D}(t)$. Nevertheless, it consistently performs worse than the SOBPCI strategy, especially for shorter durations. This is because the SOBPCI strategy is the optimal solution to the constrained problem featuring mixed preference, which more closely resembles **Constrained Problem** (see Equations (33) and (8)). Therefore, the SOBPCI strategy consistently outperforms the SCPPCI strategy as an approximation of the OBPCI strategy, particularly over shorter time horizons.

Next, we examine the effect of the risk-aversion parameter γ on the performance of each strategy, with T fixed at 10, as illustrated in Figure 3(b) and *Panel B* in Table 1. Since γ represents the investor's aversion to uncertainty, a higher γ implies that the optimal investment strategy will be more conservative. As γ increases, the performance of both the SOBPCI and SCPPCI strategies improves, as they increasingly resemble the OBPCI strategy due to the growing allocation to the risk-free bond in the OBPCI strategy. This is also reflected in the fact that $\boldsymbol{\pi}_t^*$, $\boldsymbol{\pi}_t^H$ and $\boldsymbol{\pi}_t^M$ decrease for all $t \in [0, T]$ as γ increases (see Equations (43), (46), (45)). It should be noted that when γ is low, the OBPCI strategy significantly outperforms the other two alternative strategies by allowing more wealth to be invested in the risky asset.

[4] We note that the true shape of the loss function in SCPPCI strategy (with respect to T) is a U-shape starting at 0% when $T = 0$, rapidly falling to a low point and then slowly rising at T increases.To illustrate this shape, we plot the SCPPCI welfare for a different parameterization ($a = 33{,}600$ and same D_0 and W_0) in Figure A1 in Appendix 2. As a further illustration, in *Panel A* of Table 1 we show the value of T corresponding to 1 week, which shows that the loss is much closer to 0 at very small time horizons.

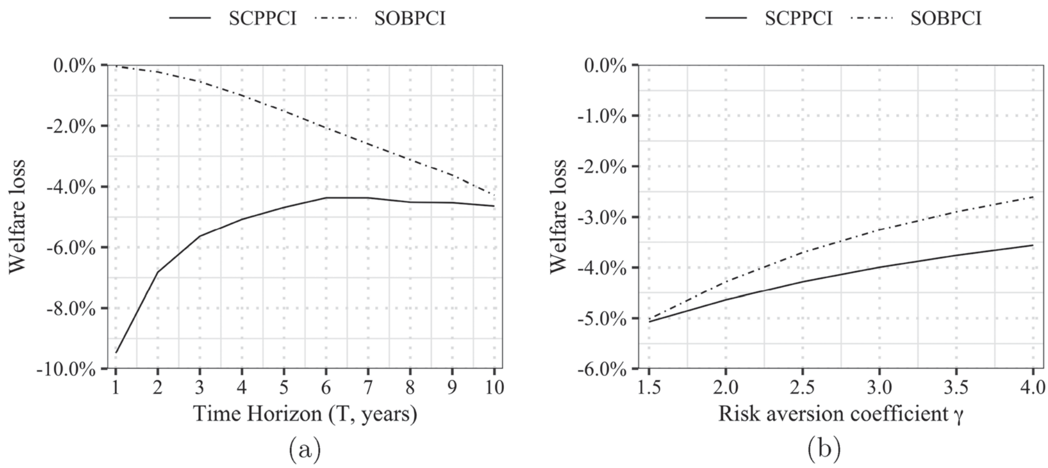

Figure 3. The welfare loss of alternative strategies. (a) The welfare loss versus time horizon T when $\gamma = 2$. (b) The welfare loss versus risk aversion parameter γ when $T = 10$.

Overall, the OBPCI strategy performs better than the SCPPCI and SOBPCI strategies when the time horizon T is extended and the risk-aversion parameter γ is relatively low. This underscores the importance of the OBPCI strategy in such scenarios and highlights our contribution to the existing literature.

5.3. Welfare loss analysis

As indicated in Section 4 following Corollary 4.1, the $\boldsymbol{\kappa}$ value estimated on the same data for the GBM and LVO-CEV models is generally different. This discrepancy implies a potential welfare loss arising from the mismatch between the optimal strategy (i.e. OBPCI) in the true market dynamics (e.g. LVO-CEV), and the optimal strategy from the wrong market dynamic (e.g. GBM). In this subsection, we investigate such welfare loss in the case where the investor follows the optimal strategy derived under the GBM model, while the actual market is governed by the LVO-CEV model.

We use the superscript notation GBM and CEV to denote parameters under the GBM and LVO-CEV models, respectively. By Proposition 3.1, the optimal consumption and terminal wealth under the LVO-CEV model are given by

$$\begin{cases} c_t^{*,\mathrm{CEV}} = \max\left(\left(\psi^{\mathrm{CEV}}\zeta_t^{\mathrm{CEV}}\right)^{-\frac{1}{\gamma}}, a(t)\right), \\ W_T^{*,\mathrm{CEV}} = \max\left(\left(\psi^{\mathrm{CEV}}\zeta_T^{\mathrm{CEV}}\right)^{-\frac{1}{\gamma}}, D(T)\right), \end{cases}$$

with ψ^{CEV} satisfying the budget constraint

$$W_0 = \mathbb{E}\left[\int_0^T \zeta_t^{\mathrm{CEV}} c_t^{*,\mathrm{CEV}}\,\mathrm{d}t + \zeta_T^{\mathrm{CEV}} W_T^{*,\mathrm{CEV}}\right].$$

As explained before, this would lead to optimal strategies, $\pi_t^{*,\mathrm{CEV}}$, infamously complex to compute.

Alternatively, if the investor derives the optimal strategy, $\pi_t^{*,\mathrm{GBM}}$, under the GBM specification, the resulting consumption and terminal wealth would be

$$\begin{cases} c_t^{*,\mathrm{GBM}} = \max\left(\left(\psi^{\mathrm{GBM}}\zeta_t^{\mathrm{GBM}}\right)^{-\frac{1}{\gamma}}, a(t)\right), \\ W_T^{*,\mathrm{GBM}} = \max\left(\left(\psi^{\mathrm{GBM}}\zeta_T^{\mathrm{GBM}}\right)^{-\frac{1}{\gamma}}, D(T)\right), \end{cases}$$

with ψ^{GBM} satisfying $W_0 = \mathbb{E}[\int_0^T \zeta_t^{\mathrm{GBM}} c_t^{*,\mathrm{GBM}}\,\mathrm{d}t + \zeta_T^{\mathrm{GBM}} W_T^{*,\mathrm{GBM}}]$.

However, if the implied GBM-based strategy ($\pi_t^{*,\mathrm{GBM}}$) were implemented in the LVO-CEV market, it would lead to a suboptimal consumption and terminal wealth, with a new multiplier, i.e. $\psi^{\mathrm{CEV-GBM}}$. This is

$$\begin{cases} c_t^{s,\mathrm{CEV-GBM}} = \max\left(\left(\psi^{\mathrm{CEV-GBM}}\zeta_t^{\mathrm{GBM}}\right)^{-\frac{1}{\gamma}}, a(t)\right), \\ W_T^{s,\mathrm{CEV-GBM}} = \max\left(\left(\psi^{\mathrm{CEV-GBM}}\zeta_T^{\mathrm{GBM}}\right)^{-\frac{1}{\gamma}}, D(T)\right), \end{cases}$$

where $\psi^{\mathrm{CEV-GBM}}$ satisfy the budget constraint evaluated under the LVO-CEV pricing kernel, namely

$$W_0 = \mathbb{E}\left[\int_0^T \zeta_t^{\mathrm{CEV}} c_t^{s,\mathrm{CEV-GBM}}\,\mathrm{d}t + \zeta_T^{\mathrm{CEV}} W_T^{s,\mathrm{CEV-GBM}}\right].$$

To measure the under-performance of the GBM strategy in the LVO-CEV market, we define the certainty equivalent loss. Specifically, W_0^{CEV} is defined as the initial wealth under $(c_t^{*,\mathrm{CEV}}, W_T^{*,\mathrm{CEV}})$ that yields the same utility as that achieved by W_0 under strategy $(c_t^{s,\mathrm{CEV-GBM}}, W_T^{s,\mathrm{CEV-GBM}})$, that is,

$$\begin{aligned} &J_C^{\mathrm{CEV}}\left(t=0, \{c_t^{*,\mathrm{CEV}}\}_{t\in[0,T]}, W_T^{*,\mathrm{CEV}}; W_0^{\mathrm{CEV}}\right) \\ &= J_C^{\mathrm{GBM}}\left(t=0, \{c_t^{s,\mathrm{CEV-GBM}}\}_{t\in[0,T]}, W_T^{s,\mathrm{CEV-GBM}}; W_0\right). \end{aligned}$$

It should be noted that the constraints $a(t)$ and $D(T)$ are omitted from the expressions of J_C in the above equation for notational simplicity. Since the GBM strategy is sub-optimal under the LVO-CEV model, it follows that $W_0^{\mathrm{CEV}} \le W_0$. As in the previous subsection, we define the welfare loss in percentage terms as

$$W_0^{\mathrm{CEV}}/W_0 - 1$$

The parameter settings for the GBM model are the same as those used in Section 5.1, with $\kappa^{\mathrm{GBM}} = 0.3$, $W_0 = \$100{,}000$, $D(0) = \$80{,}000$ and $a(t) \equiv \$8400$. For the LVO-CEV model, we consider three values of $\boldsymbol{\kappa}^{\mathrm{CEV}}$, namely 0.38, 0.41 and 0.44, to illustrate the effect of $\boldsymbol{\kappa}^{\mathrm{CEV}}$ on the welfare loss. It should be noted that $\boldsymbol{\kappa}^{\mathrm{CEV}} = 0.38$ corresponds to the estimate obtained from the Russell 2000 index in Anel and Fan (2025). Similar to Section 5.2, the certainty equivalent and welfare loss are evaluated at ten discrete time points corresponding to $T = 1, 2, \ldots, 10$ by simulations for computational convenience.

As shown in Figure 4, the (negative) welfare loss decreases smoothly from zero as the investment horizon increases for all considered values of $\boldsymbol{\kappa}^{\mathrm{CEV}}$. Moreover, a larger $\boldsymbol{\kappa}^{\mathrm{CEV}}$, indicating a greater deviation of the GBM model (with $\boldsymbol{\kappa}^{\mathrm{GBM}} = 0.3$), leads to a larger welfare loss. Nevertheless, the magnitude of the loss remains relatively small, reaching only about 1% even in the case $\kappa^{\mathrm{CEV}} = 0.44$.

Overall, the results indicate that the welfare loss induced by applying an optimal strategy derived under the GBM model to an LVO-CEV market is economically modest. This suggests that the welfare implications of the optimal consumption and investment problem are relatively robust with respect to differences in the estimated $\boldsymbol{\kappa}$ values across model specifications. From a practical perspective, these findings support the use of simpler benchmark models, such as GBM, as reasonable approximations for welfare analysis, even when the underlying market exhibits more complex dynamics.

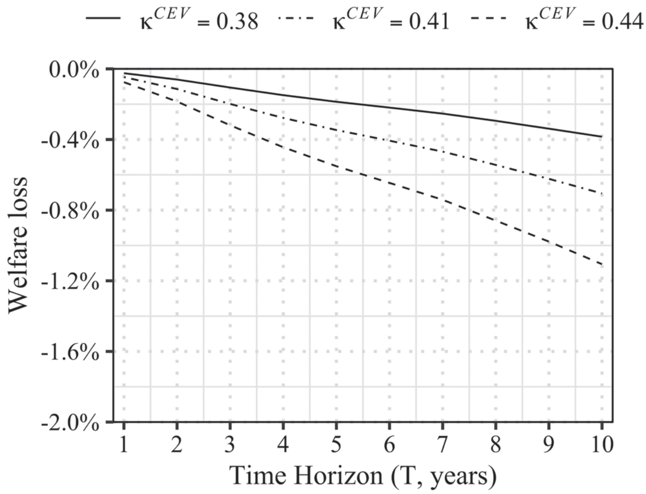

Figure 4. The welfare loss where the investor follows the optimal strategy derived under the GBM model with $\kappa^{\text{GBM}} = 0.3$, while the actual market follows the LVO-CEV model under different κ^{CEV} values, namely 0.38, 0.41 and 0.44.

6. Conclusion

We have solved the CRRA utility maximization problem with predetermined lower bounds on consumption rate and terminal wealth in a general family of covariance structures. The solution is represented by the Option-Based Portfolio and Consumption Insurance (OBPCI) strategy. This strategy implies that the investor should hedge against consumption and terminal wealth lower bounds using European put options. In contrast, the remaining wealth (a proportion λ of current wealth) is invested in the unconstrained Merton portfolio. A key advantage of the OBPCI strategy is that it can be implemented at the start of the investment period without requiring further wealth reallocation. Additionally, the OBPCI strategy has practical applications, such as providing a retirement planning solution that simultaneously addresses consumption needs and bequest objectives.

For comparison, we analyze two sub-optimal strategies: SCPPCI and SOBPCI. These correspond to optimal solutions for constrained problems with different utility functions. Specifically, the SCPPCI strategy replaces both put options in the OBPCI strategy with an investment in risk-free bonds, while the SOBPCI strategy substitutes only the inter-temporal put option. Under the GBM model, we also present the equivalent investment strategy for OBPCI and the two sub-optimal strategies without using options, where the constraint-adjusted liability captures differences.

The numerical illustrations examine the OBPCI strategy's performance as the time horizon changes under low, medium, and high consumption levels. Sequencing risk is identified as a key concern for short horizons, leading investors to allocate a relatively smaller portion of wealth to risky assets. We also compare the performance of the OBPCI strategy against the two sub-optimal strategies. Under our parameter settings, the SCPPCI and SOBPCI strategies result in a maximum welfare loss of −9.49% and −5.02%, respectively. In addition, we conduct a welfare loss analysis showing that applying the GBM-based optimal strategy in an LVO-CEV market leads to only economically modest welfare losses.

Our study may lead to further extensions, possibly, in three main directions. First, incomplete markets can be considered by employing existing derivatives to complete the market. In this setting, the OBPCI solution may be expressed as a portfolio of compound derivatives, that is, derivatives written on both assets and other derivatives. Second, richer modeling frameworks may be explored, including models that allow for jumps, stochastic covariance, or stochastic interest rates. For instance, the theoretical results developed by Michelbrink and Le (2012) can be applied to extend our analysis to incomplete markets with jump–diffusion dynamics of the underlying assets. An example of such an extension can be found in Colaneri et al. (2025) in the CPPI context. Finally, further extensions may incorporate additional constraints into the OBPCI solution, such as no short-selling or Value-at-Risk (VaR) constraints.

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Data availability statement

The data that support the findings of this study were generated through simulation. Replication R code is available from the corresponding author upon request.

Disclosure statement

The authors report there are no competing interests to declare.

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Appendices

Appendix 1. Proofs

A.1 Proof of Proposition 3.1

The Lagrangian of **Constrained Problem** (8) is given by

$$\mathcal{L} = \mathbb{E}\left[\int_0^T u\left(c_t\right)\, \mathrm{d}t + u(W_T)\right] + \psi_1\left(W_0 - \mathbb{E}\left[\int_0^T \zeta_t c_t\, \mathrm{d}t + \zeta_T W_T\right]\right)$$

$$+\int_0^T \psi_2(t)\left(c_t - a(t)\right)\,\mathrm{d}s + \psi_3\left(W_T - D(T)\right),$$

where ψ_1, $\psi_2(t)$ and ψ_3 are Lagrange multipliers. It leads to a pathwise maximization problem

$$\max_{(c_t)_{t\in[0,T]},W_T}\int_0^T \left(u\left(c_t\right) - \psi_1\zeta_t c_t + \psi_2(s)c_t\right)\,\mathrm{d}t + u(W_T) - \psi_1\zeta_T W_T + \psi_3 W_T.$$

The first-order condition for the consumption is

$$u'\left(c_t\right) - \psi_1\zeta_t + \psi_2(t) = 0. \tag{A1}$$

From the complementary slackness conditions, $\psi_2(t)(c_t^* - a(t)) = 0$, $\psi_2(t) \ge 0$, and $c_t^* \ge a(t)$, we obtain that

$$\psi_2(t) = \max\left(\psi_1\zeta_t - u'\left(a(t)\right), 0\right). \tag{A2}$$

Substituting Equation (A2) back to Equation (A1), we have

$$c_t^* = \max\left(\left(\psi_1\zeta_t\right)^{-\frac{1}{\gamma}}, a(t)\right).$$

The first-order condition for the terminal wealth is

$$u'(W_T) - \psi_1\zeta_T + \psi_3 = 0.$$

By the same logic of Equation (A2), we have

$$\psi_3 = \max\left(\psi_1\zeta_T - u'(D(T)), 0\right).$$

Then we could obtain

$$W_T^* = \max\left(\left(\psi_1\zeta_T\right)^{-\frac{1}{\gamma}}, D(T)\right).$$

Once we have got the optimal solutions about c_t and W_T, we can substitute them back into the budget constraint (5) and then we have Equation (11).

It can be found that $\psi_2(t)$ and ψ_3 are not shown in the expressions of c_t^* and W_T^* explicitly. The reason is that they only imply whether the constraints bind. If the constraints do not bind, **Constrained Problem** will become an unconstrained problem where $\psi_2(t) = 0$ and $\psi_3 = 0$. On the opposite, if the constraints bind, the supremum will be taken at the boundaries directly, i.e. $a(t)$ and $D(T)$, no matter what values of $\psi_2(t)$ and ψ_3 are.

A.2 Proof of Proposition 4.3

Given the optimal wealth expression (39), the dynamics of it could be derived according to Ito's formula:

$$\begin{aligned}
\mathrm{d}W_t^* &= \cdots\mathrm{d}t + \frac{\partial W_t^*}{\partial\zeta_t}\,\mathrm{d}\zeta_t \\
&= \cdots\mathrm{d}t + \left[-\frac{1}{\gamma}\left(\psi_1\zeta_t\right)^{-\frac{1}{\gamma}-1}\psi_1\left(\int_t^T e^{-\theta(s-t)}\Phi\left(x_1(s)\right)\,\mathrm{d}s + e^{-\theta(T-t)}\Phi(d_1)\right)\right. \\
&\quad + \int_t^T\left(\left(\psi_1\zeta_t\right)^{-\frac{1}{\gamma}}e^{-\theta(s-t)}\phi(x_1(s))\frac{\partial x_1(s)}{\partial\zeta_t} - e^{-r(s-t)}a(s)\phi\left(x_2(s)\right)\frac{\partial x_2(s)}{\partial\zeta_t}\right)\mathrm{d}s \\
&\quad \left. + \left(\left(\psi_1\zeta_t\right)^{-\frac{1}{\gamma}}e^{-\theta(T-t)}\phi(d_1)\frac{\partial d_1}{\partial\zeta_t} - e^{-r(T-t)}D(T)\phi\left(d_2\right)\frac{\partial d_2}{\partial\zeta_t}\right)\right]\mathrm{d}\zeta_t.
\end{aligned} \tag{A3}$$

The last two terms inside the square bracket in Equation (A3) are shown to be 0 as follows: Since we have

$$\begin{aligned}
e^{\frac{1}{2}\left(x_1(s)^2 - x_2(s)^2\right)} &= e^{\frac{1}{2}(x_1(s)+x_2(s))(x_1(s)-x_2(s))} \\
&= \frac{\left(\psi_1\zeta_t\right)^{-\frac{1}{\gamma}}e^{-\theta(s-t)}}{a(s)e^{-r(s-t)}},
\end{aligned}$$

the integrand in Equation (A3) can be simplified:

$$\begin{aligned}
&\left(\psi_1\zeta_t\right)^{-\frac{1}{\gamma}}e^{-\theta(s-t)}\phi(x_1(s))\frac{\partial x_1(s)}{\partial\zeta_t} - e^{-r(s-t)}a(s)\phi\left(x_2(s)\right)\frac{\partial x_2(s)}{\partial\zeta_t} \\
&= \frac{1}{\sqrt{2\pi}}\frac{\partial x_1(s)}{\partial\zeta_t}\left[\left(\psi_1\zeta_t\right)^{-\frac{1}{\gamma}}e^{-\theta(s-t)}e^{-\frac{x_1(s)^2}{2}} - e^{-r(s-t)}a(s)e^{-\frac{x_2(s)^2}{2}}\right]
\end{aligned}$$

$$= 0.$$

By the same logic, we have

$$\begin{aligned}
&(\psi_1\zeta_t)^{-\frac{1}{\gamma}} e^{-\theta(T-t)}\phi(d_1)\frac{\partial d_1}{\partial \zeta_t} - e^{-r(T-t)}D(T)\phi(d_2)\frac{\partial d_2}{\partial \zeta_t} \\
&= \frac{1}{\sqrt{2\pi}}\frac{\partial d_1}{\partial \zeta_t}\left[(\psi_1\zeta_t)^{-\frac{1}{\gamma}} e^{-\theta(T-t)}e^{-\frac{d_1^2}{2}} - e^{-r(T-t)}D(T)e^{-\frac{d_2^2}{2}}\right] \\
&= 0,
\end{aligned}$$

by noticing that

$$\begin{aligned}
e^{\frac{1}{2}(d_1{}^2-d_2{}^2)} &= e^{\frac{1}{2}(d_1+d_2)(d_1-d_2)} \\
&= \frac{(\psi_1\zeta_t)^{-\frac{1}{\gamma}} e^{-\theta(T-t)}}{D(T)e^{-r(T-t)}}.
\end{aligned}$$

Therefore, Equation (A3) can be further simplified:

$$\begin{aligned}
\mathrm{d}W_t^* &= \cdots \mathrm{d}t - \frac{1}{\gamma}(\psi_1\zeta_t)^{-\frac{1}{\gamma}-1}\psi_1\left(\int_t^T e^{-\theta(s-t)}\Phi\left(x_1(s)\right)\,\mathrm{d}s + e^{-\theta(T-t)}\Phi(d_1)\right)\mathrm{d}\zeta_t \\
&= \cdots \mathrm{d}t + \frac{1}{\gamma}(\psi_1\zeta_t)^{-\frac{1}{\gamma}}\left(\int_t^T e^{-\theta(s-t)}\Phi\left(x_1(s)\right)\,\mathrm{d}s + e^{-\theta(T-t)}\Phi(d_1)\right)\boldsymbol{\kappa}^\top\,\mathrm{d}\boldsymbol{Z}_t \\
&= \cdots \mathrm{d}t + \frac{1}{\gamma}A_t\boldsymbol{\kappa}^\top\,\mathrm{d}\boldsymbol{Z}_t. \qquad \text{(A4)}
\end{aligned}$$

Their drift and diffusion terms must be equal almost everywhere for two identical Ito processes. Thus, letting the diffusion term of Equation (A3) equal that of Equation (2), we have the following.

$$\frac{1}{\gamma}A_t\boldsymbol{\kappa}^\top\boldsymbol{\sigma}^{-1} = W_t^*\boldsymbol{\pi}_t^{*\top}, \qquad \text{(A5)}$$

where the optimal investment strategy $\boldsymbol{\pi}_t^*$ could be solved.

Appendix 2. The illustration of the welfare loss of the SCPPCI strategy in a short time horizon

In Section 5.2, with $a(t) \equiv 8400$, a significant sharp drop in the welfare loss of the SCPPCI strategy around $T = 0$ can be seen in *Panel A* in Table 1. In this specific illustration, to make the initial drop in the welfare loss of the SCPPCI better visualized, we let $a(t) \equiv 33{,}600$ (which is 4 times as large as 8400) and select the evaluation points $T = 0.1, 0.2, \ldots, 1$, while keeping the other parameters used in Section 5.2 unchanged.

Figure A1 shows that the welfare loss of the SCPPCI strategy is close to 0% for extremely short T (e.g. T less than 0.05) and there is a change from a downward trend to an upward trend at about $T = 0.4$.

The reasons for such a u-shape curve are given below:

The objective function is

$$\mathbb{E}\left[\int_0^T u\left(c_t\right)\,\mathrm{d}t + u\left(W_T\right)\right], \qquad \text{(A6)}$$

which is maximized by the OBPCI strategy (c_t^*, W_T^*).

For an extremely short horizon T, the consumption $c_t^{H,*}$ in the SCPPCI strategy is significantly lower than the consumption c_t^* in the OBPCI strategy due to their respective formulations, contributing to a potential large scale of welfare loss. However, since T is extremely small, the integral of total consumption is relatively small, leading to only a minor difference in terminal wealth between the SCPPCI and OBPCI strategies, i.e. $W_T^{H,*}$ and W_T^*. In the objective function (A6), the integral $\int_0^T u(c_t^{H,*})\,\mathrm{d}t$ does not deviate significantly from $\int_0^T u(c_t^*)\,\mathrm{d}t$, while $u(W_T^{H,*})$ is close to $u(W_T^*)$, again due to the extremely short horizon T. This effect of extremely small T dominates initially, which explains why the welfare loss is subtle when T is close to 0, as seen in Figure A1.

However, it should be noted that the deviation between $c_t^{H,*}$ and c_t^* decreases as T increases. Thus, this decreasing pointwise deviation in consumption eventually dominates, leading to an upward trend of the welfare loss in the SCPPCI strategy after some certain point (i.e. $T = 0, 4$ in Figure A1).

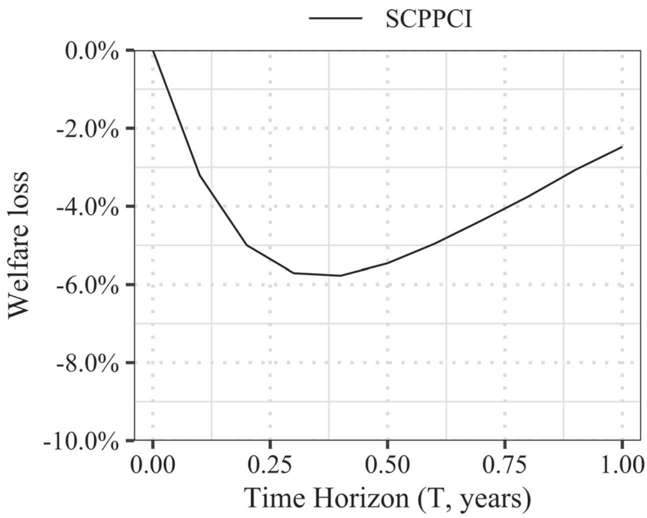

Figure A1. The welfare loss of the SCPPCI strategy over a short time horizon T when $a(t) \equiv 33{,}600$ and $\gamma = 2$.

In conclusion, the initial U-shaped welfare loss of the SCPPCI strategy around $T = 0$ depends on both the length of the time horizon T and the magnitude of the deviation between the SCPPCI and OBPCI strategies in terms of consumption and terminal wealth.