Physical Realizability and Coherent LQG Control of Linear Quantum Systems

Rebbecca Tze Yean Thien

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Declaration

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To all control theory enthusiast

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Abstract

Linear quantum systems are a special class of quantum systems whose dynamics are described by the laws of quantum mechanics. Quantum mechanics serves as a platform for comprehending and explaining the workings of the universe at the atomic scale. Control problems in the quantum domain are often more challenging compared to their classical counterparts, primarily due to the additional constraints imposed by quantum mechanics. A linear quantum system generally need not correspond to a physically meaningful system unless it satisfies some additional constraints in order to be a physically realizable quantum system. One way to implement a linear time-invariant (LTI) system as a physically realizable quantum system is to include additional quantum vacuum noise channels. The presence of quantum vacuum noise channels in a quantum controller places limits on its performance. Hence it is desirable to minimize the number (or effect) of these noises.

The first part of this thesis is to improve current approaches for implementing physically realizable quantum systems. In this context, we present an optimal method to implement a strictly proper, LTI system as a physically realizable quantum system. This method focuses on the extent to which the additional quantum noise affects the system output. We also give a necessary and sufficient condition for when a quantum system corresponding to a given LTI controller can be made physically realizable in the presence of both direct feedthrough quantum vacuum noise and additional quantum vacuum noise such that the additional quantum noise does not affect the controller output. Additionally, we give a frequency domain condition to physically realize a given transfer function matrix using only direct feedthrough quantum vacuum noise.

Coherent quantum control is a unique feedback control paradigm with no counterpart in classical control systems. Physical realizability and coherent quantum control are closely related concepts since the condition for a quantum controller to be considered coherent is that the controller must be physically realizable. The second part of this thesis considers the quantum equalization problem. We have proposed a method to find a physically realizable suboptimal coherent linear quadratic Gaussian (LQG) controller that minimizes a cost function related to the system equalization error. Subsequently, we have implemented a gradient descent approach in searching for an optimal solution to the quantum equalization problem.

Notation

$\mathbb{C}$	set of complex numbers
$(.)^{\#}$	complex conjugate of the complex vector or matrix of operators
$(.)^{\dagger}$	adjoint transpose of a vector or matrix of operators
$(.)^{T}$	transpose of a vector or matrix of operators
$(.)^{*}$	adjoint of a Hilbert space operator
$[A, B]$	commutator $AB - BA$ of two operators acting on the same Hilbert space
J	real skew-symmetric matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
$\mathrm{diag}(J, J, \ldots, J)$	a block diagonal matrix with $J, J, \ldots, J$ on its diagonal blocks
$\mathcal{O}$	observability matrix
$\mathcal{U}$	unobservable subspace
$\mathcal{N}(.)$	null space of the matrix
$\mathcal{R}(.)$	range space of the matrix
$\mathrm{sym}(.)$	$\frac{(.)+(.)^T}{2}$
$\mathrm{asym}(.)$	$\frac{(.)-(.)^T}{2}$

List of Acronyms

ARE	algebraic Riccati equation
ALE	algebraic Lyapunov equation
BLM	bilinear model
CCR	canonical commutation relation
LTI	linear time-invariant
LMI	linear matrix inequality
LQG	linear quadratic Gaussian
MME	markovian master equation
NMR	nuclear magnetic resonance
QSDE	quantum stochastic differential equation
SME	stochastic master equation

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Chapter 1

Introduction

Quantum mechanics is a framework for describing and understanding the universe at the atomic level [20] and a quantum system is a system whose dynamics are described by the laws of quantum mechanics [29, 60, 89]. This thesis focuses on the modelling and control of a special class of quantum systems which will be referred to as linear dynamical quantum systems or linear quantum systems for short. Linear quantum systems are alternatively known as linear quantum stochastic systems in some literature, with the qualifier stochastic included to highlight the role of quantum stochastic processes in the dynamics of these systems [50]. The distinctive characteristics of linear quantum systems have shown benefits for the advancement of quantum technology [15, 16, 63]. Quantum technology is an interdisciplinary field that investigates the engineering of devices by exploiting their quantum characteristics [90]. Developments such as quantum communication [34, 66], quantum optics [2, 76], and quantum computation [55, 88] provide opportunities for systems and control researchers, e.g., developing practical applications through estimation and control theories for linear quantum systems [14]. The dynamics of a linear quantum system in the Heisenberg picture of quantum mechanics can be described in terms of a quartet of matrices A, B, C, and D. However, the matrices A, B, C, and D in a linear quantum system cannot be arbitrary but must satisfy a set of constraints imposed by quantum mechanics and these constraints are referred to as physical realizability constraints [30].

This thesis considers two topics in the field of linear quantum systems. The first topic is related to physical realizability where we look into various conditions in implementing an LTI system as a physically meaningful linear quantum system. Physical realizability and coherent quantum control are closely related concepts since the condition for a quantum controller to be considered coherent is that the controller must be physically realizable [29]. This leads to the second topic of this thesis, coherent quantum control, where we use such controllers to solve real-world problems. This chapter serves to give an overview and motivations of both topics and a summary of the thesis's contributions and chapters.

1.1 Physical Realizability

The first topic addressed is the issue of physical realizability of a given linear quantum system represented by a linear quantum stochastic differential equations (QSDEs) [1, 51] with arbitrary constant coefficients.

The authors in [26] derive the QSDEs which describe the unitary evolution of open quantum systems. Then [19] developed a framework to consider quantum systems in terms of input and output relations which paves the way for systems and control theory for quantum systems. Linear QSDEs represent the Heisenberg evolution of pairs of conjugate operators in a multimode open quantum harmonic oscillator, which is coupled to external bosonic fields acting on a boson Fock space and are quantum analogues of classical stochastic differential equations that typically arise in the area of quantum optics [76]. Note that in this thesis, we will use the term "classical" to refer to objects that are not quantum mechanical, i.e., not subject to the laws of quantum mechanics.

A linear quantum system generally need not correspond to a physically meaningful system. Physical linear quantum systems must satisfy some additional constraints that limit the permissible values for the system matrices defining the QSDEs to be considered a meaningful physical system, e.g., the interconnections of various quantum optical devices such as optical cavities, beam splitters, and optical amplifiers [2]. The laws of quantum mechanics dictate that linear quantum systems evolve unitarily, implying that (in the Heisenberg picture) certain canonical commutation relations (CCR) are satisfied at all times.

To describe physically meaningful systems, a formal notion of a physically realizable quantum system has been introduced in [30] where the authors derive necessary and sufficient conditions for such systems in terms of constraints on their system matrices. Physical realizability is crucial in quantum engineering since the fact that only linear quantum systems constructed from tangible, physical components are of use in real-life applications [79, 80], which is in contrast to classical systems which may be regarded as always being physically realizable at least approximately via classical analog or digital electronics [30].

In [30], the authors formulate and solve an H^∞ controller synthesis problem for a class of noncommutative linear stochastic systems. The class of systems considered includes systems that are quantum, classical and quantum-classical (i.e., hybrid). The authors also demonstrated that it is always possible to implement an arbitrary, strictly proper, LTI system as a physically realizable quantum system if additional quantum vacuum noise channels are permitted in the implementation. It is straightforward to obtain upper and lower bounds on the number of introduced quantum noises that are necessary to obtain physical realizability, e.g., the number of introduced quantum noises necessary to physically realize a proper LTI system must be at least as many as the output dimension.

As much as the presence of quantum noise [19] is necessary to physically realize a proper LTI system, introducing quantum noises will present fundamental limits on controller performance, therefore it is desirable to minimize the number of these introduced noises. Therefore further developments are given in [75] where the authors look into physically realizing a proper LTI system that only introduces a minimal number of quantum noises. The authors in [75] give a numerical method to determine the numbers of introduced quantum noises necessary to implement a given, strictly proper, LTI system and also give an algorithm to obtain a physically realizable implementation of a given system.

In [41, 42, 43, 61], the authors also present several key results concerning the physical realizability of linear quantum systems. However, unlike [30, 75] which uses a framework that describes linear quantum systems in terms of position and momentum operators, the results in [41, 42, 43, 61] use an equivalent framework that describes linear quantum systems in terms of creation and annihilation operators.

In [41, 42, 43] the class of systems considered is restricted to systems that can be described using only annihilation operators corresponding to passive systems. The authors in [41] developed a theory that provides a comprehensive understanding of the dynamics of interest and allows a deeper insight into the notion of physical realizability [30] and its connection to the bounded real property. In contrast to [30], the authors in [41] relate the question of physical realizability to the lossless bounded real property of the system without requiring the introduction of additional quantum noises but with the added restriction of only using annihilation operators.

The results in [42] build on the results of another paper by the authors [41]. The work in [42] derives necessary and sufficient conditions of physical realizability for given controller matrices, written in terms of the complex annihilation operator. The physical realizability conditions considered in [42] are directly related to linear systems theory and are simplified as compared to the matrix condition given in [30]. In [43], the authors construct physical realizability conditions for nonlinear quantum systems described by nonlinear quantum stochastic differential equations which is a generalization of the annihilation operator-only linear quantum systems considered in [41, 42].

In [61], the authors extend the physical realizability results of [30] to the most general class of complex linear QSDEs, i.e., in terms of both creation and annihilation operators with complex coefficients. The authors in [61] showed that the property of physical realizability is equivalent to the system theory notion of (J, J)-unitary systems which originally arose in the chain scattering approach to H^∞ control [35]. Related work such as in [64] where the physical realizability condition is equivalent to the frequency domain (J, J)-unitarity of the input-output transfer function. In contrast to [61], the framework that describes linear quantum systems considered in [64] is in terms of position and momentum operators and the authors have simplified the proof which avoids the technical assumptions required in [61].

The outcomes of the first part of this thesis move in the direction where various conditions (related explicitly to quantum vacuum noise) have been developed for the implementation of a physically realizable quantum controller, which is of significance from a quantum engineering point of view.

1.2 Coherent Quantum Control

The second topic addressed is related to coherent quantum control.

Quantum control refers to the control of physical systems whose behaviour is dominated by the laws of quantum mechanics [29] and is a rapidly growing area [4, 9, 10, 25, 38, 44, 46, 58, 68, 69, 73, 74]. Quantum control has a wide range of objectives, spanning from feedback stabilization with various practical applications such as laser cooling to coherent manipulation of quantum states, quantum process engineering and decoherence control [60]. The objective of quantum control theory is to establish a strong theoretical footing and formulate systematic methods for the active manipulation and control of quantum systems [46]. This goal is nontrivial [13] since quantum control is usually taken to apply specifically to systems whose behaviour does not admit an accurate classical description, for example, stable high-powered lasers, and sophisticated cooling techniques have made the quantum regime much more accessible and controllable than before [39, 54].

Quantum control can be broadly categorized into three areas: open-loop control, measurement-based feedback control, and coherent feedback control. In classical systems, open-loop controllers act without knowing the underlying state of the system, i.e., the controller is provided with some information about the system's initial state, but obtains no further details during the control process [39]. It is similar in the quantum case where quantum open-loop control [81] is principally concerned with Hamiltonian control [21]. Quantum open-loop control is a time-dependent control Hamiltonian constructed predetermined to drive a quantum system to a desired quantum state. Quantum open-loop control does not involve any feedback mechanisms [45], see Figure 1.1.

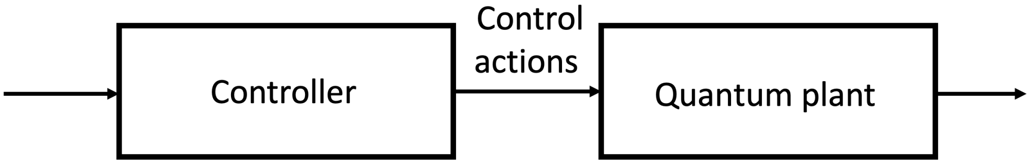

Figure 1.1: Open-loop control.

Two useful approaches for constructing a quantum open-loop controller are quantum optimal control [82] and quantum Lyapunov control [11]. In quantum optimal control, the control objective involves formulating a problem of searching for a set of relevant controls to minimize a cost function related to practical requirements. This has been applied in Nuclear Magnetic Resonance (NMR) experiments to improve the sensitivity of these systems in the presence of relaxation [32, 33, 70]. For quantum Lyapunov control, due to the difficulty of obtaining feedback information without disturbing the system, it is typical to first complete the feedback control design by simulation and then apply the control sequence to the system in an open-loop form [36]. However, quantum Lyapunov control has limitations such as slow convergence and may not be robust against uncertainties and noise [14].

The concept of feedback is fundamental and well-accepted within the realm of classical systems and control. However, in the realm of quantum control, due to the unique characteristics of quantum mechanics, developing quantum feedback control theory [77, 78, 86, 91] is a non-trivial problem [14]. The initial stages of quantum feedback control involved Markovian quantum feedback [83, 85] (which is described by a Markovian Master equation (MME) and the measurement record is fed back onto the system to alter the system dynamics, and is assumed memoryless) and Bayesian quantum feedback [12] (which involves two steps of state estimation and feedback control) [15]. Bayesian feedback is typically more advantageous than Markovian feedback since it uses more information. However, it is harder to implement Bayesian feedback than Markovian feedback due to the existence of the estimation step [84].

When the measurement is taken into account in the feedback loop, the feedback mechanism is typically called measurement-based feedback. In measurement-based quantum feedback control, the information about the system is obtained through measurement which will unavoidably disturb the state of the system [60]: even a non-demolition measurement, that leaves the quantum system in the state in which it was measured, typically alters the state of the quantum system prior to the measurement [39]. A measurement-based quantum feedback controller is a controller in which discrete or continuous measurements of some output channel of an open quantum system are used to adjust the control actions in real-time [45]. The controller can be described classically where it processes and feeds back classical information but the system itself must be described by the laws of quantum mechanics [39], see Figure 1.2.

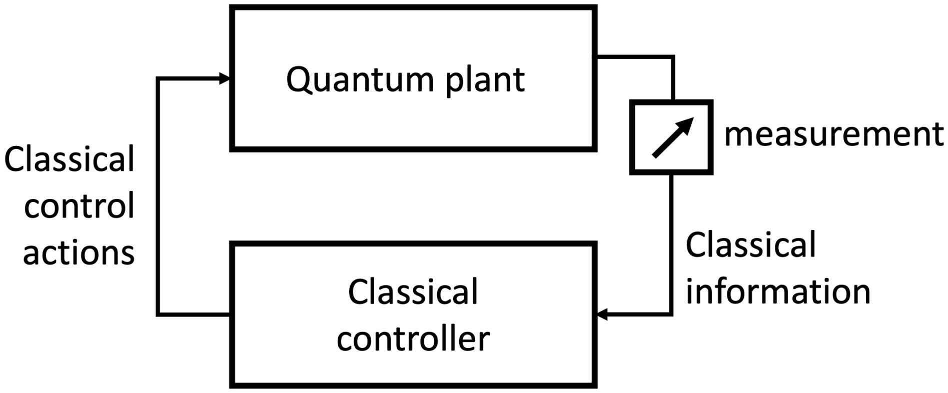

Figure 1.2: Measurement-feedback control.

The classical controller processes the results obtained from measuring an observable state of the quantum system to determine the classical control actions that are applied to control the behaviour of the quantum system [90].

Measurement-based feedback control has been used to improve control performance in a number of different areas, e.g., quantum entanglement [59, 87], quantum state reduction [57], quantum optics [57, 86], and superconducting linear quantum systems [71]. The dynamics of such quantum feedback control systems are often described by quantum stochastic master equations, which can be obtained from quantum filtering theory [7, 59, 86] and are usually challenging to analyze and simulate for complex linear quantum systems [14].

In this thesis, the coherent feedback control is of interest and will be referred to as coherent quantum control. Open-loop and measurement-based feedback controls are entirely analogous to classical open-loop and real-time feedback control, while coherent feedback control is a unique feedback control paradigm that has no counterpart in classical control systems [60]. Coherent quantum control presents additional challenges in that the controller must be physically realizable. However, it has been established for specific quantum systems that quantum feedback is more advantageous than open-loop control in dealing with uncertainties in initial states and it has also been demonstrated via simulation that feedback control is still better than open-loop control in dealing with decoherence [56].

For coherent quantum control, the controller itself is also a quantum system that processes quantum information [14], see Figure 1.3. Coherent quantum control does not involve any classical actuators or measurements, rather, it relies on indirect control of a target quantum system through its coherent interaction with another quantum system acting as the controller and aims to offset the impacts of unforeseen disturbances on a system under control [14, 45]. Coherent quantum control and measurement-based feedback control are fundamentally different schemes owing to the non-trivial effect of the measurement process [21]. Comparisons have been made concerning the performance of measurement-based feedback control and coherent quantum control [27].

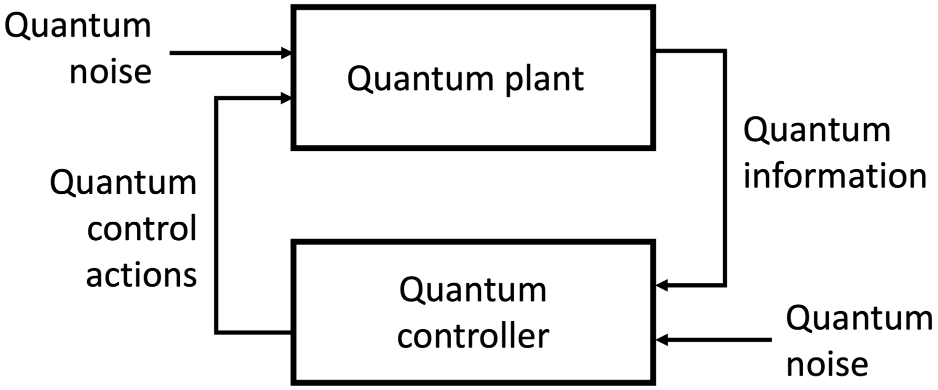

Figure 1.3: Coherent feedback control.

The feedback control of linear quantum systems has attracted considerable interest (e.g., in quantum photonics, for example, where it is possible to use cavities, mirrors, beam splitters, and waveguides to build optical networks that could be used to control the state of atoms or quantum dots [60]) since the use of a quantum controller may lead to improved performance of the system, ease of implementation, or greater controller bandwidth. Moreover, a coherent quantum controller may have a similar time scale as the plant and probably would be much faster than using classical signal processing [90]. Some prominent experimental results presented in [45, 48] have been successful in realizing a coherent quantum control system, especially in [45] where the authors verify the theory proposed in [30].

LQG control problem is one of the most fundamental optimal control theories with the aim of finding an optimal feedback control law for a stochastic linear system by optimising a quadratic cost function [37]. A significant aspect of LQG control is the separation theorem, which applies when the system is linear, the cost function is quadratic in the system variables and the noises are Gaussian [13]. In certain instances, results from classical LQG control can be applied to quantum systems [23]. In the quantum LQG control problem, the optimal control is also a linear feedback controller and the controller may be a classical controller [62, 75] or a quantum controller [40, 45, 49].

The authors in [62] formulate and solve a guaranteed cost control problem for a class of uncertain linear stochastic quantum systems by first establishing a connection with an associated classical system. In [75], the authors provide an algorithm for obtaining a suboptimal solution to a coherent quantum LQG problem. The idea in [75] is to first design a classical LQG controller and then implement the controller as a physically realizable quantum system by introducing additional quantum noise channels.

In contrast, to [62, 75], the work [40, 45, 49] formulates a quantum LQG control problem for linear quantum systems where the controller itself may also be a quantum system and the plant output signal can be fully quantum. Specifically, the authors in [40], consider a coherent LQG control problem for a class of linear quantum systems that can be defined by complex quantum stochastic differential equations in terms of annihilation operators only and provide a method for designing an LQG controller that satisfies the physical realizability conditions.

The results of the second part of this thesis have progressed to the point that coherent LQG control has been applied to solve a real-world problem.

1.3 Summary of Contributions

1. Additional noise in a quantum system can be detrimental to the performance of a coherent quantum control system and these quantum noises are needed to implement a physically realizable coherent quantum control. Since these quantum noises place limits on the achievable controller performance, it is desirable to minimize the number of introduced noises. We propose a Linear Matrix Inequality (LMI) approach in **Chapter 3** to construct an optimal quantum realization of a given LTI system. An optimal method is proposed for solving this problem in terms of a finite horizon quadratic performance index, which is related to the amount of quantum noise appearing at the system's output. This cost function provides a measure of how much the additional quantum noise in the coherent controller will alter the feedback

control system. Our proposed method showed that the performance of the system improves in terms of the specified cost function that is related to the covariance of the system output while maintaining physical realizability.

2. Related also to the number of limiting quantum noises needed in implementing physically realizable coherent quantum control, we extend the results from Chapter 3 in **Chapter 4** by providing an easily testable necessary and sufficient condition on whether it is possible to implement an LTI system as a physically realizable quantum system so that the additional quantum noise has no effect on the system output signal. We also provide an algorithm to obtain such a linear quantum system.

3. When designing LTI quantum controllers it is typical that the transfer function of the controller rather than a specific state space realization governs the closed-loop performance. Consequently, the question of whether a particular transfer function can be implemented as a physically realizable quantum system is often of interest. We introduce a condition in **Chapter 5** for realizing a given transfer function matrix in terms of a J-spectral factorization problem which also leads to a necessary frequency response condition. In order to achieve this, we use a characterization of the existence of a solution to an algebraic Riccati equation (ARE) [75] in terms of the J-spectral factorization of a rational matrix following the approach of [47].

4. We consider a quantum equalization problem in **Chapter 6** where we propose a novel approach to this problem by converting the quantum equalization problem into a coherent LQG problem. This approach provides a simple and systematic way to solve the quantum equalization problem for both passive and active systems. Using our proposed approach, we can design a suboptimal coherent LQG controller (fulfilling the role of an equalization filter) that optimizes the performance of the communication system while minimizing the impact of quantum noises and distortions.

5. Finally in **Chapter 7**, we extend the results in Chapter 6 to find an optimal controller to solve the quantum equalization problem leading to an active controller. We propose applying a gradient descent approach that involves adjusting the controller parameters with a negative gradient flow so that the cost function moves towards a local minimum. Therefore, we demonstrate the application of a gradient descent approach [65] to find an optimal coherent LQG controller.

1.4 Organization of the Thesis

The organization of this thesis is as follows. It is divided into two main parts, with Part I dealing with the first topic (Chapters 3 to 5) and Part II (Chapters 6 and Chapters 7) dealing with the second topic.

Chapter 2: Linear Quantum System Theory
This chapter introduces some useful background material on linear quantum systems theory, specifically the mathematical models and an in-depth understanding of physical realizability for both active and passive systems.

Chapter 3: Optimal Physically Realizable Implementations of State Space Representations
This chapter considers a quantum realization problem that is particularly important in designing coherent quantum feedback controllers. Here, we construct an optimal quantum realization of a given LTI system using an LMI approach. An optimal method is proposed for solving this problem in terms of a finite horizon quadratic performance index, which is related to the amount of quantum noise appearing at the system's output. This cost function provides a measure of how much the additional quantum noise in the coherent controller will alter the feedback control system.

Chapter 4: Decoupled Additional Quantum Noise Conditions for Physically Realizable Implementations of State Space LTI Systems
This chapter also considers a quantum realization problem. In this chapter, we have shown that it is possible to obtain a physically realizable quantum system such that the additional quantum noise does not affect the output.

We provide a necessary and sufficient condition for when a quantum system corresponding to a given LTI controller can be made physically realizable in the presence of both direct feedthrough quantum vacuum noise and additional quantum vacuum noise such that the additional quantum noise has no effect on the controller output.

Chapter 5: A J-Spectral Factorization Condition for the Physical Realizability of a Transfer Function Matrix with only Direct Feedthrough Quantum Noise
This chapter develops a frequency domain condition to physically realize a given transfer function matrix. This involves the relationship between spectral factorization and the solution of an ARE which then allows us to formulate a J-spectral factorization condition for the implementation of a strictly proper transfer function matrix as a physically realizable quantum system using only direct feedthrough quantum noise. Then following from that, we construct a necessary frequency response condition for this property.

Chapter 6: A Suboptimal Coherent LQG approach to Quantum Equalization
This chapter considers a quantum equalization problem. We have proposed a method to find a physically realizable suboptimal coherent LQG quantum controller that minimizes a cost function related to the system equalization error. The method involves reformulating the problem as an LQG control problem. We first design a classical LQG controller and then implement it as a physically realizable quantum system using the algorithm in Chapter 3.

Chapter 7: A Gradient Descent Coherent LQG approach to Quantum Equalization
This chapter extends the results in Chapter 6 in searching for an optimal solution to solve the quantum equalization problem. We have implemented a gradient descent approach to find a physically realizable coherent LQG quantum controller that minimizes a closed-loop cost function related to the system equalization error.

Chapter 8: Conclusion and Future Research

This chapter provides a summary of the thesis and potential directions for future research.

Chapter 2

Linear Quantum System Theory

Linear quantum systems described by the laws of quantum mechanics, take the specific form of a set of linear QSDEs. This kind of linear quantum system model has been applied in the physics and mathematical physics literature dating back to the 1980s [18, 51, 52, 83]. Note that there are four types of models used in quantum control: bilinear models (BLMs), Markovian master equations (MMEs), stochastic master equations (SMEs), and linear QSDEs. The BLMs, MMEs, and SMEs models use the Schrödinger picture of quantum mechanics where equations describing the time dependence of quantum states are given [13].

In this thesis, the QSDE model is being used and a brief description will be given in this chapter. Generally, quantum linear stochastic systems represented by the linear QSDEs with arbitrary constant coefficients need not correspond to physically meaningful systems. Physical quantum systems must satisfy some additional constraints that restrict the allowable values for the system matrices defining the QSDEs. The laws of quantum mechanics dictate that closed quantum systems evolve unitarily, implying that (in the Heisenberg picture) certain canonical observables satisfy the so-called canonical commutation relation (CCR) at all times.

To characterize physically meaningful systems, work in [30] has introduced a formal notion of physically realizable quantum linear stochastic systems in terms of the position and momentum operators and derives a pair of necessary and sufficient conditions for such systems in terms of constraints on their system matrices. In [41], the physical realizability results of [30] have been extended to annihilation operator linear quantum systems. It is shown that this class of linear quantum systems corresponds to the class of real linear quantum systems considered in [30] via a suitable state transformation [53].

The chapter continues as follows. In Section 2.1 and 2.3, we describe annihilation operator linear quantum systems and position and momentum operator linear quantum systems, respectively. Here we focus on some fundamental results of linear quantum systems, the system model and the physical realizability conditions, that are essential to the results in the subsequent chapters. The relationship of the two linear quantum systems discussed is presented in Section 2.2.

2.1 Annihilation Operator Linear Quantum System

For a thorough explanation of annihilation operator linear quantum systems, the reader is referred to [41, 53].

2.1.1 Linear Quantum System Model

Annihilation operator linear quantum systems (also known as a passive system in Chapter 6) can be described using non-commutative or quantum probability theory [7]. This class of linear quantum systems can be used to model passive quantum optical devices such as optical cavities, beam splitters, phase shifters, and interferometers [53].

In particular, the systems under consideration are described in terms of complex annihilation operators satisfying the linear QSDEs

$$\begin{aligned} da(t) &= Fa(t)dt + Gdw(t); \qquad a(0) = a_0 \\ dy(t) &= Ha(t)dt + Kdw(t) \end{aligned} \tag{2.1}$$

where $F \in \mathbb{C}^{n\times n}$, $G \in \mathbb{C}^{n\times n_w}$, $H \in \mathbb{C}^{n_y\times n}$, $K \in \mathbb{C}^{n_y\times n_w}$ (n, n_y, n_w are positive integers). Here

$$a(t) = \begin{bmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \end{bmatrix}$$

is a vector of linear combinations of annihilation operators on an underlying Hilbert space [30, 53].

The quantity $w(t)$ describes the input variables and is assumed to admit the decomposition

$$dw(t) = \beta_w(t)dt + d\tilde{w}(t)$$

where $\tilde{w}(t)$ is the noise part of $w(t)$ and $\beta_w(t)$ is an adapted process [3, 26, 51]. The noise $\tilde{w}(t)$ is an operator-valued process with a vector of quantum Wiener processes with a quantum Ito table

$$d\tilde{w}(t)d\tilde{w}^{\dagger}(t) = F_{\tilde{w}}dt$$

where $F_{\tilde{w}}$ is a nonnegative Hermitian matrix [3, 26, 51]. Here, the notation $(.)^{\dagger}$ represents the adjoint transpose of a vector or a matrix of operators.

It is also assumed that the following communication relations hold for the noise components:

$$[d\tilde{w}(t), d\tilde{w}(t)^{\dagger}] \triangleq d\tilde{w}(t)d\tilde{w}(t)^{\dagger} - (d\tilde{w}(t)d\tilde{w}(t)^{T})^{T} = T_w dt \tag{2.2}$$

where T_w is a Hermitian commutation matrix and the notation $(.)^{T}$ denotes the transpose of a vector or matrix of operators [3, 26, 51]. The noise processes can be represented as operators on an appropriate Fock space [18]. The process $\beta_w(t)$ represents variables of other systems which may be passed to the system (2.1) via an interaction. Therefore, it is required that $\beta_w(0)$ be an operator on a Hilbert space distinct from that of $a(0) = a_0$ and the noise processes. $\beta_w(t)$ is also assumed to commute with $a(t)$ and $d\tilde{w}(t)$ for all $t \geq 0$ since $\beta_w(t)$ is an adapted process.

For the annihilation operator linear quantum system (2.1), the initial system variables $a(0) = a_0$ consist of operators satisfying the commutation relations

$$\left[a_j(0), a_k^*(0)\right] = \Theta_{jk}, \quad j, k = 1, \ldots, n. \tag{2.3}$$

The notation $(.)^*$ denotes the corresponding vector of adjoint operators. Here, the commutator is defined by

$$\left[a_j, a_k^*\right] \triangleq a_j a_k^* - a_k^* a_j = \Theta_{jk}$$

where Θ is a complex matrix with elements Θ_{jk}. With

$$a^T = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix},$$

the relations (2.3) can be written as

$$\left[a, a^\dagger\right] \triangleq aa^\dagger - (a^* a^T)^T = \Theta.$$

The following theorem provides an algebraic characterization of when the annihilation operator linear quantum system (2.1) preserves the commutation relations (2.3) as time evolves.

Theorem 2.1. *[41, Theorem 4.1] For the annihilation operator linear quantum system (2.1), we have that*

$$\left[a_j(0), a_k^*(0)\right] = \Theta_{jk}$$

implies

$$\left[a_j(t), a_k^*(t)\right] = \Theta_{jk}$$

for all $t \geq 0$ *with* j, $k = 1, \ldots, n$ *if and only if*

$$F\Theta + \Theta F^\dagger + GT_wG^\dagger = 0 \tag{2.4}$$

where T_w *is defined as in (2.2).*

Here, the notation $(.)^\dagger$ refers to the complex conjugate transpose of a complex matrix.

The canonical commutation relations correspond to conditions $[a_j, a_j^*] = 1$ and $[a_i, a_j^*] = 0$ for $i \neq j$. In the canonical case, $F_{\tilde{w}}$, T_w, and Θ are defined as the identity and the canonical commutation relations are preserved if and only if

$$F + F^\dagger + GG^\dagger = 0. \tag{2.5}$$

In the generalized canonical case, Θ is a positive definite Hermitian matrix while $F_{\tilde{w}}$ and T_w are defined as identity and the generalized canonical commutation relations are preserved if and only if

$$F\Theta + \Theta F^\dagger + GG^\dagger = 0. \tag{2.6}$$

Theorem 2.2. *[41, Theorem 4.2] Suppose the system (2.1) satisfies (2.6) with* $\Theta = \Theta^\dagger > 0$. *Then there exists a change of system variables* $\tilde{a} = Za$ *such that the corresponding transformed systems satisfies (2.5) with* $\Theta = I$ *and* $T_w = I$. *Here* Z *is a nonsingular* $n \times n$ *complex matrix.*

2.1.2 Physical Realizability

An annihilation operator linear quantum system of the form (2.1) is considered to be physically realizable if the system corresponds to a complex open quantum harmonic oscillator. The class of complex open harmonic oscillators under consideration [7, 17, 51] are defined by n_w measurement channels coupled via an operator

$$L = \Lambda a$$

where Λ is a $n_w \times n$ complex matrix and a Hamiltonian $\mathcal{H}$,

$$\mathcal{H} = a^\dagger M a$$

where M is a $n \times n$ complex Hermitian matrix [41].

We now consider the problem of deriving a system of the form (2.1) from a complex open harmonic oscillator defined by L and $\mathcal{H}$. In [17], the Lindblad generator corresponding to the coupling operator L and the Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{L}[a] = i[\mathcal{H}, a] + \frac{1}{2}(L^\dagger[a, L] + [L^\dagger, a]L)$$

where

$$L^\dagger[a, L] = 0, \qquad [L^\dagger, a]L = -\Theta\Lambda^\dagger\Lambda a, \qquad i[\mathcal{H}, a] = i\Theta M a.$$

Thus,

$$\mathcal{L}[a] = -\Theta(iM + \frac{1}{2}\Lambda^\dagger\Lambda)a.$$

The quantum Langevin equation corresponding to the coupling operator L and the Hamiltonian $\mathcal{H}$ is given by

$$da = \mathcal{L}[a]dt + [a, L]dw^* - [a, L^\dagger]dw$$

where $[a, L] = 0$ and $[a, L^\dagger] = \Theta\Lambda^\dagger$. Hence,

$$da = -\Theta(iM + \frac{1}{2}\Lambda^\dagger\Lambda)a dt - \Theta\Lambda^\dagger dw.$$

Thus, an open quantum oscillator is a system of the form (2.1) in which

$$F = -\Theta(iM + \frac{1}{2}\Lambda^\dagger\Lambda), \qquad G = -\Theta\Lambda^\dagger.$$

Also, extending the approach of [17] to consider both quadratures of the output channel [41] gives

$$dy(t) = Ldt + Idw$$

where $L = \Lambda a$. Now, the open quantum harmonic oscillator is a system of the form (2.1) in which

$$\begin{aligned} H &= \Lambda \\ K &= I. \end{aligned}$$

In this situation, a linear quantum system defined by the matrices F, G, H, and K is a representation of the open quantum harmonic oscillator defined by the coupling operator matrix Λ and the Hamiltonian matrix M with the commutation matrix Θ.

This leads to the formal definition of physical realizability for annihilation operator linear quantum systems.

Definition 2.1. *[41, Definition 5.1] An annihilation operator linear quantum system of the form (2.1) is said to be canonically physically realizable if the vector of operators $a(t)$ satisfies the canonical commutation relations and the system is a representation of a complex open harmonic oscillator with $\Theta = I$.*

Definition 2.2. *[41, Definition 5.2] An annihilation operator linear quantum system of the form (2.1) is said to be physically realizable if the vector of operators $a(t)$ satisfies the generalized commutation relations with $\Theta =$*

$\Theta^\dagger > 0$, $T_w = I$, and the system is a representation of a complex harmonic oscillator with commutation matrix $\Theta > 0$.

The following theorem provides necessary and sufficient conditions for a quantum system of the form (2.1) to be physically realizable.

Theorem 2.3. *[41, Theorem 5.1] An annihilation operator linear quantum system of the form (2.1) is physically realizable if and only if there exists a matrix $\Theta = \Theta^\dagger > 0$ such that*

$$F\Theta + \Theta F^\dagger + GG^\dagger = 0; \tag{2.7}$$

$$G = -\Theta H^\dagger; \tag{2.8}$$

$$K = I. \tag{2.9}$$

In this case, the corresponding Hamiltonian matrix M is given by

$$M = \frac{i}{2}(\Theta^{-1}F - F^\dagger\Theta^{-1}) \tag{2.10}$$

and the corresponding coupling matrix Λ is given by

$$\Lambda = H. \tag{2.11}$$

Note that M is a complex Hermitian matrix. The system (2.1) is canonically physically realizable if and only if the conditions (2.7)-(2.9) are satisfied with $\Theta = I$.

In this case, the corresponding Hamiltonian matrix and coupling matrix is given as in (2.10) and (2.11) with $\Theta = I$.

Theorem 2.4. *[41, Theorem 5.2] If the system (2.1) is physically realizable then there exists a change of system variables*

$$\tilde{a} = Za$$

such that the resulting transformed system is canonically physically realizable with $\Theta = I$. Here Z is a nonsingular $n \times n$ complex matrix.

2.2 Relationship between Position and Momentum Operator and Annihilation Operator Linear Quantum Systems

For a thorough explanation of the relationship between position and momentum operator and annihilation operator linear quantum systems, the reader is referred to [53].

Note that a quantum system of the form (2.1) is generally complex. It is possible to apply a particular change of variables so that all the matrices in the resulting transformed QSDEs are real. This change of variables is defined as follows

$$\begin{bmatrix} q \\ p \end{bmatrix} = \Phi \begin{bmatrix} a \\ a^{\#} \end{bmatrix}; \tag{2.12}$$

$$\begin{bmatrix} \mathcal{Q}^{in}(t) \\ \mathcal{P}^{in}(t) \end{bmatrix} = \Phi \begin{bmatrix} w(t) \\ w(t)^{\#} \end{bmatrix}; \tag{2.13}$$

$$\begin{bmatrix} \mathcal{Q}^{out}(t) \\ \mathcal{P}^{out}(t) \end{bmatrix} = \Phi \begin{bmatrix} y(t) \\ y(t)^{\#} \end{bmatrix} \tag{2.14}$$

where the matrices Φ have the form

$$\Phi = \begin{bmatrix} I & I \\ -iI & iI \end{bmatrix} \tag{2.15}$$

and have the appropriate dimensions. Here q is a vector of the self-adjoint position operators for the system of harmonic oscillators and p is a vector of momentum operators. Also, $\mathcal{Q}^{in}(t)$ and $\mathcal{P}^{in}(t)$ are the vectors of position and momentum operators for the quantum noise fields acting on the system of harmonic oscillators. Furthermore, $\mathcal{Q}^{out}(t)$ and $\mathcal{P}^{out}(t)$ are the vectors of position and momentum operators for the output quantum noise fields.

With the transformation applied to the quantum linear system (2.1), this leads to the following real quantum linear system:

$$\begin{bmatrix} dq(t) \\ dp(t) \end{bmatrix} = A \begin{bmatrix} q(t) \\ p(t) \end{bmatrix} dt + B \begin{bmatrix} d\mathcal{Q}^{in}(t) \\ d\mathcal{P}^{in}(t) \end{bmatrix}; \tag{2.16}$$

$$\begin{bmatrix} d\mathcal{Q}^{out}(t) \\ d\mathcal{P}^{out}(t) \end{bmatrix} = C \begin{bmatrix} q(t) \\ p(t) \end{bmatrix} dt + D \begin{bmatrix} d\mathcal{Q}^{in}(t) \\ d\mathcal{P}^{in}(t) \end{bmatrix}, \tag{2.17}$$

where

$$\begin{aligned} A &= \Phi F \Phi^{-1} \\ &= \frac{1}{2}\begin{bmatrix} F + F^{\#} & i(F - F^{\#}) \\ -i(F - F^{\#}) & F + F^{\#} \end{bmatrix}; \\ B &= \Phi G \Phi^{-1} \\ &= \frac{1}{2}\begin{bmatrix} G + G^{\#} & i(G - G^{\#}) \\ -i(G - G^{\#}) & G + G^{\#} \end{bmatrix}; \\ C &= \Phi H \Phi^{-1} \\ &= \frac{1}{2}\begin{bmatrix} H + H^{\#} & i(H - H^{\#}) \\ -i(H - H^{\#}) & H + H^{\#} \end{bmatrix}; \\ D &= \Phi K \Phi^{-1} \\ &= \frac{1}{2}\begin{bmatrix} K + K^{\#} & i(K - K^{\#}) \\ -i(K - K^{\#}) & K + K^{\#} \end{bmatrix}. \end{aligned}$$

Here, the matrices A, B, C, and D are real.

2.3 Position and Momentum Operator Linear Quantum Systems

For a thorough explanation of the position and momentum operator linear quantum systems, the reader is referred to [30, 53].

2.3.1 Linear Quantum System Model

Position and momentum operator linear quantum systems (referred to as active systems in Chapters 6 and 7) can be described by the following linear QSDEs [7, 26, 30]

$$\begin{aligned} dx(t) &= Ax(t)dt + Bdw_x(t) \\ dy(t) &= Cx(t)dt + Ddw_x(t) \end{aligned} \tag{2.18}$$

where A, B, C, and D are real matrices in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times n_{w_x}}$, $\mathbb{R}^{n_y\times n}$, and $\mathbb{R}^{n_y\times n_{w_x}}$ (n, n_{w_x}, n_y are positive integers), respectively. Moreover

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

is a column vector of self-adjoint, possibly non-commutative, system variables.

The vector quantity w_x describes the input signals and is assumed to admit the decomposition

$$dw_x(t) = \beta_{w_x}(t)dt + d\tilde{w}_x(t)$$

where the self-adjoint, adapted process $\beta_{w_x}(t)$ is the signal part of $w_x(t)$ and $\tilde{w}_x(t)$ is the noise part of $w_x(t)$. Note that $\beta_{w_x}(t)$ also commutes with $d\tilde{w}_x(t)$ for all $t \geq 0$. The noise $\tilde{w}_x(t)$ is a vector of self-adjoint quantum noises with

Ito table

$$d\tilde{w}_x(t) d\tilde{w}_x^T(t) = F_{\tilde{w}_x} dt$$

where $F_{\tilde{w}_x} = S_{\tilde{w}_x} + T_{\tilde{w}_x}$ is a nonnegative Hermitian matrix [4, 51] with $S_{\tilde{w}_x}$ and $T_{\tilde{w}_x}$ being real and imaginary, respectively. $S_{\tilde{w}_x}$ describes the intensity of the quantum Wiener process and is the quantum analog of the intensity matrix for a classical Wiener process. The commutation relations for the noise components are determined by T_{w_x}:

$$[d\tilde{w}_x(t), d\tilde{w}_x(t)^\dagger] \triangleq d\tilde{w}_x(t) d\tilde{w}_x(t)^\dagger - (d\tilde{w}_x(t) d\tilde{w}_x(t)^T)^T = 2T_{w_x} dt. \tag{2.19}$$

The initial system variables $x(0) = x_0$ consist of operators on an appropriate Hilbert space satisfying the commutation relations

$$[x_j(0), x_k(0)] = 2i\Theta_{jk}, \qquad j, k = 1, \ldots, n \tag{2.20}$$

where Θ is a real antisymmetric matrix with components Θ_{jk}, and $i = \sqrt{-1}$. Here the commutator is defined by $[A, B] = AB - BA$. The commutation relations are said to be canonical (i.e., the system is fully quantum) if Θ is of the form

$$\Theta = \begin{bmatrix} J & 0 & \cdots & 0 \\ 0 & J & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J \end{bmatrix} \tag{2.21}$$

where J denotes the real skew-symmetric 2×2 matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The following theorem from [30] provides an algebraic characterization of when the system (2.18) preserves the commutation relations as time evolves.

Theorem 2.5. *[30, Theorem 2.1] The system in (2.18) will satisfy the* *commutation relations for all* $t \geq 0$ *if and only if*

$$iA\Theta + i\Theta A^T + BT_{\tilde{w}_x}B^T = 0. \tag{2.22}$$

2.3.2 Physical Realizability

The following is the definition of an open quantum harmonic oscillator by generalizing slightly the linear model [17, Section 4].

Definition 2.3. *[30, Definition 3.1] The system (2.18) (β_{w_x}=0) is said to be an open quantum harmonic oscillator if* Θ *is canonical and there exists a quadratic Hamiltonian* H

$$H = x(0)^T R x(0),$$

with a real and symmetric Hamiltonian matrix R *of dimension* $n \times n$*, and a coupling operator* $L = \Lambda x(0)$*, and complex-valued coupling matrix* Λ *of dimension* $n_w \times n$*, such that*

$$\begin{aligned} x_k(t) &= U(t)^* x_k(0) U(t); \qquad k = 1, \ldots, n; \\ y_l(t) &= U(t)^* w_l(0) U(t), \qquad l = 1, \ldots, n \end{aligned}$$

where $\{U(t); t \geq 0\}$ *is an adapted process of unitary operators satisfying the following QSDE:*

$$\begin{aligned} dU(t) &= (-iHdt - \frac{1}{2}L^\dagger L dt + [-L^\dagger \quad L^T]\Gamma dw(t))U(t); \\ U(0) &= I. \end{aligned}$$

Here $(.)^\dagger$ *denotes the complex conjugate transpose of a matrix while* $(.)^\#$ *denotes the complex conjugate of a matrix. Moreover, let* $(.)^*$ *denote the adjoint of a Hilbert space operator, i.e., the operator is a map from one Hilbert space to another.*

In this case, the matrices A*,* B*,* C*,* D *are given by*

$$A = 2\Theta(R + \Im(\Lambda^\dagger\Lambda)); \tag{2.23a}$$

$$B = 2i\Theta[-\Lambda^\dagger \quad \Lambda^T]\Gamma; \tag{2.23b}$$

$$C = P_{N_y}^T \begin{bmatrix} \Sigma_{N_y} & 0_{N_y\times N_w} \\ 0_{N_y\times N_w} & \Sigma_{N_y} \end{bmatrix} \begin{bmatrix} \Lambda + \Lambda^\# \\ -i\Lambda + i\Lambda^\# \end{bmatrix}; \tag{2.23c}$$

$$D = [I_{n_y\times n_y} \qquad 0_{n_y\times(n_w-n_y)}]. \tag{2.23d}$$

Here

$$\Gamma = P_{N_w} diag_{N_w}(M);$$
$$M = \frac{1}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix};$$
$$\Sigma_{N_y} = [I_{N_y\times N_y} \quad 0_{N_y\times(N_w-N_y)}];$$
$$P_{N_w}(a_1, a_2, ..., a_{2N_w})^T = (a_1, ..., a_{2N_w-1}, a_2, ..., a_{2N_w})^T;$$

and $diag(M)$ *is an appropriately dimensioned square block diagonal matrix with each diagonal block equal to the matrix* M*. The permutation matrix* P *has the unitary property* $PP^T = P^TP = I$*,* $N_w = n_w/2$*, and* $N_y = n_y/2$*. Note that*

$$P_m^T \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2m} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_{m+1} \\ a_2 \\ a_{m+2} \\ \vdots \\ a_{2m} \end{bmatrix}$$

where P_m *denotes a* $2m \times 2m$ *permutation matrix.*

The following formally defines physical realizability for the more general case of position and momentum operator linear quantum systems.

Definition 2.4. *[30, Definition 3.3] The system (2.18) is said to be physically realizable if Θ is canonical and (2.18) represents the dynamics of an open quantum harmonic oscillator.*

The following theorem [30] gives necessary and sufficient conditions for physical realizability.

Theorem 2.6. *[30, Theorem 3.4] The system (2.18) is physically realizable if and only if*

$$\begin{aligned} iA\Theta_n + i\Theta_n A^T + BT_{w_x}B^T &= 0; \\ B\begin{bmatrix} I_{n_y \times n_y} \\ 0_{(n_w - n_y)\times n_y} \end{bmatrix} &= \Theta C^T P_{N_y}^T \begin{bmatrix} 0_{N_y \times N_y} & I_{N_y \times N_y} \\ -I_{N_y \times N_y} & 0_{N_y \times N_y} \end{bmatrix}; \\ &= \Theta C^T diag_{N_y}(J) \end{aligned}$$

and D satisfies (2.23d).

Here, diag notation indicates a block diagonal matrix assembled from the given entries.

Moreover, for canonical Θ, the Hamiltonian matrix R, and the coupling matrix Λ are expressed as follows

$$\begin{aligned} R &= \frac{1}{4} - \Theta A + A^T\Theta; \\ \Lambda &= -\frac{1}{2}i[0_{N_w \times N_w} \quad I_{N_w \times N_w}](\Gamma^{-1})^T B^T \Theta. \end{aligned}$$

Chapter 3

Optimal Physically Realizable Implementations of State Space Representations

3.1 Introduction

Additional quantum noise [18] in a quantum system can have adverse effects on the efficacy of a coherent quantum control system. However, it is important to note that a certain level of quantum noise is essential for physically realizing an LTI system [30]. In [75], the authors considered the minimum number of additional quantum noises needed to make a given, strictly proper, LTI system physically realizable. This system would correspond to the quantum controller in Figure 1.3 (from Chapter 1) in which an LTI controller is first designed via a classical method such as an LQG [37] control and then additional quantum noises are added to make the controller physically realizable. We extend the results of [75] by focusing on the extent to which the additional quantum noise affects the system output.

In this chapter, we propose an LMI approach to construct an optimal quantum realization of a given LTI system. This involves minimizing a specific performance index, related to the covariance of the system output. The difference between our work and [75] is that the authors of [75] concentrate on minimizing the number of noises that enter the system (the dimension of the noise signal, v_2) whereas, in this chapter, we focus on evaluating the noise effect on the output signal y; see Figure 3.1.

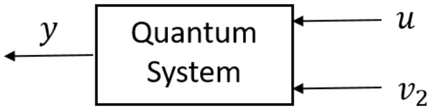

Figure 3.1: Quantum System Block Diagram.

Since our method involves minimizing a specified cost function, this cost function will be lower as compared to the previous method in [75].

The chapter continues as follows. In Section 3.2, we describe the quantum system model used. In Section 3.3, we outline the corresponding physical realizability conditions and related previous results. Then, in Sections 3.4 and 3.5, we formulate our problem and present our algorithm for its solution, respectively. Two examples are given in Section 3.6 followed by a conclusion in Section 3.7.

3.2 Linear Quantum Systems

In this work, we consider a special case of the QSDEs (2.18)

$$\begin{aligned} dx(t) &= Ax(t)dt + B_u du(t) + B_{v_1} dv_1(t) + B_{v_2} dv_2(t); \\ dy(t) &= Cx(t)dt + dv_1(t); \end{aligned} \tag{3.1}$$

see also [30, 75]. Here, $dw_x(t)$ from (2.18) has been partitioned into signal inputs, $du(t)$, direct feed through quantum vacuum noise inputs, $dv_1(t)$, and additional quantum vacuum noises, $dv_2(t)$.

The inputs to the system represents by du(t) (a column vector with n_u

components), like $dw_x(t)$, are assumed to admit the decomposition

$$du(t) = \beta_u(t)dt + d\tilde{u}(t)$$

where the self-adjoint, adapted process $\beta_u(t)$ is the signal part of $u(t)$ and $\tilde{u}(t)$ is the noise part of $u(t)$. Note that $\beta_u(t)$ also commutes with $d\tilde{u}(t)$ for all $t \geq 0$.

The noise $\tilde{u}(t)$ is a vector of self-adjoint quantum noises with Ito table

$$d\tilde{u}(t)d\tilde{u}^T(t) = F_{\tilde{u}}dt$$

where $F_{\tilde{u}} = S_{\tilde{u}} + T_{\tilde{u}}$ is a nonnegative Hermitian matrix [4, 51] with $S_{\tilde{u}}$ and $T_{\tilde{u}}$ being real and imaginary, respectively. $S_{\tilde{u}}$ describes the intensity of the quantum Wiener process and is the quantum analog of the intensity matrix for a classical Wiener process.

The commutation relations for the noise components are determined by T_u:

$$[d\tilde{u}(t), d\tilde{u}(t)^\dagger] \triangleq d\tilde{u}(t)d\tilde{u}(t)^\dagger - (d\tilde{u}(t)d\tilde{u}(t)^T)^T = 2T_u dt. \tag{3.2}$$

Also, $dv_1(t)$ and $dv_2(t)$ (column vectors with n_{v_1} and n_{v_2} components respectively) are quantum Wiener processes corresponding to the introduced vacuum noise inputs. Consequently, F_{v_1}, S_{v_1}, T_{v_1}, F_{v_2}, S_{v_2}, and T_{v_2} are defined for $dv_1(t)$ and $dv_2(t)$ respectively as $F_{\tilde{w}_x}$, $S_{\tilde{w}_x}$, and $T_{\tilde{w}_x}$ were for $d\tilde{w}_x(t)$ in (2.18).

3.3 Previous Results

In Chapter 2, the notion of physical realizability for position and momentum operator linear quantum systems was introduced and will be used in this chapter.

Now, we will restate these results for the model (3.1), introduced in the preceding section.

Definition 3.1. *([30, Definition 3.1] and Definition 2.3) The system (3.1) is physically realizable if there exists a real, symmetric,* $n \times n$ *matrix* R*, and a complex-valued* $\frac{1}{2}(n_{v_1} + n_{v_2} + n_u) \times n$ *matrix* Λ *such that the matrices* A*,* B_u*,* B_{v_1}*,* B_{v_2}*, and* C *are given by*

$$A = 2\Theta(R + \Im(\Lambda^\dagger \Lambda)) \tag{3.3a}$$

$$\begin{bmatrix} B_{v_1} & B_{v_2} & B_u \end{bmatrix} = 2i\Theta[-\Lambda^\dagger \quad \Lambda^T]\Gamma \tag{3.3b}$$

$$C = P_{N_y}^T \begin{bmatrix} \Sigma_{N_y} & 0_{N_y \times N_w} \\ 0_{N_y \times N_w} & \Sigma_{N_y} \end{bmatrix} \begin{bmatrix} \Lambda + \Lambda^\# \\ -i\Lambda + i\Lambda^\# \end{bmatrix} \tag{3.3c}$$

$$D = [I_{n_y \times n_y} \qquad 0_{n_y \times (n_w - n_y)}]. \tag{3.3d}$$

where Θ *is of the form (2.21). Here*

$$\Gamma = P\,diag(M);$$
$$M = \frac{1}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix};$$
$$\Sigma_{N_y} = [I_{\frac{1}{2}n_y \times \frac{1}{2}n_y} \quad 0_{\frac{1}{2}n_y \times \frac{1}{2}(n_{v_1} + n_{v_2} + n_u - n_y)}]$$

and P *is as defined in Definition 2.3.*

Lemma 3.1. *([30, Definition 3.4] and Definition 2.6) The system (3.1) is physically realizable if and only if*

$$iA\Theta + i\Theta A^T + B_{v_1} T_{v_1} B_{v_1}^T + B_{v_2} T_{v_2} B_{v_2}^T + B_u T_u B_u^T = 0;$$
$$B_{v_1} = \Theta C^T diag(J)$$

where Θ *is of the form (2.21).*

The authors in [30] considered the issue of physical realizability where necessary and sufficient conditions were derived for a given controller state space matrices to be physically realizable as shown in Definition 3.1 and

Lemma 3.1. Particularly, the following theorem relating to physical realizability was proved.

Theorem 3.1. *[30, Lemma 5.6] Let* A, B, *and* C *be real matrices in* $\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n_w}$, *and* $\mathbb{R}^{n_y\times n}$ $(n, n_w, n_y$ *are positive integers), respectively. Also, let* $F_{\tilde{u}}$ *be a block diagonal matrix with each diagonal block equal to* $I + iJ$ *and let* Θ *be canonical as defined in (2.21). Then, there exists an even integer* $n_{v_2} \geq 0$ *and matrices* $B_{v_1} \in \mathbb{R}^{n\times n_{v_1}}, B_{v_2} \in \mathbb{R}^{n\times n_{v_2}}$, *such that the corresponding system (3.1) is physically realizable.*

This work was extended in [75] where the minimum number of additional quantum noises n_{v_2} to make a system physically realizable was addressed through the following theorem:

Theorem 3.2. *[75, Theorem 1] A system with given* A, B_u, *and* C *matrices is considered. Then, there exist matrices* B_{v_1} *and* B_{v_2} *such that the system (3.1) is physically realizable with* $n_{v_2} = r$ *where* r *is the rank of the matrix*

$$\Theta B_u \Theta_{n_u} B_u^T \Theta - \Theta A - A^T \Theta - C^T \Theta_{n_y} C.$$

Furthermore, if there exist matrices B_{v_1} *and* B_{v_2} *such that the system (3.1) is physically realizable, then* $n_{v_2} \geq r$

In [30, 75], the matrices R, Λ, B_{v_1}, and B_{v_2} in (2.23) are constructed as follows by applying Theorem 2.6 to the system (3.1)

$$R = -\frac{1}{4}(\Theta A + A^T\Theta^T); \tag{3.4a}$$

$$\Lambda = [\frac{1}{2}C^T P^T \begin{bmatrix} I \\ iI \end{bmatrix} \Lambda_{b_1}^T \quad \Lambda_{b_2}^T]^T; \tag{3.4b}$$

$$B_{v_1} = \Theta C^T \mathrm{diag}(J); \tag{3.4c}$$

$$B_{v_2} = 2i\Theta[-\Lambda_{b_1}^\dagger \quad \Lambda_{b_1}^T] P \mathrm{diag}(M). \tag{3.4d}$$

Here, Λ_{b_1} is any complex $(1/2)n_{v_2} \times n$ matrix such that

$$\Lambda_{b_1}^\dagger \Lambda_{b_1} = \Xi_1 + i\left(\frac{A^T\Theta^T - \Theta A}{4} - \frac{1}{4}C^T P^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} PC - \Im(\Lambda_{b_2}^\dagger \Lambda_{b_2})\right), \tag{3.5}$$

and

$$\Lambda_{b_2} = -i[-\Lambda_{b_1}^\dagger \quad \Lambda_{b_1}^T]\ P\mathrm{diag}(M).$$

The matrix Ξ_1 in (3.5) is defined in [30, 75] as any real symmetric $n \times n$ matrix such that $\Lambda_{b_1}^\dagger \Lambda_{b_1}$ is nonnegative definite. According to [75], equation (3.5) can be rewritten as

$$\Xi_2 = \Xi_1 + \frac{i}{4}\tilde{S} \geq 0 \tag{3.6}$$

where

$$\tilde{S} = \Theta B_u \Theta_{n_u} B_u^T \Theta - \Theta A - A^T \Theta - C^T \Theta_{n_y} C \tag{3.7}$$

and $\tilde{S}$ is real and skew-symmetric which leads to

$$S = \frac{i}{4}\tilde{S} \tag{3.8}$$

being Hermitian.

Our main interest in this chapter is to choose B_{v_2} to minimize the effect of the noise v_2 on the system's output. Therefore, B_{v_2} is chosen to minimize the associated finite horizon quadratic performance index which is given as

$$J_{tf}(t_f) = \int_0^{t_f} \big\langle (Cx)^T(t)Cx(t)\big\rangle dt \tag{3.9}$$

for the system (3.1). Here the term $Cx(t)$ in the system output in (3.1) is taken as the performance variable in the cost function that will be optimized.

In [62], the authors presented lemmas that provide a crucial link between this quadratic cost function and the system matrices.

Lemma 3.2. *[62] The cost (3.9) can be expressed as*

$$J_{tf}(t_f) = \int_0^{t_f} Tr(CQ(t)C^T)dt \tag{3.10}$$

where $Q(t)$ is defined as

$$Q(t) = \frac{1}{2}\big\langle (x(t)x^T(t) + (x(t)x^T(t))^T\big\rangle. \tag{3.11}$$

Indeed, using equation (3.11), equation (3.10) follows from

$$\begin{aligned}\big\langle (Cx)^T Cx\big\rangle &= \big\langle Tr((Cx)^T Cx)\big\rangle;\\ &= \big\langle Tr((x^T C^T Cx)\big\rangle;\\ &= \frac{1}{2}\big\langle Tr(C^T C[xx^T + (xx^T)^T])\big\rangle;\\ &= Tr(C^T CQ);\\ &= Tr(CQC^T).\end{aligned}$$

Using the quantum Ito rule,

$$\begin{aligned} dQ(t) &= \frac{1}{2}(\langle(dx(t)x^T(t)\rangle + \langle(dx(t)x^T(t))^T\rangle) + \langle x(t)dx^T(t)\rangle \\ &+ \langle(x(t)dx^T(t))^T\rangle + (B_{v_2}F_{v_2}B_{v_2}^T + (B_{v_2}F_{v_2}B_{v_2}^T)^T)dt; \\ &= (AQ(t) + Q(t)A^T + \frac{1}{2}B_{v_2}(F_{v_2} + F_{v_2}^T)B_{v_2}^T)dt; \\ &= (AQ(t) + Q(t)A^T + B_{v_2}B_{v_2}^T) \end{aligned}$$

where $\frac{1}{2}(F_{v_2} + F_{v_2}^T) = I$*; i.e., all noises are canonical. Therefore,* $Q(t)$ *defined in equation (3.11) satisfies the differential equation*

$$\dot{Q}(t) = AQ(t) + Q(t)A^T + B_{v_2}B_{v_2}^T. \tag{3.12}$$

We consider the infinite horizon case in which we define

$$\lim_{t_f \to \infty} Q(t) = Q \tag{3.13}$$

and therefore

$$\begin{aligned} \limsup_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} Tr(CQ(t)C^T)dt &= Tr(CQC^T) \\ &= J_{cf}. \end{aligned} \tag{3.14}$$

Then equation (3.12) implies

$$A^TQ + QA + B_{v_2}B_{v_2}^T = 0. \tag{3.15}$$

In our work, we are interested in investigating the performance of the system by looking at the defined cost function while maintaining physical realizability. This differs from [75] where the authors developed an algorithm to obtain a physically realizable system with a minimal number of additional quantum noises.

3.4 Problem Formulation

By looking at equations (3.4) which follow from the requirement of physical realizability, it is straightforward to see that B_{v_1} is a fixed matrix that cannot be modified. Also, B_u from equations (3.1) is a fixed matrix and $dv_1(t)$ is the direct feed through quantum vacuum noise which therefore also cannot be avoided. Consequently, in choosing the matrix B_{v_2} we will concentrate only on the effect of the noise v_2 on the system output term Cx. Therefore, we consider the system

$$\begin{aligned} dx(t) &= Ax(t)dt + B_{v_2}dv_2(t); \\ dy(t) &= Cx(t)dt. \end{aligned} \tag{3.16}$$

Our problem can be formulated as given a fixed choice of Θ, find B_{v_2} that minimizes J_{cf} for the system (3.16) subject to the constraint (3.6) and that the system (3.1) is physically realizable; i.e., matrices A, B_u, B_{v_1}, B_{v_2} and C in equations (3.1) satisfy the corresponding conditions of Theorem 2.6.

To simplify the problem, we reformulate it as an optimization problem by first transforming (3.6) into an LMI constraint

$$\begin{bmatrix} \Xi_1 & \frac{\tilde{S}}{4} \\ -\frac{\tilde{S}}{4} & \Xi_1 \end{bmatrix} \geq 0. \tag{3.17}$$

Also, the term $B_{v_2}B_{v_2}^T$ from equations (3.15) can be rewritten as

$$\begin{aligned} B_{v_2}B_{v_2}^T &= 2i\Theta[-\Lambda_{b_1}^\dagger \quad \Lambda_{b_1}^T]Pdiag(M)diag(M^\dagger)P^T \begin{bmatrix} -\Lambda_{b_1} \\ \bar{\Lambda}_{b_1} \end{bmatrix} \Theta 2i; \\ &= -2\Theta[-\Lambda_{b_1}^\dagger \quad \Lambda_{b_1}^T] \begin{bmatrix} -\Lambda_{b_1} \\ \bar{\Lambda}_{b_1} \end{bmatrix} \Theta; \\ &= -2\Theta(\Lambda_{b_1}^\dagger\Lambda_{b_1} + \Lambda_{b_1}^T\bar{\Lambda}_{b_1})\Theta. \end{aligned}$$

Meanwhile,

$$\begin{aligned}\mathrm{Re}(\Lambda_{b_1}^{\dagger}\Lambda_{b_1}) &= \frac{\Lambda_{b_1}^{\dagger}\Lambda_{b_1} + \Lambda_{b_1}^{T}\bar{\Lambda}_{b_1}}{2} \\ &= \Xi_1 \geq 0,\end{aligned}$$

which means that $B_{v_2}B_{v_2}^T$ can be rewritten as

$$\begin{aligned}B_{v_2}B_{v_2}^T &= -4\Theta\Xi_1\Theta \\ &= 4i\Theta\Xi_1 i\Theta \geq 0.\end{aligned} \tag{3.18}$$

Finally, our problem can be reformulated as follows: We wish to minimize $J_{cf} = tr(CQC^T)$ with respect to Ξ_1 and Q subject to the constraints

$$\Xi_1 \geq 0; \tag{3.19a}$$

$$Q \geq 0; \tag{3.19b}$$

$$A^TQ + QA - 4\Theta\Xi_1\Theta = 0; \tag{3.19c}$$

$$\begin{bmatrix} \Xi_1 & \frac{\tilde{S}}{4} \\ -\frac{\tilde{S}}{4} & \Xi_1 \end{bmatrix} \geq 0. \tag{3.19d}$$

Once the optimal solution Ξ_1 is obtained from this LMI problem, we can construct the required matrix B_{v_2} from equation (3.18) using straightforward matrix factorization of a positive semi-definite matrix.

3.5 Algorithm

In this section, we describe our algorithm to obtain an optimal physically realizable (as per Theorem 2.6) implementation of the system

$$\begin{aligned}dx(t) &= Ax(t)dt + B_u du(t) \\ dy(t) &= Cx(t)dt,\end{aligned} \tag{3.20}$$

where A is taken to be a stable matrix. Furthermore, the algorithm optimizes B_{v_2} so as to minimize the defined cost function (3.14). The algorithm is as follows:

1. Beginning with the matrices A, B_u, and C in (3.20), construct $\tilde{S}$ (3.7)

   $$\tilde{S} = \Theta B_u \Theta_{n_u} B_u^T \Theta - \Theta A - A^T \Theta - C^T \Theta_{n_y} C$$

   and S (3.8)

   $$S = \frac{i}{4}\tilde{S}$$

   (see [75] for further details).

2. Find Ξ_1 by solving the LMI problem (3.19) using an optimization tool; e.g., CVX [22].

3. Evaluate the cost function (3.14)

   $$\limsup_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} Tr(CQ(t)C^T)dt = Tr(CQC^T) = J_{cf}.$$

4. Find B_{v_1} using equation (3.4c)

   $$B_{v_1} = \Theta C^T \text{diag}(J).$$

5. Find $\Lambda_{b_1}^\dagger \Lambda_{b_1}$ using equation (3.5)

   $$\Lambda_{b_1}^\dagger \Lambda_{b_1} = \Xi_1 + i\left(\frac{A^T\Theta^T - \Theta A}{4} - \frac{1}{4}C^T P^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} PC - \Im(\Lambda_{b_2}^\dagger \Lambda_{b_2})\right).$$

6. Find Λ_{b_1} using a positive semi-definite matrix factorization method; e.g., eigen-decomposition [5].

7. Then, find B_{v_2} using equation (3.4d)

   $$B_{v_2} = 2i\Theta[-\Lambda_{b_1}^\dagger \quad \Lambda_{b_1}^T]P\text{diag}(M).$$

3.6 Examples

In this section, we demonstrate our results with two examples. We apply the algorithm proposed in Section 3.5 to implement an LTI system as a physically realizable quantum system. Furthermore, we compare the performance of our algorithm (in terms of the cost function (3.14)) with the method proposed by [75].

3.6.1 Example 1

The example considered here was adapted from [30, 75] where we consider an example of quantum H^∞ controller synthesis. The controller in [30] is implemented as a degenerate canonical controller with both classical and quantum degrees of freedom. In our examples, the implementations have only quantum degrees of freedom.

The controller from [30, 75] is of the form (3.20) with

$$A = \begin{bmatrix} -1.389 & 0 & -0.447 & 0 \\ 0 & -1.389 & 0 & -0.447 \\ -0.200 & 0 & -0.250 & 0 \\ 0 & -0.200 & 0 & -0.250 \end{bmatrix};$$

$$B_u = \begin{bmatrix} -0.447 & 0 \\ 0 & -0.447 \\ 0 & 0 \\ 0 & 0 \end{bmatrix};$$

$$C = \begin{bmatrix} -0.447 & 0 & 0 & 0 \\ 0 & -0.447 & 0 & 0 \end{bmatrix}.$$

We now begin by applying the algorithm in Section 3.5.

1. Construct the matrices

$$
\begin{aligned}
\tilde{S} &= \Theta B_u \Theta_{n_u} B_u^T \Theta - \Theta A - A^T \Theta - C^T \Theta_{n_y} C \\
&= \begin{bmatrix} 0 & 2.379 & 0 & 0.647 \\ -2.379 & 0 & -0.647 & 0 \\ 0 & 0.647 & 0 & 0.500 \\ -0.647 & 0 & -0.500 & 0 \end{bmatrix}; \\
S &= \frac{i}{4}\tilde{S} \\
&= \begin{bmatrix} 0 & 0.595i & 0 & 0.162i \\ -0.595i & 0 & -0.162i & 0 \\ 0 & 0.162i & 0 & 0.125i \\ -0.162i & 0 & -0.125i & 0 \end{bmatrix}.
\end{aligned}
$$

2. Solving the corresponding LMI problem (3.19) using CVX [22], we obtain

$$
\Xi_1 = \begin{bmatrix} 0.595 & 0 & 0.162 & 0 \\ 0 & 0.595 & 0 & 0.162 \\ 0.162 & 0 & 0.125 & 0 \\ 0 & 0.162 & 0 & 0.125 \end{bmatrix}.
$$

3. Evaluating the optimal value of the cost function gives

$$
\begin{aligned}
J_{cf} &= Tr(CQC^T) \\
&= 0.339.
\end{aligned}
$$

4. Finding the matrix B_{v_1} gives

$$\begin{aligned} B_{v_1} &= \Theta C^T \mathrm{diag}(J) \\ &= \begin{bmatrix} 0.447 & 0 \\ 0 & 0.447 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

5. Finding $\Lambda_{b_1}^\dagger \Lambda_{b_1}$ gives

$$\begin{aligned} \Lambda_{b_1}^\dagger \Lambda_{b_1} &= \Xi_1 + i\left(\frac{A^T\Theta^T - \Theta A}{4} - \frac{1}{4}C^T P^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} PC - \Im(\Lambda_{b_2}^\dagger \Lambda_{b_2})\right) \\ &= \begin{bmatrix} 0.595 & 0.595i & 0.162 & 0.162i \\ -0.595i & 0.595 & -0.162i & 0.162 \\ 0.162 & 0.162i & 0.125 & 0.125i \\ -0.162i & 0.162 & -0.125i & 0.125 \end{bmatrix}. \end{aligned}$$

6. Finding Λ_{b_1} and B_{v_2} gives

$$\begin{aligned} \Lambda_{b_1} &= \begin{bmatrix} 0.081i & -0.081 & -0.261i & 0.261 \\ -0.767i & 0.767 & -0.239i & 0.239 \end{bmatrix}; \\ B_{v_2} &= 2i\Theta[-\Lambda_{b_1}^\dagger \quad \Lambda_{b_1}^T]P\mathrm{diag}(M) \\ &= \begin{bmatrix} 0 & -0.162 & 0 & 1.534 \\ 0.162 & 0 & -1.534 & 0 \\ 0 & 0.522 & 0 & 0.477 \\ -0.522 & 0 & -0.477 & 0 \end{bmatrix}. \end{aligned}$$

Consider the system (3.20) with A, B_u, and C as given above, and with the calculated B_{v_1} and B_{v_2}. This then defines the physically realizable system (3.1) with two direct feedthrough quantum noises and four additional quantum noises. In addition, the evaluated cost function is 0.339.

3.6.2 Example 2

For this example, the matrices A (where A is Hurwitz), B_u, and C are randomly generated. The matrices are as follows

$$A = \begin{bmatrix} -0.480 & -3.586 & 0.332 & 3.117 \\ 3.253 & -0.752 & -2.146 & -2.383 \\ -0.774 & 2.525 & -0.809 & -1.634 \\ -3.392 & 1.277 & 2.220 & -0.887 \end{bmatrix};$$

$$B_u = \begin{bmatrix} 0 & 0.318 \\ 0.536 & 0.163 \\ 0.693 & 1.159 \\ -0.688 & 0.128 \end{bmatrix};$$

$$C = \begin{bmatrix} 1.040 & -0.648 & 0 & -0.056 \\ 0 & -0.038 & 0.882 & -0.841 \end{bmatrix}.$$

1. Construct the matrices

$$\begin{aligned} \tilde{S} &= \Theta B_u \Theta_{n_u} B_u^T \Theta - \Theta A - A^T \Theta - C^T \Theta_{n_y} C \\ &= \begin{bmatrix} 0 & 1.442 & -2.344 & 4.540 \\ -1.442 & 0 & 2.399 & 0.270 \\ 2.344 & -2.399 & 0 & 0.761 \\ -4.540 & -0.270 & -0.761 & 0 \end{bmatrix}; \\ S &= \frac{i}{4}\tilde{S} \\ &= \begin{bmatrix} 0 & 0.360i & -0.586i & 1.135i \\ -0.360i & 0 & 0.600i & 0.067i \\ 0.586i & -0.600i & 0 & 0.190i \\ -1.135i & -0.067i & -0.190i & 0 \end{bmatrix}. \end{aligned}$$

2. Solving the corresponding LMI problem (3.20) using CVX [22], we

obtain

$$\Xi_1 = \begin{bmatrix} 3.045 & -1.917 & 2.549 & -1.375 \\ -1.917 & 2.136 & -1.956 & 1.496 \\ 2.549 & -1.956 & 2.656 & -1.155 \\ -1.375 & 1.496 & -1.155 & 1.755. \end{bmatrix}.$$

3. Evaluating the optimal value of the cost function gives

$$\begin{aligned} J_{cf} &= Tr(CQC^T) \\ &= 7.211. \end{aligned}$$

4. Finding the matrix B_{v_1} gives

$$\begin{aligned} B_{v_1} &= \Theta C^T \mathrm{diag}(J) \\ &= \begin{bmatrix} 0.038 & -0.648 \\ 0 & -1.040 \\ 0.841 & -0.056 \\ 0.882 & 0 \end{bmatrix}. \end{aligned}$$

5. Finding $\Lambda_{b_1}^\dagger \Lambda_{b_1}$ gives

$$\begin{aligned} \Lambda_{b_1}^\dagger \Lambda_{b_1} &= \Xi_1 + i \left(\frac{A^T\Theta^T - \Theta A}{4} - \frac{1}{4} C^T P^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} PC - \Im(\Lambda_{b_2}^\dagger \Lambda_{b_2}) \right) \\ &= \begin{bmatrix} 3.045 & -1.917 + 0.360i & 2.549 - 0.586i & -1.375 + 1.135i \\ -1.917 - 0.360i & 2.136 & -1.956 + 0.600i & 1.496 + 0.067i \\ 2.549 + 0.586i & -1.956 - 0.600i & 2.656 & -1.155 + 0.190i \\ -1.375 - 1.135i & 1.496 - 0.067i & -1.155 - 0.190i & 1.754 \end{bmatrix}. \end{aligned}$$

6. Finding Λ_{b_1} and B_{v_2} gives

$$\Lambda_{b_1} = \begin{bmatrix} 0.238 - 0.403i & 0.197 - 0.574i & 0.541 + 0.069i & 0.809 \\ -1.494 - 0.771i & 1.274 + 0.378i & -1.518 - 0.235i & 1.049 \end{bmatrix};$$

$$\begin{aligned} B_{v_2} &= 2i\Theta[-\Lambda_{b_1}^{\dagger} \quad \Lambda_{b_1}^{T}]P\text{diag}(M) \\ &= \begin{bmatrix} 1.149 & 0.394 & -0.757 & 2.548 \\ -0.805 & -0.475 & -1.543 & 2.987 \\ 0 & 1.617 & 0 & 2.098 \\ 0.139 & -1.082 & -0.469 & 3.035 \end{bmatrix}. \end{aligned}$$

Consider the system (3.20) with A, B_u, and C as given above, and with the calculated B_{v_1} and B_{v_2}. This then defines the physically realizable system (3.1) with two direct feedthrough quantum noises and four additional quantum noises. In addition, the evaluated cost function is 7.211.

3.6.3 Comparison with the algorithm from the previous method

We now compare the performance of the algorithm of this work with the algorithm proposed in [75] in terms of the cost function (3.14). The results are tabulated in Table 3.1. In our first example, both algorithms gave the same value of the cost function (3.14). However, in our second example, the algorithm proposed here performs better than the method of [75].

	Example 1	Example 2
Our method	0.339	7.211
[75] method	0.339	12.258

Table 3.1: Calculated cost function from both examples using two different methods.

3.7 Conclusions

Physical realizability is an essential element for realizing meaningful physical quantum systems. This is particularly important in designing coherent feedback control systems in which the controller is required to be a physically realizable quantum system, but the performance of the system is also crucial.

In this work, we have proposed an optimal method for solving the quantum realization problem in terms of a finite horizon quadratic performance index, which is related to the amount of quantum noise appearing at the system's output. This cost function provides a measure of how much the additional quantum noise in the coherent controller will alter the feedback control system. Our proposed algorithm is demonstrated through examples and shows that the evaluated cost function may give better performance than the previous method [75].

Chapter 4

Decoupled Additional Quantum Noise Conditions for Physically Realizable Implementations of State Space LTI Systems

4.1 Introduction

In this chapter, we extend the results from Chapter 3 where here we focus on the case where additional quantum noise is required for physically realizing an LTI system but it can be implemented in such a way that it has no effect on the controller output. The chapter continues as follows. In Sections 4.2 and 4.3, we describe the quantum linear system models under consideration and define the corresponding notion of physical realizability. Then, in Section 4.4, we present our main results. An algorithm that implements our main result and an example is given in Sections 4.5 and 4.6 respectively. Then a conclusion is given in Section 4.7.

4.2 Linear Quantum System

In this work, we consider a special case of (2.18) which is a similar model to (3.1):

$$\begin{aligned} dx(t) &= Ax(t)dt + B_u du(t) + B_{v_1} dv_1(t) + B_{v_2} dv_2(t); \\ dy(t) &= Cx(t)dt + dv_1(t). \end{aligned} \tag{4.1}$$

Here $du(t)$ is a vector of signal inputs, $dv_1(t)$ is a vector of direct feedthrough quantum vacuum noise inputs and $dv_2(t)$ is a vector of additional quantum vacuum noises.

The inputs to the system represented by $du(t)$ (a column vector with n_u components), are assumed to admit the decomposition

$$du(t) = \beta_u(t)dt + d\tilde{u}(t)$$

where the self-adjoint, adapted process $\beta_u(t)$ is the signal part of $u(t)$ and $\tilde{u}(t)$ is the noise part of $u(t)$. Note that $\beta_u(t)$ also commutes with $d\tilde{u}(t)$ for all $t \geq 0$.

The noise $\tilde{u}(t)$ is a vector of self-adjoint quantum noises with Ito table

$$d\tilde{u}(t)d\tilde{u}^T(t) = F_{\tilde{u}}dt$$

where $F_{\tilde{u}} = S_{\tilde{u}} + T_{\tilde{u}}$ is a nonnegative Hermitian matrix [4, 51] with $S_{\tilde{u}}$ and $T_{\tilde{u}}$ being real and imaginary, respectively. $S_{\tilde{u}}$ describes the intensity of the quantum Wiener process and is the quantum analog of the intensity matrix for a classical Wiener process.

The commutation relations for the noise components are determined by T_u:

$$[d\tilde{u}(t), d\tilde{u}(t)^\dagger] \triangleq d\tilde{u}(t)d\tilde{u}(t)^\dagger - (d\tilde{u}(t)d\tilde{u}(t)^T)^T = 2T_u dt. \tag{4.2}$$

Also, $dv_1(t)$ and $dv_2(t)$ (column vectors with n_{v_1} and n_{v_2} components respectively) are quantum Wiener processes corresponding to the introduced vacuum noise inputs. Consequently, F_{v_1}, S_{v_1}, T_{v_1}, F_{v_2}, S_{v_2}, and T_{v_2} are defined for $dv_1(t)$ and $dv_2(t)$ respectively as $F_{\tilde{w}_x}$, $S_{\tilde{w}_x}$, and $T_{\tilde{w}_x}$ were for $d\tilde{w}_x(t)$ in (2.18).

4.3 Physical Realizability

The notion of physical realizability [30] of a quantum system is similar to Definition 3.1 and Lemma 3.1. Then the work in [30] was extended by [75] and in a similar way to the proof of [75, Theorem 2], we obtain the following lemma relating to the physical realizability of the system (4.1) that is related to our problem:

Lemma 4.1. *Suppose*

$$B_{v_1} = \Theta C^T \Theta_{v_1}. \tag{4.3}$$

Then there exists a matrix B_{v_2} such that the system (4.1) is physically realizable if and only if there exists a matrix

$$\Xi_1 = \Xi_1^T \geq 0$$

such that

$$\begin{bmatrix} -4\Theta\Xi_1\Theta & -\Theta\tilde{S}\Theta \\ \Theta\tilde{S}\Theta & -4\Theta\Xi_1\Theta \end{bmatrix} \geqslant 0 \tag{4.4}$$

where

$$\tilde{S} = \Theta B_u \Theta_{n_u} B_u^T \Theta - \Theta A - A^T \Theta - C^T \Theta_{n_y} C. \tag{4.5}$$

Furthermore if the condition (4.4) is satisfied, the matrix B_{v_2} is defined as follows:

Let Λ_{b_1} be any complex matrix such that

$$\Lambda_{b_1}^\dagger \Lambda_{b_1} = \Xi_1 + \frac{i}{4}\tilde{S} \geq 0.$$

Then

$$B_{v_2} = 2i\Theta[-\Lambda_{b_1}^{\dagger} \quad \Lambda_{b_1}^{T}]P\mathrm{diag}(M). \tag{4.6}$$

This matrix also satisfies

$$B_{v_2}B_{v_2}^{T} = -4\Theta\Xi_1\Theta. \tag{4.7}$$

Also, if there exists a B_{v_2} such that the system (4.1) is physically realizable, then the matrix Ξ_1 satisfying (4.4) will be such that (4.7) is satisfied.

Proof. Suppose there exists a matrix B_{v_2} such that the system (4.1) is physically realizable. Then, it follows from (2.23b) that there exists a complex matrix

$$\Lambda = \begin{bmatrix} \Lambda_0 \\ \Lambda_{b_1} \\ \Lambda_{b_2} \end{bmatrix} \tag{4.8}$$

such that

$$[B_{v_1} \quad B_{v_2} \quad B_u] = 2i\Theta[-\Lambda \quad \Lambda^{T}]P\mathrm{diag}(M). \tag{4.9}$$

Then as in the proof of [75, Theorem 2], it follows that

$$\Xi_1 + \frac{i}{4}\tilde{S} \geqslant 0 \tag{4.10}$$

where $\Xi_1 = Re(\Lambda_{b_1}^{\dagger}\Lambda_{b_1}) \geq 0$. Now, pre and post multiplying by Θ in equation (4.10), we obtain the following

$$-4\Theta\Xi_1\Theta - i\Theta\tilde{S}\Theta \geqslant 0 \tag{4.11}$$

which is equivalent to equation (4.4).

Also, it follows from (4.6) that equation (4.7) will be satisfied. Indeed,

$$\begin{aligned} B_{v_2}B_{v_2}^T &= 2i\Theta[-\Lambda_{b_1}^{\dagger}\Lambda_{b_1}^{T}]Pdiag(M)diag(M^{\dagger})P^T\begin{bmatrix}-\Lambda_{b_1}\\ \bar{\Lambda}_{b_1}\end{bmatrix}\Theta i2;\\ &= -2\Theta[-\Lambda_{b_1}^{\dagger}\Lambda_{b_1}^{T}]\begin{bmatrix}-\Lambda_{b_1}\\ \bar{\Lambda}_{b_1}\end{bmatrix}\Theta;\\ &= -2\Theta(\Lambda_{b_1}^{\dagger}\Lambda_{b_1}+\Lambda_{b_1}^{T}\bar{\Lambda}_{b_1})\Theta. \end{aligned}$$

Meanwhile,

$$\begin{aligned} \mathrm{Re}(\Lambda_{b_1}^{\dagger}\Lambda_{b_1}) &= \frac{\Lambda_{b_1}^{\dagger}\Lambda_{b_1}+\Lambda_{b_1}^{T}\bar{\Lambda}_{b_1}}{2}\\ &= \Xi_1\\ &\geq 0 \end{aligned}$$

which means $B_{v_2}B_{v_2}^T$ can be rewritten as

$$B_{v_2}B_{v_2}^T = -4\Theta\Xi_1\Theta.$$

Conversely, suppose (4.4) holds and hence (4.11) is also satisfied. We will show that this implies the existence of matrices R and Λ such that equations (3.3a)-(3.3d) are satisfied and hence the system (4.1) is physically realizable. Pre and post multiplying by Θ in equation (4.11), we obtain

$$\Xi_1 + i\frac{\tilde{S}}{4} \geq 0. \tag{4.12}$$

Then, we construct the matrix Λ_{b_1} such that

$$\Lambda_{b_1}^{\dagger}\Lambda_{b_1} = \Xi_1 + i\frac{\tilde{S}}{4}. \tag{4.13}$$

Also, we let

$$\Lambda_{b_2} = -i[I \quad 0]P\mathrm{diag}(M)B_u^T\Theta$$

and

$$\Lambda_{b_0} = \frac{1}{2}[I \quad iI]PC.$$

Matrix Λ is defined as in (4.8). Also, we define

$$R = -\frac{1}{4}(\Theta A + A^T\Theta).$$

Then it follows as in the proof of [75, Theorem 2] that equations (3.3a)-(3.3d) are satisfied with B_{v_1} and B_{v_2} defined as in (4.3) and (4.6) respectively. Also, it follows from (4.6) that (4.7) will be satisfied. □

4.4 Main Result

Following from the requirement of physical realizability given in Lemma 4.1, B_{v_1} is a fixed matrix defined by equation (4.3) that cannot be modified and $dv_1(t)$ is the direct feed through quantum vacuum noise which therefore also cannot be avoided. Therefore, in choosing the matrix B_{v_2} we will concentrate only on the effect of the additional noise dv_2 on the output signal.

In this section, we consider the problem of whether is it possible for the system (4.1) to be physically realizable in such a way that the additional quantum noise dv_2 has no effect on the system output y. This would correspond to the transfer function $C(sI - A)^{-1}B_{v_2}$ being zero.

The following theorem is our main result of this chapter, which provides an easily testable condition to determine if this is possible.

Theorem 4.1. *Let*

$$\tilde{S} = \Theta B_u \Theta_{n_u} B_u^T \Theta - \Theta A - A^T\Theta - C^T\Theta_{n_y}C \tag{4.14}$$

where the matrices A, B_u and C are given. Then there exist matrices B_{v_1}

and B_{v_2} *such that the system*

$$\begin{aligned} dx(t) &= Ax(t)dt + B_u du(t) + B_{v_1} dv_1(t) + B_{v_2} dv_2(t); \\ dy(t) &= Cx(t)dt + dv_1(t) \end{aligned} \tag{4.15}$$

is physically realizable and

$$C(sI - A)^{-1} B_{v_2} \equiv 0 \tag{4.16}$$

if and only if

$$\mathcal{O}\Theta\tilde{S}\Theta\mathcal{O}^T = 0 \tag{4.17}$$

where $\mathcal{O}$ *is the observability matrix associated with the pair (A, C) and* $\tilde{S}$ *is determined as in (4.5).*

In order to prove this theorem, we will use Finsler's Theorem which is restated as follows.

Lemma 4.2. *(see [28] for proof). Let* P *and* Q *be symmetric matrices such that*

$$P \geqslant 0$$

and

$$x^* Q x \geqslant 0$$

for all $x \in \mathbb{R}^n$ *such that* $x^* P x = 0$*. Then there exists a constant* $\alpha > 0$ *such that*

$$Q + \alpha P \geqslant 0 \quad \textit{(positive semi-definite).} \tag{4.18}$$

Proof of Theorem 4.1. The proof is structured as follows: we first prove the sufficiency of (4.17) and then prove the necessity of (4.17).

Let $\mathcal{U}$ denote the unobservable subspace of (A, C); i.e., $\mathcal{U} = \mathcal{N}(\mathcal{O})$. Here $\mathcal{N}(\mathcal{O})$ denotes the null space of the matrix $\mathcal{O}$. Construct a matrix M such that its columns are a basis for $\mathcal{U}$. Let

$$\mathrm{U} = MM^T. \tag{4.19}$$

Suppose that x_1 and x_2 are state vectors in the observable subspace; i.e., $x_1, x_2 \in \mathcal{R}(\mathcal{O}^T)$. Here $\mathcal{R}(\mathcal{O}^T)$ denotes the range space of the matrix $\mathcal{O}^T$. Then, we can write

$$x_1 = \mathcal{O}^T \mathrm{u}_1, \quad x_2 = \mathcal{O}^T \mathrm{u}_2$$

for some u_1 and u_2.

Hence,

$$2x_1^T \Theta \tilde{S} \Theta x_2 = 2\mathrm{u}_1^T \mathcal{O} \Theta \tilde{S} \Theta \mathcal{O}^T \mathrm{u}_2^T. \tag{4.20}$$

Now suppose (4.17) is satisfied. Then

$$2x_1^T \Theta \tilde{S} \Theta x_2 = 0. \tag{4.21}$$

Hence,

$$x^T \begin{bmatrix} 0 & -\theta \tilde{S} \Theta \\ \Theta \tilde{S} \Theta & 0 \end{bmatrix} x = 0 \tag{4.22}$$

for all $x_1, x_2 \in \mathcal{R}(\mathcal{O}^T)$.

However,

$$\begin{aligned} \mathcal{R}(\mathcal{O}^T) &= \mathcal{N}(\mathcal{O})^\perp \\ &= \mathcal{R}(M)^\perp \\ &= \mathcal{N}(M^T) \\ &= \mathcal{N}(MM^T) \\ &= \mathcal{N}(\mathrm{U}). \end{aligned}$$

Hence,

$$x^T \begin{bmatrix} 0 & -\theta\tilde{S}\theta \\ \theta\tilde{S}\theta & 0 \end{bmatrix} x = 0 \tag{4.23}$$

for all $x \in \mathbb{R}^n$ such that $x^* \mathrm{U} x = 0$. Using Lemma 4.2, it follows that there exists a constant $\alpha > 0$ such that

$$\begin{bmatrix} \alpha\mathrm{U} & -\Theta\tilde{S}\Theta \\ \Theta\tilde{S}\Theta & \alpha\mathrm{U} \end{bmatrix} \geqslant 0. \tag{4.24}$$

Let $\Xi_1 = \frac{\alpha}{4}\Theta U \Theta$. That is

$$\alpha U = -4\Theta\Xi_1\Theta.$$

Then

$$\begin{bmatrix} -4\Theta\Xi_1\Theta & -\Theta\tilde{S}\Theta \\ \Theta\tilde{S}\Theta & -4\Theta\Xi_1\Theta \end{bmatrix} \geqslant 0 \tag{4.25}$$

and the system (4.15) is physically realizable using Lemma 4.1 provided that we choose B_{v_1} and B_{v_2} according to equations (4.3) and (4.6) respectively. It also follows from Lemma 4.1 that

$$\begin{aligned} B_{v_2} B_{v_2}^T &= -4\Theta\Xi_1\Theta \\ &= \alpha U. \end{aligned}$$

Furthermore,

$$\begin{aligned} \mathcal{R}(B_{v_2} B_{v_2}^T) &= \mathcal{R}(\mathrm{U}) \\ &= \mathcal{N}(\mathcal{O}) \\ &= \mathcal{U} \end{aligned}$$

which implies that the condition (4.16) is satisfied. This completes the first part of the proof.

We now establish the necessity part of the theorem. Suppose the system (4.15) is physically realizable and that condition (4.16) is satisfied. Then by Lemma 4.1, we have that there exists a $\Xi_1 \geq 0$ such that

$$\begin{bmatrix} -4\Theta\Xi_1\Theta & -\Theta\tilde{S}\Theta \\ \Theta\tilde{S}\Theta & -4\Theta\Xi_1\Theta \end{bmatrix} = \begin{bmatrix} B_{v_2}B_{v_2}^T & -\Theta\tilde{S}\Theta \\ \Theta\tilde{S}\Theta & B_{v_2}B_{v_2}^T \end{bmatrix} \geq 0. \tag{4.26}$$

Furthermore, condition (4.16) implies that

$$\mathcal{R}(B_{v_2}B_{v_2}^T) = \mathcal{R}(B_{v_2}) \subset \mathcal{N}(\mathcal{O}).$$

Equivalently [5],

$$\mathcal{R}(\mathcal{O}^T) = \mathcal{N}(\mathcal{O})^{\perp} \subset \mathcal{R}(B_{v_2})^{\perp} = \mathcal{N}(B_{v_2}^T).$$

Therefore, given

$$x_1, x_2 \in \mathcal{R}(\mathcal{O}^T),$$

then

$$x_1, x_2 \in \mathcal{N}(B_{v_2}^T).$$

Hence, it follows from equation (4.26) that

$$[x_1^T \; x_2^T] \begin{bmatrix} 0 & -\Theta\tilde{S}\Theta \\ \Theta\tilde{S}\Theta & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geqslant 0$$

for all $x_1, x_2 \in \mathcal{R}(\mathcal{O}^T)$. This implies

$$2x_2^T\Theta\tilde{S}\Theta x_1 = 0 \tag{4.27}$$

for all $x_1, x_2 \in \mathcal{R}(\mathcal{O}^T)$.

However,

$$x_1, x_2 \in \mathcal{R}(\mathcal{O}^T)$$

implies that there exists u_1 and u_2 such that

$$x_1 = \mathcal{O}^T \mathrm{u}_1$$

and

$$x_2 = \mathcal{O}^T \mathrm{u}_2.$$

Hence, equation (4.27) can be rewritten as

$$2\mathrm{u}_2^T \mathcal{O}\Theta\tilde{S}\Theta\mathcal{O}^T \mathrm{u}_1 = 0$$

for all u_1 and u_2. This equality holds only when

$$\mathcal{O}\Theta\tilde{S}\Theta\mathcal{O}^T = 0.$$

□

4.5 Algorithm

We now give an algorithm for realizing a quantum system using the method arising from the proof of Thereom 4.1:

1. Beginning with the matrices A, B_u, and C in (4.15), compute the observability matrix $\mathcal{O}$ and check its rank.
2. Construct $\tilde{S}$ as in (4.14).
3. Construct $\mathcal{O}\Theta\tilde{S}\Theta\mathcal{O}^T$ and check that if it is zero. If so, the system can be made physically realizable such that dv_2 has no effect on the system output.
4. In this case, construct M such that its column are a basis of $\mathcal{U}$.
5. Construct $\mathrm{U} = MM^T$.
6. Choose $\alpha > 0$ so that equation (4.24) is satisfied.
7. Find B_{v_1} using equation (4.3).

8. Find Ξ_1 from $-4\Theta\Xi_1\Theta = \alpha U$. Next, find $\Lambda_{b_1}^{\dagger}\Lambda_{b_1}$ using equation (4.13).
9. Find Λ_{b_1} using a positive semi-definite matrix factorization method; e.g., eigen-decomposition [5]. Then, find B_{v_2} using equation (4.6).
10. Check the condition (4.16).
11. Check the physical realizability conditions of Lemma 3.1 are satisfied.

Note that when the system (4.15) is observable, (4.17) implies that $\tilde{S}$ defined in (4.14) will be zero then the main result in [75] will imply that the system can be physically realized without any additional noise.

In this work, we have shown that an unobservable system is physically realizable without any additional noise when (4.17) is satisfied. In this case, if (4.17) is satisfied, then it can be shown that the matrix $\tilde{S}$ defined in (4.14) for the observable subsystem will be zero.

4.6 Example

In this section, we demonstrate our results with an example. Given matrices A, B_u, and C (where A is Hurwitz and diagonal)

$$A = \begin{bmatrix} -0.794 & 0 & 0 & 0 \\ 0 & -0.792 & 0 & 0 \\ 0 & 0 & -0.847 & 0 \\ 0 & 0 & 0 & -0.847 \end{bmatrix};$$

$$B_u = \begin{bmatrix} -0.116 & 0 \\ 0 & -0.116 \\ 0 & -0.042 \\ -0.042 & 0 \end{bmatrix};$$

$$C = \begin{bmatrix} -1.255 & 0 & 0 & 0 \\ 0 & -1.253 & 0 & 0 \end{bmatrix}.$$

1. Compute the observability matrix

$$\mathcal{O} = \begin{bmatrix} -1.255 & 0 & 0 & 0 \\ 0 & -1.253 & 0 & 0 \\ 0.997 & 0 & 0 & 0 \\ 0 & 0.993 & 0 & 0 \\ -0.792 & 0 & 0 & 0 \\ 0 & -0.787 & 0 & 0 \\ 0.629 & 0 & 0 & 0 \\ 0 & 0.623 & 0 & 0 \end{bmatrix};$$

   that has rank

$$rank(\mathcal{O}) = 2.$$

   This shows that the system is an unobservable system.

2. Construct the matrix

$$\tilde{S} = \begin{bmatrix} 0 & 0 & 0.005 & 0 \\ 0 & 0 & 0 & -0.005 \\ -0.005 & 0 & 0 & 1.696 \\ 0 & 0.005 & -1.696 & 0 \end{bmatrix}$$

   Note that since the calculated $\tilde{S}$ is not equal to zero, the method proposed in [75] cannot be applied. Then, we calculate $\mathcal{O}\Theta\tilde{S}\Theta\mathcal{O}^T$ to be zero within the accuracy of our calculations.

3. Construct the matrix

$$M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

4. Find the matrix U

$$MM^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

5. Choose $\alpha > 0$ to satisfy (4.24) to obtain

$$\alpha = 8;$$

see Figure 4.1.

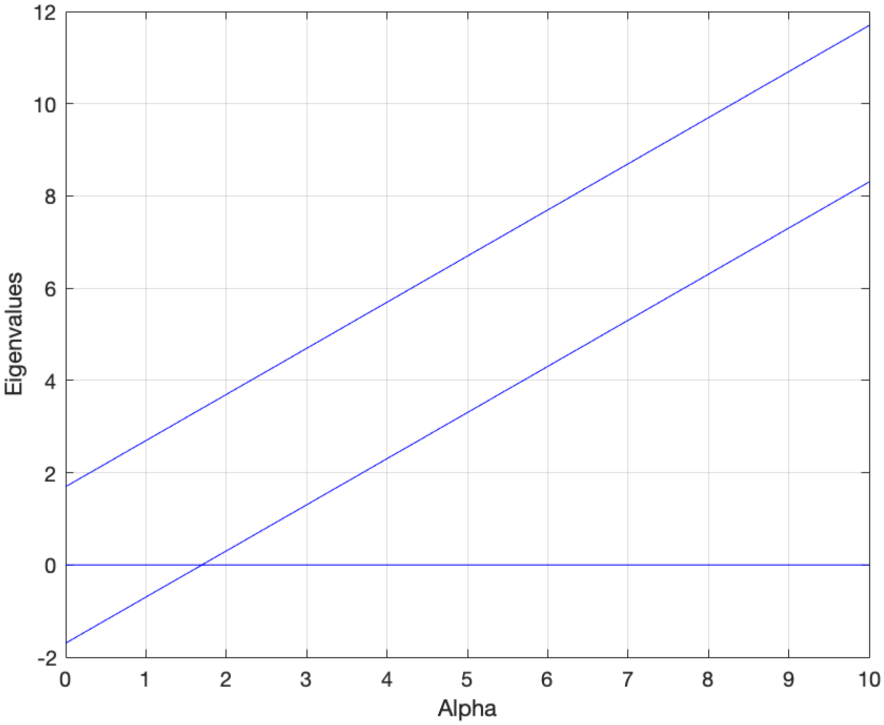

Figure 4.1: Eigenvalues of $\begin{bmatrix} \alpha \mathrm{U} & -\Theta\tilde{S}\Theta \\ \Theta\tilde{S}\Theta & \alpha \mathrm{U} \end{bmatrix}$ as a function of α.

6. Find the matrix B_{v_1} using equation (4.3)

$$B_{v_1} = \begin{bmatrix} 1.253 & 0 \\ 0 & 1.255 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

7. Find Ξ_1 and $\Lambda_{b_1}^{\dagger}\Lambda_{b_1}$ using $-4\Theta\Xi_1\Theta = \alpha U$ and equation (4.13) respectively

$$\Xi_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix};$$

$$\Xi_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0.424i \\ 0 & 0 & -0.424i & 2 \end{bmatrix}.$$

8. Find Λ_{b_1} using eigenvalue decomposition to obtain

$$\Lambda_{b_1} = \begin{bmatrix} -0.001 & -0.001i & -0.888i & -0.888 \\ -0.001 & 0.001i & -1.101i & 1.101 \end{bmatrix}.$$

9. Find B_{v_2} using equation (4.6)

$$B_{v_2} = \begin{bmatrix} 0.001 & 0 & 0.001 & 0 \\ 0 & 0.001 & 0 & 0.001 \\ 0 & -1.775 & 0 & 2.202 \\ -1.775 & 0 & -2.202 & 0 \end{bmatrix}.$$

10. Check (4.16)

$$C(sI - A)^{-1}B_{v_2} \cong 0.$$

11. We check the condition of Lemma 3.1 to be approximately zero with the accuracy of 10^{-5}.

This then defines the physically realizable system (4.15) such that the additional noise dv_2 does not affect the output.

4.7 Conclusions

The performance of the closed loop system is important in designing coherent feedback control systems in which the controller is required to be a physically realizable quantum system.

In this work, we have shown that it is possible to obtain a physically realizable quantum system such that the additional quantum noise does not affect the output. We provide a necessary and sufficient condition as to when this is possible. An example shows the implementation of our result.

Chapter 5

A J-Spectral Factorization Condition for the Physical Realizability of a Transfer Function Matrix with only Direct Feedthrough Quantum Noise

5.1 Introduction

The authors in [75] provided a condition given in terms of a non-standard ARE for physically realizing a given transfer function matrix by only introducing direct feedthrough quantum noises. It is well-known [47] that there is a relationship between spectral factorization and the solution of an ARE. This motivates us to consider a J-spectral factorization approach to physically realize a given transfer function matrix. In this chapter, we present a condition for realizing a given transfer function matrix in terms of a J-spectral factorization problem [35, 67]. This also leads to a necessary frequency response condition. The chapter continues as follows. In Sections 5.2 and 5.3, we describe the quantum linear system models under consideration and define the corresponding notion of physical realizability. Then, in Section 5.4, we present our main results. Examples are given in Section 5.5 followed by a conclusion in Section 5.6.

5.2 Linear Quantum System

In this work, we consider a special case of (2.18)

$$\begin{aligned} dx(t) &= Ax(t)dt + B_u du(t) + B_v dv(t); \\ dy(t) &= Cx(t)dt + dv(t). \end{aligned} \tag{5.1}$$

Here $du(t)$ is the signal input vector, and $dv(t)$ is a vector of direct feedthrough quantum vacuum noise inputs.

The inputs to the system represented by $du(t)$ (a column vector with n_u components), are assumed to admit the decomposition

$$du(t) = \beta_u(t)dt + d\tilde{u}(t)$$

where the self-adjoint, adapted process $\beta_u(t)$ is the signal part of $u(t)$ and $\tilde{u}(t)$ is the noise part of $u(t)$. Note that $\beta_u(t)$ also commutes with $d\tilde{u}(t)$ for all $t \geq 0$.

The noise $\tilde{u}(t)$ is a vector of self-adjoint quantum noises with Ito table

$$d\tilde{u}(t)d\tilde{u}^T(t) = F_{\tilde{u}}dt$$

where $F_{\tilde{u}} = S_{\tilde{u}} + T_{\tilde{u}}$ is a nonnegative Hermitian matrix [4, 51] with $S_{\tilde{u}}$ and $T_{\tilde{u}}$ being real and imaginary, respectively. $S_{\tilde{u}}$ describes the intensity of the quantum Wiener process and is the quantum analog of the intensity matrix for a classical Wiener process.

The commutation relations for the noise components are determined by

T_u:

$$[d\tilde{u}(t), d\tilde{u}(t)^\dagger] \triangleq d\tilde{u}(t)d\tilde{u}(t)^\dagger - (d\tilde{u}(t)d\tilde{u}(t)^T)^T = 2T_u dt. \tag{5.2}$$

Also, $dv(t)$ (a column vector with n_v components) is a quantum Wiener process corresponding to the introduced vacuum noise inputs. Consequently, F_v, S_v, and T_v are defined for $dv(t)$ as $F_{\tilde{w}_x}$, $S_{\tilde{w}_x}$, and $T_{\tilde{w}_x}$ were defined for $d\tilde{w}_x(t)$ in (2.18).

5.3 Physical Realizability

The notion of physical realizability [30] of a quantum system is similar to Definition 3.1 and Lemma 3.1.

Here $(.)^\dagger$ denotes the complex conjugate transpose of a matrix while $(.)^\#$ denotes the complex conjugate of a matrix. We consider a strictly proper $n_y \times n_u$ transfer function matrix $G(s)$ with McMillan degree [31] n where n, n_u, and n_y are all even.

Definition 5.1. *Consider an $n_y \times n_u$ strictly proper transfer function matrix $G(s)$ of McMillan degree n where n, n_u, and n_y are all even. $G(s)$ is said to be physically realizable with only direct feedthrough quantum noise if there exists a minimal realization $G(s) = C(sI - A)^{-1}B_u$ and a matrix B_v such that the system (5.1) is physically realizable.*

The following theorem from [75] gives a state-space condition in the form of a non-standard ARE under which a strictly proper transfer function matrix can be implemented as a physically realizable quantum system, which only introduces direct feedthrough quantum noise.

Theorem 5.1. *Consider an $n_y \times n_u$ strictly proper transfer function matrix*

$G(s)$ of McMillan degree n with minimal state-space realization

$$G(s) = C(sI - A)^{-1}B_u \tag{5.3}$$

where n, n_u, and n_y are all even. This transfer function matrix is physically realizable with only direct feedthrough quantum noise if and only if the ARE

$$A^T X + XA - XB_u\Theta_{n_u}B_u^T X + C^T\Theta_{n_y}C = 0 \tag{5.4}$$

has a non-singular, real, skew-symmetric solution X.

Proof. Sufficiency follows directly from [75, Theorem 2]. Necessity is straightforward to verify from [75, Theorem 1] using a suitable state-space transformation. □

Associated with the state-space realization (5.3) and the Riccati equation (5.4) is the Hamiltonian matrix

$$H = \begin{bmatrix} A & -B_u\Theta_{n_u}B_u^T \\ -C^T\Theta_{n_y}C & -A^T \end{bmatrix} \tag{5.5}$$

where Θ_{n_u} and Θ_{n_y} are defined as in (2.21) and are of appropriate dimensions.

Remark 5.1. *Note that H and $-H^T$ are similar whereby λ_H is an eigenvalue of H if and only if $-\lambda_H^\dagger$ is also an eigenvalue of H; e.g., see [92, pp. 327-328].*

In this chapter, instead of a state-space condition, we want to find a frequency domain condition such that a given transfer function matrix is a physically realizable quantum system with only direct feedthrough quantum noise. To achieve this, we use a characterization of the existence of a solution to the non-standard ARE (5.4) in terms of the J-spectral factorization of a rational matrix following the approach of [47].

5.4 Main Result

In this section, we will show that a given transfer function matrix $G(s)$ is physically realizable with only direct feedthrough noise if and only if there exists a J-spectral factorization of a certain transfer function matrix $\Phi_J(s)$.

This $n_u \times n_u$ matrix $\Phi_J(s)$ is defined as

$$\Phi_J(s) = \Theta_{n_u} - G^{\sim}(s)\Theta_{n_y}G(s) \tag{5.6}$$

where

$$G^{\sim}(s) = G(-s)^T.$$

We now consider some assumptions on $G(s)$. For a given minimal realization $G(s) = C(sI - A)^{-1}B_u$, we assume the following:

A1. *The matrix A is Hurwitz;*

A2. *The matrix A and the Hamiltonian matrix defined in (5.5) H have no common eigenvalues.*

Note, it follows from the property of Hamiltonian matrices given in Remark 5.1 that Assumption A2 also implies that the matrix $-A^T$ and the matrix H will have no common eigenvalues.

5.4.1 A J-Spectral Factorization Problem

The J-spectral factorization problem considered in this chapter is defined as follows.

Definition 5.2. *An $n_u \times n_u$ rational matrix $N(s)$ defines a J-spectral factorization of $\Phi_J(s)$ if the following conditions hold:*

C1. $\Phi_J(s) = N^{\sim}(s)\Theta_{n_u}N(s)$*;*

C2. $N(s)$ *is analytic in Re* $s \geq 0$*;*

C3. $N^{-1}(s)$ *has no poles in common with* $N(s)$*;*

C4. $\lim_{s\to\infty} N(s) = I$*.*

Theorem 5.2. *Let $G(s)$ be a given $n_y \times n_u$ strictly proper transfer function matrix with McMillan degree n and minimal realization $G(s) = C(sI - A)^{-1}B_u$ where n, n_u, and n_y are all even. Also, let $\Phi_J(s)$ be defined as in (5.6) and suppose Assumptions A1- A2 are satisfied. Then $G(s)$ is physically realizable with only direct feedthrough quantum noise if and only if $\Phi_J(s)$ has a J-spectral factorization.*

Proof. The proof is structured as follows and follows [47]: We prove necessity and sufficiency for the existence of a skew-symmetric solution X to the non-standard ARE (5.4) and then apply Theorem 5.1.

Necessity: Suppose $G(s)$ is physically realizable with only direct feedthrough quantum noise. It follows from Theorem 5.1 that there exists an X which is a skew-symmetric solution of the non-standard ARE (5.4).

Let $N(s)$ be defined as follows

$$N(s) = I + \Theta_{n_u}B_u^T X(sI - A)^{-1}B_u. \tag{5.7}$$

We first show $N(s)$ satisfies condition C1 in Definition 5.2. Indeed

$$\begin{aligned}
&N^{\sim}(s)\Theta_{n_u}N(s)\\
&= [I + B_u^T(-sI - A^T)^{-1}XB_u\Theta_{n_u}]\Theta_{n_u} \times [I + \Theta_{n_u}B_u^T X(sI - A)^{-1}B_u]\\
&= \Theta_{n_u} - B_u^T X(sI - A)^{-1}B_u - B_u^T(-sI - A^T)^{-1}XB_u\\
&- B_u^T(-sI - A^T)^{-1}XB_u\Theta_{n_u}B_u^T X(sI - A)^{-1}B_u.
\end{aligned} \tag{5.8}$$

Also, the non-standard ARE (5.4) implies

$$-(-sI - A^T)X - X(sI - A) + C^T\Theta_{n_y}C = XB_u\Theta_{n_u}B_u^T X$$

for any $s \in \mathbb{C}$ and hence

$$\begin{aligned}
&- X(sI - A)^{-1} - (-sI - A^T)^{-1}X + (-sI - A^T)^{-1}C^T\Theta_{n_y}C(sI - A)^{-1}\\
&= (-sI - A^T)^{-1}XB_u\Theta_{n_u}B_u^T X(sI - A)^{-1}.
\end{aligned}$$

Substituting this result into equation (5.8), it follows that

$$\begin{aligned}
N^{\sim}(s)\Theta_{n_u}N(s) &= \Theta_{n_u} - B_u^T(-sI - A^T)^{-1}C^T\Theta_{n_u}C\ (sI - A)^{-1}B_u\\
&= \Phi_J(s).
\end{aligned}$$

Thus, we have established condition C1 of Definition 5.2.

In order to establish condition C2 of the definition, note that Assumption A1 implies that the $N(s)$ is analytic in $Re\ s \geq 0$.

To establish condition C3, note that $\Phi_J(s)$ in (5.6) can be rewritten in the form

$$\Phi_J(s) = \Theta_{n_u} + \begin{bmatrix} 0 & B_u^T \end{bmatrix} \begin{bmatrix} sI - A & 0 \\ -C^T\Theta_{n_y}C & sI + A^T \end{bmatrix}^{-1} \begin{bmatrix} B_u \\ 0 \end{bmatrix}.$$

Taking the determinant of the above equation and using the determinant relation from [5, p. 135, Fact 2.14.13], it follows that

$$\begin{aligned}
&\det\left(\begin{bmatrix} sI - A & 0 \\ -C^T\Theta_{n_y}C & sI + A^T \end{bmatrix}\right)\det(\Phi_J(s)) \\
&= \det(\Theta_{n_u})\det\left(\begin{bmatrix} sI - A & 0 \\ -C^T\Theta_{n_y}C & sI + A^T \end{bmatrix} + \begin{bmatrix} B_u \\ 0 \end{bmatrix}\Theta_{n_u}^{-1}\begin{bmatrix} 0 & B_u^T \end{bmatrix}\right)
\end{aligned}$$

and hence

$$\begin{aligned}
&\det(sI + A^T)\det(sI - A)\det(\Phi_J(s)) \\
&= \det(\Theta_{n_u})\det\left(\begin{bmatrix} sI - A & -B_u\Theta_{n_u}B_u^T \\ -C^T\Theta_{n_y}C & sI + A^T \end{bmatrix}\right),
\end{aligned}$$

where

$$\det(\Theta_{n_u}) = 1.$$

Now, considering the Hamiltonian matrix (5.5), we obtain

$$\det(\Phi_J(s)) = \frac{\det(sI - H)}{\det(sI + A^T)\det(sI - A)}. \tag{5.9}$$

Furthermore, using the fact that there exists a skew-symmetric solution X of the non-standard ARE (5.4), it follows that $\Phi_J(s)$ can be represented as

$$\begin{aligned}
\Phi_J(s) &= \Theta_{n_u} - B_u^T(-sI - A^T)^{-1}C^T\Theta_{n_y}C\ (sI - A)^{-1}B_u \\
&= \Theta_{n_u} - B_u^T[X(sI - A)^{-1} + (-sI - A^T)^{-1}X \\
&+ (-sI - A^T)^{-1}XB_u\Theta_{n_u}B_u^TX(sI - A)^{-1}]B_u \\
&= \Theta_{n_u} - B_u^TX(sI - A)^{-1}B_u - B_u^T(-sI - A^T)^{-1}XB_u \\
&- B_u^T(-sI - A^T)^{-1}XB_u\Theta_{n_u}B_u^TX(sI - A)^{-1}B_u \\
&= [I + B_u^T(-sI - A^T)^{-1}XB_u\Theta_{n_u}]\Theta_{n_u}[I + \Theta_{n_u}B_u^TX(sI - A)^{-1}B_u].
\end{aligned}$$

This implies

$$\begin{aligned}\det(\Phi_J(s)) = \det(I + (-B_u^T)(sI + A^T)^{-1}XB_u\Theta_{n_u})& \\ \times \det(\Theta_{n_u})\det(I + \Theta_{n_u}B_u^TX(sI - A)^{-1}B_u).& \quad (5.10)\end{aligned}$$

Since $\det(\Theta_{n_u}) = 1$, then

$$\begin{aligned}\det(\Phi_J(s)) = \det(I + (-B_u^T)(sI + A^T)^{-1}XB_u\Theta_{n_u})& \\ \times \det(I + \Theta_{n_u}B_u^TX(sI - A)^{-1}B_u).&\end{aligned}$$

Now, considering the right-hand side of this equation using the determinant relation from [5, p. 135, Fact 2.14.13], it follows that

$$\begin{aligned}&\det(sI + A^T)\det(I + (-B_u^T)(sI + A^T)^{-1}XB_u\Theta_{n_u}) \\ &= \det(I)\det(sI + A^T - XB_u\Theta_{n_u}I^{-1}B_u^T) \\ &= \det(sI + A^T - XB_u\Theta_{n_u}B_u^T)\end{aligned}$$

and hence

$$\det(I + (-B_u^T)(sI + A^T)^{-1}XB_u\Theta_{n_u}) = \frac{\det(sI + A^T - XB_u\Theta_{n_u}B_u^T)}{\det(sI + A^T)}.$$

Similarly,

$$\begin{aligned}&\det(sI - A)\det(I + \Theta_{n_u}B_u^TX(sI - A)^{-1}B_u) \\ &= \det(I)\det(sI - A + B_uI^{-1}\Theta_{n_u}B_u^TX) \\ &= \det(sI - A + B_u\Theta_{n_u}B_u^TX)\end{aligned}$$

and hence

$$\det(I + \Theta_{n_u}B_u^TX(sI - A)^{-1}B_u) = \frac{\det(sI - A + B_u\Theta_{n_u}B_u^TX)}{\det(sI - A)}. \quad (5.11)$$

Substituting equations (5.4.1) and (5.11) into equation (5.10), it follows that

$$\det(\Phi_J(s)) = \frac{\det(sI - A + B_u\Theta_{n_u}B_u^T X)}{\det(sI - A)}\frac{\det(sI + A^T - XB_u\Theta_{n_u}B_u^T)}{\det(sI + A^T)}.$$

Substituting for $\det(\Phi_J(s))$ using (5.9), it follows that

$$\begin{aligned}&\frac{\det(sI - H)}{\det(sI + A^T)\det(sI - A)}\\ &= \frac{\det(sI - A + B_u\Theta_{n_u}B_u^T X)}{\det(sI - A)}\frac{\det(sI + A^T - XB_u\Theta_{n_u}B_u^T)}{\det(sI + A^T)}.\end{aligned} \tag{5.12}$$

Now, using the Matrix Inversion Lemma [5], the inverse of $N(s)$ can be calculated as

$$N^{-1}(s) = I - \Theta_{n_u}B_u^T X(sI - A + B_u\Theta_{n_u}B_u^T X)^{-1}B_u. \tag{5.13}$$

This shows that the poles of $N(s)$ and $N^{-1}(s)$ serve as part of the denominator and numerator of equation (5.12), respectively.

Using Assumption A2 and considering Remark 5.1, it follows that there can be no pole-zero cancellation on the left-hand side of equation (5.12). Therefore, no pole-zero cancellation can exist in the right-hand side of equation (5.12). Thus, condition C3 has been established.

Now observe that condition C4 of Definition 5.2 follows directly from equation (5.7). Thus, we have demonstrated that $N(s)$ defined in (5.7) indeed defines a J-spectral factorization of $\Phi_J(s)$.

Sufficiency: Conversely, suppose that $\Phi_J(s)$ has a J-spectral factorization

$$\Phi_J(s) = N^{\sim}(s)\Theta_{n_u}N(s).$$

We will show there exists a skew-symmetric solution X of the non-standard ARE (5.4). Firstly, recall equation (5.9) and consider Assumptions A1 and A2; it follows that there will be no pole-zero cancellations in this expression. Thus, $\det(\Phi_J(s))$ must be of McMillan degree $2n$.

We now establish some useful claims to aid the proof.

Claim 5.1. *The matrix $N(s)$ is of McMillan degree n.*

To establish this claim, first let

$$\det(N(s)) = \frac{\pi_n(s)}{\pi_d(s)}$$

where $\pi_n(s)$ and $\pi_d(s)$ (relatively prime) are of the same degree. Then, from equation (5.9) and condition C1 of Definition 5.2 implies

$$\begin{aligned}\det(\Phi_J) &= \frac{\det(sI - H)}{\det(sI + A^T)\det(sI - A)} \\ &= \frac{\pi_{n^\sim}(s)\pi_n(s)}{\pi_{d^\sim}(s)\pi_d(s)}\end{aligned}$$

with a McMillan degree $2n$. From C3 of Definition 5.2, we obtain

$$\begin{aligned}\pi_{n^\sim}(s)\pi_n(s) &= \det(sI - H) \\ \pi_{d^\sim}(s)\pi_d(s) &= \det(sI + A^T)\det(sI - A).\end{aligned}$$

We can write

$$\begin{aligned}\deg(\pi_{d^\sim}(s)) &= n_a, \\ \deg(\pi_d(s)) &= n_b.\end{aligned}$$

Note that $\deg(p(s))$ denotes the degree of a polynomial $p(s)$.

The corresponding McMillan degrees are related by [31]

$$2n \leq n_a + n_b.$$

Hence, $n_a = n_b = n$ and $\pi_d(s)$ will be of degree of n and thus $N(s)$ will have McMillan degree n. This completes the proof of the claim.

We now define the matrix L to be a skew-symmetric solution to the Lyapunov equation

$$A^T L + LA + C^T \Theta_{n_y} C = 0. \tag{5.14}$$

Claim 5.2. *The matrix Φ_J can be written in the form*

$$\Phi_J(s) = \Theta_{n_u} - B_u^T(-sI - A^T)^{-1} L B_u - B_u^T L (sI - A)^{-1} B_u$$

such that the realizations of

$$-B_u^T(-sI - A^T)^{-1} L B_u$$

and

$$-B_u^T L (sI - A)^{-1} B_u$$

are minimal. Hence, the pair (A, B_u) is controllable.

To establish this claim, first observe that (5.14) implies

$$L(sI - A)^{-1} - (-sI - A^T)^{-1} L = (-sI - A^T)^{-1} C^T \Theta_{n_y} C (sI - A)^{-1}.$$

Hence, we can write

$$\begin{aligned}\Phi_J(s) &= \Theta_{n_u} - B_u^T(-sI - A^T)^{-1} C^T \Theta_{n_u} C \; (sI - A)^{-1} B_u \\ &= \Theta_{n_u} - B_u^T [L(sI - A)^{-1} - (-sI - A^T)^{-1} L] B_u \\ &= \Theta_{n_u} - B_u^T(-sI - A^T)^{-1} L B_u - B_u^T L (sI - A)^{-1} B_u \end{aligned} \tag{5.15}$$

as required. Also, it follows from this expression for $\Phi_J(s)$ that its McMillan degree satisfies the inequality

$$\begin{aligned}\delta(\Phi_J(s)) &\leq \delta(-B_u^T(-sI - A^T)^{-1} L B_u) + \delta(-B_u^T L (sI - A)^{-1} B_u) \\ &\leq n + n.\end{aligned}$$

We know that $\delta(\Phi_J(s)) = 2n$ and hence equality (5.15) must hold. Therefore,

$$\delta(-B_u^T(-sI - A^T)^{-1} L B_u) = n$$

and

$$\delta(-B_u^T L(sI-A)^{-1}B_u) = n.$$

That is the realizations

$$-B_u^T(-sI-A^T)^{-1}LB_u$$

and

$$-B_u^T L(sI-A)^{-1}B_u$$

are minimal.

It remains to show that L is a skew-symmetric solution to (5.14). Adding equation (5.14) and the transpose of (5.14), we obtain

$$A^T L + LA + C^T\Theta_{n_y}C + A^T L^T + L^T A - C^T\Theta_{n_y}C = 0$$

which implies

$$A^T(L+L^T) + (L+L^T)A = 0.$$

Since A is Hurwitz according to Assumption A1, this implies that

$$L + L^T = 0;$$

i.e., $L^T = -L$. This completes the proof of the claim.

Claim 5.3. *There exists a minimal realization of* $N(s)$ *of the form*

$$N(s) = I + N_A(sI-A)^{-1}B_u.$$

To establish this claim, first recall condition C4 of Definition 5.2. Hence, $N(s)$ will have a minimal realization of the form

$$N(s) = I + N_A(sI-F)^{-1}E$$

where F is a $n \times n$ matrix. Also, since we know $N(s)$ is analytic in Re $s \geq 0$ it follows that the matrix F will be Hurwitz. We can now define the matrix

Y to be the solution to the Lyapunov equation

$$F^T Y + YF + N_A^T \Theta_{n_u} N_A = 0. \tag{5.16}$$

Using this equation, it follows that

$$Y(sI - F)^{-1} + (-sI - F^T)^{-1} Y = (-sI - F^T)^{-1} N_A^T \Theta_{n_u} N_A (sI - F)^{-1}.$$

Thus,

$$\begin{aligned} \Phi_J(s) &= N^{\sim}(s)\Theta_{n_u} N(s) \\ &= (I + N_A(sI - F)^{-1}E)^{\sim}\Theta_{n_u}(I + N_A(sI - F)^{-1}E) \\ &= \Theta_{n_u} + \Theta_{n_u} N_A(sI - F)^{-1}E + E^T(-sI - F^T)^{-1}N_A^T\Theta_{n_u} \\ &+ E^T(-sI - F^T)^{-1}N_A^T\Theta_{n_u}N_A(sI - F)^{-1}E \\ &= \Theta_{n_u} + \Theta_{n_u} N_A(sI - F)^{-1}E + E^T(-sI - F^T)^{-1}N_A^T\Theta_{n_u} \\ &+ E^T[Y(sI - F)^{-1} + (-sI - F^T)^{-1}Y]E \\ &= \Theta_{n_u} + \Theta_{n_u} N_A(sI - F)^{-1}E + E^T(-sI - F^T)^{-1}N_A^T\Theta_{n_u} \\ &+ E^TY(sI - F)^{-1}E + E^T(-sI - F^T)^{-1}YE \\ &= \Theta_{n_u} + (\Theta_{n_u}N_A + E^TY)(sI - F)^{-1}E \\ &+ E^T(-sI - F^T)^{-1}(N_A^T\Theta_{n_u} + YE) \end{aligned} \tag{5.17}$$

Using this equation, it follows that the McMillan degree of $F(s)$ satisfies the inequality

$$\begin{aligned} \delta(\Phi_J(s)) &\leq \delta((\Theta_{n_u}N_A + E^TY)(sI - F)^{-1}E) \\ &+ \delta(E^T(-sI - F^T)^{-1}(N_A^T\Theta_{n_u} + YE)) \\ &\leq n + n. \end{aligned}$$

However, we know that $\delta(\Phi_J(s)) = 2n$ and hence equality must hold.

Thus,

$$\delta((\Theta_{n_u}N_A + E^TY)(sI - F)^{-1}E) = n$$

and

$$\delta(E^T(-sI - F^T)^{-1}(N_A^T \Theta_{n_u} + YE)) = n.$$

That is, the realizations

$$(\Theta_{n_u} N_A + E^T Y)(sI - F)^{-1} E)$$

and

$$E^T(-sI - F^T)^{-1}(N_A^T \Theta_{n_u} + YE)$$

are minimal.

We now compare the expressions for $\Phi_J(s)$ obtained in Claim 5.2 and equation (5.17). It follows that

$$\begin{aligned}\Phi_J(s) &= \Theta_{n_u} - B_u^T(-sI - A^T)^{-1} L B_u - B_u^T L(sI - A)^{-1} B_u \\ &= \Theta_{n_u} + (\Theta_{n_u} N_A + E^T Y)(sI - F)^{-1} E \\ &+ E^T(-sI - F^T)^{-1}(N_A^T \Theta_{n_u} + YE).\end{aligned}$$

Equating stable and anti-stable terms in this equation, it follows that

$$B_u^T(-sI - A^T)^{-1}(-L B_u) = E^T(-sI - F^T)^{-1}(N_A^T \Theta_{n_u} + YE) \tag{5.18}$$

and

$$-B_u^T L(sI - A)^{-1} B_u = (\Theta_{n_u} N_A + E^T Y)(sI - F)^{-1} E. \tag{5.19}$$

However, both sides of equation (5.19) are minimal realizations. Thus, we conclude that there exists a minimal realization of $N(s)$ such that $F = A$ and $E = B_u$.

In a similar manner to Claim 5.2, it remains to show that Y is a skew-symmetric solution to (5.16). Adding equation (5.16) to its transpose, we obtain

$$F^T Y + YF + N_A^T \Theta_{n_u} N_A + F^T Y^T + Y^T F - N_A^T \Theta_{n_u} N_A = 0$$

which implies

$$F^T(Y+Y^T)+(Y+Y^T)F=0.$$

Since F is Hurwitz, this implies that $Y+Y^T=0$ i.e., $Y^T=-Y$. This completes the proof of the claim.

Now, returning to the proof of the theorem, substitute $F=A$ and $E=B_u$ into equations (5.18) and (5.19). Thus

$$B_u^T(-sI-A^T)^{-1}(-LB_u)=B_u^T(-sI-A^T)^{-1}(N_A^T\Theta_{n_u}+YB_u)$$

and

$$-B_u^TL(sI-A)^{-1}B_u=(\Theta_{n_u}N_A+B_u^TY)(sI-A)^{-1}B_u.$$

However, we know from Claim 5.2 that equations (5.18) and (5.19) define minimal realizations. Therefore, it follows from the result above that

$$\begin{aligned}-LB_u&=N_A^T\Theta_{n_u}+YB_u\\-B_u^TL&=\Theta_{n_u}N_A+B_u^TY.\end{aligned}$$

Defining $X=L+Y$, then

$$\begin{aligned}-XB_u&=N_A^T\Theta_{n_u}\\-B_u^TX&=\Theta_{n_u}N_A,\end{aligned}$$

which provides the desired realizations. It now remains to show that X is a solution of the non-standard ARE (5.4). Adding (5.14) and (5.16) gives

$$A^TL+LA+C^T\Theta_{n_y}C+F^TY+YF+N_A^T\Theta_{n_u}N_A=0.$$

Hence, substitute $X=L+Y$ and the result above gives

$$A^TX+XA+C^T\Theta_{n_y}C-XB_u\Theta_{n_u}B_u^TX=0$$

as required. In addition, from Claim 5.2 and 5.3, we know that L and Y are skew-symmetric solution. Therefore, X is a skew-symmetric solution of the non-standard ARE (5.4). It now follows from Theorem 5.1 that $G(s)$ is physically realizable with only direct feedthrough quantum noise. □

5.4.2 Frequency Response Condition

The following corollary gives a necessary frequency response condition for physical realizability with only direct feedthrough quantum noise.

Corollary 5.1. *Suppose $G(s)$ is physically realizable with only direct feedthrough quantum noise. Then the following frequency response condition*

$$\det(\Theta_{n_u} - G^{\dagger}(j\omega)\Theta_{n_y}G(j\omega)) > 0 \tag{5.20}$$

holds for all ω.

Proof. From Theorem 5.2, it follows that $\Phi_J(s)$ has a J-spectral factorization. $\Phi_J(s)$ can also be written as

$$\Phi_J(s) = \Theta_{n_u} - G^{\sim}(s)\Theta_{n_y}G(s).$$

Now, we let $s = j\omega$

$$\Phi_J(j\omega) = \Theta_{n_u} - G^{\dagger}(j\omega)\Theta_{n_y}G(j\omega) \tag{5.21}$$

and take the determinant of equation (5.21)

$$\det(\Phi_J(j\omega)) = \det(\Theta_{n_u} - G^{\dagger}(j\omega)\Theta_{n_y}G(j\omega)). \tag{5.22}$$

First, look at the right-hand side of this expression as $\omega \to \infty$, and note that since $G(s)$ is strictly proper

$$\det(G^{\dagger}(j\omega)\Theta_{n_y}G(j\omega)) \to 0$$

as $\omega \to \infty$. Hence,

$$\det(\Phi_J(j\omega)) \to \det(\Theta_{n_u}) = 1 > 0$$

as $\omega \to \infty$.

Also, it follows from condition C1 in Definition 5.2 that

$$\det(\Phi_J(j\omega)) = \det(N^\dagger(j\omega)\Theta_{n_u}N(j\omega))$$

for all ω.

Furthermore, the matrix

$$iN^\dagger(j\omega)\Theta_{n_u}N(j\omega)$$

is congruent to the matrix $i\Theta_{n_u}$ which has $\frac{n_u}{2}$ positive eigenvalues and $\frac{n_u}{2}$ negative eigenvalues. Hence, using the Inertia Theorem [24, pp. 281-282, Definition 4.5.6], it follows that the eigenvalues of

$$iN^\dagger(j\omega)\Theta_{n_u}N(j\omega)$$

will have this same property. From this, it follows that the $\det(N^\dagger(j\omega)\Theta_{n_u}N(j\omega))$ must be real and non-zero for any ω. Hence, it now follows that

$$\det(\Phi_J(j\omega)) > 0$$

for all ω. □

5.5 Examples

5.5.1 Example 1

We first illustrate our results through an example that considers a system from [30]. Consider matrices A, B_u, and C and the corresponding transfer function matrix $G(s) = C(sI - A)^{-1}B_u$. Here A is Hurwitz and has no common eigenvalues with the Hamiltonian H:

$$\begin{aligned} A &= \begin{bmatrix} -1.3894 \times I & -0.4472 \times I \\ -0.2 \times I & -0.25 \times I \end{bmatrix}; \\ B_u &= \begin{bmatrix} -0.4472 \times I \\ 0_{2\times 2} \end{bmatrix}; \\ C &= \begin{bmatrix} -0.4472 \times I & 0_{2\times 2} \end{bmatrix}. \end{aligned}$$

J-Spectral Factorization

First, form $G(s) = C(sI - A)^{-1}B_u$ to obtain

$$G(s) = \begin{bmatrix} \frac{0.20s^3+0.38s^2+0.133s+0.01}{s^4+3.279s^3+3.203s^2+0.8456s+0.07} & 0 \\ 0 & \frac{0.20s^3+0.38s^2+0.133s+0.01}{s^4+3.279s^3+3.203s^2+0.8456s+0.07} \end{bmatrix}.$$

and construct $\Phi_J(s) = \Theta_u - G^{\sim}(s)\Theta_{n_y}G(s)$ to obtain

$$\Phi_J(s) = \begin{bmatrix} 0 & \frac{s^4+3.28s^3+3.16s^2+0.826s+0.06}{s^4+3.28s^3+3.20s^2+0.85s+0.07} \\ \frac{-s^4-3.28s^3-3.16s^2-0.83s-0.06}{s^4+3.28s^3+3.20s^2+0.85s+0.07} & 0 \end{bmatrix}. \tag{5.23}$$

After a process of trial and error, we find a suitable $N(s)$ as

$$N(s) = \begin{bmatrix} \frac{s^2+1.62s+0.25}{s^2+1.64s+0.26} & 0 \\ 0 & \frac{s^2+1.62s+0.25}{s^2+1.64s+0.26} \end{bmatrix}.$$

It is straightforward to verify that this $N(s)$ satisfies the conditions in Definition 5.2. Now using the method described in [75], the calculated solution X of (5.4) is

$$X = \begin{bmatrix} 0.0000 & 0.0763 & 0.0000 & -0.0270 \\ -0.0763 & 0.0000 & 0.0270 & 0.0000 \\ 0.0000 & -0.0270 & 0.0000 & 0.0486 \\ 0.0270 & 0.0000 & -0.0486 & 0.0000 \end{bmatrix}.$$

It is straightforward to verify that this value of X is real, non-singular and skew-symmetric. Next, we check (5.4) and find

$$A^T X + XA - XB_u \Theta_{n_u} B_u^T X + C^T \Theta_{n_y} C = 0.$$

Thus, the calculated X above solves the non-standard ARE (5.4). From Theorem 5.2, we conclude that $G(s)$ is physically realizable using only direct feedthrough quantum noise where we calculate the matrix B_v using the approach of [75] as

$$\begin{aligned} B_v &= \Theta C^T diag(J) \\ &= \begin{bmatrix} 0.4472 & 0 \\ 0 & 0.4472 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

This formula which comes from Theorem 2.6, [75], uses a state space transformation so that $X \to \Theta_n$. The matrix C would need to be transformed via the same state space transformation.

Frequency Response Condition

In the above example, we showed that $G(s)$ is physically realizable with only direct feedthrough quantum noise. Therefore, according to Corollary 5.1, the frequency response condition (5.20) should hold. This can be seen in Figure 5.1.

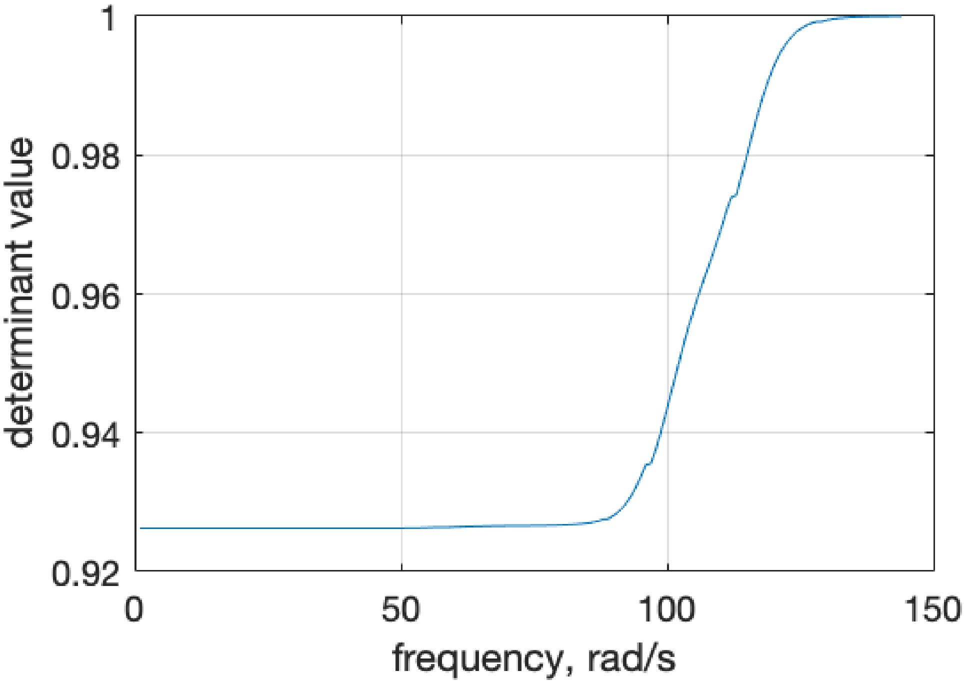

Figure 5.1: Plot of $\det(\Theta_{n_u} - G^{\sim}(j\omega)\Theta_{n_u} G(j\omega))$ versus ω for Example 1.

5.5.2 Example 2

We now give an example where the frequency response condition (5.20) holds but the condition for a transfer function matrix to be physically realizable in Theorem 5.1 fails. Consider the matrices

$$A = \begin{bmatrix} 0.8844 & 0.4385 & 0.3249 & 0.5466 \\ 0.7209 & 0.4378 & 0.2462 & 0.5619 \\ 0.0186 & 0.1170 & 0.3427 & 0.3958 \\ 0.6748 & 0.8147 & 0.3757 & 0.3981 \end{bmatrix};$$

$$B_u = \begin{bmatrix} 0.5154 & 0.4001 \\ 0.6575 & 0.8319 \\ 0.9509 & 0.1343 \\ 0.7223 & 0.0605 \end{bmatrix};$$

$$C = \begin{bmatrix} 0.0842 & 0.3242 & 0.0117 & 0.0954 \\ 0.1639 & 0.3017 & 0.5399 & 0.1465 \end{bmatrix}.$$

Figure 5.2 gives a plot of the corresponding frequency response condition (5.20). It can be seen in Figure 5.2 that the frequency response condition (5.20) holds.

Now following the steps in [75] to find a solution for the Riccati equation (5.4), we calculate the eigenvalues of the Hamiltonian matrix H (5.5) and find that this matrix has no purely imaginary eigenvalues. We form the matrix $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ of eigenvectors corresponding to the eigenvalues, λ of H

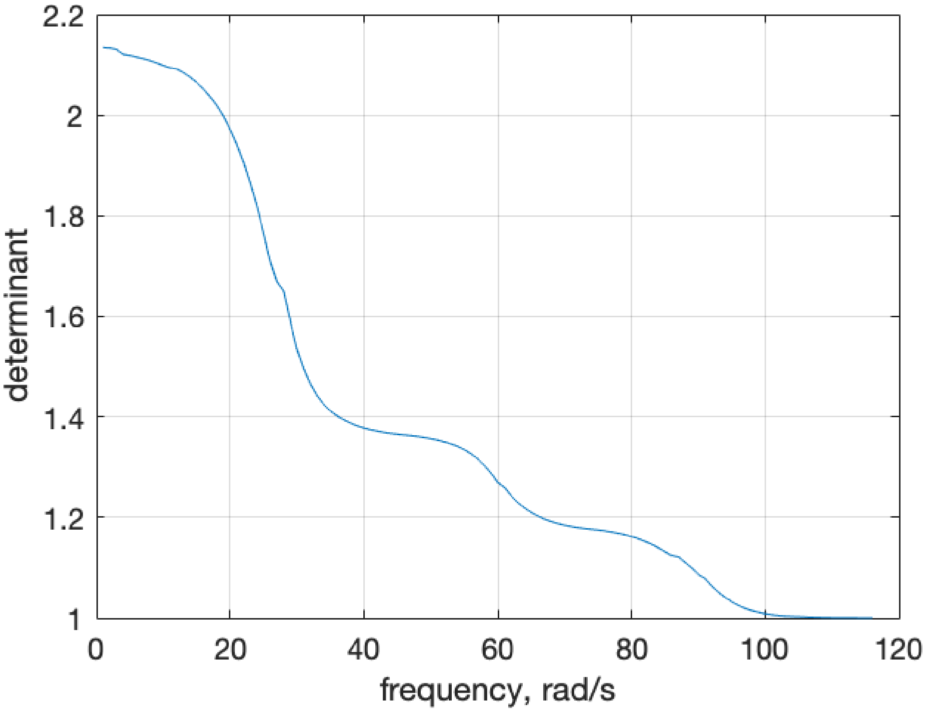

Figure 5.2: Plot of $\det(\Theta_{n_u} - G^{\sim}(j\omega)\Theta_{n_u}G(j\omega))$ versus ω for Example 2.

satisfying $Re(\lambda) < 0$. In particular, the matrix X_1 is found as

$$X_1 = \begin{bmatrix} 0.0731 + 0.0000i & 0.1216 + 0.0000i & 0.1166 - 0.0240i & 0.1166 + 0.0240i \\ 0.2869 + 0.0000i & 0.5130 + 0.0000i & 0.1546 - 0.0660i & 0.1546 + 0.0660i \\ -0.1730 + 0.0000i & 0.3364 + 0.0000i & 0.2213 + 0.2812i & 0.2213 - 0.2812i \\ -0.2628 + 0.0000i & -0.6759 + 0.0000i & -0.5983 + 0.0000i & -0.5983 + 0.0000i \end{bmatrix}.$$

It is straightforward to verify that this matrix X_1 is singular. Hence, it follows as in [75] that the Riccati equation (5.4) does not have a solution. Therefore, it follows from Theorem 5.1 that this transfer function matrix $G(s) = C(sI - A)^{-1}B_u$ is not physically realizable with only direct feedthrough quantum noise.

Thus, we have shown that the frequency response condition (5.20) is not sufficient for physical realizability with only direct feedthrough quantum noise as defined in Definition 5.1.

5.6 Conclusions

In [75], the existence of a non-singular, real, and skew-symmetric solution X to the non-standard ARE (5.4) guarantees a strictly proper transfer function can be implemented as a physically realizable quantum system with only direct feedthrough noise. In this work, we have shown that the J-spectral factorization of $\Phi_J(s)$ gives a necessary and sufficient condition for $G(s)$ to be physically realizable with only direct feedthrough quantum noise. We also present a necessary frequency response condition.

Chapter 6

A Suboptimal Coherent LQG approach to Quantum Equalization

6.1 Introduction

Communication systems are necessary for transmitting information over long distances. However, this often results in degradation of signal quality. In this situation, the implementation of equalization can be beneficial. The goal of equalization is to estimate the transmitted signal from the received signal, compensating for the effects of noise and distortion. This is typically done by designing a filter that maps the received signal to an estimate of the original signal [6]. In the case of quantum communication systems, the laws of quantum mechanics limit their capacity to transfer information. Hence, the problem of correcting distortions in quantum communication systems is complex compared to its classical counterpart [68]. Similar to the classical equalization problem, a problem termed the quantum equalization problem [69] can be modelled and is depicted in Figure 6.1.

In [69], the authors aim to demonstrate an application of an optimization

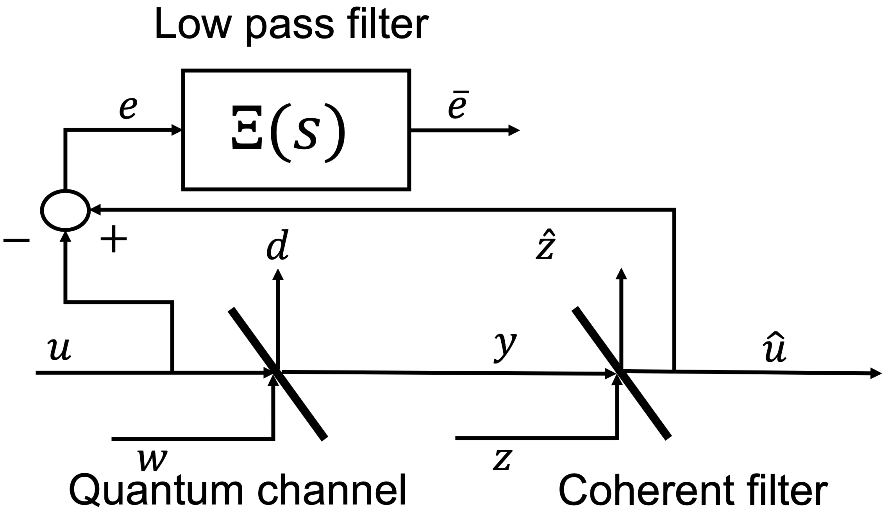

Figure 6.1: A quantum optical communication system consisting of two beam splitters acting as a channel and a filter, respectively.

paradigm in the derivation of coherent equalisation filters for quantum systems, i.e., filters that themselves can be realized as quantum systems. Figure 6.1 illustrates this situation. The quantum optical beam splitter on the left represents a network comprising quantum optical devices functioning as a quantum communication channel. The channel transmits a message u that undergoes distortion due to the nature of the channel and its interaction with the physical environment w. The device on the right is another quantum physical device serving as a coherent equalisation filter with the goal of mitigating this distortion while being subjected to its own environment z. Coherent filters are highly desirable in quantum engineering because coherent filters eliminate the need for traditional non-quantum measurement devices and, thereby are able to deliver technological advantages in quantum information processing [69]. The authors in [69] introduce a H^∞-like methodology for coherent filtering in the equalization of passive linear quantum systems.

In this chapter, we propose a novel approach to solving the equalization problem by converting it into a coherent LQG problem. The main difference between our work and [69], is that we are using the coherent LQG control, while [69] uses a H_∞-like methodology. It is not possible to do a direct comparison between H_∞ control methods and LQG control methods, since their performance indices are measuring different quantities [53]. Our proposed approach adapts results from [75] and [41]. By converting the equalization problem into a coherent LQG problem, we can then design coherent filters that optimize the performance of the communication system while minimizing the impact of noise and distortion.

The chapter continues as follows: in Section 6.2 and 6.3, we describe the quantum linear system models under consideration and define the corresponding notion of physical realizability, respectively. Then, we formulate our problem in Section 6.4 and propose our algorithm in Section 6.5. Examples are given in Section 6.6 followed by a conclusion in Section 6.7.

6.2 Linear Quantum Systems

We consider both passive and active linear quantum systems. Here, passive means that the system is defined in terms of annihilation operators only. Passive quantum systems are a class of systems that can be described using non-commutative or quantum probability theory [7]. In particular, the systems under consideration are described in terms of complex annihilation operators satisfying the linear QSDE as given (2.1):

$$\begin{aligned} da(t) &= Fa(t)dt + Gdw(t); \\ dA^{out}(t) &= Ha(t)dt + Kdw(t). \end{aligned} \tag{6.1}$$

where $F \in \mathbb{C}^{n\times n}$, $G \in \mathbb{C}^{n\times n_w}$, $H \in \mathbb{C}^{n_y\times n}$, $K \in \mathbb{C}^{n_y\times n_w}$ (n, n_y, n_w are positive integers).

The quantity $w(t)$ describes the input variables and is assumed to admit the decomposition

$$dw(t) = \beta_w(t)dt + d\tilde{w}(t)$$

where $\tilde{w}(t)$ is the noise part of $w(t)$ and $\beta_w(t)$ is an adapted process [3, 26, 51]. The noise $\tilde{w}(t)$ is an operator-valued process consisting of a vector of quantum Wiener processes with quantum Ito table

$$d\tilde{w}(t)d\tilde{w}^{\dagger}(t) = F_{\tilde{w}}dt$$

where $F_{\tilde{w}}$ is a nonnegative Hermitian matrix [3, 26, 51]. Here, the notation $(.)^{\dagger}$ represents the adjoint transpose of a vector of operators.

It is also assumed that the following communication relations hold for the noise components:

$$[d\tilde{w}(t), d\tilde{w}(t)^{\dagger}] \triangleq d\tilde{w}(t)d\tilde{w}(t)^{\dagger} - (d\tilde{w}(t)d\tilde{w}(t)^{T})^{T} = T_{w}dt \tag{6.2}$$

where T_w is a Hermitian commutation matrix and the notation $(.)^T$ denotes the transpose of a vector or matrix of operators [3, 26, 51]. The noise processes can be represented as operators on an appropriate Fock space [18]. The process $\beta_w(t)$ represents variables of other systems which may be passed to the system (2.1) via an interaction. Therefore, it is required that $\beta_w(0)$ be an operator on a Hilbert space distinct from that of $a(0) = a_0$ and the noise processes. $\beta_w(t)$ is also assumed to commute with $a(t)$ and $d\tilde{w}(t)$ for all $t \geq 0$ since $\beta_w(t)$ is an adapted process.

Meanwhile, active means that the system is defined in terms of annihilation and creation operators or position and momentum operators. For an active system, we use position and momentum operators for convenience, so that we can directly use the results of Chapter 3. In this chapter, we consider a special case of (2.18)

$$\begin{aligned} dx(t) &= Ax(t)dt + B_u du(t) + B_v dv(t); \\ dy(t) &= Cx(t)dt + dv(t). \end{aligned} \tag{6.3}$$

Here $du(t)$ is the signal input, and $dv(t)$ is a vector of the direct feedthrough quantum vacuum noise inputs. The inputs to the system represented by $du(t)$ (a column vector with n_u components), are assumed to admit the decomposition

$$du(t) = \beta_u(t)dt + d\tilde{u}(t)$$

where the self-adjoint, adapted process $\beta_u(t)$ is the signal part of $u(t)$ and $\tilde{u}(t)$ is the noise part of $u(t)$. Note that $\beta_u(t)$ also commutes with $d\tilde{u}(t)$ for all $t \geq 0$.

The noise $\tilde{u}(t)$ is a vector of self-adjoint quantum noises with Ito table

$$d\tilde{u}(t)d\tilde{u}^T(t) = F_{\tilde{u}}dt$$

where $F_{\tilde{u}} = S_{\tilde{u}} + T_{\tilde{u}}$ is a nonnegative Hermitian matrix [4, 51] with $S_{\tilde{u}}$ and $T_{\tilde{u}}$ being real and imaginary, respectively. $S_{\tilde{u}}$ describes the intensity of the quantum Wiener process and is the quantum analog of the intensity matrix for a classical Wiener process.

The commutation relations for the noise components are determined by T_u:

$$[d\tilde{u}(t), d\tilde{u}(t)^\dagger] \triangleq d\tilde{u}(t)d\tilde{u}(t)^\dagger - (d\tilde{u}(t)d\tilde{u}(t)^T)^T = 2T_u dt. \tag{6.4}$$

Also, $dv(t)$ (column vectors with n_v components) is a vector of quantum Wiener processes corresponding to the introduced vacuum noise inputs. Consequently, F_v, S_v, and T_v is defined for $dv(t)$ as $F_{\tilde{w}_x}$, $S_{\tilde{w}_x}$, and $T_{\tilde{w}_x}$ were for $d\tilde{w}_x(t)$ in (2.18).

6.3 Physical Realizability

In [41], the notion of physical realizability for passive quantum systems is developed around the concept of a complex open quantum harmonic oscillator. We consider a passive quantum plant described by the following equations which are in terms of annihilation operators:

$$\begin{aligned} da(t) &= Fa(t)dt + \begin{bmatrix} G_0 & G_1 & G_2 \end{bmatrix} \begin{bmatrix} dv(t)^T \\ dw(t)^T \\ du(t)^T \end{bmatrix}; \\ dz(t) &= H_1 a(t)dt + K_{12} du(t); \\ dy(t) &= H_2 a(t)dt + \begin{bmatrix} K_{20} & K_{21} & 0_{n_y \times n_y} \end{bmatrix} \begin{bmatrix} dv(t)^T \\ dw(t)^T \\ du(t)^T \end{bmatrix} \end{aligned} \tag{6.5}$$

where $\mathbb{F} \in \mathbb{C}^{n \times n}$, $\mathbb{G}_0 \in \mathbb{C}^{n \times n_v}$, $\mathbb{G}_1 \in \mathbb{C}^{n \times n_w}$, $\mathbb{G}_2 \in \mathbb{C}^{n \times n_u}$, $\mathbb{H}_1 \in \mathbb{C}^{n_z \times n_w}$, $\mathbb{K}_{12} \in \mathbb{C}^{n_z \times n_u}$, $\mathbb{H}_2 \in \mathbb{C}^{n_y \times n_n}$, $\mathbb{K}_{20} \in \mathbb{C}^{n_y \times n_v}$ and $\mathbb{K}_{21} \in \mathbb{C}^{n_y \times n_w}$.

Similarly, a controller is defined as follows:

$$\begin{aligned} d\xi(t) &= F_c \xi(t)dt + \begin{bmatrix} G_{c_0} & G_{c_1} & G_c \end{bmatrix} \begin{bmatrix} dw_{c_0}(t) \\ dw_{c_1}(t) \\ dy(t) \end{bmatrix}; \\ du(t) &= H_c \xi(t)dt + dw_{c_0} \end{aligned} \tag{6.6}$$

where $\xi(t) = [\xi_1(t) \dots \xi_n(t)]^T$ is a vector of controller annihilation operator variables. We now define the notion of physically realizable for this class of systems.

Definition 6.1. *[42, Definition 3.1] The matrices F_c, G_c, H_c are said to define a physically realizable controller of the form (6.6) if there exist matrices G_{c_0}, G_{c_1}, H_{c_1} and H_{c_2} such that the following quantum system of the* *form (6.1)*

$$\begin{aligned} d\xi(t) &= F_c\xi(t)dt + \begin{bmatrix} G_{c_0} & G_{c_1} & G_c \end{bmatrix} \begin{bmatrix} dw_{c_0} \\ dw_{c_1} \\ dy \end{bmatrix}; \\ \begin{bmatrix} du \\ du_1 \\ du_2 \end{bmatrix} & \begin{bmatrix} H_c \\ H_{c_1} \\ H_{c_2} \end{bmatrix} \xi(t)dt + \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} dw_{c_0} \\ dw_{c_1} \\ dy \end{bmatrix} \end{aligned} \tag{6.7}$$

is physically realizable when $T_y = K_{20}T_vK_{20}^\dagger + K_{21}T_wK_{21}^\dagger = I$.

Theorem 6.1. *[42, Theorem 3.2] Suppose the matrices* F_c, G_c, *and* H_c *are such the corresponding system is minimal [41]. Then the matrices* F_c, G_c, *and* H_c *define a physically realizable controller of the form (6.6) if and only if* F_c *is Hurwitz and*

$$\|H_c(sI - F_c)^{-1}G_c\|_\infty \leq 1;$$

i.e., the corresponding system is bounded real [41]. In this case, the matrices G_{c_1} *and* H_{c_1} *in (6.7) can be taken as zero.*

In [30], the notion of physical realizability for an active system is based on the concept of an open quantum harmonic oscillator. The formal definition is as in Definition 3.1 and Lemma 3.1.

In this chapter, we consider the LMI version of the physical realizability which is similar to the approach in [30, 75] but reformulated into an LMI problem as seen in Chapter 3.

6.4 Problem Formulation

The problem formulation described in this chapter is similar to [49, 75], with some minor differences.

Suppose we have passive quantum channel dynamics described by the QSDEs

$$\begin{aligned} da_c(t) &= F_c a_c(t)dt + G_{c_u} d\mathcal{A}_u(t) + G_{c_\omega} d\mathcal{A}_\omega(t); \\ d\mathcal{A}^{out}(t) &= H_c a_c(t)dt + K_{c_u} d\mathcal{A}_u(t) + K_{c_\omega} d\mathcal{A}_\omega(t) \end{aligned} \tag{6.8}$$

and passive low-pass filter dynamics

$$\begin{aligned} da_f(t) &= F_f a_f(t)dt + G_{f_k} d\mathcal{A}_k(t) - G_{f_u} d\mathcal{A}_u(t); \\ \bar{e}_p(t) &= a_f(t). \end{aligned} \tag{6.9}$$

The low pass filter (6.9) is introduced so that the corresponding cost function will be well defined. This is justified since in practice, the equalization filter only needs to work over a finite bandwidth rather than an infinite bandwidth.

Then the passive quantum plant can be obtained through the combination of the dynamics of the quantum channel (6.8) and the low-pass filter (6.9) as follows

$$\begin{aligned} \begin{bmatrix} da_c(t) \\ da_f(t) \end{bmatrix} &= \begin{bmatrix} F_c & 0 \\ 0 & F_f \end{bmatrix} \begin{bmatrix} a_c(t) \\ a_f(t) \end{bmatrix} dt + \begin{bmatrix} G_{c_u} & G_{c_\omega} \\ -G_{f_u} & 0 \end{bmatrix} \begin{bmatrix} d\mathcal{A}_u(t) \\ d\mathcal{A}_\omega(t) \end{bmatrix} + \begin{bmatrix} 0 \\ G_{f_k} \end{bmatrix} d\mathcal{A}_k(t); \\ d\mathcal{A}^{out}(t) &= \begin{bmatrix} H_c & 0 \end{bmatrix} \begin{bmatrix} a_c(t) \\ a_f(t) \end{bmatrix} dt + \begin{bmatrix} K_{c_u} & K_{c_\omega} \end{bmatrix} \begin{bmatrix} d\mathcal{A}_u(t) \\ d\mathcal{A}_\omega(t) \end{bmatrix}. \end{aligned} \tag{6.10}$$

Note that the quantum plant (6.10) is in the passive complex form.

Then based on [53], we can transform the quantum plant into an active complex form as follows:

$$\begin{bmatrix} da_c(t) \\ da_f(t) \\ da_c(t)^{\#} \\ da_f(t)^{\#} \end{bmatrix} = \begin{bmatrix} F_c & 0 & 0 & 0 \\ 0 & F_f & 0 & 0 \\ 0 & 0 & F_c^{\#} & 0 \\ 0 & 0 & 0 & F_f^{\#} \end{bmatrix} \begin{bmatrix} a_c(t) \\ a_f(t) \\ a_c(t)^{\#} \\ a_f(t)^{\#} \end{bmatrix} dt + \begin{bmatrix} G_{c_u} & G_{c_\omega} & 0 & 0 \\ -G_{f_u} & 0 & 0 & 0 \\ 0 & 0 & G_{c_u}^{\#} & G_{c_\omega}^{\#} \\ 0 & 0 & -G_{f_u}^{\#} & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} d\mathcal{A}_u(t) \\ d\mathcal{A}_\omega(t) \\ d\mathcal{A}_u(t)^{\#} \\ d\mathcal{A}_\omega(t)^{\#} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ G_{f_k} & 0 \\ 0 & 0 \\ 0 & G_{f_k}^{\#} \end{bmatrix} \begin{bmatrix} d\mathcal{A}_k(t) \\ d\mathcal{A}_k(t)^{\#} \end{bmatrix};$$

$$\begin{bmatrix} d\mathcal{A}^{out}(t) \\ d\mathcal{A}^{out}(t)^{\#} \end{bmatrix} = \begin{bmatrix} H_c & 0 & 0 & 0 \\ 0 & 0 & H_c^{\#} & 0 \end{bmatrix} \begin{bmatrix} a_c(t) \\ a_f(t) \\ a_c(t)^{\#} \\ a_f(t)^{\#} \end{bmatrix} dt + \begin{bmatrix} K_{c_u} & K_{c_\omega} & 0 & 0 \\ 0 & 0 & K_{c_u}^{\#} & K_{c_\omega}^{\#} \end{bmatrix} \begin{bmatrix} d\mathcal{A}_u(t) \\ d\mathcal{A}_\omega(t) \\ d\mathcal{A}_u(t)^{\#} \\ d\mathcal{A}_\omega(t)^{\#} \end{bmatrix}. \tag{6.11}$$

By applying a particular change of variables (2.12) [53], the equations (6.11) can be transformed into real matrices as follows

$$\begin{aligned} \begin{bmatrix} dq_c(t) \\ dq_f(t) \\ dp_c(t) \\ dp_f(t) \end{bmatrix} &= A \begin{bmatrix} q_c(t) \\ q_f(t) \\ p_c(t) \\ p_f(t) \end{bmatrix} dt + B_{w_1} \begin{bmatrix} d\mathcal{Q}_c(t) \\ d\mathcal{Q}_\omega(t) \\ d\mathcal{P}_c(t) \\ d\mathcal{P}_\omega(t) \end{bmatrix} + B_{\hat{u}} \begin{bmatrix} d\mathcal{Q}_k(t) \\ d\mathcal{P}_k(t) \end{bmatrix}; \\ \begin{bmatrix} d\mathcal{Q}^{out}(t) \\ d\mathcal{P}^{out}(t) \end{bmatrix} &= C \begin{bmatrix} q_c(t) \\ q_f(t) \\ p_c(t) \\ p_f(t) \end{bmatrix} dt + D_{w_1} \begin{bmatrix} d\mathcal{Q}_c(t) \\ d\mathcal{Q}_\omega(t) \\ d\mathcal{P}_c(t) \\ d\mathcal{P}_\omega(t) \end{bmatrix}. \end{aligned} \tag{6.12}$$

Alternatively, these equations can be rewritten as

$$\begin{aligned}
dx(t) &= Ax(t)dt + B_{w_1}dw_1(t) + B_{\hat{u}}d\hat{u}(t);\\
dy(t) &= Cx(t) + D_{w_1}dw_1(t)\\
\bar{e}(t) &= x_f(t) = \begin{bmatrix} q_f(t) \\ p_f(t) \end{bmatrix}
\end{aligned}$$

where dw_1=$[du^T \quad dw^T]^T$ and

$$A = \frac{1}{2}\begin{bmatrix} \begin{bmatrix} F_c & 0 \\ 0 & F_f \end{bmatrix} + \begin{bmatrix} F_c & 0 \\ 0 & F_f \end{bmatrix}^{\#} & -i\left(\begin{bmatrix} F_c & 0 \\ 0 & F_f \end{bmatrix} - \begin{bmatrix} F_c & 0 \\ 0 & F_f \end{bmatrix}^{\#}\right) \\ i\left(\begin{bmatrix} F_c & 0 \\ 0 & F_f \end{bmatrix} - \begin{bmatrix} F_c & 0 \\ 0 & F_f \end{bmatrix}^{\#}\right) & \begin{bmatrix} F_c & 0 \\ 0 & F_f \end{bmatrix} + \begin{bmatrix} F_c & 0 \\ 0 & F_f \end{bmatrix}^{\#} \end{bmatrix};$$

$$B_{w_1} = \frac{1}{2}\begin{bmatrix} \begin{bmatrix} G_{c_u} & G_{c_\omega} \\ -G_{f_u} & 0 \end{bmatrix} + \begin{bmatrix} G_{c_u} & G_{c_\omega} \\ -G_{f_u} & 0 \end{bmatrix}^{\#} & -i\left(\begin{bmatrix} G_{c_u} & G_{c_\omega} \\ -G_{f_u} & 0 \end{bmatrix} - \begin{bmatrix} G_{c_u} & G_{c_\omega} \\ -G_{f_u} & 0 \end{bmatrix}^{\#}\right) \\ i\left(\begin{bmatrix} G_{c_u} & G_{c_\omega} \\ -G_{f_u} & 0 \end{bmatrix} - \begin{bmatrix} G_{c_u} & G_{c_\omega} \\ -G_{f_u} & 0 \end{bmatrix}^{\#}\right) & \begin{bmatrix} G_{c_u} & G_{c_\omega} \\ -G_{f_u} & 0 \end{bmatrix} + \begin{bmatrix} G_{c_u} & G_{c_\omega} \\ -G_{f_u} & 0 \end{bmatrix}^{\#} \end{bmatrix};$$

$$B_{\hat{u}} = \frac{1}{2}\begin{bmatrix} \begin{bmatrix} 0 \\ G_{f_k} \end{bmatrix} + \begin{bmatrix} 0 \\ G_{f_k} \end{bmatrix}^{\#} & -i\left(\begin{bmatrix} 0 \\ G_{f_k} \end{bmatrix} + \begin{bmatrix} 0 \\ G_{f_k} \end{bmatrix}^{\#}\right) \\ i\left(\begin{bmatrix} 0 \\ G_{f_k} \end{bmatrix} + \begin{bmatrix} 0 \\ G_{f_k} \end{bmatrix}^{\#}\right) & \begin{bmatrix} 0 \\ G_{f_k} \end{bmatrix} + \begin{bmatrix} 0 \\ G_{f_k} \end{bmatrix}^{\#} \end{bmatrix};$$

$$C = \frac{1}{2}\begin{bmatrix} \begin{bmatrix} H_c & 0 \end{bmatrix} + \begin{bmatrix} H_c & 0 \end{bmatrix}^{\#} & -i\left(\begin{bmatrix} H_c & 0 \end{bmatrix} + \begin{bmatrix} H_c & 0 \end{bmatrix}^{\#}\right) \\ i\left(\begin{bmatrix} H_c & 0 \end{bmatrix} + \begin{bmatrix} H_c & 0 \end{bmatrix}^{\#}\right) & \begin{bmatrix} H_c & 0 \end{bmatrix} + \begin{bmatrix} H_c & 0 \end{bmatrix}^{\#} \end{bmatrix};$$

$$D_{w_1} = \frac{1}{2}\begin{bmatrix} \begin{bmatrix} K_{c_u} & K_{c_\omega} \end{bmatrix} + \begin{bmatrix} K_{c_u} & K_{c_\omega} \end{bmatrix}^{\#} & -i\left(\begin{bmatrix} K_{c_u} & K_{c_\omega} \end{bmatrix} + \begin{bmatrix} K_{c_u} & K_{c_\omega} \end{bmatrix}^{\#}\right) \\ i\left(\begin{bmatrix} K_{c_u} & K_{c_\omega} \end{bmatrix} + \begin{bmatrix} K_{c_u} & K_{c_\omega} \end{bmatrix}^{\#}\right) & \begin{bmatrix} K_{c_u} & K_{c_\omega} \end{bmatrix} + \begin{bmatrix} K_{c_u} & K_{c_\omega} \end{bmatrix}^{\#} \end{bmatrix}.$$

The passive quantum equalization problem is as follows: given a passive quantum plant of the form (6.10), design a classical LQG controller of the form

$$\begin{aligned} da_k(t) &= F_k a_k(t)dt + G_y d\mathcal{A}^{out}(t); \\ d\mathcal{A}_k(t) &= H_k a_k(t)dt \end{aligned} \tag{6.13}$$

that minimizes the cost function

$$J_{cost_p} = \lim_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} \left\langle \bar{e}_p(t)^T R_{1_p} \bar{e}_p(t) + \mu \mathcal{A}_k(t)^T R_{2_p} \mathcal{A}_k(t) \right\rangle dt \tag{6.14}$$

which is then implemented as a physically realizable LQG quantum controller of the form (6.7).

Similarly, the active quantum equalization problem is as follows: given an active quantum plant of the form (6.12), design a classical LQG controller of the form

$$\begin{aligned} dx(t) &= A_k x_k(t)dt + B_y dy(t); \\ d\hat{u}(t) &= C_k x_k(t)dt. \end{aligned} \tag{6.15}$$

that minimizes the cost function

$$J_{cost_a} = \lim_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} \left\langle \bar{e}(t)^T R_{1_a} \bar{e}(t) + \mu \hat{u}(t)^T R_{2_a} \hat{u}(t) \right\rangle dt \tag{6.16}$$

which is then implemented as a physically realizable LQG quantum controller of the form

$$\begin{aligned} dx(t) &= A_k x_k(t)dt + B_y dy(t) + B_{v_1} d_{v_1}(t) + B_{v_2} d_{v_2}(t); \\ d\hat{u}(t) &= C_k x_k(t)dt + dv_1(t). \end{aligned} \tag{6.17}$$

Note that R_{1_p}=$4R_{1_a}$ and R_{2_p}=$4R_{2_a}$.

6.5 Coherent LQG Control

The main idea of our algorithm is to design a classical LQG controller and then use the results in [42] (for the passive case) or Chapter 3 (for the active case) to implement this controller as a physically realizable quantum system.

The algorithm will be presented here specifically for the active case, as the passive case is very similar. To begin with, we consider a classical LQG problem. This is defined by the quantum plant (6.12) and the classical controller (6.15). This classical LQG problem can be solved in the usual manner [37, Theorem 5]. The solution is the controller (6.15) with

$$\begin{aligned} A_k &= A - K_a C - B_{\hat{u}} + K_a D_{\hat{u}} F_a; \\ B_y &= K_a; \\ C_k &= -F_a. \end{aligned}$$

The matrices F_a and K_a can be obtained as follows:

$$F_a = R_{2_a}^{-1} B_{\hat{u}}^T P$$

where $P \geq 0$ is the solution to the ARE:

$$A^T P + PA + PB_{\hat{u}} R_{2_a}^{-1} B_{\hat{u}}^T P + R_{1_a} = 0,$$

and

$$K_a = (QC^T + V_{12}) V_2^{-1}$$

where $Q \geq 0$ is the solution to the ARE:

$$(A - V_{12} V_2^{-1} C) Q + Q (A - V_{12} V_2^{-1} C)^T - QC^T V_2^{-1} Q + V_1 - V_{12} V_2^{-1} V_{12}^T = 0.$$

Note that,

$$\mathbb{E} \begin{bmatrix} B_u & B_w \\ D_u & D_w \end{bmatrix} \begin{bmatrix} du \\ dw \end{bmatrix} \begin{bmatrix} du \\ dw \end{bmatrix}^T \begin{bmatrix} B_u & B_w \\ D_u & D_w \end{bmatrix}^T = \begin{bmatrix} V_1 & V_{12} \\ V_{12}^T & V_2 \end{bmatrix} dt.$$

Next, we obtain a coherent LQG controller of the form (6.17) by applying the method from Chapter 3 to the classical controller (6.15) with A_k, B_y, and C_k calculated above. To evaluate the cost (6.16) explicitly, we consider the closed loop system:

$$d\zeta(t) = A_{cl}\zeta(t)dt + B_{cl}dw_{cl}(t); \tag{6.18}$$

where

$$\zeta = \begin{bmatrix} x \\ x_k \end{bmatrix}; \quad w_{cl} = \begin{bmatrix} dw_1 \\ dv_1 \\ dv_2 \end{bmatrix}$$

and

$$J_{cl} = Tr(\bar{R}\bar{Q}) \tag{6.19}$$

where $\bar{Q}$ is the unique symmetric positive definite solution of the Lyapunov equation

$$A\bar{Q} + \bar{Q}A^T + BB^T = 0$$

and

$$\bar{R} = \begin{bmatrix} R_1 & 0 \\ 0 & C_k^T R_2 C_k \end{bmatrix}.$$

That is, the cost function (6.16) is evaluated using the expression (6.19).

Our proposed algorithm can be summarized as follows:

1. Beginning with matrices A, $B_{\hat{u}}$, B_w, C and D_w in (6.12), we design a classical LQG controller (6.15) using the standard approach [37, Theorem 5] and obtain A_k, B_y and C_k.

2. Implement (6.15) as a coherent quantum controller using the method in Chapter 3.

3. Form the closed loop system (6.18) and evaluate the cost function (6.19).

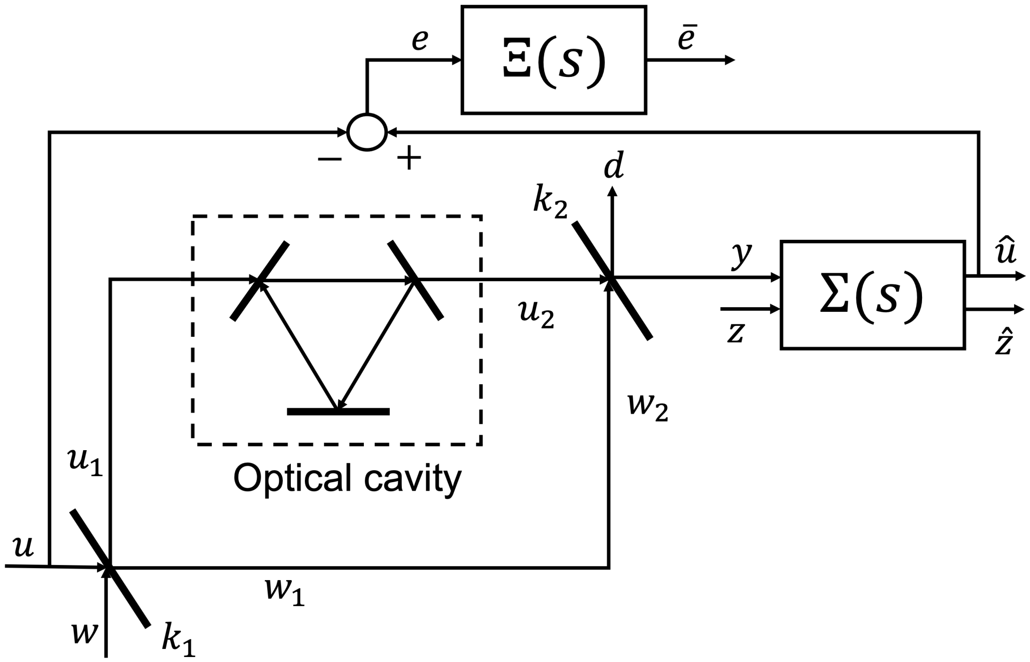

Figure 6.2: Equalization of an optical cavity system.

6.6 Example of an Equalization Problem

We now consider a modified example of the equalization problem from [69], as shown in Figure 6.2. The channel consists of an optical cavity and two optical beam splitters. The following are the constants used

$$\kappa = 5, \quad k = 0.4, \quad m = \sqrt{1-k^2}, \quad \Omega = 10 \text{ and } \tau = 0.1.$$

The constants are adapted from [69, Section 6.2]. We will consider both passive and active systems for the equalization filter $\Sigma(s)$ in subsections (6.6.1) and (6.6.2), respectively. We will then comment on their relative performance.

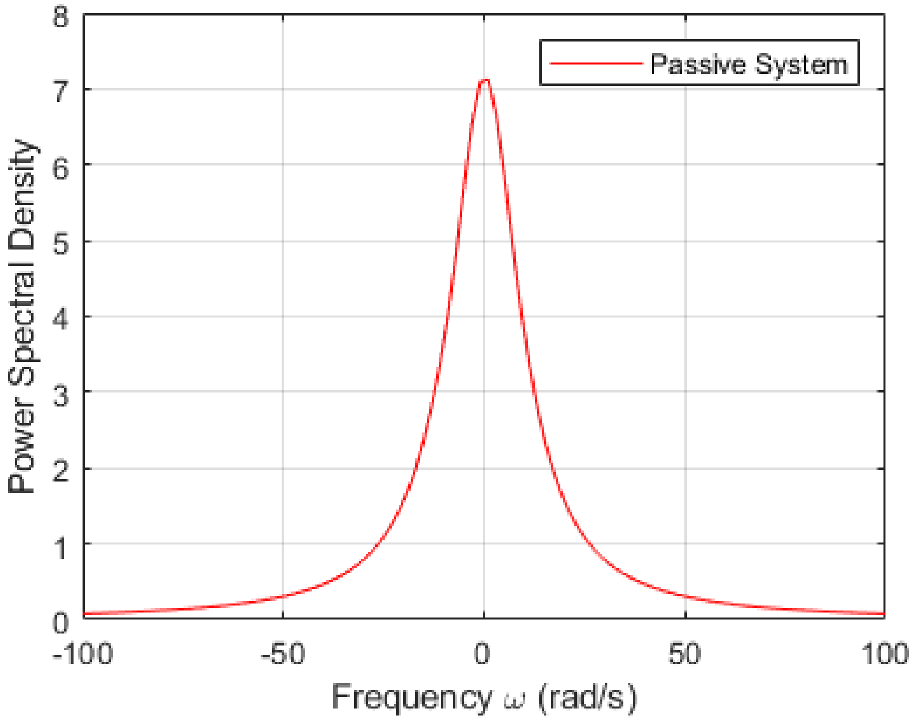

Figure 6.3: Closed loop power spectral density for passive controller design.

6.6.1 Coherent LQG Control of a Passive Quantum System

Here our plant is of the form (6.8) with

$$F = \begin{bmatrix} -k + i\Omega & 0 \\ 0 & -\frac{1}{\tau} \end{bmatrix}; \tag{6.20a}$$

$$G_\omega = \begin{bmatrix} 0 \\ \frac{1}{\tau} \end{bmatrix}; \tag{6.20b}$$

$$G_u = \begin{bmatrix} -k\sqrt{2\kappa} & -m\sqrt{2\kappa} \\ -\frac{1}{\tau} & 0 \end{bmatrix}; \tag{6.20c}$$

$$H = \begin{bmatrix} k\sqrt{2\kappa} & 0 \end{bmatrix}; \tag{6.20d}$$

$$K_u = \begin{bmatrix} k^2 - m^2 & 2km \end{bmatrix} \tag{6.20e}$$

and we choose R_{1_p}, R_{2_p}, and μ_p of (6.14) to be

$$\begin{aligned} R_{1_p} &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \\ R_{2_p} &= 1; \\ \mu_p &= 0.1. \end{aligned}$$

The evaluated cost function (6.19) is 36.5105 and this is reflected in Figure 6.3 which gives the closed loop power spectral density of the quantity

$$\begin{bmatrix} R_{1_p}^{\frac{1}{2}} \bar{e}_p \\ R_{2_p}^{\frac{1}{2}} \mathcal{A}_k \end{bmatrix}.$$

6.6.2 Coherent LQG Control of an Active Quantum System

For the active case, we use a similar plant as in equations (6.20a-6.20e) obtained by applying the conversion matrix [53, Equation 22]

$$\Phi = \begin{bmatrix} I & I \\ -iI & iI \end{bmatrix}$$

with appropriate dimensions to obtain matrices A, $B_{\hat{u}}$, B_{w_1}, C, and D_{w_1} as follows:

$$\begin{aligned} A &= \Phi F \Phi^{-1}; \\ B_{\hat{u}} &= \Phi G_{\omega} \Phi^{-1}; \\ B_{w_1} &= \Phi G_u \Phi^{-1}; \\ C &= \Phi H \Phi^{-1}; \\ D_{w_1} &= \Phi K_u \Phi^{-1} \end{aligned}$$

and R_{1_a}, R_{2_a}, and μ_a expands accordingly

$$\begin{aligned} R_{1_a} &= 4 \times \begin{bmatrix} 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & I_{2\times2} \end{bmatrix}; \\ R_{2_a} &= 4 \times I_{2\times2}; \\ \mu_a &= 0.1. \end{aligned}$$

Now, the evaluated cost function (6.19) is 38.8176 and this is reflected in Figure 6.4 which gives the closed loop power spectral density of the quantity

$$\begin{bmatrix} R_{1_a}^{\frac{1}{2}} \bar{e} \\ R_{2_a}^{\frac{1}{2}} \hat{u} \end{bmatrix}.$$

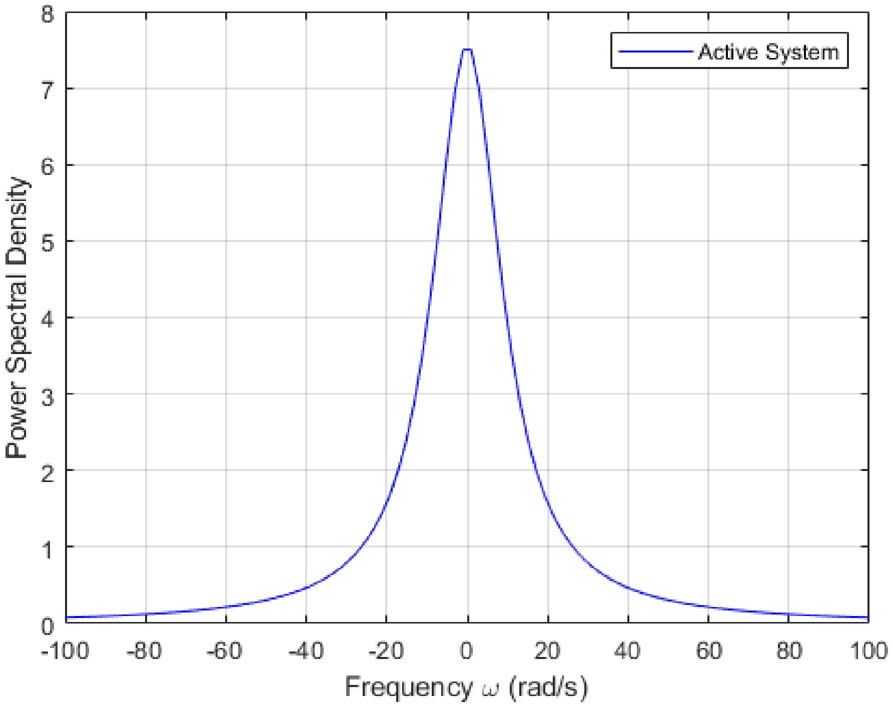

Figure 6.4: Closed loop power spectral density for active controller design.

6.6.3 Comparison of Controller System Performance

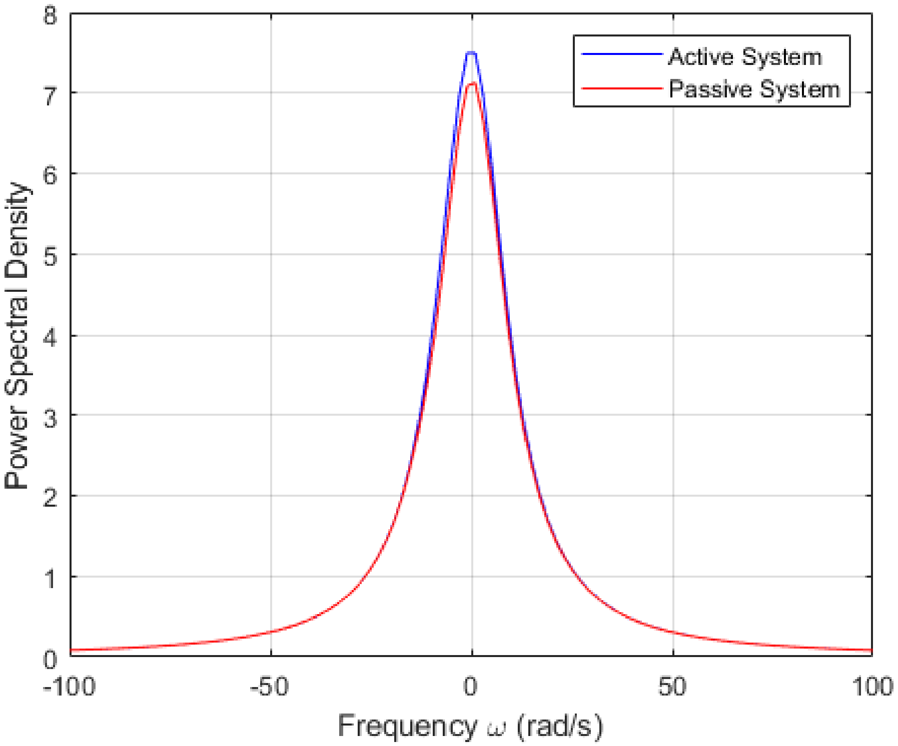

Figure 6.5: Closed loop power spectral density for both active and passive controller designs.

The performance of the passive and active system's cost function (6.19) is illustrated in Figure 6.5 in terms of the power spectral density graph. For this example, the passive system gave a marginal improvement in comparison to the active system. However, we might expect that the active system will perform at least as well as the passive system since the passive system is a special case of the active system. We conjecture that the discrepancy is due to the suboptimal nature of our approach.

6.7 Conclusions

The general idea of quantum equalization is to design a feedback controller that is a physically realizable quantum system and can compensate for the error in a quantum communication channel. In this work, we have proposed a method to find a physically realizable coherent LQG quantum controller that minimizes a cost function related to the system equalization error. Examples are shown for both passive and active linear quantum equalizers.

Chapter 7

A Gradient Descent Coherent LQG approach to Quantum Equalization

7.1 Introduction

The authors in [65] proposed an approach concerning coherent quantum LQG controller design using the gradient descent method [8]. The problem considered in [65] is to find a stabilizing coherent quantum controller for a quantum plant to minimize the mean square cost associated with the fully quantum closed-loop system. A line-search gradient descent algorithm with adaptive stepsize selection is proposed for the numerical solution of the coherent quantum LQG control problem in [65]. The algorithm in [65] aims to find a local minimum of the LQG cost over the parameters of the Hamiltonian and coupling operators of a stabilizing coherent quantum controller, thus taking the physical realizability constraints into account.

An advantageous characteristic of the gradient descent method for coherent quantum LQG control is that at intermediate steps it produces stabilizing physically realizable quantum controllers [65]. These controllers can be regarded as gradually improving suboptimal solutions to the problem. Furthermore, a locally optimal solution, if it exists, is attained asymptotically by traversing anti-gradient directions with appropriately chosen step sizes.

In this chapter, we propose a modified gradient descent approach to find an optimal coherent LQG controller to solve the quantum channel equalization problem that is first considered in [69]. The main difference between our work and [65], is that our considered quantum plant is not necessarily physically realizable, while [65] assumed both the quantum plant and the quantum controller satisfy the physical realizability conditions [30].

The chapter continues as follows: in Section 7.2 and 7.3, we describe the quantum linear system models under consideration and define the corresponding notion of physical realizability, respectively. Then, we formulate our problem in Section 7.4 and propose our algorithm in Section 7.5. Examples are given in Section 7.6 followed by a conclusion in Section 7.7.

7.2 Linear Quantum Systems

In contrast to Chapter 6, in this chapter, we consider the active linear quantum systems only. Here, active means that the system is defined in terms of annihilation and creation operators or position and momentum operators.

For an active system, we use position and momentum operators for convenience, so that we can directly use the results of Chapter 3. In this chapter, we consider a special case of (2.18):

$$\begin{aligned} dx(t) &= Ax(t)dt + B_u du(t) + B_v dv(t); \\ dy(t) &= Cx(t)dt + dv(t). \end{aligned} \tag{7.1}$$

Here $du(t)$ is the signal input, and $dv(t)$ is a vector of the direct feedthrough quantum vacuum noise inputs.

The inputs to the system represented by $du(t)$ (a column vector with n_u

components), are assumed to admit the decomposition

$$du(t) = \beta_u(t)dt + d\tilde{u}(t)$$

where the self-adjoint, adapted process $\beta_u(t)$ is the signal part of $u(t)$ and $\tilde{u}(t)$ is the noise part of $u(t)$. Note that $\beta_u(t)$ also commutes with $d\tilde{u}(t)$ for all $t \geq 0$.

The noise $\tilde{u}(t)$ is a vector of self-adjoint quantum noises with Ito table

$$d\tilde{u}(t)d\tilde{u}^T(t) = F_{\tilde{u}}dt$$

where $F_{\tilde{u}} = S_{\tilde{u}} + T_{\tilde{u}}$ is a nonnegative Hermitian matrix [4, 51] with $S_{\tilde{u}}$ and $T_{\tilde{u}}$ being real and imaginary, respectively. $S_{\tilde{u}}$ describes the intensity of the quantum Wiener process and is the quantum analog of the intensity matrix for a classical Wiener process.

The commutation relations for the noise components are determined by T_u:

$$[d\tilde{u}(t), d\tilde{u}(t)^\dagger] \triangleq d\tilde{u}(t)d\tilde{u}(t)^\dagger - (d\tilde{u}(t)d\tilde{u}(t)^T)^T = 2T_u dt. \tag{7.2}$$

Also, $dv(t)$ (column vectors with n_v components) is a vector of quantum Wiener processes corresponding to the introduced vacuum noise inputs. Consequently, F_v, S_v, and T_v is defined for $dv(t)$ as $F_{\tilde{w}_x}$, $S_{\tilde{w}_x}$, and $T_{\tilde{w}_x}$ were for $d\tilde{w}_x(t)$ in (2.18).

7.3 Physical Realizability

In [30], the notion of physical realizability for an active system is based on the concept of an open quantum harmonic oscillator. The formal definition is as in Definition 3.1 and Lemma 3.1.

In this chapter, we consider the LMI version of the physical realizability which is similar to the approach in [30, 75] but reformulated into an LMI problem as seen in Chapter 3.

7.4 Coherent LQG Control Problem

The problem formulation described in this work is similar to [65]. However, we relax certain conditions related to physical realizability. In particular, in [65], the authors considered the case where both the quantum plant and quantum controller are physically realizable while in our case, we only require the quantum controller to be physically realizable.

Suppose we have a quantum plant [53] that is a combination of the dynamics of the quantum channel and a low pass filter which is described by the following QSDEs (similar to Chapter 6.4):

$$\begin{aligned} dx(t) &= Ax(t)dt + Bd\omega(t) + Ed\hat{u}(t); \\ dy(t) &= Cx(t)dt + Ddw(t) \end{aligned} \tag{7.3}$$

where A, B, E, C, and D are real matrices in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times n_w}$, $\mathbb{R}^{n\times n_u}$, $\mathbb{R}^{n_y\times n}$, and $\mathbb{R}^{n_y\times n_w}$ (n, n_w, n_u, n_y are even positive integers), respectively and the vector $d\omega = [du \quad dw]^T$. Note that the low pass filter is included in the quantum plant (7.3) to ensure that the cost (7.10) is finite. This is justified by the fact that in practice, equalization will only be required over a finite bandwidth.

Suppose also that we have a controller of the form:

$$\begin{aligned} dx_k(t) &= ax_k(t)dt + bd\omega_k(t) + edy(t); \\ d\hat{u}(t) &= cx_k(t)dt + dd\omega_k(t). \end{aligned} \tag{7.4}$$

where a, b, e, c, and d are real matrices in $\mathbb{R}^{n_k\times n_k}$, $\mathbb{R}^{n_k\times n_{\omega_k}}$, $\mathbb{R}^{n_k\times n_y}$, $\mathbb{R}^{n_u\times n_k}$, and $\mathbb{R}^{n_u\times n_{\omega_k}}$. The proposed gradient descent algorithm by [65] requires for its initialisation a stabilizing physically realizable quantum controller. Therefore, for our work, our modified gradient descent algorithm is initialised with a physically realizable quantum controller designed in Chapter 6.

The combination of equations (7.3) and (7.4) describes the quantum closed-loop system. In the minimisation of a quadratic cost, adopted in quantum control [49, 72] settings from classical LQG control [37], the performance of the coherent quantum controller will be described to an r-dimensional quantum process

$$\mathscr{Z} = Fx(t). \tag{7.5}$$

Here $F \in \mathbb{R}^{r\times n}$ is the given weighting matrix which is chosen so that $\mathscr{Z}(t)$ corresponds to the filtered error signal $\bar{e}$ as in Figure 6.1. The process $\mathscr{Z}$ in (7.5) is linearly related to the $2n$-dimensional vector of dynamic variables

$$\mathscr{X} = \begin{bmatrix} x \\ x_k \end{bmatrix} \tag{7.6}$$

of the closed-loop system whose dynamics are governed by the QSDEs

$$\begin{aligned} d\mathscr{X} &= \mathscr{A}\mathscr{X}dt + \mathscr{B}d\mathscr{W}; \\ \mathscr{Z} &= \mathscr{C}\mathscr{X}dt \end{aligned} \tag{7.7}$$

which is driven by the combined quantum Wiener process

$$\mathscr{W} = \begin{bmatrix} \omega \\ \omega_k \end{bmatrix}. \tag{7.8}$$

The state-space matrices $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ of the closed-loop system in (7.7) are obtained by combining equations (7.3), (7.4) with (7.5), (7.6), (7.8) and depend on the controller matrices a, b, c, e in an affine manner:

$$\mathscr{A} = \begin{bmatrix} A & Ec \\ eC & a \end{bmatrix}; \tag{7.9a}$$

$$\mathscr{B} = \begin{bmatrix} B & Ed \\ eD & b \end{bmatrix}; \tag{7.9b}$$

$$\mathscr{C} = \begin{bmatrix} F & 0 \end{bmatrix}. \tag{7.9c}$$

Then, the problem under consideration is as follows: given a quantum plant and coherent controller of the forms (7.3) and (7.4), use the gradient descent [65] approach to find an optimal coherent controller that minimizes the cost function

$$J_{cost} = \frac{1}{2} \lim_{t \to \infty} E(\mathscr{Z}(t)^T \mathscr{Z}(t)) = \frac{1}{2} \langle \mathscr{C}^T \mathscr{C}, P \rangle \tag{7.10}$$

where P coincides with the controllability Gramian [37] of the pair $(\mathscr{A}, \mathscr{B})$ which is the unique solution of the algebraic Lyapunov equation (ALE)

$$\mathscr{A} P + P \mathscr{A}^T + \mathscr{B} \mathscr{B}^T = 0. \tag{7.11}$$

The cost function (7.10) is a function of the triple

$$u = (R, b, e)$$

which parameterize physically realizable quantum controllers (7.4) through the formulas

$$a = 2\Theta_2 R - \frac{1}{2}(eDJ_1D^Te^T + bJ_2b^T)\Theta_2^{-1}$$

and

$$c = -dJ_2b^T\Theta_2^{-1},$$

with the controller noise feedthrough matrix d being fixed and satisfying

$$dd^T = I.$$

Note that J_1, J_2, and Θ_2 are similar to matrix Θ (2.21) with appropriate dimensions.

7.5 Algorithm

The main idea of our algorithm is to find an optimal LQG controller that minimizes the cost function (7.10). The gradient descent approach involves adjusting the controller parameters with a negative gradient flow so that the cost function moves towards a local minimum.

We consider the coherent LQG control approach [65] which gives the following controller gradients

$$\partial_R J_{cost} = -2\mathrm{sym}(\Theta_2 H_{22}); \tag{7.12a}$$
$$\partial_b J_{cost} = Q_{21}Ed + Q_{22}b - \psi b J_2 - \chi d J_2; \tag{7.12b}$$
$$\partial_e J_{cost} = H_{21}C^T + Q_{21}BD^T + Q_{22}e - \psi e D J_1 D^T \tag{7.12c}$$

where ψ and χ are auxiliary matrices defined by

$$\psi = \mathrm{asym}(H_{22}\Theta_2^{-1}),$$
$$\chi = \Theta_2^{-1}(H_{12}^T E + P_{21}F^T G + P_{22}c^T G^T G).$$

Here, the parameters R, b and e define a physically realizable controller as in Chapter 6. Also, Q is denoted as the observability Gramian of the pair $(\mathscr{A}, \mathscr{C})$ which is the unique solution of the ALE

$$\mathscr{A}^T Q + Q\mathscr{A} + \mathscr{C}^T\mathscr{C} = 0 \tag{7.13}$$

where $\mathscr{A}$ is Hurwitz. In addition, the matrix H is defined as $H = QP$.

Note that $(2n \times 2n)$ matrices X (such as P, Q, and H) are partitioned into $n \times n$ blocks as

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

Moreover, sym(.) and asym(.) are denoted as $\frac{(.)+(.)^T}{2}$ and $\frac{(.)-(.)^T}{2}$, respectively.

7.6 Application to a Quantum Equalization Problem

We now describe the application of gradient descent coherent LQG method to a quantum equalization problem. Consider the quantum equalization problem from Chapter 6, as shown in Figure 7.1.

The original version of this equalization problem is given in [69]. However, in [69] attention is limited to passive controllers whereas our method allows

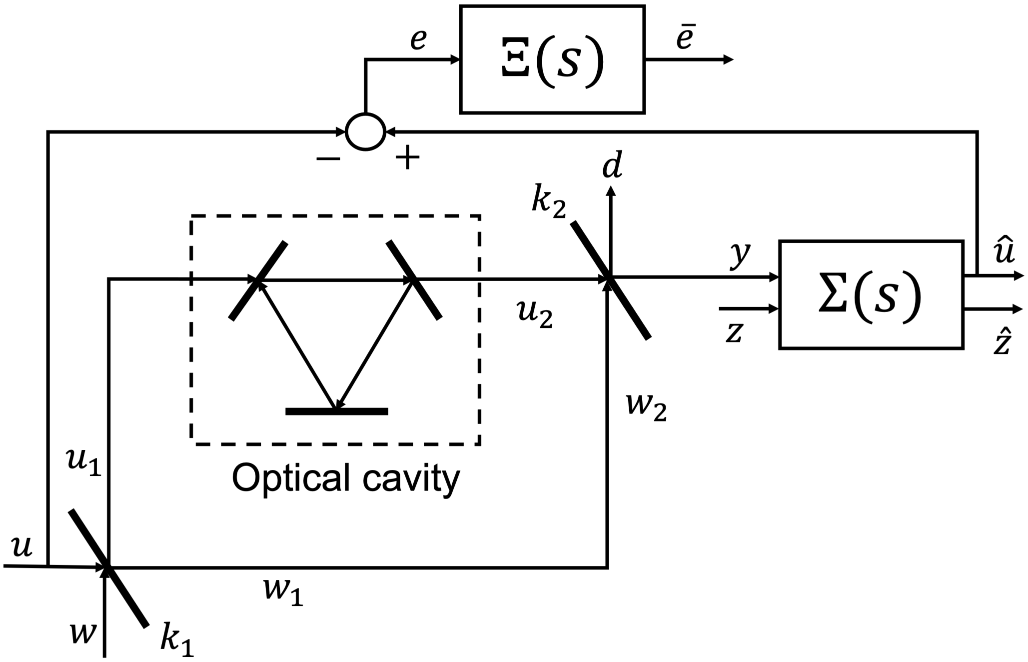

Figure 7.1: Equalization of an optical cavity system.

for active controllers. The channel consists of an optical cavity and two optical beam splitters with the constants as follows: $\kappa = 5$, $k = 0.4$, $m = \sqrt{1-k^2}$ and $\Omega = 10$. Also, the time constant of the low pass filter is chosen as $\tau = 0.1$. The quantum plant and quantum controller are adapted from Chapter 6, while the constants are adapted from [69, Section 6.2].

Applying the gradient descent method to this quantum equalization problem, the local minimum value of the cost is 9.8598 as shown in Figure 7.2. This

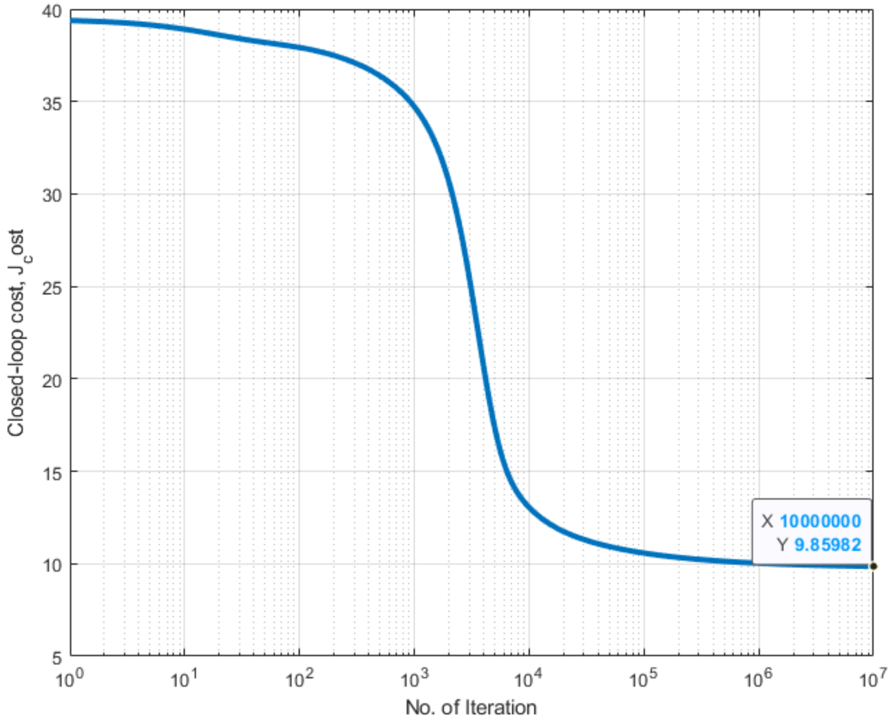

Figure 7.2: Closed loop Cost Function.

is achieved with the following controller parameters:

$$
\begin{aligned}
R &= \begin{bmatrix} 2.6459 & 2.6667 & -0.3977 & -1.2539 \\ 2.6672 & -0.9135 & 1.2654 & 0.0868 \\ -0.3974 & 1.2652 & 2.4148 & 2.5863 \\ -1.2537 & 0.0867 & 2.5867 & -0.5650 \end{bmatrix} \\
b &= \left[\begin{matrix} 7.5474 & -45.4974 & -0.0001 & 0.0000 & 0.5171 \\ 44.9196 & 7.5767 & 0.0000 & 0.0000 & 1.7191 \\ 42.2906 & 3.2957 & 0.0000 & 0.0000 & -1.9469 \\ -3.1259 & 41.5932 & -0.0001 & 0.0000 & 0.3873 \end{matrix}\right. \\
&\qquad \left.\begin{matrix} -1.7852 & 1.4669 & 8.9850 & -4.6647 & -3.8767 \\ 0.7833 & -3.4738 & -2.3761 & 1.3236 & -10.6587 \\ -0.6870 & 3.6853 & 1.7305 & -0.9976 & 12.1382 \\ -1.9803 & 1.8982 & 9.8170 & -5.1056 & -3.1174 \end{matrix}\right]; \\
e &= \begin{bmatrix} 4.5916 & -45.7648 \\ 45.4293 & 4.5173 \\ 42.2991 & 0.5437 \\ -0.2921 & 41.8492 \end{bmatrix}.
\end{aligned}
$$

The norm of the gradients for R, b, and e can be observed in Figure 7.3. The gradient descent algorithm was initialised with a coherent controller designed using the method of Chapter 6.

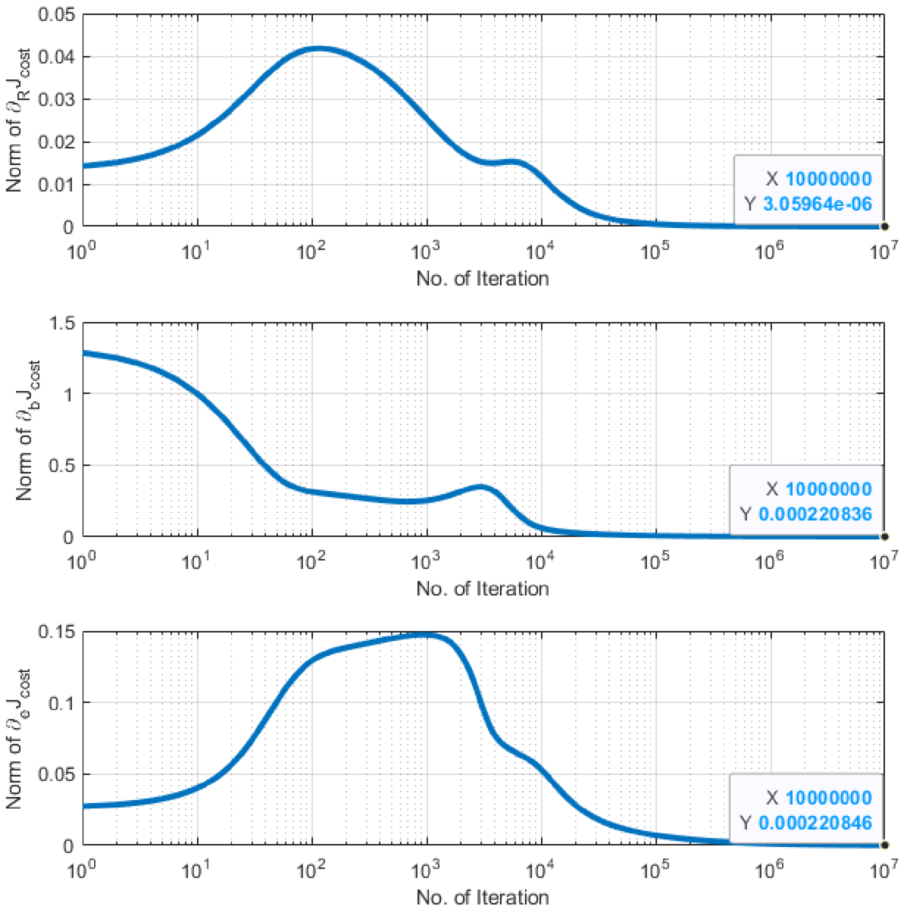

Figure 7.3: Norm of Gradients of R, b,and e.

7.7 Conclusions

In this work, we have implemented a gradient descent approach to find a physically realizable coherent LQG quantum controller that minimizes a closed-loop cost function related to the system equalization error. We apply this approach to a quantum equalization problem derived from an example that was first considered in [69].

Chapter 8

Conclusion and Future Research

8.1 Conclusions

In this thesis, we have focused on physical realizability and coherent control of linear quantum systems, concentrating on the dynamics within the framework of quantum mechanics. It highlights the challenges in dealing with linear control problems in the quantum realm and emphasizes the need for additional constraints to make linear quantum systems physically realizable. The introduction of additional quantum vacuum noise channels in LTI systems adds complexity and is a non-trivial problem.

The first part of the thesis focuses on extending and improving existing methods for implementing LTI systems as a physically realizable quantum system. It introduces novel ways to implement LTI systems, paying attention to minimizing the impact of additional quantum noise on the system output. This brings necessary and sufficient conditions for physical realizability when dealing with both direct feedthrough quantum vacuum noise and additional quantum vacuum noise, along with a frequency domain condition for achieving a given transfer function matrix using only direct feedthrough quantum vacuum noise.

The latter part of the thesis explores coherent quantum control, a unique feedback control paradigm with no counterpart in classical control systems. It specifically addresses the quantum equalization problem. A method is proposed to find a physically realizable suboptimal LQG controller, minimizing a cost related to system equalization error. Then the implementation of the gradient descent approach enhances the search for the best solution to the quantum equalization problem.

In conclusion, this thesis contributes to deepening our understanding of linear quantum systems and provides methods for making LTI systems physically meaningful linear quantum systems. The application of quantum equalization highlights the practical implications of these findings in practical linear quantum systems, emphasizing the importance of minimizing quantum vacuum noises for improving overall control performance.

8.2 Future Work

A summary of potential directions for future research:

1. In Chapter 3, we have proposed a method to find a physically realizable system that minimizes a cost function related to the amount of additional quantum noise that affects the system output. Therefore, to extend this work, it is worthwhile to consider whether examples can be found where our method does not give the minimum number of noises. In particular, to investigate the conjecture that minimizing the cost function also minimizes the number of noises.

2. The frequency response condition presented in Chapter 5 is only a necessary frequency response condition. Exploring the extension of this condition to a necessary and sufficient criterion, perhaps by investigating into the concept of physical realizability with only direct feedthrough noise, could be interesting.

3. The suboptimal nature of the method to implement physically realizable active and passive LTI systems used in Chapter 6 is probably the cause for the discrepancy seen in the calculated cost function. Therefore, looking into this direction, it is worthwhile to consider developing a general method to physically realize both passive and active LTI systems. It would also be of interest to extend the results in Chapter 3 to the case when purely passive systems are considered.

4. It would be interesting to explore conditions to determine when it is better to use passive or active coherent quantum controllers. Understanding the specific conditions for each type of system and their intended purpose can contribute to an improved understanding of coherent LQG problems.

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