

New progress on Grothendieck duality, explained to those familiar with category theory and with algebraic geometry

Amnon Neeman

ABSTRACT

Much has been written about Grothendieck duality. This survey will make the point that most of this literature is now obsolete: there is a brilliant 1968 article by Verdier with the right idea on how to approach the subject. Verdier’s article was largely forgotten for two decades until Lipman resurrected it, reworked it and developed the ideas to obtain the right statements for what had before been a complicated theory.

For the reader’s benefit, Sections 1 through 5, which present the (large) portion of the theory that can nowadays be obtained from formal nonsense about rigidly compactly generated tensor triangulated categories, are all post-Verdier. The major landmarks in developing this approach were a 1996 article by me which was later generalized and improved on by Balmer, Dell’Ambrogio and Sanders, and a much more recent article of mine about improvements to the Verdier base-change theorem and the functor $f^!$. Section 6 is where Verdier’s 1968 ideas still play a pivotal role, but in the cleaned-up version due to Lipman and with new, short and direct proofs.

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1. Introduction

This article is about Grothendieck’s duality theorem, but for the sake of the introduction we begin with an older result due to Serre [14]. First we establish

NOTATION 1.1. Let X be a compact, connected, n -dimensional complex manifold and let $\mathcal{V}$ be a holomorphic vector bundle on X . We can form the twisted dual vector bundle $\mathcal{H}om(\mathcal{V}, \Omega_X^n)$, where Ω_X^n is the line bundle of holomorphic n -forms on X . The cup product in cohomology gives the first map in the composite

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$$\begin{array}{ccc}
H^i(\mathcal{V}) \otimes H^{n-i}[\mathcal{H}\text{om}(\mathcal{V}, \Omega_X^n)] & \xrightarrow{\cup} & H^n[\mathcal{V} \otimes \mathcal{H}\text{om}(\mathcal{V}, \Omega_X^n)] \\
& & \downarrow H^n(\text{ev}) \\
\mathbb{C} & \xleftarrow{\text{residue}} & H^n(\Omega_X^n)
\end{array}$$

The second map is induced by applying the functor H^n to the natural evaluation map $\text{ev} : \mathcal{V} \otimes \mathcal{H}\text{om}(\mathcal{V}, \Omega_X^n) \rightarrow \Omega_X^n$, and the map $H^n(\Omega_X^n) \rightarrow \mathbb{C}$ is the homomorphism taking a meromorphic differential form to its residue.

EXPLANATION 1.2. We need to explain to the reader what is the traditional definition of the ‘residue map’ of Notation 1.1. The modern explanation will come later in this article — in some ways this is the crux of the recent progress, we have finally arrived at a clear understanding of the residue map.

Traditionally the approach was to define a large vector space V , whose elements are finite sums of meromorphic n -forms at the points of X . And then one needs to do two things to define the residue map $H^n(\Omega_X^n) \rightarrow \mathbb{C}$.

- (i) Produce a surjection of $\mathbb{C}$ -vector spaces $\varphi : V \rightarrow H^n(\Omega_X^n)$.
- (ii) Show that the map taking an element of V to its residue kills the kernel of φ and therefore induces a well-defined map $H^n(\Omega_X^n) \rightarrow \mathbb{C}$.

For the sake of clarity, in (ii) above, an element of V is a sum, over finitely many points $x \in X$, of meromorphic forms which in local coordinates can be written as $\mathbb{C}$ -linear combinations of the basis elements

$$\left(\frac{dx_1}{x_1^{m_1}}\right) \wedge \left(\frac{dx_2}{x_2^{m_2}}\right) \wedge \cdots \wedge \left(\frac{dx_{n-1}}{x_{n-1}^{m_{n-1}}}\right) \wedge \left(\frac{dx_n}{x_n^{m_n}}\right),$$

where $m_i \in \mathbb{Z}$ are arbitrary. The residue of an element of V is defined to be the (finite) sum over the relevant points $x \in X$ of the residue at x . And the residue at x of the meromorphic n -form which expands, in local coordinates, to a $\mathbb{C}$ -linear combination of the basis elements

$$\left(\frac{dx_1}{x_1^{m_1}}\right) \wedge \left(\frac{dx_2}{x_2^{m_2}}\right) \wedge \cdots \wedge \left(\frac{dx_{n-1}}{x_{n-1}^{m_{n-1}}}\right) \wedge \left(\frac{dx_n}{x_n^{m_n}}\right)$$

is just the coefficient of the basis vector where $m_i = 1$ for all $1 \leq i \leq n$.

With the notation under our belt, we arrive at Serre’s duality theorem.

THEOREM 1.3. *The pairing $H^i(\mathcal{V}) \otimes H^{n-i}[\mathcal{H}\text{om}(\mathcal{V}, \Omega_X^n)] \rightarrow \mathbb{C}$ of Notation 1.1, with the residue map as in Explanation 1.2, is non-degenerate. Thus the finite-dimensional vector spaces $H^i(\mathcal{V})$ and $H^{n-i}[\mathcal{H}\text{om}(\mathcal{V}, \Omega_X^n)]$ are canonically dual to each other.*

Serre’s theorem works for any compact, connected complex manifold X . As far as I know the existing literature on Grothendieck duality deals only with the algebraic setting, meaning that Serre’s theorem is not strictly speaking a special case of any of the documented results[†]. But it almost is: the idea of Grothendieck duality is to come up with a relative version of Serre’s theorem, that is a version that applies to morphisms of schemes $f : X \rightarrow Y$, and such that the scheme version of Theorem 1.3 is the special case where $Y = \text{Spec}(\mathbb{C})$ and f is smooth and proper.

[†]This should change soon, the work of Clausen and Scholze on analytic geometry develops machinery that promises to allow Grothendieck duality to be generalized to the complex analytic setting.

The author has written a much longer and gentler survey introducing the cleaned-up, modern approach to Grothendieck duality. If, after finishing the current, short survey, the reader would like to see how Serre's old theorem follows from the more general theory, then the place to look is [13, Section 2.3].

It is time to introduce Grothendieck duality in something approaching the right generality.

2. Rigidly compactly generated tensor triangulated categories and their properties

Since not all readers will be familiar with this, we briefly recall the formal properties of rigidly compactly generated tensor triangulated categories — the reader can find much more in Balmer, Dell'Ambrogio and Sanders [3]. This falls within the more general area of tensor triangular geometry, for surveys of this rich field the reader is referred to Balmer [1, 2], with the caveat that there continues to be considerable progress in this active discipline.

But first we remember the older and even more basic notion of compact generation in triangulated categories.

REMINDER 2.1. A triangulated category $\mathcal{T}$ is called *compactly generated* if:

- (i) $\mathcal{T}$ contains a coproduct of any set of its objects;
- (ii) there is a set G of compact objects of $\mathcal{T}$ such that every non-zero object $t \in \mathcal{T}$ admits a non-zero map $g \rightarrow t$ with $g \in G$.

Remember that an object $g \in \mathcal{T}$ is *compact* if the functor $\mathrm{Hom}_{\mathcal{T}}(g, -) : \mathcal{T} \rightarrow \mathcal{A}b$ respects coproducts.

REMINDER 2.2. Now suppose $\mathcal{T}$ is a monoidal category. Recall that this means there is a tensor product $\mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$, which takes a pair of objects $t_1, t_2 \in \mathcal{T}$ to the object $t_1 \otimes t_2 \in \mathcal{T}$, in such a way that associativity holds up to canonical isomorphism and for some unit $\mathbb{1} \in \mathcal{T}$ we have $\mathbb{1} \otimes t \cong t \cong t \otimes \mathbb{1}$. If $\mathcal{T}$ is both triangulated and monoidal, we can ask for the two structures to be compatible, meaning that both $t \otimes (-)$ and $(-) \otimes t$ are functors taking triangles to triangles. And we will make the extra assumption that our tensor products are all symmetric.

Any such triangulated category is called a *tensor triangulated category* and, as we have already mentioned, these have been and are being studied extensively by a growing community of tensor-triangular geometers; we refer the reader again to Balmer's surveys [1, 2].

In any tensor triangulated category, one has the full subcategory of *rigid* objects (also called *strongly dualizable objects*). An object $t \in \mathcal{T}$ is rigid if there exists an object $t^\vee$ and morphisms $\rho : \mathbb{1} \rightarrow t \otimes t^\vee$ and $\sigma : t^\vee \otimes t \rightarrow \mathbb{1}$ so that the two composites below are identities

$$t \xrightarrow{\rho \otimes \mathrm{id}} t \otimes t^\vee \otimes t \xrightarrow{\mathrm{id} \otimes \sigma} t \qquad t^\vee \xrightarrow{\mathrm{id} \otimes \rho} t^\vee \otimes t \otimes t^\vee \xrightarrow{\sigma \otimes \mathrm{id}} t^\vee$$

A tensor triangulated category $\mathcal{T}$ is called *rigidly compactly generated* if it is compactly generated as in Reminder 2.1, if the tensor product commutes with coproducts, and in addition if the compact objects in $\mathcal{T}$ are precisely the rigid objects.

EXAMPLE 2.3. If X is a quasicompact, quasiseparated scheme, then the category $\mathbf{D}_{\mathrm{qc}}(X)$ is a rigidly compactly generated tensor triangulated category. The objects in $\mathbf{D}_{\mathrm{qc}}(X)$ are the cochain complexes of sheaves of $\mathcal{O}_X$ -modules on X with quasicoherent cohomology. And the compact objects, which are the same as the rigid objects, are precisely the perfect complexes on X .

It is interesting to study functors between such categories. The following is an easy consequence of the results that may be found in Balmer, Dell’Ambrogio and Sanders [3]; see also [11] for an older account.

LEMMA 2.4. Suppose $\mathcal{S}$ and $\mathcal{T}$ are both rigidly compactly generated tensor triangulated categories. Let $f^* : \mathcal{S} \rightarrow \mathcal{T}$ be a strong monoidal functor[†] respecting coproducts. Then f^* has a right adjoint f_* which has a right adjoint $f^\times$. In symbols we have adjunctions $f^* \dashv f_* \dashv f^\times$. Moreover, the following facts are formal.

(i) The projection formula holds for the pair (f^*, f_*) . This means that, for any $s \in \mathcal{S}$ and $t \in \mathcal{T}$, the natural map $s \otimes f_* t \rightarrow f_*(f^* s \otimes t)$ is an isomorphism.

(ii) Suppose the functor $f_* : \mathcal{T} \rightarrow \mathcal{S}$ sends compact objects in $\mathcal{T}$ to compact objects in $\mathcal{S}$. Then the functor $f^\times$ respects coproducts, and moreover there is an isomorphism, natural in $(-) \in \mathcal{S}$, of the form $f^\times(\mathbb{1}) \otimes f^*(-) \rightarrow f^\times(-)$.

(iii) Write $\varepsilon : f_* f^\times \rightarrow \text{id}$ for the counit of the adjunction $f_* \dashv f^\times$, and let $s \in \mathcal{S}$ be any object. Then the following pentagon commutes

$$\begin{array}{ccc}
 & f_* f^\times s & \xrightarrow{\varepsilon} s \\
 f_*(\text{ii}) \nearrow & & \downarrow \wr \\
 f_*(f^\times \mathbb{1} \otimes f^* s) & & \\
 \nwarrow \text{(i)} & (f_* f^\times \mathbb{1}) \otimes s & \xrightarrow{\varepsilon \otimes \text{id}} \mathbb{1} \otimes s
 \end{array}$$

where the vertical map is the obvious isomorphism. The arrow labeled (i) is obtained by applying the isomorphism $f_*(A) \otimes B \rightarrow f_*(A \otimes f^* B)$ of (i) above, with $A = f^\times \mathbb{1}$ and with $B = s$. The arrow labeled $f_*(\text{ii})$ is by applying f_* to the isomorphism $f^\times \mathbb{1} \otimes f^* s \rightarrow f^\times s$ of (ii) above.

SUMMARY 2.5. In this section, we have worked in the very general setting of ‘rigidly compactly generated tensor triangulated categories’, and strong monoidal functors between them that respect coproducts. Given any such functor $f^* : \mathcal{S} \rightarrow \mathcal{T}$, we may form adjoints $f^* \dashv f_* \dashv f^\times$. And if f_* respects compact objects, we have a canonical isomorphism $f^\times(-) \cong f^\times(\mathbb{1}) \otimes f^*(-)$. Moreover the pentagon of Lemma 2.4(iii) tells us that, up to the natural isomorphisms given by the vertical and slanted arrows, the top row is isomorphic to the bottom row. That is the counit of adjunction $\varepsilon : f_* f^\times s \rightarrow s$ is canonically isomorphic to the tensor product of s with the counit of adjunction $\varepsilon : f_* f^\times \mathbb{1} \rightarrow \mathbb{1}$.

EXAMPLE 2.6. Let $f : X \rightarrow Y$ be a morphism of quasicompact, quasiseparated schemes, and let $f^* : \mathbf{D}_{\text{qc}}(Y) \rightarrow \mathbf{D}_{\text{qc}}(X)$ be the derived pullback functor. Then f^* is an example, the hypotheses of Lemma 2.4 hold. Therefore the functor f^* admits a right adjoint with a right adjoint, in other words we obtain adjoints $f^* \dashv f_* \dashv f^\times$.

Suppose now that X and Y are Noetherian schemes, and $f : X \rightarrow Y$ is proper and of finite Tor-dimension. Then there is an old theorem of Illusie telling us that f_* takes perfect complexes to perfect complexes — see [6, Proposition 3.7]. Thus for proper morphisms of

[†]A strong monoidal functor $f^* : \mathcal{S} \rightarrow \mathcal{T}$ is a functor respecting the monoidal structure, meaning it comes with isomorphisms $f^*(s \otimes s') \cong f^*(s) \otimes f^*(s')$ natural in $s, s' \in \mathcal{S}$, and an isomorphism $f^*(\mathbb{1}) \cong \mathbb{1}$, and these satisfy the obvious compatibilities. There is also something called a ‘lax monoidal functor’ and something called an ‘oplax monoidal functor’, but these play no role in the current article.

Noetherian schemes $f : X \rightarrow Y$, of finite Tor-dimension, the functors $f^* \dashv f_* \dashv f^\times$ satisfy parts (i), (ii) and (iii) of Lemma 2.4.

The fact that the projection formula holds, that is part (i) of Lemma 2.4, is classical and easy — meaning that so far the general nonsense approach has told us nothing new in the setting of algebraic geometry. But parts (ii) and (iii) are remarkable: the functor $f^\times$ has always been viewed as mysterious, and (ii) and (iii) tell us that certain properties of $f^\times$ are best viewed as a special case of the appropriate abstract nonsense.

3. Digression: a reminder of ‘mates’

Let us begin at the formal level. Suppose we are given four categories $\mathcal{W}$, $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, as well as pairs of adjoint functors

$$\gamma : \mathcal{Y} \rightleftarrows \mathcal{W} : \Gamma \quad \text{and} \quad \beta : \mathcal{Z} \rightleftarrows \mathcal{X} : B.$$

Recall that this is shorthand for the assertion that $\Gamma : \mathcal{W} \rightarrow \mathcal{Y}$ is right adjoint to $\gamma : \mathcal{Y} \rightarrow \mathcal{W}$, and $B : \mathcal{X} \rightarrow \mathcal{Z}$ is right adjoint to $\beta : \mathcal{Z} \rightarrow \mathcal{X}$. Assume further that we are given a pair of functors

$$\alpha : \mathcal{X} \longrightarrow \mathcal{W} \quad \text{and} \quad \delta : \mathcal{Z} \longrightarrow \mathcal{Y}.$$

Then there is a canonical bijection between natural transformations

$$\begin{array}{ccc} \mathcal{W} & \xleftarrow{\alpha} & \mathcal{X} \\ \gamma \uparrow & \nearrow \rho & \uparrow \beta \\ \mathcal{Y} & \xleftarrow{\delta} & \mathcal{Z} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{W} & \xleftarrow{\alpha} & \mathcal{X} \\ \Gamma \downarrow & \nwarrow \sigma & \downarrow B \\ \mathcal{Y} & \xleftarrow{\delta} & \mathcal{Z} \end{array}$$

The formula is explicit: if $\eta(\gamma \dashv \Gamma)$ and $\varepsilon(\gamma \dashv \Gamma)$ are, respectively, the unit and counit of the adjunction $\gamma \dashv \Gamma$, while $\eta(\beta \dashv B)$ and $\varepsilon(\beta \dashv B)$ are, respectively, the unit and counit of the adjunction $\beta \dashv B$, then the formulas are that the bijection takes $\rho : \gamma\delta \rightarrow \alpha\beta$ and $\sigma : \delta B \rightarrow \Gamma\alpha$, respectively, to

$$\begin{aligned} \delta B &\xrightarrow{\eta(\gamma \dashv \Gamma)} \Gamma\gamma\delta B \xrightarrow{\rho} \Gamma\alpha\beta B \xrightarrow{\varepsilon(\beta \dashv B)} \Gamma\alpha \\ \gamma\delta &\xrightarrow{\eta(\beta \dashv B)} \gamma\delta B\beta \xrightarrow{\sigma} \gamma\Gamma\alpha\beta \xrightarrow{\varepsilon(\gamma \dashv \Gamma)} \alpha\beta \end{aligned}$$

The standard terminology in category theory is that ρ and σ are *mates* of each other. More explicitly ρ is the *left mate* of σ , and σ is the *right mate* of ρ .

4. Back to rigidly compactly generated tensor triangulated categories

DISCUSSION 4.1. Now we apply the formalism of the previous section to the following situation. We assume given four tensor triangulated categories $\mathcal{W}$, $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, as well as four pairs of adjoint functors

$$\begin{aligned} \alpha : \mathcal{X} &\rightleftarrows \mathcal{W} : A \quad \text{and} \quad \delta : \mathcal{Z} \rightleftarrows \mathcal{Y} : \Delta \\ \gamma : \mathcal{Y} &\rightleftarrows \mathcal{W} : \Gamma \quad \text{and} \quad \beta : \mathcal{Z} \rightleftarrows \mathcal{X} : B. \end{aligned}$$

Recall that this is shorthand for the assertion that we have adjoint pairs $\alpha \dashv A$, $\beta \dashv B$, $\gamma \dashv \Gamma$ and $\delta \dashv \Delta$. And now assume the functors $\alpha, \beta, \gamma, \delta$ are all strong monoidal functors. Suppose furthermore that we are given a 2-isomorphism ρ as below

$$\begin{array}{ccc} \mathcal{W} & \xleftarrow{\alpha} & \mathcal{X} \\ \gamma \uparrow & \nearrow \rho & \uparrow \beta \\ \mathcal{Y} & \xleftarrow{\delta} & \mathcal{Z} \end{array}$$

Taking right mates of ρ^{-1} and ρ we obtain, respectively, the 2-morphisms θ and σ below

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{A} & \mathcal{X} \\ \gamma \uparrow & \nwarrow \theta & \uparrow \beta \\ \mathcal{Y} & \xrightarrow{\Delta} & \mathcal{Z} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{W} & \xleftarrow{\alpha} & \mathcal{X} \\ \Gamma \downarrow & \nwarrow \sigma & \downarrow B \\ \mathcal{Y} & \xleftarrow{\delta} & \mathcal{Z} \end{array}$$

For the purpose of a number of proofs that will follow, starting with the very next one, we recall the following useful fact.

REMARK 4.2. We have been assuming throughout that our triangulated categories are all rigidly compactly generated. It is worth remembering one of the standard tricks that one uses when dealing with such categories: suppose we are given a diagram

$$\begin{array}{ccc} & f & \\ \mathcal{S} & \xrightarrow{\quad} & \mathcal{T} \\ & g & \end{array} \quad \begin{array}{c} \parallel \\ \theta \\ \parallel \end{array}$$

meaning $\mathcal{S}, \mathcal{T}$ are rigidly compactly generated tensor triangulated categories, f and g are coproduct-preserving triangulated functors and $\theta : f \rightarrow g$ is a natural transformation.

Then proving that θ is an isomorphism can be reduced to checking on compact objects, which are the same as rigid objects. That is, it suffices to show that $\theta(c) : f(c) \rightarrow g(c)$ is an isomorphism for all compact=rigid objects $c \in \mathcal{S}$.

Remember the reason for this is that the full subcategory $S \subset \mathcal{S}$, of all objects s for which $\theta(s) : f(s) \rightarrow g(s)$ is an isomorphism, is triangulated and closed under coproducts. Under the assumption of rigid compact generation, if S contains the rigids, then it must be all of $\mathcal{S}$.

And now we are ready to state the main result of this section — to the best of the author's knowledge this result is new.

LEMMA 4.3. *With the notation as in Discussion 4.1, assume that the categories $\mathcal{W}, \mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ are all rigidly compactly generated tensor triangulated categories and the functors α, β, γ and δ are all strong monoidal functors respecting coproducts. In Section 2, we learned that the right adjoints A, B, Γ and Δ exist and are all coproduct-preserving.*

We assert that the natural transformation $\sigma : \delta B \rightarrow \Gamma \alpha$ is an isomorphism if and only if $\theta : \beta \Delta \rightarrow A \gamma$ is.

Proof. By Remark 4.2, it is enough to check that, for every pair of rigid objects $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, the natural maps

$$\begin{aligned} \mathrm{Hom}(x, \beta\Delta(y)) &\xrightarrow{\theta} \mathrm{Hom}(x, A\gamma(y)) \\ \mathrm{Hom}(y, \delta B(x)) &\xrightarrow{\sigma} \mathrm{Hom}(y, \Gamma\alpha(x)) \end{aligned}$$

are isomorphisms together.

The right-hand sides simplify, respectively, to

$$\mathrm{Hom}(\mathbf{1}_{\mathcal{W}}, \alpha(x^\vee) \otimes \gamma(y)), \quad \mathrm{Hom}(\mathbf{1}_{\mathcal{W}}, \gamma(y^\vee) \otimes \alpha(x))$$

For the left-hand sides, observe the isomorphisms

$$\begin{aligned} \mathrm{Hom}(x, \beta\Delta(y)) &\cong \mathrm{Hom}(\mathbf{1}_{\mathcal{X}}, x^\vee \otimes \beta\Delta(y)) \\ &= \mathrm{Hom}(\beta(\mathbf{1}_{\mathcal{X}}), x^\vee \otimes \beta\Delta(y)) \\ &\cong \mathrm{Hom}(\mathbf{1}_{\mathcal{X}}, B(x^\vee \otimes \beta\Delta(y))) \\ &\cong \mathrm{Hom}(\mathbf{1}_{\mathcal{X}}, B(x^\vee) \otimes \Delta(y)) \end{aligned}$$

where the last isomorphism is by the projection formula, see Lemma 2.4(i). Similarly, we obtain an isomorphism

$$\mathrm{Hom}(y, \delta B(x)) \cong \mathrm{Hom}(\mathbf{1}_{\mathcal{X}}, \Delta(y^\vee) \otimes B(x)).$$

Thus the two maps we need to show are isomorphisms together, for all rigid x and y , rewrite as

$$\begin{aligned} \mathrm{Hom}(\mathbf{1}_{\mathcal{X}}, B(x^\vee) \otimes \Delta(y)) &\xrightarrow{\theta} \mathrm{Hom}(\mathbf{1}_{\mathcal{W}}, \alpha(x^\vee) \otimes \gamma(y)) \\ \mathrm{Hom}(\mathbf{1}_{\mathcal{X}}, \Delta(y^\vee) \otimes B(x)) &\xrightarrow{\sigma} \mathrm{Hom}(\mathbf{1}_{\mathcal{W}}, \gamma(y^\vee) \otimes \alpha(x)). \end{aligned}$$

So all that needs to be checked is that they are the same map up to the symmetry of the tensor product. Which is not difficult; as the reader can check herself, or look for hints in the Appendix, up to the symmetry of the tensor product and replacing x and y by $x^\vee$ and/or $y^\vee$ where necessary, the map is nothing other than the composite

$$\begin{array}{ccc} \mathrm{Hom}(\mathbf{1}_{\mathcal{X}}, B(x) \otimes \Delta(y)) & \xrightarrow{\gamma\delta} & \mathrm{Hom}(\gamma\delta(\mathbf{1}_{\mathcal{X}}), \gamma\delta[B(x) \otimes \Delta(y)]) \\ & & \parallel \\ & & \mathrm{Hom}(\mathbf{1}_{\mathcal{W}}, \gamma\delta[B(x)] \otimes \gamma\delta[\Delta(y)]) \\ & & \downarrow \rho \\ \mathrm{Hom}(\mathbf{1}_{\mathcal{W}}, \alpha(x) \otimes \gamma(y)) & \xleftarrow{\varepsilon \otimes \varepsilon} & \mathrm{Hom}(\mathbf{1}_{\mathcal{W}}, \alpha\beta[B(x)] \otimes \gamma\delta[\Delta(y)]) \end{array}$$

where the first arrow is just by applying the functor $\gamma\delta$, the equality is from the canonical isomorphism that we have because the functor $\gamma\delta$ is strong monoidal, the map labeled ρ is by applying the natural transformation $\rho: \gamma\delta \rightarrow \alpha\beta$ to the object $B(x) \in \mathcal{X}$ and the map labeled $\varepsilon \otimes \varepsilon$ is obtained from the counits of adjunction $\varepsilon: \beta B \rightarrow \mathrm{id}$ and $\varepsilon: \delta\Delta \rightarrow \mathrm{id}$. $\square$

EXAMPLE 4.4. In this paper, we are most interested in the case that arises out of algebraic geometry, meaning we begin with a commutative square of quasicompact, quasiseparated schemes

$$\begin{array}{ccc} W & \xrightarrow{u} & X \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{v} & Z \end{array}$$

giving rise to the square

$$\begin{array}{ccc} \mathbf{D}_{\mathbf{qc}}(W) & \xleftarrow{u^*} & \mathbf{D}_{\mathbf{qc}}(X) \\ f^* \uparrow & \xrightarrow{\rho} & \uparrow g^* \\ \mathbf{D}_{\mathbf{qc}}(Y) & \xleftarrow{v^*} & \mathbf{D}_{\mathbf{qc}}(Z) \end{array}$$

with ρ the canonical isomorphism. Taking the right mates of ρ^{-1} and ρ , we deduce, respectively, the 2-morphisms

$$\begin{array}{ccc} \mathbf{D}_{\mathbf{qc}}(W) & \xrightarrow{u_*} & \mathbf{D}_{\mathbf{qc}}(X) \\ f^* \uparrow & \xleftarrow{\theta} & \uparrow g^* \\ \mathbf{D}_{\mathbf{qc}}(Y) & \xrightarrow{v_*} & \mathbf{D}_{\mathbf{qc}}(Z) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{D}_{\mathbf{qc}}(W) & \xleftarrow{u^*} & \mathbf{D}_{\mathbf{qc}}(X) \\ f_* \downarrow & \xleftarrow{\sigma} & \downarrow g_* \\ \mathbf{D}_{\mathbf{qc}}(Y) & \xleftarrow{v^*} & \mathbf{D}_{\mathbf{qc}}(Z) \end{array}$$

and Lemma 4.3 informs us that the natural transformation σ is an isomorphism if and only if θ is.

So much for formal nonsense. Next we will return to the example from algebraic geometry, but first we recall a definition.

REMINDER 4.5. A cartesian square of schemes

$$\begin{array}{ccc} W & \xrightarrow{u} & X \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{v} & Z \end{array}$$

is called *Tor-independent* if, for every point $p \in W$, we have

$$\mathrm{Tor}_i^{\mathcal{O}_{Z, v f(p)}}(\mathcal{O}_{X, u(p)}, \mathcal{O}_{Y, f(p)}) = 0 \quad \text{for all } i > 0.$$

In words the stalks $\mathcal{O}_{X, u(p)}$ and $\mathcal{O}_{Y, f(p)}$, of (respectively) the structure sheaf $\mathcal{O}_X$ at the point $u(p)$ and the structure sheaf $\mathcal{O}_Y$ at the point $f(p)$, have vanishing higher Tor-groups over the stalk of the structure sheaf $\mathcal{O}_Z$ at the point $v f(p) = g u(p)$.

And now we are ready for a statement with some content.

PROPOSITION 4.6. With the notation as in Example 4.4, assume that the morphisms of schemes f, g, u, v are all separated. Then the natural transformations σ and θ , of Example 4.4, are isomorphisms if and only if:

(i) the Cartesian square

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\pi_1} & X \\ \pi_2 \downarrow & & \downarrow g \\ Y & \xrightarrow{v} & Z \end{array}$$

is Tor-independent; and

(ii) the natural map $w : W \rightarrow X \times_Z Y$ satisfies the condition that the unit of the adjunction $w^* \dashv w_*$ takes the object $\mathbb{1} \in \mathbf{D}_{\text{qc}}(X \times_Z Y)$ to an isomorphism $\eta : \mathbb{1} \rightarrow w_* w^* \mathbb{1}$.

Proof. Let us begin with the ‘if’ direction. If $w : W \rightarrow X \times_Z Y$ is an isomorphism and the (cartesian) square of Example 4.4 is Tor-independent, then we are in the easy, classical case: that σ and θ are isomorphisms follows from the facts.

(i) The derived pushforward functors, f_*, g_*, u_*, v_* can all be computed using Čech complexes of K -flat resolutions of the given objects in the derived categories.

(ii) For K -flat complexes over $\mathcal{O}_X$ and $\mathcal{O}_Y$, which are Tor-independent over $\mathcal{O}_Z$, the ordinary tensor product over $\mathcal{O}_Z$ agrees with the derived tensor product.

Next we continue proving the ‘if’ direction, but dropping the assumption that w is an isomorphism. From the classical result explained above, we know that the natural map $\sigma : v^* g_* \rightarrow \pi_{2*} \pi_1^*$ is an isomorphism. Hence it remains to show that the natural map

$$\pi_{2*} \pi_1^* \xrightarrow{\eta} \pi_{2*} w_* w^* \pi_1^* \xlongequal{\quad} f_* u^*$$

is also an isomorphism, for which it suffices to show that $\eta : \text{id} \rightarrow w_* w^*$ is an isomorphism. But $w_* w^*(-) = w_*[\mathbb{1} \otimes w^*(-)] \cong w_*(\mathbb{1}) \otimes (-)$ by the projection formula, and our hypothesis that $w_*(\mathbb{1}) = \mathbb{1}$ gives us the isomorphism $\text{id} \cong w_* w^*$.

Now we prove the ‘only if’ part. We assume therefore that the maps $\sigma : v^* g_* \rightarrow f_* u^*$ and (equivalently) $\theta : g^* v_* \rightarrow u_* f^*$ are isomorphisms. We need to show (1) that the morphism $w : W \rightarrow X \times_Z Y$ has the property that $\mathbb{1} \rightarrow w_*(\mathbb{1})$ is an isomorphism, and (2) that the Cartesian square

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\pi_1} & X \\ \pi_2 \downarrow & & \downarrow g \\ Y & \xrightarrow{v} & Z \end{array}$$

is Tor-independent. Both assertions are local in $X \times_Z Y$. Pick a point $p \in X \times_Z Y$; we need to find an open neighborhood of p over which the map w satisfies $\mathbb{1} = w_*(\mathbb{1})$, and such that $\mathcal{O}_{X, \pi_1(p)}$ and $\mathcal{O}_{Y, \pi_2(p)}$ have vanishing torsion over $\mathcal{O}_{Z, v\pi_2(p)}$.

Fix therefore the point $p \in X \times_Z Y$. First we choose an affine open subset $Z' \subset Z$ containing $v\pi_2(p) = g\pi_1(p)$. Then we choose an affine open subset Y' of $v^{-1}(Z') \subset Y$ containing $\pi_2(p)$. Now form the Cartesian square

$$\begin{array}{ccc} W' & \xrightarrow{u'} & W \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{v'} & Y \end{array}$$

The square is Cartesian and Tor-independent (the Tor-independence is because v' is an open immersion, hence flat), and the first part of the proof tells us that $\sigma' : v'^* f_* \rightarrow f'_* u'^*$ is an isomorphism. In the concatenation of the two squares

$$\begin{array}{ccccc} W' & \xrightarrow{u'} & W & \xrightarrow{u} & X \\ f' \downarrow & & \downarrow f & & \downarrow g \\ Y' & \xrightarrow{v'} & Y & \xrightarrow{v} & Z \end{array}$$

we have that both $\sigma' : v'^* f_* \rightarrow f'_* u'^*$ and $\sigma : v^* g_* \rightarrow f_* u^*$ are isomorphisms, hence so is the composite

$$v'^* v^* g_* \xrightarrow{v'^* \sigma} v'^* f_* u^* \xrightarrow{\sigma' u^*} f'_* u'^* u^*.$$

Since $Y' \times_Z X$ is an open subset of $Y \times_Z X$ containing p , we may without loss replace the given square by

$$\begin{array}{ccc} W' & \xrightarrow{uu'} & X \\ f' \downarrow & & \downarrow g \\ Y' & \xrightarrow{vv'} & Z \end{array}$$

and hence assume that Y is affine and $v(Y)$ is contained in our chosen affine open subset $Z' \subset Z$. And now symmetry allows us to shrink X too: replacing X by an open subset $X' \subset X$ containing $\pi_1(p)$ and contained in $g^{-1}(Z')$, we may assume that X is also affine and that $g(X) \subset Z'$. And since both X and Y have their images inside $Z' \subset Z$, we may shrink Z as well; thus we may assume that X , Y and Z are all affine. This forces $X \times_Z Y$ to be affine.

With our reduction to the affine case complete, let $Z = \operatorname{Spec}(R)$, $X = \operatorname{Spec}(S)$, $Y = \operatorname{Spec}(T)$ and $X \times_Z Y = \operatorname{Spec}(S \otimes_R T)$. In the derived categories $\mathbf{D}_{\text{qc}}(Z) \cong \mathbf{D}(R)$, $\mathbf{D}_{\text{qc}}(X) \cong \mathbf{D}(S)$, $\mathbf{D}_{\text{qc}}(Y) \cong \mathbf{D}(T)$ and $\mathbf{D}_{\text{qc}}(W) \cong \mathbf{D}(S \otimes_R T)$, we compute that

$$f_* u^* \mathcal{O}_X = f_* \mathcal{O}_W = \pi_{2*} w_* \mathcal{O}_W, \quad \text{while} \quad v^* g_* \mathcal{O}_X = S \otimes_R^{\mathbf{L}} T.$$

The base-change map $v^* g_* \mathcal{O}_X \rightarrow f_* u^* \mathcal{O}_X$ is therefore a morphism in $\mathbf{D}(S)$ of the form

$$S \otimes_R^{\mathbf{L}} T \xrightarrow{\sigma} \pi_{2*} w_* \mathcal{O}_W$$

and the first thing to note is that $S \otimes_R^{\mathbf{L}} T \in \mathbf{D}(S)^{\leq 0}$ while $\pi_{2*} w_* \mathcal{O}_W \in \mathbf{D}(S)^{\geq 0}$. Since σ is an isomorphism, we deduce that all the negative cohomology of $S \otimes_R^{\mathbf{L}} T$ must vanish, giving the Tor-independence. And since the map π_2 is affine, the functor π_{2*} must be conservative, but it takes the morphism $\mathcal{O}_{X \times_Z Y} \rightarrow w_* \mathcal{O}_W$ to an isomorphism. It follows that the map $\mathcal{O}_{X \times_Z Y} \rightarrow w_* \mathcal{O}_W$ must be an isomorphism. That is the natural map $\mathbb{1}_{\mathbf{D}_{\text{qc}}(X \times_Z Y)} \rightarrow w_* \mathbb{1}_{\mathbf{D}_{\text{qc}}(W)}$ is an isomorphism. $\square$

REMARK 4.7. Precursors of Proposition 4.6 exist in the literature, see, for example, Kuznetsov [8, Corollary 2.21]. What is new is a proof by reduction to the affine case, which is made possible by the symmetry obtained from Lemma 4.3 and allows us to obtain a sharp ‘if and only if’ statement.

REMARK 4.8. Note that the map $w : W \rightarrow X \times_Z Y$ does not have to be the identity. For example, if $\mathbb{P}_{\mathbb{Z}}^n$ is the n -dimensional projective space, we could let $W = \mathbb{P}_{\mathbb{Z}}^n \times X \times_Z Y$ and let $w : W \rightarrow X \times_Z Y$ be the projection. This map satisfies $\mathbb{1} = w_*(\mathbb{1})$.

5. Separating the formal nonsense from the part that has content

LEMMA 5.1. Suppose we are given a square

$$\begin{array}{ccc} \mathcal{W} & \xleftarrow{\alpha} & \mathcal{X} \\ \gamma \uparrow & \nearrow \rho & \uparrow \beta \\ \mathcal{Y} & \xleftarrow{\delta} & \mathcal{Z} \end{array}$$

where $\mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are all rigidly compactly generated tensor triangulated categories, where $\alpha, \beta, \gamma, \delta$ are all coproduct-preserving strong monoidal functors, and where the natural transformation ρ is an isomorphism.

If $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are compact objects, then so is $\alpha(x) \otimes \gamma(y)$. Moreover, the following are equivalent.

(i) In the category $\mathcal{W}$ the compact objects

$$\{\alpha(x) \otimes \gamma(y) \mid x \in \mathcal{X}^c \text{ and } y \in \mathcal{Y}^c\}$$

generate.

- (ii) If $w \in \mathcal{W}$ is non-zero, then there exists a compact object $y \in \mathcal{Y}$ with $A(\gamma(y^\vee) \otimes w) \neq 0$.
- (iii) If $w \in \mathcal{W}$ is non-zero, then there exists a compact object $x \in \mathcal{X}$ with $\Gamma(\alpha(x^\vee) \otimes w) \neq 0$.

Proof. Since both α and γ are strong monoidal functors, they must take rigid objects to rigid objects. If $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are compact, they must be rigid, hence so are $\alpha(x)$ and $\gamma(y)$. But the tensor product in $\mathcal{W}$ of these two rigid objects is rigid, meaning $\alpha(x) \otimes \gamma(y)$ is rigid in $\mathcal{W}$ and hence compact.

It remains to prove the equivalence of (i), (ii) and (iii), and by symmetry it suffices to prove the equivalence of (i) and (ii).

Suppose therefore that $w \in \mathcal{W}$ is an object. For all compact $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{W}}[\alpha(x) \otimes \gamma(y), w] &= \mathrm{Hom}_{\mathcal{W}}[\alpha(x), \gamma(y^\vee) \otimes w] \\ &= \mathrm{Hom}_{\mathcal{X}}(x, A[\gamma(y^\vee) \otimes w]). \end{aligned}$$

The first equality uses the relation between y and $y^\vee$, and the second equality is by the adjunction $\alpha \dashv A$. Now assume (i) holds and $w \in \mathcal{W}$ is a non-zero object. Then there exists a pair of compact objects $x \in \mathcal{X}$, $y \in \mathcal{Y}$ with $\mathrm{Hom}_{\mathcal{W}}[\alpha(x) \otimes \gamma(y), w] \neq 0$, and hence with $A[\gamma(y^\vee) \otimes w] \neq 0$. This proves (i) $\implies$ (ii). If (ii) holds and $w \in \mathcal{W}$ is non-zero, then there exists a compact $y \in \mathcal{Y}$ with $A[\gamma(y^\vee) \otimes w] \neq 0$, hence there will exist a compact $x \in \mathcal{X}$ with $\mathrm{Hom}_{\mathcal{X}}(x, A[\gamma(y^\vee) \otimes w]) \neq 0$, in other words we can find a pair x, y with $\mathrm{Hom}_{\mathcal{W}}[\alpha(x) \otimes \gamma(y), w] \neq 0$. This proves (ii) $\implies$ (i). $\square$

DISCUSSION 5.2. We are most interested in the case where the square

$$\begin{array}{ccc} \mathcal{W} & \xleftarrow{\alpha} & \mathcal{X} \\ \gamma \uparrow & \nearrow \rho & \uparrow \beta \\ \mathcal{Y} & \xleftarrow{\delta} & \mathcal{Z} \end{array}$$

is such that the mates σ and θ of Discussion 4.1 are both isomorphisms. And here I have a confession to make: I do not know a clean, category-theoretical way to describe necessary and sufficient conditions under which (i), (ii) and (iii) of Lemma 5.1 hold.

The example from algebraic geometry tells us that some condition will be necessary. Let k be a field and consider the commutative square of schemes

$$\begin{array}{ccc} \mathbb{P}_k^n & \xrightarrow{u} & \mathrm{Spec}(k) \\ f \downarrow & & \downarrow g \\ \mathrm{Spec}(k) & \xrightarrow{v} & \mathrm{Spec}(k) \end{array}$$

Remark 4.8 informs us that the corresponding commutative diagram of derived categories

$$\begin{array}{ccc} \mathbf{D}_{\mathbf{qc}}(\mathbb{P}_k^n) & \xleftarrow{u^*} & \mathbf{D}_{\mathbf{qc}}(\mathrm{Spec}(k)) \\ f^* \uparrow & \nearrow \rho & \uparrow g^* \\ \mathbf{D}_{\mathbf{qc}}(\mathrm{Spec}(k)) & \xleftarrow{v^*} & \mathbf{D}_{\mathbf{qc}}(\mathrm{Spec}(k)) \end{array}$$

has the property that σ and θ are both isomorphisms. But clearly objects of the form $u^*(x) \otimes f^*(y)$ do not generate $\mathbf{D}_{\mathbf{qc}}(\mathbb{P}_k^n)$.

Because I have not figured out how to correctly turn the next statement into tensor triangulated formalism, we simply quote the (easy) result from algebraic geometry.

PROPOSITION 5.3. *Suppose we are given a Tor-independent Cartesian square of quasicompact, quasiseparated schemes*

$$\begin{array}{ccc} W & \xrightarrow{u} & X \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{v} & Z \end{array}$$

Then in the square of derived categories

$$\begin{array}{ccc} \mathbf{D}_{\mathbf{qc}}(W) & \xleftarrow{u^*} & \mathbf{D}_{\mathbf{qc}}(X) \\ f^* \uparrow & \nearrow \rho & \uparrow g^* \\ \mathbf{D}_{\mathbf{qc}}(Y) & \xleftarrow{v^*} & \mathbf{D}_{\mathbf{qc}}(Z) \end{array}$$

we have that the objects of the form $u^*(x) \otimes f^*(y)$, with x a compact object in $\mathbf{D}_{\mathbf{qc}}(X)$ and y a compact object in $\mathbf{D}_{\mathbf{qc}}(Y)$, generate the category $\mathbf{D}_{\mathbf{qc}}(W)$.

Proof. See [4, Lemma 3.4.1]. □

DEFINITION 5.4. In view of Example 4.4, Proposition 4.6 and Proposition 5.3, a square as below

$$\begin{array}{ccc} \mathcal{W} & \xleftarrow{\alpha} & \mathcal{X} \\ \gamma \uparrow & \nearrow \rho & \uparrow \beta \\ \mathcal{Y} & \xleftarrow{\delta} & \mathcal{Z} \end{array}$$

will be called a *Tor-independent Cartesian square* if $\mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are all rigidly compactly generated tensor triangulated categories, if $\alpha, \beta, \gamma, \delta$ are all coproduct-preserving strong monoidal functors, if the natural transformation ρ is an isomorphism, if the right mates of ρ^{-1} and of ρ — that is the natural transformations θ and σ of Discussion 4.1 — are both isomorphisms, and if the equivalent conditions of Lemma 5.1 (i), (ii) and (iii) all hold.

6. The Verdier base-change theorem

LEMMA 6.1. Let $f^* : \mathcal{Y} \rightarrow \mathcal{W}$ be a coproduct-preserving strong monoidal functor of rigidly compactly generated tensor triangulated categories. Suppose that the right adjoint $f_* : \mathcal{W} \rightarrow \mathcal{Y}$ takes compact objects to compact objects. Then for a compact object $w \in \mathcal{W}$ and an arbitrary object $y \in \mathcal{Y}$ there is an isomorphism, natural in w and y ,

$$(f_* w)^\vee \otimes y \cong f_*(w^\vee \otimes f^\times y)$$

Proof. Let $s \in \mathcal{Y}$ be arbitrary. We have isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{Y}}[s, (f_* w)^\vee \otimes y] &\cong \mathrm{Hom}_{\mathcal{Y}}[s \otimes f_* w, y] \\ &\cong \mathrm{Hom}_{\mathcal{Y}}[f_*(f^* s \otimes w), y] \\ &\cong \mathrm{Hom}_{\mathcal{W}}[f^* s \otimes w, f^\times y] \\ &\cong \mathrm{Hom}_{\mathcal{W}}[f^* s, w^\vee \otimes f^\times y] \\ &\cong \mathrm{Hom}_{\mathcal{Y}}[s, f_*(w^\vee \otimes f^\times y)], \end{aligned}$$

where the second isomorphism comes from the $s \otimes f_* w \cong f_*(f^* s \otimes w)$ of the projection formula, see Lemma 2.4(i), while the other isomorphisms are all by adjunction. The isomorphism between the first and last term is an isomorphism of representable functors of s , hence must come from an isomorphism

$$(f_* w)^\vee \otimes y \cong f_*(w^\vee \otimes f^\times y)$$

natural in w and y . We leave it to the interested reader to compute explicitly what this isomorphism is. $\square$

NOTATION 6.2. Suppose

$$\begin{array}{ccc} \mathcal{W} & \xleftarrow{u^*} & \mathcal{X} \\ f^* \uparrow & \nearrow \rho & \uparrow g^* \\ \mathcal{Y} & \xleftarrow{v^*} & \mathcal{Z} \end{array}$$

is a Tor-independent Cartesian square as in Definition 5.4. We are accustomed to considering the natural isomorphism σ below, which is the right mate of ρ . But now we want to also consider the natural transformation Φ which is the right mate of σ^{-1} , see below.

$$\begin{array}{ccc} \mathcal{W} & \xleftarrow{u^*} & \mathcal{X} \\ f_* \downarrow & \nwarrow \sigma & \downarrow g_* \\ \mathcal{Y} & \xleftarrow{v^*} & \mathcal{Z} \end{array} \quad \text{with } \Phi \text{ the right mate of } \sigma^{-1} \quad \begin{array}{ccc} \mathcal{W} & \xleftarrow{u^*} & \mathcal{X} \\ f^\times \uparrow & \nwarrow \Phi & \uparrow g^\times \\ \mathcal{Y} & \xleftarrow{v^*} & \mathcal{Z} \end{array}$$

PROPOSITION 6.3. Suppose

$$\begin{array}{ccc} \mathcal{W} & \xleftarrow{u^*} & \mathcal{X} \\ f^* \uparrow & \nearrow \rho & \uparrow g^* \\ \mathcal{Y} & \xleftarrow{v^*} & \mathcal{Z} \end{array}$$

is a Tor-independent cartesian square as in Definition 5.4, and assume further that the right adjoints $f_* : \mathcal{W} \rightarrow \mathcal{Y}$ and $g_* : \mathcal{X} \rightarrow \mathcal{Z}$ take compact objects to compact objects.

Then the natural transformation $\Phi : u^* g^\times \longrightarrow f^\times v^*$ of Notation 6.2 is an isomorphism.

Proof. Let $x \in \mathcal{X}$ be a compact object and $z \in \mathcal{Z}$ an arbitrary object. We have the string of isomorphisms

$$\begin{aligned} f_*[u^*(x^\vee) \otimes u^* g^\times(z)] &\cong f_* u^*[x^\vee \otimes g^\times(z)] \\ &\cong v^* g_*[x^\vee \otimes g^\times(z)] \\ &\cong v^*[(g_* x)^\vee \otimes z] \\ &\cong (v^* g_* x)^\vee \otimes v^*(z) \\ &\cong (f_* u^* x)^\vee \otimes v^*(z) \\ &\cong f_*[(u^*(x^\vee) \otimes f^\times v^*(z))]. \end{aligned}$$

The second and fifth isomorphisms come from $\sigma : v^* g_* \longrightarrow f_* u^*$, the third and sixth are applications of Lemma 6.1, and the remaining isomorphisms are obvious.

In total we learn that, for every compact object $x \in \mathcal{X}$, the functor $f_*[u^*(x^\vee) \otimes (-)]$ takes the morphism $\Phi(z) : u^* g^\times(z) \longrightarrow f^\times v^*(z)$ to an isomorphism, and hence takes the mapping cone to zero. But now as Lemma 5.1(iii) holds, we learn that $\Phi(z) : u^* g^\times(z) \longrightarrow f^\times v^*(z)$ must be an isomorphism. Since this is true for every object $z \in \mathcal{Z}$, it follows that Φ is a natural isomorphism. $\square$

EXAMPLE 6.4. Suppose now that we are given a Tor-independent Cartesian square of Noetherian schemes

$$\begin{array}{ccc} W & \xrightarrow{u} & X \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{v} & Z \end{array}$$

inducing a Tor-independent cartesian square of derived categories

$$\begin{array}{ccc} \mathbf{D}_{\mathbf{qc}}(W) & \xleftarrow{u^*} & \mathbf{D}_{\mathbf{qc}}(X) \\ f^* \uparrow & \nearrow \rho & \uparrow g^* \\ \mathbf{D}_{\mathbf{qc}}(Y) & \xleftarrow{v^*} & \mathbf{D}_{\mathbf{qc}}(Z) \end{array}$$

If f and g are both proper and of finite Tor-dimension, then Example 2.6 tells us that f_* and g_* both respect compact objects. The hypotheses of Proposition 6.3 hold, and hence so does the conclusion. It teaches us that $\Phi : u^* g^\times \longrightarrow f^\times v^*$ is an isomorphism.

With a bit more work one can show that it suffices for g to be proper and f of finite Tor-dimension. And a slightly different generalization allows us to prove Verdier's original version of this Proposition, asserting that if g is proper, then $\Phi(z)$ is an isomorphism for any $z \in \mathbf{D}_{\mathbf{qc}}^+(Z)$. The reader might wish to look at Verdier [15, Theorem 2] for the original presentation, where the proof occupies half the paper and is done by reducing to special cases.

The abstract-nonsense approach to the Verdier base-change theorem, which we presented in this section, first appeared in my article [10]. Although the earlier, short incarnations of [10] are better and less technical than the most recent, long version that works in the generality of algebraic stacks.

7. Coming down to earth

We have seen a tremendous amount of abstract nonsense, mostly because I wanted to show the reader that, up to now, everything is formal and nearly devoid of real algebro-geometric content. But now we want to specialize to the case of interest: we will assume given a proper morphism $f : X \rightarrow Y$ of Noetherian schemes, and in the remainder of the article we will assume f to be of finite Tor-dimension. By Example 2.6, there are induced functors of derived categories

$$\begin{array}{ccc} & \mathbf{D}_{\mathbf{qc}}(X) & \\ f^* \uparrow & \downarrow f_* & \uparrow f^\times \\ & \mathbf{D}_{\mathbf{qc}}(Y) & \end{array}$$

with $f^* \dashv f_* \dashv f^\times$, and such that f_* respects compact objects. The functors f^* and f_* are computable, they are (respectively) the left derived pullback functor and the right derived pushforward functor. The functor $f^\times$ is the mysterious one.

DISCUSSION 7.1. With the notation as above, Lemma 2.4 (ii) and (iii) tell us that there is an isomorphism $f^\times(-) \cong f^\times(\mathbf{1}) \otimes f^*(-)$, and that this isomorphism is such that the counit of adjunction $\varepsilon : f_* f^\times(s) \rightarrow s$ is naturally isomorphic to the tensor product of s with the counit of adjunction $\varepsilon : f_* f^\times(\mathbf{1}) \rightarrow \mathbf{1}$. In other words, we would be happy if we could compute:

- (i) the object $f^\times(\mathbf{1})$;
- (ii) the morphism $\varepsilon : f_* f^\times(\mathbf{1}) \rightarrow \mathbf{1}$.

Now the honest truth is that, within epsilon, all that is known about this problem is how to compute these in the special cases where f is either a closed immersion or is smooth.

The case of closed immersions is easy. The case of smooth and proper maps was traditionally fiendishly difficult, and this article surveys the new progress, referring the reader to the relevant literature for more detail. And the reader should note that the computational methods that underpin the recent breakthrough might well provide new information, beyond the scope of what was attainable classically. Almost half of the (much longer) survey [13] is devoted to open questions.

OBSERVATION 7.2. Suppose therefore that $f : X \rightarrow Y$ is a proper morphism of Noetherian schemes, and assume f to be smooth and of relative dimension $n \geq 0$. The brilliant idea of Verdier [15, Proof of Theorem 3] was to study the following commutative diagram of schemes

$$\begin{array}{ccccc} X & & & & \\ & \searrow \delta & & & \\ & X \times_Y X & \xrightarrow{\pi_1} & X & \\ & \pi_2 \downarrow & & \downarrow f & \\ & X & \xrightarrow{f} & Y & \end{array}$$

where the square is Cartesian and $\delta : X \rightarrow X \times_Y X$ is the diagonal map. Example 6.4 gives a natural isomorphism $\Phi : \pi_1^* f^\times \rightarrow \pi_2^* f^*$. Now Verdier already noticed that by applying the functor $\delta^\times$ to this isomorphism one is led to a computation of $f^\times(\mathbf{1})$. Let us recall how this goes.

By Example 6.4, we have an isomorphism $\Phi(\mathbb{1}) : \pi_1^* f^\times(\mathbb{1}_Y) \longrightarrow \pi_2^\times f^*(\mathbb{1}_Y)$, and applying $\delta^\times$ we deduce the first isomorphism in the composite

$$\delta^\times \pi_1^* f^\times(\mathbb{1}) \xrightarrow{\delta^\times \Phi} \delta^\times \pi_2^\times f^*(\mathbb{1}_Y) = f^*(\mathbb{1}_Y) = \mathbb{1}_X$$

The first equality comes about because $\delta^\times \pi_2^\times = (\pi_2 \delta)^\times = \text{id}^\times = \text{id}$, and the second equality is obvious. Expanding out this isomorphism gives

$$\mathbb{1}_X \cong \delta^\times [\pi_1^* f^\times(\mathbb{1})] \cong \delta^\times [\mathbb{1}_{X \times_Y X}] \otimes \delta^* [\pi_1^* f^\times(\mathbb{1})] \cong \delta^\times [\mathbb{1}_{X \times_Y X}] \otimes f^\times(\mathbb{1})$$

The second isomorphism comes by applying to $s = \pi_1^* f^\times(\mathbb{1})$ the isomorphism $\delta^\times(s) \cong \delta^\times(\mathbb{1}) \otimes \delta^*(s)$ of Lemma 2.4(ii). The third isomorphism is because $\delta^* \pi_1^* = (\pi_1 \delta)^* = \text{id}^* = \text{id}$. And now we appeal to the fact that δ is a closed immersion, and for closed immersions $\delta^\times(\mathbb{1})$ is easy to compute. Since f is smooth, the closed immersion δ is even a regular immersion—it is locally given by a regular sequence. For such closed immersions the computation of $\delta^\times$ is especially simple, see Hartshorne [5, III.7.2]. The argument uses local coordinates and is not obviously global and functorial, but one does obtain an isomorphism $f^\times(\mathbb{1}) \cong \Omega_f^n[n]$. So far we have sketched the argument proving Verdier [15, Theorem 3].

Lipman cleaned this up, he observed that it is better to apply the functor δ^* , rather than the functor $\delta^\times$, to the isomorphism $\Phi : \pi_1^* f^\times \longrightarrow \pi_2^\times f^*$, this leads to a computation of $f^\times(\mathbb{1})$ which is coordinate-free, global and functorial. (Arguably this is not such a crucial point, it is not so difficult to study how Verdier’s construction behaves under a change of coordinates, and prove independence that way).

Anyway the reader can find Lipman’s argument presented in [13, Construction 3.1.6]: Lipman constructed, in the category $\mathbf{D}_{\text{qc}}(X)$, a morphism $\theta : \Omega_f^n[n] \longrightarrow f^\times(\mathbb{1})$ which is manifestly coordinate-free, global and functorial. And then [13, Theorem 3.1.7] is the assertion that Lipman’s map θ is an isomorphism. Much of [13, Section 3.2] explains that recent progress, due to a collaborative effort of Srikanth Iyengar, Joe Lipman and the author [7], achieved a simple and direct proof of [13, Theorem 3.1.7]. It worked by reducing to a computation of certain Hochschild homology-cohomology constructs. And since the problem is étale-local in X , one reduces to the case where $X = \mathbb{A}_Y^n$. The reader can find the actual computation in [12, Section 1].

REMARK 7.3. Up to this point, the careful algebraic geometer would be justified in being unimpressed. Verdier already had the isomorphism $\Omega_f^n[n] \cong f^\times(\mathbb{1})$ in his [15, Theorem 3], Lipman’s objection to Verdier’s argument was that the isomorphism was not coordinate-free, global and functorial — at least not obviously. Lipman found a map θ which had the virtue of being coordinate-free, global and functorial, but he did not have a direct proof that his θ was an isomorphism. Now we have such a proof — big deal.

DISCUSSION 7.4. And now we come to the really major new breakthrough, the understanding of the map $\varepsilon : f_* f^\times(\mathbb{1}) \longrightarrow \mathbb{1}$. The problem with this map is that it is a global construct, and it is far from clear how to compute it locally. Grothendieck’s solution to this was clumsy, it amounted to artificially defining local data and showing that these glue. The end result should be simple, we met it in the Introduction, we should end up with the map taking a meromorphic differential to its residue. But Grothendieck gave a description in terms of ‘residue symbols’ and their functorial properties. If the reader looks at [15, the paragraphs between the end of the proof of Theorem 3 and the beginning of the proof of Theorem 2], she will learn that Verdier did not have a simplification of Grothendieck’s statements. He only offered simpler proofs of Grothendieck’s impenetrable theorems.

Lipman found a much more elegant formulation, for a recent account in his own words, see his book [9]. In simple words, Lipman looks not at the counit of adjunction $\varepsilon : f_* f^\times(\mathbb{1}) \longrightarrow \mathbb{1}$

but at a close relative, the composite given in the displayed formula below. And this composite turns out to be the right thing to look at: its definition is intrinsic, global, coordinate-free and invariant under all reasonable natural operations. And the theorem becomes that Lipman's map commutes with flat base-change in Y and is computable étale-locally in X . And in local coordinates Lipman's map comes down, very naturally, to the map taking a meromorphic form to its residue.

In more detail, let Γ_W be the usual 'local cohomology' functor, the endofunctor $\Gamma_W : \mathbf{D}_{\text{qc}}(X) \rightarrow \mathbf{D}_{\text{qc}}(X)$ taking an K -injective complex to the subcomplex of those cycles whose support is generically finite over Y . For any K -injective complex I , there is an inclusion map $\Gamma_W(I) \rightarrow I$, which gives a natural transformation $i : \Gamma_W \rightarrow \text{id}$. Lipman's map is the composite

$$f_*\Gamma_W\Omega_f^n[n] \xrightarrow{\theta} f_*\Gamma_W f^\times(\mathbb{1}) \xrightarrow{i} f_*f^\times(\mathbb{1}) \xrightarrow{\varepsilon} \mathbb{1}$$

The expression on the left, meaning $f_*\Gamma_W\Omega_f^n[n]$, has a standard Čech-complex description as a quotient of the space of meromorphic differentials. And the right formulation of the duality theorem, which the reader can find in [13, Theorem 2.2.5], is that the composite above is just the map sending a meromorphic differential to its residue.

If $Y = \text{Spec}(k)$, with k an algebraically closed field, then [13, Theorem 2.2.5] means what you think it does — this is explained in [13, Example 2.2.6]. For Y more general one needs to be more careful in defining what is meant by residues, but this is not so mysterious, hence let me try to explain the key point by looking at the example where $Y = \text{Spec}(k)$ but we no longer make the simplifying assumption that the field k is algebraically closed. By [13, Example 2.2.4], Lipman's composite simplifies to the composite

$$\bigoplus_{p \in X} H_p^n(\Omega_X^n) \xrightarrow{i} H^n(\Omega_X^n) \xrightarrow{\varepsilon} k,$$

where the first term is the direct sum, over all closed points $p \in X$, of the n th local cohomology at p of the canonical bundle Ω_X^n .

Pick a closed point $p \in X$ and choose a system of parameters at p ; this means that, in the stalk $\mathcal{O}_{X,p}$ of the structure sheaf $\mathcal{O}_X$ at the closed point $p \in X$, we choose n elements $\{x_1, x_2, \dots, x_n\}$ which generate an ideal containing a power of the maximal ideal. These define a morphism $g : U \rightarrow \mathbb{A}_k^n$, where U is an open subset of X containing p and g is finite and flat at p . The functoriality of Lipman's map tells us that there is a commutative triangle

$$\begin{array}{ccc} H_p^n(\Omega_X^n) & \xrightarrow{L} & k \\ \varepsilon_g \downarrow & & \nearrow L \\ H_{g(p)}^n(\Omega_{\mathbb{A}_k^n}^n) & \xrightarrow{L} & k \end{array}$$

where the vertical map is a morphism which (for now) we pretend we do not understand[†], but the slanted morphisms are Lipman composites. Since the point $g(p)$ is k -rational, the computation of [13, Example 2.2.6] applies, we know all about the bottom Lipman map, that is the bottom slanted arrow of the triangle. And formal non-sense about local cohomology at closed points tells us that the vertical map is unchanged by completing.

Therefore, for free, we discover that the vertical map, and hence the entire composite, may be computed after first completing at p . Without explicitly telling the reader the formula for

[†]There is literature about this map, but for now we ignore this.

Lipman's residue map, I have shown that it cannot be too complicated, after all what we have sketched above is an algorithm for computing Lipman's residue map.

8. The future

The new methods that allowed us (meaning Iyengar, Lipman, and I) to make the recent breakthrough are Hochschild-homological. It turns out that $f^\times(\mathbb{1})$ can locally be computed by Hochschild techniques, and in [13, Section 5] there is an extensive discussion of the fact that these Hochschild techniques are (in principle) available much more generally than in the smooth case where I was able to carry out the computation. This opens the door for potential future progress.

In the Acknowledgements, we mention that this survey originated in private correspondence with Scholze. More precisely, when I was reading the fascinating account of 'condensed mathematics', jointly developed by Clausen and Scholze, I was surprised to learn that the construction of Grothendieck's[‡] functor $f^!$, and a computation of $f^!(\mathbb{1})$ for f smooth, was supposed to be the highlight.

I emailed Scholze to let him know that we already had a simple and direct computation of $f^!(\mathbb{1})$ for f smooth, in fact that much was already in Verdier's 1968 paper. And in the ensuing email conversation this survey was born — I am very grateful to Scholze, both for his interest and for zeroing in so quickly on what was really new and major about the work surveyed in [13].

Anyway in this section, it is only right to mention condensed mathematics, which might well play a major role in future developments. When I wrote the survey [13] the condensed mathematics approach to Grothendieck duality had yet to be born, but in the current survey it clearly should be given equal time. There are two new tools introduced to the field, the Hochschild computational techniques due originally to Avramov and Iyengar, and Clausen and Scholze's condensed mathematics.

At some level, the progress is all about being able to compute locally. The condensed mathematics approach has the virtue that the functor $f^!$ is local by construction, whereas the traditional developments all define it in terms of compactifications. This is amazing, and even allows one to define a functor $f_!$. And who knows, there might even be a way to pass directly from the condensed mathematics definition of $f^!$ to the Hochschild-homological, computational formulas.

At the moment condensed mathematics is still work-in-progress, with the current notes available on Scholze's web page.

Appendix. More detail on the proof of Lemma 4.3

In the proof of Lemma 4.3, there were assertions about certain composites agreeing. We wanted to show that, for every pair of rigid objects $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, the natural maps

$$\begin{aligned} \mathrm{Hom}(x, \beta\Delta(y)) &\xrightarrow{\mathrm{Hom}(x, \theta)} \mathrm{Hom}(x, A\gamma(y)) \\ \mathrm{Hom}(y, \delta B(x)) &\xrightarrow{\mathrm{Hom}(y, \sigma)} \mathrm{Hom}(y, \Gamma\alpha(x)) \end{aligned}$$

are isomorphisms together and, after writing down some diagrams and allowing ourselves to freely replace x and y by $x^\vee$ and $y^\vee$, we asserted that both are really the same map, namely the composite

[‡]Grothendieck's $f^!$ agrees with the functor $f^\times$ when f is proper, but not for general morphisms of schemes. The interested reader is referred to [13, Reminder 5.1.2 and Remark 5.1.4] for a fuller discussion.

$$\begin{array}{ccc}
 \mathrm{Hom}(\mathbb{1}_{\mathcal{X}}, B(x) \otimes \Delta(y)) & \xrightarrow{\gamma\delta} & \mathrm{Hom}(\gamma\delta(\mathbb{1}_{\mathcal{X}}), \gamma\delta[B(x) \otimes \Delta(y)]) \\
 & & \parallel \\
 & & \mathrm{Hom}(\mathbb{1}_{\mathcal{W}}, \gamma\delta[B(x)] \otimes \gamma\delta[\Delta(y)]) \\
 & & \downarrow \rho \\
 \mathrm{Hom}(\mathbb{1}_{\mathcal{W}}, \alpha(x) \otimes \gamma(y)) & \xleftarrow{\varepsilon \otimes \varepsilon} & \mathrm{Hom}(\mathbb{1}_{\mathcal{W}}, \alpha\beta[B(x)] \otimes \gamma\delta[\Delta(y)]).
 \end{array}$$

This appendix is to sketch out a little more detail.

From symmetry it suffices to study the map labeled $\mathrm{Hom}(y, \sigma)$, and exhibit that it agrees up to isomorphism with the composite we have written above. Now recall the discussion of mates of Section 3, we are given the pair of mates

$$\begin{array}{ccc}
 \mathcal{W} & \xleftarrow{\alpha} & \mathcal{X} \\
 \uparrow \gamma & \nearrow \rho & \uparrow \beta \\
 \mathcal{Y} & \xleftarrow{\delta} & \mathcal{Z}
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 \mathcal{W} & \xleftarrow{\alpha} & \mathcal{X} \\
 \downarrow \Gamma & \nwarrow \sigma & \downarrow B \\
 \mathcal{Y} & \xleftarrow{\delta} & \mathcal{Z}
 \end{array}$$

and the formula for σ in terms of ρ , given in Section 3, says that σ is the composite

$$\delta B \xrightarrow{\eta(\gamma \dashv \Gamma)} \Gamma \gamma \delta B \xrightarrow{\rho} \Gamma \alpha \beta B \xrightarrow{\varepsilon(\beta \dashv B)} \Gamma \alpha$$

Applying the functor $\mathrm{Hom}(y^\vee, -)$ to the definition of the map σ yields the commutative square

$$\begin{array}{ccc}
 \mathrm{Hom}(y^\vee, \delta B(x)) & \xrightarrow{\mathrm{Hom}(y^\vee, \sigma)} & \mathrm{Hom}(y^\vee, \Gamma \alpha(x)) \\
 \mathrm{Hom}(y^\vee, \eta) \downarrow & & \uparrow \mathrm{Hom}(y^\vee, \varepsilon) \\
 \mathrm{Hom}(y^\vee, \Gamma \gamma \delta B(x)) & \xrightarrow{\mathrm{Hom}(y^\vee, \rho)} & \mathrm{Hom}(y^\vee, \Gamma \alpha \beta B(x))
 \end{array}$$

Now recalling the natural isomorphism $\mathrm{Hom}(-, \Gamma(?)) \cong \mathrm{Hom}(\gamma(-), ?)$ of the adjunction $\gamma \dashv \Gamma$, we can rewrite this commutative square as

$$\begin{array}{ccc}
 \mathrm{Hom}(y^\vee, \delta B(x)) & \xrightarrow{\mathrm{Hom}(y^\vee, \sigma)} & \mathrm{Hom}(\gamma(y^\vee), \alpha(x)) \\
 \gamma \downarrow & & \uparrow \mathrm{Hom}(\gamma(y^\vee), \varepsilon) \\
 \mathrm{Hom}(\gamma(y^\vee), \gamma \delta B(x)) & \xrightarrow{\mathrm{Hom}(\gamma(y^\vee), \rho)} & \mathrm{Hom}(\gamma(y^\vee), \alpha \beta B(x)),
 \end{array}$$

where the fact that the vertical maps on the left agree is by the definition of the unit η of the adjunction $\gamma \dashv \Gamma$. But now the fact that γ is strong monoidal allows us to rewrite this commutative square as

$$\begin{array}{ccc}
 \mathrm{Hom}(\mathbb{1}_{\mathcal{Y}}, y \otimes \delta B(x)) & \xrightarrow{\mathrm{Hom}(y^\vee, \sigma)} & \mathrm{Hom}(\mathbb{1}_{\mathcal{W}}, \gamma(y) \otimes \alpha(x)) \\
 \gamma \downarrow & & \uparrow \mathrm{Hom}(\mathbb{1}_{\mathcal{W}}, \mathrm{id} \otimes \varepsilon) \\
 \mathrm{Hom}(\mathbb{1}_{\mathcal{W}}, \gamma(y) \otimes \gamma \delta B(x)) & \xrightarrow{\mathrm{Hom}(\mathbb{1}_{\mathcal{W}}, \mathrm{id} \otimes \rho)} & \mathrm{Hom}(\mathbb{1}_{\mathcal{W}}, \gamma(y) \otimes \alpha \beta B(x))
 \end{array}
 \quad (\dagger)$$

It remains to simplify the term $\mathrm{Hom}(\mathbb{1}_{\mathcal{Y}}, y \otimes \delta B(x))$.

For this, we recall the projection formula of Lemma 2.4(i), for the adjoint pair $\delta \dashv \Delta$; it gives an isomorphism

$$\Delta(y) \otimes B(x) \xrightarrow{(i)} \Delta[y \otimes \delta B(x)].$$

The definition of the projection map (i) is such that the diagram below commutes

$$\begin{array}{ccc}
 \mathrm{Hom}(\mathbb{1}_{\mathcal{X}}, \Delta(y) \otimes B(x)) & \xrightarrow{\delta} & \mathrm{Hom}(\delta(\mathbb{1}_{\mathcal{X}}), \delta\Delta(y) \otimes \delta B(x)) \\
 \mathrm{Hom}(\mathbb{1}_{\mathcal{X}}, (i)) \downarrow & & \downarrow \mathrm{Hom}(\mathrm{id}, \varepsilon \otimes \mathrm{id}) \\
 (\dagger\dagger) \quad \mathrm{Hom}(\mathbb{1}_{\mathcal{X}}, \Delta[y \otimes \delta B(x)]) & \xrightarrow{\sim} & \mathrm{Hom}(\delta(\mathbb{1}_{\mathcal{X}}), y \otimes \delta B(x)) \\
 & & \parallel \\
 & & \mathrm{Hom}(\mathbb{1}_{\mathcal{X}}, y \otimes \delta B(x)),
 \end{array}$$

where the horizontal isomorphism $\sim$ is from the adjunction $\delta \dashv \Delta$. In particular, the composite from top left to bottom right in the diagram $(\dagger\dagger)$ is an isomorphism. This is obvious if we follow the path where each of the maps is an isomorphism, but must also be true when we follow the other path. And now composing the commutative square $(\dagger)$, with the isomorphism $(\dagger\dagger)$ of the projection formula, shows that the map

$$\mathrm{Hom}(y^{\vee}, \delta B(x)) \xrightarrow{\mathrm{Hom}(y^{\vee}, \sigma)} \mathrm{Hom}(y^{\vee}, \Gamma\alpha(x))$$

that we began with identifies, under the isomorphism of the projection formula, with the more symmetric composite of the proof of Lemma 4.3.

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Amnon Neeman
 Centre for Mathematics and its Applications
 Mathematical Sciences Institute
 Building 145, The Australian National University
 Canberra ACT 2601
 Australia

amnon.neeman@anu.edu.au