The boundedness of the Riesz transform on a metric cone

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Declaration

The work in this thesis is my own except where otherwise stated.

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Abstract

In this thesis we study the boundedness, on $L^p(M)$, of the Riesz transform T associated to a Schrödinger operator with an inverse square potential $V = \frac{V_0}{r^2}$ on a metric cone M defined by

$$T = \nabla\left(\Delta + \frac{V_0(y)}{r^2}\right)^{-\frac{1}{2}}.$$

Here $M = Y \times [0, \infty)_r$ has dimension $d \geq 3$, and the smooth function V_0 on Y is restricted to satisfy the condition $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2 > 0$, where Δ_Y is the Laplacian on the compact Riemannian manifold Y.

The definition of T involves the Laplacian Δ on the cone M. However, the cone is not a manifold at the cone tip, so we initially define the Laplacian away from the cone tip, and then consider its self-adjoint extensions. The Friedrichs extension is adopted as the definition of the Laplacian.

Using functional calculus, T can be written as an integral involving the expression $(\Delta + \frac{V_0(y)}{r^2} + \lambda^2)^{-1}$. Therefore if we understand the resolvent kernel of the Schrödinger operator $\Delta + \frac{V_0(y)}{r^2}$, we have information about T. We construct and at the same time collect information about this resolvent kernel, and then use the information to study the boundedness of T.

The two most interesting parts in the construction of the resolvent kernel are the behaviours of the kernel as $r, r' \to 0$ and $r, r' \to \infty$. To study them, a process called the blow-up is performed on the domain of the kernel. We use the b-calculus to study the kernel as $r, r' \to 0$, while the scattering calculus is used as $r, r' \to \infty$.

The main result of this thesis provides a necessary and sufficient condition on p for the boundedness of T on $L^p(M)$. The interval of boundedness depends on V_0 through the first and second eigenvalues of $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2$.

- When the potential function V is positive, we have shown that the lower

threshold is 1, and the upper threshold is strictly greater than the dimension d.

- When the potential function V is negative, we have shown that the lower threshold is strictly greater than 1, and the upper threshold is strictly between 2 and d.
- Our results for $p \leq 2$ are contained in the work of J. Assaad, but we use different methods in this thesis. Our boundedness results for $p \geq \frac{d}{2}$ for positive inverse square potentials, and for $p > 2$ for negative inverse square potentials, are new.

Contents

Chapter 1

Introduction

1.1 The main results of the thesis

The Riesz transform T on the Euclidean space $\mathbb{R}^d$ is defined by

$$T = \nabla \Delta_{\mathbb{R}^d}^{-\frac{1}{2}}, \tag{1.1}$$

where $\Delta_{\mathbb{R}^d}$ is the Laplacian operator. In this thesis we study the Riesz transform T in the more general setting of metric cones, which preserve the good dilation properties of Euclidean spaces. A metric cone M is of the form $Y \times [0, \infty)$, where (Y, h) is a compact Riemannian manifold with dimension $d - 1$. The cone M is equipped with the metric $g = dr^2 + r^2 h$, and is usually illustrated as in Figure 1.1, like an ice-cream cone. The Euclidean space $\mathbb{R}^d$ provides the simplest example of a metric cone, with the compact Riemannian manifold $Y = \mathbb{S}^{d-1}$.

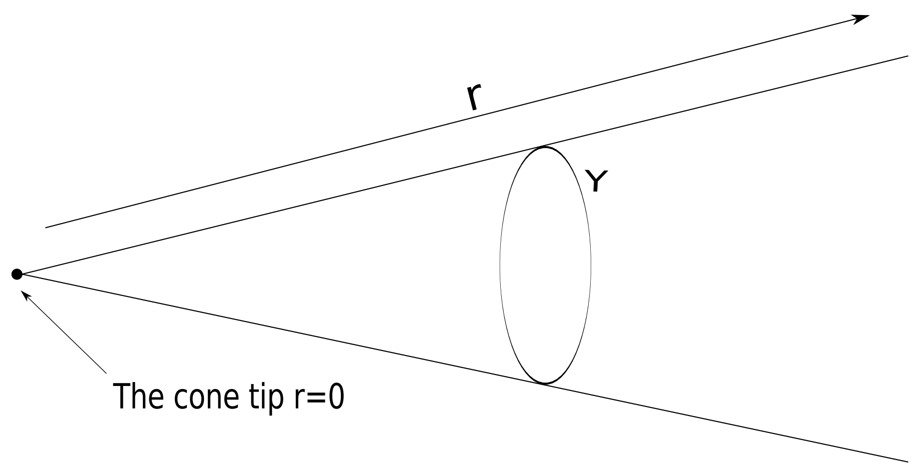


Figure 1.1: The metric cone $M = Y \times [0, \infty)$

The Laplacian on the cone expressed in polar coordinates is

$$\Delta = -\partial_r^2 - \frac{d-1}{r}\partial_r + \frac{1}{r^2}\Delta_Y, \tag{1.2}$$

where Δ_Y is the Laplacian on the compact Riemannian manifold Y. Since the cone is not a manifold at the cone tip, we initially define the Laplacian by formula (1.2) away from the cone tip, and then consider its self-adjoint extensions. The canonical Friedrichs extension is adopted as the definition of the Laplacian. For the details, see Chapter 2.

Then the Riesz transform T on the cone M is defined by

$$T = \nabla\Delta^{-\frac{1}{2}}. \tag{1.3}$$

The question of the boundedness of T in the setting of cones, ie for what p the Riesz transform T is bounded on $L^p(M)$, was answered by H.-Q. Li in [HQL]. The characterisation of the boundedness, stated in Theorem 1.1.1, is in terms of the second smallest eigenvalue of the operator $\Delta_Y + (\frac{d-2}{2})^2$. A different proof to this result will be provided in Chapter 6 of this thesis.

Theorem 1.1.1. *(Theorem 6.5.3) Let $d \geq 3$, and M be a metric cone with dimension d and cross section Y. The Riesz transform $T = \nabla\Delta^{-\frac{1}{2}}$ is bounded on $L^p(M)$ if and only if p is in the interval*

$$\left(1, \frac{d}{\max(\frac{d}{2} - \mu_1, 0)}\right), \tag{1.4}$$

where $\mu_1 > 0$ is the square root of the second smallest eigenvalue of the operator $\Delta_Y + (\frac{d-2}{2})^2$.

More importantly, the methods used in this thesis to prove Theorem 1.1.1 can be applied to study the boundedness properties of a more generalised class of operators, obtained by introducing an inverse square potential to the Riesz transform. Let $V_0 : Y \to \mathbb{R}$ be a smooth function on Y, then the Riesz transform T associated to a Schrödinger operator with the inverse square potential $V = \frac{V_0}{r^2}$ is defined by

$$T = \nabla\left(\Delta + \frac{V_0(y)}{r^2}\right)^{-\frac{1}{2}}.$$

The Laplacian Δ is homogeneous of degree -2, and adding an inverse square potential term preserves this homogeneity. That's why we consider inverse square potentials. In order to guarantee that $\Delta + \frac{V_0(y)}{r^2}$ is positive, the function V_0 is restricted to satisfy the condition $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2 > 0$, ie the eigenvalues of $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2$ are all strictly positive. This allows our potential $V = \frac{V_0}{r^2}$ to be "a bit negative". H.-Q. Li uses heat kernel estimates in [HQL]. It's difficult to apply this technique to negative potentials.

The goal of this thesis is to find the exact interval for p on which the Riesz transform T with an inverse square potential $V = \frac{V_0}{r^2}$ is bounded on $L^p(M)$, where M is a metric cone with dimension $d \geq 3$.

A necessary condition, stated in Theorem 1.1.2, for the boundedness was found in [GH] by C. Guillarmou and A. Hassell, but in a slightly different setting - asymptotically conic manifolds. Here M_0 is the interior of a compact manifold with boundary M, with boundary defining function x. The coordinate x is analogous to r^{-1}, and the potential V is in $x^2 C^\infty(M)$, ie V decreases as r^{-2} as $r \to \infty$.

Theorem 1.1.2. *([GH, Theorem 1.5]) Let $d \geq 3$, and (M_0, g) be an asymptotically conic manifold with dimension d. Consider the operator $P = \Delta_g + V$ with V satisfying*

$$\Delta_{\partial M} + V_0 + \left(\frac{d-2}{2}\right)^2 > 0 \text{ where } V_0 = \left.\frac{V}{x^2}\right|_{\partial M}.$$

Suppose that P has no zero modes or zero resonances, and that $V_0 \not\equiv 0$, then the Riesz transform $\nabla P^{-\frac{1}{2}}$ is unbounded on $L^p(M)$ if p is outside the interval

$$\left(\frac{d}{\min(\frac{d}{2} + 1 + \mu_0, d)}, \frac{d}{\max(\frac{d}{2} - \mu_0, 0)}\right), \tag{1.5}$$

where $\mu_0 > 0$ is the square root of the smallest eigenvalue of the operator $\Delta_{\partial M} + V_0 + (\frac{d-2}{2})^2$.

The counter-example used in [GH] to show the unboundedness of T can be easily adapted to the context of metric cones, so a similar result also holds for metric cones. Therefore the task now is to find a sufficient condition for the boundedness. We will see, in Chapter 6 of this thesis, that the sufficient condition involves the same interval (1.5) as in Theorem 1.1.2, so this interval gives us a complete characterisation of the boundedness of T with $V \not\equiv 0$. The main result of this thesis is as follows.

Theorem 1.1.3. *(Theorem 6.5.1) Let $d \geq 3$, and M be a metric cone with dimension d and cross section Y. Let V_0 be a smooth function on Y that satisfies $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2 > 0$. The Riesz transform T with the inverse square potential $V = \frac{V_0}{r^2}$ is bounded on $L^p(M)$ for p in the interval*

$$\left(\frac{d}{\min(1 + \frac{d}{2} + \mu_0, d)}, \frac{d}{\max(\frac{d}{2} - \mu_0, 0)} \right), \tag{1.6}$$

where $\mu_0 > 0$ is the square root of the smallest eigenvalue of the operator $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2$.

Moreover, for any $V \not\equiv 0$, the interval (1.6) characterises the boundedness of T, ie T is bounded on $L^p(M)$ if and only if p is in the interval (1.6).

- For a positive potential V, ie $V \geq 0$ and $V \not\equiv 0$, the lower threshold for the L^p boundedness is 1, and the upper threshold is strictly greater than d.
- For a negative potential V, ie $V \leq 0$ and $V \not\equiv 0$, the lower threshold for the L^p boundedness is strictly greater than 1, and the upper threshold is strictly between 2 and d.
- For the Euclidean space $\mathbb{R}^d$, the lower threshold was obtained by J. Assaad in [JA], but in that paper she didn't show boundedness for any $p > 2$ for inverse square potentials; see the end of Section 1.3 for further discussion.

An immediate application of Theorem 1.1.3 is to show that the converse of the second part of [GH, Theorem 1.5], ie the converse of Theorem 1.1.2, is also true. According to [GH, Remark 1.7], Theorem 1.1.3 is exactly the missing ingredient. Therefore we have the following result.

Theorem 1.1.4. *(Theorem 6.5.2) In the setting of Theorem 1.1.2, the Riesz transform $\nabla P^{-\frac{1}{2}}$ is bounded on $L^p(M)$ if and only if p is in the interval*

$$\left(\frac{d}{\min(\frac{d}{2} + 1 + \mu_0, d)}, \frac{d}{\max(\frac{d}{2} - \mu_0, 0)} \right), \tag{1.7}$$

where $\mu_0 > 0$ is the square root of the smallest eigenvalue of the operator $\Delta_{\partial M} + V_0 + (\frac{d-2}{2})^2$.

From Theorem 1.1.3 we can quickly obtain the following result on the Riesz transforms with constant non-zero V_0, in which the boundedness interval is written in terms of the constant.

Proposition 1.1.5. *(Proposition 6.5.6) Let $d \geq 3$, and M be a metric cone with dimension d and cross section Y. The Riesz transform $T = \nabla(\Delta + \frac{c}{r^2})^{-\frac{1}{2}}$, where $c > -(\frac{d-2}{2})^2$ and $c \neq 0$, is bounded on $L^p(M)$ if and only if p is in the interval*

$$\left(\frac{2d}{\min(d + 2 + \sqrt{(d-2)^2 + 4c}, 2d)}, \frac{2d}{\max(d - \sqrt{(d-2)^2 + 4c}, 0)} \right). \tag{1.8}$$

1.2 The main ideas in the thesis

Using functional calculus, we can express T as,

$$T = \frac{2}{\pi} \int_0^\infty \nabla \left(\Delta + \frac{V_0(y)}{r^2} + \lambda^2 \right)^{-1} d\lambda.$$

From this expression, and due to homogeneity, we know that if we understand the properties of $(\Delta + \frac{V_0(y)}{r^2} + 1)^{-1}$, we know about the operator T. Therefore the task is transformed into constructing and hence collecting the properties of the resolvent kernel of $H = \Delta + \frac{V_0(y)}{r^2}$, ie we need to study the operator $P = H + 1$.

Clearly, the two interesting parts are when $r \to 0$ and $r \to \infty$. Figure 1.2 illustrates the domain of the kernel of P. For convenience, in this thesis we use the same letter to denote an operator and its kernel. The two interesting parts correspond to the bottom left corner and the top right corner.

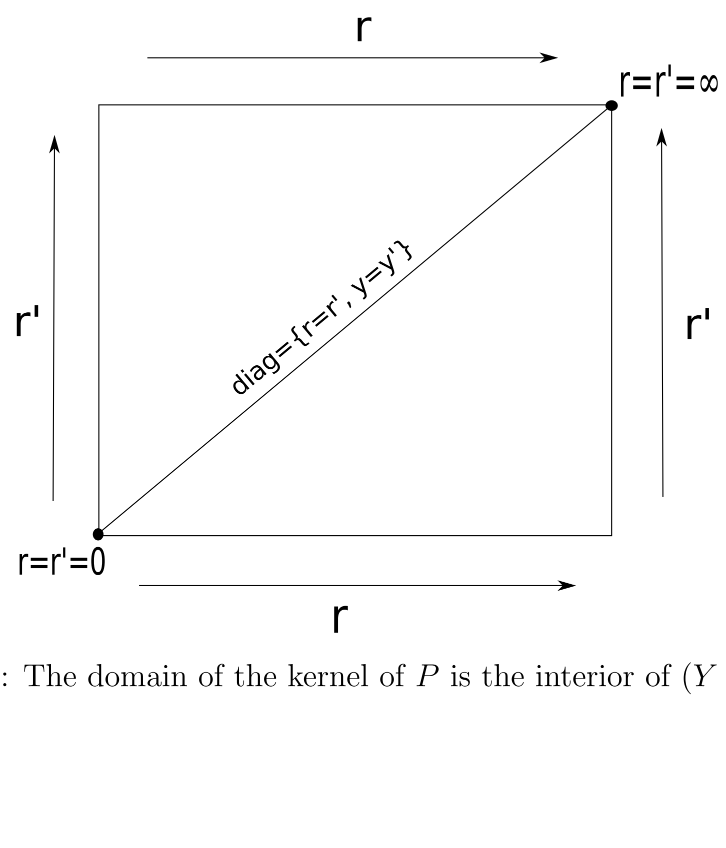


Figure 1.2: The domain of the kernel of P is the interior of $(Y \times [0, \infty]_r)^2$

Note that the variables of the kernel are r, r', y and y', but Figure 1.2 only shows r and r'. The diagonal illustrated is especially deceitful because the definition of it also includes $y = y'$.

The kernel of P behaves nicely at most parts of the domain shown in Figure 1.2, so we focus on its behaviours when $r, r' \to 0$ and $r, r' \to \infty$. For that, we perform a process called the blow-up to the two points $r = r' = 0$ and $r = r' = \infty$. The intuition comes form the process of changing from Cartesian coordinates to polar coordinates on $\mathbb{R}^2$, which essentially "blows up" the origin into a circle. The precise definition of blow-ups will be introduced in Chapter 3.

The blow-ups we perform on the two points are different. When $r \to 0$, the operator rPr is elliptic as a b-differential operator. We blow up the point $r = r' = 0$ in Figure 1.2 to a hypersurface, denoted by zf, and use the b-calculus near zf. The definition and the properties of the b-calculus will also be introduced in Chapter 3, and its application to our problem is in Chapter 5. A lot of the materials on the b-calculus are covered in the book [RM93] by R. Melrose.

When $r \to \infty$, the operator P is elliptic as a scattering differential operator. Intuitively the cone becomes so flat that locally it behaves like the Euclidean space. To study the behaviour of P here, we perform a double blow-up to the point $r = r' = \infty$ in Figure 1.2, and obtain the hypersurfaces bf and sf. Near sf, we use the scattering calculus. Double blow-ups and the scattering calculus are discussed in Chapter 4. Their applications to our problem are in Chapter 5.

After the blow-ups, we have obtained a space called the *blown-up space*. It is illustrated by Figure 1.3.

Because the kernel behaves differently in different parts of the blown-up space, and especially because we use different calculi near the two hypersurfaces zf and sf, we must break the blown-up space into different regions, and construct the resolvent kernel in each region separately using different tools and techniques. In the end we patch up the constructions in these different regions to obtain the overall resolvent kernel. This construction of the resolvent kernel of H, ie the kernel of P^{-1}, is done in Chapter 5.

Equipped with the knowledge on the behaviours of the kernel of P^{-1} at different parts of the blown-up space, collected through its construction in Chapter 5, we can finally in Chapter 6 tackle the problem of the boundedness of the Riesz transform T. We again break up the blown-up space into several regions, because different regions give different restrictions on p for the boundedness of T

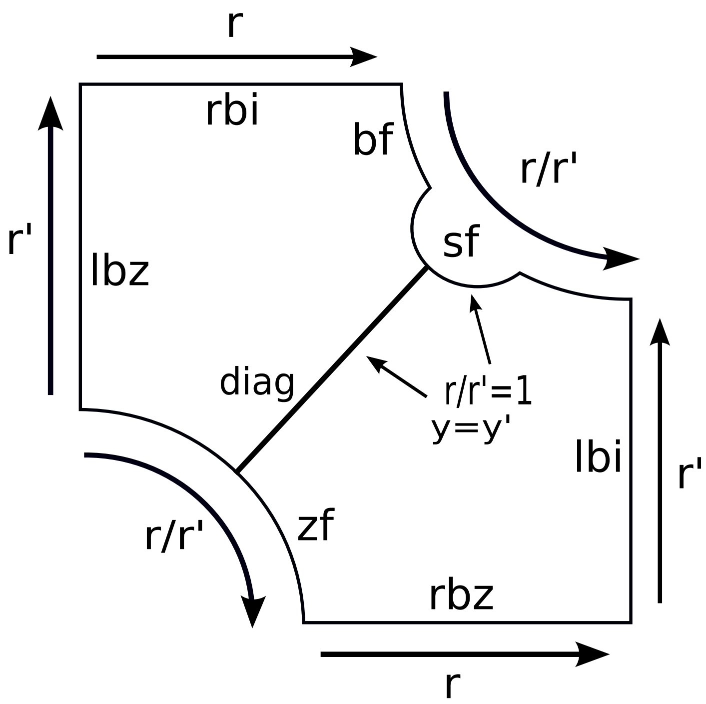


Figure 1.3: The blown-up space

on $L^p(M)$. However, there's no reason why we must break up the space into the same regions as those during the construction of the resolvent kernel in Chapter 5. Indeed, we break up the blown-up space differently in Chapter 6, in a way that suits the purpose of analysing the boundedness of T most. We combine all the restrictions on p given by these different regions to obtain a sufficient condition for the boundedness. From there we quickly arrive at the main result of this thesis, ie Theorem 1.1.3.

1.3 Literature review

Cones have been studied since the 19th century, particularly the problem of wave diffraction from a cone point which is important in applied mathematics, for example in [ASO] by A. Sommerfeld. Other papers include [FGF] and [FGF2] by F. G. Friedlander and [BK] by A. Blank and J. B. Keller. The Laplacians defined on cones were studied by J. Cheeger and M. Taylor in [CT] and [CT2]. A study on the self-adjoint extensions of the Laplacians on cones can be found in [AGHH] by S. Albeverio, F. Gesztesy, R. Høegh-Krohn and H. Holden. The Laplacian on compact Riemannian manifolds with cone-like singularities, ie singularities behave like cone tips, has been studied in [JC] by J. Cheeger and in [EM] by E. Mooers; while in [BS], J. Brüning and R. Seeley studied the Laplacian on manifolds with an asymptotically conic singularity.

The classical case of the Riesz transform on the Euclidean space $\mathbb{R}^d$ goes back to the 1920s, and the case of one dimension, which is called the Hilbert transform, was studied by M. Riesz in [MR]. The paper [RSS] by R. S. Strichartz is the first paper that studies the Riesz transform on a complete Riemannian manifold. In [CD] T. Coulhon and X. T. Duong proved that the Riesz transform on a complete Riemannian manifold, satisfying the doubling condition and the diagonal bound on the heat kernel, is of weak type $(1, 1)$, and hence is bounded on L^p for $1 < p \leq 2$. In the same paper they also showed that the Riesz transform defined on the connected sum of two copies of $\mathbb{R}^d$ is unbounded on L^p for $p > d$. The boundedness of the Riesz transform defined on the connected sum of a finite number of $\mathbb{R}^d$ on L^p for $1 < p < d$ was shown by G. Carron, T. Coulhon and A. Hassell in [CCH]. Riesz transforms on connected sums were further studied by G. Carron in [GC2]. In [HS], A. Hassell and A. Sikora studied the boundedness of the Riesz transform where the Laplacian is defined on $\mathbb{R}$ or $\mathbb{R}_+$, with respect to the measure $|r|^{d-1}dr$ where dr is Lebesgue measure and $d > 1$ can be any real number.

Many papers have been written on Schrödinger operators with an inverse square potential. We only mention a few of the most relevant ones here. In [XPW], X. P. Wang studied the perturbations of such operators. In [GC], G. Carron studied Schrödinger operators with potentials that are homogeneous of degree -2 near infinity. In [BPSTZ] by N. Burq, F. Planchon, J. G. Stalker and A. S. Tahvildar-Zadeh, the authors generalised the corresponding standard Strichartz estimates of the Schrödinger equation and the wave equation to the case in which an additional inverse square potential is present.

Now let's turn to past results on the main problem of this thesis, the boundedness of the Riesz transform T with a potential V on metric cones. As mentioned in Section 1.1, the case with $V \equiv 0$ was answered by H.-Q. Li in [HQL]. Li's result is stated in Theorem 1.1.1. In Section 1.1 we mentioned that in [GH] C. Guillarmou and A. Hassell found a necessary condition for the boundedness, but in the slightly different setting of asymptotically conic manifolds. Their result is stated in Theorem 1.1.2. According to [GH, Remark 1.7], the main result of this thesis, ie Theorem 1.1.3, provides the missing ingredient of proving that the converse of Theorem 1.1.2 is also true. In [GH2] the two authors performed a similar analysis but allowed zero modes and zero resonances. In [ABA], P. Auscher and B. Ben Ali obtained a result on $\mathbb{R}^d$, stated in Theorem 1.3.1, which involves the reverse Hölder's condition. It is an improvement of the earlier results by Z.W. Shen in [ZWS].

Theorem 1.3.1. *([ABA, Theorem 1.1]) Let $1 < q \leq \infty$. If $V \in B_q$ then for some $\varepsilon > 0$ depending only on V the Riesz transform with potential V is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < q + \varepsilon$.*

The reverse Hölder's condition, $V \in B_q$, means that $V \in L^q_{loc}(\mathbb{R}^d)$, $V > 0$ almost everywhere and there exists a constant C such that for all cube Q of $\mathbb{R}^d$,

$$\left(\frac{1}{|Q|}\int_Q V^q(x)dx\right)^{\frac{1}{q}} \leq \frac{C}{|Q|}\int_Q V(x)dx. \tag{1.9}$$

The main result of this thesis, ie Theorem 1.1.3, doesn't contain Theorem 1.3.1, because the class of potential functions given by the reverse Hölder's conidion is more general than the class of potential functions allowed by us. However, P. Auscher and B. Ben Ali's result doesn't cover Theorem 1.1.3 either. The setting of Theorem 1.1.3 is on metric cones, so more general. But even on the Euclidean space $\mathbb{R}^d$ with $V = \frac{V_0}{r^2}$ satisfying our condition $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2 > 0$, since in this case V is not in $L^{\frac{d}{2}}_{loc}(\mathbb{R}^d)$, P. Auscher and B. Ben Ali's result gives the boundedness interval $(1, \frac{d}{2})$; while according to the second bullet point following the proof of Theorem 6.5.1, our result shows that the upper threshold for the boundedness is strictly greater than d.

The most recent papers, and also the most relevant to this thesis, are [JA] by J. Assaad and [AO] by J. Assaad and E. M. Ouhabaz. J. Assaad's result in [JA] is stated below. She also generalized it to manifolds where the Sobolev inequality does not necessarily hold.

Theorem 1.3.2. *Let M be a non-compact complete Riemannian manifold with dimension $d \geq 3$. Suppose that the function $V \leq 0$ satisfies $\Delta + (1+\varepsilon)V \geq 0$, the Sobolev inequality*

$$||f||_{L^{\frac{2d}{d-2}}(M)} \lesssim |||\nabla f|||_{L^2(M)},$$

holds for all $f \in C_0^\infty(M)$, and that M is of homogeneous type, ie for all $x \in M$ and $r > 0$,

$$\mu\big(B(x,2r)\big) \lesssim \mu\big(B(x,r)\big),$$

where μ is the measure on M. Then the Riesz transform $T = \nabla(\Delta + V)^{-\frac{1}{2}}$ is bounded on $L^p(M)$ for all p in the interval

$$\left(\frac{2d}{d+2+(d-2)\sqrt{\frac{\varepsilon}{\varepsilon+1}}}, \quad 2 \right]. \tag{1.10}$$

For Riesz transforms of the form $T = \nabla(\Delta + \frac{c}{r^2})^{-\frac{1}{2}}$, where the constant c satisfies $-(\frac{d-2}{2})^2 < c < 0$, the lower threshold in (1.8) given by our result Proposition 1.1.5 is the same as the lower threshold in (1.10) given by J. Assaad's result, and she also gave a counter-example showing that T is unbounded for p greater than our upper threshold; for the details, see Remark 6.5.8. In [JA], J. Assaad also showed boundedness on $(1, d)$ for positive potentials that are in $L^{\frac{d}{2}}(M)$, but note that this space just fails to include inverse square potentials, which are in $L^{\frac{d}{2},\infty}(M)$. In [JA] and [AO], J. Assaad and E. M. Ouhabaz obtained some boundedness results for $p > 2$ for negative potentials, but the conditions of those results exclude negative inverse square potentials.

Chapter 2

Self-adjoint extensions of the Laplacian in $\mathbb{R}^d$

2.1 Introduction

To study the Riesz transform on a metric cone M, we need to define the Laplacian on it. Since the cone is not a manifold at the cone tip P, we start with a definition of the Laplacian $\tilde{\Delta}$ on the cone away from the cone tip by formula (1.2). However, this $\tilde{\Delta}$ defined on $C_c^\infty(M \backslash \{P\})$ is not always essentially self-adjoint. Therefore in this chapter we will make a study of its self-adjoint extensions. For simplicity, we will only study this on $\mathbb{R}^d$ from Section 2.3 onwards, as $\mathbb{R}^d$ is the simplest example of metric cones, and the results we obtain also apply to general metric cones. The results in this chapter can be found in [AGHH], but proved with a different method.

2.2 Friedrichs extension

The self-adjoint extension of $\tilde{\Delta}$ may not be unique. However, there is always a canonical one called the *Friedrichs extension*, denoted by Δ. Conisider the quadratic form q associated with $\tilde{\Delta}$:

$$q(\varphi, \psi) = \int_M \nabla\varphi \cdot \overline{\nabla\psi} dx.$$

The Friedrichs extension is the only self-adjoint extensioin whose domain is contained in the form domain of the closure of q. From here we know that functions in the domain of Δ must have derivatives in $L^2(M)$. For more information on the Friedrichs extension, see [RS, Sec. X.3].

The Friedrichs extension is the definition of the Laplacian adopted in this thesis.

2.3 The operator L and its closure

We work in the Hilbert space $L^2(\mathbb{R}^d)$, and start with the unbounded operator $L = -\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ with domain $D(L) = C_c^\infty(\mathbb{R}^d \backslash \{0\})$. Note that L is symmetric, which can be shown by integration by parts, but it is not self-adjoint as the domain of its adjoint $D(L^*)$ contains $C_c^\infty(\mathbb{R}^d)$ hence is strictly bigger than $D(L)$. Recall that the deficiency subspaces $\mathscr{K}_+$, $\mathscr{K}_-$ of L are the null spaces of the operators $i - L^*$, $i + L^*$ respectively. By applying von Neumann's theorem to L, we know that the two deficiency subspaces of L have the same dimension hence L has self-adjoint extensions. To find out how many self-adjoint extensions L has and what they look like, we use the following proposition due to von Neumann, for the proof see [RS, Sec. X.1]:

Proposition 2.3.1. *Let A be a closed symmetric operator with equal deficiency indices. Then there is a one-one correspondence between self-adjoint extensions of A and unitary maps from $\mathscr{K}_+$ onto $\mathscr{K}_-$. If U is such a unitary map, the corresponding self-adjoint extension A_U has domain*

$$D(A_U) = \{\varphi + \psi + U\psi : \varphi \in D(A), \psi \in \mathscr{K}_+\},$$

and

$$A_U(\varphi + \psi + U\psi) = A\varphi + i\psi - iU\psi.$$

The above proposition requires a closed symmetric operator. To find the domain of $\overline{L}$ means to complete $C_c^\infty(\mathbb{R}^d \backslash \{0\})$ under the norm $||\cdot|| + ||L(\cdot)||$. (In this thesis, the norm notation $||\cdot||$ always denotes the L^2-norm.) This norm is equivalent to the $W^{2,2}$-norm hence we have $D(\overline{L}) = W_0^{2,2}(\mathbb{R}^d \backslash \{0\}) \subseteq W^{2,2}(\mathbb{R}^d)$. The situation differs as the dimension varies. We first treat the case $d > 4$.

2.3.1 Case d>4

Proposition 2.3.2. *For $\mathbb{R}^d$ with $d > 4$, we have*

$$D(\overline{L}) = W^{2,2}(\mathbb{R}^d).$$

Proof. From above, we already know that $D(\overline{L}) \subseteq W^{2,2}(\mathbb{R}^d)$. Since $C_c^\infty(\mathbb{R}^d)$ is dense in $W^{2,2}(\mathbb{R}^d)$, we pick any $\psi \in C_c^\infty(\mathbb{R}^d)$, and approximate it with a sequence of functions in $C_c^\infty(\mathbb{R}^d\backslash\{0\})$ that converges to ψ under the $W^{2,2}$-norm. Let φ be a smooth function such that $\varphi = 1$ on $B_{\frac{1}{2}}(0)$ and $\varphi = 0$ outside $B_1(0)$. Then define $\varphi_\epsilon(x) = 1 - \varphi(\frac{x}{\epsilon})$. (So $\varphi_\epsilon(x) \to 1$ as $\epsilon \to 0$ for all $x \neq 0$.) We now show that $\psi\varphi_\epsilon$ converges to ψ under the $W^{2,2}$-norm when ϵ approaches 0. Indeed, $\psi\varphi_\epsilon \in C_c^\infty(\mathbb{R}^d\backslash\{0\})$, and as $\epsilon \to 0$,

$$||\psi\varphi_\epsilon - \psi||^2 = \int_{\mathbb{R}^d} |\psi(x)\varphi(\frac{x}{\epsilon})|^2 dx \leq M_1^2 \int_{\mathbb{R}^d} |\varphi(\frac{x}{\epsilon})|^2 dx = \epsilon^d M_1^2 ||\varphi||^2 \to 0,$$

where M_1 is the maximum value of ψ.

Now consider $||L(\psi\varphi_\epsilon - \psi))||$. The product rule creates three terms. As $\epsilon \to 0$,

$$\begin{aligned} ||(L\psi)(\varphi_\epsilon - 1)||^2 &= \int_{\mathbb{R}^d} |(L\psi)(x)\varphi(\frac{x}{\epsilon})|^2 dx \\ &\leq M_2^2 \int_{\mathbb{R}^d} |\varphi(\frac{x}{\epsilon})|^2 dx = \epsilon^d M_2^2 ||\varphi||^2 \to 0, \end{aligned}$$

where M_2 is the maximum value of $L\psi$.

Since $d - 2 > 0$, as $\epsilon \to 0$,

$$\begin{aligned} ||\nabla\psi \cdot \nabla(\varphi_\epsilon - 1)||^2 &\leq M_3^2 \int_{\mathbb{R}^d} |\nabla\varphi(\frac{x}{\epsilon})|^2 dx \\ &= \epsilon^{-2} M_3^2 \int_{\mathbb{R}^d} |(\nabla\varphi)(\frac{x}{\epsilon})|^2 dx = \epsilon^{d-2} M_3^2 |||\nabla\varphi|||^2 \to 0, \end{aligned}$$

the partial derivatives of ψ take values smaller than M_3.

At last, since $d - 4 > 0$, as $\epsilon \to 0$,

$$\begin{aligned} ||\psi L(\varphi_\epsilon - 1)||^2 &\leq M_1^2 \int_{\mathbb{R}^d} |L\varphi(\frac{x}{\epsilon})|^2 dx \\ &= M_1^2 \epsilon^{-4} \int_{\mathbb{R}^d} |(L\varphi)(\frac{x}{\epsilon})|^2 dx = M_1^2 \epsilon^{d-4} ||L\varphi||^2 \to 0. \end{aligned}$$

We have established in the case $d > 4$,

$$D(\overline{L}) = W^{2,2}(\mathbb{R}^d).$$

$\square$

2.3.2 Case d=4

We can see for the case $d \leq 4$, the above proof fails because $M_1^2 \epsilon^{d-4} ||L\varphi||^2$ doesn't converge to 0. In fact, when $d < 4$, the domain of $\overline{L}$ is smaller than $W^{2,2}(\mathbb{R}^d)$. For the case $d = 4$, we still have $D(\overline{L}) = W^{2,2}(\mathbb{R}^4)$, but we can't use the same φ_ϵ as defined in the above proof because $||L(\varphi_\epsilon - 1)|| = ||L\varphi(\frac{x}{\epsilon})||$ is independent of ϵ. So to get convergence we need to define φ_ϵ in a way that breaks the scaling invariance of $||L(\varphi_\epsilon - 1)||$.

Proposition 2.3.3. *For $\mathbb{R}^4$, we have*

$$D(\overline{L}) = W^{2,2}(\mathbb{R}^4).$$

Proof. As before, we pick any arbitrary $\psi \in C_c^\infty(\mathbb{R}^4)$. Let $\varphi : [0, \infty) \to [0, 1]$ be a smooth function such that $\varphi\big([0, \frac{1}{2}]\big) = 1$ and $\varphi\big([1, \infty)\big) = 0$. Then define $\varphi_\epsilon(x) := 1 - \varphi\big((\frac{|x|}{\epsilon})^\epsilon\big)$. We now verify that as $\epsilon \to 0$, $\psi\varphi_\epsilon$ approximates ψ under the $W^{2,2}$-norm. Indeed, $\psi\varphi_\epsilon \in C_c^\infty(\mathbb{R}^4 \backslash \{0\})$, and

$$||\psi\varphi_\epsilon - \psi||^2 = \int_{\mathbb{R}^4} |\psi(x)\varphi\big((\frac{|x|}{\epsilon})^\epsilon\big)|^2 dx.$$

We show that the above norm tends to 0 when ϵ approaches 0. Consider any x, when $\epsilon \leq |x|$, we have $(\frac{|x|}{\epsilon})^\epsilon \geq 1$, hence $|\psi(x)\varphi\big((\frac{|x|}{\epsilon})^\epsilon\big)|^2 \to 0$ pointwise. Moreover, $|\psi(x)\varphi\big((\frac{|x|}{\epsilon})^\epsilon\big)|^2$ is bounded above by the L^1 function $|\psi|^2$, so we can conclude $||\psi\varphi_\epsilon - \psi|| \to 0$ as $\epsilon \to 0$.

Again as before, the term $||L(\psi\varphi_\epsilon - \psi)||$ produces three terms: $||(L\psi)(\varphi_\epsilon - 1)||$, $||\nabla\psi \cdot \nabla(\varphi_\epsilon - 1)||$ and $||\psi L(\varphi_\epsilon - 1)||$. Here we just show how to deal with the most difficult term $||\psi L(\varphi_\epsilon - 1)||$ where both derivatives fall on the cutoff function, as the computations for the others are similar, but easier. We will use the full strength of $d = 4$. As before, we just need to estimate $||L(\varphi_\epsilon - 1)||$. We will first calculate this norm for general $1 \le d \le 4$, then substitute the dimension $d = 4$,

$$||L(\varphi_\epsilon - 1)||^2 = \int_{\mathbb{R}^d} |L\varphi\big((\frac{|x|}{\epsilon})^\epsilon\big)|^2 dx.$$

After computation, we obtain the expression

$$\begin{aligned} L\varphi\big((\frac{|x|}{\epsilon})^\epsilon\big) &= (d-2)\epsilon^{1-\epsilon}|x|^{\epsilon-2}\varphi'\big((\frac{|x|}{\epsilon})^\epsilon\big) \\ &\quad + \epsilon^{2-2\epsilon}|x|^{2\epsilon-2}\varphi''\big((\frac{|x|}{\epsilon})^\epsilon\big) + \epsilon^{2-\epsilon}|x|^{\epsilon-2}\varphi'\big((\frac{|x|}{\epsilon})^\epsilon\big). \end{aligned} \tag{2.1}$$

We will use it in the proofs for all the dimensions $1 \le d \le 4$. In particular here for $d = 4$, it becomes

$$\begin{aligned} L\varphi\big((\frac{|x|}{\epsilon})^\epsilon\big) &= 2\epsilon^{1-\epsilon}|x|^{\epsilon-2}\varphi'\big((\frac{|x|}{\epsilon})^\epsilon\big) \\ &\quad + \epsilon^{2-2\epsilon}|x|^{2\epsilon-2}\varphi''\big((\frac{|x|}{\epsilon})^\epsilon\big) + \epsilon^{2-\epsilon}|x|^{\epsilon-2}\varphi'\big((\frac{|x|}{\epsilon})^\epsilon\big). \end{aligned}$$

Because both φ' and φ'' are bounded, we just need to work with the three terms $\epsilon^{1-\epsilon}|x|^{\epsilon-2}$, $\epsilon^{2-2\epsilon}|x|^{2\epsilon-2}$, $\epsilon^{2-\epsilon}|x|^{\epsilon-2}$. Because φ' and φ'' vanish outside the ball $B(0,1)$, we only integrate over the ball $B(0,\epsilon)$. The square of these terms are $\epsilon^{2-2\epsilon}|x|^{2\epsilon-4}$, $\epsilon^{4-4\epsilon}|x|^{4\epsilon-4}$, $\epsilon^{4-2\epsilon}|x|^{2\epsilon-4}$. The biggest is $\epsilon^{2-2\epsilon}|x|^{2\epsilon-4}$, so we only need to estimate this one. Using polar coordinates, as $\epsilon \to 0$,

$$\int_{B(0,\epsilon)} \epsilon^{2-2\epsilon}|x|^{2\epsilon-4}dx = 2\pi^2\epsilon^{2-2\epsilon}\int_0^\epsilon r^{2\epsilon-4}r^3 dr = 2\pi^2\epsilon^{2-2\epsilon}\int_0^\epsilon r^{2\epsilon-1}dr = \pi^2\epsilon \to 0.$$

This completes the proof. □

Remark 2.3.4. Note that this convergence is only logarithmic as a function of the "spread" of $\varphi\big((\frac{|x|}{\epsilon})^\epsilon\big)$, as one would expect.

2.3.3 Case d=3

We already know the domain of $\overline{L}$ lies in $W^{2,2}(\mathbb{R}^3)$. By the Sobolev Embedding Theorem we know that $W^{2,2}(\mathbb{R}^3)$ can be embedded into $C^{\frac{1}{2}}(\mathbb{R}^3)$, therefore it makes sense to talk about the value of one of these functions at a single point, in this case the origin. Any function in $D(\overline{L})$ is the limit of a sequence of continuous functions that take 0 at the origin under the $W^{2,2}$-norm, hence it must also take

0 at the origin. Therefore we know that $D(\overline{L}) \subseteq \{\psi \in W^{2,2}(\mathbb{R}^3) | \psi(0) = 0\}$. In fact, they are equal.

Proposition 2.3.5. *In $\mathbb{R}^3$, we have*

$$D(\overline{L}) = \{\psi \in W^{2,2}(\mathbb{R}^3) | \psi(0) = 0\}.$$

Proof. We pick any arbitrary $\psi \in C_c^\infty(\mathbb{R}^3)$ with $\psi(0) = 0$. We use the same sequence as the case $d = 4$ to approximate ψ, and the calculations are similar as well. The difference here is we get one less power of r when we change into polar coordinates, but this is compensated with the Hölder condition. We show $||\psi\varphi_\epsilon - \psi||, ||(L\psi)(\varphi_\epsilon - 1)||, ||\nabla\psi \cdot \nabla(\varphi_\epsilon - 1)|| \to 0$ as $\epsilon \to 0$ exactly the same as before. The only substantially different term is $||\psi L(\varphi_\epsilon - 1)||$. By the Hölder condition, we have $|\psi(x)|^2 \leq C|x|$ for some constant C. Then it follows that

$$\begin{aligned} ||\psi L(\varphi_\epsilon - 1)||^2 &= \int_{\mathbb{R}^3} |\psi(x) L\varphi\big((\frac{|x|}{\epsilon})^\epsilon\big)|^2 dx \\ &= \int_{B(0,\epsilon)} |\psi(x) L\varphi\big((\frac{|x|}{\epsilon})^\epsilon\big)|^2 dx \leq C \int_{B(0,\epsilon)} |x||L\varphi\big((\frac{|x|}{\epsilon})^\epsilon\big)|^2 dx. \end{aligned}$$

Substitute $d = 3$ into (2.1) to obtain the expression for $L\varphi\big((\frac{|x|}{\epsilon})^\epsilon\big)$,

$$L\varphi\big((\frac{|x|}{\epsilon})^\epsilon\big) = \epsilon^{1-\epsilon}|x|^{\epsilon-2}\varphi'\big((\frac{|x|}{\epsilon})^\epsilon\big) + \epsilon^{2-2\epsilon}|x|^{2\epsilon-2}\varphi''\big((\frac{|x|}{\epsilon})^\epsilon\big) + \epsilon^{2-\epsilon}|x|^{\epsilon-2}\varphi'\big((\frac{|x|}{\epsilon})^\epsilon\big).$$

It is the same as the case $d = 4$ except the coefficient for the first term goes down by 1. (Later on when we discuss the case $d = 2$, this term will disappear.) The three terms we need to work with are still $\epsilon^{1-\epsilon}|x|^{\epsilon-2}$, $\epsilon^{2-2\epsilon}|x|^{2\epsilon-2}$ and $\epsilon^{2-\epsilon}|x|^{\epsilon-2}$. Square these terms and multiply with $|x|$, we get $\epsilon^{2-2\epsilon}|x|^{2\epsilon-3}$, $\epsilon^{4-4\epsilon}|x|^{4\epsilon-3}$ and $\epsilon^{4-2\epsilon}|x|^{2\epsilon-3}$. Parallel to before, we integrate them over the ball $B(0,\epsilon)$, and we check the worst term to complete the proof. As $\epsilon \to 0$,

$$\int_{B(0,\epsilon)} \epsilon^{2-2\epsilon}|x|^{2\epsilon-3} dx = 4\pi\epsilon^{2-2\epsilon} \int_0^\epsilon r^{2\epsilon-3} r^2 dr = 4\pi\epsilon^{2-2\epsilon} \int_0^\epsilon r^{2\epsilon-1} dr = 2\pi\epsilon \to 0.$$

$\square$

2.3.4 Case d=2

Here by the Sobolev Embedding Theorem, we have $W^{2,2}(\mathbb{R}^2) \subseteq C^\gamma(\mathbb{R}^2)$ for any $\gamma < 1$. Unfortunately we don't get 1, so the increase in the Hölder exponent is not enough to compensate the loss in one power when changing into polar coordinates. Instead here for any $\psi \in D(\overline{L})$, we have $|\psi(x)| = |\psi(x) - \psi(0)| \leq C|x|(\ln \frac{1}{|x|})^{\frac{1}{2}}$, for some constant $C > 0$ and for small $|x|$; see [MT, Sec. 4.1].

Proposition 2.3.6. *For $\mathbb{R}^2$, we have*

$$D(\overline{L}) = \{\psi \in W^{2,2}(\mathbb{R}^2) | \psi(0) = 0\}.$$

Proof. The terms $||\psi\varphi_\epsilon - \psi||$, $||(L\psi)(\varphi_\epsilon - 1)||$ and $||\nabla\psi \cdot \nabla(\varphi_\epsilon - 1)||$ are estimated like before. Now we work on the term $||\psi L(\varphi_\epsilon - 1)||$. As stated above, here we have

$$|\psi(x)| \leq C|x|(\ln \frac{1}{|x|})^{\frac{1}{2}},$$

for some constant $C > 0$. Then

$$||\psi L(\varphi_\epsilon - 1)||^2 = \int_{\mathbb{R}^3} |\psi(x) L\varphi\big((\frac{|x|}{\epsilon})^\epsilon\big)|^2 dx \leq C^2 \int_{B(0,\epsilon)} |x|^2 \ln \frac{1}{|x|} |L\varphi\big((\frac{|x|}{\epsilon})^\epsilon\big)|^2 dx.$$

Again as before, we substitute $d = 2$ into (2.1) to obtain the expression for $L\varphi\big((\frac{|x|}{\epsilon})^\epsilon\big)$,

$$L\varphi\big((\frac{|x|}{\epsilon})^\epsilon\big) = \epsilon^{2-2\epsilon}|x|^{2\epsilon-2}\varphi''\big((\frac{|x|}{\epsilon})^\epsilon\big) + \epsilon^{2-\epsilon}|x|^{\epsilon-2}\varphi'\big((\frac{|x|}{\epsilon})^\epsilon\big).$$

Note that in this dimension the first term in (2.1), $\epsilon^{1-\epsilon}|x|^{\epsilon-2}\varphi'\big((\frac{|x|}{\epsilon})^\epsilon\big)$, is gone, which is good news as it was previously the biggest term. Because φ' and φ'' are bounded, we estimate $\epsilon^{2-2\epsilon}|x|^{2\epsilon-2}$ and $\epsilon^{2-\epsilon}|x|^{\epsilon-2}$. We square them and then multiply with $|x|^2 \ln \frac{1}{|x|}$ to obtain $\epsilon^{4-4\epsilon}|x|^{4\epsilon-2} \ln \frac{1}{|x|}$ and $\epsilon^{4-2\epsilon}|x|^{2\epsilon-2} \ln \frac{1}{|x|}$. The second term is bigger,

$$\int_{B(0,\epsilon)} \epsilon^{4-2\epsilon}|x|^{2\epsilon-2} \ln \frac{1}{|x|} dx = -2\pi\epsilon^{4-2\epsilon} \int_0^\epsilon r^{2\epsilon-1} \ln r dr = -\pi\epsilon^3 \ln \epsilon + \frac{\pi\epsilon^2}{2}.$$

This expression goes to 0 as ϵ approaches 0. This completes the proof. □

2.3.5 Case d=1

In this case the power of r obtained from changing into polar coordinates no longer exists. Instead we have the Hölder condition on the derivatives as compensation. This means the domain is different from the above cases as we must impose a condition on the derivatives. Here, by the Sobolev Embedding Theorem, not only $W^{2,2}(\mathbb{R}) \subseteq C^{\frac{1}{2}}(\mathbb{R})$ we also have $W^{1,2}(\mathbb{R}) \subseteq C^{\frac{1}{2}}(\mathbb{R})$, which means it also makes sense to talk about the derivative of a function at a single point in the domain. We will skip the proof of the following proposition as it is similar to the other cases.

Proposition 2.3.7. *For $\mathbb{R}$, we have*

$$D(\overline{L}) = \{\psi \in W^{2,2}(\mathbb{R}) | \psi(0) = 0, \psi'(0) = 0\}.$$

2.4 Self-adjoint extensions of L

After finding the closures we are now in the position to calculate the self-adjoint extensions.

2.4.1 Case d≥4

In this case $\overline{L}$ is already self-adjoint. We know that by calculating the deficiency indices of $\overline{L}$. Suppose that $u \in \mathscr{K}_+$, then it means

$$\big(u, (\overline{L} + i)f\big) = 0 \text{ for all } f \in W^{2,2}(\mathbb{R}^d).$$

In the Fourier space, we have $\big(\hat{u}, (|\xi|^2 + i\big)\hat{f}\big) = 0$ for all $f \in W^{2,2}(\mathbb{R}^d)$. Then $\big((|\xi|^2 - i)\hat{u}, \hat{f}\big) = 0$ for all $f \in W^{2,2}(\mathbb{R}^d)$. Since $\mathscr{F}(W^{2,2}(\mathbb{R}^d))$ is dense in $L^2(\mathbb{R}^d)$, we have $(|\xi|^2 - i)\hat{u} = 0$. It follows that $u = 0$, ie $\mathscr{K}_+ = \{0\}$. Similarly, $\mathscr{K}_- = \{0\}$.

2.4.2 Case d=2, 3

Here for $u \in \mathscr{K}_+$, we have

$$\big(u, (\overline{L} + i)f\big) = 0, \text{ for all } f \in W^{2,2}(\mathbb{R}^d) \text{ such that } f(0) = 0.$$

Again we work in the Fourier space, then the condition becomes

$$\big(\hat{u}, (|\xi|^2 + i)\hat{f}\big) = 0, \text{ for all } \hat{f} \in (1 + |\xi|^2)^{-1} L^2(\mathbb{R}^d) \text{ such that } \int_{\mathbb{R}^d} \hat{f}(\xi) d\xi = 0.$$

Let φ be a fixed function in $W^{2,2}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \hat{\varphi}(\xi) d\xi = 1$. Then for any $f \in W^{2,2}(\mathbb{R}^d)$, the function $\hat{f} - \int_{\mathbb{R}^d} \hat{f}(x) dx \hat{\varphi}$ is in $(1 + |\xi|^2)^{-1} L^2(\mathbb{R}^d)$, and its integral over $\mathbb{R}^d$ is 0. Hence we have

$$\Big(\hat{u}, \big(|\xi|^2 + i\big)\big(\hat{f} - \int_{\mathbb{R}^d} \hat{f}(x) dx \hat{\varphi}\big)\Big) = 0.$$

That means

$$\Big((|\xi|^2 - i)\hat{u} - \big((|\xi|^2 - i)\hat{u}, \hat{\varphi}\big), \hat{f}\Big) = 0.$$

Since $W^{2,2}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, we conclude that $(|\xi|^2 - i)\hat{u} = c$ where $c = ((|\xi|^2 - i)\hat{u}, \hat{\varphi})$ is a constant. Thus $\hat{u} = \frac{c}{|\xi|^2 - i}$, and so we know $\mathscr{K}_+$ is the one dimensional complex space spanned by $\mathcal{F}^{-1}(\frac{1}{|\xi|^2 - i})$. Similarly, $\mathscr{K}_-$ is the one dimensional complex space spanned by $\mathcal{F}^{-1}(\frac{1}{|\xi|^2 + i})$. Since both $\mathscr{K}_+$ and $\mathscr{K}_-$ are one dimensional, and $\frac{1}{|\xi|^2 - i}$, $\frac{1}{|\xi|^2 + i}$ have the same norm, the unitary maps from $\mathscr{K}_+$ onto $\mathscr{K}_-$ can be parametrized by the unit circle in the complex plane, such that for each $\theta \in [-\pi, \pi)$, $\mathcal{F}^{-1}(\frac{1}{|\xi|^2 - i})$ is mapped to $e^{i\theta}\mathcal{F}^{-1}(\frac{1}{|\xi|^2 + i})$. Therefore we have the following proposition.

Proposition 2.4.1. *The self-adjoint extensions of L in $\mathbb{R}^d$, $d = 2, 3$, can be parametrized by a circle $\theta \in [-\pi, \pi)$, with $D(L_\theta)$ equal to*

$$\{\varphi + \beta\mathcal{F}^{-1}(\frac{1}{|\xi|^2 - i}) + e^{i\theta}\beta\mathcal{F}^{-1}(\frac{1}{|\xi|^2 + i}) : \varphi \in W^{2,2}(\mathbb{R}^d), \varphi(0) = 0 \text{ and } \beta \in \mathbb{C}\},$$

and

$$\begin{aligned} &L_\theta\big(\varphi + \beta\mathcal{F}^{-1}(\frac{1}{|\xi|^2 - i}) + e^{i\theta}\beta\mathcal{F}^{-1}(\frac{1}{|\xi|^2 + i})\big) \\ =&\overline{L}\varphi + i\beta\mathcal{F}^{-1}(\frac{1}{|\xi|^2 - i}) - ie^{i\theta}\beta\mathcal{F}^{-1}(\frac{1}{|\xi|^2 + i}). \end{aligned}$$

Remark 2.4.2. Notice that $\frac{1}{|\xi|^2 - i}$, $\frac{1}{|\xi|^2 + i}$ are in $L^2(\mathbb{R}^d)$ if and only if $d < 4$, hence they cannot be in the deficiency subspaces of $\overline{L}$ for any $d \geq 4$.

Among all the self-adjoint extensions of L, $L_{-\pi}$ is the special one. We denote it by Δ, and call it the Laplacian. It can be easily shown that

$$D(\Delta) = W^{2,2}(\mathbb{R}^d), \quad d = 2, 3.$$

It is the only self-adjoint extension whose domain is contained in the form domain of the closure of the quadratic form q associated with L, see Section 2.2 for the expression of q. Note that $D(\Delta) = W^{2,2}(\mathbb{R}^d)$ is contained in $Q(\hat{q}) = W^{1,2}(\mathbb{R}^d)$, and it follows from [RS, Sec. X.3] that Δ is the Friedrichs extension.

For $d = 3$, let's look at another way to parametrize these self-adjoint extensions by considering the Taylor expansions at the origin of the functions in the domains of these extensions. The parameter is the ratio between the constant term and the coefficient of the $\frac{1}{|x|}$ term in the expansions. For that, we use the following well-known result,

$$\mathscr{F}^{-1}\left(\frac{1}{|\xi|^2 + \lambda^2}\right) = \frac{1}{4\pi} \frac{e^{-\lambda|x|}}{|x|}, \tag{2.2}$$

for $\lambda \in \mathbb{C}\backslash i\mathbb{R}$. From this we obtain the following two expansions at the origin,

$$\mathscr{F}^{-1}\left(\frac{1}{|\xi|^2 - i}\right) = \frac{1}{4\pi} \frac{e^{-\operatorname{cis}(-\frac{\pi}{4})|x|}}{|x|} = \frac{1}{4\pi|x|} - \frac{1}{4\pi} e^{-\frac{\pi}{4}i} + O(|x|),$$

$$\mathscr{F}^{-1}\left(\frac{1}{|\xi|^2 + i}\right) = \frac{1}{4\pi} \frac{e^{-\operatorname{cis}(\frac{\pi}{4})|x|}}{|x|} = \frac{1}{4\pi|x|} - \frac{1}{4\pi} e^{\frac{\pi}{4}i} + O(|x|).$$

We will use the above expansions in the proof of the following proposition.

Proposition 2.4.3. *The self-adjoint extensions of L in $\mathbb{R}^3$ can be parametrized by $\mu \in \mathbb{R} \cup \{\infty\}$ with*

$$D(L^\mu) = \{\varphi = \chi\beta(\frac{1}{|x|} + \mu) + \phi : \phi \in W^{2,2}(\mathbb{R}^3), \phi(0) = 0, \beta \in \mathbb{C}\},$$

where χ is a function in $C_c^\infty(\mathbb{R}^3)$ and $\chi \equiv 1$ near 0, and

$$D(L^\infty) = W^{2,2}(\mathbb{R}^3).$$

Moreover, the relationship with the previous parametrisation is

$$L^\mu = L_\theta \textit{ iff } \mu(\theta) = \frac{\sqrt{2}}{2}\big(\tan(\frac{\theta}{2}) - 1\big), \quad \theta \in (-\pi, \pi),$$

and

$$L^\infty = L_{-\pi} = \Delta.$$

Remark 2.4.4. Note that the case $\theta = -\pi$ corresponds to functions with only the constant term but no $\frac{1}{|x|}$ term in the expansion at the origin. The case $\theta = \frac{\pi}{2}$ corresponds to functions with only the $\frac{1}{|x|}$ term but no constant term in the expansion at the origin.

Proof. We focus on the first two terms in the expansion. From Proposition 2.4.1 and equation (2.2) we know they are a multiple of

$$(1+e^{i\theta})\frac{1}{|x|} - (e^{-\frac{\pi}{4}i} + e^{(\theta+\frac{\pi}{4})i}).$$

For $\theta = -\pi$, the $\frac{1}{|x|}$ term disappears, and we have a non-zero constant $\sqrt{2}$. This means we can get any value at the origin, which is consistent with what we already know. In the new parametrisation, we label this operator L^∞.

Then we consider $\theta \neq -\pi$. Here the first two terms in the expansion are a multiple of

$$\frac{1}{|x|} - \frac{e^{-\frac{\pi}{4}i} + e^{(\theta+\frac{\pi}{4})i}}{1+e^{i\theta}}.$$

Denote

$$\mu(\theta) = -\frac{e^{-\frac{\pi}{4}i} + e^{(\theta+\frac{\pi}{4})i}}{1+e^{i\theta}} = -e^{\frac{\pi}{4}i}(\frac{e^{i\theta}-i}{e^{i\theta}+1}), \tag{2.3}$$

and in the new parametrisation this operator is denoted by L^μ. After some calculations, we know that

$$\mu(\theta) = \frac{\sqrt{2}}{2}\big(\tan(\frac{\theta}{2}) - 1\big), \quad \theta \in (-\pi, \pi). \tag{2.4}$$

From this expression, we can see $\mu(\theta)$ ranges across the real line. When θ increases from $-\pi$ to π, $\mu(\theta)$ moves rightwards along the real line from $-\infty$ to ∞, and it changes sign when $\theta = \frac{\pi}{2}$. $\square$

Remark 2.4.5. The μ-parametrization corresponds to the usual way self-adjoint extensions of the Laplacian on a cone are defined, that is, in terms of the expansions of harmonic functions at the cone point; see [EM].

2.5 Resolvent kernel

2.5.1 Resolvent kernel

From now on we focus on $\mathbb{R}^3$. In this section we calculate the resonances and eigenvalues of the self-adjoint extensions of L. For that we need to determine the resolvent kernel of these various self-adjoint extensions. We start with Δ. Since the spectrum of a self-adjoint operator is a subset of $\mathbb{R}$, for any $\lambda \notin \mathbb{R}$, λ^2 is in the resolvent set, ie $(\Delta - \lambda^2)^{-1}$ exists. Let's determine the kernel of $(\Delta - \lambda^2)^{-1}$ for $\lambda \in \mathbb{C}$ with $Im(\lambda) > 0$. Suppose $(\Delta - \lambda^2)u = f$. By taking the Fourier transform we have $(|\xi|^2 - \lambda^2)\hat{u} = \hat{f}$. Hence, $\hat{u} = \frac{\hat{f}}{|\xi|^2 - \lambda^2}$, and so

$$(\Delta - \lambda^2)^{-1} f = u = \mathscr{F}^{-1}(\frac{\hat{f}}{|\xi|^2 - \lambda^2}) = f * \mathscr{F}^{-1}(\frac{1}{|\xi|^2 - \lambda^2}) = \frac{1}{4\pi} f * \frac{e^{i\lambda|x|}}{|x|}.$$

Hence we know the kernel of the operator $(\Delta - \lambda^2)^{-1}$ is

$$K_{free}(\lambda, x, y) = \frac{e^{i\lambda|x-y|}}{4\pi|x-y|}. \tag{2.5}$$

Now we determine the kernel of $(L^\mu - \lambda^2)^{-1}$, $\mu \in (-\infty, \infty)$. Besides the term $K_{free}(\lambda, x, y)$, the kernel here has another term $K_{extra}(\mu, \lambda, x, y)$. We guess that

$$K_{extra}(\mu, \lambda, x, y) = b(\mu, \lambda) \frac{e^{i\lambda(|x|+|y|)}}{|x||y|},$$

for some $b(\mu, \lambda)$. With this guess, we first determine what $b(\mu, \lambda)$ must be, then verify it is indeed the kernel what we are after. Denote

$$K(\mu, \lambda, x, y) = K_{free}(\lambda, x, y) + K_{extra}(\mu, \lambda, x, y) = \frac{e^{i\lambda|x-y|}}{4\pi|x-y|} + b(\mu, \lambda)\frac{e^{i\lambda(|x|+|y|)}}{|x||y|}.$$

If $K(\mu, \lambda, x, y)$ is indeed the kernel of $(L^\mu - \lambda^2)^{-1}$, it must lie in the domain of L^μ when y is fixed. So we fix y and consider the expansion of the function at $x = 0$,

$$\frac{e^{i\lambda|y|}}{|y|}\Big(b(\mu, \lambda)\frac{1}{|x|} + \big(\frac{1}{4\pi} + i\lambda b(\mu, \lambda)\big)\Big).$$

When $\mu \neq \infty$, $b(\mu, \lambda) \neq 0$, and being in the domain of L^μ means

$$\frac{\frac{1}{4\pi} + i\lambda b(\mu, \lambda)}{b(\mu, \lambda)} = \mu.$$

Solve for $b(\mu, \lambda)$,

$$b(\mu, \lambda) = \frac{i}{4\pi(\lambda + i\mu)}.$$

Therefore

$$K_{extra}(\mu, \lambda, x, y) = \frac{ie^{i\lambda(|x|+|y|)}}{4\pi|x||y|(\lambda + i\mu)},$$

and the ansatz for the kernel of $(L^\mu - \lambda^2)^{-1}$ is

$$K(\mu, \lambda, x, y) = \frac{e^{i\lambda|x-y|}}{4\pi|x-y|} + \frac{ie^{i\lambda(|x|+|y|)}}{4\pi|x||y|(\lambda + i\mu)}. \tag{2.6}$$

We are left to verify this is the correct kernel. Indeed, it maps $L^2(\mathbb{R}^3)$ into $D(L^\mu)$, and away from the origin, we have

$$(L^\mu - \lambda^2)K_{\mu,\lambda}f = (\Delta - \lambda^2)K_{\mu,\lambda}f = f,$$

where $K_{\mu,\lambda}$ denotes the operator which corresponds to the kernel $K(\mu, \lambda, x, y)$. Since the origin has measure 0, it means in $L^2(\mathbb{R}^3)$ we have

$$(L^\mu - \lambda^2)K_{\mu,\lambda}f = f,$$

which shows that we have found the correct kernel.

2.5.2 Resonance

In Section 2.5.1 we determine the kernel of $(\Delta - \lambda^2)^{-1}$ for $\lambda \in \mathbb{C}$ with $Im(\lambda) > 0$. In this region, we have exponential decay, so the operator maps L^2-functions to L^2-functions. The kernel $K(\mu, \lambda, x, y)$, as a function of λ, clearly has a meromorphic continuation to the whole complex plane $\mathbb{C}$. The continuation is defined by the same expression, so for convenience, we use the same name $K(\mu, \lambda, x, y)$ to denote the continuation. We see from (2.6) that $K(\mu, \lambda, x, y)$ has a single pole at $\lambda = -i\mu$. For $\mu < 0$, this pole is in the physical half of the plane, and as we will see in Section 2.5.3, its square is an eigenvalue of the operator L^μ. For $\mu \geq 0$, the pole is in the non-physical half of the plane, but it still has physical significance as shown in Section 2.6, and in this case the pole is called a resonance.

Remark 2.5.1. Note that the Laplacian Δ is the only self-adjoint extension of L without either an eigenvalue or a resonance.

2.5.3 Eigenvalue

We start with an arbitrary λ, and try to find an eigenfunction in $D(L^*)$, then determine whether it lies in the domain of any self-adjoint extension of L. Suppose φ is an eigenfunction in $D(L^*)$ with eigenvalue λ, that means it lies in $\mathrm{Ker}(L^* - \lambda) = \mathrm{Ran}(\overline{L} - \lambda)^{\perp}$, and hence

$$\big(\varphi, (\overline{L} - \lambda)\psi\big) = 0 \qquad \text{for all } \psi \in D(\overline{L}).$$

We can solve the above equation similar as before. It has a non-trivial solution only when λ is negative, therefore we know none of the self-adjoint extensions has a non-negative eigenvalue. For $\lambda < 0$, the eigenspace corresponding to λ is spanned by

$$\mathscr{F}^{-1}\Big(\frac{1}{|\xi|^2 - \lambda}\Big) = \frac{1}{4\pi}\frac{e^{-\sqrt{-\lambda}|x|}}{|x|} = \frac{1}{4\pi|x|} - \frac{\sqrt{-\lambda}}{4\pi} + O(|x|).$$

Equation (2.2) is used to obtain this expansion. By Proposition 2.4.3 we know this function is in $D(L^{-\sqrt{-\lambda}})$.

From the above discussion we know that for $\mu \in (-\infty, 0)$, the extension L^{μ} has an eigenvalue $-\mu^2$, and the eigenspace is spanned by the function

$$v_\mu(x) = \frac{e^{\mu|x|}}{|x|}. \tag{2.7}$$

Since μ is negative, v_μ is in $L^2(\mathbb{R}^3)$. We compute that $||v_\mu||_2^2 = -\frac{2\pi}{\mu}$, so a normalised eigenfunction corresponding to the eigenvalue $-\mu^2$ is

$$\sqrt{\frac{-\mu}{2\pi}}\frac{e^{\mu|x|}}{|x|}.$$

When $\mu \in [0, \infty)$, that is when $\theta \in [\frac{\pi}{2}, \pi)$, the expression of the eigenfunction v_μ in the above case, ie expression (2.7), is no longer in $L^2(\mathbb{R}^3)$. As mentioned earlier, in this case the pole $a(\mu) = -i\mu$ is a resonance. To summarise, the spectra of the self-adjoint extensions are

$$\sigma(L^\mu) = \{-\mu^2\} \cup [0, \infty), \quad \mu \in (-\infty, 0),$$
$$\sigma(L^\mu) = [0, \infty), \quad \mu \in [0, \infty].$$

Remark 2.5.2. As pointed out by one of the examiners, the resolvents $(L^\mu + \lambda)^{-1}$ and $(L^\infty + \lambda)^{-1}$ differ from a rank one operator, hence from the formula (2.6), one gets a formula for the spectral shift function. This gives immediately the result about the resonance and the eigenvalue.

2.6 A wave equation involving L^μ

For $\mu \in [0, \infty)$, the resonance $-i\mu$ doesn't result in an eigenvalue, and it is in the non-physical half of the complex plane as it means exponential growth of the kernel $K(\mu, \lambda, x, y)$. But this resonance still has physical significance, and as we will see, it appears in the wave kernel involving L^μ; see [LP]. The wave equation we consider here is

$$\begin{cases} \partial_t^2 u + L^\mu u = 0, \\ u|_{t=0} = f, \\ \partial_t u|_{t=0} = g, \end{cases}$$

where $\mu \in \mathbb{R} \cup \{\infty\}$, and $f, g \in C_c^\infty(\mathbb{R}^3 \backslash \{0\})$.

We know that the solution for the system

$$\begin{cases} \partial_t^2 u + a^2 u = 0, \\ u|_{t=0} = f, \\ \partial_t u|_{t=0} = g, \end{cases}$$

where $a \in \mathbb{R}$, is

$$u(t) = \cos(at) f + \frac{\sin(at)}{a} g.$$

So by functional calculus, the solution for the system we are interested in is

$$u(t) = \cos(t\sqrt{L^\mu}) f + \frac{\sin(t\sqrt{L^\mu})}{\sqrt{L^\mu}} g.$$

Here if $f \in D(L^\mu)$ and $g \in D(\sqrt{L^\mu})$, we have a strong solution, ie $u(t) \in D(L^\mu)$ for each t, and u is continuous as a function of t with values in $D(L^\mu)$.

We proceed to calculate the kernel of $\frac{\sin(t\sqrt{L^\mu})}{\sqrt{L^\mu}}$, and the kernel of $\cos(t\sqrt{L^\mu})$ is given by its time derivative.

Proposition 2.6.1. *For any $\mu \in \mathbb{R}$, the kernel of $\frac{\sin(t\sqrt{L^\mu})}{\sqrt{L^\mu}}$ for $t \geq 0$ is*

$$\frac{1}{4\pi}\big(\delta(t^2 - |x-y|^2) + \frac{1}{|x||y|}H(t - |x| - |y|)e^{\mu(|x|+|y|-t)}\big), \tag{2.8}$$

where H is the Heaviside function.

Remark 2.6.2. We know that away from the origin, L^μ is the same as Δ. Also, due to finite propagation speed, the minimum time required to travel from x to y through the origin is $|x| + |y|$. Therefore for $t < |x| + |y|$, we would expect that $\frac{\sin(t\sqrt{L^\mu})}{\sqrt{L^\mu}}$ has the same kernel as $\frac{\sin(t\sqrt{\Delta})}{\sqrt{\Delta}}$. Our kernel (2.8) satisfies this, which is a check on its correctness.

Remark 2.6.3. The second term in (2.8) can be interpreted as a "diffracted" wave from the origin thought of as a cone point. The strength of the singularity is 1 order weaker than the incident singularity, as is the case for a diffracted wave; see [CT] and [MW].

Remark 2.6.4. When $\mu < 0$, the second term of (2.8) is exponentially growing in time as $t \to \infty$, but exponentially decaying in space as $|x|, |y| \to \infty$. This is due to the negative eigenvalue. On the other hand, when $\mu > 0$, this term is exponentially decaying in time as $t \to \infty$, but exponentially growing in space as $|x|, |y| \to \infty$. It corresponds to a term in the "resonance expansion" for solutions to the wave equation on a compact set; see [LP] and [TZ].

Remark 2.6.5. See [LH] for a different approach to obtain the kernel by solving an auxiliary problem.

Proof. We have,

$$\frac{\sin(t\sqrt{L^\mu})}{\sqrt{L^\mu}} = \int_{-\infty}^{\infty} \frac{\sin(t\sqrt{\sigma})}{\sqrt{\sigma}} dP_\sigma = \lim_{R\to\infty} \int_{-\infty}^{\infty} \frac{\sin(t\sqrt{\sigma})}{\sqrt{\sigma}} \varphi(\frac{\sigma}{R}) dP_\sigma,$$

where $\varphi : \mathbb{R} \to [0,1]$ is a smooth function such that $\varphi = 1$ on $B_1(0)$ and $\varphi = 0$ outside $B_2(0)$, and the limit converges under the strong operator topology. Then depending on whether L^μ has an eigenvalue, we have two possibilities. First for

$\mu \in [0, \infty)$, we have

$$\frac{\sin(t\sqrt{L^\mu})}{\sqrt{L^\mu}} = \lim_{R\to\infty} \int_{-\infty}^{\infty} \frac{\sin(t\sqrt{\sigma})}{\sqrt{\sigma}} \varphi(\frac{\sigma}{R}) dP_\sigma = \lim_{R\to\infty} \int_0^{\infty} \frac{\sin(t\sqrt{\sigma})}{\sqrt{\sigma}} \varphi(\frac{\sigma}{R}) dP_\sigma.$$

For $\mu \in (-\infty, 0)$, the eigenvalue contributes an extra term,

$$\begin{aligned}\frac{\sin(t\sqrt{L^\mu})}{\sqrt{L^\mu}} &= \lim_{R\to\infty} \int_{-\infty}^{\infty} \frac{\sin(t\sqrt{\sigma})}{\sqrt{\sigma}} \varphi(\frac{\sigma}{R}) dP_\sigma \\ &= \lim_{R\to\infty} \int_0^{\infty} \frac{\sin(t\sqrt{\sigma})}{\sqrt{\sigma}} \varphi(\frac{\sigma}{R}) dP_\sigma + \frac{\sin(i\mu t)}{i\mu} P_{-\mu^2},\end{aligned}$$

where $P_{-\mu^2}$ is the orthogonal projection onto the eigenspace of $-\mu^2$.

In either case we need to calculate the term $\lim_{R\to\infty} \int_0^\infty \frac{\sin(t\sqrt{\sigma})}{\sqrt{\sigma}} \varphi(\frac{\sigma}{R}) dP_\sigma$. For each $R > 0$, we apply integration by parts twice to evaluate the integral, and also by using Stone's formula, we obtain,

$$\begin{aligned}&\int_0^{\infty} \frac{\sin(t\sqrt{\sigma})}{\sqrt{\sigma}} \varphi(\frac{\sigma}{R}) dP_\sigma \\ =&\frac{1}{2\pi i} \int_0^{\infty} \frac{\sin(t\sqrt{\sigma})}{\sqrt{\sigma}} \varphi(\frac{\sigma}{R}) \lim_{\epsilon\downarrow 0} \big((L^\mu - \sigma - i\epsilon)^{-1} - (L^\mu - \sigma + i\epsilon)^{-1}\big) d\sigma.\end{aligned}$$

Recall from Section 2.5 that the kernel of $(L^\mu - \lambda^2)^{-1}$, $\mu \in \mathbb{R} \cup \{\infty\}$, is denoted by $K(\mu, \lambda, x, y)$. Hence for any function $f \in D(\sqrt{L^\mu})$ we have

$$\begin{aligned}&\int_0^{\infty} \frac{\sin(t\sqrt{\sigma})}{\sqrt{\sigma}} \varphi(\frac{\sigma}{R}) dP_\sigma\big(f(x)\big) \\ =&\frac{1}{2\pi i} \int_0^{\infty} \frac{\sin(t\sqrt{\sigma})}{\sqrt{\sigma}} \varphi(\frac{\sigma}{R}) \int_{-\infty}^{\infty} \big(K(\mu, \sqrt{\sigma}, x, y) - K(\mu, -\sqrt{\sigma}, x, y)\big) f(y) dy d\sigma \\ =&\frac{1}{\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin(t\lambda) K(\mu, \lambda, x, y) \varphi(\frac{\lambda^2}{R}) f(y) d\lambda dy \quad \text{(substitute } \lambda = \sqrt{\sigma}).\end{aligned} \tag{2.9}$$

Again from Section 2.5, we know that

$$K(\mu, \lambda, x, y) = K_{free}(\lambda, x, y) + K_{extra}(\mu, \lambda, x, y) = \frac{e^{i\lambda|x-y|}}{4\pi|x-y|} + \frac{ie^{i\lambda(|x|+|y|)}}{4\pi|x||y|(\lambda + i\mu)}.$$

The free resolvent kernel $K_{free}(\lambda, x, y)$ substituted to the last line of (2.9) gives the free wave kernel $\frac{1}{4\pi}\delta(t^2 - |x-y|^2)$, so from now on we concentrate on the term contributed by $K_{extra}(\mu, \lambda, x, y)$,

$$\frac{1}{\pi i}\lim_{R\to\infty}\int_{-\infty}^{\infty}\sin(t\lambda)K_{extra}(\mu,\lambda,x,y)\varphi(\frac{\lambda^2}{R})d\lambda. \tag{2.10}$$

We continue this computation in different cases depending on the sign of μ.

Case 1: $\mu = 0$

In this case equation (2.10) becomes

$$\begin{aligned}
&\frac{1}{4\pi^2|x||y|}\lim_{R\to\infty}\int_{-\infty}^{\infty}\frac{1}{\lambda}\sin(t\lambda)e^{i\lambda(|x|+|y|)}\varphi(\frac{\lambda^2}{R})d\lambda\\
=&\frac{1}{4\pi^2|x||y|}\lim_{R\to\infty}\int_{-\infty}^{\infty}\frac{1}{\lambda}\sin(t\lambda)\cos\big(\lambda(|x|+|y|)\big)\varphi(\frac{\lambda^2}{R})d\lambda\\
=&\frac{1}{8\pi^2|x||y|}\lim_{R\to\infty}\Big(\int_{-\infty}^{\infty}\frac{1}{\lambda}\sin\big(\lambda(t+|x|+|y|)\big)\varphi(\frac{\lambda^2}{R})d\lambda\\
&\qquad\qquad+\int_{-\infty}^{\infty}\frac{1}{\lambda}\sin\big(\lambda(t-|x|-|y|)\big)\varphi(\frac{\lambda^2}{R})d\lambda\Big).
\end{aligned}\tag{2.11}$$

We now make a substitution, and split into three sub-cases:

(i) When $t-|x|-|y|>0$, (2.11) becomes

$$\begin{aligned}
\frac{1}{8\pi^2|x||y|}\Bigg(&\lim_{R\to\infty}\int_{-\infty}^{\infty}\frac{\sin\lambda}{\lambda}\varphi\Big(\frac{\lambda^2}{R(t+|x|+|y|)^2}\Big)d\lambda\\
&+\lim_{R\to\infty}\int_{-\infty}^{\infty}\frac{\sin\lambda}{\lambda}\varphi\Big(\frac{\lambda^2}{R(t-|x|-|y|)^2}\Big)d\lambda\Bigg).
\end{aligned}\tag{2.12}$$

Note that we have two Dirichlet integrals, each of which equals π, so the above expression equals

$$\frac{1}{4\pi|x||y|.}$$

(ii) When $t-|x|-|y|<0$, after the substitution we get the same expression as (2.12) except the sign of the second integral is negative. Hence the two integrals cancel each other, and therefore in this sub-case (2.11) equals 0.

(iii) When $t-|x|-|y|=0$, the second integral in the last line of equation (2.11) is 0, hence we are left with only the first integral in expression (2.12). Therefore here (2.11) equals $\frac{1}{8\pi|x||y|}$.

Combine all three sub-cases, the kernel of $\frac{\sin(t\sqrt{L^0})}{\sqrt{L^0}}$ contributed by K_{extra} for

$t \geq 0$ is

$$\frac{1}{4\pi|x||y|}H(t-|x|-|y|).$$

Case 2: $\mu \neq 0$

Since $\sin(t\lambda) = \frac{e^{it\lambda}-e^{-it\lambda}}{2i}$, and from (2.10), in this case we have to deal with the following two terms:

$$\frac{-i}{8\pi^2|x||y|}\lim_{R\to\infty}\int_{Im(\lambda)=0}\frac{e^{i\lambda(|x|+|y|+t)}}{\lambda+i\mu}\varphi(\frac{\lambda^2}{R})d\lambda, \tag{2.13}$$

and

$$\frac{i}{8\pi^2|x||y|}\lim_{R\to\infty}\int_{Im(\lambda)=0}\frac{e^{i\lambda(|x|+|y|-t)}}{\lambda+i\mu}\varphi(\frac{\lambda^2}{R})d\lambda. \tag{2.14}$$

We firstly work on (2.13),

$$\begin{aligned}
&\lim_{R\to\infty}\int_{Im(\lambda)=0}\frac{e^{i\lambda(|x|+|y|+t)}}{\lambda+i\mu}\varphi(\frac{\lambda^2}{R})d\lambda\\
=&\frac{1}{i(|x|+|y|+t)}\lim_{R\to\infty}\int_{Im(\lambda)=0}\frac{d}{d\lambda}(e^{i\lambda(|x|+|y|+t)})\frac{\varphi(\frac{\lambda^2}{R})}{\lambda+i\mu}d\lambda \qquad (2.15)\\
=&\frac{-1}{i(|x|+|y|+t)}\lim_{R\to\infty}\int_{Im(\lambda)=0}e^{i\lambda(|x|+|y|+t)}\frac{d}{d\lambda}(\frac{\varphi(\frac{\lambda^2}{R})}{\lambda+i\mu})d\lambda.
\end{aligned}$$

The last equality is established by integration by parts. By applying the quotient rule we then get two integrals. The first one is

$$\lim_{R\to\infty}\frac{2}{R}\int_{Im(\lambda)=0}\frac{\lambda\varphi'(\frac{\lambda^2}{R})e^{i\lambda(|x|+|y|+t)}}{\lambda+i\mu}d\lambda.$$

This limit is 0, hence (2.15) equals the second integral obtained from the application of the quotient rule, which is

$$\begin{aligned}
&\frac{1}{i(|x|+|y|+t)}\lim_{R\to\infty}\int_{Im(\lambda)=0}\frac{e^{i\lambda(|x|+|y|+t)}\varphi(\frac{\lambda^2}{R})}{(\lambda+i\mu)^2}d\lambda\\
=&\frac{1}{i(|x|+|y|+t)}\int_{Im(\lambda)=0}\frac{e^{i\lambda(|x|+|y|+t)}\lim_{R\to\infty}\varphi(\frac{\lambda^2}{R})}{(\lambda+i\mu)^2}d\lambda\\
&\qquad\qquad\text{(by the Dominated Convergence Theorem)}\\
=&\frac{1}{i(|x|+|y|+t)}\int_{Im(\lambda)=0}\frac{e^{i\lambda(|x|+|y|+t)}}{(\lambda+i\mu)^2}d\lambda.
\end{aligned}$$

Therefore (2.13) becomes

$$\frac{-1}{8\pi^2|x||y|(|x|+|y|+t)}\int_{Im(\lambda)=0}\frac{e^{i\lambda(|x|+|y|+t)}}{(\lambda+i\mu)^2}d\lambda. \tag{2.16}$$

Now we can shift the contour $Im(\lambda) = 0$ upwards to $Im(\lambda) = M$ for any $M > 0$, and the integral should stay the same except when the contour moves across a pole. When $M \to \infty$, the integral approaches zero.

Similarly, (2.14) becomes

$$\frac{1}{8\pi^2|x||y|(|x|+|y|-t)}\int_{Im(\lambda)=0}\frac{e^{i\lambda(|x|+|y|-t)}}{(\lambda+i\mu)^2}d\lambda. \tag{2.17}$$

In the region $t \leq |x| + |y|$, we can shift the contour $Im(\lambda) = 0$ upwards, while in the region $t \geq |x| + |y|$, we can shift it downwards. As before, the integral stays the same except when the contour moves across a pole, and the integral approaches zero when the contour is shifted further and further away. Note that for $t = |x| + |y|$, we have a choice between shifting it upwards or downwards so we can always avoid the pole, hence we know the integral is 0. We continue the computation in two sub-cases.

Sub-case 2(a): $\mu > 0$

In this case the pole, which is the resonance $a(\mu) = -i\mu$, lies on the negative imaginary axis. We shift the contour upwards for the integral (2.16), and the integral goes to 0; while for the integral (2.17), it depends on the sign of $|x|+|y|-t$.

(i) In the region $t \leq |x| + |y|$, the contour is also shifted upwards, so it also goes to 0. Hence (2.10) equals 0.

(ii) In the region $t > |x| + |y|$, the contour is shifted downwards hence across the pole $-i\mu$. The residue of the integrand at the pole $\lambda = -i\mu$ is

$$i(|x| + |y| - t)e^{\mu(|x|+|y|-t)}.$$

Since the winding number is -1, by the Residue Theorem, we know that (2.17), ie (2.10) equals

$$\frac{e^{\mu(|x|+|y|-t)}}{4\pi|x||y|}.$$

Combining these two cases using a single expression, the kernel of $\frac{\sin(t\sqrt{L^\mu})}{\sqrt{L^\mu}}$

contributed by K_{extra} for $\mu > 0$ and $t \geq 0$ is

$$\frac{H(t-|x|-|y|)e^{\mu(|x|+|y|-t)}}{4\pi|x||y|}.$$

Sub-case 2(b): $\mu < 0$

In this sub-case we have a negative eigenvalue, and

$$\frac{\sin(t\sqrt{L^\mu})}{\sqrt{L^\mu}} = \lim_{R\to\infty}\int_0^\infty \frac{\sin(t\sqrt{\sigma})}{\sqrt{\sigma}}\varphi(\frac{\sigma}{R})dP_\sigma + \frac{\sin(i\mu t)}{i\mu}P_{-\mu^2}. \tag{2.18}$$

For the $\frac{\sin(i\mu t)}{i\mu}P_{-\mu^2}$ term, we calculate the kernel of $P_{-\mu^2}$. As discussed in Section 2.5.3, the eigenspace of $-\mu^2$ is one dimensional, and a normalised eigenfunction is

$$v_\mu(x) = \sqrt{\frac{-\mu}{2\pi}}\frac{e^{\mu|x|}}{|x|}.$$

So the kernel of $P_{-\mu^2}$ is

$$v_\mu(x)\overline{v_\mu(y)} = \frac{-\mu e^{\mu(|x|+|y|)}}{2\pi|x||y|}.$$

Therefore the kernel of $\frac{\sin(i\mu t)}{i\mu}P_{-\mu^2}$ is

$$\frac{\sin(i\mu t)}{i\mu}\frac{-\mu e^{\mu(|x|+|y|)}}{2\pi|x||y|} = \frac{\sinh(\mu t)}{\mu}\frac{-\mu e^{\mu(|x|+|y|)}}{2\pi|x||y|} = \frac{-e^{\mu(|x|+|y|+t)}}{4\pi|x||y|} + \frac{e^{\mu(|x|+|y|-t)}}{4\pi|x||y|}. \tag{2.19}$$

Now we turn to the other term in (2.18). That is, to work out what (2.10) is in this sub-case. The pole, ie the resonance $a(\mu) = -i\mu$, now lies on the positive imaginary axis, hence the integral (2.16) may contribute some value. The residue of its integrand at the pole $\lambda = -i\mu$ is

$$i(|x|+|y|+t)e^{\mu(|x|+|y|+t)}.$$

Since the winding number is 1, by the Residue Theorem, the integral (2.16) equals

$$\frac{e^{\mu(|x|+|y|+t)}}{4\pi|x||y|}. \tag{2.20}$$

To deal with (2.17), we split into two cases as before in the Sub-case 2(a).

(i) In the region $t < |x|+|y|$, the contour is shifted upwards, hence across the pole $-i\mu$. The residue of the integrand at the pole $\lambda = -i\mu$ is

$$i(|x| + |y| - t)e^{\mu(|x|+|y|-t)}.$$

The winding number is 1, so by the Residue Theorem, we know (2.17) equals

$$\frac{-e^{\mu(|x|+|y|-t)}}{4\pi|x||y|}. \tag{2.21}$$

Adding (2.19), (2.20) and (2.21) gives us 0.

(ii) In the region $t \geq |x| + |y|$, the contour is shifted downwards, so it doesn't move across any pole, hence (2.17) is 0 here. Adding (2.19) and (2.20) gives us

$$\frac{e^{\mu(|x|+|y|-t)}}{4\pi|x||y|}.$$

Combining the above two cases into a single expression, the kernel of $\frac{\sin(t\sqrt{L^\mu})}{\sqrt{L^\mu}}$ contributed by K_{extra} for $\mu < 0$ and $t \geq 0$ is

$$\frac{H(t - |x| - |y|)e^{\mu(|x|+|y|-t)}}{4\pi|x||y|}.$$

We have completed the proof.

$\square$

Remark 2.6.6. In the three cases of different signs of μ, we have different expressions for the kernel for $t = |x| + |y|$. This is fine because we can ignore the value of the kernel on a set of measure 0, but we do need to know that it is finite there so that we can be sure there is no distribution supported on this set.

Chapter 3

b-calculus

3.1 b-differential operators

3.1.1 Manifold with corners

The b-differential operators which we are going to introduce later are defined on manifolds with corners. A manifold with corners is like a manifold except that some of the local coordinates are restricted to be non-negative.

Definition 3.1.1. A *differential manifold with corners* M with dimension d is a topological manifold such that each point has a neighbourhood equipped with local coordinates $x_1, ..., x_k, y_1, ..., y_{d-k}$ such that the point is the origin and the first k coordinates $x_1, ..., x_k$ are non-negative, and these local coordinates are compatible with each other in the same way as a usual differential manifold.

A p-submanifold is similar to a usual submanifold but in the context of a manifold with corners.

Definition 3.1.2. A *p-submanifold*, S, of a manifold with corners, M, is a subset such that for each point of S, there is a neighbourhood U of M, with local coordinates $x_1, ..., x_k, y_1, ..., y_{d-k}$ as described in Definition 3.1.1, and there are $s \leq k$ and $t \leq d-k$ such that $S \cap U = \{x_i = 0, y_j = 0, \text{ for } 1 \leq i \leq s \text{ and } 1 \leq j \leq t\}$.

The letter p is short for product since there is a local decomposition of M as a product of intervals and half-intervals in terms of which S is a product.

More useful definitions.

Definition 3.1.3. Let M be a manifold with corners. Its boundary ∂M is the union of the *boundary hypersurfaces* of M which are themselves manifolds with corners. For a boundary hypersurface H of M, a *boundary defining function* for the hypersurface H is a function $\rho : M \to [0, \infty)$ such that $\rho^{-1}(0) = H$, ρ is smooth up to the boundary, and $d\rho \neq 0$ on H.

A boundary defining function gives us the coordinate in terms of which the asymptotic behaviour of functions when approaching that boundary hypersurface can be described. The definition for "smooth up to the boundary" is given here.

Definition 3.1.4. A function u on M is *smooth up to the boundary* if all derivatives of all orders of u are bounded on bounded subsets of the interior of M.

Every manifold with corners M can be embedded in some manifold N. Functions smooth up to the boundary of M are simply smooth functions on N restricted to M. This coincides with the above intrinsic definition, shown by *Seeley's extension theorem.*

Theorem 3.1.5. *[RMPRE, Theorem 1.4.1] If $\Omega' \subset \mathbb{R}^d$ is open and $\Omega = \Omega' \cap (\mathbb{R}^k_+ \times \mathbb{R}^{d-k})$, then there is a continuous linear extension map $E : \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega')$ such that $E(f)|_\Omega = f$ for all $f \in \mathcal{C}^\infty(\Omega)$.*

Let's illustrate these concepts by a simple example.

Example 3.1.6. *The simplest example of a manifold with corners is $\mathbb{R}^2_+$, where $\mathbb{R}_+ = [0, \infty)$. Its boundary consists of two hypersurfaces, $lb = \{0\} \times [0, \infty)$ and $rb = [0, \infty) \times \{0\}$. (The notations lb, rb are used here just to be consistent with notations introduced later on.) Use (x, x') as the coordinates. One boundary defining function for lb is $\rho_{lb} = x$, while one boundary defining function for rb is $\rho_{rb} = x'$. Any function that's smooth on $\mathbb{R}^2$ is smooth up to the boundary when restricted to be a function on $\mathbb{R}^2_+$.*

There is a parallel concept for a p-submanifold similar to the boundary defining function for a hypersurface.

Definition 3.1.7. Let M be a manifold with corners, and let S be a p-submanifold of M. A *quadratic defining function* for S is a smooth function $a : M \to [0, \infty)$ such that $a^{-1}(0) = S$ and its Hessian is positive definite in the normal direction of S.

An example of a quadratic defining function is given later on by Proposition 5.2.1.

Remark 3.1.8. In fact we often only define a quadratic defining function near the p-submanifold. Similarly, we often only define a boundary defining function near the hypersurface.

3.1.2 b-differential operators

As $r \to 0$ on a metric cone, the operators we will study in later chapters involve expressions like $r\partial_r$, ∂_{y_i}, $1 \leq i \leq d-1$. These are b-vector fields.

Definition 3.1.9. Let M be a manifold with corners with dimension d. A smooth vector field on M is a *b-vector field* if it is tangent to the boundary. Clearly, in terms of local coordinates defined in Definition 3.1.1, a b-vector field has the form

$$V = a_1 x_1 \partial_{x_1} + \cdots + a_k x_k \partial_{x_k} + a_{k+1} \partial_{y_1} + \cdots + a_d \partial_{y_{d-k}},$$

where the coefficients $a_1, ..., a_d$ are smooth functions of x and y. The set of all b-vector fields on M is denoted by $\mathcal{V}_b(M)$.

These b-vector fields motivate the definition of b-differential operators, which will be generalised to define the b-calculus later on.

Definition 3.1.10. A differential operator P on a manifold with boundary M is a *b-differential operator* of order m, if it has smooth coefficients, and, in local coordinates (x, y) around a boundary point x, it has the form

$$P = \sum_{j+|\alpha| \leq m} a_{j\alpha}(x, y)(x\partial_x)^j \partial_y^\alpha, \tag{3.1}$$

with the coefficients $a_{j\alpha}$ smooth up to the boundary.

Remark 3.1.11. These b-differential operators are basically compositions of b-vector fields. That's why the expression in the above definition involves the terms $x\partial_x$ and ∂_y.

Remark 3.1.12. For a b-differential operator as in (3.1), the principal symbol is

$$p(x, y, \lambda, \eta) = \sum_{j+|\alpha|=m} a_{j\alpha}(x, y)\lambda^j \eta^\alpha.$$

Ellipticity means that $p(x, y, \lambda, \eta) \neq 0$ whenever $(\lambda, \eta) \neq 0$, $x \geq 0$.

In order to define the b-calculus, which is used to study the behaviour of the Riesz transform as $r \to 0$, we need to introduce the concept of blow-ups.

3.2 Blow-ups

Definition 3.2.1. Let S be a closed p-submanifold of a manifold with corners M. Then the *blow-up of M along S* is a new manifold with corners obtained by introducing polar coordinates in the normal directions to S at S. The blown-up manifold, denoted by $[M; S]$, is the "resolution space" of the polar coordinates, ie the space in which the polar coordinates are non-singular.

More useful definitions related to blow-ups.

Definition 3.2.2. The *front face* $\mathrm{ff}[M; S]$ of a blown-up manifold $[M; S]$ is the hypersurface in $[M; S]$ that results from the blow-up of S, ie the inward pointing spherical normal bundle of S; the blown-up manifold has a natural *blow-down map* $\beta : [M; S] \to M$, which is a diffeomorphism of the complement of $\mathrm{ff}[M; S]$ onto the complement of S, and it is a smooth map; smooth functions on $[M; S]$ are simply functions on M which are smooth expressed in terms of local polar coordinates in the normal directions to S.

Polar coordinates on a plane is the simplest example of a blow-up. Example 3.2.3 is the continuation of Example 3.1.6.

Example 3.2.3. *Consider the manifold with corners $M = \mathbb{R}^2_+$ and its corner $O = (0, 0)$. Introducing polar coordinates in the normal directions of O at O gives the blown-up space $[M; O] = \mathbb{R}_+ \times [0, \frac{\pi}{2}]$. The coordinates in the blown-up space are (r, θ), where $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \tan^{-1}(\frac{x_2}{x_1})$ when $x_1 \neq 0$, and $\theta = \frac{\pi}{2}$ when $x_1 = 0$. The front face ff$[M; O]$ is $\{0\} \times [0, \frac{\pi}{2}]$. The blow-down map $\beta : [M; O] \to M$ obviously maps (r, θ) to $(r\cos\theta, r\sin\theta)$. The blow-up transformation and the blow-down map β are illustrated in Figure 3.1.*

Remark 3.2.4. Unlike the usual polar coordinates, where the whole front face is considered as a single point, here the front face consists of different points as θ varies.

Example 3.2.5 (Projective coordinates)**.** *As in Example 3.2.3, on the same manifold with corners $M = \mathbb{R}^2_+$, but now consider what happens to the polar coordinates r, θ when $x_2 \ll x_1$. In this region we have $r = \sqrt{x_1^2 + x_2^2} \approx x_1$ and $\theta = \tan^{-1}(\frac{x_2}{x_1}) \approx \frac{x_2}{x_1}$. This suggests that x_1 and $s = \frac{x_2}{x_1}$ can be coordinates. Indeed, x_1, s define local coordinates for $x_1 \neq 0$. They are called the projective coordinates. Similarly, another set of projective coordinates can be defined if we start with $x_1 \ll x_2$.*

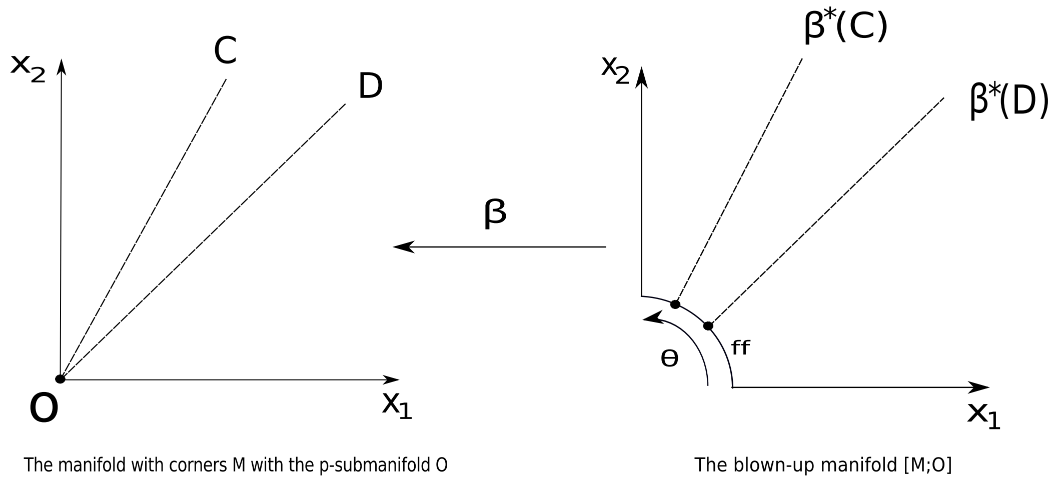


Figure 3.1: Blowing up M along O

The last concept on blow-ups that will be used later on is *the lift*, which acts like the inverse of the blow-down map.

Definition 3.2.6. *The lift* β^* maps closed subsets of M to closed subsets of the blown-up manifold $[M;S]$, and it is defined in terms of the blow-down map β. Suppose the closed subset C of M is the closure of $C\backslash S$, $\beta^*(C)$ is defined to be the closure of $\beta^{-1}(C\backslash S)$ in $[M;S]$; if C is a subset of S, then $\beta^*(C)$ is defined to be $\beta^{-1}(C)$; otherwise it is not defined.

Example 3.2.7. *Figure 3.1 also illustrates that closed subsets C and D of M are lifted to $\beta^*(C)$ and $\beta^*(D)$ in the blown-up space $[M;O]$, where*

$$C = \{(x_1, 2x_1) : x_1 \geq 0\}, \quad \beta^*(C) = \{\big(r, \tan^{-1}(2)\big) : r \geq 0\};$$
$$D = \{(x_1, x_1) : x_1 \geq 0\}, \quad \beta^*(D) = \{(r, \frac{\pi}{4}) : r \geq 0\}.$$

3.3 The b-double space

3.3.1 Definition of the b-double space

We will use blow-ups to study the kernels of pseudo-differential operators. Let M be a manifold with a connected boundary ∂M. Consider an operator T on distributions on M. For simplicity of notation, we use the same letter T to denote its kernel. The kernel $T(x, x')$ is a distribution living on the manifold with corners M^2. To study the behaviour near $(\partial M)^2$, we blow up M^2 along $(\partial M)^2$ to obtain the blown-up manifold $M_b^2 = [M^2; (\partial M)^2]$, called the b-double space. The front face and the diagonal are illustrated in the diagram below. The boundary lb is when the left variable x equals 0, and similarly the boundary rb is when the right variable x' equals 0. We call the front face here the b-front face, denoted by bf.

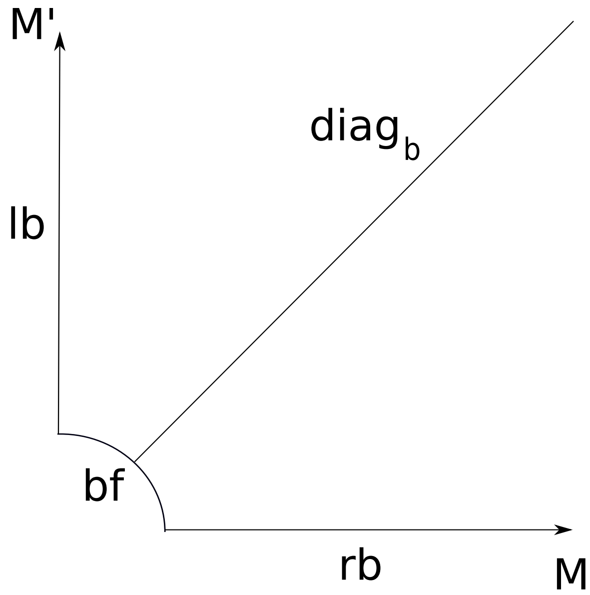


Figure 3.2: The b-double space M_b^2

Why do we perform this blow-up? Firstly, in the b-double space M_b^2, the diagonal diag_b is a p-submanifold, which is the setting of the definition of conormality later on; see [RM93, Lemma 4.3] for the proof. Secondly, in M_b^2, b-vector fields such as $x\partial_x$ is transverse to the diagonal diag_b, uniformly down to the hypersurface bf; see [RM93, Lemma 4.5].

3.3.2 Densities and half-densities

We will introduce b-half-densities in this subsection. We firstly start with densities on a vector space.

Definition 3.3.1. Let V be a vector space with dimension d, then the space of *s-densities* on V, denoted by $\Omega^s V$, for $s \in \mathbb{R}$, is

$$\Omega^s V = \{u : \Lambda^d V^* \backslash \{0\} \to \mathbb{R} : u(t\alpha) = |t|^s u(\alpha), \text{ for all } \alpha \in \Lambda^d V^* \backslash \{0\}, t \neq 0\},$$

where Λ^n denotes the n-th exterior power. Easily from the definition we have the following isomorphism of the tensor product,

$$\Omega^s V \otimes \Omega^t V \equiv \Omega^{s+t} V, \tag{3.2}$$

for all $s, t \in \mathbb{R}$.

Let M be a manifold, then

$$\Omega_x^s = \Omega^s(T_x^* M)$$

generates a bundle over M, denoted by $\Omega^s M$. When $s = 1$, we normally just write ΩM.

Definition 3.3.2. A *smooth density* on a manifold M is a smooth function from M into ΩM such that each $x \in M$ is mapped into Ω_x^1. When expressed in local coordinates, we use the notation $u(x)|dx|$, where u is a smooth function on M.

Smooth densities can be integrated, and integration is a linear functional.

We now continue to define half-densities. The advantage is there's a natural pairing for two half-densities.

Definition 3.3.3. A *smooth half-density* on a manifold M is a smooth function from M into $\Omega^{\frac{1}{2}} M$ such that each $x \in M$ is mapped into $\Omega_x^{\frac{1}{2}}$. When expressed in local coordinates, we use the notation $u(x)|dx|^{\frac{1}{2}}$, where u is a smooth function on M.

By the isomorphism (3.2), we know that when we multiply two smooth half-densities, we obtain a smooth density. And since integration is a linear functional on the space of smooth densities, it gives us a sesquilinear pairing of half-densities,

$$\langle \mu, \nu \rangle = \int_M \mu \overline{\nu},$$

where μ, ν are half-densities on M.

Definition 3.3.4. *Distributional densities* and *distributional half-densities* are defined exactly the same as definitions 3.3.2 and 3.3.3, except that u is allowed to be a distribution.

Definition 3.3.5. Let M be a manifold with corners with dimension d, and let $x_1, ..., x_k, y_1, ..., y_{d-k}$ be coordinates as in Definition 3.1.1. A *smooth b-density* on M has the form $\mu = u\left|\frac{dx_1}{x_1} ... \frac{dx_k}{x_k} dy_1 ... dy_{d-k}\right|$, where u is a smooth function on M. Similarly, a *smooth b-half-density* has the form $\mu = u\left|\frac{dx_1}{x_1} ... \frac{dx_k}{x_k} dy_1 ... dy_{d-k}\right|^{\frac{1}{2}}$, where u again is a smooth function on M.

Remark 3.3.6. The expression $\frac{dx_1}{x_1} ... \frac{dx_k}{x_k} dy_1 ... dy_{d-k}$ may look strange but it occurs because

$$\frac{dx_1}{x_1}, ..., \frac{dx_k}{x_k}, dy_1, ..., dy_{d-k}$$

is the dual basis for $x_1\partial_{x_1}, ..., x_k\partial_{x_k}, \partial_{y_1}, ..., \partial_{y_{d-k}}$.

A smooth b-half-density may not be a smooth half-density; while a distributional b-half-density is just a distributional half-density written in a different form.

Definition 3.3.7. Let M be a manifold with corners with dimension d, and let $x_1, ..., x_k, y_1, ..., y_{d-k}$ be coordinates as in Definition 3.1.1. A distributional *b-density* on M is just a distributional density, except that we write it as $\mu = u\left|\frac{dx_1}{x_1}...\frac{dx_k}{x_k}dy_1...dy_{d-k}\right|$, where u is a distribution on M. Similarly, a distributional *b-half-density* has the form $\mu = u\left|\frac{dx_1}{x_1}...\frac{dx_k}{x_k}dy_1...dy_{d-k}\right|^{\frac{1}{2}}$, where u is again a distribution on M.

Later on in this chapter we will introduce the b-calculus. When we use the b-calculus to study operators, the convention is to write the kernels in the form of b-half-densities. We use the symbol ${}^b\Omega^{\frac{1}{2}}(M_b^2)$ to denote the bundle of b-half-densities over the b-double space M_b^2.

3.3.3 An example: the identity operator

In this subsection we will use the identity operator as an example to illustrate the definitions introduced in the previous subsection. Let's consider $M = \mathbb{R}_+^2$ with the projective coordinates defined in Example 3.2.5. Let's write the kernel of the identity operator Id in terms of the concepts defined in Subsection 3.3.2. Firstly we write the operator as a half-density on M^2,

$$Id = \delta(x - x')\left|dxdx'\right|^{\frac{1}{2}}.$$

Then as a b-half-density on M^2,

$$Id = x'\delta(x - x')\left|\frac{dx}{x}\frac{dx'}{x'}\right|^{\frac{1}{2}},$$

here we use $x'^{\frac{1}{2}}\delta(x - x') = x^{\frac{1}{2}}\delta(x - x')$.

Now we write it as a half-density on M_b^2, ie in terms of the coordinates $s = \frac{x}{x'}$ and x'. The transformation of the measure is,

$$dxdx' = x'dsdx'.$$

We also use the the property that δ is homogeneous of degree -1. Therefore,

$$Id = \delta(x - x')\left|dxdx'\right|^{\frac{1}{2}} = \delta\big(x'(s-1)\big)\left|x'dsdx'\right|^{\frac{1}{2}}$$
$$= \frac{1}{x'}\delta(s-1)x'^{\frac{1}{2}}\left|dsdx'\right|^{\frac{1}{2}} = x'^{-\frac{1}{2}}\delta(s-1)\left|dsdx'\right|^{\frac{1}{2}}.$$

We will get a nicer expression if we write the operator as a b-half-density on M_b^2, because then the transformation of the measure is particularly simple,

$$\frac{dx}{x}\frac{dx'}{x'} = \frac{ds}{s}\frac{dx'}{x'}.$$

We will again use the homogeneity property of δ,

$$Id = x'\delta(x - x')\left|\frac{dx}{x}\frac{dx'}{x'}\right|^{\frac{1}{2}} = x'\delta\big(x'(s-1)\big)\left|\frac{ds}{s}\frac{dx'}{x'}\right|^{\frac{1}{2}} = \delta(s-1)\left|\frac{ds}{s}\frac{dx'}{x'}\right|^{\frac{1}{2}}.$$

3.4 Small b-calculus

3.4.1 b-differential operators as b-half-densities

Our purpose is to use blow-ups to study pseudo-differential operators. We firstly start with an intuitive category of operators, the b-differential operators, then generalise it to define the b-calculus.

The following theorem tells us what the kernels of b-differential operators on $\mathbb{R}_+$ look like as b-half-densities.

Theorem 3.4.1. *([RM93, Lemma 4.21]) Let $M = \mathbb{R}_+$. The Schwartz kernels of b-differential operators on b-half-densities on M_b^2 are precisely the Dirac distributions, ie a finite combination of the derivatives of the Dirac δ functions, on* diag_b *which are smooth up to the boundary.*

We will generalise this characterisation of the kernels to define the small b-calculus. In particular, one property that will be generalised is the conormality of the Dirac distributions with respect to diag_b. We will define conormality in the next subsection.

3.4.2 Conormality

Definition 3.4.2. Let M be a manifold and Y be a submanifold of M. A distribution u on M is *conormal of order m* with respect to Y if, for some $m \in \mathbb{R}$, u is smooth on $M \backslash Y$, and in any local coordinate system $x : U \subset M \to \mathbb{R}^d$ sending $Y \cap U$ to $\mathbb{R}^k \times \{0\}^{d-k}$, there is a representation

$$u(y,z) = \int_{\mathbb{R}^{d-k}} e^{iz\xi} a(y,\xi) d\xi,$$

where $y = (x_1, ..., x_k)$, $z = (x_{k+1}, ..., x_d)$, and a is a symbol of order m, ie a is a smooth function on $(Y \cap U) \times \mathbb{R}^{d-k}$ satisfying

$$\left|\partial_\xi^\alpha \partial_y^\beta a(y,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|},$$

for all $(y,\xi) \in (Y \cap U) \times \mathbb{R}^{d-k}$, for all multi-indices α, β, and for some constants $C_{\alpha,\beta}$.

Intuitively, the first k coordinates in the above definition gives the coordinates in Y, while the last $d-k$ coordinates parametrize the neighbourhood of Y.

Remark 3.4.3. The same definition makes sense if M is a manifold with corners and Y is a p-submanifold transverse to the boundary of M.

Example 3.4.4. *We again go back to the example of the projective coordinates defined in Example 3.2.5. The manifold with corners is the b-double space M_b^2, and the p-submanifold* diag_b *is transverse to the boundary of M_b^2. The function $\delta(s-1)$ is conormal with respect to* diag_b. *In fact it's constant along* diag_b.

From Subsection 3.3.3, we know that written as a b-half-density, the expression for the identity operator is $\delta(s-1)|\frac{ds}{s}\frac{dx'}{x'}|^{\frac{1}{2}}$, which is non-degenerate, ie conormal with respect to diag_b *and smooth up to the boundary.*

3.4.3 Small b-calculus

We are now in the position to define the small b-calculus.

Definition 3.4.5 (small b-calculus). Let M be a manifold with a connected boundary. The *small b-calculus* $\Psi_b^m(M)$, $m \in \mathbb{R}$, is defined as the set of b-half-density-valued distributions u on M_b^2 satisfying

(i) u is conormal of order m with respect to diag_b, smoothly up to the hypersurface bf;

(ii) u vanishes to infinite order at lb and rb.

We also define

$$\Psi_b^{-\infty}(M) = \bigcap_m \Psi_b^m(M).$$

Remark 3.4.6. It can be shown that $\Psi_b^{-\infty}(M)$ are simply smooth b-half-densities that vanish at lb and rb.

The good property of small b-calculus is that it's closed under composition. See [RM93, Prop 5.20] for the proof of the following result.

Proposition 3.4.7. *If M is a compact manifold with a connected boundary then*

$$\Psi_b^m(M) \circ \Psi_b^{m'}(M) \subset \Psi_b^{m+m'}(M),$$

where $m, m' \in \mathbb{R}$.

Since our purpose is to study pseudo-differential operators, it's important to know about parametrix constructions under the small b-calculus. The following proposition is analogous to [LH85, Theorem 18.1.24].

Proposition 3.4.8. *Let M be a manifold with a connected boundary. If P is an elliptic partial differential operator of order k, then there exists an operator G in the small b-calculus of order $-k$ such that*

$$Id - PG \in \Psi_b^{-\infty}(M),$$

and G with this property is unique up to an element of $\Psi_b^{-\infty}(M)$. Similarly, there exists an operator G' in the small b-calculus of order $-k$ such that

$$Id - G'P \in \Psi_b^{-\infty}(M),$$

and G' with this property is unique up to an element of $\Psi_b^{-\infty}(M)$.

This inversion property is not good enough for our purpose, as the error terms $Id - PG$, $Id - G'P$ are in $\Psi_b^{-\infty}(M)$, whose elements are not in general compact. To investigate when an element in the small b-calculus is compact, we define *the indicial operator*.

3.4.4 Indicial operator

Operators in the small b-calculus are defined over the whole b-double space. Since the behaviour at the front face bf is the most interesting, for each operator, we will define a corresponding indicial operator that focuses its behaviour on the front face. It is simply the restriction of the kernel to bf.

Definition 3.4.9. Let M be a manifold with a connected boundary ∂M and dimension d. Let $x, y_1, ..., y_{d-1}$ be local coordinates near ∂M, where x is a boundary defining function, and $y = (y_1, ..., y_{d-1})$ is a coordinate system on ∂M. Then in the b-double space the local coordinates are $s = \frac{x}{x'}, x', y, y'$. Consider the b-half-density

$$A = A(s, x', y, y)\left|\frac{ds}{s}\frac{dx'}{x'}dydy'\right|^{\frac{1}{2}},$$

where $A(s, x', y, y')$ is a distribution. The *restriction* of A to the hypersurface bf is defined as

$$A\big|_{bf} = A(s, 0, y, y')\left|\frac{ds}{s}dydy'\right|^{\frac{1}{2}}.$$

The following proposition tells us that the restriction of the kernel of an element in the small b-calculus to the front face bf can be interpreted as an operator on the cylinder $\partial M \times \mathbb{R}$.

Proposition 3.4.10. *[RM93, Prop. 4.44] Let M be a manifold with a connected boundary ∂M. For any element A in the small b-calculus $\Psi_b^m(M)$, the restriction of the kernel of A to the hypersurface bf defines an element $I_b(A)$ in the small b-calculus $\Psi_b^m(\partial M \times \mathbb{R})$. The operator $I_b(A)$, called the indicial operator of A, is translation invariant on the cylinder $\partial M \times \mathbb{R}$.*

For elements in the full b-calculus, which will be introduced in Section 3.5, the indicial operators can be defined similarly.

We have a good composition property for the indicial operators. From [RM93, Lemma 4.45], we know that for a b-differential operator P and an operator A in the small-b calculus,

$$I_b(PA) = I_b(P)I_b(A).$$

As mentioned at this end of Subsection 3.4.3, the compactness of an operator is linked to its indicial operator.

Proposition 3.4.11. *[RM93, Ex. 4.40] Suppose that M is a manifold with a connected boundary, and $A \in \Psi_b^m(M)$. If A is compact on $L^2(M)$, then $I_b(A) = 0$.*

Remark 3.4.12. The converse of this proposition is also true provided the "optimal order" of A is negative. That is, for any $A \in \Psi_b^m(M)$, it is compact on $L^2(M)$, if and only if, $I_b(A) = 0$ and $A \in \Psi_b^n(M)$ for some $n < 0$.

When inverting an elliptic partial differential operator in the small b-calculus, as in Proposition 3.4.8, the error term may not be compact. That's because the error term may have a non-zero indicial operator. In order to obtain an error term whose indicial operator is 0, we have to expand the small b-calculus into a bigger calculus, called the full b-calculus.

3.5 Full b-calculus

3.5.1 Polyhomogeneous conormal functions

To define polyhomogeneous conormal functions, we need the notion of an index set, because a function is polyhomogeneous conormal with respect to certain index set.

Definition 3.5.1. An *index set* is a discrete subset $F \subset \mathbb{C} \times \mathbb{N}_0$ such that every "left segment" $F \cap \{(z,p) : Rez < N\}$, $N \in \mathbb{R}$, is a finite set. Also, it is assumed that if $(z,p) \in F$ and $p \geq q$, we also have $(z,q) \in F$.

We start with a manifold with boundary first, and then consider a manifold with corners. These polyhomogeneous conormal functions behave like sums of products of powers and logarithms.

Definition 3.5.2. Let M be a manifold with boundary H, and let x, y be coordinates near H as defined in Definition 3.1.1. Given an index set F, a smooth function u defined on the interior of M is called *polyhomogeneous conormal* as it approaches the boundary H with respect to F if, on a tubular neighborhood $[0,1) \times H$ of H, one has

$$u(x,y) \sim \sum_{(z,p)\in F} a_{z,p}(y) x^z \log^p x$$

as $x \to 0$ with $a_{z,p}$ smooth on H.

To define the notion on a manifold with corners we simply require such expansion at each boundary hypersurface, but with possibly different index sets. For that we need the notion of an index family, which basically assigns each hypersurface a index set.

Definition 3.5.3. An *index family* $\mathcal{E}$ for a manifold with corners M is an assignment of an index set to each boundary hypersurface. A function u on M is polyhomogeneous conormal with respect to the index family $\mathcal{E}$ if for each boundary hypersurface H and its assigned index set F by $\mathcal{E}$, u is polyhomogeneous conormal as it approaches H with respect to F as described in Definition 3.5.2.

3.5.2 Full b-calculus

The main difference between the full b-calculus and the small b-calculus is, apart from conormality with respect to the diagonal, the full b-calculus also requires the kernel to be polyhomogeneous conormal as it approaches the two boundary hypersurfaces lb and rb.

In this subsection let M be a manifold with a connected boundary ∂M.

Definition 3.5.4 (Full b-calculus). The *full b-calculus* $\Psi_b^{m,\mathcal{E}}$ on M, where m is a real number and $\mathcal{E} = (E_{lb}, E_{rb})$ is an index family for M^2, is defined as follows. A distribution u on M_b^2 is in $\Psi_b^{m,\mathcal{E}}(M)$ if and only if $u = u_1 + u_2 + u_3$ with

(i) u_1 is in the small calculus Ψ_b^m;

(ii) u_2 is polyhomogeneous conormal with respect to the index family $(E_{lb}, 0, E_{rb})$, where $0 := \{(n, 0) : n \in \mathbb{N}_0\}$, and the index sets E_{lb}, 0 and E_{rb} are assigned to the boundary hypersurfaces lb, bf and rb respectively;

(iii) $u_3 = \beta^* v$, where $\beta : M_b^2 \to M^2$ is the blow-down map and v is polyhomogeneous conormal with respect to the index family $\mathcal{E}$.

The number m is called the *differential order*, or simply the *order* of u.

Remark 3.5.5. The condition that u_2 being polyhomogeneous conormal as it approaches the front face bf with respect to the index set 0 just means that u_2 is smooth up to bf.

By expanding the small b-calculus into the full b-calculus, we've lost the good closure property. The composition of two operators in the full b-calculus only stay in the full b-calculus with a restriction. To formulate this restriction and the composition result in the full b-calculus, we need the following definitions on the index sets.

Definition 3.5.6. Let E and F be two index sets. We define

$$\inf E = \min\{Re z : (z, 0) \in E\}.$$

The *extended union* of E and F, denoted by $E\overline{\cup}F$, is the index set defined by

$$E \cup F \cup \{(z,k) : \text{there exist } (z,l_1) \in E \text{ and } (z,l_2) \in F \text{ with } k = l_1 + l_2 + 1\}.$$

We can now state the composition result in the full b-calculus.

Theorem 3.5.7. *[RM93, Theorem 5.53] If $\mathcal{E} = (E_{lb}, E_{rb})$ and $\mathcal{F} = (F_{lb}, F_{rb})$ are index families for M^2 with $\inf E_{rb} + \inf F_{lb} > 0$, then*

$$\Psi_b^{m,\mathcal{E}}(M) \circ \Psi_b^{m',\mathcal{R}}(M) \subset \Psi_b^{m+m',\mathcal{G}}(M),$$

where $m, m' \in \mathbb{R}$, and $\mathcal{G} = (G_{lb}, G_{rb})$ is an index family defined as

$$G_{lb} = E_{lb}\overline{\cup}F_{lb}, \quad G_{rb} = E_{rb}\overline{\cup}F_{rb}.$$

The loss of the closure property doesn't matter for our purpose, as in Chapter 6 we are only going to use the following property, which says the composition of an element in the the full b-calculus with an element in the small b-calculus is in the full b-calculus; see [RM93, prop 5.46].

Proposition 3.5.8. *The full b-calculus on M is a two-sided module over the small b-calculus, ie*

$$\Psi_b^{m,\mathcal{E}}(M) \circ \Psi_b^{m'}(M) \subset \Psi_b^{m+m',\mathcal{E}}(M),$$

and

$$\Psi_b^{m'}(M) \circ \Psi_b^{m,\mathcal{E}}(M) \subset \Psi_b^{m+m',\mathcal{E}}(M),$$

where $m, m' \in \mathbb{R}$, and $\mathcal{E}$ is an index family.

The gain, which is also the reason to introduce the full b-calculus, is that the error term we get under the parametrix construction in the full b-calculus is compact. For the proof of the following proposition, see [RM93, Prop 5.59].

Proposition 3.5.9. *Let P be an elliptic b-differential operator of order k, then there exists an operator G, in the full b-calculus of order $-k$, with respect to some index family, such that the error term $Id - PG$ is compact on $L^2(M)$ and has a smooth kernel. Similarly, a left inverse with the same properties also exists.*

Chapter 4

Scattering calculus

4.1 Scattering vector fields

As $r \to \infty$ on a metric cone, the operators we will study in later chapters involve expressions like $x^2\partial_x$, $x\partial_{y_i}$, $1 \leq i \leq d-1$, where $x = \frac{1}{r} \to 0$. These are called the scattering vector fields.

Definition 4.1.1. Let M be a manifold with a connected boundary ∂M and dimension d, with local coordinates $x, y_1, ..., y_{d-1}$, where x is a boundary defining function for ∂M. A smooth vector field V on M is a *scattering vector field* if it is x times a b-vector field on M, ie it has the form

$$V = a_0 x^2 \partial_x + a_1 x \partial_{y_1} + \cdots + a_{d-1} x \partial_{y_{d-1}},$$

where the coefficients $a_0, ..., a_{d-1}$ are smooth functions of x and y. The set of all scattering vector fields on M is denoted by $\mathcal{V}_{sc}(M)$.

Unlike b-vector fields, a scattering vector field vanishes at the hypersurface bf in the b-double space. To address that, we perform a second blow-up to obtain the *scattering double space*, with a front face on which the scattering vector fields don't vanish.

4.2 Scattering double space

Suppose M is manifold with a connected boundary ∂M and dimension d. Let $x, y_1, ..., y_{d-1}$ be local coordinates near ∂M, where x is a boundary defining function, and $y = (y_1, ..., y_{d-1})$ is a coordinate system on ∂M. We are going to perform two blow-ups on the double space M^2. The resulting space is called the *scattering double space*. We are particularly interested in what the scattering vector fields will become. So we are going to trace

$$x^2\partial_x,\ x\partial_{y_1},\ ...,\ x\partial_{y_{d-1}}. \tag{4.1}$$

Through the first blow-up we obtain the b-double space, as in Section 3.3.1,

$$M_b^2 = [M^2; (\partial M)^2].$$

Recall that the boundary ∂M is lifted to the hypersurface bf, while the diagonal of M^2 is lifted to a p-submanifold of M_b^2, denoted by diag_b. The local coordinates are $s = \frac{x}{x'}, x', y, y'$, and vector fields in (4.1) are lifted to

$$x's^2\partial_s,\ x's\partial_{y_1},\ ...,\ x's\partial_{y_{d-1}}. \tag{4.2}$$

They clearly vanish at bf, so we need a second blow-up. The boundary of the b-diagonal $\partial(\mathrm{diag}_b)$ lies inside the front face bf. We perform the second blow-up along this boundary $\partial(\mathrm{diag}_b)$ to obtain the scattering double space,

$$M_{sc}^2 = [M_b^2; \partial(\mathrm{diag}_b)].$$

In this second blow-up, $\partial(\mathrm{diag}_b)$ in M_b^2 is lifted to a hypersurface in the blown-up manifold M_{sc}^2. This new hypersurface is called the *scattering front face*, denoted by sf. The b-diagonal of M_b^2 is lifted to a p-submanifold of M_{sc}^2, denoted by diag_{sc}. The local coordinates near $\partial(\mathrm{diag}_{sc})$ are

$$X = \frac{s-1}{x'},\ x',\ Y_1 = \frac{y_1 - y_1'}{x'},\ ...,\ Y_{d-1} = \frac{y_{d-1} - y_{d-1}'}{x'},\ y_1',\ ...,\ y_{d-1}'. \tag{4.3}$$

Because both x and x' are boundary defining functions for bf, we can replace the x' in (4.3) by x to obtain another set of valid local coordinates.

Now on the scattering double space M_{sc}^2 the vector fields in (4.2) become

$$(1 + x'X)^2\partial_X,\ (1 + x'X)\partial_{Y_1},\ ...,\ (1 + x'X)\partial_{Y_{d-1}}, \tag{4.4}$$

which don't vanish at the sf-front face, as $1 + x'X = s = 1$ at sf.

Let's use a simple example to illustrate how this double blow-up works. Let

M be the simplest possible case, $[0,\infty)$. The two blow-ups, including an example of coordinates for each space, are illustrated in Figure 4.1.

4.3 The scattering face

As mentioned in the last section, the local coordinates near the sf-face are

$$X=\frac{s-1}{x'},\ x',\ Y_1=\frac{y_1-y_1'}{x'},\ ...,\ Y_{d-1}=\frac{y_{d-1}-y_{d-1}'}{x'},\ y_1',\ ...,\ y_{d-1}'.$$

We restrict our attention to the interior of the sf-face. Here the second coordinate x' disappears. Further more, if we fix $y'=u=(u_1,...,u_{d-1})$, there is a vector space Ω_u with dimension d associated with it, with coordinates

$$X=\frac{s-1}{x'},\ Y_1=\frac{y_1-u_1}{x'},\ ...,\ Y_{d-1}=\frac{y_{d-1}-u_{d-1}}{x'}.$$

We need to check that $X,Y_1,...,Y_{d-1}$ are well defined linear coordinates.

Proposition 4.3.1. *The interior of the scattering face sf in the scattering double space M^2_{sc} is a bundle over ∂M, and each fibre Ω_y, $y\in\partial M$, has a natural vector space structure.*

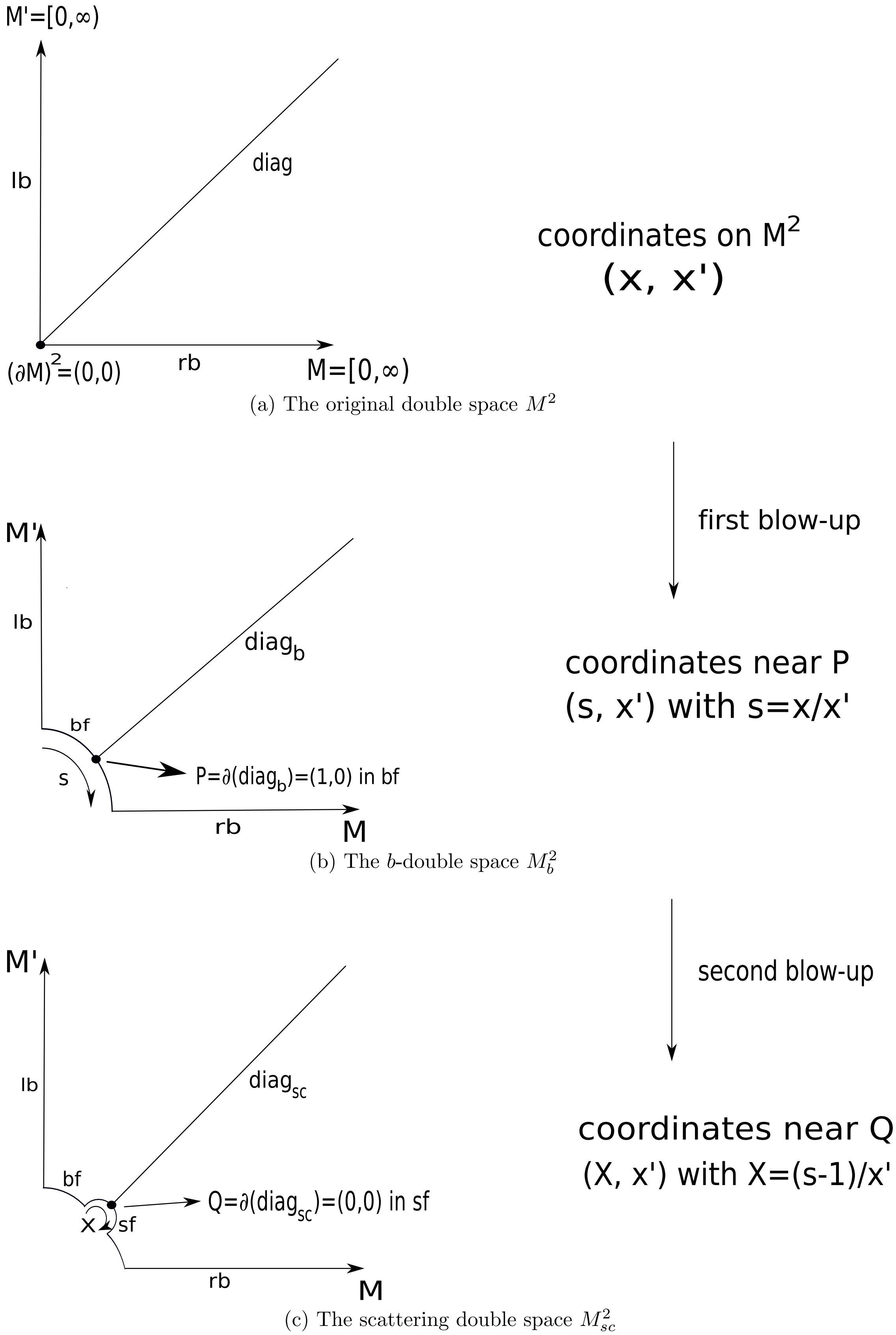
Proof. As discussed above, we only need to check that $X,Y_1,...,Y_{d-1}$ are well defined linear coordinates. For that we consider another set of local coordinates $\tilde{x},\tilde{y}_1,...,\tilde{y}_{d-1}$ near ∂M, where $\tilde{x}$ is another boundary defining function, and $\tilde{y}=(\tilde{y}_1,...,\tilde{y}_{d-1})$ gives another coordinate system on ∂M. Denote $\tilde{s}=\frac{\tilde{x}}{\tilde{x}'}$. The corresponding local coordinates in the scattering double space near $\partial(\mathrm{diag}_{sc})$ are

$$\tilde{X}=\frac{\tilde{s}-1}{\tilde{x}}=\frac{\tilde{x}-\tilde{x}'}{\tilde{x}\tilde{x}'},\ \ \tilde{Y}_1=\frac{\tilde{y}_1-\tilde{u}_1}{\tilde{x}},\ \ ...,\ \ \tilde{Y}_{d-1}=\frac{\tilde{y}_{d-1}-\tilde{u}_{d-1}}{\tilde{x}},$$

where $\tilde{u}=(\tilde{u}_1,...,\tilde{u}_{d-1})$ is the same point as $u=(u_1,...,u_{d-1})$ in ∂M but in terms of the new coordinates. What remains is to verify that the new coordinates $\tilde{X},\tilde{Y}_1,...,\tilde{Y}_{d-1}$ are linear functions of the old coordinates $X,Y_1,...,Y_{d-1}$. We know that

$$\tilde{x}=a(x,y)x,\quad \tilde{y}_i=\tilde{y}_i(x,y),$$

where a and $\tilde{y}_i$, $1\le i\le d-1$, are smooth functions, a is non-zero, and



(a) The original double space M^2

(b) The b-double space M_b^2

(c) The scattering double space M_{sc}^2

Figure 4.1: A double blow-up

$\det(\frac{\partial \tilde{y}_i}{\partial y_j})|_{x=0} \neq 0$. Then we have

$$\tilde{u}_i = \tilde{y}_i(0,u), \quad 1 \leq i \leq d-1.$$

Note that $x = 0$ on sf, and by applying the Taylor series, at Ω_u we have

$$\begin{aligned} a(0,y) &= a(0,u) + \sum_{i=1}^{d-1}(y_i - u_i)\frac{\partial a}{\partial y_i}(0,u) + O(|y-u|^2) \\ &= a(0,u) + x\sum_{i=1}^{d-1} Y_i \frac{\partial a}{\partial y_i}(0,u) + x^2 O(|Y|^2) \\ &= a(0,u). \quad (\text{As } x = 0 \text{ on } sf.) \end{aligned} \tag{4.5}$$

This means $a(x,y)$ is constant on Ω_u, with the value $a(0,u)$. Let's now express the new coordinates on Ω_u in terms of the old ones,

$$\tilde{X} = \frac{\tilde{x} - \tilde{x}'}{\tilde{x}\tilde{x}'} = \frac{X}{a(0,y)} = \frac{X}{a(0,u)},$$

and for $1 \leq i \leq d-1$,

$$\begin{aligned} \tilde{Y}_i &= \frac{\tilde{y}_i - \tilde{u}_i}{\tilde{x}} \\ &= \frac{\tilde{y}_i(0,y) - \tilde{y}_i(0,u)}{a(0,u)x} \\ &= \frac{\sum_{j=1}^{d-1}(y_j - u_j)\frac{\partial \tilde{y}_i}{\partial y_j}(0,u) + O(|y-u|^2)}{a(0,u)x} \\ &= \frac{x\sum_{j=1}^{d-1} Y_j \frac{\partial \tilde{y}_i}{\partial y_j}(0,u) + x^2 O(|Y|^2)}{a(0,u)x} \\ &= \frac{1}{a(0,u)}\sum_{j=1}^{d-1}\frac{\partial \tilde{y}_i}{\partial y_j}(0,u)Y_j + \frac{x}{a(0,u)}O(|Y|^2) \\ &= \frac{1}{a(0,u)}\sum_{j=1}^{d-1}\frac{\partial \tilde{y}_i}{\partial y_j}(0,u)Y_j. \quad (\text{As } x = 0 \text{ on } sf.) \end{aligned}$$

As $a(0,u)$ and $\frac{\partial \tilde{y}_i}{\partial y_j}(0,u)$ are constants with $a(0,u) \neq 0$ and $\det(\frac{\partial \tilde{y}_i}{\partial y_j})|_{x=0} \neq 0$, we have shown the linear relationship between the two sets of coordinates on the fibre Ω_u. $\square$

4.4 Scattering-half-densities

The dual basis for $x^2\partial_x, x\partial_{y_1}, ..., x\partial_{y_{d-1}}, x'^2\partial_{x'}, x'\partial_{y'_1}, ..., x'\partial_{y'_{d-1}}$ is

$$\frac{dx}{x^2},\ \frac{dy_1}{x},\ ...,\ \frac{dy_{d-1}}{x},\ \frac{dx'}{x'^2},\ \frac{dy'_1}{x'},\ ...,\ \frac{dy'_{d-1}}{x'}.$$

Therefore we make the following definition.

Definition 4.4.1. Let M be a manifold with a connected boundary ∂M and dimension d, and let $x, y_1, ..., y_{d-1}$ be local coordinates near the boundary with x being a boundary defining function. A *smooth scattering-half-density* on M^2_{sc} has the form

$$v\left|\frac{dx}{x^2}\frac{dy_1}{x}\cdots\frac{dy_{d-1}}{x}\frac{dx'}{x'^2}\frac{dy'_1}{x'}\cdots\frac{dy'_{d-1}}{x'}\right|^{\frac{1}{2}} = v\left|\frac{dxdydx'dy'}{x^{d+1}x'^{d+1}}\right|^{\frac{1}{2}},$$

where v is a smooth function on M^2_{sc}. A *distributional scattering-half-density* on M^2_{sc} has the same form except that v is allowed to be a distribution.

Let's focus on the half density

$$\left|\frac{dxdydx'dy'}{x^{d+1}x'^{d+1}}\right|^{\frac{1}{2}}.$$

Note that x and x' are boundary defining functions for lb and rb respectively, and we write $\rho_{lb} = x$ and $\rho_{rb} = x'$. Their powers are both $-\frac{1}{2}(d+1)$. Let's work out what this half-density looks like in terms of coordinates in the scattering double space X^2_{sc} near $\partial(\mathrm{diag}_{sc})$. Firstly we change into the coordinates $s = \frac{x}{x'}$, x', y and y' near $\partial(\mathrm{diag}_b)$ in the b-double space,

$$\left|\frac{dxdydx'dy'}{x^{d+1}x'^{d+1}}\right|^{\frac{1}{2}} = \left|\frac{x'dsdydx'dy'}{x^{d+1}x'^{d+1}}\right|^{\frac{1}{2}} = \left|\frac{dsdydx'dy'}{x^{d+1}x'^{d}}\right|^{\frac{1}{2}} = \left|\frac{dsdydx'dy'}{s^{d+1}x'^{2d+1}}\right|^{\frac{1}{2}}. \tag{4.6}$$

The boundary defining function for bf is x', ie $\rho_{bf} = x'$. Note that its power is $-\frac{1}{2}(2d+1)$.

We then change into the coordinates $X = \frac{s-1}{x'}$, $Y = \frac{y-y'}{x'}$, x' and y' near $\partial(\mathrm{diag}_{sc})$ in the scattering double space M^2_{sc},

$$\left|\frac{dsdydx'dy'}{x^{d+1}x'^{d}}\right|^{\frac{1}{2}} = \left|\frac{x'^{d}dXdYdx'dy'}{x^{d+1}x'^{d}}\right|^{\frac{1}{2}} = \left|\frac{dXdYdx'dy'}{x^{d+1}}\right|^{\frac{1}{2}}. \tag{4.7}$$

Near $\partial(\mathrm{diag}_{sc})$ both x and x' are boundary defining functions for sf. Here we use $\rho_{sf} = x$ and note that its power is $-\frac{1}{2}(d+1)$.

In light of the above computation, we make an equivalent definition for the scattering-half-densities.

Definition 4.4.2. Let M be a manifold with a connected boundary ∂M and dimension d. A *smooth scattering-half-density* on M^2_{sc} has the form

$$(\rho_{lb}\rho_{rb}\rho_{sf})^{-\frac{1}{2}(d+1)}\rho_{bf}^{-\frac{1}{2}(2d+1)}\upsilon,$$

where υ is a smooth half-density on M^2_{sc}, and $\rho_{lb}, \rho_{rb}, \rho_{sf}, \rho_{bf}$ are boundary defining functions for lb, rb, sf and bf respectively. A *distributional scattering-half-density* on M^2_{sc} has the same form except that υ is allowed to be a distributional half-density.

We denote the bundle of scattering-half-densities over M^2_{sc} by the symbol ${}^{sc}\Omega^{\frac{1}{2}}(M^2_{sc})$.

Example 4.4.3. *Let's work out the expression of the identity operator Id in terms of the above concepts. Consider a manifold M with a connected boundary ∂M and dimension d. Let (x, y) be local coordinates near ∂M, with x being a boundary defining function, and $y = (y_1, ..., y_{d-1})$ being the tangential coordinates. Then as a half-density, we have*

$$Id(x, x', y, y') = \delta(x - x')\delta(y - y')\big|dxdx'dydy'\big|^{\frac{1}{2}}. \tag{4.8}$$

As a scattering-half-density, we have

$$Id(x, x', y, y') = \delta(x - x')(y - y')x^{d+1}\Big|\frac{dxdydx'dy'}{x^{d+1}x'^{d+1}}\Big|^{\frac{1}{2}}, \tag{4.9}$$

and note that here $x = x'$.

In terms of the coordinates $s = \frac{x}{x'}$ and x' near $\partial(\mathrm{diag}_b)$ in the b-double space M^2_b, (4.8) becomes,

$$Id(s, x', y, y') = \delta(s - 1)\delta(y - y')x'^{-\frac{1}{2}}\big|dsdx'dydy'\big|^{\frac{1}{2}}. \tag{4.10}$$

In terms of the coordinates $X = \frac{s-1}{x'}$, $Y = \frac{y-y'}{x'}$, x' and y' near $\partial(\mathrm{diag}_{sc})$ in *the scattering double space M^2_{sc}, we have*

$$\begin{aligned} Id(X,Y,x',y') &= \delta(X)\delta(Y)x'^{-d-\frac{1}{2}}\left|x'^{d}dXdYdx'dy'\right|^{\frac{1}{2}} \\ &= \delta(X)\delta(Y)x'^{-\frac{1}{2}(d+1)}\left|dXdYdx'dy'\right|^{\frac{1}{2}}. \end{aligned} \tag{4.11}$$

Here Id has the particularly simple expression of $\delta(X)\delta(Y)$ times a smooth non-zero scattering-half-density.

We will introduce the *scattering calculus* in the next section. When we use the scattering calculus to study operators, the convention is to write the kernels in the form of scattering-half-densities.

4.5 Scattering calculus

As in the case of the b-differential operators, which are generated by the b-vector fields, the scattering differential operators are generated by the scattering vector fields.

Definition 4.5.1. Let M be a manifold with a connected boundary. A differential operator P on M is a scattering differential operator of order m, if it has smooth coefficients, and, in local coordinates (x, y) around a boundary point x, it has the form

$$P = \sum_{j+|\alpha|\leq m} a_{j\alpha}(x,y)(x^2\partial_x)^j(x\partial_y)^\alpha,$$

with the coefficients $a_{j\alpha}$ smooth up to the boundary.

Remark 4.5.2. Note that if v is a b-differential operator of order m, then $x^m v$ is a scattering differential operator of order m.

As in the case of b-calculus, we generalise these scattering differential operators to define the scattering calculus, and it is defined in terms of the behaviours of the kernel at different parts of the scattering double space.

Definition 4.5.3 (Scattering calculus). Let M be a manifold with a connected boundary. The *scattering calculus* $\Psi^{m,l}_{sc}(M)$ of order (m, l) is defined as the set of scattering-half-densities $v \in {}^{sc}\Omega^{\frac{1}{2}}(M^2_{sc})$ satisfying

(i) $\rho_{sf}^{-l}v$ is conormal of order m with respect to diag_{sc} uniformly up to sf, where ρ_{sf} is a boundary defining function for sf;

(ii) v vanishes to infinite order at lb, rb and bf.

The order m is called the *differential order* of v.

Remark 4.5.4. Note that the calculus $\Psi_{sc}^{m,l}$ becomes bigger when m increases, ie $\Psi_{sc}^{m,l} \subseteq \Psi_{sc}^{m',l}$ for any $m' > m$; while it becomes smaller when l increases, ie $\Psi_{sc}^{m,l'} \subseteq \Psi_{sc}^{m,l}$ for any $l' > l$.

Remark 4.5.5. A scattering differential operator of order m is in the scattering calculus of order $(m, 0)$.

Scattering calculus is closed under composition.

Proposition 4.5.6. *[RM95, Eqn. 6.12] Let M be a manifold with a connected boundary, and $m, l, m', l' \in \mathbb{R}$, then*

$$\Psi_{sc}^{m,l}(M) \circ \Psi_{sc}^{m',l'}(M) \subset \Psi_{sc}^{m+m',l+l'}(M).$$

Like Proposition 3.4.8 on the parametrix constructions in the small b-calculus, in the scattering calculus we also have a proposition that's analogous to [LH85, Theorem 18.1.24].

Proposition 4.5.7. *Let M be a manifold with a connected boundary. Suppose that $P \in \Psi_{sc}^{k,0}(M)$ is elliptic, then there exist $G, G' \in \Psi_{sc}^{-k,0}(M)$ such that*

$$PG - Id,\ \ G'P - Id \in \Psi_{sc}^{-\infty,0}(M).$$

4.6 Normal operators

Similar to the indicial operators in Section 3.4.4, we restrict the kernel of an element u in the scattering calculus $\Psi_{sc}^{m,l}$ to the hypersurface sf. How do we interpret this restriction?

Proposition 4.6.1. *Let M be a manifold with a connected boundary ∂M and dimension d. The restriction of a smooth scattering-half-density on M_{sc}^2 to sf gives (in a canonical way) a smooth function on ∂M valued in densities on each fibre.*

Proof. From Definition 4.4.1, we know that a smooth scattering-half-density has the form

$$v(x, y, x', y')\left|\frac{dxdydx'dy'}{x^{d+1}x'^{d+1}}\right|^{\frac{1}{2}}. \tag{4.12}$$

Let's consider a fibre Ω_u of the interior of sf, on which $x' = 0$ and $y' = u$. We want to write $\left|\frac{dxdydx'dy'}{x^{d+1}x'^{d+1}}\right|^{\frac{1}{2}}$ in terms of only X and Y. From the calculations in (4.6) and (4.7), we know that

$$\left|\frac{dxdy}{x'^{d+1}}\right|^{\frac{1}{2}} = |dXdY|^{\frac{1}{2}}.$$

Since x, y and x', y' are completely symmetrical, we also have

$$\left|\frac{dx'dy'}{x^{d+1}}\right|^{\frac{1}{2}} = |dXdY|^{\frac{1}{2}}.$$

With this, and substitute $x' = 0$, $y' = u$ into $v(x, y, x', y')$, (4.12) becomes

$$v(X, Y, 0, u)\big|dXdY\big|, \tag{4.13}$$

which proves the proposition. □

Similarly we have the following proposition.

Proposition 4.6.2. *Let M be a manifold with a connected boundary ∂M and dimension d. The restriction of a scattering-half-density on M^2_{sc} to sf gives (in a canonical way) a smooth function on ∂M valued in densities on each fibre, and the densities are conormal with respect to the origin in each fibre.*

Let's introduce some notation so that we can describe this more effectively. Let Ω_{fib} denote the bundle of fibre-densities on sf, ie the smooth sections of Ω_{fib} are the smooth functions on ∂M valued in densities on each fibre. Let $I^m\big(sf, \partial(\mathrm{diag}_{sc}), \Omega_{fib}\big)$ denote the space of smooth sections of Ω_{fib} with the condition that on each fibre the density is conormal of order $m + \frac{1}{4}$ with respect to $\partial(\mathrm{diag}_{sc})$, ie the origin of the fibre, and it decreases rapidly at infinity. The following proposition says the normal operator maps the scattering calculus onto this space; see [RM94, Prop. 20].

Proposition 4.6.3. *Let M be a manifold with a connected boundary. The normal operator $N_{sc} : \Psi^{m,l}_{sc}(M) \to I^m\big(sf, \partial(\mathrm{diag}_{sc}), \Omega_{fib}\big)$ is an epimorphism for each m and l, with kernel $\Psi^{m,l+1}_{sc}(M)$.*

The fact that the restriction gives a density on each fibre, and the vector space structure of the fibre, allow us to define a convolution operator on the fibre. For example, the density (4.13), $v(X, Y, 0, u)|dXdY|$, on the fibre Ω_u, is associated with the convolution operator that maps the function $f(X, Y)$ to

$$\int v(X - X', Y - Y', 0, u) f(X', Y')|dX'dY'|.$$

Therefore we have the following proposition.

Proposition 4.6.4. *For each element μ in the scattering calculus $\Psi_{sc}^{m,l}$, the normal operator $N_{sc}(\mu)$, can be interpreted as a family of convolution operators, one assigned for each fibre of the bundle sf. The assignment varies smoothly across ∂M.*

This allows us to compose normal operators. Suppose A and B are elements in the scattering calculus, then for each $y \in \partial M$, $N_{sc}(A)(y)$ and $N_{sc}(B)(y)$ are convolution operators on the fibre Ω_y, hence we can compose them. The composition of $N_{sc}(A)$ and $N_{sc}(B)$ are simply point-wise composition at each $y \in \partial M$, ie

$$\big(N_{sc}(A) \circ N_{sc}(B)\big)(y) = N_{sc}(A)(y) \circ N_{sc}(B)(y).$$

With this definition of the composition, the normal operator is multiplicative under composition.

Proposition 4.6.5. *[RM94, Eqn. 5.14] Let A and B be elements in the scattering calculus, then*

$$N_{sc}(AB) = N_{sc}(A)N_{sc}(B).$$

There is an analogous result of Remark 3.4.12 on the normal operators. It's as follows.

Proposition 4.6.6. *Let M be a manifold with a connected boundary. Suppose $A \in \Psi_{sc}^{m,0}(M)$. Then A is compact on $L^2(M)$, if and only if, $N_{sc}(A) = 0$ and $A \in \Psi_{sc}^{n,0}(M)$ for some $n < 0$.*

Remark 4.6.7. The main reason behind Proposition 4.6.6 is that a convolution operator is compact if and only if it is identically zero.

Chapter 5

Resolvent construction

For the last two chapters, we only consider $d \geq 3$.

5.1 The Riesz transform T

5.1.1 The operator H

Let Y be a compact Riemannian manifold with dimension $d-1$. The metric cone over Y is $M = Y \times [0, \infty)$, and the cone tip P is the point with $r = 0$. The cone M is equipped with the metric $r^2 h + dr^2$, where h is the metric on Y.

Let $\operatorname{dist}(z, z')$ denote the distance between z and z' on M. Write $z = (r, y)$ and $z' = (r', y')$, with $y, y' \in Y$, then we have the formula

$$\operatorname{dist}(z, z')^2 = \begin{cases} r^2 + r'^2 - 2rr' \cos\big(\operatorname{dist}_Y(y, y')\big), & \text{if } \operatorname{dist}_Y(y, y') \leq \pi, \\ (r + r')^2, & \text{if } \operatorname{dist}_Y(y, y') > \pi, \end{cases} \tag{5.1}$$

where dist_Y denotes the distance on Y.

The Laplacian on the cone M expressed in polar coordinates is

$$\Delta = -\partial_r^2 - \frac{d-1}{r}\partial_r + \frac{1}{r^2}\Delta_Y,$$

where Δ_Y is the Laplacian on Y. This operator Δ is initially defined away from the cone tip, then extended to the whole cone. Here we consider the Friedrichs extension, see Section 2.2.

For any smooth function $V_0 : Y \to \mathbb{C}$, we define the operator

$$H_{V_0} = \Delta + \frac{V_0(y)}{r^2}.$$

This is a natural class of operators: as both Δ and $\frac{V_0(y)}{r^2}$ are homogeneous of degree -2, the operator H_{V_0} has the same homogeneity. The term $V = \frac{V_0}{r^2}$ is called the *potential*. For simplicity of notation, we will write H_{V_0} simply as H.

The following proposition tells us for what V_0, the operator H is positive.

Proposition 5.1.1. *Suppose that $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2$ is a strongly positive operator on $L^2(Y)$, ie all its eigenvalues are strictly positive, then the operator H is positive.*

Proof. We work in polar coordinates. Consider the isometry $U : L^2(M; r^{d-1}drdy) \to L^2(M; r^{-1}drdy)$ defined by

$$Uf = r^{\frac{d}{2}} f. \tag{5.2}$$

For $f \in L^2(M; r^{-1}drdy)$, let's compute

$$UHU^{-1}f = r^{\frac{d}{2}}\Big(- \partial_r^2 - \frac{d-1}{r}\partial_r + \frac{1}{r^2}\Delta_Y + \frac{V_0(y)}{r^2}\Big) r^{-\frac{d}{2}} f.$$

Calculate the four terms separately,

$$\begin{aligned}
r^{\frac{d}{2}}(-\partial_r^2) r^{-\frac{d}{2}} f &= -\frac{d(d+2)}{4r^2} f + \frac{d}{r}\partial_r f - \partial_r^2 f,\\
r^{\frac{d}{2}}\Big(- \frac{d-1}{r}\partial_r\Big) r^{-\frac{d}{2}} f &= \frac{d(d-1)}{2r^2} f - \frac{d-1}{r}\partial_r f,\\
r^{\frac{d}{2}}\Big(\frac{1}{r^2}\Delta_Y\Big) r^{-\frac{d}{2}} f &= \frac{1}{r^2}\Delta_Y f,\\
r^{\frac{d}{2}}\frac{V_0(y)}{r^2} r^{-\frac{d}{2}} f &= \frac{V_0(y)}{r^2} f.
\end{aligned}$$

Adding the above terms, and we obtain

$$UHU^{-1}f = \Big(\frac{d(d-4)}{4} + V_0(y)\Big)\frac{1}{r^2} f + \frac{1}{r}\partial_r f - \partial_r^2 f + \frac{1}{r^2}\Delta_Y f.$$

Therefore,

$$\begin{aligned}
&\left(\frac{1}{r}\Big(- (r\partial r)^2 + \Delta_Y + \Big(\frac{d-2}{2}\Big)^2 + V_0(y)\Big)\frac{1}{r}\right)f \\
=&-\partial_r\Big(r\partial_r\big(\frac{1}{r}f\big)\Big) + \frac{1}{r^2}\Delta_Y f + \Big(\Big(\frac{d-2}{2}\Big)^2 + V_0(y)\Big)\frac{1}{r^2}f \\
=&\partial_r\big(\frac{1}{r}f\big) - \partial_r^2 f + \frac{1}{r^2}\Delta_Y f + \Big(\Big(\frac{d-2}{2}\Big)^2 + V_0(y)\Big)\frac{1}{r^2}f \\
=&-\frac{1}{r^2}f + \frac{1}{r}\partial_r f - \partial_r^2 f + \frac{1}{r^2}\Delta_Y f + \Big(\Big(\frac{d-2}{2}\Big)^2 + V_0(y)\Big)\frac{1}{r^2}f \\
=&\Big(\frac{d(d-4)}{4} + V_0(y)\Big)\frac{1}{r^2}f + \frac{1}{r}\partial_r f - \partial_r^2 f + \frac{1}{r^2}\Delta_Y f \\
=&UHU^{-1}f.
\end{aligned}$$

Hence we have established

$$UHU^{-1} = \frac{1}{r}\Big(- (r\partial_r)^2 + \Delta_Y + V_0(y) + \Big(\frac{d-2}{2}\Big)^2\Big)\frac{1}{r}. \tag{5.3}$$

Make a substitution $s = \ln r$, then the space $L^2(M; r^{-1}drdy)$ becomes $L^2(M; dsdy)$, and we have

$$UHU^{-1} = e^{-s}\Big(- \partial_s^2 + \Delta_Y + V_0(y) + \Big(\frac{d-2}{2}\Big)^2\Big)e^{-s}.$$

From here we can clearly see that the operator H is positive if $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2 > 0$. This completes the proof. $\square$

Remark 5.1.2. Note that since we have $d \geq 3$, the condition $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2 > 0$ means that the potential $V = \frac{V_0}{r^2}$ is allowed to be "a bit negative".

Remark 5.1.3. Note that U maps a smooth scattering-half-density to a smooth b-half-density. We will discuss more on half-densities in Section 5.2.

5.1.2 The Riesz transform T

Let V_0 be a smooth function on Y which satisfies $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2 > 0$, the Riesz transform T with the inverse square potential $V = \frac{V_0}{r^2}$ is defined to be

$$T = \nabla H^{-\frac{1}{2}},$$

where H is defined in Subsection 5.1.1.

The goal of this thesis is to find out for what p, the Riesz transform T with an inverse square potential $V = \frac{V_0}{r^2}$ is bounded on $L^p(M)$. The paper [GH] by C. Guillarmou and A. Hassell gives a necessary condition on p for the boundedness in the setting of asymptotically conic manifolds. This necessary condition also holds for cones. In this thesis we will determine a sufficient condition and hence a complete characterisation of the boundedness. The paper [HQL] by H.-Q. Li gives a characterisation for the special case $V \equiv 0$. In this thesis we will provide a different proof of the same result using the b-calculus and the scattering calculus.

Using functional calculus, we can express T as,

$$T = \frac{2}{\pi} \int_0^\infty \nabla (H + \lambda^2)^{-1} d\lambda.$$

We see from this expression that in order to understand T, we need to know the properties of $(H + \lambda^2)^{-1}$. Because H is homogeneous of degree -2, we only need to compute $(H + 1)^{-1}$, and then use scaling for $(H + \lambda^2)^{-1}$. The relationship between $(H + \lambda^2)^{-1}$ and $(H + 1)^{-1}$ will be discussed and used in the proof of Proposition 6.1.1.

Denote $P = H + 1$, and we now proceed to study P and P^{-1}.

5.2 The blown-up space

When $r \to 0$, the operator rPr is elliptic as a b-differential operator; see Section 5.3.1 for the details. Therefore we can expect the kernel of its inverse to be in the b-calculus. Starting with the space $(Y \times [0, \infty]_r)^2$, which is a compactification of M^2, we blow up the point $r = r' = 0$ to obtain the zf-face. Here zf means the "zero face" as it is the front face with $r, r' = 0$. For more details of the b-double space and the b-calculus, see Chapter 3. Recall that one possible set of the local coordinates near the zf-face are

$$s = \frac{r}{r'}, \;\; r', \;\; y, \;\; y'. \tag{5.4}$$

When $r \to \infty$, the operator P is elliptic as a scattering differential operator. Hence we use the scattering calculus here; see Section 5.5 for the details. We blow up the point $r = r' = \infty$ in $(Y \times [0, \infty]_r)^2$ to obtain the bf-face. In the bf-face at where it meets the b-diagonal, we make a second blow-up to obtain the

sf-face. More information on the double blow-ups and the scattering calculus can be found in Chapter 4. Recall from (4.3) that one possible set of the local coordinates near the sf-face are

$$\frac{s-1}{x},\ \ x,\ \ \frac{y-y'}{x},\ \ y, \tag{5.5}$$

where $x = \frac{1}{r}$, $x' = \frac{1}{r'}$ and $s = \frac{x}{x'}$, and we can replace the x in these coordinates by x'. In terms of r and r', we have the local coordinates

$$r'-r,\ \ \frac{1}{r},\ \ r(y-y'),\ \ y. \tag{5.6}$$

Since the x in (5.5) can be replaced by x', we can replace the $r(y-y')$ in (5.6) by $r'(y-y')$.

The resulting space after the blow-ups at $r = r' = 0$ and $r = r' = \infty$ is called *the blown-up space*. The diagonal of the blown-up space, simply denoted by diag, is the lift in the blown-up space of the diagonal $\{\frac{r}{r'} = 1, y = y'\}$ in the original double space $(Y \times [0,\infty]_r)^2$. In terms of the local coordinates near zf, the diagonal is defined by $\frac{r}{r'} = 1$ and $y = y'$; while in terms of the local coordinates near sf, the diagonal is defined by $r'-r = 0$, and $r(y-y') = 0$ or $r'(y-y') = 0$. The following result about the diagonal is useful for a later proof.

Proposition 5.2.1. *Let $\varphi : [0,\infty) \to [0,1]$ be an increasing smooth function such that $\varphi(x) = x$ for $x \in [0,\frac{1}{2}]$ and $\varphi(x) = 1$ for $x \in [1,\infty)$. Then the function*

$$a_{\mathrm{diag}}(z,z') = \frac{\mathrm{dist}(z,z')^2}{\varphi^2(r')},$$

where $z = (r,y)$ and $z' = (r',y')$, is a quadratic defining function for the diagonal in the blown-up space.

Proof. By formula (5.1), near the diagonal we have

$$\mathrm{dist}(z,z')^2 = r^2 + r'^2 - 2rr'\cos\big(\mathrm{dist}_Y(y,y')\big),$$

where dist_Y denotes the distance on Y. Therefore near the diagonal we have

$$\begin{aligned} \mathrm{dist}(z,z')^2 &= (r-r')^2 + 2rr'\Big(1-\cos\big(d_Y(y,y')\big)\Big) \\ &= (r-r')^2 + rr'\Big(\mathrm{dist}_Y(y,y')^2 + O\big(\mathrm{dist}_Y(y,y')^4\big)\Big). \end{aligned} \tag{5.7}$$

Near the sf-face, we have

$$a_{\mathrm{diag}}(z,z') = \mathrm{dist}(z,z')^2 = (r-r')^2 + rr'\Big(\mathrm{dist}_Y(y,y')^2 + O\big(\mathrm{dist}_Y(y,y')^4\big)\Big),$$

which is good for a quadratic defining function for the diagonal. To see that we recall from the discussion in this section just before this proposition that near sf the diagonal is defined by $r'-r=0$, and $r(y-y')=0$ or $r'(y-y')=0$, and we also recall the standard fact that $\mathrm{dist}_Y(y,y')^2$ is a quadratic defining function for the diagonal of Y^2 for any closed Riemannian manifold Y.

Near the zf-face, we have

$$a_{\mathrm{diag}}(z,z') = \frac{\mathrm{dist}(z,z')^2}{r'^2} = \big(\frac{r}{r'}-1\big)^2 + \frac{r}{r'}\Big(\mathrm{dist}_Y(y,y')^2 + O\big(\mathrm{dist}_Y(y,y')^4\big)\Big),$$

which is again good for a quadratic defining function for the diagonal, as here the diagonal is instead defined by $\frac{r}{r'}=1$ and $y=y'$. □

The following lemma needed in Chapter 6 can be shown; see Proposition 1 in Section 4 Chapter VI of [ES].

Lemma 5.2.2. *For any $k>-d$, if an operator is conormal of order k with respect to the diagonal in the blown-up space then its kernel is bounded by $a_{\mathrm{diag}}^{-\frac{d+k}{2}}$ near the diagonal, where a_{diag} is a quadratic defining function and d is the dimension of the cone.*

The blown-up space is illustrated by Figure 5.1. The four boundaries are labelled as *lbz*, *rbz*, *lbi*, *rbi*, with *lbz* meaning the left variable $r=0$, *rbz* meaning the right variable $r'=0$, *lbi* meaning the left variable $r=\infty$, and *rbi* meaning the right variable $r'=\infty$. The diagonal is also shown.

A b-half-density on the blown-up space has the form

$$u(r,r',y,y')\left|\frac{dr}{r}\frac{dr'}{r'}dhdh'\right|^{\frac{1}{2}},$$

where u is a distribution.

Let $x=\frac{1}{r}$ and $x'=\frac{1}{r'}$. Since x and x' are boundary defining functions for sf, a scattering-half-density has the form,

$$v(x,x',y,y')\left|\frac{dxdx'dhdh'}{x^{d+1}x'^{d+1}}\right|^{\frac{1}{2}},$$

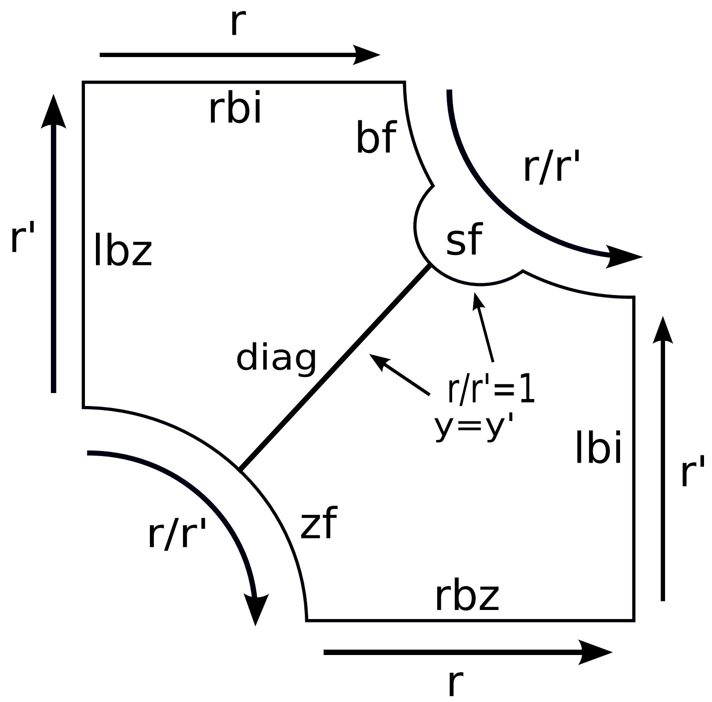


Figure 5.1: The blown-up space

where v is a distribution. In terms of r and r' it becomes,

$$v(r,r',y,y')\left|r^{d+1}r'^{d+1}d\Big(\frac{1}{r}\Big)d\Big(\frac{1}{r'}\Big)dhdh'\right|^{\frac{1}{2}} = v(r,r',y,y')\left|r^{d-1}r'^{d-1}drdr'dhdh'\right|^{\frac{1}{2}}.$$

Remark 5.2.3. The scattering-half-density $|r^{d-1}r'^{d-1}drdr'dhdh'|^{\frac{1}{2}}$ has length 1. That's why we adopt the convention that any kernel on the blown-up space is written in the form

$$v(r,r',y,y')\left|r^{d-1}r'^{d-1}drdr'dhdh'\right|^{\frac{1}{2}},$$

where v is a distribution. A kernel written in this form is called a *Riemannian half-density*. However, when we study the properties of a kernel near the zf-face, we write it as a b-half-density.

5.3 A formula for the resolvent

5.3.1 Determining the formula

We now proceed to find an explicit formula for P^{-1}. However, as we will discuss later in Subsection 5.3.2, this formula only has good convergence properties in certain region of the blown-up space.

From equation (5.3) in the proof of Proposition 5.1.1 we know that

$$H = r^{-\frac{d}{2}-1}\Big(-(r\partial_r)^2 + \Delta_Y + \Big(\frac{d-2}{2}\Big)^2 + V_0(y)\Big) r^{\frac{d}{2}-1}.$$

Therefore

$$P = H + 1 = r^{-\frac{d}{2}-1}\Big(-(r\partial_r)^2 + \Delta_Y + V_0(y) + r^2 + \Big(\frac{d-2}{2}\Big)^2\Big) r^{\frac{d}{2}-1}.$$

Let P' denote the differential operator consisting of the terms in the middle. That is,

$$P' = -(r\partial_r)^2 + \Delta_Y + V_0(y) + r^2 + \Big(\frac{d-2}{2}\Big)^2. \tag{5.8}$$

We take P' to act on half-densities, using the flat connection that annihilates the Riemannian half-density $|r^{d-1}drdh|^{\frac{1}{2}}$ on M. Now let $\tilde{P}$ be the differential operator given by the same expression (5.8), but endowed with the flat connection on half-densities annihilating the b-half-density $|\frac{dr}{r}dh|^{\frac{1}{2}}$. Recall that U, defined by (5.2), maps this b-half-density to the Riemannian half-density, hence these two differential operators are related by

$$\tilde{P} = U^{-1}P'U. \tag{5.9}$$

Therefore,

$$P = r^{-1}\tilde{P}r^{-1}. \tag{5.10}$$

Since P is self-adjoint, (5.10) shows that $\tilde{P}$ is also self-adjoint. (Note that for operators on half-densities there is an invariant notion of self-adjointness, since the inner product on half-densities is invariantly defined.) Denote $G = P^{-1}$, $\tilde{G} = \tilde{P}^{-1}$; the Schwartz kernels of G and $\tilde{G}$ are related by

$$G = rr'\tilde{G}. \tag{5.11}$$

Again, we emphasize that this is an identity involving half-densities: if we write $G = K|r^{d-1}r'^{d-1}dhdh'|^{\frac{1}{2}}$ and $\tilde{G} = \tilde{K}|\frac{dr}{r}\frac{dr'}{r'}dhdh'|^{\frac{1}{2}}$ then we have

$$K = r^{1-\frac{d}{2}}r'^{1-\frac{d}{2}}\tilde{K}. \tag{5.12}$$

So to determine G, we just need to determine $\tilde{G}$, and then (5.11) gives us G.

Let's now work out an expression for $\tilde{G}$. Let (μ_j^2, u_j) be the eigenvalues and the corresponding L^2-normalized eigenfunctions of the positive operator $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2$. We also let Π_j denote the projection onto the u_j-eigenspace. Then we have

$$\tilde{P} = \sum_j \Pi_j \tilde{T}_j, \tag{5.13}$$

and

$$Id = \sum_j \delta(\frac{r}{r'} - 1)\Pi_j,$$

where

$$\tilde{T}_j = -(r\partial_r)^2 + r^2 + \mu_j^2 = -r^2\partial_r^2 - r\partial_r + \mu_j^2. \tag{5.14}$$

In order to find the inverse of $\tilde{T}_j$, let's solve for $a_j(r)$ such that

$$\tilde{T}_j a_j(r) = \delta(\frac{r}{r'} - 1) = r'\delta(r - r').$$

When $r \neq r'$, the solution space is spanned by the modified Bessel functions $I_{\mu_j}(r)$ and $K_{\mu_j}(r)$; see [AS, Sec. 9.6]. That means, we must have

$$\tilde{T}_j^{-1}(r, r') = (aI_{\mu_j} + bK_{\mu_j})(r)(cI_{\mu_j} + dK_{\mu_j})(r')\left|\frac{dr}{r}\frac{dr'}{r'}\right|^{\frac{1}{2}}, \quad r < r',$$

for some constants a, b, c and d. We don't get all the terms due to the restriction which comes from the fact that our choice of the Laplacian is the Friedrichs extension. The domain of the Friedrichs extension requires the derivatives of $rr'\tilde{T}_j^{-1}(r, r')$ to be in $L^2(M)$, see Section 2.2.

Let's explore what this restriction means when $r < r'$ and r approaches 0. For this we need the limiting forms for small arguments from [AS, Sec. 9.6], that is when $r \to 0$,

$$I_{\mu_j}(r) = \frac{r^{\mu_j}}{2^{\mu_j}\Gamma(\mu_j + 1)} + O(r^{\mu_j+2}), \tag{5.15}$$

and

$$K_{\mu_j}(r) = \frac{2^{\mu_j - 1}\Gamma(\mu_j)}{r^{\mu_j}} + O(r^{-\mu_j+2}). \tag{5.16}$$

To be precise, for the expansion of $K_{\mu_j}(r)$, when $\mu_j = 1$, the error term is of order $r \log r$.

We first work out the power of r in $||\frac{\partial}{\partial r}\big(rr' I_{\mu_j}(r)\big)||_{L^2}$. The power of r in $rr' I_{\mu_j}(r)$ is $\mu_j + 1$. Differentiating with respect to r decrease the power to μ_j. Taking L^2-norm with respect to the half-density $\left|\frac{dr}{r}\right|^{\frac{1}{2}}$ changes the power to $2\mu_j - 1$, which is integrable since $\mu_j > 0$. Similarly, we work out the power of r in $||\frac{\partial}{\partial r}\big(rr' K_{\mu_j}(r)\big)||_{L^2}$, which is $-2\mu_j - 1$, which is not integrable. Therefore we conclude that for $r < r'$, our $\tilde{T}_j^{-1}$ can contain the $I_{\mu_j}(r)$ term but not the $K_{\mu_j}(r)$ term.

We now explore what the restriction means when $r < r'$ and r' approaches infinity. Both I_{μ_j} and its derivative grows exponentially at infinity, and both K_{μ_j} and its derivative decays exponentially at infinity; see [AS, Sec. 9.7]. From this we know our $\tilde{T}_j^{-1}$ can contain the $K_{\mu_j}(r')$ term but not the $I_{\mu_j}(r')$ term.

At last, $I_{\mu_j}(r)$ and $K_{\mu_j}(r')$ cannot exist in $\tilde{T}_j^{-1}$ for $r > r'$ by symmetry. Therefore, we know for each j, the kernel for the operator $\tilde{T}_j^{-1}$ is

$$\tilde{T}_j^{-1}(r, r') = \begin{cases} C I_{\mu_j}(r) K_{\mu_j}(r') \left|\frac{dr}{r}\frac{dr'}{r'}\right|^{\frac{1}{2}}, & r < r', \\ C K_{\mu_j}(r) I_{\mu_j}(r') \left|\frac{dr}{r}\frac{dr'}{r'}\right|^{\frac{1}{2}}, & r > r', \end{cases}$$

where $C > 0$ is some constant. The Wroskian of $K_{\mu_j}(r)$ and $I_{\mu_j}(r)$, given by [AS, Equation 9.6.15], is $\frac{1}{r}$, and from this we can determine that $C = 1$. Hence

$$\tilde{T}_j^{-1}(r, r') = \begin{cases} I_{\mu_j}(r) K_{\mu_j}(r') \left|\frac{dr}{r}\frac{dr'}{r'}\right|^{\frac{1}{2}}, & r < r', \\ K_{\mu_j}(r) I_{\mu_j}(r') \left|\frac{dr}{r}\frac{dr'}{r'}\right|^{\frac{1}{2}}, & r > r', \end{cases}$$

We know that

$$\tilde{G} = \sum_j \Pi_j \tilde{T}_j^{-1}.$$

In terms of the kernels, we have

$$\tilde{G}(r, r', y, y') = \sum_j u_j(y)\overline{u_j(y')}\tilde{T}_j^{-1}(r, r')\left|dh dh'\right|^{\frac{1}{2}},$$

Hence

$$\tilde{G}(r, r', y, y') = \begin{cases} \sum_j u_j(y)\overline{u_j(y')} I_{\mu_j}(r) K_{\mu_j}(r') \left|\frac{dr}{r}\frac{dr'}{r'} dhdh'\right|^{\frac{1}{2}}, & r < r', \\ \sum_j u_j(y)\overline{u_j(y')} K_{\mu_j}(r) I_{\mu_j}(r') \left|\frac{dr}{r}\frac{dr'}{r'} dhdh'\right|^{\frac{1}{2}}, & r > r'. \end{cases}$$

This is our first attempt to determine P^{-1}, so let's call it $\tilde{G}_0$, and rewrite it more compactly as

$$\begin{aligned} \tilde{G}_0(r, r', y, y') = \sum_j u_j(y)\overline{u_j(y')} \Big(& I_{\mu_j}(r) K_{\mu_j}(r') \chi_{\{r<r'\}} \\ & + K_{\mu_j}(r) I_{\mu_j}(r') \chi_{\{r>r'\}} \Big) \left| \frac{dr}{r}\frac{dr'}{r'} dhdh' \right|^{\frac{1}{2}}, \end{aligned} \tag{5.17}$$

where $\chi_{\{r<r'\}}$ and $\chi_{\{r>r'\}}$ are characteristic functions for the regions $r < r'$ and $r > r'$ respectively.

Now we have obtained an explicit formula, ie $\tilde{G}_0$, for the resolvent. However, as mentioned before, we can't just use it for the whole blown-up space, because it has poor convergence near $r = r'$. The convergence properties of the formula $\tilde{G}_0$ is the topic of the next subsection.

5.3.2 Convergence of the formula

Firstly consider $r < r'$; here we work with the sum,

$$\sum_j u_j(y)\overline{u_j(y')} I_{\mu_j}(r) K_{\mu_j}(r'). \tag{5.18}$$

From [AS, Sec. 9.6], we have the representations

$$I_{\mu_j}(r) = \frac{2^{-\mu_j} r^{\mu_j}}{\pi^{\frac{1}{2}}\Gamma(\mu_j + \frac{1}{2})} \int_{-1}^{1} (1 - t^2)^{\mu_j - \frac{1}{2}} e^{-rt} dt, \tag{5.19}$$

and

$$K_{\mu_j}(r') = \frac{\pi^{\frac{1}{2}} 2^{-\mu_j} r'^{\mu_j}}{\Gamma(\mu_j + \frac{1}{2})} \int_{1}^{\infty} e^{-r't} (t^2 - 1)^{\mu_j - \frac{1}{2}} dt. \tag{5.20}$$

We now estimate each of these integrals in a way that is uniform as $\mu \to \infty$. When $r \leq 1$, the integral in the expression for I_μ is uniformly bounded in $\mu > 0$, and hence we see that

$$\big|I_\mu(r)\big| \le C\frac{2^{-\mu}r^\mu}{\Gamma(\mu+\frac{1}{2})} \text{ when } r \le 1, \tag{5.21}$$

where C is independent of r and μ. On the other hand, for $r \ge 1$, we estimate $e^{-r}r^{1/2}I_\mu(r)$:

$$\begin{aligned}
e^{-r}r^{1/2}I_\mu(r) &= \frac{2^{-\mu}r^{\mu+\frac{1}{2}}}{\pi^{\frac{1}{2}}\Gamma(\mu+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{\mu-\frac{1}{2}}e^{-r(t+1)}dt \\
&\le C\frac{2^{-\mu}r^{\mu+\frac{1}{2}}}{\Gamma(\mu+\frac{1}{2})}\int_{-1}^{1} e^{-r(t+1)}\,dt \\
&\le C\frac{2^{-\mu}r^{\mu+\frac{1}{2}}}{\Gamma(\mu+\frac{1}{2})}\int_{0}^{2r} e^{-t}\,\frac{dt}{r} \\
&\le C\frac{2^{-\mu}r^{\mu-\frac{1}{2}}}{\Gamma(\mu+\frac{1}{2})},
\end{aligned}$$

with C independent of μ, for $\mu \ge \frac{1}{2}$. This gives rise to an estimate of the form

$$\big|I_\mu(r)\big| \le C\frac{2^{-\mu}r^{\mu-1}e^r}{\Gamma(\mu+\frac{1}{2})} \text{ when } r \ge 1. \tag{5.22}$$

We next estimate K_μ in a similar way. For $r \le 1$, we estimate

$$\begin{aligned}
K_\mu(r) &= \frac{\pi^{\frac{1}{2}}2^{-\mu}r^\mu}{\Gamma(\mu+\frac{1}{2})}\int_1^\infty e^{-rt}\big(t^2-1\big)^{\mu-\frac{1}{2}}\,dt \\
&\le \frac{\pi^{\frac{1}{2}}2^{-\mu}r^\mu}{\Gamma(\mu+\frac{1}{2})}\int_0^\infty e^{-rt}t^{2\mu-1}\,dt \\
&\le C\frac{2^{-\mu}r^\mu}{\Gamma(\mu+\frac{1}{2})}\int_0^\infty e^{-t}t^{2\mu-1}r^{-2\mu}\,dt \\
&= C\frac{2^{-\mu}r^{-\mu}\Gamma(2\mu)}{\Gamma(\mu+\frac{1}{2})}.
\end{aligned}$$

On the other hand, for $r \ge 1$, we compute

$$\begin{aligned}
e^{r}r^{1/2}K_\mu(r) &= \frac{\pi^{\frac{1}{2}}2^{-\mu}r^{\mu+\frac{1}{2}}}{\Gamma(\mu+\frac{1}{2})}\int_1^\infty e^{-r(t-1)}(t^2-1)^{\mu-\frac{1}{2}}\,dt \\
&= \frac{\pi^{\frac{1}{2}}2^{-\mu}r^{\mu+\frac{1}{2}}}{\Gamma(\mu+\frac{1}{2})}\int_0^\infty e^{-t}\big(t(2r+t)\big)^{\mu-\frac{1}{2}}r^{-2\mu}\,dt,
\end{aligned}$$

where we made a substitution $t \to r(t-1)$ in the integral. We now estimate

$$(2r+t)^{\mu-\frac{1}{2}} \leq 2^{\mu-\frac{1}{2}} \max\Big((2r)^{\mu-\frac{1}{2}},\ t^{\mu-\frac{1}{2}}\Big),$$

which gives rise to an estimate

$$\Big|K_\mu(r)\Big| \leq C\frac{e^{-r}r^{-\mu}}{\Gamma(\mu+\frac{1}{2})}\max\Big((2r)^{\mu-\frac{1}{2}}\Gamma\big(\mu+\frac{1}{2}\big),\ \Gamma(2\mu)\Big). \tag{5.23}$$

Now to absorb the factor $r^{\mu-\frac{1}{2}}$ in the first argument of the maximum function, we sacrifice half of our exponential decay: we estimate $e^{-\frac{r}{2}}r^{\mu-\frac{1}{2}}$ by bounding it by the value where it achieves its maximum in r, which is when $r = 2\mu - 1$:

$$e^{-\frac{r}{2}}r^{\mu-\frac{1}{2}} \leq e^{-\frac{2\mu-1}{2}}(2\mu-1)^{\mu-\frac{1}{2}} \leq Ce^{-\mu}2^\mu\mu^{\mu-\frac{1}{2}} \leq C2^\mu\Gamma(\mu).$$

Then we can use this in (5.23) to estimate

$$\Big|K_\mu(r)\Big| \leq C\frac{e^{-\frac{r}{2}}r^{-\mu}}{\Gamma(\mu+\frac{1}{2})}\max\Big(2^{2\mu}\Gamma(\mu)\Gamma(\mu+\frac{1}{2}),\ \Gamma(2\mu)\Big). \tag{5.24}$$

Finally using the identity

$$\Gamma(2\mu) = \frac{2^{2\mu-1}}{\sqrt{\pi}}\Gamma(\mu)\Gamma(\mu+\frac{1}{2}), \tag{5.25}$$

we obtain

$$\Big|K_\mu(r)\Big| \leq Ce^{-\frac{r}{2}}r^{-\mu}2^{2\mu}\Gamma(\mu). \tag{5.26}$$

Hence, when $r' \geq 4r$, $I_\mu(r)K_\mu(r')$ is bounded above by:

(i) for $0 \leq r \leq r' \leq 1$,

$$C\big(\frac{r}{r'}\big)^\mu\frac{2^{-2\mu}\Gamma(2\mu)}{\big(\Gamma(\mu+\frac{1}{2})\big)^2};$$

(ii) for $r \leq 1 \leq r'$,

$$C2^\mu\big(\frac{r}{r'}\big)^\mu e^{-\frac{r'}{2}}\frac{\Gamma(\mu)}{\Gamma(\mu+\frac{1}{2})};$$

(iii) and for $1 \leq r \leq r'$,

$$C2^\mu\big(\frac{r}{r'}\big)^\mu e^{-\frac{r'}{4}}\frac{\Gamma(\mu)}{\Gamma(\mu+\frac{1}{2})}.$$

We emphasize that the constant C is independent of $\mu \geq \frac{1}{2}$, r and r' here. Noting that the combination of Γ factors is uniformly bounded in each case (using (5.25) again), we find that for $r' \geq 4r$, $I_\mu(r)K_\mu(r')$ is bounded above by

(i) for $0 \leq r \leq r' \leq 1$,
$$C\big(\frac{r}{r'}\big)^{\mu};$$

(ii) for $r \leq 1 \leq r'$,
$$C\big(\frac{2r}{r'}\big)^{\mu}e^{-\frac{r'}{2}};$$

(iii) and for $1 \leq r \leq r'$,
$$C\big(\frac{2r}{r'}\big)^{\mu}e^{-\frac{r'}{4}}.$$

By Hörmander's L^∞-estimate, see [LH68], we know that $||u_j||_\infty \leq C\mu_j^{\frac{d-1}{2}}$. Therefore each term in the series is bounded above by $C\mu_j^{d-1}(\frac{2r}{r'})^{\mu_j}e^{-\frac{r'}{4}}$. To continue the discussion on convergence, we need the following lemma.

Lemma 5.3.1. *Suppose that μ_j^2 are the eigenvalues of $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2$, then for any $0 < \beta < 1$, and any $M, N \geq 0$, the sum*
$$\sum_{\mu_j \geq M} \mu_j^N \alpha^{\mu_j - M}$$
converges for all $0 < \alpha \leq \beta$, and it is bounded uniformly in α.

Proof. Note that for any $\mu_j \geq 2M$, we have
$$\mu_j - M = M + (\mu_j - 2M) \geq M + \Big(\frac{\mu_j - 2M}{2}\Big) = \frac{\mu_j}{2}.$$
Therefore,
$$\sum_{\mu_j \geq 2M} \mu_j^N \alpha^{\mu_j - M} \leq \sum_{\mu_j \geq 2M} \mu_j^N \alpha^{\frac{\mu_j}{2}} \leq \sum_{\mu_j \geq 2M} \mu_j^N \beta^{\frac{\mu_j}{2}}.$$
We know that there is an integer $N_1(\beta, N) > 2M$ such that for all $j \geq N_1(\beta, N)$, $j^N \leq \beta^{-\frac{j}{4}}$. It follows that
$$\begin{aligned}\sum_{\mu_j \geq 2M} \mu_j^N \beta^{\frac{\mu_j}{2}} &\leq \sum_{2M \leq \mu_j < N_1(\beta,N)} \mu_j^N \beta^{\frac{\mu_j}{2}} + \sum_{\mu_j \geq N_1(\beta,N)} \beta^{-\frac{\mu_j}{4}}\beta^{\frac{\mu_j}{2}} \\ &\leq \big|\{j : \mu_j < N_1(\beta, N)\}\big| N_1(\beta, N)^N + \sum_{\mu_j \geq N_1(\beta,N)} \beta^{\frac{\mu_j}{4}} \qquad (5.27) \\ &\leq CN_1(\beta, N)^{d+N-1} + \sum_{\mu_j \geq N_1(\beta,N)} \beta^{\frac{\mu_j}{4}},\end{aligned}$$

where the constant $C > 0$ comes from the Weyl's estimate, see [CS, Corollary 5.1.2], and it satisfies that

$$\big|\{j : \mu_j \le \mu\}\big| \le C\mu^{d-1}, \tag{5.28}$$

for any $\mu > 0$. Now estimate the part of the summation that's greater than $N_1(\beta, N)$. An implication of (5.28) is, for any $j \in \mathbb{N}$, we have

$$\mu_j \ge \left(\frac{j}{C}\right)^{\frac{1}{d-1}}.$$

Therefore,

$$\sum_{\mu_j \ge N_1(\beta,N)} \beta^{\frac{\mu_j}{4}} \le \sum_{\mu_j \ge N_1(\beta,N)} \beta^{\frac{1}{4}(\frac{j}{C})^{\frac{1}{d-1}}} \le \sum_{j\ge 0} \beta^{\frac{1}{4}(\frac{j}{C})^{\frac{1}{d-1}}}.$$

There is $N_2(\beta, C) \in \mathbb{N}$ such that for all $j \ge N_2(\beta, C)$, we have $\frac{1}{4}(\frac{j}{C})^{\frac{1}{d-1}} > \log_\gamma j$, where $\gamma = \beta^{-\frac{1}{2}} > 1$. Then

$$\begin{aligned}\sum_{j\ge 1} \beta^{\frac{1}{4}(\frac{j}{C})^{\frac{1}{d-1}}} &\le \sum_{0\le j<N_2(\beta,C)} \beta^{\frac{1}{4}(\frac{j}{C})^{\frac{1}{d-1}}} + \sum_{j\ge N_2(\beta,C)} \beta^{\log_\gamma j} \\ &\le N_2(\beta, C) + \sum_{j\ge N_2(\beta,C)} j^{-2} \\ &\le N_2(\beta, C) + \frac{\pi^2}{6}.\end{aligned} \tag{5.29}$$

The remaining part of the summation is from M to $2M$,

$$\sum_{M\le \mu_j<2M} \mu_j^N \alpha^{\mu_j - M} \le \big|\{j : \mu_j < 2M\}\big|(2M)^N \le C(2M)^{d-1}(2M)^N = C(2M)^{d+N-1}.$$

Bringing all the parts together, we have

$$\sum_{\mu_j\ge M} \mu_j^N \alpha^{\mu_j - M} \le C(2M)^{d+N-1} + CN_1(\beta, N)^{d+N-1} + N_2(\beta, C) + \frac{\pi^2}{6} < \infty. \tag{5.30}$$

Note the finite constant depends on M, N, C, β but not α, therefore we have uniform boundedness in α. $\square$

Proposition 5.3.2. *The expansion* (5.18) *is polyhomogeneous conormal at lbz.*

Proof. Since the functions $I_\mu(r)$ and $K_\mu(r)$ have expansions in powers at $r = 0$ (including logarithms in the case of K_μ when μ is an integer), the individual terms in the series are polyhomogeneous conormal. So consider the tail of the series. Lemma 5.3.1 implies that the sum of the tail of the series, that is over $\mu_j \geq M$ is bounded by $Cr^M e^{-r'/4}$ for small r. We can apply the same argument to derivatives of the series. In fact, the derivatives of I_μ and K_μ can be treated as above, showing that for $r \leq 1$, $\mu^{-k}(r\partial_r)^k I_\mu$ and $\mu^{-k}(r\partial_r)^k K_\mu$, and for $r \geq 1$, $\mu^{-k}\partial_r^k I_\mu$ and $\mu^{-k}\partial_r^k K_\mu$ satisfy the same estimates as I_μ and K_μ. Moreover, we have a Hörmander estimate $\|\nabla^{(k)} u_j\|_\infty \leq C_k \mu_j^{(n-1)/2+k}$ for derivatives of u_k. Thus derivatives only give us extra powers of μ, which are harmless as Lemma 5.3.1 applies with arbitrary powers of μ. □

Proposition 5.3.2 implies, in particular, that $\tilde{G}$ decays exponentially, with all its derivatives, as $r' \to \infty$, ie when approaching the boundary rbi. Similarly in the region $\frac{r}{r'} \geq 4$, as $r \to \infty$, ie when approaching lbi, the kernel is also exponentially decreasing. Therefore we cut off $\tilde{G}_0$ to restrict it away from the $r = r'$ to obtain a well-defined operator $\tilde{G}_1$ with the kernel

$$\tilde{G}_1(r, r', y, y') = \tilde{G}_0(r, r', y, y')\left(\chi\left(\frac{8r}{r'}\right) + \chi\left(\frac{8r'}{r}\right)\right). \tag{5.31}$$

Here $\chi : [0, \infty) \to [0, 1]$ is a smooth cutoff function such that $\chi\big([0, 1]\big) = 1$ and $\chi\big([2, \infty)\big) = 0$. The support of $\tilde{G}_1$ is shown in Figure 5.2.

At last, similar to (5.11), we define

$$G_1 = rr'\tilde{G}_1. \tag{5.32}$$

5.4 Near diagonal

The formula obtained in the previous section doesn't converge near $r = r'$, so in this section we construct an operator G_{nd}, which is good near the diagonal. The subscript nd means "near diagonal".

Near the zf-face we consider the b-elliptic operator $\tilde{P}$. In order to keep it away from the sf-face, we multiply it with a cutoff function. Hence we consider $\tilde{P}\chi(r)$, where $\chi : [0, \infty) \to [0, 1]$ is a smooth cutoff function such that $\chi\big([0, 1]\big) = 1$ and $\chi\big([2, \infty)\big) = 0$. By the ellipticity of $\tilde{P}$ near the zf-face, and by Proposition 3.4.8,

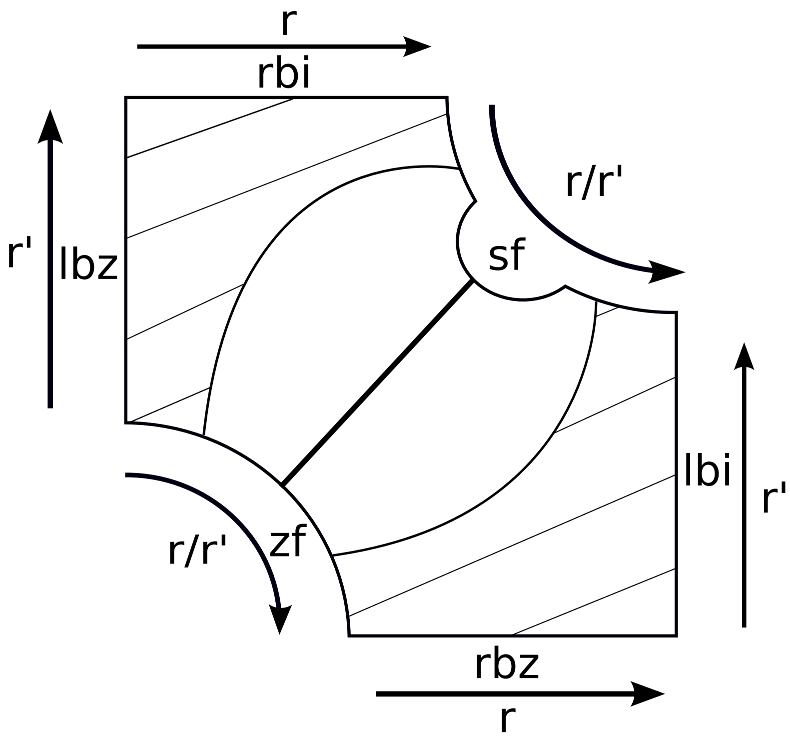


Figure 5.2: The support of $\tilde{G}_1$

there is $\tilde{G}^{zf}_{nd}$ in the small b-calculus such that

$$\tilde{P}\tilde{G}^{zf}_{nd}\chi(r) = \chi(r) + E_{zf}, \tag{5.33}$$

where E_{zf} is smooth across the diagonal and vanishes at zf. Let

$$G^{zf}_{nd} = rr'\tilde{G}^{zf}_{nd},$$

then we have

$$PG^{zf}_{nd}\chi(r) = \chi(r) + E_{zf}.$$

Near the sf-face the operator P is elliptic in the scattering calculus. We multiply it with $1-\chi(r)$ to keep it away from the zf-face, ie we consider the operator $P\big(1-\chi(r)\big)$. Since $P\big(1-\chi(r)\big)$ is elliptic near the sf-face, by Proposition 4.5.7, there is G^{sf}_{nd} in the scattering calculus such that

$$PG^{sf}_{nd}\big(1-\chi(r)\big) = 1-\chi(r) + E_{sf},$$

where again the error term E_{sf} is smooth across the diagonal.

At last, we define our G_{nd} using G_{nd}^{zf} and G_{nd}^{sf}. We use cutoff functions to keep the support of G_{nd} near the diagonal.

$$G_{nd} = (G_{nd}^{zf} + G_{nd}^{sf})\chi\Big(\frac{r}{2r'}\Big)\chi\Big(\frac{r'}{2r}\Big)\chi\big(\operatorname{dist}(z,z')\big).$$

Then we have

$$PG_{nd} = Id + E_{nd},$$

where the error term E_{nd} is smooth across the diagonal. The supports of $\chi(\frac{r}{2r'})\chi(\frac{r'}{2r})$, $\chi\big(\operatorname{dist}(z,z')\big)$ and G_{nd} are shown in Figure 5.3.

For the construction of the overall final parametrix G, we must add various terms to G_{nd} so that it is consistent with formula (5.31) and has the desirable decaying properties at each boundary. We start with the sf-face.

5.5 sf-face

Here we investigate what happens when $r, r' \to \infty$ and $d(z,z')$ remains uniformly bounded, ie the sf-face. We use the scattering calculus. In the previous section we constructed G_{nd} such that $PG_{nd} = Id + E_{nd}$. Now we want to construct an operator G_{sf}^1 in the scattering calculus with differential order $-\infty$, supported near the sf-face, such that

$$N_{sc}(P)N_{sc}(G_{sf}^1) = -N_{sc}(E_{nd}),$$

where N_{sc} is the normal operator defined in Section 4.6, and recall that it is multiplicative under composition. We then have

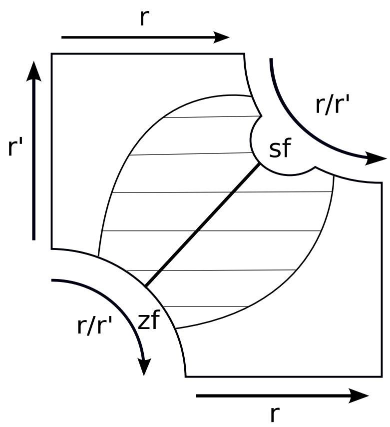
$$N_{sc}\big(P(G_{nd} + G_{sf}^1) - Id\big) = 0. \tag{5.34}$$

The reason we want property (5.34) is that, according to Proposition 4.6.6, it implies that $G_{nd} + G_{sf}^1$ approximates the inverse of P apart from a compact error term which vanishes to the first order at sf. That is,

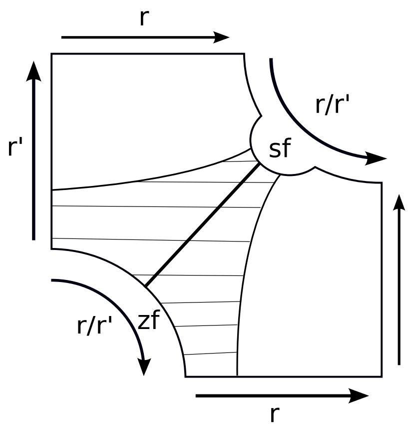
$$P(G_{nd} + G_{sf}^1) = Id + \frac{1}{r'}E_1,$$

for some E_1.

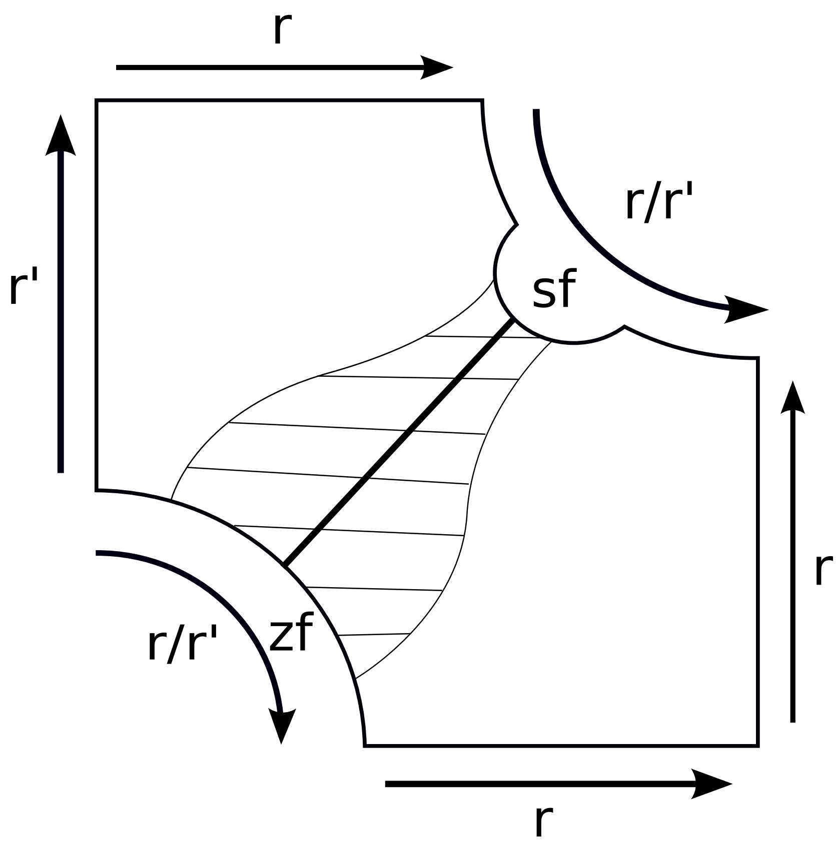
Let's now define this G_{sf}^1. As each fibre of the interior of sf has the Euclidean



(a) The support of $\chi(\frac{r}{2r'})\chi(\frac{r'}{2r})$



(b) The support of $\chi\big(\operatorname{dist}(z,z')\big)$



(c) The support of G_{nd}

Figure 5.3: The supports of $\chi(\frac{r}{2r'})\chi(\frac{r'}{2r})$, $\chi\big(\operatorname{dist}(z,z')\big)$ and G_{nd}

structure, see Section 4.3, $N_{sc}(P)$ behaves like $\Delta_{\mathbb{R}^d}+1$. From the explicit formula for $(\Delta_{\mathbb{R}^d}+1)^{-1}$, we know that $N_{sc}(P)$ is conormal of order -2 with respect to the origin and decreases rapidly at infinity. Therefore,

$$-N_{sc}(P)^{-1}N_{sc}(E_{nd})$$

is also conormal with respect to the origin and decreases rapidly at infinity. By subjectivity of the normal operator, see Proposition 4.6.3, there is a G^1_{sf} in the scattering calculus such that

$$N_{sc}(G^1_{sf}) = -N_{sc}(P)^{-1}N_{sc}(E_{nd}).$$

Since only the value near the sf-face matters, we can choose our G^1_{sf} with support near the sf-face.

We continue to define, inductively, G^j_{sf}, for each $j \in \mathbb{N}$. Suppose we have defined $G^1_{sf}, ..., G^k_{sf}$ and the error terms $E_1, ..., E_k$ which satisfy

$$P\big(G_{nd} + G^1_{sf} + \frac{1}{r'}G^2_{sf} \cdots + \frac{1}{r'^{k-1}}G^k_{sf}\big) = Id + \frac{1}{r'^k}E_k.$$

We then construct an G^{k+1}_{sf}, which is supported near the sf-face, and satisfies

$$N_{sc}(P)N_{sc}(G^{k+1}_{sf}) = -N_{sc}(E_k).$$

The construction of G^{k+1}_{sf} is completely parallel to the construction of G^1_{sf} described above. Then we have

$$N_{sc}\bigg(r'^k P\big(G_{nd} + G^1_{sf} + \frac{1}{r'}G^2_{sf} \cdots + \frac{1}{r'^{k-1}}G^k_{sf}\big) + PG^{k+1}_{sf} - r'^k Id\bigg) = 0.$$

Again by Proposition 4.6.6, we know this means

$$r'^k P\big(G_{nd} + G^1_{sf} + \frac{1}{r'}G^2_{sf} \cdots + \frac{1}{r'^{k-1}}G^k_{sf}\big) + PG^{k+1}_{sf} = r'^k Id + \frac{1}{r'}E_{k+1},$$

for some E_{k+1}. Divide both sides by r'^k, we have

$$P\big(G_{nd} + G^1_{sf} + \frac{1}{r'}G^2_{sf} \cdots + \frac{1}{r'^{k-1}}G^k_{sf} + \frac{1}{r'^k}G^{k+1}_{sf}\big) = Id + \frac{1}{r'^{k+1}}E_{k+1}.$$

This completes the inductive definition of G^j_{sf}, $j \in \mathbb{N}$. We define our G_{sf} to be an operator in the scattering calculus which has the Borel sum

$$\sum_{j\in\mathbb{N}} \frac{1}{r'^{j-1}}G^j_{sf}.$$

Then G_{sf} satisfies

$$P(G_{nd} + G_{sf}) = Id + E_{sf},$$

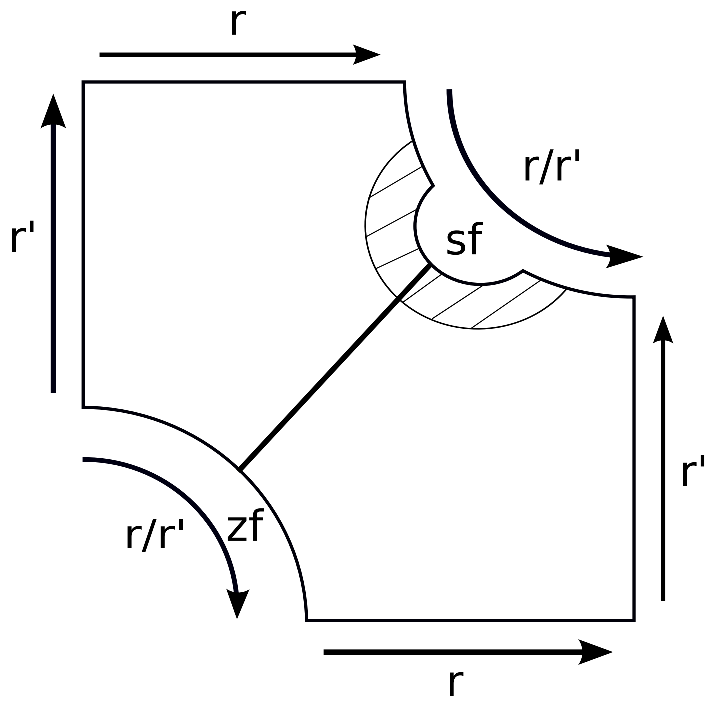


Figure 5.4: The support of G_{sf}

where the error term E_{sf} is compact on $L^2(M)$, and vanishes to infinite order at the sf-face. The support of G_{sf} is illustrated in Figure 5.4.

Remark 5.5.1. The first approximation G^1_{sf} only gave us an error term with the vanishing order 1 at the sf-face. But we want an error term that vanishes to infinite order at sf. That's why we continued to inductively define G^j_{sf}, $j \in \mathbb{N}$.

5.6 zf-face

5.6.1 Defining G_{zf}

This is the hypersurface when $r, r' \to 0$. The operator $\tilde{P}$, defined in Subsection 5.3.1, is b-elliptic near zf. In order to keep away from the sf-face, we consider $\tilde{P}\chi(r)$, where χ is the same cutoff function as in (5.31). According to Proposition 3.5.9, there is an operator G_b of order -2 in the full b-calculus, supported in $r \leq 2$, such that $\chi(r) - \tilde{P}G_b\chi(r)$ is smooth on $L^2(M)$. From (5.33) we know that $\chi(r) - \tilde{P}\tilde{G}_{nd}\chi(r)$ is smooth. That means $\tilde{P}(G_b - \tilde{G}_{nd})\chi(r)$ is smooth, and because $\tilde{P}$, lifted to the blown up space, is elliptic at the conormal bundle of the diagonal, we know that $G_b - \tilde{G}_{nd}$ is smooth across the diagonal on the support of χ.

Define

$$\tilde{G}_{zf} = (G_b - \tilde{G}_{nd})\chi(r)\Big(1 - \chi\Big(\frac{8r}{r'}\Big) - \chi\Big(\frac{8r'}{r}\Big)\Big), \tag{5.35}$$

and

$$G_{zf} = rr'\tilde{G}_{zf}. \tag{5.36}$$

The support of $\tilde{G}_{zf}$ is shown in Figure 5.5. By Remark 3.4.12, we know that

$$P(G_{nd} + G_{zf}) = Id + E_{zf},$$

where E_{zf} is a compact error term that vanishes to the first order at the zf-face.

Note that G_{zf} is supported near zf and near $r = r'$. The cutoff functions are chosen in a way that when we add $\tilde{G}_{zf}$ and $\tilde{G}_1$ later on, the sum is the same as $\tilde{G}_{zf}$ near $r = r'$ and the same as $\tilde{G}_1$ away from $r = r'$. The cutoff functions in definitions 5.31 and 5.35 are chosen so that they add up to 1 near the zf.

We need to check that near zf in the region where the supports of $\tilde{G}_1$ and $\tilde{G}_{zf}$ intersect, ie the region

$$\{r \le 2 : \frac{1}{8} \le \frac{r}{r'} \le \frac{1}{4} \text{ or } 4 \le \frac{r}{r'} \le 8\},$$

the two definitions are compatible. That means to check for $\frac{1}{8} \le \frac{r}{r'} \le \frac{1}{4}$ or $4 \le \frac{r}{r'} \le 8$, the restriction of $\tilde{G}_0$ to the zf-face is the same as $I_b(G_b)$. This will be done in Subsection 5.6.3.

5.6.2 The expression of $I_b(G_b)$ away from $r = r'$

Again we are working with the operator $\tilde{P}$, b-elliptic near zf. Recall from Chapter 3 that $r\partial_r$ is non-vanishing as a b-vector field at zf; while the term r^2 vanishes at zf. Therefore the indicial operator of $\tilde{P}\chi(r)$ is

$$I_b\big(\tilde{P}\chi(r)\big) = \tilde{P} - r^2 = -(r\partial_r)^2 + \Delta_Y + V_0(y) + \Big(\frac{d-2}{2}\Big)^2.$$

It describes the leading behaviour of $\tilde{P}$ near zf.

Now to determine $I_b(G_b)$. Let μ_j^2, u_j, Π_j be the same as defined in Section 5.3.1, but here instead of (5.13) and (5.14) we have

$$I_b\big(\tilde{P}\chi(r)\big) = \sum_j \Pi_j S_j,$$

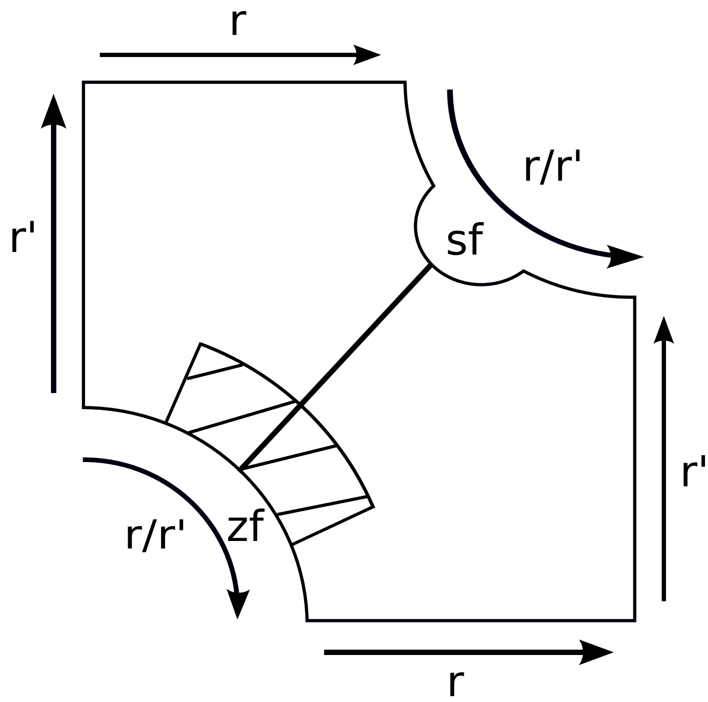


Figure 5.5: The support of $\tilde{G}_{zf}$

where

$$S_j = -(r\partial_r)^2 + \mu_j^2.$$

In order to find the inverse of S_j, we need to solve for $a_j(r)$ such that

$$S_j a_j(r) = \delta(\frac{r}{r'} - 1) = r'\delta(r - r').$$

When $r \neq r'$, the solution space is spanned by r^{μ_j} and $r^{-\mu_j}$. Then, similar to Section 5.3.1, the kernel S_j^{-1} is

$$S_j^{-1}(r, r') = \begin{cases} \frac{1}{2\mu_j}(\frac{r}{r'})^{\mu_j} \left|\frac{dr}{r}\frac{dr'}{r'}\right|^{\frac{1}{2}}, & r < r', \\ \frac{1}{2\mu_j}(\frac{r'}{r})^{\mu_j} \left|\frac{dr}{r}\frac{dr'}{r'}\right|^{\frac{1}{2}}, & r > r'. \end{cases}$$

Note that S_j^{-1} is compact and self-adjoint, and

$$I_b(G_b) = \sum_j \Pi_j S_j^{-1}.$$

In terms of the coordinates on zf, ie $s = \frac{r}{r'}, y, y'$, we have

$$I_b(G_b)(s,y,y') = \sum_j u_j(y)\overline{u_j(y')}S_j^{-1}(r,r')\big|_{zf}\left|dhdh'\right|^{\frac{1}{2}},$$

where $S_j^{-1}(r,r')\big|_{zf}$ means the restriction of the b-half-density to zf, as in Definition 3.4.9. Hence

$$I_b(G_b)(s,y,y') = \begin{cases} \frac{1}{2}\sum_j \frac{1}{\mu_j}u_j(y)\overline{u_j(y')}s^{\mu_j}\left|\frac{ds}{s}dhdh'\right|^{\frac{1}{2}}, & s>1, \\ \frac{1}{2}\sum_j \frac{1}{\mu_j}u_j(y)\overline{u_j(y')}s^{-\mu_j}\left|\frac{ds}{s}dhdh'\right|^{\frac{1}{2}}, & s<1. \end{cases} \tag{5.37}$$

The convergence of this sum can be discussed similarly as in Section 5.3.2. In particular it has good convergence properties for $s \le \frac{1}{4}$ or $s \ge 4$.

5.6.3 Compatibility of G_1 and G_{zf}

We only consider the case $\frac{1}{8} \le \frac{r}{r'} \le \frac{1}{4}$ here, as the case $4 \le \frac{r}{r'} \le 8$ is completely parallel. Recall from expression (5.17), for $\frac{1}{8} \le \frac{r}{r'} \le \frac{1}{4}$ we have,

$$\tilde{G}_0(r,r',y,y') = \sum_j u_j(y)\overline{u_j(y')}I_{\mu_j}(r)K_{\mu_j}(r')\left|\frac{dr}{r}\frac{dr'}{r'}dhdh'\right|^{\frac{1}{2}}. \tag{5.38}$$

We use the limiting forms (5.15) and (5.16) for small arguments again. We need to estimate the error terms uniformly in μ. Using (5.19), we see that

$$I_\mu(r) - \frac{r^\mu}{2^\mu\Gamma(\mu+1)} = \frac{2^{-\mu}r^\mu}{\pi^{\frac{1}{2}}\Gamma(\mu+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{\mu-\frac{1}{2}}\Big(e^{-rt}-(1-rt)\Big)dt \tag{5.39}$$

(the rt term makes no contribution in the integral due to the evenness of $(1-t^2)^{\mu-\frac{1}{2}}$). We may estimate $|e^{-rt}-(1-rt)| \le er^2t^2/2$ for $|t| \le 1, r \le 1$. Then integrating by parts we have

$$\int_{-1}^{1}(1-t^2)^{\mu-\frac{1}{2}}t^2\,dt = \frac{1}{2\mu+1}\int_{-1}^{1}(1-t^2)^{\mu+\frac{1}{2}}\,dt$$

leading to a uniform estimate

$$\left|I_\mu(r) - \frac{r^\mu}{2^\mu\Gamma(\mu+1)}\right| \le \frac{e}{2\mu+1}\frac{r^{\mu+2}}{2^{\mu+1}\Gamma(\mu+1)}, \quad r \le 1. \tag{5.40}$$

As for K_μ, we change variable in the integral (5.20) to $\tau = r\sqrt{t^2-1}$ to obtain

$$K_\mu(r) - \frac{2^{\mu-1}\Gamma(\mu)}{r^\mu} = \frac{\pi^{\frac{1}{2}}2^{-\mu}r^{-\mu}}{\Gamma(\mu+\frac{1}{2})}\int_0^\infty \frac{e^{-\sqrt{\tau^2+r^2}}}{\sqrt{\tau^2+r^2}}\tau^{2\mu}\,d\tau.$$

We estimate

$$\left|\frac{e^{-\sqrt{\tau^2+r^2}}}{\sqrt{\tau^2+r^2}} - \frac{e^\tau}{\tau}\right| = \left|\int_0^1 \frac{d}{ds}\frac{e^{-\sqrt{\tau^2+sr^2}}}{\sqrt{\tau^2+sr^2}}\,ds\right| \leq \frac{r^2}{2}\Big(\frac{e^{-\tau}}{\tau^2}+\frac{e^{-\tau}}{\tau^3}\Big). \tag{5.41}$$

For $\mu \geq 2$ the contribution of the second term on the right hand side is not greater than the contribution of the first term. This leads to an estimate for $\mu \geq 2$

$$\left|K_\mu(r) - \frac{2^{\mu-1}\Gamma(\mu)}{r^\mu}\right| \leq \frac{\pi^{\frac{1}{2}}2^{-\mu}r^{-\mu+2}}{\Gamma(\mu+\frac{1}{2})}\Gamma(2\mu-1) = \frac{2^{\mu-1}\Gamma(\mu)r^{-\mu+2}}{2\mu-1},$$

where we used (5.25) in the last step. From these estimates and Lemma 5.3.1, we can replace I_{μ_j} and K_{μ_j} in (5.38) by their leading asymptotic at $r=0$ up to error terms which are summable in j, leading to an estimate

$$\tilde{G}_0(r,r',y,y') = \frac{1}{2}\sum_j \frac{1}{\mu_j}u_j(y)\overline{u_j(y')}\Big(\frac{r}{r'}\Big)^{\mu_j}\left|\frac{dr}{r}\frac{dr'}{r'}dhdh'\right|^{\frac{1}{2}} + O(r'^2). \tag{5.42}$$

Note that we are working in the region where $r < r'$, hence only the $O(r'^2)$ term appears. Since $\frac{dr}{r}\frac{dr'}{r'} = \frac{ds}{s}\frac{dr'}{r'}$, the expression (5.42) of $\tilde{G}_0(r,r',y,y')$, when restricted to the zf-face, is consistent with the expression (5.37) of $I_b(G_b)(s,y,y')$.

5.7 Construction of P^{-1}

Let $G_a = G_1 + G_{nd} + G_{zf} + G_{sf}$, then

$$PG_a = Id + E. \tag{5.43}$$

We give the operator G_a the subscript a because it approximates the inverse of P, with the error term E. The operators G_a and E behave desirably at the boundaries of the blown-up space, and we will solve away E to obtain our final $G = P^{-1}$. We then obtain the properties of G at the various boundaries from those of G_a.

Let's summarise the properties of G_a and E.

Proposition 5.7.1. *As a multiple of the Riemannian half-density, ie the scattering-half-density $|r^{d-1}r'^{d-1}drdr'dhdh'|^{\frac{1}{2}}$, on the blown-up space, the kernel G_a is the sum of two terms. One is G_{nd}, supported near the diagonal, and is such that $\rho_{zf}^{d-2}G_{nd}$ is conormal of order -2 with respect to the diagonal uniformly up to both zf and sf, where ρ_{zf} is any boundary defining function for zf. The other term $G_a - G_{nd}$ satisfies:*

(i) *it is smooth at the diagonal, and polyhomogeneous conormal at all boundary hypersurfaces;*

(ii) *it vanishes to infinite order at lbi, rbi and bf;*

(iii) *it vanishes to order $1 - \frac{d}{2} + \mu_0$ at lbz and rbz;*

(iv) *it vanishes to order $2 - d$ at zf.*

Proof. Only (iii) and (iv) are not so clear. We obtain the vanishing property at lbz from equations (5.31) and (5.32). Since r is the boundary defining function for lbz, we need to work out its power. Clearly one power of r comes from (5.32), while $I_{\mu_0}(r)$ in (5.31) gives us the power r^{μ_0}. Then the difference between the b-half-density and the Riemannian half-density gives us a power of $r^{-\frac{d}{2}}$. Combining these we conclude that the vanishing order at lbz is $1 - \frac{d}{2} + \mu_0$. The vanishing order at rbz is similar.

Now (iv). Since both r and r' vanish at zf, to obtain the vanishing order of G_a at zf, as a scattering-half-density, we add the powers of r and r' in (5.36) with the factor $(rr')^{-\frac{d}{2}}$ involved in the change from a b-half-density to the Riemannian half-density. That is, $1 + 1 - \frac{d}{2} - \frac{d}{2} = 2 - d$. □

Remark 5.7.2. In the case of the potential $V \equiv 0$, we have $\mu_0 = \frac{d}{2} - 1$. That means the vanishing order given by item (iii) of Proposition 5.7.1 is 0. We need more information than this for the proof Theorem 6.5.3 in Chapter 6. Therefore we have to look at the second term of the expansion of $I_{\mu_0}(r)$ and the first term of the expansion of $I_{\mu_1}(r)$. By (5.15), the first option gives us a vanishing order of $1 - \frac{d}{2} + (\mu_0 + 2) = 2$; while the second option gives us a vanishing order of $1 - \frac{d}{2} + \mu_1$.

Proposition 5.7.3. *The error term E has the following properties on the blown-up space:*

(i) *it is compact, and in fact Hilbert-Schmidt, on* $L^2(M)$*;*

(ii) *it is smooth;*

(iii) *it vanishes to the first order at the* zf*-face;*

(iv) *it vanishes to infinite order at other boundaries.*

Proof. These properties are obtained through the construction process of G_a. □

We proceed to solve away E. To achieve that, we would like to invert $Id+E$. But it might not be invertible, in which case we perturb G_a so that the right hand side of (5.43) becomes invertible.

Since E is compact, $Id+E$ is Fredholm of index 0, and its null space and the complement of its range both have the same finite dimension, say N. Removing the null space gives us an invertible operator, and to achieve that we add a rank N operator to G_a. To construct this rank N operator, we need the following lemma.

Lemma 5.7.4. *There exist smooth functions* $\psi_1, ..., \psi_N$*,* $\phi_1, ..., \phi_N$ *on* M *such that*

(i) $\psi_1, ..., \psi_N$ *span the null space of* $Id+E$*, and* $P\phi_1, ..., P\phi_N$ *span a space supplementary to the range of* $Id+E$*;*

(ii) *they are* $O(r^\infty)$ *as* $r \to 0$ *and* $O(r^{-\infty})$ *as* $r \to \infty$*.*

Proof. We choose the ψ_i to be any basis of the null space of $Id+E$. To obtain property (ii) for the ψ_i, we note that $\psi_i = -E(\psi_i)$, hence iterating, we have $\psi_i = E^{2N}\psi_i$ for each $N \geq 1$. Now we consider mapping properties of the operator E^N. First, writing $E = E_b + E_{sc}$ as in the proof of Proposition 5.7.3, it is easy to see that E_{sc} and ∇E_{sc} map $L^2(M)$ to $\langle r\rangle^{-K}L^2(M)$ for arbitrary K. (Here ∇ is shorthand for the vector of derivatives $(\partial_r, r^{-1}\partial_{y_i})$.) As for E_b, since it has negative order in the b-calculus and vanishes to first order at zf, we see that E_b maps $L^2(M)$ to $rL^2(M)$. Since the kernel $(r/r')^a E$ has the same properties as E listed in Proposition 5.7.3, it follows that E_b maps $r^aL^2(M)$ to $r^{a+1}L^2(M)$ for any a. Also, applying a derivative $\nabla = (\partial_r, r^{-1}\partial_{y_i})$ to E_b, it is still of negative order in the b-calculus, though no longer vanishing at zf, so we see that ∇E_b

maps $r^a L^2(M)$ to $r^a L^2(M)$ for any a. Summarizing, we have

$$\begin{aligned} &E \text{ boundedly maps } r^a L^2(M) \to r^{a+1}\langle r\rangle^{-K} L^2(M), \\ &\nabla E \text{ boundedly maps } r^a L^2(M) \to r^a\langle r\rangle^{-K} L^2(M). \end{aligned} \tag{5.44}$$

Applying these properties of E iteratively, we see that E^{2N} maps $L^2(M)$ to $r^N\langle r\rangle^{-2N} H^N(M)$ for any N. Hence, using Sobolev embeddings, ψ is smooth and has rapid decay both as $r \to 0$ and $r \to \infty$.

As for the ϕ_i, to show that we can choose functions $\phi_1, \dots, \phi_N$ as above, it is sufficient to show that the range of P on the subspace $\mathcal{S}$ of smooth half-densities satisfying (ii) is dense on $L^2(M)$. If this were not true, then there would be a nonzero half-density $f \in L^2(M)$ orthogonal to the range of P on such half-densities: that is, we would have

$$\langle Pu, f\rangle = 0, \text{ for all } u \in \mathcal{S}.$$

Since $\mathcal{S}$ is a dense subspace, this implies that $Pf = 0$ distributionally. By elliptic regularity this means that f is smooth and $Pf = 0$ strongly, but since P is invertible on L^2 this implies $f = 0$, a contradiction. Therefore we can choose the $\phi_i \in \mathcal{S}$ as desired. □

The rank N operator is

$$Q = \sum_{i=1}^{N} \phi_i \langle \psi_i, \cdot\rangle,$$

where $\langle \psi_i, \cdot\rangle$ means the inner product with ψ_i. The functions $\psi_1, ..., \psi_N$, $\phi_1, ...\phi_N$ are chosen as in Lemma 5.7.4. Then we have,

$$P(G_a + Q) = Id + E + PQ,$$

which is invertible. From here we obtain

$$P^{-1} = (G_a + Q)(Id + E + PQ)^{-1}.$$

Note that the functions $\psi_1, ..., \psi_N, \phi_1, ..., \phi_N$ are C^∞, $O(r^\infty)$ as $r \to 0$ and $O(r^{-\infty})$ as $r \to \infty$. From this we know that $G_a + Q$ behaves the same as G_a at the boundaries, ie it has those properties listed in Proposition 5.7.1. It also guarantees that PQ is "nice" enough so that $E + PQ$ has those properties of E that are listed in Proposition 5.7.3. Denote $E+PQ$ by E', and consider the operator S defined as

$$S = (Id + E')^{-1} - Id.$$

Then we can write

$$P^{-1} = (G_a + Q)(Id + S).$$

In order to continue with our analysis, we need to know the properties of S.

Lemma 5.7.5. *The operator S has the properties (i)-(iv) listed in Proposition 5.7.3.*

Remark 5.7.6. A similar analysis was done in [GHS, Sec. 5.4].

Proof. We have

$$\begin{aligned} S &= -E'(Id + E')^{-1} \\ &= -E' \sum_{j=0}^{4N-1} (-1)^j E'^j - E'(Id + E')^{-1} E'^{4N} \\ &= \sum_{j=1}^{4N} (-1)^j E'^j + E'^{2N} S E'^{2N}. \end{aligned} \tag{5.45}$$

The term $\sum_{j=1}^{4N} (-1)^j E'^j$ clearly has all the properties listed in Proposition 5.7.3, so we focus on the term $E'^{2N} S E'^{2N}$. This term is Hilbert-Schmidt because E' is Hilbert-Schmidt and S is bounded. It then follows that S is Hilbert-Schmidt. Note that $E'^{2N} S E'^{2N}$ vanishes to order N at lbz, rbz and zf, to infinite order at the other boundaries, and is C^N everywhere. To summarise, we have written S as a sum of two parts, where the first part satisfies

(i) it vanishes to the first order at the zf-face;

(ii) it vanishes to infinite order at the other boundaries;

(iii) it is C^∞ everywhere;

while the second part satisfies

(i) it vanishes to order N at lbz, rbz and zf;

(ii) it vanishes to infinite order at the other boundaries;

(iii) it is C^N everywhere.

Since we can do this for any N, we can conclude that S has the properties listed in Proposition 5.7.3. □

To summarise, we have

$$G = P^{-1} = (G_a + Q)(Id + S),$$

where $G_a + Q$ has those properties listed in Proposition 5.7.1, $Id + S$ is a compact operator, and S has those properties listed in Proposition 5.7.3.

We summarise the information about $G = P^{-1}$ at the various parts of the blown-up space, obtained through the construction, in the following theorem.

Theorem 5.7.7. *As a multiple of the Riemannian half-density, ie the scattering-half-density $|r^{d-1}r'^{d-1}drdr'dhdh'|^{\frac{1}{2}}$, on the blown-up space, the kernel G is the sum of two terms. The first term G_c is supported near the diagonal, and is such that $\rho_{zf}^{d-2}G_c$ is conormal of order -2 with respect to the diagonal uniformly up to both zf and sf, where ρ_{zf} is any boundary defining function for zf. The other term G_s satisfies properties (i)-(iv) listed in Proposition 5.7.1.*

Remark 5.7.8. The subscripts c and s are chosen to indicate that G_c is the part of G which is conormal at the diagonal, while G_s is the part of G which is smooth at the diagonal.

Proof. Note that $G = (G_a + Q)(Id + S) = G_a + Q + QS + G_aS$. Since Q and QS are smooth and vanish to infinite order at the boundaries, we just need to check that G_aS has the same properties as G_a, then by Proposition 5.7.1 we obtain the results in this theorem.

Let $\chi : [0, \infty) \to \mathbb{R}$ be a smooth cutoff function such that $\chi\big([0, 1]\big) = 0$ and $\chi\big([2, \infty)\big) = 1$. We write G_a as a sum of two parts. The first part $\chi(r)G_a\chi(r')$ is in the scattering calculus. Note that $\chi(2r')S$ is also in the scattering calculus. Therefore by Proposition 4.5.6,

$$\big(\chi(r)G_a\chi(r')\big)S = \big(\chi(r)G_a\chi(r')\big)\big(\chi(2r')S\big)$$

is in the scattering calculus. The second part $G_a - \chi(r)G_a\chi(r')$ is in the full b-calculus. Note that S is in the small b-calculus. Therefore by Proposition 3.5.8, $\big(G_a - \chi(r)G_a\chi(r')\big)S$ is the full b-calculus. Therefore the required properties for G_c and property (i) for G_s follow.

With regard to the vanishing orders at the boundaries, ie properties (ii)-(iv) listed in Proposition 5.7.1, we use the properties of S from Lemma 5.7.5, ie that S vanishes to infinite order at all the boundaries apart from to the first order at zf. That means it doesn't change the vanishing orders of G_a at the boundaries when composed with it. □

Remark 5.7.9. In the case of the potential $V \equiv 0$, we have $\mu_0 = \frac{d}{2} - 1$. So the vanishing order given by item (iii) of Theorem 5.7.7 is 0. It's consistent with the case when the cone is $\mathbb{R}^d$ and the potential $V \equiv 0$, in which case the cone tip can be chosen arbitrarily, and G is smooth everywhere.

Remark 5.7.10. As mentioned in Remark 5.7.2, in the case of $V \equiv 0$, in order to obtain more useful information than item (iii) of Theorem 5.7.7, in the proof of Theorem 6.5.3 we will have to look at the second term of the expansion of $I_{\mu_0}(r)$, which gives us a vanishing order of 2, and the first term of the expansion of $I_{\mu_1}(r)$, which gives us a vanishing order of $1 - \frac{d}{2} + \mu_1$.

The vanishing orders of $G = P^{-1}$ at the various boundaries of the blown-up space are shown in Figure 5.6.

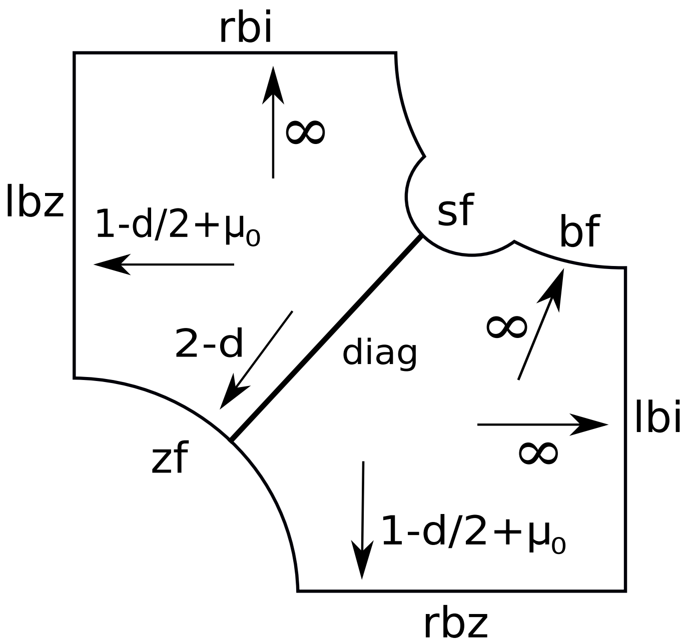


Figure 5.6: The vanishing orders at the various boundaries

Remark 5.7.11. The construction of $G = P^{-1}$ in this chapter is present in the paper [GH] by C. Guillarmou and A. Hassell but some details are lacking. It's not fully justified in [GH] that the kernel is in the scattering calculus near sf, and in the b-calculus near zf. It's for this reason we have given complete details in this chapter.

Chapter 6

The boundedness of the Riesz transform

6.1 Estimate on the kernel

Recall that the Riesz transform T with the inverse square potential $V = \frac{V_0}{r^2}$, defined in Chapter 5, can be expressed as

$$T = \frac{2}{\pi} \int_0^\infty \nabla \big(H + \lambda^2\big)^{-1} d\lambda,$$

where

$$H = \Delta + \frac{V_0(y)}{r^2},$$

and recall that H is homogenous of degree -2. From here we obtain an important estimate on the kernel $T(z, z')$.

Proposition 6.1.1. *We have the following estimate on the kernel of T,*

$$|T(z, z')| \lesssim \int_0^\infty \lambda^{d-2} \big| \nabla \big(G(\lambda z, \lambda z') \big) \big| d\lambda,$$

where $G = (H + 1)^{-1}$, with properties stated in Theorem 5.7.7.

Proof. This estimate follows from the relationship between $(H + \lambda^2)^{-1}$ and $G = (H + 1)^{-1}$, which is

$$(H + \lambda^2)^{-1}(z, z') = \lambda^{d-2} (H + 1)^{-1} (\lambda z, \lambda z').$$

The power -2 of λ appears because H is homogenous of degree -2. Note that these kernels are Riemannian half-densities, hence we get the power d of λ as follows,

$$|(\lambda r)^{d-1}(\lambda r')^{d-1}d(\lambda r)d(\lambda r')dhdh'|^{\frac{1}{2}} = \lambda^d |r^{d-1}r'^{d-1}drdr'dhdh'|^{\frac{1}{2}}.$$

□

6.2 Boundedness on $L^2(M)$

Proposition 6.2.1. *The Riesz transform T with the inverse square potential $V = \frac{V_0}{r^2}$ is bounded on $L^2(M)$.*

Proof. Our assumption is $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2 > 0$, ie $\Delta + \frac{1}{r^2}V_0(y) > 0$. Hence there is $\varepsilon > 0$ such that $\Delta + \frac{1}{(1-\varepsilon)r^2}V_0(y) > 0$. That means, $\Delta + \frac{1}{r^2}V_0(y) > \varepsilon\Delta$. From here, for any $f \in C_c^\infty\big(Y \times (0,\infty)\big)$,

$$\begin{aligned}
\langle Tf, Tf\rangle &= \langle \Delta\big(\Delta + \frac{1}{r^2}V_0(y)\big)^{-\frac{1}{2}}f, \big(\Delta + \frac{1}{r^2}V_0(y)\big)^{-\frac{1}{2}}f\rangle \\
&\leq \langle \varepsilon^{-1}\big(\Delta + \frac{1}{r^2}V_0(y)\big)\big(\Delta + \frac{1}{r^2}V_0(y)\big)^{-\frac{1}{2}}f, \big(\Delta + \frac{1}{r^2}V_0(y)\big)^{-\frac{1}{2}}f\rangle \\
&= \varepsilon^{-1}\langle\big(\Delta + \frac{1}{r^2}V_0(y)\big)^{\frac{1}{2}}f, \big(\Delta + \frac{1}{r^2}V_0(y)\big)^{-\frac{1}{2}}f\rangle \\
&= \varepsilon^{-1}\langle f, f\rangle.
\end{aligned}$$

Therefore T is bounded on $L^2(M)$. □

6.3 The region R_1

As before, we break up the blown-up space into different regions. But now as we are studying the boundedness properties of the kernel, there's no reason why we must break up the blown-up space into the same regions as before when we were constructing the kernel. In fact, we will break it up differently.

Define

$$T_1 = \frac{2}{\pi}\int_0^\infty \nabla\Big((H+\lambda^2)^{-1}\Big(1 - \chi\big(\frac{8r}{r'}\big) - \chi\big(\frac{8r'}{r}\big)\Big)\Big)d\lambda, \tag{6.1}$$

where χ is the same cutoff function as in (5.31).

Let

$$G_1 = G\Big(1 - \chi(\frac{8r}{r'}) - \chi(\frac{8r'}{r})\Big).$$

(Note that the G_1 here is different from the G_1 is Chapter 5.) Let R_1 denote the support of G_1. This region is illustrated in Figure 6.1.

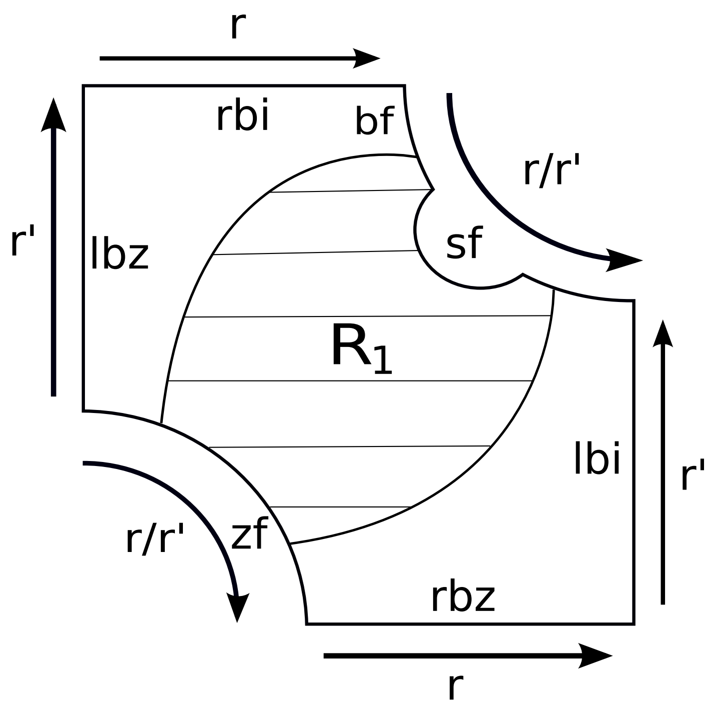


Figure 6.1: The region R_1

The kernel $T_1(z, z')$ satisfies the estimate

$$|T_1(z, z')| \lesssim \int_0^\infty \lambda^{d-2} \big|\nabla_z\big(G_1(\lambda z, \lambda z')\big)\big| d\lambda.$$

With this estimate we can show that T_1 is of weak type $(1, 1)$. For that we first need to estimate the derivatives of G_1.

Lemma 6.3.1. *Let* $\operatorname{dist}(z, z')$ *denote the distance between z and z' on M. In R_1 near bf, we have $\rho_{bf} \lesssim$* $\operatorname{dist}(z, z')^{-1}$*, where ρ_{bf} is a boundary defining function for bf.*

Proof. Let $z = (r, y)$ and $z' = (r', y')$. Observe from formula (5.1) that $\operatorname{dist}(z, z')$ is bounded above by $r + r'$. Therefore in the region R_1 we have

$$\operatorname{dist}(z, z')^{-1} \geq (r + r')^{-1} = r'^{-1}(1 + \frac{r}{r'})^{-1} \geq \frac{1}{9} r'^{-1}.$$

Note that r'^{-1} is a boundary defining function for bf. □

Lemma 6.3.2. *The kernel G_1 satisfies the estimate that for any integer $j \geq 0$, we have*

$$|\nabla_z^j G_1(z,z')| \lesssim \begin{cases} \mathrm{dist}(z,z')^{2-d-j}, & \mathrm{dist}(z,z') \leq 1, \\ \mathrm{dist}(z,z')^{-N}, & \mathrm{dist}(z,z') \geq 1, \end{cases}$$

for any $N > 0$.

Proof. Note that G_1 is supported in the region R_1, illustrated in Figure 6.1. By Theorem 5.7.7, $\rho_{zf}^{d-2} G_1$ is conormal of order -2 with respect to the diagonal, where ρ_{zf} is any boundary defining function for zf. Then by Lemma 5.2.2, near the diagonal we have

$$|G_1(z,z')| \lesssim \rho_{zf}^{2-d} a_{\mathrm{diag}}^{\frac{2-d}{2}}, \tag{6.2}$$

where

$$a_{\mathrm{diag}} = \frac{\mathrm{dist}(z,z')^2}{\varphi^2(r')}$$

is the quadratic defining function defined in Proposition 5.2.1. Note that $\varphi(r')$ is a boundary defining function for zf, so by substituting $\rho_{zf} = \varphi(r')$ into (6.2) we have

$$|G_1(z,z')| \lesssim \rho_{zf}^{2-d} a_{\mathrm{diag}}^{\frac{2-d}{2}} = \varphi(r')^{2-d} \left(\frac{\mathrm{dist}(z,z')}{\varphi(r')} \right)^{2-d} = \mathrm{dist}(z,z')^{2-d}.$$

Now let's consider the behaviour of G_1 near bf. By Theorem 5.7.7, we know that it vanishes to infinite order at bf. By Lemma 6.3.1, we know that $\rho_{bf} \lesssim \mathrm{dist}(z,z')^{-1}$. Therefore near bf we have

$$|G_1(z,z')| \lesssim \mathrm{dist}(z,z')^{-N},$$

for any $N > 0$.

The rest of R_1 is easy because after we take away the neighbourhoods near zf, sf, bf and the diagonal, we are left with a compact set, on which both G_1 and $\mathrm{dist}(z,z')^{-1}$ are continuous with $\mathrm{dist}(z,z')^{-1}$ being non-zero. Therefore we can conclude that

$$|G_1(z,z')| \lesssim \begin{cases} \mathrm{dist}(z,z')^{2-d}, & \mathrm{dist}(z,z') \leq 1, \\ \mathrm{dist}(z,z')^{-N}, & \mathrm{dist}(z,z') \geq 1, \end{cases}$$

for any $N > 0$. From here, by using the conormality of G at the diagonal and polyhomogeneous conormality of G at the boundary hypersurfaces, we obtain the estimates on $\nabla_z^j G_1(z,z')$, $j \geq 0$. □

Proposition 6.3.3. *The operator T_1 maps $L^1(M)$ into $L^{1,weak}(M)$.*

Proof. We just apply the Calderón-Zygmund theory; see [LG, Section 8.1.1]. It is sufficient to verify the following conditions:

(i) T_1 is bounded on $L^2(M)$;

(ii) $|T_1(z,z')| \leq \frac{C}{\big(\operatorname{dist}(z,z')\big)^d}$;

(iii) $|\nabla_z T_1(z,z')| \leq \frac{C}{\big(\operatorname{dist}(z,z')\big)^{d+1}}$ and $|\nabla_{z'} T_1(z,z')| \leq \frac{C}{\big(\operatorname{dist}(z,z')\big)^{d+1}}$,

for some constant $C > 0$.

We already know from Proposition 6.2.1 that T is bounded on $L^2(M)$. So to verify condition (i), we just need to show $T - T_1$ is bounded on $L^2(M)$, which is done in Proposition 6.4.8 in Section 6.4.

Now let's show conditions (ii) and (iii). By Lemma 6.3.2 we know that the kernel G_1 satisfies, for any $\lambda > 0$ and any $N > 0$,

$$\begin{aligned}\Big|\nabla_z\big(G_1(\lambda z,\lambda z')\big)\Big| \leq& \lambda\Big|(\nabla_z G_1)(\lambda z,\lambda z')\Big| \\ \lesssim& \begin{cases}\lambda^{2-d}\operatorname{dist}(z,z')^{1-d}, & \lambda\operatorname{dist}(z,z') \leq 1,\\ \lambda^{-N+1}\operatorname{dist}(z,z')^{-N}, & \lambda\operatorname{dist}(z,z') \geq 1,\end{cases}\end{aligned}$$

and

$$\begin{aligned}\Big|\nabla_z^2\big(G_1(\lambda z,\lambda z')\big)\Big| \leq& \lambda^2\Big|(\nabla_z^2 G_1)(\lambda z,\lambda z')\Big| \\ \lesssim& \begin{cases}\lambda^{2-d}\operatorname{dist}(z,z')^{-d}, & \lambda\operatorname{dist}(z,z') \leq 1,\\ \lambda^{-N+2}\operatorname{dist}(z,z')^{-N}, & \lambda\operatorname{dist}(z,z') \geq 1.\end{cases}\end{aligned}$$

We use this to estimate $T_1(z,z')$,

$$
\begin{aligned}
|T_1(z,z')| &\lesssim \int_0^\infty \lambda^{d-2}\big|\nabla_z\big(G_1(\lambda z,\lambda z')\big)\big|d\lambda \\
&\lesssim \int_0^{\frac{1}{\operatorname{dist}(z,z')}} \operatorname{dist}(z,z')^{1-d}d\lambda + \int_{\frac{1}{\operatorname{dist}(z,z')}}^\infty \lambda^{d-N-1}\operatorname{dist}(z,z')^{-N}d\lambda \\
&= \operatorname{dist}(z,z')^{-d} + \operatorname{dist}(z,z')^{-N}\int_{\frac{1}{\operatorname{dist}(z,z')}}^\infty \lambda^{d-N-1}d\lambda \\
&= \operatorname{dist}(z,z')^{-d} + \operatorname{dist}(z,z')^{-d} \qquad\qquad (\text{Choose } N = d+1) \\
&= \frac{2}{\operatorname{dist}(z,z)^d}.
\end{aligned}
$$

Now estimate the derivative with respect to z. The z' case is similar.

$$
\begin{aligned}
|\nabla_z T_1(z,z')| &= \frac{2}{\pi}\left|\int_0^\infty \lambda^{d-2}\nabla_z\nabla_z\big(G_1(\lambda z,\lambda z')\big)d\lambda\right| \\
&\lesssim \int_0^\infty \lambda^{d-2}\big|\nabla_z^2\big(G_1(\lambda z,\lambda z')\big)\big|d\lambda \\
&\lesssim \int_0^{\frac{1}{\operatorname{dist}(z,z')}} \operatorname{dist}(z,z')^{-d}d\lambda + \int_{\frac{1}{\operatorname{dist}(z,z')}}^\infty \lambda^{d-N}\operatorname{dist}(z,z')^{-N}d\lambda \\
&= \operatorname{dist}(z,z')^{-d-1} + \operatorname{dist}(z,z')^{-N}\int_{\frac{1}{\operatorname{dist}(z,z')}}^\infty \lambda^{d-N}d\lambda \\
&= \operatorname{dist}(z,z')^{-d-1} + \operatorname{dist}(z,z')^{-d-1} \qquad\qquad (\text{Choose } N = d+2) \\
&= \frac{2}{\operatorname{dist}(z,z')^{d+1}}.
\end{aligned}
$$

This completes the proof. $\square$

By interpolation, we obtain the following proposition.

Proposition 6.3.4. *The operator T_1 is bounded on $L^p(M)$ for any $p > 1$.*

Proof. By Marcinkiewicz Interpolation Theorem, we know that T_1 is bounded on $L^p(M)$ for all $1 < p \leq 2$. Then by considering the adjoint operator, we obtain the boundedness for all $p > 1$. $\square$

6.4 Regions R_2 and R_3

We make the following definitions

$$T_2 = \frac{2}{\pi}\int_0^\infty \nabla\Big((H+\lambda^2)^{-1}\chi\Big(\frac{8r}{r'}\Big)\Big)d\lambda,$$

$$T_3 = \frac{2}{\pi}\int_0^\infty \nabla\Big((H+\lambda^2)^{-1}\chi\Big(\frac{8r'}{r}\Big)\Big)d\lambda,$$

and

$$G_2 = G\chi\Big(\frac{8r}{r'}\Big),$$

$$G_3 = G\chi\Big(\frac{8r'}{r}\Big),$$

where χ is the same cutoff function as in (5.31) and (6.1). Let R_2 and R_3 denote the supports of G_2 and G_3 respectively. The two regions are illustrated in Figure 6.2.

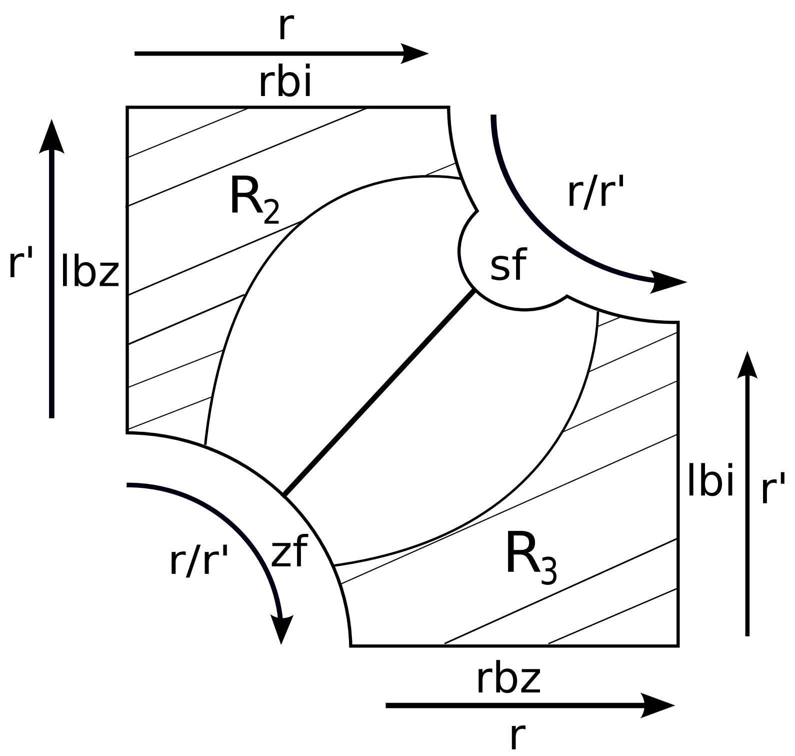


Figure 6.2: The regions R_2 and R_3

To study the boundedness of T_2 and T_3, the following lemmas will be useful. They are similar to [HS, Lemma 5.4] but not covered by it.

Lemma 6.4.1. *Consider the kernel $K(r,r')$ defined by*

$$K(r,r') = \begin{cases} r^{-\alpha} r'^{-\beta}, & r \le r', \\ 0, & r > r'. \end{cases}$$

If $\alpha+\beta = d$, $\beta > 0$, and p satisfies

$$p < \frac{d}{\max(\alpha, 0)}, \tag{6.3}$$

then K is bounded as an operator on $L^p(\mathbb{R}_+; r^{d-1}dr)$.

Proof. To find out for what p the operator with kernel $K(r,r')$ is bounded on $L^p(\mathbb{R}_+, r^{d-1}dr)$, we consider the isometry $M : L^p(\mathbb{R}_+, r^{d-1}dr) \to L^p(\mathbb{R}_+, r^{-1}dr)$ defined by

$$(Mf)(r) = r^{\frac{d}{p}} f(r).$$

Then the kernel of the operator $\tilde{K} = MKM^{-1} : L^p(\mathbb{R}_+, r^{-1}dr) \to L^p(\mathbb{R}_+, r^{-1}dr)$ is

$$\tilde{K}(r,r') = r^{\frac{d}{p}} r'^{d-\frac{d}{p}} K(r,r') = (\frac{r}{r'})^{-\alpha+\frac{d}{p}} \chi_{\{r \le r'\}}.$$

Perform the substitutions $s = \ln r$ and $s' = \ln r'$, then $\tilde{K}(s,s')$ is an operator on $L^p(\mathbb{R}, ds)$, and

$$\tilde{K}(s,s') = e^{(-\alpha+\frac{d}{p})(s-s')} \chi_{\{s-s' \le 0\}}.$$

This is a convolution operator, so it is bounded provided the kernel is an L^1-function with variable $s-s'$. Since $s - s' \le 0$, it means we want $-\alpha + \frac{d}{p} > 0$. That is,

$$p < \frac{d}{\max(\alpha, 0)}.$$

Since we also want $p > 1$, we require $\alpha < d$, ie $\beta > 0$. $\square$

Corollary 6.4.2. *Let $K(r,r',y,y')$ be a kernel on the cone M satisfying*

$$|K(r,r',y,y')| \le \begin{cases} r^{-\alpha} r'^{-\beta}, & r \le r', \\ 0, & r > r'. \end{cases}$$

If $\alpha+\beta = d$, $\beta > 0$, and p satisfies

$$p < \frac{d}{\max(\alpha, 0)}, \tag{6.4}$$

then K is bounded as an operator on $L^p(M; r^{d-1}drdh)$.

Proof. It follows from Lemma 6.4.1 and the fact that the cross section Y has finite volume. $\square$

Lemma 6.4.3. *Consider the kernel $K(r, r')$ defined by*

$$K(r, r') = \begin{cases} 0 & r \leq r', \\ r^{-\alpha} r^{-\beta}, & r > r'. \end{cases}$$

If $\alpha + \beta = d$, $\alpha > 0$, and p satisfies

$$p > \frac{d}{\min(\alpha, d)}, \tag{6.5}$$

then K is bounded as an operator on $L^p(\mathbb{R}_+; r^{d-1}dr)$.

Proof. By duality and Lemma 6.4.1. $\square$

Similarly as before, Lemma 6.4.3 also has a corollary about the boundedness of operators on the cone M.

Corollary 6.4.4. *Let $K(r, r', y, y')$ be a kernel on the cone M satisfying*

$$|K(r, r', y, y')| \leq \begin{cases} 0 & r \leq r', \\ r^{-\alpha} r^{-\beta}, & r > r'. \end{cases}$$

If $\alpha + \beta = d$, $\alpha > 0$, and p satisfies

$$p > \frac{d}{\min(\alpha, d)}, \tag{6.6}$$

then K is bounded as an operator on $L^p(M; r^{d-1}drdh)$.

Proposition 6.4.5. *The operator T_2 is bounded on $L^p(M)$ for*

$$p < \frac{d}{\max(\frac{d}{2} - \mu_0, 0)},$$

where $\mu_0 > 0$ is the square root of the smallest eigenvalue of the operator $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2$.

Proof. Consider G_2, which is supported in the region R_2, shown in Figure 6.2. We will use the following three boundary defining functions

$$\rho_{zf} = r', \;\; \rho_{lbz} = \frac{r}{r'}, \;\; \rho_{rbi} = \langle r' \rangle^{-1}.$$

By Theorem 5.7.7, we have

$$|G_2(r, r', y, y')| \lesssim \rho_{zf}^{2-d} \rho_{lbz}^{1-\frac{d}{2}+\mu_0} \rho_{rbi}^{\infty} = r^{1-\frac{d}{2}+\mu_0} r'^{1-\frac{d}{2}-\mu_0} \langle r' \rangle^{-\infty}, \tag{6.7}$$

where $\mu_0 > 0$ is the square root of the smallest eigenvalue of the operator $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2$. From here we know that

$$|G_2(\lambda r, \lambda r', y, y')| \lesssim \begin{cases} \lambda^{2-d} r^{1-\frac{d}{2}+\mu_0} r'^{1-\frac{d}{2}-\mu_0}, & \lambda \leq \frac{1}{r'}, \\ \lambda^{1-\frac{d}{2}+\mu_0-N} r^{1-\frac{d}{2}+\mu_0} r'^{-N}, & \lambda \geq \frac{1}{r'}, \end{cases}$$

for all $N > 0$. That means, by polyhomogeneous conormality of G_2, if $\mu_0 \neq \frac{d}{2} - 1$, we have

$$\left| \nabla_{(r,y)} \big(G_2(\lambda r, \lambda r', y, y') \big) \right| \lesssim \begin{cases} \lambda^{2-d} r^{-\frac{d}{2}+\mu_0} r'^{1-\frac{d}{2}-\mu_0}, & \lambda \leq \frac{1}{r'}, \\ \lambda^{1-\frac{d}{2}+\mu_0-N} r^{-\frac{d}{2}+\mu_0} r'^{-N}, & \lambda \geq \frac{1}{r'}, \end{cases} \tag{6.8}$$

for all $N > 0$. For the case $\mu_0 = \frac{d}{2} - 1$, the right hand side of (6.7) doesn't depend on r at all, so when we differentiate it with respect to r, it becomes 0. However in this case (6.8) still holds.

We can now estimate T_2,

$$
\begin{aligned}
|T_2(r,r',y,y')| &\lesssim \int_0^\infty \lambda^{d-2}\big|\nabla_{(r,y)}\big(G_2(\lambda r,\lambda r',y,y')\big)\big|d\lambda \\
&\lesssim \int_0^{\frac{1}{r'}} \lambda^{d-2}\big(\lambda^{2-d} r^{-\frac{d}{2}+\mu_0} r'^{1-\frac{d}{2}-\mu_0}\big)d\lambda \\
&\qquad + \int_{\frac{1}{r'}}^{\frac{1}{r}} \lambda^{d-2}\big(\lambda^{1-\frac{d}{2}+\mu_0-N} r^{-\frac{d}{2}+\mu_0} r'^{-N}\big)d\lambda \\
&= r^{-\frac{d}{2}+\mu_0} r'^{1-\frac{d}{2}-\mu_0}\int_0^{\frac{1}{r'}} d\lambda + r^{-\frac{d}{2}+\mu_0} r'^{-N}\int_{\frac{1}{r'}}^{\frac{1}{r}} \lambda^{\frac{d}{2}+\mu_0-N-1}d\lambda \\
&= r^{-\frac{d}{2}+\mu_0} r'^{-\frac{d}{2}-\mu_0} + \frac{1}{\frac{d}{2}+\mu_0-N}\big(r^{N-d} r'^{-N} - r^{-\frac{d}{2}+\mu_0} r'^{-\frac{d}{2}-\mu_0}\big) \\
&\lesssim r^{-\frac{d}{2}+\mu_0} r'^{-\frac{d}{2}-\mu_0} \\
&\quad \text{(Choose } N=\mu_0+d \text{ so that the coefficient } \tfrac{1}{\frac{d}{2}+\mu_0-N} \text{ is negative.)} \\
&= \left(\frac{r}{r'}\right)^{\mu_0-\frac{d}{2}} r'^{-d}.
\end{aligned}
\tag{6.9}
$$

Since the support of T_2 is contained in the region $r \le r'$, we have

$$|T_2(r,r',y,y')| \lesssim \left(\frac{r}{r'}\right)^{\mu_0-\frac{d}{2}} r'^{-d}\chi_{\{r\le r'\}}.$$

By Corollary 6.4.2, we conclude that T_2 is bounded on $L^p(M)$ provided that

$$p < \frac{d}{\max(\frac{d}{2}-\mu_0, 0)}.$$

□

Remark 6.4.6. The special case $\mu_0 = \frac{d}{2} - 1$ corresponds to the case when the Riesz transform T has potential $V \equiv 0$. Later on in Section 6.5.2 we will improve estimate (6.8) to obtain a better result for this special case.

Proposition 6.4.7. *The operator T_3 is bounded on $L^p(M)$ for*

$$p > \frac{d}{\min(1+\frac{d}{2}+\mu_0, d)},$$

where $\mu_0 > 0$ is the square root of the smallest eigenvalue of the operator $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2$.

Proof. Consider G_3, which is supported in the region R_3, shown in Figure 6.2. We will use the following three boundary defining functions

$$\rho_{zf} = r, \ \ \rho_{rbz} = \frac{r'}{r}, \ \ \rho_{lbi} = \langle r \rangle^{-1}.$$

By Theorem 5.7.7, we have

$$|G_3(r, r', y, y')| \lesssim \rho_{zf}^{2-d} \rho_{rbz}^{1-\frac{d}{2}+\mu_0} \rho_{lbi}^{\infty} = r^{1-\frac{d}{2}-\mu_0} r'^{1-\frac{d}{2}+\mu_0} \langle r \rangle^{-\infty}, \tag{6.10}$$

where $\mu_0 > 0$ is the square root of the smallest eigenvalue of the operator $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2$. It follows that, as in the proof of Proposition 6.4.5, by the polyhomogeneous conormality of G_3,

$$\left|\nabla_{(r,y)}\big(G_3(\lambda r, \lambda r', y, y')\big)\right| \lesssim \begin{cases} \lambda^{2-d} r^{-\frac{d}{2}-\mu_0} r'^{1-\frac{d}{2}+\mu_0}, & \lambda \leq \frac{1}{r}, \\ \lambda^{-\frac{d}{2}+\mu_0-N+1} r^{-N-1} r'^{1-\frac{d}{2}+\mu_0}, & \lambda \geq \frac{1}{r}, \end{cases}$$

for all $N > 0$. Then

$$\begin{aligned}
|T_3(r, r', y, y')| &\lesssim \int_0^\infty \lambda^{d-2} \left|\nabla_{(r,y)}\big(G_3(\lambda r, \lambda r', y, y')\big)\right| d\lambda \\
&\lesssim \int_0^{\frac{1}{r}} \lambda^{d-2} \big(\lambda^{2-d} r^{-\frac{d}{2}-\mu_0} r'^{1-\frac{d}{2}+\mu_0}\big) d\lambda \\
&\qquad + \int_{\frac{1}{r}}^{\frac{1}{r'}} \lambda^{d-2} \big(\lambda^{-\frac{d}{2}+\mu_0-N+1} r^{-N-1} r'^{1-\frac{d}{2}+\mu_0}\big) d\lambda \\
&= r^{-\frac{d}{2}-\mu_0} r'^{1-\frac{d}{2}+\mu_0} \int_0^{\frac{1}{r}} d\lambda + r^{-N-1} r'^{1-\frac{d}{2}+\mu_0} \int_{\frac{1}{r}}^{\frac{1}{r'}} \lambda^{\frac{d}{2}+\mu_0-N-1} d\lambda \\
&= r^{-1-\frac{d}{2}-\mu_0} r'^{1-\frac{d}{2}+\mu_0} \\
&\qquad + \frac{1}{\frac{d}{2}+\mu_0-N} \big(r^{-N-1} r'^{N-d+1} - r^{-1-\frac{d}{2}-\mu_0} r'^{1-\frac{d}{2}+\mu_0}\big) \\
&\lesssim r^{-1-\frac{d}{2}-\mu_0} r'^{1-\frac{d}{2}+\mu_0} \\
&\text{(Choose } N = \mu_0 + d \text{ so that the coefficient } \tfrac{1}{\frac{d}{2}+\mu_0-N} \text{ is negative.)} \\
&= \left(\frac{r'}{r}\right)^{\mu_0-\frac{d}{2}+1} r^{-d}.
\end{aligned}$$

Since T_3 is supported in the region $r \geq r'$, we have

$$|T_3(r, r', y, y')| \lesssim \left(\frac{r'}{r}\right)^{\mu_0-\frac{d}{2}+1} r^{-d} \chi_{\{r \geq r'\}}.$$

By applying Corollary 6.4.4, we conclude that T_3 is bounded on $L^p(M)$ provided that

$$p > \frac{d}{\min(1 + \frac{d}{2} + \mu_0, d)}.$$

$\square$

Proposition 6.4.8. *The operator T_1 is bounded on $L^2(M)$.*

Proof. Since 2 satisfies the boundedness criteria in both Proposition 6.4.5 and Proposition 6.4.7, the operator $T_2 + T_3 = T - T_1$ is bounded on $L^2(M)$. The operator T is bounded on $L^2(M)$ by Proposition 6.2.1, and from here the boundedness of T_1 on $L^2(M)$ follows. $\square$

Remark 6.4.9. Proposition 6.4.8 completes the missing part in the proof of Proposition 6.3.3.

6.5 Main results

6.5.1 The characterisation of the boundedness of the Riesz transform T

Theorem 6.5.1. *Let M be a metric cone with dimension d and cross section Y. The Riesz transform T with the inverse square potential $V = \frac{V_0}{r^2}$ is bounded on $L^p(M)$ for p in the interval*

$$\left(\frac{d}{\min(1 + \frac{d}{2} + \mu_0, d)}, \frac{d}{\max(\frac{d}{2} - \mu_0, 0)} \right), \tag{6.11}$$

where $\mu_0 > 0$ is the square root of the smallest eigenvalue of the operator $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2$.

Moreover, for any $V \not\equiv 0$, the interval (6.11) characterises the boundedness of T, ie T is bounded on $L^p(M)$ if and only if p is in the interval (6.11).

Proof. Since $T = T_1 + T_2 + T_3$, we just combine Proposition 6.3.4, Proposition 6.4.5 and Proposition 6.4.7 to prove the first part of this theorem.

For the second part, with $V \not\equiv 0$, for any p outside the interval (6.11), we use a counter-example to show the unboundedness of T on $L^p(M)$. The manifold M in [GH, Theorem 1.5] is asymptotically conic, but the counter-example in its proof for the unboundedness can also be used here. The counter-example is the following function,

$$(\ln r)^{-1} r^{\mu_0 - \frac{d}{2}} \chi(r),$$

where $\chi : [0, \infty) \to [0, 1]$ is a smooth cutoff function such that $\chi\big([0, 2]\big) = 0$ and $\chi\big([3, \infty)\big) = 1$. For more details, see [GH, Section 5.2], and note that the variables x and x' in [GH] correspond to $\frac{1}{r}$ and $\frac{1}{r'}$ in this thesis. $\square$

- For constant zero potential, ie $V \equiv 0$, we have $\mu_0 = \frac{d}{2} - 1$, so the first part of Theorem 6.5.1 gives us the boundedness interval $(1, d)$. However, this is not the optimal interval. We will obtain a bigger interval in terms of μ_1, the square root of the second smallest eigenvalue of the operator $\Delta_Y + (\frac{d-2}{2})^2$, in Section 6.5.2.

- For a positive potential, ie $V \geq 0$ and $V \not\equiv 0$, we have $\mu_0 > \frac{d}{2} - 1$, so Theorem 6.5.1 tells us the lower threshold for the L^p boundedness is 1, while the upper threshold is strictly greater than d, and it is ∞ if $\mu_0 \geq \frac{d}{2}$.

- Now turn to the case of a negative potential, ie $V \leq 0$ and $V \not\equiv 0$. In this case we have $\mu_0 < \frac{d}{2} - 1$. So Theorem 6.5.1 tells us the lower threshold for the L^p boundedness is strictly greater than 1, and the upper threshold is strictly between 2 and d. For the Euclidean space $\mathbb{R}^d$, the lower threshold was obtained by J. Assaad in [JA], but in that paper she didn't obtain boundedness for any $p > 2$ for negative inverse square potentials; see the end of Section 1.3 for further discussion.

An immediate application of Theorem 6.5.1 is to show that the converse of the second part of [GH, Theorem 1.5], ie the converse of Theorem 1.1.2, is also true. As suggested by [GH, Remark 1.7], Theorem 6.5.1, combined with other results in [GH], gives us the converse. Therefore we have Theorem 6.5.2. Here M_0 is the interior of a compact manifold with boundary M, with boundary defining function x. The metric on M has the form

$$\frac{dx^2}{x^4} + \frac{h(x)}{x^2},$$

in a collar neighbourhood of ∂M, where $h(x)$ is a family of metrics on ∂M. Here $r = \frac{1}{x}$ behaves like the radial coordinate on the cone over ∂M: the metric in terms of r reads $g = dr^2 + r^2 h(\frac{1}{r})$, so is asymptotic to the conic metric $dr^2 + r^2 h(0)$ as

$r \to \infty$. In [GH], potentials of the form $V \in x^2 C^\infty(M)$ were considered; that is, the potentials decay as r^{-2} at infinity, and the limiting "potential at infinity" V_0 was defined by $V_0 := x^{-2}V|_{\partial M}$.

Theorem 6.5.2. *Let $d \geq 3$, and (M_0, g) be an asymptotically conic manifold with dimension d. Consider the operator $P = \Delta_g + V$ with V satisfying*

$$\Delta_{\partial M} + V_0 + \left(\frac{d-2}{2}\right)^2 > 0 \text{ where } V_0 = \left.\frac{V}{x^2}\right|_{\partial M}.$$

Suppose that P has no zero modes or zero resonances, and that $V_0 \not\equiv 0$, then the Riesz transform $\nabla P^{-\frac{1}{2}}$ is bounded on $L^p(M)$ if and only if p is in the interval

$$\left(\frac{d}{\min(\frac{d}{2} + 1 + \mu_0, d)}, \frac{d}{\max(\frac{d}{2} - \mu_0, 0)}\right), \tag{6.12}$$

where $\mu_0 > 0$ is the square root of the smallest eigenvalue of the operator $\Delta_{\partial M} + V_0 + (\frac{d-2}{2})^2$.

6.5.2 The case $V \equiv 0$

If the potential $V \equiv 0$, then the square root of the first eigenvalue $\mu_0 = \frac{d}{2} - 1$, and the first eigenfunction u_0 is a constant function. In this case we can obtain a bigger interval for the boundedness of T than (6.11). The following result has been proved in [HQL] by H.-Q. Li. Here we will provide a different proof, using the methods introduced in this thesis.

Theorem 6.5.3. *Let M be a metric cone with dimension d and cross section Y. The Riesz transform $T = \nabla\Delta^{-\frac{1}{2}}$ is bounded on $L^p(M)$ if and only if p is in the interval*

$$\left(1, \frac{d}{\max(\frac{d}{2} - \mu_1, 0)}\right), \tag{6.13}$$

where $\mu_1 > 0$ is the square root of the second smallest eigenvalue of the operator $\Delta_Y + (\frac{d-2}{2})^2$.

Remark 6.5.4. If M is the Euclidean space, ie $M = \mathbb{R}^d$, then $Y = \mathbb{S}^{d-1}$. Since the second smallest eigenvalue of $\Delta_{\mathbb{S}^{d-1}}$ is $d-1$, we know that in this case $\mu_1^2 = d - 1 + (\frac{d-2}{2})^2 = \frac{d^2}{4}$, ie $\mu_1 = \frac{d}{2}$. Then Theorem 6.5.3 tells us that the Riesz transform on $\mathbb{R}^d$ with potential $V \equiv 0$ is bounded on $L^p(\mathbb{R}^d)$ for all $1 < p < \infty$.

Proof. For the unboundedness, just use the same counter-example mentioned in the proof of Theorem 6.5.1, so we focus on showing that for any p in the interval (6.13), T is bounded on $L^p(M)$. The lower threshold 1 is obtained by applying Theorem 6.5.1 with $\mu_0 = \frac{d}{2} - 1$. For the upper threshold, we again consider G_2, as in the proof of Proposition 6.4.5. However, here estimate (6.7) becomes

$$|G_2(r, r', y, y')| \lesssim r'^{2-d} \langle r' \rangle^{-\infty},$$

as the power of r is 0. Since the right hand side doesn't depend on r at all, when we differentiate it with respect to r, instead of getting an estimate involving r like (6.8), the right hand side simply becomes 0, which is not useful.

Let's improve estimate (6.7) for this case $\mu_0 = \frac{d}{2} - 1$, so that we can obtain a more useful estimate on $\nabla_{(r,y)}\big(G_2(\lambda r, \lambda r', y, y')\big)$. As suggested in Remark 5.7.10, we should look at the second term of the expansion of $I_{\mu_0}(r)$, and the first term of the expansion of $I_{\mu_1}(r)$. They give us vanishing orders 2 and $1 - \frac{d}{2} + \mu_1$ respectively. Therefore the improved estimate on G_2 is,

$$\begin{aligned} |G_2(r, r', y, y')| &\lesssim \rho_{zf}^{2-d} \rho_{lbz}^{2} \rho_{rbi}^{\infty} + \rho_{zf}^{2-d} \rho_{lbz}^{1-\frac{d}{2}+\mu_1} \rho_{rbi}^{\infty} \\ &= r^2 r'^{-d} \langle r' \rangle^{-\infty} + r^{1-\frac{d}{2}+\mu_1} r'^{1-\frac{d}{2}-\mu_1} \langle r' \rangle^{-\infty}. \end{aligned}$$

Hence instead of estimate (6.8), we now have

$$\big|\nabla_{(r,y)}\big(G_2(\lambda r, \lambda r', y, y')\big)\big| \lesssim \begin{cases} \lambda^{2-d} r r'^{-d} + \lambda^{2-d} r^{-\frac{d}{2}-\mu_1} r'^{1-\frac{d}{2}-\mu_1}, & \lambda \le \frac{1}{r'}, \\ \lambda^{2-N} r r'^{-N} + \lambda^{1-\frac{d}{2}+\mu_1-N} r^{-\frac{d}{2}+\mu_1} r'^{-N}, & \lambda \ge \frac{1}{r'}, \end{cases} \tag{6.14}$$

for all $N > 0$.

We use estimate (6.14) to perform a calculation similar to (6.9), and we get

$$|T_2(r, r', y, y')| \lesssim \left(\left(\frac{r}{r'}\right) r'^{-d} + \left(\frac{r}{r'}\right)^{\mu_1 - \frac{d}{2}} r'^{-d} \right) \chi_{\{r \le r'\}}.$$

Then by Corollary 6.4.2, we obtain that T_2 is bounded on $L^p(M)$ provided that

$$p < \frac{d}{\max(\frac{d}{2} - \mu_1, 0)}.$$

This completes the proof. $\square$

Remark 6.5.5. Since $u_1 > u_0 = \frac{d}{2} - 1$, we have $\frac{d}{2} - \mu_1 < 1$. That means the upper threshold for p is strictly greater than d, so T is bounded on $L^d(M)$. Also note that if $u_1 \geq \frac{d}{2}$, T is a bounded operator on $L^p(M)$ for all $1 < p < \infty$.

6.5.3 Constant V_0

An especially interesting case is when the function V_0 is a constant function, ie $V_0 \equiv c$ for some $c \in \mathbb{R}$. We have already discussed the case $c = 0$ in Section 6.5.2. Here we consider $c \neq 0$, and write the boundedness interval in terms of the constant c.

Proposition 6.5.6. *Let M be a metric cone with dimension d and cross section Y. The Riesz transform $T = \nabla(\Delta + \frac{c}{r^2})^{-\frac{1}{2}}$, where $c > -(\frac{d-2}{2})^2$ and $c \neq 0$, is bounded on $L^p(M)$ if and only if p is in the interval*

$$\left(\frac{2d}{\min(d+2+\sqrt{(d-2)^2+4c}, 2d)}, \frac{2d}{\max(d-\sqrt{(d-2)^2+4c}, 0)} \right). \tag{6.15}$$

Proof. Apply Theorem 6.5.1, and note that $\mu_0^2 = (\frac{d-2}{2})^2 + c$. □

Remark 6.5.7. From the expression (6.15) we can easily see that

- when $c \geq d-1$, T is bounded on $L^p(M)$ for all $p \in (1, \infty)$;
- for $0 < c < d-1$, the lower threshold for the L^p boundedness is 1, and the upper threshold is strictly greater than d;
- while for $-(\frac{d-2}{2})^2 < c < 0$, the lower threshold is strictly greater than 1, and the upper threshold is strictly between 2 and d.

Remark 6.5.8. For the case $-(\frac{d-2}{2})^2 < c < 0$, our lower threshold in (6.15) is the same as the lower threshold in (1.10) obtained by J. Assaad in [JA]. To see that, take

$$\varepsilon = -\frac{c + (\frac{d-2}{2})^2}{c} > 0.$$

That means

$$\Delta_Y + (1+\varepsilon)c + \left(\frac{d-2}{2}\right)^2 \geq 0.$$

By the calculations in the proof of Proposition 5.1.1 we know that means

$$\Delta + (1+\varepsilon)\frac{c}{r^2} \geq 0.$$

Therefore ε satisfies the condition of Theorem 1.3.2, with $V = \frac{c}{r^2}$, and it's obviously the biggest such ε, so we substitute it into the lower threshold in (1.10),

$$\frac{2d}{d+2+(d-2)\sqrt{\frac{\varepsilon}{\varepsilon+1}}} = \frac{2d}{d+2+(d-2)\sqrt{1+c(\frac{2}{d-2})^2}} = \frac{2d}{d+2+\sqrt{(d-2)^2+4c}},$$

which is the same as our lower threshold in (6.15). In the same paper, J. Assaad also gave a counter-example showing that T is unbounded for p greater than our upper threshold in (6.15). However, our boundedness result for $p > 2$ for negative inverse square potentials is new.

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