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An Equivariant Atiyah–Patodi–Singer Index Theorem for Proper Actions I: The Index Formula

Peter Hochs[1], Bai-Ling Wang[2] and Hang Wang[3]

[1]Institute for Mathematics, Astrophysics and Particle Physics, Radboud University, PO Box 9010, 6500 GL, Nijmegen, The Netherlands, [2]Mathematical Sciences Institute, Australian National University, Canberra, ACT 0200, Australia, and [3]School of Mathematical Sciences, East China Normal University, Shanghai 200241, China

*Correspondence to be sent to: e-mail: p.hochs@math.ru.nl

Consider a proper, isometric action by a unimodular locally compact group G on a Riemannian manifold M with boundary, such that M/G is compact. For an equivariant, elliptic operator D on M, and an element $g \in G$, we define a numerical index $\mathrm{index}_g(D)$, in terms of a parametrix for D and a trace associated to g. We prove an equivariant Atiyah–Patodi–Singer index theorem for this index. We first state general analytic conditions under which this theorem holds, and then show that these conditions are satisfied if $g = e$ is the identity element; if G is a finitely generated, discrete group, and the conjugacy class of g has polynomial growth; and if G is a connected, linear, real semisimple Lie group, and g is a semisimple element. In the classical case, where M is compact and G is trivial, our arguments reduce to a relatively short and simple proof of the original Atiyah–Patodi–Singer index theorem. In part II of this series, we prove that, under certain conditions, $\mathrm{index}_g(D)$ can be recovered from a K-theoretic index of D via a trace defined by the orbital integral over the conjugacy class of g.

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1 Introduction

1.1 Background

For a Dirac-type operator D on a compact, even-dimensional manifold M with boundary N, the Atiyah–Patodi–Singer (APS) index theorem [5] states that

$$\mathrm{index}_{\mathrm{APS}}(D) = \int_M \mathrm{AS}(D) - \frac{\dim\ker(D_N) + \eta(D_N)}{2}. \tag{1.1}$$

Here $\mathrm{index}_{\mathrm{APS}}(D)$ is the analytic index of D as an operator between spaces of sections satisfying the APS boundary conditions; $\mathrm{AS}(D)$ is the Atiyah–Singer integrand corresponding to D; D_N is a Dirac operator on the boundary N, compatible with D, and $\eta(D_N)$ is its η-*invariant*.

This η-invariant is the key additional ingredient compared to the Atiyah–Singer index theorem for the case of closed manifolds [8]. It measures the spectral asymmetry of D_N. If $\{\lambda_j\}_{j=1}^{\infty}$ are the nonzero eigenvalues of D_N with multiplicities, then one considers the series

$$\sum_{j=1}^{\infty} \frac{\mathrm{sgn}(\lambda_j)}{|\lambda_j|^z},$$

for z with large enough real part so that the series converges. The meromorphic continuation of this expression is regular at $z = 0$, and $\eta(D_N)$ is its value at $z = 0$. An equivalent expression is

$$\eta(D_N) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \mathrm{Tr}(D_N e^{-t^2 D_N^2})\, \mathrm{d}t,$$

where Tr is the operator trace. The integral converges in many cases, for example for twisted Spin–Dirac operators [13]. Otherwise, one replaces the lower integration limit by $\varepsilon > 0$ and defines the integral as the constant term in the asymptotic expansion in powers of ε, as $\varepsilon \downarrow 0$.

The equality (1.1) has been generalised in several directions. Space does not permit us to give a complete overview of the literature here, but we briefly mention some of the earlier work that is most relevant to the present paper, where we study equivariant indices on noncompact manifolds.

Donnelly [25] proved an equivariant version of (1.1), where a compact group acts on M, generalising the Atiyah–Segal–Singer fixed point theorem [6, 7] to manifolds with boundary. This was generalised to manifolds that do not have a product form near the boundary by Braverman–Maschler [14].

On noncompact manifolds, APS index theory of Callias-type Dirac operators was developed by Braverman and Shi [15, 16, 48, 49].

After this paper appeared, Piazza, Posthuma, Song, and Tang generalised our results in the case of semisimple Lie groups [42]. Their definition of the index involves b-calculus and is different from ours, but likely yields the same number.

Various authors refined (1.1) by considering the action by the fundamental group $G = \pi_1(M)$ on the universal cover $\overline{M}$ of M. A first version of the η-invariant that applies in this setting is Cheeger–Gromov's L^2-η-invariant [19]. This corresponds to the identity element of G in the sense of [35]. This type of η-invariant was used by Ramachandran [45] to generalise (1.1) to a statement about the lift $\overline{D}$ of D to $\overline{M}$.

To include information corresponding to nontrivial conjugacy classes of G, Lott developed the notions of *higher* and *delocalised η-invariants* [35, 37]. (See also the $\hat{\eta}$-form of [12].) The higher η-invariant lies in a homology space of an algebra of rapidly decaying functions on G. These invariants are interesting for their own sakes and were also used to state and prove generalisations of (1.1). Generalisations of Ramachandran's result involving this higher η-invariant were proved in the context of b-calculus by Leichtnam–Piazza [32–34]. These constructions and results involve assumptions on G and its elements used, such as polynomial growth of conjugacy classes.

Xie–Yu [53] worked in the context of coarse geometry and showed that Lott's delocalised η-invariant can be recovered from a K-theoretic ρ-invariant. They generalised (1.1) to a version for nontrivial elements of G whose conjugacy classes have polynomial growth, for the action by G on $\overline{M}$. Since this action is free, and the first term on the right hand side of (1.1) becomes an integral over the fixed point set of the group element in question, that term vanishes in this context. A version for higher cyclic cocycles was proved by Chen–Wang–Xie–Yu, see Theorem 7.3 in [20].

In this paper, we work in the more general setting of proper actions by locally compact groups, instead of just fundamental groups acting freely on universal covers. We obtain an equivariant version of (1.1) in this case.

1.2 The main results

Consider a unimodular, locally compact group G acting properly and isometrically on a Riemannian manifold M, with boundary N, such that M/G is compact. Let D be a G-equivariant Dirac-type operator on a G-equivariant, $\mathbb{Z}_2$-graded Hermitian vector bundle $E = E_+ \oplus E_- \to M$. Suppose that all structures have a product form near N.

In particular, suppose that near N, the restriction of D to sections of E_+ equals

$$\sigma\Big(-\frac{\partial}{\partial u}+D_N\Big),$$

where $\sigma: E_+|_N \to E_-|_N$ is an equivariant vector bundle isomorphism, u is the coordinate in $(0,1]$ in a neighbourhood of N equivariantly isometric to $N\times(0,1]$, and D_N is a Dirac operator on $E_+|_N$.

Let $g\in G$, and let Z be its centraliser in G. Suppose that G/Z has a G-invariant measure $d(hZ)$. We will define an index of D in terms of the *g-trace* of a G-equivariant operator T on $L^2(E)$, defined as follows. Let χ be a nonnegative function on M whose support has compact intersections with all G-orbits, such that for all $m\in M$,

$$\int_G \chi(gm)^2\,\mathrm{d}g = 1. \tag{1.2}$$

A G-equivariant operator T on $L^2(E_+)$ or $L^2(E_-)$ is said to be *g-trace class* if it has a smooth kernel κ, and the integral

$$\int_{G/Z}\int_M \chi(hgh^{-1}m)^2\mathrm{tr}(hgh^{-1}\kappa(hg^{-1}h^{-1}m,m))\,\mathrm{d}m\,\mathrm{d}(hZ)$$

converges absolutely. This integral is then its g-trace, denoted by $\mathrm{Tr}_g(T)$. An equivariant, possibly unbounded operator F from $L^2(E_+)$ to $L^2(E_-)$ is *g-Fredholm* if it has a parametrix R such that the operators $1-RF$ and $1-FR$ are g-trace class. The *g-index* of F, denoted by $\mathrm{index}_g(F)$, is then the difference of the g-traces of these operators. For odd, self-adjoint operators on $L^2(E)$, we define the g-index as the g-index of their restrictions to even-graded sections.

The g-trace and g-index can be defined analogously for operators on N. The *g-delocalised η-invariant*of D_N is

$$\eta_g(D_N) := \frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{Tr}_g(D_N e^{-t^2D_N^2})\,\mathrm{d}t, \tag{1.3}$$

whenever the integrand is defined for all $t>0$ and the integral converges. (The term *delocalised* is often reserved for the case $g\neq e$. We will use it for $g=e$ as well.)

Let $\tilde{M}$ be the double of M, let $\tilde{E}\to\tilde{M}$ be the extension of E, and let $\tilde{D}$ be the extension of D to $\tilde{M}$. (To be precise, on the second copy of M inside $\tilde{M}$, both the orientation on M and the grading on E are reversed.) We write $\tilde{D}_\pm$ for the restrictions of $\tilde{D}$ to sections of $\tilde{E}_\pm$, respectively. We state the results here under the condition that D_N

is invertible for simplicity; the results generalise to the case where 0 is isolated in the spectrum of D_N.

Theorem 1.1. Suppose that D_N is invertible and has a well-defined g-delocalised η-invariant. Suppose that the conditions listed in Theorem 2.13 hold. Then the operators D and $\tilde{D}$ are g-Fredholm, and

$$\operatorname{index}_g(D) = \frac{1}{2}\operatorname{index}_g(\tilde{D}) - \frac{1}{2}\eta_g(D_N). \tag{1.4}$$

See Theorem 2.13. The manifold $\tilde{M}$ has no boundary, so that the first term on the right hand side of (1.4) is the contribution of the interior of M.

We make Theorem 1.1 more concrete if g is the identity element e of an arbitrary unimodular locally compact group, and for more general elements g of two relevant classes of groups. We do this by showing that the conditions of Theorem 1.1 hold, and giving a topological expression for the first term on the right hand side of (1.4). This has the following consequence.

Corollary 1.2. Suppose that D is a Spin^c-Dirac operator twisted by a vector bundle, and that D_N is invertible. Suppose furthermore that either

- G is any unimodular locally compact group, and $g = e$;
- G is discrete and finitely generated, and the conjugacy class of g has polynomial growth; or
- G is a connected, linear, real semisimple Lie group, and g is semisimple.

Then D is g-Fredholm, D_N has a well-defined g-delocalised η-invariant, and

$$\operatorname{index}_g(D) = \int_{M^g} \chi_g^2 \operatorname{AS}_g(D) - \frac{1}{2}\eta_g(D_N),$$

where $\operatorname{AS}_g(D)$ is the integrand in the Atiyah–Segal–Singer fixed point formula, and χ_g is defined analogously to (1.2), for the action by Z on the fixed-point set M^g.

See Corollaries 2.16, 2.19, and 2.21. The topological expressions for the contributions from $\tilde{M}$, that is, the contributions of the interior of M, are the main results in [29, 51, 52].

Theorem 1.1 and Corollary 1.2 generalise to the case where D_N is not necessarily invertible, but 0 is isolated in its spectrum. See Theorem 6.2 and Corollary 6.3. These results simultaneously generalise the main results in [5, 25] to the noncompact case, the main result in [45] and Theorem 5.3 in [53] to the case of general proper actions, by a larger class of groups, and the main results in [29, 51, 52] to the case of manifolds with boundary.

In part II of this series [28], we prove that, under certain conditions, $\mathrm{index}_g(D)$ can be recovered from a K-theoretic index of D via a trace defined by the orbital integral over the conjugacy class of g. Then Theorem 1.1 and Corollary 1.2 can be used to study that index as well.

1.3 Features of the proofs and future work

Our approach to proving an equivariant APS index theorem for proper actions is to work with general locally compact groups as long as possible, and to find general conditions under which such an index theorem holds. This leads to Theorem 1.1. We then check these conditions $g = e$ and for discrete and semisimple groups and obtain the more concrete Corollary 1.2. We then discuss how to weaken the assumption that D_N is invertible to the assumption that zero is isolated in its spectrum, if it is in the spectrum.

As expected, the techniques used to show that the conditions of Theorem 1.1 hold are very different for discrete groups and semisimple Lie groups. Nevertheless, one of our aims here is to develop a unified approach to this type of index theorem as far as possible. For other classes of groups, such index theorems can now be obtained by checking the conditions of Theorem 1.1. And the fact that the explicit index formula in Corollary 1.2 holds for two quite different classes of groups suggests that this formula should hold more generally.

Our proofs of Theorem 1.1 and Corollary 1.2 follow a different approach from proofs of earlier results. In the technical heart of our proof, we apply the g-trace to a sum of operators that individually are not trace-class for this trace. By adding and subtracting another operator in a suitable way, we are able to split up that trace into two computable terms. “Regularising” traces in this way is related to the b-calculus approach to the APS index theorem, as Richard Melrose pointed out to us. In part II [28], we link the approach in this paper to an equivariant index defined in terms of coarse geometry and Roe algebras. We have not found the combination of coarse index theory and such techniques related to b-calculus elsewhere in the literature, and this is a potentially novel feature of our approach.

The notion of g-trace class operators that we use is a weak one and does reduce to the usual notion of trace class operators if G is trivial, for example. We use this weak notion, because the operators $1 - RF$ and $1 - FR$, as below (1.2), which are relevant in our setting, have this property, and not (obviously) a stronger property. This is enough for the proof of Theorem 1.1. A price we pay for using this generalised notion of g-trace class operators is that compositions, or even squares, of g-trace class operators are not obviously g-trace class. This is one of the technical points that is solved in part II [28], to obtain a link with a K-theoretic index.

In the classical case, where M is compact and G is trivial (or compact), many technical parts of our proofs are unnecessary. Then only simplified versions of the arguments in Subsections 5.2–5.5 and parts of Section 6 are needed to prove (1.1). This leads to a proof of the original APS index theorem that, to the authors, seems considerably shorter and simpler than the proof in [5]. As pointed out in the previous paragraph, this proof involves a regularisation procedure for traces of initially non-trace class operators, analogous to the b-trace in b-calculus [38].

Part of our motivation for proving Corollary 1.2 is potential future applications to Borel–Serre compactifications of noncompact locally symmetric spaces. These spaces are finite-volume orbifolds of the form $\Gamma\backslash G/K$, where G is a semisimple Lie group, $K < G$ is maximal compact, and $\Gamma < G$ is a cofinite-volume lattice. If $\Gamma\backslash G/K$ is noncompact, a natural compactification is its Borel–Serre compactification, which in general is an orbifold with corners. By either generalising Corollary 1.2 to manifolds with corners, or, initially, considering cases where the Borel–Serre compactification is a manifold with boundary, for example, $G = SL(2,\mathbb{R}), K = SO(2)$, and $\Gamma = SL(2,\mathbb{Z})$, we intend to investigate applications of the results in this paper to equivariant index theory for the action by Γ on the space whose quotient is the Borel–Serre compactification of $\Gamma\backslash G/K$, and associated trace formulas. Such applications would be equivariant refinements of [3, 4]. This may require an extension of Corollary 1.2 to conjugacy classes that do not have polynomial growth; see Remark 2.20. More generally, the case of Corollary 1.2 for discrete groups has potential applications to orbifolds with boundary.

1.4 Outline of this paper

In Section 2, we introduce the g-trace and the g-index and Lott's delocalised η-invariants and state our main results: Theorem 2.13 and Corollaries 2.16, 2.19, and 2.21. In Section 3, we discuss some properties of the g-trace and the g-index. We use these properties in Section 4 to prove convergence of delocalised η-invariants in several cases, and then to prove the main results in Section 5. Up to this point, we assume that the Dirac operator D_N on the boundary is invertible. We discuss how to remove that assumption in Section 6. In Appendix A, we show how the proof of the index theorem simplifies in the original compact case.

2 Preliminaries and Results

2.1 The g-trace and the g-index

Let G be a locally compact group, with a left Haar measure dg. Let $g \in G$, and let Z be its centraliser. Suppose that G/Z has a G-invariant measure $d(hZ)$. Let M be a complete Riemannian manifold, on which G acts properly and isometrically. Let $\chi \in C^\infty(M)$ be nonnegative, and such that for all $m \in M$,

$$\int_G \chi(gm)^2 \, \mathrm{d}g = 1.$$

Let $E \to M$ be a G-equivariant, Hermitian vector bundle. We will write

$$\mathrm{End}(E) := E \boxtimes E^* \to M \times M.$$

Definition 2.1. A section $\kappa \in \Gamma^\infty(\mathrm{End}(E))^G$ is *g-trace class* if the integral

$$\int_{G/Z} \int_M \chi(hgh^{-1}m)^2 |\mathrm{tr}(hgh^{-1}\kappa(hg^{-1}h^{-1}m, m))| \, \mathrm{d}m \, \mathrm{d}(hZ) \tag{2.1}$$

converges. Then

$$\int_{G/Z} \int_M \chi(hgh^{-1}m)^2 \mathrm{tr}(hgh^{-1}\kappa(hg^{-1}h^{-1}m, m)) \, \mathrm{d}m \, \mathrm{d}(hZ) \tag{2.2}$$

is the *g-trace* of κ, denoted by $\mathrm{Tr}_g(\kappa)$.

An operator $T \in \mathcal{B}(L^2(E))^G$ is g-trace class if it has a smooth Schwartz kernel κ that is g-trace class. Then we write $\mathrm{Tr}_g(T) := \mathrm{Tr}_g(\kappa)$.

Lemma 2.2. If G is compact, and T is a trace-class operator on $L^2(E)$ with a smooth kernel, then T is e-trace class and $\mathrm{Tr}_e(T) = \mathrm{Tr}(T)$.

Proof. If G is compact, we normalise dg so that G has unit volume and take $\chi \equiv 1$. Then the claim follows from the definitions. ■

Lemma 2.2 applies for example if T is orthogonal projection onto the finite-dimensional kernel of an elliptic operator.

Remark 2.3. Definition 2.1 is a relatively weak condition. For example, if G is trivial, then an e-trace class kernel need not be trace class in the usual sense. And, more relevantly to us in part II [28], it is not necessarily true that a product of g-trace class operators is again g-trace class. This is why in [28], it will take some work to show that the squares of certain g-trace class operators are again g-trace class under certain conditions.

The reason why we use the weak definition of g-trace class as in Definition 2.1 is that the operators S_0 and S_1 in (5.4) are g-trace class in this sense, but not obviously in stronger senses one could think of.

From now on, fix a G-invariant $\mathbb{Z}_2$-grading on E. Let E_+ be the even part of E for this grading and E_- the odd part. We will apply Definition 2.1 to operators on $L^2(E_\pm)$ as well.

Definition 2.4. Let D be an odd-graded, G-equivariant, elliptic differential operator on E. Let D_+ be its restriction to sections of E_+. Then D is *g-Fredholm* if D_+ has a parametrix R such that the operators $S_0 := 1_{E_+} - RD_+$ and $S_1 := 1_{E_-} - D_+R$ are g-trace class.

Lemma 2.5. Suppose that D is a g-Fredholm operator, and let S_0 and S_1 be as in Definition 2.4. Then the number

$$\mathrm{Tr}_g(S_0) - \mathrm{Tr}_g(S_1)$$

is independent of the choice of the parametrix R.

Definition 2.6. The *g-index* of a g-Fredholm operator D is the number

$$\mathrm{index}_g(D) := \mathrm{Tr}_g(S_0) - \mathrm{Tr}_g(S_1), \tag{2.3}$$

with S_0 and S_1 as in Definition 2.4.

In the rest of this subsection, we mention some special cases of the g-index. Some of these will be used later in this paper, but most are included to show that this index is a natural number to study.

Lemma 2.7. Suppose that D is a g-Fredholm operator. Let $P_\pm$ be the orthogonal projections onto the kernels of D_+ and D_+^*. If P_+ and P_- are g-trace class, then

$$\mathrm{index}_g(D) = \mathrm{Tr}_g(P_+) - \mathrm{Tr}_g(P_-). \tag{2.4}$$

Lemmas 2.5 and 2.7 are proved in Subsection 3.2.

Lemma 2.7 shows that index_e generalises Atiyah's L^2-index [2], also used in [22, 52]. Ramachandran [45] obtained an Atiyah–Patodi–Singer type index theorem, using the right hand side of (2.4) for $g = e$ to define an index. Lemma 2.7 shows that this index is a special case of the g-index. This is also noted in the proof of Theorem 7.11 on page 343 of [45].

Lemmas 2.2 and 2.7 have the following immediate consequence. (We will use this in the discussion of the compact case of the index theorem in Appendix A.)

Lemma 2.8. If G is compact, and D is an e-Fredholm operator with finite-dimensional kernel, then

$$\mathrm{index}_e(D) = \dim(\ker(D_+)) - \dim(\ker(D_+^*)).$$

Lemma 2.9. Suppose that D is a g-Fredholm operator with finite-dimensional kernel, and that G is compact. Let $\chi_\pm$ be the characters of the finite-dimensional representations $\ker(D_+)$ and $\ker(D_+^*)$ of G. Then

$$\mathrm{index}_g(D) = \chi_+(g) - \chi_-(g).$$

Proof. If G is compact, then one may take $\chi = 1$ in (2.2). It then follows that $\mathrm{Tr}_g(P_\pm) = \chi_\pm(g)$, so the claim follows from Lemma 2.7. ■

In part II of this series [28], we relate the g-index to a K-theoretic index of D. This was done before in cases where M/G is compact. (We will need the case where M/G is noncompact as well, since we apply the g-index to the manifold $\hat{M}$ in (2.7).) If M/G is compact, then the natural K-theoretic index of D to use the the analytic assembly map [9]. This yields

$$\mathrm{index}_G(D) \in K_*(C_r^*(G)).$$

Here $C_r^*(G)$ is the *reduced group C^*-algebra* of G: the closure in $\mathcal{B}(L^2(G))$ of the algebra of left convolution operators by functions in $L^1(G)$. The relation with the g-index is based on the *orbital integral trace*

$$\tau_g(f) = \int_{G/Z} f(hgh^{-1})\,\mathrm{d}(hZ), \tag{2.5}$$

for functions f on G for which the integral converges. If that is the case for f in a dense subalgebra of $C_r^*(G)$, closed under holomorphic functional calculus, then τ_g defines a map $\tau_g\colon K_0(C_r^*(G)) \to \mathbb{C}$.

Proposition 2.10. Suppose that M/G is compact, and that either

- $g = e$;
- G is discrete and τ_g extends to a dense subalgebra of $C^*_r(G)$, closed under holomorphic functional calculus; or
- G is a real semisimple Lie group and g is a semisimple element.

Then D is g-Fredholm, and

$$\mathrm{index}_g(D) = \tau_g(\mathrm{index}_G(D)).$$

Proof. The first case is Propositions 3.6 and 4.4 in [52]. The second case is Propositions 3.20 and 5.9 in [51]. The last case is Lemma 3.5 and Proposition 3.6 in [29]. ■

In cases where the g-index can be recovered from the K-theoretic index, as in Proposition 2.10 and in the main result in [28], homotopy invariance of the latter index implies homotopy invariance of the former. See Corollary 2.9 in [28].

2.2 Manifolds with boundary

We now specialise to the case we are interested in in this paper. Slightly changing notation from the previous subsection, we let M be a Riemannian manifold with boundary N. We still suppose that G acts properly and isometrically on M, preserving N, such that M/G is compact. We assume that a G-invariant neighbourhood U of N is G-equivariantly isometric to a product $N \times (0, \delta]$, for a $\delta > 0$. To simplify notation, we assume that $\delta = 1$; the case for general δ is entirely analogous.

As before, let $E = E_+ \oplus E_- \to M$ be a $\mathbb{Z}_2$-graded G-equivariant, Hermitian vector bundle. We assume that E is a *Clifford module*, in the sense that there is a G-equivariant vector bundle homomorphism c, the Clifford action, from the Clifford bundle of TM to the endomorphism bundle of E, mapping odd-graded elements of the Clifford bundle to odd-graded endomorphisms. We also assume that there is a G-equivariant isomorphism of Clifford modules $E|_U \cong E|_N \times (0, 1]$.

Let D be a Dirac-type operator on E, of the form

$$D = c \circ \nabla \tag{2.6}$$

for a Hermitian Clifford connection ∇ on E. Let D_+ be the restriction of D to sections of E_+. Suppose that

$$D_+|_U = \sigma\Big(-\frac{\partial}{\partial u} + D_N\Big),$$

where $\sigma\colon E_+|_N \to E_-|_N$ is a G-equivariant vector bundle isomorphism, u is the coordinate in the factor $(0,1]$ in $U = N\times(0,1]$, and D_N is an (ungraded) Dirac-type operator on $E_+|_N$. We initially assume that D_N is *invertible* and show how to remove this assumption in Section 6.

Consider the cylinder $C := N\times[0,\infty)$, equipped with the product of the metric on M restricted to N, and the Euclidean metric. Because the metric, group action, Clifford module, and Dirac operator have a product form on U, all these structures extend naturally to C. Let D_C be the extension of $D|_U$ to C, acting on sections of the extension of $E|_U$ to C. We form the complete manifold

$$\hat{M} := (M \sqcup C)/\sim, \tag{2.7}$$

where $m\sim(n,u)$ if $m=(n,u)\in U = N\times(0,1]$. Let $\hat{E}\to\hat{M}$ and $\hat{D}$ be the extensions of E and D to $\hat{M}$, respectively, obtained by gluing the relevant objects on M and C together along U.

2.3 Delocalised η-invariants

If M, and hence N, is compact, then D_N has discrete spectrum. In [5], Atiyah, Patodi, and Singer consider the sum

$$\sum_{\lambda\in\mathrm{spec}(D_N)\setminus\{0\}} \mathrm{sgn}(\lambda)|\lambda|^{-s},$$

where the sum is over the nonzero eigenvalues of D_N taken with multiplicities, and the real part of $s\in\mathbb{C}$ is large enough so that the sum converges. This sum has a meromorphic continuation to $\mathbb{C}$ as a function of s, which is regular at $s=0$. The η-invariant of D_N, denoted by $\eta(D_N)$, is the value of this continuation at $s=0$.

We will denote the smooth kernel of $D_N e^{-tD_N^2}$ by $\tilde{\kappa}_t$. Bismut and Freed proved that for twisted Spin–Dirac operators, there is a function $b\in C^\infty(N)$ such that

$$\mathrm{tr}(\tilde{\kappa}_t(n,n)) = b(n)t^{1/2} + O(t^{3/2}) \tag{2.8}$$

as $t\downarrow 0$, uniformly for n in compact subsets of N. See Theorem 2.4 in [13]. If D_N has the property (2.8), then we have the equivalent definition

$$\eta(D_N) = \frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{Tr}(D_N e^{-t^2 D_N})\,\mathrm{d}t, \tag{2.9}$$

where Tr is the operator trace. See Section II(d) in [13].

For general Dirac-type operators D_N, the equality (2.9) still holds if we replace the lower integration limit by $\varepsilon > 0$, asymptotically expand the integral as $\varepsilon \to 0$, and take the right hand side of (2.9) to be the constant term in this expansion. In the compact case, we can take $\chi \equiv 1$, so that $\mathrm{Tr}_e = \mathrm{Tr}$. (See also Lemma 2.2.) Hence the following definition is a generalisation of the definition of the η-invariant in the compact case.

Definition 2.11. The operator D_N *has a g-delocalised η-invariant* if the operator $D_N e^{-tD_N^2}$ is g-trace class for all $t > 0$, and the integral

$$\frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}_g(D_N e^{-t^2 D_N^2})\, \mathrm{d}t \tag{2.10}$$

converges. The value of that integral is then the *g-delocalised η-invariant* of D_N, and denoted by $\eta_g(D_N)$.

For the case of the fundamental group of a compact manifold acting on the manifold's universal cover, delocalised η-invariants were defined by Lott [35, 37]. In that setting, the case for $g = e$ was introduced by Cheeger and Gromov [19], and applied to index theory by Ramachandran [45].

The integral (2.10) does not always converge, see for example Section 3 of [43]. However, in the case of fundamental groups of compact manifolds acting on their universal covers, and for g not a central element, a sufficient condition for the convergence of (2.10) is that D_N has a sufficiently large spectral gap at zero, see Theorem 1.1 in [20]. We discuss this convergence in Section 4 and find concrete classes of situations for discrete and semisimple groups when (2.10) converges.

Convergence of the integral (2.10) means convergence of the two integrals

$$\int_0^1 \mathrm{Tr}_g(D_N e^{-t^2 D_N^2})\, \mathrm{d}t \qquad \text{and} \qquad \int_1^\infty \mathrm{Tr}_g(D_N e^{-t^2 D_N^2})\, \mathrm{d}t. \tag{2.11}$$

Definition 2.12. If the first integral in (2.11) converges, then we say that $\eta_g(D_N)$ *converges for small t*. If the second integral in (2.11) converges, then we say that $\eta_g(D_N)$ *converges for large t*.

2.4 The index theorem

We first state the general form of our index theorem, and then make this more concrete for discrete groups and semisimple Lie groups in Subsection 2.5. In the general form of the index theorem, Theorem 2.13, we list technical conditions that imply a general index formula. The content of the specific forms for discrete and semisimple groups, Corollaries 2.19 and 2.21, is that those technical conditions hold in these settings, and that the general index formula then has a concrete topological expression.

Let X be a proper, isometric G-manifold, and, for $t > 0$, let κ_t be the smooth kernel of a G-equivariant, g-trace class operator on a space of smooth sections of a Hermitian G-vector bundle over X, such that κ_t depends smoothly on t. Let $\chi_g \in C^\infty(X)$ be nonnegative, and such that for all $m \in X$,

$$\int_Z \chi_g(zm)^2 \, dz = 1. \tag{2.12}$$

We say that κ_t *localises at* X^g *as* $t \downarrow 0$, if for all Z-invariant neighbourhoods V of X^g,

$$\lim_{t\downarrow 0} \int_{X\setminus V} \chi_g(x)^2 |\mathrm{tr}(g\kappa_t(g^{-1}x, x))| \, dx = 0. \tag{2.13}$$

Consider the setting of Subsection 2.2. Let $\tilde{M}$ be the double of M, and let $\tilde{E} = \tilde{E}_+ \oplus \tilde{E}_-$ and $\tilde{D}$ be the extensions of E and D to $\tilde{M}$, respectively. More explicitly, as on page 55 of [5], $\tilde{M}$ is obtained from M by gluing together a copy of M and a copy of M with reversed orientation, while $\tilde{E}$ is obtained by gluing together a copy of E and a copy of E with reversed grading. To glue these copies of E together along N, we use the isomorphism σ. Let $\tilde{D}_\pm$ be the restrictions of $\tilde{D}$ to the sections of $\tilde{E}_\pm$.

Theorem 2.13. Suppose that D_N is invertible, and furthermore that

- the operators $e^{-t\tilde{D}^2}$ and $e^{-t\tilde{D}^2}\tilde{D}$ are g-trace class for all $t > 0$;
- the Schwartz kernels of $e^{-t\tilde{D}^2}\tilde{D}$ and $\varphi e^{-tD_C^2} D_C \psi$ localise at M^g as $t \downarrow 0$, for all G-invariant $\varphi, \psi \in C^\infty(C)$, supported in the relatively cocompact neighbourhood $U \cong N \times (0, 1]$ of N; and
- $\eta_g(D_N)$ converges for large t.

Then operators $\hat{D}$ and $\tilde{D}$ are g-Fredholm, D_N has a g-delocalised η-invariant, and

$$\mathrm{index}_g(\hat{D}) = \frac{1}{2}\mathrm{index}_g(\tilde{D}) - \frac{1}{2}\eta_g(D_N). \tag{2.14}$$

We generalise this result to the case where D_N may not be invertible, but zero is isolated in its spectrum, in Section 6; see Theorem 6.2.

Remark 2.14. The condition in Theorem 2.13 that either g has no fixed points or D is a twisted Spin^c–Dirac operator is used to prove convergence of the integral in (2.10) for small t; see the proof of Proposition 4.1 in Subsection 4.1.

Remark 2.15. Reversing both the orientation on M and the grading on E on the second copy of M inside $\tilde{M}$ will ensure that the topological expression for the g-trace of the index of $\tilde{D}$ is twice the contribution of the interior of M to $\mathrm{index}_g(\hat{D})$. In particular, this choice of orientation and grading implies that the first term on the right hand side of (2.14) is not (always) zero.

In fact, the use of the double of M is not essential to our arguments, and we may replace $\tilde{M}$ by *any* manifold without boundary to which E, D and the action by G extend in suitable ways, such that $\tilde{M}/G$ is compact. Then the first term on the right hand side of (2.14) should be replaced by a difference of g-traces of heat operators on $\tilde{M}$ composed with functions supported in M. See Lemma 5.3 and its proof.

For more concrete versions of Theorem 2.13, we will show that its conditions hold in relevant cases and give a topological expression for the difference of g-traces on the right hand side of (2.14). Such a topological expression takes the following form. We specialise to the case of twisted Spin^c–Dirac operators. The case of general Dirac-type operators is similar; see for example Section 6.4 in [10]. Suppose that $E = S \otimes V$, for the spinor bundle $S \to M$ of a G-equivariant Spin^c-structure, and a G-equivariant, Hermitian vector bundle $V \to M$. Let R_V be the curvature of a G-invariant, Hermitian connection on V. Let $L \to M$ be the determinant line bundle of S.

If $g = e$, then we obtain a result for general unimodular locally compact groups.

Corollary 2.16. If D_N is invertible, then

$$\mathrm{index}_e(\hat{D}) = \int_M \chi^2 \hat{A}(M) e^{c_1(L)/2} \mathrm{tr}(e^{-R_V/2\pi i}) - \frac{1}{2}\eta_e(D_N).$$

This is a generalisation of the main result in [45] to arbitrary proper actions by locally compact groups, and a generalisation of the main result in [52] to manifolds with boundary.

For more general elements $g \in G$, let M^g be the fixed point set of g, and $\mathcal{N} \to M^g$ its normal bundle in M. Let $R^{\mathcal{N}}$ be the curvature of the restriction of the Levi–Civita connection of M to $\mathcal{N}$. Let $\chi_g \in C_c(M^g)$ be a function with the property (2.12). As before, we denote all extensions to $\tilde{M}$ by tildes.

Question 2.17. Under what conditions does the formula

$$\mathrm{index}_g(\tilde{D}) = \int_{\tilde{M}^g} \tilde{\chi}_g^2 \frac{\hat{A}(\tilde{M}^g)e^{c_1(L|_{\tilde{M}^g})/2}\mathrm{tr}(ge^{-R_{\tilde{V}}|_{\tilde{M}^g}/2\pi i})}{\det(1-ge^{-R_{\tilde{\mathcal{N}}}/2\pi i})^{1/2}} \tag{2.15}$$

hold?

This question is formulated here for the operator $\tilde{D}$ on the double $\tilde{M}$ of M; it can be asked more generally of course, for twisted Spin^c-Dirac operators on any Riemannian manifold with a proper, cocompact, isometric action. We will use affirmative answers to this question (the index theorems proved in [29, 51]) to make Theorem 2.13 more concrete in the next subsection. The fact that the answer is "yes" for the two very different classes of groups we will discuss suggests that (2.15) should hold more generally.

2.5 Discrete groups and semisimple groups

The conditions of Theorem 2.13 are satisfied, and the answer to Question 2.17 is "yes," in the two settings we will now describe.

First of all, suppose that G is discrete, and finitely generated. Let d_G be a word length metric on G. Let (g) be the conjugacy class of g.

Definition 2.18. The conjugacy class (g) has *polynomial growth* if there exist $C, d > 0$ such that for all $k \in \mathbb{N}$,

$$\#\{h \in (g); d_G(e,h) = k\} \leq Ck^d. \tag{2.16}$$

Corollary 2.19. If D_N is invertible, G is discrete and finitely generated, and (g) has polynomial growth, then a twisted Spin^c-Dirac operator D is g-Fredholm, and

$$\mathrm{index}_g(\hat{D}) = \int_{M^g} \chi_g^2 \frac{\hat{A}(M^g)e^{c_1(L|_{M^g})/2}\mathrm{tr}(ge^{-R_V|_{M^g}/2\pi i})}{\det(1-ge^{-R_{\mathcal{N}}/2\pi i})^{1/2}} - \frac{1}{2}\eta_g(D_N).$$

Remark 2.20. The assumption in Corollary 2.19 that (g) has polynomial growth is used to prove large t convergence of $\eta_g(D_N)$, see Proposition 4.14. It is possible that this assumption can be removed if one uses other criteria for the convergence of this delocalised η-invariant, such as the one in [20]. See Remark 4.15.

Next, we consider semisimple Lie groups.

Corollary 2.21. Suppose that D_N is invertible. Let G be a linear, connected, real semisimple Lie group. Let $K < G$ be a maximal compact subgroup. If G/K is even-dimensional, then for all semisimple $g \in G$, the operator D is g-Fredholm, and

$$\mathrm{index}_g(\hat{D}) = \int_{M^g} \chi_g^2 \frac{\hat{A}(M^g) e^{c_1(L|_{M^g})/2} \mathrm{tr}(g e^{-R_V|_{M^g}/2\pi i})}{\det(1 - g e^{-R_{\mathcal{N}}/2\pi i})^{1/2}} - \frac{1}{2}\eta_g(D_N). \tag{2.17}$$

Theorem 2.13 and Corollaries 2.16, 2.19, and 2.21 are proved in Subsection 5.5. Just like Theorem 2.13, Corollaries 2.19 and 2.21 generalise to the case where D_N is not necessarily invertible. See Corollary 6.3.

3 Properties of the g-Trace and the g-Index

3.1 The trace property

Let $\chi_G \in C^\infty(G)$ be any nonnegative function that has compact support on every orbit of the action by Z on G by right multiplication, and such that for all $x \in G$,

$$\int_Z \chi_G(xz^{-1})^2 \,\mathrm{d}z = 1.$$

Define the nonnegative function χ_g on M by

$$\chi_g(m)^2 = \int_G \chi_G(x)^2 \chi(xgm)^2 \,\mathrm{d}x. \tag{3.1}$$

It has the property (2.12).

Lemma 3.1. If T is a g-trace class operator on $L^2(E)$ with smooth kernel κ, then

$$\mathrm{Tr}_g(T) = \int_M \chi_g(m)^2 \mathrm{tr}(g\kappa(g^{-1}m, m)) \,\mathrm{d}m.$$

Proof. For all $h \in G$ and $m \in M$,

$$\mathrm{tr}(hgh^{-1}\kappa(hg^{-1}h^{-1}m, m)) = \mathrm{tr}(g\kappa(g^{-1}h^{-1}m, h^{-1}m)),$$

by G-invariance of κ and the trace property of tr. So, substituting $m' = h^{-1}m$, we find that (2.2) equals

$$\int_{G/Z}\int_{Z}\chi_G(hz^{-1})^2\int_M \chi(hgh^{-1}m)^2\mathrm{tr}(hgh^{-1}\kappa(hg^{-1}h^{-1}m,m))\,\mathrm{d}m\;\mathrm{d}z\,\mathrm{d}(hZ)$$
$$= \int_M \chi_g(m')^2\mathrm{tr}(g\kappa(g^{-1}m',m'))\,\mathrm{d}m'.$$

■

The following trace property is an analogue of Proposition 3.18 in [51] and Lemma 3.2 in [29].

Lemma 3.2. Let $S,T \in \mathcal{B}(L^2(E))^G$. Suppose that

- S has a distributional kernel;
- T has a smooth kernel;
- ST and TS are g-trace-class.

Then $\mathrm{Tr}_g(ST) = \mathrm{Tr}_g(TS)$.

Proof. For notational simplicity, we consider the case where E is the trivial line bundle, so that $\kappa_T \in C^\infty(M\times M)$ and $\kappa_S \in \mathcal{D}'(M\times M)$.

For $m \in M$, the restriction of κ_S to $\{m\}\times M$ is a well-defined distribution in $\mathcal{D}'(M)$, because the wave front set of κ_S is normal to the diagonal. We denote this restriction by $\kappa^{(2)}_{S,m}$. For $m' \in M$, we also write $\kappa^{(1)}_{T,m'} \in C^\infty(M)$ for the restriction of κ_T to $M\times\{m'\}$. The pairing between $\kappa^{(2)}_{S,m}$ and $\kappa^{(1)}_{T,m'}$ is well-defined for all $m,m' \in M$, because the compositions ST and TS are.

By Lemma 3.1,

$$\mathrm{Tr}_g(ST) = \int_M \chi_g(m)^2\kappa^{(2)}_{S,g^{-1}m}(\kappa^{(1)}_{T,m})\,\mathrm{d}m,$$

and

$$\mathrm{Tr}_g(TS) = \int_M \kappa^{(2)}_{S,g^{-1}m}(\chi_g^2\kappa^{(1)}_{T,m})\,\mathrm{d}m.$$

Using (2.12), unimodularity of Z, and Z-equivariance of S and T, one computes

$$\begin{aligned}
\int_M \chi_g(m)^2 \kappa^{(2)}_{S,g^{-1}m}(\kappa^{(1)}_{T,m})\, \mathrm{d}m &= \int_M \chi_g(m)^2 \kappa^{(2)}_{S,g^{-1}m}\Big(\int_Z z^{-1}\cdot \chi_g^2\, \mathrm{d}z\, \kappa^{(1)}_{T,m}\Big)\, \mathrm{d}m \\
&= \int_Z \int_M \chi_g(m)^2 \kappa^{(2)}_{S,g^{-1}m}\big(z^{-1}\cdot (\chi_g^2 \kappa^{(1)}_{T,zm})\big)\, \mathrm{d}m\, \mathrm{d}z \\
&= \int_Z \int_M \chi_g(z^{-1}m)^2 \kappa^{(2)}_{S,g^{-1}z^{-1}m}\big(z^{-1}(\chi_g^2 \kappa^{(1)}_{T,m})\big)\, \mathrm{d}m\, \mathrm{d}z \\
&= \int_Z \int_M \chi_g(z^{-1}m)^2 \kappa^{(2)}_{S,g^{-1}m}(\chi_g \kappa^{(1)}_{T,m})\, \mathrm{d}m\, \mathrm{d}z \\
&= \int_M \kappa^{(2)}_{S,g^{-1}m}(\chi_g^2 \kappa^{(1)}_{T,m})\, \mathrm{d}m.
\end{aligned}$$

■

3.2 Properties of the g-index

Lemma 3.2 allows us to prove Lemmas 2.5 and 2.7.

Proof of Lemma 2.5. Let R and R' be parametrices of D_+ such that such that the operators

$$\begin{aligned}
S_0 &:= 1 - RD_+; \\
S_1 &:= 1 - D_+R; \\
S_0' &:= 1 - R'D_+; \\
S_1' &:= 1 - D_+R'
\end{aligned}$$

are g-trace class. Then

$$\begin{aligned}
S_0 - S_0' &= (R' - R)D_+; \\
S_1 - S_1' &= D_+(R' - R).
\end{aligned} \tag{3.2}$$

The operator $D_+(R' - R)$ is smoothing, because S_1 and S_1' are assumed to be. So elliptic regularity implies that $R' - R$ is smoothing as well. Hence $R' - R$ has a smooth kernel. The operators $S_0 - S_0'$ and $S_1 - S_1'$ are g-trace class by assumption. The operator D_+ has a distributional kernel. So Lemma 3.2 and (3.2) imply that

$$(\mathrm{Tr}_g(S_0) - \mathrm{Tr}_g(S_1)) - (\mathrm{Tr}_g(S_0') - \mathrm{Tr}_g(S_1')) = \mathrm{Tr}_g((R' - R)D_+) - \mathrm{Tr}_g(D_+(R' - R)) = 0.$$

■

Proof of Lemma 2.7. This can be proved as on pages 46–47 of [2], where the trace property of the operator trace is replaced by Lemma 3.2. ■

3.3 A trace-like map on Banach algebras

The results and constructions in this subsection are analogues and generalisations of pages 18–19 of [37]. The discussion that follows can be simplified if we first consider the case of the trivial line bundle, and then note that one can reduce the general case to this one via local frames. We have chosen to be explicit here, at the cost of some notational inconvenience.

Let M be a *cocompact* G-manifold, and $E \to M$ a Hermitian G-vector bundle. Fix an open cover $\{U_j\}$ of open sets $U_j \subset M$ on which E admits local orthonormal frames$\{e_{j,k}\}_{k=1}^r$. Let $\{\zeta_j\}$ be a partition of unity subordinate to $\{U_j\}$. For $\kappa \in \Gamma(\mathrm{End}(E))^G$ and $m_1, m_2 \in M$, for j_1, j_2 such that $m_1 \in U_{j_1}$ and $m_2 \in U_{j_2}$, and for $k_1, k_2 = 1, \dots r$, consider the function $F_{j_1,j_2,k_1,k_2}(\kappa; m_1, m_2)$ on G, mapping $x \in G$ to

$$\left(e_{j_1,k_1}(m_1), x\kappa(x^{-1}m_1, m_2)e_{j_2,k_2}(m_2)\right)_{E_{m_1}}. \tag{3.3}$$

Let A be a Banach algebra (with respect to convolution) of functions on G. Let $\mathrm{End}^0_A(\Gamma^\infty(E))^G$ be the space of bounded operators on $L^2(E)$ with continuous kernels $\kappa \in \Gamma(\mathrm{End}(E))^G$ such that all of the functions $F_{j_1,j_2,k_1,k_2}(\kappa; m_1, m_2)$ lie in A, and such that the maps

$$(m_1, m_2) \mapsto F_{j_1,j_2,k_1,k_2}(\kappa; m_1, m_2) \tag{3.4}$$

from $M \times M$ to A are continuous.

Let χ be a cutoff function on M, then $\mathrm{supp}(\chi)$ is compact. For $\kappa \in \mathrm{End}_A(\Gamma^\infty(E))^G$, set

$$\|\kappa\|_A := \mathrm{vol}(\mathrm{supp}(\chi))\|\chi\|^2_\infty \max_{m_1,m_2 \in \mathrm{supp}(\chi)} \sum_{j_1,j_2} \sum_{k_1,k_2=1}^{r} \zeta_{j_1}(m_1)\zeta_{j_2}(m_2)\|F_{j_1,j_2,k_1,k_2}(\kappa; m_1, m_2)\|_A. \tag{3.5}$$

Continuity of the maps (3.4) implies that the norms of these functions take maximum values on the compact set $\mathrm{supp}(\chi)$. Furthermore, the sums are locally finite, so $\|\kappa\|_A$ is well-defined. Different choices of a partition of unity and local orthonormal frames result in equivalent norms, but we will not need this.

Lemma 3.3. The norm $\|\cdot\|_A$ is submultiplicative.

Proof. Let $\kappa, \lambda \in \mathrm{End}^0_A(\Gamma^\infty(E))^G$. Then for all $m_1, m_2 \in M$ and $x \in G$,

$$x(\kappa\lambda)(x^{-1}m_1, m_2) = \int_M \int_G \chi(ym_3)^2 x\kappa(x^{-1}m_1, m_3)\lambda(m_3, m_2)\,\mathrm{d}y\,\mathrm{d}m_3$$
$$= \int_M \chi(m_3)^2 \int_G xy^{-1}\kappa(yx^{-1}m_1, m_3)y\lambda(y^{-1}m_3, m_2)\,\mathrm{d}y\,\mathrm{d}m_3.$$

For the first equality, we used the definition of cutoff functions. For the second, we substituted ym_3 for m_3 and used G-invariance of κ.

So for all relevant j_1, j_2, k_1, and k_2, and $x \in G$,

$$F_{j_1,j_2,k_1,k_2}(\kappa\lambda; m_1, m_2)(x) =$$
$$\int_M \chi(m_3)^2 \int_G \left(e_{j_1,k_1}(m_1), xy^{-1}\kappa(yx^{-1}m_1, m_3)y\lambda(y^{-1}m_3, m_2)e_{j_2,k_2}(m_2)\right)_{E_{m_1}}\,\mathrm{d}y\,\mathrm{d}m_3 =$$
$$\sum_{j_3}\sum_{k_3=1}^{r} \int_M \chi(m_3)^2\zeta_{j_3}(m_3) \int_G \left(e_{j_1,k_1}(m_1), xy^{-1}\kappa(yx^{-1}m_1, m_3)e_{j_3,k_3}(m_3)\right)_{E_{m_1}}$$
$$\left(e_{j_3,k_3}(m_3)y\lambda(y^{-1}m_3, m_2)e_{j_2,k_2}(m_2)\right)_{E_{m_3}}\,\mathrm{d}y\,\mathrm{d}m_3 =$$
$$\sum_{j_3}\sum_{k_3=1}^{r} \int_M \chi(m_3)^2\zeta_{j_3}(m_3)\left(F_{j_1,j_3,k_1,k_3}(\kappa; m_1, m_3) * F_{j_3,j_2,k_3,k_2}(\lambda; m_3, m_2)\right)(x)\,\mathrm{d}m_3.$$

By submultiplicativity of the norm $\|\cdot\|_A$ on A, we find that

$$\|F_{j_1,j_2,k_1,k_2}(\kappa\lambda; m_1, m_2)\|_A \leq$$
$$\sum_{j_3}\sum_{k_3=1}^{r} \int_M \chi(m_3)^2\zeta_{j_3}(m_3)\|F_{j_1,j_3,k_1,k_3}(\kappa; m_1, m_3)\|_A\|F_{j_3,j_2,k_3,k_2}(\lambda; m_3, m_2)\|_A\,\mathrm{d}m_3 \leq$$
$$\mathrm{vol}(\mathrm{supp}(\chi))\|\chi\|_\infty^2 \max_{m_3,m_4\in\mathrm{supp}(\chi)} \sum_{j_3,j_4}\sum_{k_3,k_4=1}^{r} \zeta_{j_3}(m_3)\zeta_{j_4}(m_4)$$
$$\|F_{j_1,j_3,k_1,k_3}(\kappa; m_1, m_3)\|_A\|F_{j_4,j_2,k_4,k_2}(\lambda; m_4, m_2)\|_A.$$

This implies the claim by the definition (3.5) of the norm $\|\cdot\|_A$. ■

Let $\mathrm{End}_A(\Gamma^\infty(E))^G$ be the completion of $\mathrm{End}^0_A(\Gamma^\infty(E))^G$ (that space may already be complete in some cases). By Lemma 3.3, $\mathrm{End}_A(\Gamma^\infty(E))^G$ is a Banach algebra.

Lemma 3.4. Setting

$$\mathrm{TR}(\kappa)(x) = \int_M \chi(xm)^2 \mathrm{tr}(x\kappa(x^{-1}m, m))\, dm, \tag{3.6}$$

for $\kappa \in \mathrm{End}^0_A(\Gamma^\infty(E))^G$ and $x \in G$, defines a continuous map $\mathrm{TR}\colon \mathrm{End}_A(\Gamma^\infty(E))^G \to A$ with respect to the norm $\|\cdot\|_A$.

Proof. Let $\kappa \in \mathrm{End}^0_A(\Gamma^\infty(E))^G$. Then by a substitution $xm \mapsto m$, equivariance of κ, and the trace property, we have for all $x \in G$,

$$\begin{aligned}
\mathrm{TR}(\kappa)(x) &= \int_M \chi(m)^2 \mathrm{tr}(x\kappa(x^{-2}m, x^{-1}m))\, dm \\
&= \int_M \chi(m)^2 \mathrm{tr}(x\kappa(x^{-1}m, m))\, dm \\
&= \sum_j \sum_{k=1}^r \int_M \chi(m)^2 \zeta_j(m) F_{j,j,k,k}(\kappa; m, m)(x)\, dm.
\end{aligned}$$

So

$$\begin{aligned}
\|\mathrm{TR}(\kappa)\|_A &\leq \sum_j \sum_{k=1}^r \int_M \chi(m)^2 \zeta_j(m) \|F_{j,j,k,k}(\kappa; m, m)\|_A\, dm \\
&\leq \mathrm{vol}(\mathrm{supp}(\chi)) \|\chi\|_\infty^2 \max_{m \in \mathrm{supp}(\chi)} \sum_j \sum_{k=1}^r \zeta_j(m) \|F_{j,j,k,k}(\kappa; m, m)\|_A \\
&\leq \|\kappa\|_A.
\end{aligned}$$

■

The motivation for Lemmas 3.3 and 3.4 is the following. Suppose that τ_g, defined by (2.5), defines a continuous functional on A. It is immediate from the definitions that

$$\mathrm{Tr}_g = \tau_g \circ \mathrm{TR}. \tag{3.7}$$

So by Lemma 3.4, Tr_g is a continuous functional on $\mathrm{End}_A(\Gamma^\infty(E))^G$.

Remark 3.5. The map TR generalises the map Tr_N in (3.11) in [29] and the map TR on page 429 of [36].

4 Convergence of Delocalised η-Invariants

In the proof of Theorem 2.13, we assume large t convergence of $\eta_g(D_N)$ and deduce small t convergence from the index formula. In this section, we study large t convergence, which will be used to verify the condition in Theorem 2.13 in the special cases in Corollaries 2.16, 2.19, and 2.21. We also comment on small t convergence of $\eta_g(D_N)$ in Subsection 4.2, but these results are only used in the case where D_N is not invertible in Section 6.

Building on arguments by Lott [37], we show that the integrand in (2.10) is rapidly decreasing as t goes to infinity, if an algebra $\mathcal{A} \subset C^*_r(G)$ with certain properties exists. We then show that such algebras exist for discrete groups and semisimple Lie groups.

4.1 Projective limits of Banach algebras and large t convergence

Suppose that $\mathcal{A}$ is the projective limit of a sequence of Banach algebras $A_j \subset C^*_r(G)$ with norms $\|\cdot\|_j$, on which the orbital integral trace (2.5) defines a continuous functional. Let $\mathrm{End}_{A_j}(\Gamma^\infty(E|_N))^G$, τ_g and TR be as in Subsection 3.3. Let $\mathrm{End}_{\mathcal{A}}(\Gamma^\infty(E|_N))^G$ be the projective limit of the algebras $\mathrm{End}_{A_j}(\Gamma^\infty(E|_N))^G$. In this subsection, we prove the following extension of Proposition 8 in [37].

Proposition 4.1. Suppose that $\mathcal{A}$ is closed under holomorphic functional calculus. If D_N is invertible and $D_N e^{-tD_N^2} \in \mathrm{End}_{\mathcal{A}}(\Gamma^\infty(E|_N))^G$ for all $t > 0$, then $\eta_g(D_N)$ converges for large t.

In Subsections 4.4 and 4.5, we give examples of cases where the conditions of this proposition hold. In fact, in those examples, the algebra $\mathcal{A}$ is a Banach algebra itself, rather than just a projective limit of Banach algebras. We have included the possibility that $\mathcal{A}$ is only a projective limit of Banach algebras for the sake of generality, for example to include the algebras used in [37].

We prove Proposition 4.1 by extending the proof of Proposition 8 in [37] to our setting.

Lemma 4.2. Suppose that $\mathcal{A}$ is the projective limit of a sequence of Banach algebras, and that $D_N e^{-t^2D_N^2} \in \mathrm{End}_{\mathcal{A}}(\Gamma^\infty(E|_N))^G$. Then $\mathrm{Tr}_g(D_N e^{-t^2D_N^2})$ decreases at least as fast as a Gaussian function in t as $t \to \infty$.

Proof. Continuity of τ_g on $\mathcal{A}$ means that it is continuous on A_j for all j. So, by Lemma 3.4, the composition $\mathrm{Tr}_g = \tau_g \circ \mathrm{TR}$ is a continuous functional on $\mathrm{End}_{\mathcal{A}}(\Gamma^\infty(E))^G$. By construction of projective limits, this continuity implies that there exist j and $C > 0$ such that for all t,

$$\tau_g(\mathrm{TR}(D_N e^{-t^2 D_N^2})) \le C\|D_N e^{-t^2 D_N^2}\|_j. \tag{4.1}$$

Now Lemma 3.3 implies that

$$\|D_N e^{-t^2 D_N^2}\|_j \le \|D_N e^{-D_N^2}\|_j \|e^{(t^2-1)D_N^2}\|_j. \tag{4.2}$$

Since D_N is invertible, the spectrum of D_N^2 has a positive smallest element λ_0. By the spectral mapping theorem, the spectral radius of $e^{-tD_N^2}$ as an operator on $L^2(E)$ is $e^{-t\lambda_0}$. This is in fact also the spectral radius $\rho_{A_j}(e^{-tD_N^2})$ of $e^{-tD_N^2}$ as an element of the Banach algebra $\mathrm{End}_{A_j}(\Gamma^\infty(E))^G$. This follows from the fact that $\mathcal{A}$ is closed under holomorphic functional calculus; the details of this argument are as in Proposition 19 on page 38 of [37] and Lemmas 2 and 3 on pages 40–41 of [37].

By Theorem 1.22 in [24] (a general fact about one-parameter semigroups in Banach algebras), the limit

$$a := \lim_{t\to\infty} t^{-1} \ln \|e^{-tD_N^2}\|_j$$

exists, and $\rho_{A_j}(e^{-tD_N^2}) = e^{at}$ for all t. So $a = -\lambda_0$. Hence there is a $t_0 > 0$ such that for all $t > t_0$,

$$t^{-1} \ln \|e^{-tD_N^2}\|_j < -\lambda_0/2.$$

Hence for all such t,

$$\|e^{-tD_N^2}\|_j < e^{-t\lambda_0/2}.$$

(Compare also Corollary 3 on page 41 of [37].) We conclude that if $t^2 - 1 > t_0$,

$$\|D_N e^{-t^2 D_N^2}\|_j < \|D_N e^{-D_N^2}\|_j e^{-(t^2-1)\lambda_0/2},$$

a Gaussian function in t. ■

Proposition 4.1 follows immediately from Lemma 4.2.

Remark 4.3. Lemma 4.2, and hence Proposition 4.1, generalise to the case where D_N is not invertible, but zero is isolated in its spectrum. This can be proved by considering the compression of D_N to the orthogonal complement to its kernel, as on page 20 of [37].

4.2 Small t convergence

Small t convergence of $\eta_g(D_N)$ will be deduced from the index formula in the proof of Theorem 2.13. But in this subsection, we give independent proofs of small t convergence in some cases. The material in this subsection is only used in Section 6 to replace a regularised delocalised η-invariant by a usual delocalised η-invariant, and even there these direct proofs can probably be avoided (see Remark 6.8).

Proposition 4.4.

(a) If g has no fixed points in N, then $\eta_g(D_N)$ converges for small t.

(b) Without the assumption that g has no fixed points in N, suppose that either

- (b1) G is any unimodular locally compact group, and G/Z is compact;
- (b2) G is discrete and finitely generated; or
- (b3) G is a connected, real semisimple Lie group, and g is a semisimple element.

Suppose furthermore that D_N is invertible and a twisted Spin^c-Dirac operator, and that $D_N e^{-tD_N^2} \in \mathrm{End}_{\mathcal{A}}(\Gamma^\infty(E|_N))^G$ for all $t > 0$. Then $\eta_g(D_N)$ converges for small t, and hence D_N has a g-delocalised η-invariant.

From now on, we denote the Schwartz kernel of $D_N e^{-tD_N^2}$ by $\tilde{\kappa}_t$. The Bismut–Freed estimate (2.8) has an equivariant generalisation due to Zhang.

Proposition 4.5. Suppose that D_N is a twisted Spin^c-Dirac operator. For all compact subsets $K_1 \subset G$ and $K_2 \subset N$, there is a $C > 0$ such that for all $x \in K_1$ and $t \in (0, 1]$,

$$\left| \int_{K_2} \mathrm{tr}(x\tilde{\kappa}_t(x^{-1}n, n))\, \mathrm{d}n \right| \leq C\sqrt{t}. \tag{4.3}$$

Proof. By (2.2) in [54], we can select a constant $C_x > 0$ for every $x \in G$ such that (4.3) holds with C replaced by C_x. In [54], compact manifolds are considered, but it is shown that the computation localises near the fixed point set of g. The same arguments apply with the fixed point set in a compact manifold replaced by the intersection of the fixed point set of g with the compact set K_2.

By going through the proof of (2.2) in [54], one sees that the constants in the estimates that are used (in particular, the constant C_1 in Lemma 2.17 in [54]) can be chosen continuously in x, so that we can choose a single C such that (4.3) holds for all $x \in K_1$. ∎

Lemma 4.6. Suppose that D_N is a twisted Spin^c-Dirac operator. Then for any compact subset $A \subset G/Z$, the integral

$$\int_0^1 \int_A \int_N \chi(hgh^{-1}n)^2 \mathrm{tr}(hgh^{-1}\tilde{\kappa}_t(hg^{-1}h^{-1}n, n))\, \mathrm{d}n\, \mathrm{d}(hZ)\, \mathrm{d}t \tag{4.4}$$

converges absolutely.

Proof. By compactness of A and $\mathrm{supp}(\chi)$, we can apply Proposition 4.5 to find a $C > 0$ such that for all $h \in A$,

$$\left| \int_N \chi(hgh^{-1}n)^2 \mathrm{tr}(hgh^{-1}\tilde{\kappa}_t(hg^{-1}h^{-1}n, n))\, \mathrm{d}n \right| \leq C\sqrt{t}.$$

So the integrand in the outer integrals over $[0, 1]$ and A in (4.4) is bounded. ∎

Lemma 4.7. Suppose that G is discrete and finitely generated. Then there is a compact (i.e., finite) subset $A \subset G/Z$ such that the integrals and sum

$$\int_0^1 \sum_{h \in (G/Z)\setminus A} \int_N \chi(hgh^{-1}n)^2 \mathrm{tr}(hgh^{-1}\tilde{\kappa}_t(hg^{-1}h^{-1}n, n))\, \mathrm{d}n\, \mathrm{d}t \tag{4.5}$$

converge absolutely.

Proof. Let l be the word length function for a fixed, finite generating set of G. By the Švarc–Milnor lemma, there are a finite subset $\tilde{A} \subset G$ and a $c > 0$ such that for all $x \in G \setminus \tilde{A}$ and $n \in \mathrm{supp}(\chi)$,

$$d(xn, n) \geq cl(x). \tag{4.6}$$

Furthermore, by standard Gaussian estimates for heat kernels (see e.g., Proposition 4.2(i) in [18] or (31) in [35]), there are $a_1, a_2, a_3 > 0$ such that for all $t \in (0, 1]$ and $n, n' \in N$,

$$\|\tilde{\kappa}_t(n, n')\| \leq a_1 t^{-a_2} e^{-a_3 d(n,n')^2/t}. \tag{4.7}$$

By (4.6) and (4.7),

$$
\begin{aligned}
&\int_0^1 \sum_{x\in G\setminus A} \int_N \chi(xn)^2 |\mathrm{tr}(x\tilde{\kappa}_t(x^{-1}n, n))| \,\mathrm{d}n\,\mathrm{d}t \\
&\le a_1 \int_0^1 t^{-a_2} \sum_{k=1}^{\infty} \sum_{\substack{x\in G\setminus A\\ k\le l(x)<k-1}} \int_N \chi(xn)^2\,\mathrm{d}n\, e^{-ca_3 l(x)^2/t}\,\mathrm{d}t \\
&= a_1 \int_N \chi(n)^2\,\mathrm{d}n \int_0^1 t^{-a_2} \sum_{k=1}^{\infty} \big(\#\{x\in G; k\le l(x) < k-1\}\big) e^{-ca_3 k^2/t}\,\mathrm{d}t.
\end{aligned}
\tag{4.8}
$$

Now there is a $b > 0$ such that for every $k \ge 1$,

$$
\int_0^1 t^{-a_2} e^{-ca_3 k^2/t}\,\mathrm{d}t \le b e^{-ca_3 k^2}.
$$

(This is a calculus exercise, see Lemma 4.2 in [28].) Because the number $\#\{x \in G;$ $k \le l(x) < k-1\}$ grows at most exponentially in k, we find that

$$
\sum_{k=1}^{\infty} \big(\#\{x\in G; k\le l(x) < k-1\}\big) \int_0^1 t^{-a_2} e^{-ca_3 k^2/t}\,\mathrm{d}t
$$

converges absolutely. This implies that the left hand side of (4.8) converges absolutely by Tonelli's theorem. By restricting the sum to the conjugacy class of g, we find that (4.5) converges absolutely as well. ■

Proof of Proposition 4.4. In case (a), Proposition 4.25 in [42] and Lemma 3.1 imply that the integrand in (2.10) decreases like a function of the form $t^{-a}e^{-b/t}$ as $t \downarrow 0$, for $a, b > 0$. Hence we have convergence of (2.10) in case (a).

Convergence of (2.10) for small t follows directly from Lemma 4.6 in case (b1) of Proposition 4.1. In case (b2), this follows from Lemmas 4.6 and 4.7. In case (b3), this follows from Proposition 4.29 in [42]. ■

Remark 4.8. In Proposition 4.29 in [42], used in the proof of case (b3) of Proposition 4.4, it is not assumed that D_N is invertible. So this assumption is not necessary in that case.

4.3 Kağan Samurkaş' algebra

In this subsection and the next, we assume that G is a finitely generated discrete group, and that (g) has polynomial growth (Definition 2.18) from now on. We briefly review Kağan Samurkaş' construction of an algebra on which the trace τ_g defines a continuous functional.

For $S \subset G$ and $f \in l^2(G)$, let $\|f\|_S$ be the norm of $f|_S$ in $l^2(S)$. For $r > 0$, write

$$B_r(S) := \{h \in G; \forall s \in S, d_G(e, hs^{-1}) < r\}.$$

For $a \in \mathcal{B}(l^2(G))$ and $r > 0$, define

$$\mu_a(r) := \inf\{C > 0; \forall f \in l^2(G), \|af\|_{G\setminus B_r(\mathrm{supp}(f))} \le C\|f\|_G\}.$$

Fix $C > 0$ and $d \in \mathbb{N}$ such that (2.16) holds for all k. For $a \in \mathcal{B}(l^2(G))$, define

$$\|a\|_g := \inf\{A > 0; \forall r > 0, \mu_a(r) \le Ar^{-(d/2+2)}\}. \tag{4.9}$$

Let $\|\cdot\|_{\mathcal{B}(l^2(G))}$ be the operator norm on $\mathcal{B}(l^2(G))$. For $f \in \mathbb{C}G$, we also denote the convolution operator by f from the left by $f \in \mathcal{B}(l^2(G))$

Lemma 4.9. The expression (4.9) defines a seminorm on $\mathbb{C}G$, and for all $a, b \in \mathbb{C}G$,

$$\|ab\|_g \le 2^{d/2+2}(\|a\|_{\mathcal{B}(l^2(G))} + \|a\|_g)(\|b\|_{\mathcal{B}(l^2(G))} + \|b\|_g).$$

Proof. The first claim is Lemma 3.6 in [47]. In the proof of Lemma 3.10 in [47], it is proved that for all $a, b \in \mathbb{C}G$,

$$\|ab\|_g \le \max\big(2^{d/2+3}\|a\|_{\mathcal{B}(l^2(G))}\|b\|_g + 2^{d/2+2}\|a\|_g\|b\|_{\mathcal{B}(l^2(G))} + 2^{d/2+2}\|a\|_g\|b\|_g,$$
$$\|a\|_{\mathcal{B}(l^2(G))}\|b\|_{\mathcal{B}(l^2(G))}\big).$$

This implies the second claim. ∎

Let $C_g^{\mathrm{pol}}(G)$ be the completion of $\mathbb{C}G$ in the norm

$$\|f\| := \|f\|_{\mathcal{B}(l^2(G))} + \|f\|_g.$$

By Lemma 4.9, this is a Banach algebra. By Lemma 3.12 in [47], it is closed under holomorphic functional calculus. Theorems 3.18 and 3.19 in [47] state that τ_g defines a continuous trace on $C_g^{\mathrm{pol}}(G)$.

We will use the following equality.

Lemma 4.10. For all $f \in l^1(G)$,

$$\mu_f(r) = \|f\|_{G \setminus B_r(e)}.$$

Proof. Let $f \in l^1(G)$. Since G is discrete, we also have $f \in l^2(G)$. For all $\varphi \in \mathbb{C}G$, we have

$$\begin{aligned}
\|f * \varphi\|^2_{G\setminus B_r(\mathrm{supp}(\varphi))} &= \sum_{x \in G\setminus B_r(\mathrm{supp}\varphi)} \Big| \sum_{y \in \mathrm{supp}\varphi} f(xy^{-1})\varphi(y)\Big|^2 \\
&\leq \sum_{x \in G\setminus B_r(\mathrm{supp}\varphi)} \sum_{y \in \mathrm{supp}\varphi} |f(xy^{-1})|^2 |\varphi(y)|^2 \\
&\leq \sum_{z \in G\setminus B_r(e)} |f(z)|^2 \sum_{y \in \mathrm{supp}\varphi} |\varphi(y)|^2.
\end{aligned}$$

Here we used the fact that for all $x \in G \setminus B_r(\mathrm{supp}\varphi)$ and $y \in \mathrm{supp}\varphi$, we have $xy^{-1} \in G \setminus B_r(e)$. So $\mu_f(r) \leq \|f\|_{G\setminus B_r(e)}$.

And if $\varphi = \delta_e$, then

$$\|f * \varphi\|_{G\setminus B_r(\mathrm{supp}(\varphi))} = \|f\|_{G\setminus B_r(e)}.$$

So $\mu_f(r) \geq \|f\|_{G\setminus B_r(e)}$. ■

4.4 Discrete groups

We continue to assume that G is a finitely generated discrete group, and that (g) has polynomial growth. From now on, we denote the Schwartz kernel of $e^{-tD_N^2}$ by κ_t, and recall that we denote the Schwartz kernel of $D_N e^{-tD_N^2}$ by $\tilde{\kappa}_t$. We first show that the integrand of (2.10) is well-defined. In the proof of the following lemma, and in other places, we denote the operator norm of an endomorphism A of a finite-dimensional normed vector space by $\|A\|$.

Lemma 4.11. For all $t > 0$, $D_N e^{-tD_N^2}$ is g-trace class.

Proof. It is noted on page 11 of [37] that for all $m \in M$,

$$\sum_{x \in G} \|x\tilde{\kappa}_t(x^{-1}m, m)\| < \infty.$$

In [37] the action considered is free; the argument extends to the proper case because stabilisers are finite. See Theorem 3.4 in [51], the proof of which also applies to $D_N e^{-tD_N^2}$, via the inequality in the third displayed equation on page 307 of [51]. (That theorem is an analogue of Theorem 4.3 in [25] for non-scalar Laplacians, which also applies to derivatives of heat kernels as noted at the bottom of page 491 in [25].) Hence also

$$\sum_{x \in (g)} \|x\tilde{\kappa}_t(x^{-1}m, m)\| < \infty.$$

In this case, the integral over G/Z is just the sum over (g) and the integral over M with weight function χ^2 is the integral over a compact fundamental domain. Hence the right hand side of (2.1) converges. ■

Proposition 4.12. For all $t > 0$, the Schwartz kernels κ_t and $\tilde{\kappa}_t$ lie in $\mathrm{End}_{C_g^{\mathrm{pol}}(G)}(\Gamma^\infty(E|_N))^G$.

Lemma 4.13. Fix $t > 0$ and $n, n' \in N$, and consider the functions $\psi := F_{j_1,j_2,k_1,k_2}(\kappa_t; n, n')$ and $\tilde{\psi} := F_{j_1,j_2,k_1,k_2}(\tilde{\kappa}_t; n, n')$ defined as in (3.3). The functions μ_ψ and $\mu_{\tilde{\psi}}$ decrease faster than any rational function.

Proof. The function ψ lies in $l^1(G)$; see for example Section 3.2, page 308 of [51]. And so does $\tilde{\psi}$, as pointed out in the proof of Lemma 4.11. So by Lemma 4.10, the claim is that the numbers $\|\psi\|_{G\setminus B_r(e)}$ and $\|\tilde{\psi}\|_{G\setminus B_r(e)}$ decrease faster than any rational function of r, as $r \to \infty$.

For $k \in \mathbb{N}$, write

$$G(k) := \{x \in G; k-1 < d(x^{-1}m, m) \leq k\}.$$

It was shown on page 307 of [51] that there are $A_1, b_1 > 0$ such that for all $k \in \mathbb{N}$,

$$\#G(k) \leq A_1 e^{b_1 k}. \tag{4.10}$$

Furthermore, for fixed $t > 0$, standard estimates for heat kernels show that there are $A_2, b_2 > 0$ such that for all $m_1, m_2 \in M$,

$$\|\kappa_t(m_1, m_2)\| \leq A_2 e^{-b_2 d(m_1,m_2)^2}.$$

Similar estimates hold for the derivatives of κ_t. This implies that there are $A_3, b_3 > 0$ such that for all $m_1, m_2 \in M$,

$$\|\tilde{\kappa}_t(m_1, m_2)\| \leq A_3 e^{-b_3 d(m_1,m_2)^2}. \tag{4.11}$$

From now on, we will consider $\tilde{\psi}$ and $\tilde{\kappa}_t$. The same arguments apply to ψ and κ_t. For fixed $r > 0$, we have

$$\|\tilde{\psi}\|^2_{G\backslash B_r(e)} \leq \sum_{k=1}^{\infty} \sum_{x \in G(k); d_G(e,x) > r} \|x\tilde{\kappa}_t(x^{-1}n, n')\|^2.$$

By (4.11) and the triangle inequality, this is at most equal to

$$A_3^2 e^{-2b_3 d(n,n')^2} \sum_{k=1}^{\infty} \#\{x \in G(k); d_G(e,x) > r\} e^{-2b_3 k^2}. \tag{4.12}$$

By the Švarc–Milnor Lemma (see e.g., Lemma 2 in [40], or [17]), there are constants $A_4, b_4 > 0$ such that for all $x \in G$,

$$d(x^{-1}n, n) \geq A_4 d_G(e, x) - b_4.$$

So if $x \in G(k)$ and $d_G(e, x) > r$, then $k \geq A_4 r - b_4$. Together with (4.10), this implies that (4.12) is at most equal to

$$A_1 A_3^2 e^{-2b_3 d(n,n')^2} \sum_{k=\lfloor A_4 r - b_4 \rfloor}^{\infty} e^{b_1 k - 2b_3 k^2}.$$

This decreases exponentially as $r \to \infty$. ■

Proof of Proposition 4.12. Under the conditions in the proposition, the functions ψ and $\tilde{\psi}$ decrease faster than any rational function by Lemma 4.13. So in particular, there are constants $A, \tilde{A} > 0$ such that for all $r > 0$,

$$\mu_\psi(r) \leq Ar^{-(d/2+2)};$$
$$\mu_{\tilde{\psi}}(r) \leq \tilde{A}r^{-(d/2+2)}.$$

Hence $\|\psi\|_g$ and $\|\tilde{\psi}\|_g$ are finite. We saw in the proof of Lemma 4.13 that $\psi, \tilde{\psi} \in l^1(G) \subset C^*_r(G)$. There we also saw that the constant A in (4.9) can be chosen continuously in n and n'. This finishes the proof. ■

Propositions 4.1, 4.4, and 4.12 have the following consequence.

Proposition 4.14. Suppose that G is a discrete, finitely generated group and (g) has polynomial growth. If D_N is invertible, then $\eta_g(D_N)$ converges for large t. If, furthermore, either g has no fixed points or D is a twisted Spin^c-Dirac operator, then $\eta_g(D_N)$ also converges for small t, so D_N has a g-delocalised η-invariant.

Remark 4.15. For finitely generated discrete groups, a general criterion for convergence of delocalised η-invariants was obtained in [20]. For hyperbolic groups, Puschnigg [44] shows that an algebra $\mathcal{A}$ to which τ_g extends exists. It may be possible to use this to obtain a version of Proposition 4.14 for such groups.

4.5 Real semisimple groups

Suppose, in this subsection, that G is a real, linear, connected semisimple Lie group, and that $g \in G$ is semisimple (i.e., $\mathrm{Ad}(g)$ is diagonalisable). We continue to write κ_t for the Schwartz kernel of $e^{-tD_N^2}$, and $\tilde{\kappa}_t$ for the Schwartz kernel of $D_N e^{-tD_N^2}$.

Lemma 4.16. If G is a real, linear, connected, semisimple Lie group and g is a semisimple element, then $D_N e^{-tD_N^2}$ is g-trace class for all $t > 0$.

Proof. By Abels' theorem [1], connectedness of G implies that N decomposes G-equivariantly as $N = G \times_K Y$, for a maximal compact subgroup $K < G$ and a K-invariant submanifold $Y \subset N$. Then κ_t lies in

$$\left(C^\infty(G \times G) \otimes \Gamma^\infty(\mathrm{End}(E|_Y))\right)^{G \times K} \cong \left(C^\infty(G) \otimes \Gamma^\infty(\mathrm{End}(E|_Y))\right)^K.$$

Lemma 3.5 in [29] states that this heat kernel in fact lies in

$$\left(\mathcal{C}(G) \otimes \Gamma^\infty(\mathrm{End}(E|_Y))\right)^K,$$

where $\mathcal{C}(G)$ is Harish-Chandra's Schwartz algebra. The operator D_N decomposes into an operator on G and one on Y, see for example, (4.2) in [29]. The operator on G in this decomposition is defined in terms of the left (or right) regular representation by $\mathfrak{g}$ on $C^\infty(G)$, which preserves $\mathcal{C}(G)$. As pointed out in (3.7) in [41], this implies that also

$$\tilde{\kappa}_t \in \left(\mathcal{C}(G) \otimes \Gamma^\infty(\mathrm{End}(E|_Y))\right)^K.$$

The trace Tr_g corresponds to the trace τ_g on $\mathcal{C}(G)$ under this identification, combined with a trace on $\Gamma^\infty(\mathrm{End}(E|_Y))$, which converges by compactness of Y, see Lemma 3.1(c) in [29]. (This is a special case of (3.7).) Hence the claim follows. ∎

Lafforgue [31] constructed an algebra satisfying the conditions of Proposition 4.1. In fact, this algebra is a Banach algebra itself, rather than just a projective limit of Banach algebras. Furthermore, it contains heat kernels of Dirac operators. It is shown how to generalise the construction to reductive Lie groups in Section 4.2 of [31], but we will work with linear semisimple groups for simplicity.

Consider the spherical function $\varphi := \varphi_0^G$, as in (7.41) in [30]. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition for a maximal compact subgroup $K < G$, and let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace. Fix a positive restricted root system Σ^+ for $(\mathfrak{g}, \mathfrak{a})$, and write

$$\rho := \sum_{\lambda \in \Sigma^+} \dim(\mathfrak{g}_\lambda)\lambda.$$

Define the function $d_G \colon G \to [0, \infty)$ by

$$d_G(k \exp(H) k') = \langle \rho, H \rangle,$$

for $k, k' \in K$, and $H \in \mathfrak{a}$ positive with respect to Σ^+. Then $d_G(x)$ is the distance from xK to eK in G/K. For $t \in \mathbb{R}$, let $\mathcal{S}_t(G)$ be the completion of $C_c(G)$ in the norm

$$\|f\|_t := \sup_{g \in G} |f(g)| \varphi(g)^{-1} (1 + d_G(g))^t.$$

By Proposition 4.1.2 in [31], the algebra $\mathcal{S}_t(G)$ is a Banach algebra, dense in $C^*_r(G)$ and closed under holomorphic functional calculus, for large enough t.

Lemma 4.17. For every semisimple element $g \in G$, and for t large enough, the trace τ_g defines a continuous functional on $\mathcal{S}_t(G)$.

Proof. Let $g \in G$ be semisimple. By Theorem 6 in [26], the integral

$$\int_{G/Z} (1 + d_G(xgx^{-1}))^{-t} \varphi(xgx^{-1}) \mathrm{d}(xZ)$$

converges for t large enough. So, for such t, and $f \in \mathcal{S}_t(G)$,

$$|\tau_g(f)| \leq \|f\|_t \int_{G/Z} (1 + d_G(xgx^{-1}))^{-t} \varphi(xgx^{-1}) \mathrm{d}(xZ).$$

■

Lemma 4.18. For all $s > 0$, the Schwartz kernels κ_s and $\tilde{\kappa}_s$ lie in $\mathrm{End}_{\mathcal{S}_t(G)}(\Gamma^\infty(E|_N))^G$.

Proof. As in the proof of Lemma 4.16, we see that these kernels lie in

$$\left(\mathcal{C}(G) \otimes \Gamma^\infty(\mathrm{End}(E|_Y))\right)^K,$$

where $\mathcal{C}(G)$ is Harish-Chandra's Schwartz algebra, and we write $N = G \times_K Y$ as in Abels' theorem [1]. The claim follows from the fact that $\mathcal{C}(G) \subset \mathcal{S}_t(G)$ for $t \geq 0$. ■

Propositions 4.1 and 4.4 and the lemmas in this subsection have the following consequence.

Proposition 4.19. Suppose that G is a real, linear, connected semisimple Lie group and $g \in G$ is semisimple. If D_N is invertible, then $\eta_g(D_N)$ converges for large t. If, furthermore, either g has no fixed points or D is a twisted Spin^c-Dirac operator, then $\eta_g(D_N)$ also converges for small t, so D_N has a g-delocalised η-invariant.

5 Proof of the index theorem

After discussing convergence of delocalised η-invariants in Section 4, we are ready to prove Theorem 2.13 and Corollaries 2.16, 2.19, and 2.21. We do this by defining a suitable parametrix of D, depending on a positive parameter t, and splitting up the right hand side of (2.3) as a sum of several terms, and computing the limits of these terms as $t \downarrow 0$. These terms involve different parametrices of the Dirac operators on $\hat{M}$, $\tilde{M}$, and C.

5.1 Parametrices

Consider the setting of Subsection 2.2. Let $\psi_1 \colon (0,\infty) \to [0,1]$ be a smooth function such that ψ_1 equals 1 on $(0,\varepsilon)$ and 0 on $(1-\varepsilon,\infty)$, for some $\varepsilon \in (0,1/2)$. Set $\psi_2 := 1-\psi_1$. Let $\varphi_1, \varphi_2 \colon (0,\infty) \to [0,1]$ be smooth functions such that φ_1 equals 1 on $(0, 1-\varepsilon/2)$ and 0 on $(1,\infty)$, while φ_2 equals 0 on $(0,\varepsilon/4)$ and 1 on $(\varepsilon/2,\infty)$. Then $\varphi_j\psi_j = \psi_j$ for $j=1,2$, and φ_j' and ψ_j have disjoint supports.

We pull back the functions φ_j and ψ_j to C along the projection onto $(0,\infty)$ and extend these functions smoothly to $\hat{M}$ by setting ψ_1 and φ_1 equal to 1 on $M \setminus U$, and ψ_2 and φ_2 equal to 0 on $M \setminus U$. We denote the resulting functions by the same symbols ψ_j and φ_j. (No confusion is possible in what follows, because we will always use these symbols to denote the functions on $\hat{M}$.) We denote the derivatives of these functions in the $(0,\infty)$ directions by φ_j' and ψ_j', respectively. These derivatives are only defined on $N \times (0,\infty) \subset \hat{M}$, so they will only be used there.

As before, let $\tilde{M}$ be the double of M, $\tilde{E}$, and $\tilde{D}$ the extensions of E and D to $\tilde{M}$. Fix $t>0$, and consider the parametrix

$$\tilde{Q} := \frac{1-e^{-t\tilde{D}_-\tilde{D}_+}}{\tilde{D}_-\tilde{D}_+}\tilde{D}_- \tag{5.1}$$

of $\tilde{D}_+$. (The part without the last factor $\tilde{D}_-$ is formed via functional calculus, by an application of the function $x \mapsto \frac{1-e^{-tx}}{x}$ to $\tilde{D}_-\tilde{D}_+$; this does not require invertibility of $\tilde{D}_-\tilde{D}_+$.) Set

$$\begin{aligned} \tilde{S}_0 &:= 1-\tilde{Q}\tilde{D}_+ = e^{-t\tilde{D}_-\tilde{D}_+};\\ \tilde{S}_1 &:= 1-\tilde{D}_+\tilde{Q} = e^{-t\tilde{D}_+\tilde{D}_-}. \end{aligned} \tag{5.2}$$

The extension of D_C to the complete manifold $N \times \mathbb{R}$ is essentially self-adjoint, and its square is positive. Hence its self-adjoint closure is invertible. Let Q_C be the inverse of that closure, restricted to odd-graded sections. Set

$$\begin{aligned} R &:= \varphi_1\tilde{Q}\psi_1 + \varphi_2 Q_C\psi_2.\\ R' &:= \psi_1\tilde{Q}\varphi_1 + \psi_2 Q_C\varphi_2. \end{aligned} \tag{5.3}$$

Note that the operator $\tilde{Q}$ is well-defined on the supports of φ_1 and ψ_1, and that Q_C is well-defined on the supports of φ_2 and ψ_2.

A parametrix like R was also defined at the bottom of page 55 of [5]. In [5], it was only used to prove that D is Fredholm on spaces of sections with the APS-boundary conditions. In this paper, it plays a central role in the proof of Theorem 2.13 as well. This is possible because, contrary to [5], we first consider the case where D_C is invertible, so that we can use its inverse in the definition of R.

Set

$$\begin{aligned} S_0 &:= 1 - R\hat{D}_+; \\ S_0' &:= 1 - R'\hat{D}_+; \\ S_1 &:= 1 - \hat{D}_+ R. \end{aligned} \tag{5.4}$$

Lemma 5.1. We have

$$\begin{aligned} S_0 &= \varphi_1 \tilde{S}_0 \psi_1 + \varphi_1 \tilde{Q} \sigma \psi_1' + \varphi_2 Q_C \sigma \psi_2'; \\ S_0' &= \psi_1 \tilde{S}_0 \varphi_1 + \psi_1 \tilde{Q} \sigma \varphi_1' + \psi_2 Q_C \sigma \varphi_2'; \\ S_1 &= \varphi_1 \tilde{S}_1 \psi_1 - \varphi_1' \sigma \tilde{Q} \psi_1 - \varphi_2' \sigma Q_C \psi_2. \end{aligned} \tag{5.5}$$

Proof. These equalities follow from straightforward computations. ■

Lemma 5.2. The operators S_0 and S_1 have smooth kernels.

Proof. Because $\mathrm{supp}(\psi_j) \cap \mathrm{supp}(\varphi_j') = \emptyset$ for $j = 1, 2$ and the Schwartz kernels of the pseudo-differential operators $\tilde{Q}, Q_C$ are smooth off the respective diagonals in $\tilde{M}$ and C, the operators S_0', S_1 have smooth Schwartz kernels. Hence so does

$$S_0 = S_0' + (R'S_1 - S_0'R)\hat{D}_+.$$

■

5.2 The interior contribution

Lemma 5.3. If $e^{-t\tilde{D}^2}$ is g-trace class, then so are the operators S_0' and S_1 in (5.4). And the function ψ_1 can be chosen such that

$$\mathrm{Tr}_g(S_0') - \mathrm{Tr}_g(S_1) = \frac{1}{2}\big(\mathrm{Tr}_g(e^{-t\tilde{D}_-\tilde{D}_+}) - \mathrm{Tr}_g(e^{-t\tilde{D}_+\tilde{D}_-})\big).$$

Proof. In Lemma 5.1, the terms involving $\tilde{S}_0$ and $\tilde{S}_1$ are g-trace class because $e^{-t\tilde{D}^2}$ is. The functions φ_j' and ψ_j have disjoint, G-invariant supports. This implies that the terms in Lemma 5.1 involving those functions are g-trace class, with g-trace equal to zero. (Here we use the fact that the operators $\tilde{Q}$ and Q_C are pseudo-differential operators, and hence have smooth kernels off the diagonals of $\tilde{M}$ and C, respectively.) We find that S_0' and S_1 are g-trace class, and

$$\mathrm{Tr}_g(S_0') - \mathrm{Tr}_g(S_1) = \mathrm{Tr}_g(\psi_1\tilde{S}_0\varphi_1) - \mathrm{Tr}_g(\varphi_1\tilde{S}_1\psi_1).$$

By G-invariance of ψ_1 and φ_1, and the fact that $\psi_1\varphi_1 = \psi_1$, we have

$$\mathrm{Tr}_g(\psi_1\tilde{S}_1\varphi_1) = \mathrm{Tr}_g(\tilde{S}_1\psi_1);$$
$$\mathrm{Tr}_g(\varphi_1\tilde{S}_0\psi_1) = \mathrm{Tr}_g(\tilde{S}_0\psi_1).$$

Choosing ψ_1 such that $1_{\tilde{M}} - \psi_1$ equals the mirror image of ψ_1 on the double $\tilde{M}$, we have

$$\mathrm{Tr}_g(\tilde{S}_j) = \mathrm{Tr}_g(\tilde{S}_j\psi_1) + \mathrm{Tr}_g(\tilde{S}_j(1_{\tilde{M}} - \psi_1)) = 2\mathrm{Tr}_g(\tilde{S}_j\psi_1).$$

Hence the claim follows. ■

5.3 Another parametrix on C

Next, we show how the boundary contribution $-\frac{1}{2}\eta_g(D_N)$ appears in Theorem 2.13; see Proposition 5.8. We start by defining another parametrix for D_C, and showing that, in a suitable sense, its difference with $\tilde{Q}$ restricted to U has zero g-trace in the limit $t \downarrow 0$. This is Lemma 5.5.

Let $\tilde{\kappa}_t$ be the Schwartz kernel of $e^{-t\tilde{D}_-\tilde{D}_+}\tilde{D}_-$, and $\kappa_{C,t}$ the Schwartz kernel of $e^{-tD_{C,-}D_{C,+}}D_{C,-}$.

Lemma 5.4. For any $\varphi, \psi \in C^\infty(\hat{M})^G$ supported in $N \times (0, 1)$,

$$(\varphi \otimes \psi)(\tilde{\kappa}_t - \kappa_{C,t}) \sim 0,$$

as an asymptotic expansion in t, with respect to the C^k-norm on compact subsets of $\hat{M}$, for all k.

Proof. The Schwartz kernels of the heat operators $e^{-t\tilde{D}_-\tilde{D}_+}$ and $e^{-tD_{C,-}D_{C,+}}$ have asymptotic expansions in t, with respect to the C^k-norm on compact subsets of $\hat{M}$, in which the coefficient of t^{j+1} is the solution of a differential equation in terms of the local geometry of the manifold and the Dirac operator in question, and the coefficient of t^j. See for example Theorem 2.26 in [10], or the proof of Theorem 7.15 in [46]. Because the operators $\tilde{D}$ and D_C are equal on $N \times (0,1)$, their heat kernels therefore have the same asymptotic expansion there. Since these asymptotic expansions may be differentiated term-wise, and $\tilde{D}\psi = D_C\psi$, the claim follows. ∎

Let

$$Q'_C := \frac{1 - e^{-tD_{C,-}D_{C,+}}}{D_{C,-}D_{C,+}} D_{C,-} \tag{5.6}$$

be an alternative parametrix for $D_{C,+}$ on the cylindrical end C. This operator is defined in terms functional calculus applied to the self-adjoint closure of the extension of D_C to $N \times \mathbb{R}$.

When we wish to emphasise the dependence of operators on t, we write $\tilde{Q}(t)$ for $\tilde{Q}$, etc.

Lemma 5.5. Suppose that, for all $\varphi, \psi \in C^\infty(N \times \mathbb{R})$ with supports in $N \times (0,1)$, the operators $\tilde{D}e^{-t\tilde{D}^2}$ and $\varphi D_C e^{-tD_C^2}\psi$ are g-trace class for all $t > 0$, and that their Schwartz kernels localise at M^g as $t \downarrow 0$ in the sense of (2.13). Then the operator $(\varphi_1\tilde{Q} - \varphi_2 Q'_C)\sigma\psi'_1$ is of g-trace class for all $t > 0$, and

$$\lim_{t\downarrow 0} \mathrm{Tr}_g((\varphi_1\tilde{Q}(t) - \varphi_2 Q'_C(t))\sigma\psi'_1) = 0. \tag{5.7}$$

Proof. Choose a function $\varphi \in C^\infty(\hat{M})$ such that for $j = 1, 2$, φ equals 1 on the support of ψ'_j, and zero outside the support of $1 - \varphi_j$. With $\varepsilon > 0$ as at the start of Subsection 5.1, this is the case if φ equals 0 outside $N \times (\varepsilon/2, 1-\varepsilon/2)$ and 1 on $N \times (\varepsilon, 1-\varepsilon)$. Then $\varphi\varphi_j = \varphi$, and ψ'_1 and $1 - \varphi$ have disjoint supports. As in the proof of Lemma 5.3, this implies that

$$(\varphi_1\tilde{Q} - \varphi_2 Q'_C)\sigma\psi'_1 = \varphi(\tilde{Q} - Q'_C)\sigma\psi'_1 + (1-\varphi)(\varphi_1\tilde{Q} - \varphi_2 Q'_C)\sigma\psi'_1,$$

where the second term is g-trace class, with g-trace zero. As on page 369 of [23], we write

$$\frac{1 - e^{-tA^*A}}{A^*A} = -\int_0^t e^{-sA^*A}\,ds,$$

for any operator A such that A^*A is self-adjoint. Then

$$\varphi(\tilde{Q} - Q'_C)\sigma\psi'_1 = -\int_0^t \varphi(e^{-s\tilde{D}_-\tilde{D}_+}\tilde{D}_- - e^{-s(D_{C,-}D_{C,+})}D_{C,-})\sigma\psi'_1 \,\mathrm{d}s.$$

By assumption, $e^{-t\tilde{D}_-\tilde{D}_+}\tilde{D}_-$ and $\varphi e^{-tD_{C,-}D_{C,+}}D_{C,-}\psi'_1$ are g-trace class. Let $V \subset \hat{M}$ be a Z-invariant neighbourhood of U^g. We use Lemma 3.1, and note that by G-invariance of φ,

$$\int_{\hat{M}}\int_0^t \chi_g(m)^2\varphi(g^{-1}m)\psi'_1(m)\mathrm{tr}(g(\tilde{\kappa}_s - \kappa_{C,s})(g^{-1}m, m)\sigma)\,\mathrm{d}s\,\mathrm{d}m = \int_V\int_0^t \chi_g(m)^2\psi'_1(m)\mathrm{tr}(g(\tilde{\kappa}_s - \kappa_{C,s})(g^{-1}m, m)\sigma)\,\mathrm{d}s\,\mathrm{d}m + \int_{U\setminus V}\int_0^t \chi_g(m)^2\psi'_1(m)\mathrm{tr}(g(\tilde{\kappa}_s - \kappa_{C,s})(g^{-1}m, m)\sigma)\,\mathrm{d}s\,\mathrm{d}m. \quad (5.8)$$

Since $\overline{U^g}/Z$ is compact, we can and will choose V such that $\overline{V}/Z$ is compact as well. The intersection of the support of χ_g and $\overline{V}$ is then compact. So Lemma 5.4 implies that the first term on the right hand side of (8) is bounded by a constant times any power of t. The second term goes to zero as $t \downarrow 0$ by the localisation assumption on $\tilde{D}e^{-t\tilde{D}^2}$ and $\varphi D_C e^{-tD_C^2}\psi$. ■

5.4 The boundary contribution

Definition 5.6. If $\eta_g(D_N)$ converges for large t, then for $t > 0$ we set

$$\eta_g^t(D_N) := \frac{2}{\sqrt{\pi}}\int_{\sqrt{t}}^{\infty} \mathrm{Tr}_g(e^{-s^2D_N^2}D_N)\mathrm{d}s.$$

Lemma 5.7. If $\eta_g(D_N)$ converges for large t, then for all $t > 0$, the operator $\varphi_2 e^{-tD_{C,-}D_{C,+}}Q_C\sigma\psi'_2$ is g-trace class, and

$$\mathrm{Tr}_g(\varphi_2 e^{-tD_{C,-}D_{C,+}}Q_C\sigma\psi'_2) = -\frac{1}{2}\eta_g^t(D_N).$$

Proof. By definition, $D_{C,+} = \sigma\left(-\frac{\partial}{\partial u} + D_N\right)$, so $D_{C,-} = \left(\frac{\partial}{\partial u} + D_N\right)\sigma^{-1}$. Hence, using an integral representation of $e^{-tD_{C,-}D_{C,+}}(D_{C,-}D_{C,+})^{-1}$ analogous to the expression for $Q(D)$

on page 369 of [23], we have

$$\begin{aligned} e^{-tD_{C,-}D_{C,+}}Q_C = e^{-tD_{C,-}D_{C,+}}D_{C,+}^{-1} &= -\int_t^\infty e^{-sD_{C,-}D_{C,+}}D_{C,-}\,\mathrm{d}s \\ &= -\int_t^\infty e^{-s(-\frac{\partial^2}{\partial u}+D_N^2)}\left(\frac{\partial}{\partial u}+D_N\right)\sigma^{-1}\,\mathrm{d}s \\ &= -\int_t^\infty e^{s\frac{\partial^2}{\partial u^2}}e^{-sD_N^2}D_N\sigma^{-1}\,\mathrm{d}s - \int_t^\infty e^{s\frac{\partial^2}{\partial u^2}}\frac{\partial}{\partial u}e^{-sD_N^2}\sigma^{-1}\,\mathrm{d}s. \end{aligned} \tag{5.9}$$

We denote the Schwartz kernel of an operator T by κ_T. The Schwartz kernel of $e^{s\frac{\partial^2}{\partial u^2}}$ is the heat kernel of the Laplacian $-\frac{d^2}{du^2}$ on the real line:

$$\kappa_{e^{s\frac{\partial^2}{\partial u^2}}}(u,u') = \frac{1}{\sqrt{4\pi s}}e^{-\frac{|u-u'|^2}{4s}}.$$

The Schwartz kernel of $e^{s\frac{\partial^2}{\partial u^2}}\frac{\partial}{\partial u}$ is the derivative of the above heat kernel in the first variable:

$$\kappa_{e^{s\frac{\partial^2}{\partial u^2}}\frac{\partial}{\partial u}}(u,u') = \frac{-1}{\sqrt{4\pi s}}\frac{u-u'}{2s}e^{-\frac{|u-u'|^2}{4s}}.$$

Thus,

$$\int_0^\infty \varphi_2(u)\kappa_{e^{s\frac{\partial^2}{\partial u^2}}}(u,u)\psi_2'(u)\,\mathrm{d}u = \int_0^\infty \frac{1}{\sqrt{4\pi s}}\psi_2'(u)\mathrm{d}u = \frac{1}{\sqrt{4\pi s}}, \tag{5.10}$$

and

$$\int_0^\infty \varphi_2(u)\kappa_{e^{s\frac{\partial^2}{\partial u^2}}\frac{\partial}{\partial u}}(u,u)\psi_2'(u)\,\mathrm{d}u = \int_0^\infty \frac{-1}{\sqrt{4\pi s}}\frac{u-u}{2s}\psi_2'(u)\,\mathrm{d}u = 0. \tag{5.11}$$

We choose the cutoff function χ such that on $U = N\times(0,1]$, it is the pullback of a function χ_N on N. Let dn be the Riemannian density on N, so that $dm = dn\,du$ on U.

Using (5.9), (5.10), and (5.11), we then find that

$$
\begin{aligned}
&\mathrm{Tr}_g(\varphi_2 e^{-tD_{C,-}D_{C,+}} Q_C\sigma\psi_2') = \\
&\quad -\int_t^\infty \left[\int_{G/Z}\int_N \chi_N(hgh^{-1}n)^2 \mathrm{tr}\big(hgh^{-1}\kappa_{e^{-sD_N^2}D_N}(hg^{-1}h^{-1}n,n)\big)\,\mathrm{d}n\,\mathrm{d}(hZ)\right] \\
&\qquad\qquad \times\left[\int_0^\infty \varphi_2(u)\kappa_{e^{s\frac{\partial^2}{\partial u^2}}}(u,u)\psi_2'(u)\,\mathrm{d}u\right]\mathrm{d}s \\
&\quad -\int_t^\infty \left[\int_{G/Z}\int_N \chi_N(hgh^{-1}n)^2 \mathrm{tr}\big(hgh^{-1}\kappa_{e^{-sD_N^2}}(hg^{-1}h^{-1}n,n)\big)\,\mathrm{d}n\,\mathrm{d}(hZ)\right] \\
&\qquad\qquad \times\left[\int_0^\infty \varphi_2(u)\kappa_{e^{s\frac{\partial^2}{\partial u^2}}\frac{\partial}{\partial u}}(u,u)\psi_2'(u)\,\mathrm{d}u\right]\mathrm{d}s \\
&\qquad\qquad = \frac{-1}{\sqrt{4\pi}}\int_t^\infty \mathrm{Tr}_g(e^{-sD_N^2}D_N)\frac{1}{\sqrt{s}}\,\mathrm{d}s = \frac{-1}{\sqrt{\pi}}\int_{\sqrt{t}}^\infty \mathrm{Tr}_g(e^{-s^2D_N^2}D_N)\,\mathrm{d}s.
\end{aligned}
$$

In particular, this computation, with absolute values in the appropriate places, shows that $\varphi_2 e^{-tD_{C,-}D_{C,+}}Q_C\sigma\psi_2'$ is g-trace class if $\eta_g(D_N)$ converges for large t. ■

The following proposition is the key step in the proof of Theorem 2.13.

Proposition 5.8. Suppose that $e^{-t\tilde{D}^2}$ is g-trace class for all $t>0$, $\eta_g(D_N)$ converges for large t, $\tilde{D}e^{-t\tilde{D}^2}$ and $\varphi D_C e^{-tD_C^2}\psi$ are g-trace class for all $t>0$ and all $\varphi,\psi\in C^\infty(C)^G$ supported in $N\times(0,1)$, and that their Schwartz kernels localise at M^g as $t\downarrow 0$. Then $S_0(t)$ is g-trace class for all $t>0$, and

$$
\mathrm{Tr}_g S_0(t) - \mathrm{Tr}_g S_0'(t) = -\frac{1}{2}\eta_g^t(D_N) + F(t),
$$

for a function F satisfying $\lim_{t\downarrow 0}F(t)=0$.

Proof. By Lemma 5.1,

$$
S_0 - S_0' = (\varphi_1\tilde{S}_0\psi_1 - \psi_1\tilde{S}_0\varphi_1) - (\psi_1\tilde{Q}\sigma\varphi_1' + \psi_2 Q_C\sigma\varphi_2') + (\varphi_1\tilde{Q}\sigma\psi_1' + \varphi_2 Q_C\sigma\psi_2'). \tag{5.12}
$$

The first term in brackets on the right hand side is g-trace class because $e^{-t\tilde{D}^2}$ is, and its g-trace is zero by Lemma 3.2. As in the first paragraph of the proof of Lemma 5.3, the second term in brackets on the right hand side of (5.12) is g-trace class, with g-trace zero.

Using the fact that $\psi_1' + \psi_2' = 0$, we rewrite the third term in brackets on the right hand side of (5.12) as

$$\varphi_1 \tilde{Q} \sigma \psi_1' + \varphi_2 Q_C \sigma \psi_2' = (\varphi_1 \tilde{Q} - \varphi_2 Q_C') \sigma \psi_1' + \varphi_2 (Q_C - Q_C') \sigma \psi_2'. \tag{5.13}$$

Since Q_C is the exact inverse of $D_{C,+}$,

$$Q_C - Q_C' = Q_C - \frac{1 - e^{-tD_{C,-}D_{C,+}}}{D_{C,-}D_{C,+}} D_{C,-}(D_{C,+}Q_C) = e^{-tD_{C,-}D_{C,+}} Q_C.$$

So Lemmas 5.5 and 5.7 imply that (5.13) is g-trace class, and that its g-trace equals

$$\frac{1}{2}\eta_g^t(D_N) + \mathrm{Tr}_g((\varphi_1 \tilde{Q}(t) - \varphi_2 Q_C'(t))\sigma \psi_1'),$$

where the second term on the right goes to zero as $t \downarrow 0$.

Because S_0' is g-trace class by Lemma 5.3, we find that S_0 is also g-trace class, which completes the proof. ■

5.5 Proofs of index theorems

Proof of Theorem 2.13. Under the conditions in Theorem 2.13, Lemma 5.3 and Proposition 5.8 imply that $S_0 = S_0(t)$ and $S_1 = S_0(t)$ are g-trace class for all $t > 0$, so D is g-Fredholm. By Lemma 5.3 and Proposition 5.8, we have for all $t > 0$,

$$\mathrm{index}_g(\hat{D}) = \mathrm{Tr}_g(S_0(t)) - \mathrm{Tr}_g(S_1(t)) = \frac{1}{2}(\mathrm{Tr}_g(e^{-t\tilde{D}_-\tilde{D}_+}) - \mathrm{Tr}_g(e^{-t\tilde{D}_+\tilde{D}_-})) - \frac{1}{2}\eta_g^t(D_N) + F(t), \tag{5.14}$$

where $\lim_{t\downarrow 0} F(t) = 0$. The first term on the right is independent of t and equals $\frac{1}{2}\mathrm{index}_g(\tilde{D})$. The left hand side of (5.14) is also independent of t. We find that $\lim_{t\downarrow 0} \eta_g^t(D_N)$ converges, that is, $\eta_g(D_N)$ converges for small t. And (2.14) holds in the limit $t \downarrow 0$. ■

We prove Corollaries 2.16, 2.19, and 2.21 by verifying that the conditions of Theorem 2.13 hold in those settings, and applying existing index theorems for manifolds without boundary.

If $g = e$, then all conditions in Theorem 2.13 hold. The topological expression for the interior contribution is Theorem 6.10 or Theorem 6.12 in [52], so Corollary 2.16 follows.

To deduce Corollaries 2.19 and 2.21 from Theorem 2.13, we first note that if α is a volume form on (each connected component of) M^g, and $\tilde{\alpha}$ is its extension to $\tilde{M}^g$, then

$$\int_{M^g} \alpha = \frac{1}{2} \int_{\tilde{M}^g} \tilde{\alpha}, \tag{5.15}$$

if either of the integrals converges. This is because near N, the fixed point set $\tilde{M}^g$ equals the Cartesian product of N^g and an interval. So there are no components of $\tilde{M}^g$ contained in N, so that there is no "double counting" in passing from $\tilde{M}^g$ to two copies of M^g. Because both the orientation on TM and the grading on E are reversed on the second copy of M in $\tilde{M}$, the integrand on the right hand side of (2.15) is the extension of the corresponding integrand on M^g in this sense, so (5.15) applies in that case.

Let us check that the localisation property (2.13) of $e^{-t\tilde{D}^2}\tilde{D}$ and $\varphi e^{-tD_C^2} D_C \psi$ assumed in Theorem 2.13 hold in the settings of Corollaries 2.19 and 2.21.

Proposition 5.9. Suppose $\varphi, \psi \in C^\infty(C)$ are G-invariant and supported in U. If G is a finitely generated, discrete group, then the kernels of $e^{-t\tilde{D}^2}\tilde{D}$ and $\varphi e^{-tD_C^2} D_C \psi$ localise at fixed point sets, as $t \downarrow 0$.

Proof. Let $\tilde{\kappa}_t$ be the kernel of $e^{-t\tilde{D}^2}\tilde{D}$ and let V be a neighbourhood of $\tilde{M}^g$ in $\tilde{M}$ where $\bar{V}$ is Z-compact. Then, with χ_g as in (3.1),

$$\begin{aligned}
&\int_{\tilde{M}\setminus V} \chi_g(m)^2 |\mathrm{tr}(g\tilde{\kappa}_t(g^{-1}m, m))| \, \mathrm{d}m \\
&= \sum_{h\in(g)} \int_{\tilde{M}\setminus V} \chi(m)^2 |\mathrm{tr}(h\tilde{\kappa}_t(h^{-1}m, m))| \, \mathrm{d}m \\
&\leq \|\chi\|_\infty \sum_{h\in(g)} \int_{(\tilde{M}\setminus V)\cap \mathrm{supp}(\chi)} |\mathrm{tr}(h\tilde{\kappa}_t(h^{-1}m, m))| \, \mathrm{d}m.
\end{aligned}$$

The last sum of integrals converges because of the compactness of $(\tilde{M}\setminus V) \cap \mathrm{supp}(\chi)$ and the uniform convergence of $\sum_{h\in G} |\mathrm{tr}(h\tilde{\kappa}_t(h^{-1}m, m))|$. Here we use Corollary 3.5 from [51], as in Lemma 4.11. This ensures that the limit as t goes to 0 can be evaluated inside the sum and the integral. Therefore

$$\lim_{t\downarrow 0}\int_{\tilde{M}\setminus V}\chi_g(m)^2|\mathrm{tr}(g\tilde{\kappa}_t(g^{-1}m,m))|\,\mathrm{d}m=0,$$

because if $m\in\tilde{M}\setminus V$ and $h\in(g)$, then $\mathrm{tr}(h\tilde{\kappa}_t(h^{-1}m,m))\to 0$ as $t\downarrow 0$.

In a similar way, one can prove that the operator $\varphi e^{-tD_C^2}D_C\psi$ localises at fixed point sets, noting that $U\cap\mathrm{supp}(\chi)$ is relatively compact. ■

Proposition 5.10. Suppose $\varphi,\psi\in C^\infty(C)$ are G-invariant and supported in U. If G is a real, linear, connected semisimple Lie group, then the kernels of $e^{-t\tilde{D}^2}\tilde{D}$ and $\varphi e^{-tD_C^2}D_C\psi$ localise at fixed point sets, as $t\downarrow 0$.

Proof. The claim follows from cocompactness of $\tilde{M}$ and the supports of φ and ψ, by Proposition 4.6 in [29]. That result applies to the Schwartz kernels of $e^{-t\tilde{D}^2}$ and $\varphi e^{-tD_C^2}\psi$; this extends to $e^{-t\tilde{D}^2}\tilde{D}$ and $\varphi e^{-tD_C^2}D_C\psi$ because the kernels those operators have Gaussian off-diagonal decay behaviour as well. See Section 4.3 of [29], with the reference to Theorem 4 in [21] replaced by a reference to Theorem 6 in [21]. The only difference is an extra factor t^{-1} in the function bounding the kernel, and the arguments can easily be modified to incorporate this. Furthermore, the condition in [29] that group elements have "finite Gaussian orbital integrals" is shown to hold for all semisimple elements in Section 4.2 of [11]. ■

Proof of Corollary 2.19. In the setting of Corollary 2.19, Proposition 4.14 implies that D_N has a g-delocalised η-invariant. Entirely analogously to Lemma 4.11, one finds that $e^{-t\tilde{D}^2}$ and $e^{-t\tilde{D}^2}\tilde{D}$ are g-trace class. The localisation property of $e^{-t\tilde{D}^2}\tilde{D}$ and $\varphi e^{-tD_C^2}D_C\psi$ (for G-invariant $\varphi,\psi\in C^\infty(C)$, supported in U) is Proposition 5.9. The index formula (2.15) on manifolds without boundary holds by Theorem 5.10 in [51]. By (5.15) and the comments below it, the right hand side of (2.15) can be rewritten as an integral over M^g. Hence Theorem 2.13 implies Corollary 2.19. ■

Proof of Corollary 2.21. In the setting of Corollary 2.21, Proposition 4.19 implies that D_N has a g-delocalised η-invariant. Similarly to the discrete case, arguments like those in the proof of Lemma 4.16 imply that $e^{-t\tilde{D}^2}$ $e^{-t\tilde{D}^2}\tilde{D}$ are g-trace class. The localisation property of $e^{-t\tilde{D}^2}\tilde{D}$ and $\varphi e^{-tD_C^2}D_C\psi$ (for G-invariant $\varphi,\psi\in C^\infty(C)$, supported in U) is Proposition 5.10. The index formula (2.15) now holds by Theorem 2.5 in [29], and again we use (5.15) to rewrite the right hand side of (2.15) as an integral over M^g. In the application of Theorem 2.5 in [29], we use the fact that the condition that group elements have "finite Gaussian orbital integrals" holds for all semisimple elements, as shown in Section 4.2 of [11]. We conclude that Corollary 2.21 follows from Theorem 2.13. ■

6 Non-Invertible D_N

We have so far assumed the operator D_N to be invertible. We now weaken this assumption and only assume that 0 is isolated in the spectrum of D_N. We indicate how to modify the arguments in the preceding sections to obtain generalisations of Theorem 2.13 and Corollaries 2.16, 2.19, and 2.21 to that case.

6.1 An index theorem for non-invertible D_N

Let $\varepsilon > 0$ be such that $([-2\varepsilon, 2\varepsilon] \cap \mathrm{spec}(D_N)) \setminus \{0\} = \emptyset$. Let $\psi \in C^\infty(\hat{M})^G$ be a nonnegative function such that

$$\psi(n,u) = \begin{cases} u & \text{if } n \in N \text{ and } u \in (1/2, \infty); \\ 0 & \text{if } n \in N \text{ and } u \in (0, 1/4); \end{cases}$$

$$\psi(m) = 0 \quad \text{if } m \in M \setminus U.$$

(Recall that $U \cong N \times (0,1]$ is a neighbourhood of N in M.)

Consider the operator

$$\hat{D}_\varepsilon := e^{\varepsilon\psi} \hat{D} e^{-\varepsilon\psi}.$$

The operator $\hat{D}_\varepsilon$ is G-equivariant, odd-graded, and elliptic. It is symmetric on $\Gamma_c^\infty(E)$, because it equals D on $M \setminus (N \times (0, 1/4))$, and

$$\sigma\Big(-\frac{\partial}{\partial u} + D_N + \varepsilon\psi'\Big) \tag{6.1}$$

on C. It is essentially self-adjoint by a finite propagation speed argument; see for example, Proposition 10.2.11 in [27]. On $\hat{M} \setminus M$, the operator (6.1) equals

$$\sigma\Big(-\frac{\partial}{\partial u} + D_N + \varepsilon\Big). \tag{6.2}$$

The point is that the operator $D_N + \varepsilon$ is invertible, and many arguments that apply in the case where D_N itself is invertible now apply with D_N replaced by $D_N + \varepsilon$.

A difference between $\hat{D}$ and $\hat{D}_\varepsilon$, and between D_N and $D_N + \varepsilon$, is that the operators $\hat{D}_\varepsilon$ and $D_N + \varepsilon$ are not Dirac operators of the form (2.6). One implication is that $\eta_g(D_N + \varepsilon)$ may not converge for small t. To circumvent this, we use a standard regularisation technique. We note that $\eta_g(D_N+\varepsilon)$ does converge for large t, as a version of Proposition 4.1 still applies. Hence for all $t > 0$, the number $\eta_g^t(D_N)$ as in Definition 5.6 is well-defined.

Definition 6.1. For a function $f(t)$ that has an asymptotic expansion in t as $t \downarrow 0$, the *regularised limit* $\mathrm{LIM}_{t\downarrow 0} f(t)$ is the coefficient of t^0 in such an asymptotic expansion.

The *regularised g-delocalised η-invariant* of $D_N + \varepsilon$ is

$$\eta_g^{\mathrm{reg}}(D_N + \varepsilon) := \mathrm{LIM}_{t\downarrow 0}\eta_g^t(D_N + \varepsilon).$$

Let $P \colon L^2(E|_N) \to \ker(D_N)$ be the orthogonal projection. Theorem 2.13 generalises as follows.

Theorem 6.2. Suppose that

- the operators $e^{-t\tilde{D}^2}$ and $e^{-t\tilde{D}^2}\tilde{D}$ are g-trace class for all $t > 0$;
- the Schwartz kernels of $e^{-t\tilde{D}^2}\tilde{D}$ and $\varphi e^{-tD_C^2} D_C \psi$ localise at M^g as $t \downarrow 0$, for all G-invariant $\varphi, \psi \in C^\infty(C)$, supported in U;
- $\eta_g(D_N + \varepsilon)$ converges for large t;
- the projection P is g-trace class.

Then $\hat{D}_\varepsilon$ is g-Fredholm, and

$$\mathrm{index}_g(\hat{D}_\varepsilon) = \frac{1}{2}\mathrm{LIM}_{t\downarrow 0}\big(\mathrm{Tr}_g(e^{-t\tilde{D}_{\varepsilon,-}\tilde{D}_{\varepsilon,+}}) - \mathrm{Tr}_g(e^{-t\tilde{D}_{\varepsilon,+}\tilde{D}_{\varepsilon,-}})\big) - \frac{1}{2}\eta_g^{\mathrm{reg}}(D_N + \varepsilon). \tag{6.3}$$

Corollary 6.3. Suppose D is a Spin^c-Dirac operator twisted by a vector bundle V, and that either

- G is any unimodular locally compact group, and $g = e$;
- G is discrete and finitely generated, and the conjugacy class of g has polynomial growth; or
- G is a connected, linear, real semisimple Lie group, and g is semisimple.

Then $\hat{D}_\varepsilon$ is g-Fredholm, D_N has a g-delocalised η-invariant, and

$$\lim_{\varepsilon\downarrow 0} \mathrm{index}_g(\hat{D}_\varepsilon) = \frac{1}{2}\int_{\tilde{M}^g} \chi_g^2 \frac{\hat{A}(\tilde{M}^g)e^{c_1(L|_{\tilde{M}^g})/2}\mathrm{tr}(ge^{-R_{\tilde{V}}|_{\tilde{M}^g}/2\pi i})}{\det(1-ge^{-R_{\tilde{\mathcal{N}}}/2\pi i})^{1/2}} - \frac{\mathrm{Tr}_g(P)+\eta_g(D_N)}{2}. \qquad (6.4)$$

Remark 6.4. The operator $\hat{D}_\varepsilon$ is unitarily equivalent to the operator $\hat{D}$ acting on Sobolev spaces weighted by the function $e^{\varepsilon\psi}$. This point of view is related to the b-calculus approach to the APS index theorem [38]. In the setting of [5], where M is compact and G is trivial, the index of $\hat{D}$ on those weighted Sobolev spaces equals the dimension of the kernel of D_+ on L^2-sections minus the dimension of the kernel of D_- on *extended* L^2-sections. See Section 6.5 in [38]. By Proposition 3.11 in [5], that number equals the APS index of D_+ in the classical sense. Furthermore, all conditions in Theorem 6.2 trivially hold in the compact case. On the right hand side of (6.3), the term $\frac{1}{2}\mathrm{index}_e(\tilde{D})$ equals the interior contribution in the APS-index theorem, by Lemma 2.8 and the Atiyah–Singer index theorem. Hence Theorem 6.2 reduces to the original APS index theorem (Theorem 3.10 in [5]) in this case.

In the compact case, $\mathrm{index}_g(\hat{D}_\varepsilon)$ is independent of ε by a homotopy argument, so the limit $\lim_{\varepsilon\downarrow 0}$ may be omitted in (6.4). It seems likely that this applies in certain noncompact situations as well, in particular in cases where $\mathrm{index}_g(\hat{D}_\varepsilon)$ can be recovered from a K-theoretic index, as in [28, 42].

Remark 6.5. In the setting of a Galois covering (when G is discrete and acts freely), the perturbation of the boundary operator D_N is equivalent to the choice of a spectral section in the sense of Leichtnam–Piazza [33]. The notion of a spectral section was initially introduced by Melrose-Piazza [39] as an analogue of the APS boundary conditions for a family of elliptic operators on a manifold with boundary and then generalised to the higher index associated to a Galois cover of a manifold with boundary [33]. A spectral section exits if and only if the higher index of the boundary Dirac operator vanishes, that is, $\mathrm{index}_G(D_N) = 0 \in K_*(C_r^*(G))$. In our setting the bordism invariance of the index implies vanishing of the higher index of D_N, so a spectral section always exists. Translating to the case of a manifold with a cylindrical end, the existence of a spectral section is equivalent to existence of a perturbation of D_N so that the perturbed operator ($\hat{D}_\varepsilon$ in our notation) is a generalised Fredholm operator. For example, Wahl considered manifolds with cylindrical ends and proved an APS index theorem for Dirac operator on Hilbert C^*-modules by performing a perturbation on the boundary Dirac operator, which is equivalent to the choice of a boundary condition given by a spectral section [50].

6.2 Proof of Theorem 6.2

Let $\tilde{\psi}$ be any smooth, G-invariant extension of $\psi|_M$ to the double $\tilde{M}$ of M. Set

$$\tilde{D}_\varepsilon := e^{\varepsilon\tilde{\psi}}\tilde{D}e^{-\varepsilon\tilde{\psi}},$$

and

$$\tilde{Q}_\varepsilon := \frac{1 - e^{-t\tilde{D}_{\varepsilon,-}\tilde{D}_{\varepsilon,+}}}{\tilde{D}_{\varepsilon,-}\tilde{D}_{\varepsilon,+}}\tilde{D}_{\varepsilon,-}.$$

Set

$$\tilde{S}_{\varepsilon,0} := 1 - \tilde{Q}_\varepsilon\tilde{D}_{\varepsilon,+} = e^{-t\tilde{D}_{\varepsilon,-}\tilde{D}_{\varepsilon,+}};$$
$$\tilde{S}_{\varepsilon,1} := 1 - \tilde{D}_{\varepsilon,+}\tilde{Q}_\varepsilon = e^{-t\tilde{D}_{\varepsilon,+}\tilde{D}_{\varepsilon,-}}.$$

Let $D_{C,\varepsilon}$ be the restriction of $\hat{D}_\varepsilon$ to $N \times (1/2, \infty)$. This operator equals (6.2), and the self-adjoint closure of its extension to $N \times \mathbb{R}$ is invertible. Let $Q_{C,\varepsilon}$ be its inverse, restricted to sections of E_-.

Let the functions φ_j and ψ_j be as in Subsection 5.1, with the difference that they change values between 0 and 1 on the interval $(1/2, 1)$ rather than on $(0, 1)$. Set

$$R_\varepsilon := \varphi_1\tilde{Q}_\varepsilon\psi_1 + \varphi_2 Q_{C,\varepsilon}\psi_2.$$

Set

$$S_{\varepsilon,0} := 1 - R_\varepsilon\hat{D}_{\varepsilon,+};$$
$$S_{\varepsilon,1} := 1 - \hat{D}_{\varepsilon,+}R_\varepsilon.$$

From now on, the proof of Theorem 6.2 is analogous to the proof of Theorem 2.13, apart from the fact that $\tilde{D}_\varepsilon$ and $D_N + \varepsilon$ are not Dirac operators of the form (2.6). As in Lemma 5.1, we have

$$S_{\varepsilon,0} = \varphi_1\tilde{S}_{\varepsilon,0}\psi_1 + \varphi_1\tilde{Q}_\varepsilon\sigma\psi_1' + \varphi_2 Q_{C,\varepsilon}\sigma\psi_2';$$
$$S_{\varepsilon,1} = \varphi_1\tilde{S}_{\varepsilon,1}\psi_1 - \varphi_1'\sigma\tilde{Q}_\varepsilon\psi_1 - \varphi_2'\sigma Q_{C,\varepsilon}\psi_2.$$

In the same way as for S_0 and S_1, we find that $S_{\varepsilon,0}$ and $S_{\varepsilon,1}$ have smooth kernels, and they are g-trace class. And Proposition 5.8 generalises directly to $\hat{D}_\varepsilon$, and we obtain the following analogue of (5.14):

$$\mathrm{index}_g(\hat{D}_\varepsilon) = \frac{1}{2}(\mathrm{Tr}_g(e^{-t\tilde{D}_{\varepsilon,-}\tilde{D}_{\varepsilon,+}}) - \mathrm{Tr}_g(e^{-t\tilde{D}_{\varepsilon,+}\tilde{D}_{\varepsilon,-}})) - \frac{1}{2}\eta_g^t(D_N+\varepsilon) + F(t), \tag{6.5}$$

for a function F such that $\lim_{t\downarrow 0} F(t) = 0$. Here we used the fact that $\eta_g(D_N+\varepsilon)$ converges for large t as Proposition 4.1 still applies. But because $\tilde{D}_\varepsilon$ and $D_N+\varepsilon$ are not Dirac operators of the form (2.6), the first and second terms on the right hand side of (6.5) may not have well-defined limits as $t\downarrow 0$. But the left hand side is independent of t, so we obtain (6.3).

6.3 Delocalised η-invariants and projection onto $\ker D_N$

Corollary 6.3 follows from Theorem 6.2 in the same way that Corollaries 2.16, 2.19, and 2.21 follow from Theorem 2.13, with three additional ingredients.

The first is a generalisation of Proposition 4.1 to the case where D_N is not invertible, but 0 is isolated in its spectrum. That case can be deduced from the case where D_N is invertible, see Remark 4.3.

The second additional ingredient is the following fact.

Lemma 6.6. In the setting of Corollary 6.3, the projection P is g-trace class.

Proof. By the first part of the proof of Proposition 5.22 in [42], the operator P has a smooth kernel in $\mathrm{End}_{\mathcal{A}}(\Gamma^\infty(E|_N))^G$, for the relevant algebra $\mathcal{A}$ as in Subsection 4.1. Here we use that $\mathrm{End}_{\mathcal{A}}(\Gamma^\infty(E|_N))^G$ is closed under holomorphic functional calculus. By (3.7), this implies that P is g-trace class. ■

The proof of the third additional ingredient is a modification of the proof of Proposition 5.14 in [42], which, in turn, is a variation on the proof of Proposition 8.38 in [38].

Lemma 6.7. In the setting of Corollary 6.3,

$$\lim_{\varepsilon\downarrow 0} \eta_g^{\mathrm{reg}}(D_N+\varepsilon) = \mathrm{Tr}_g(P) + \eta_g^{\mathrm{reg}}(D_N).$$

Proof. We have

$$\eta_g^{\mathrm{reg}}(D_N+\varepsilon) = \mathrm{LIM}_{t\downarrow 0}\frac{1}{\sqrt{\pi}}\int_t^1 \mathrm{Tr}_g(e^{-s(D_N+\varepsilon)^2}(D_N+\varepsilon))\frac{\mathrm{d}s}{\sqrt{s}}$$
$$+\lim_{T\to\infty}\frac{1}{\sqrt{\pi}}\int_1^T \mathrm{Tr}_g(e^{-s(D_N+\varepsilon)^2}(D_N+\varepsilon))\frac{\mathrm{d}s}{\sqrt{s}}. \tag{6.6}$$

To evaluate the first term on the right hand side in the limit $\varepsilon\downarrow 0$, we note that the Schwartz kernel of $e^{-s(D_N+\varepsilon)^2}(D_N+\varepsilon)$ and the terms in its asymptotic expansion in t are continuous in ε, uniformly in s in compact intervals in $(0,\infty)$. Using this, we find that

$$\lim_{\varepsilon\downarrow 0}\mathrm{LIM}_{t\downarrow 0}\frac{1}{\sqrt{\pi}}\int_t^1 \mathrm{Tr}_g(e^{-s(D_N+\varepsilon)^2}(D_N+\varepsilon))\frac{\mathrm{d}s}{\sqrt{s}} = \mathrm{LIM}_{t\downarrow 0}\frac{1}{\sqrt{\pi}}\int_t^1 \mathrm{Tr}_g(e^{-sD_N^2}D_N)\frac{\mathrm{d}s}{\sqrt{s}}. \tag{6.7}$$

To evaluate the second term on the right hand side of (6.6), we note that

$$\begin{aligned}\frac{d}{d\varepsilon}\int_1^T \mathrm{Tr}_g(e^{-s(D_N+\varepsilon)^2}(D_N+\varepsilon))\frac{\mathrm{d}s}{\sqrt{s}} &= \int_1^T \mathrm{Tr}_g\big(e^{-s(D_N+\varepsilon)^2}\big(1-2s(D_N+\varepsilon)^2\big)\big)\frac{\mathrm{d}s}{\sqrt{s}}\\ &= \int_1^T \frac{d}{ds}\Big(2\sqrt{s}\,\mathrm{Tr}_g(e^{-s(D_N+\varepsilon)^2})\Big)\,\mathrm{d}s\\ &= 2\sqrt{T}\,\mathrm{Tr}_g(e^{-T(D_N+\varepsilon)^2}) - 2\mathrm{Tr}_g(e^{-(D_N+\varepsilon)^2}).\end{aligned}$$

We find that

$$\int_1^T \mathrm{Tr}_g(e^{-s(D_N+\varepsilon)^2}(D_N+\varepsilon))\frac{\mathrm{d}s}{\sqrt{s}} =$$
$$\int_1^T \mathrm{Tr}_g(e^{-sD_N^2}D_N)\frac{\mathrm{d}s}{\sqrt{s}} + 2\sqrt{T}\int_0^\varepsilon \mathrm{Tr}_g(e^{-T(D_N+\varepsilon')^2})\,\mathrm{d}\varepsilon' - 2\int_0^\varepsilon \mathrm{Tr}_g(e^{-(D_N+\varepsilon')^2})\,\mathrm{d}\varepsilon'. \tag{6.8}$$

The last term on the right hand side is independent of T, and the integrand is bounded in ε'. So

$$\lim_{\varepsilon\downarrow 0}\lim_{T\to\infty} 2\int_0^\varepsilon \mathrm{Tr}_g(e^{-(D_N+\varepsilon')^2})\,\mathrm{d}\varepsilon' = 0. \tag{6.9}$$

In the second term on the right hand side of (8), we note that

$$e^{-T(D_N+\varepsilon')^2} = e^{-T(D_N^2+2\varepsilon' D_N)}e^{-T\varepsilon'^2}. \tag{6.10}$$

And for all $\varepsilon' \in (0, \varepsilon)$, zero is isolated in the spectrum of the operator $D_N^2 + 2\varepsilon' D_N$, and this operator has the same kernel as D_N. Set

$$R_T := e^{-T(D_N^2 + 2\varepsilon' D_N)} - P. \tag{6.11}$$

This is the action of the heat operator of the Laplace-type operator $D_N^2 + 2\varepsilon' D_N$ on the complement of its kernel. The operator on left hand side of (6.10) is g-trace class, hence so is the first operator on the right hand side of (6.11). The operator P is g-trace class by Lemma 6.6. Hence R_T is g-trace class. By the arguments in the proof of Lemma 4.2, the g-trace of R_T goes to zero as a Gaussian function as $T \to \infty$.

Because of this decay behaviour, and the fact that P is g-trace class,

$$\lim_{T\to\infty} 2\sqrt{T} \int_0^{\varepsilon} \mathrm{Tr}_g(e^{-T(D_N+\varepsilon')^2})\,\mathrm{d}\varepsilon' = \lim_{T\to\infty} 2\sqrt{T} \int_0^{\varepsilon} e^{-T\varepsilon'^2}\,\mathrm{d}\varepsilon' \mathrm{Tr}_g(P).$$

And for all $\varepsilon > 0$,

$$\lim_{T\to\infty} 2\sqrt{T} \int_0^{\varepsilon} e^{-T\varepsilon'^2}\,\mathrm{d}\varepsilon' = \int_0^{\infty} e^{-\varepsilon''^2}\,\mathrm{d}\varepsilon'' = \sqrt{\pi}.$$

So the second term on the right hand side of (8) satisfies

$$\lim_{\varepsilon\downarrow 0} \lim_{T\to\infty} 2\sqrt{T} \int_0^{\varepsilon} \mathrm{Tr}_g(e^{-T(D_N+\varepsilon')^2})\,\mathrm{d}\varepsilon' = \sqrt{\pi}\,\mathrm{Tr}_g(P). \tag{6.12}$$

Using (8) and the expressions (6.9) and (6.12) for the relevant limits of the second and third terms on the right, and the fact that that $\eta_g(D_N)$ converges for large t by the generalisation of Proposition 4.1 mentioned at the start of this subsection, we conclude that

$$\lim_{\varepsilon\downarrow 0} \lim_{T\to\infty} \int_1^T \mathrm{Tr}_g(e^{-s(D_N+\varepsilon)^2}(D_N+\varepsilon))\frac{\mathrm{d}s}{\sqrt{s}} = \int_1^{\infty} \mathrm{Tr}_g(e^{-sD_N^2} D_N)\frac{\mathrm{d}s}{\sqrt{s}} + \sqrt{\pi}\,\mathrm{Tr}_g(P).$$

Together with (6.6) and (6.7), this implies the claim. ■

Proof of Corollary 6.3. In the situation of Corollary 6.3, Theorem 6.2 applies. And by Lemmas 6.6 and 6.7,

$$\lim_{\varepsilon\downarrow 0} \mathrm{index}_g(\hat{D}_\varepsilon) = \frac{1}{2} \lim_{\varepsilon\downarrow 0} \mathrm{LIM}_{t\downarrow 0}(\mathrm{Tr}_g(e^{-t\tilde{D}_{\varepsilon,-}\tilde{D}_{\varepsilon,+}}) - \mathrm{Tr}_g(e^{-t\tilde{D}_{\varepsilon,+}\tilde{D}_{\varepsilon,-}})) - \frac{\mathrm{Tr}_g(P) + \eta_g^{\mathrm{reg}}(D_N)}{2}. \tag{6.13}$$

By Proposition 4.4, $\eta_g(D_N)$ converges for small t in the setting of Corollary 6.3, so $\eta_g^{\mathrm{reg}}(D_N) = \eta_g(D_N)$.

Now $\mathrm{Tr}_g(e^{-t\tilde{D}_{\varepsilon,\mp}\tilde{D}_{\varepsilon,\pm}})$ and the coefficients in its asymptotic expansion are continuous in ε. So the first term on the right hand side of (6.13) equals

$$\frac{1}{2}\mathrm{LIM}_{t\downarrow 0}(\mathrm{Tr}_g(e^{-t\tilde{D}_-\tilde{D}_+}) - \mathrm{Tr}_g(e^{-t\tilde{D}_+\tilde{D}_-})) = \frac{1}{2}\lim_{t\downarrow 0}\big(\mathrm{Tr}_g(e^{-t\tilde{D}_-\tilde{D}_+}) - \mathrm{Tr}_g(e^{-t\tilde{D}_+\tilde{D}_-})\big).$$

And the right hand side equals the first term on the right hand side of (6.4) by the results cited in the proofs of Corollaries 2.16, 2.19, and 2.21 in Subsection 5.5. ■

Remark 6.8. In the proof of Corollary 6.3, we used Proposition 4.4 to find that $\eta_g^{\mathrm{reg}}(D_N) = \eta_g(D_N)$. It seems likely that this can also be deduced from the index formula, as in the proof of Theorem 2.13. An argument of this type would involve analysing the expression in Lemma 5.5 for the operator $\hat{D}_\varepsilon$ in the limit $\varepsilon \downarrow 0$. In situations where convergence of $\eta_g(D_N)$ for small t is not clear, a version of Corollary 6.3 holds with $\eta_g(D_N)$ replaced by $\eta_g^{\mathrm{reg}}(D_N)$.

A The Compact Case

Suppose that M is compact and G is trivial. Then many arguments in this paper are unnecessary, and the others become simpler. We illustrate this by giving a sketch of a proof of the original APS index theorem here. This may also help to clarify the overall idea of the proof of Theorem 2.13. The details are simplified versions of the corresponding arguments in this paper. We discuss the case where D_N is invertible here. The general case then follows as in Section 6.

We use the same notation as in the rest of this paper. In particular, consider the parametrices $\tilde{Q}$ of $\tilde{D}_+$, Q'_C of $D_{C,+}$, and R of $\hat{D}_+$ defined in (5.1), (5.3) and (5.6), respectively. In other approaches to the APS index theorem, one often works with the heat operators $e^{-tD_-D_+}$ and $e^{-tD_+D_-}$, or $e^{-t\hat{D}_-\hat{D}_+}$ and $e^{-t\hat{D}_+\hat{D}_-}$, which are not trace class. (See e.g., Section 4 of the introduction to [38] for comments on this, and a solution in terms of b-calculus.) We work with the operators $1 - R\hat{D}_+$ and $1 - \hat{D}_+R$ instead, which are e-trace class.

The APS index of D equals the index of $\hat{D}$ on L^2-sections, which by Lemma 2.8 equals

$$\mathrm{Tr}_e(1 - R\hat{D}_+) - \mathrm{Tr}_e(1 - \hat{D}_+R) = \mathrm{Tr}_e(\varphi_1(1 - \tilde{Q}\tilde{D}_+)\psi_1) - \mathrm{Tr}_e(\varphi_1(1 - \tilde{D}_+\tilde{Q})\psi_1)$$
$$+ \mathrm{Tr}_e((\varphi_1\tilde{Q} - \varphi_2 Q'_C)\sigma\psi'_1) + \mathrm{Tr}_e(\varphi_2(D_{C,+}^{-1} - Q'_C)\sigma\psi'_2). \quad \text{(A.1)}$$

The first two terms on the right hand side together equal $\frac{1}{2}\mathrm{index}(\tilde{D})$, the contribution from the interior of M. This is because the operators $\varphi_1(1-\tilde{Q}\tilde{D}_+)\psi_1$ and $\varphi_1(1-\tilde{D}_+\tilde{Q})\psi_1$ are trace class, so their e-traces equal their traces by Lemma 2.2.

The third term on the right hand side of (1) equals

$$-\operatorname{Tr}_e\Big(\int_0^t (\varphi_1 e^{-s\tilde{D}_-\tilde{D}_+} - \varphi_2 e^{-sD_{C,-}D_{C,+}})D_{C,+}\sigma\psi_1'\,\mathrm{d}s\Big).$$

This goes to zero as $t\downarrow 0$, because the kernels of $e^{-s\tilde{D}_-\tilde{D}_+}$ and $e^{-sD_{C,-}D_{C,+}}$ have the same asymptotic expansion on the support of ψ_1', and φ_1 and φ_2 equal 1 on the support of ψ_1'.

The fourth term on the right hand side of (1) equals

$$\begin{aligned}\operatorname{Tr}_e(\varphi_2 e^{-tD_{C,-}D_{C,+}}D_{C,+}^{-1}\sigma\psi_2') &= -\operatorname{Tr}_e\Big(\int_t^\infty \varphi_2 e^{-sD_{C,-}D_{C,+}}D_{C,-}\sigma\psi_2'\,\mathrm{d}s\Big)\\ &= -\operatorname{Tr}_e\Big(\int_t^\infty \varphi_2 e^{-s\frac{\partial^2}{\partial u^2}}\frac{\partial}{\partial u}e^{-sD_N^2}\psi_2'\,\mathrm{d}s\Big) - \operatorname{Tr}_e\Big(\int_t^\infty \varphi_2 e^{-s\frac{\partial^2}{\partial u^2}}e^{-sD_N^2}D_N\psi_2'\,\mathrm{d}s\Big).\end{aligned}$$

Using the explicit form of the heat kernel on the real line, one shows that the first term on the right hand side equals zero, while the second term equals

$$-\int_t^\infty \frac{1}{\sqrt{4\pi s}}\operatorname{Tr}_e(D_N e^{-sD_N^2})\,\mathrm{d}s. \tag{A.2}$$

The operator $D_N e^{-sD_N^2}$ is trace class, so its e-trace equals its trace. Hence (A.2) tends to $-\frac{1}{2}\eta(D_N)$ as $t\downarrow 0$ (a priori we obtain the regularised η-invariant, but it then follows from the index formula that this is the actual η-invariant). The conclusion is that the APS-index formula holds:

$$\mathrm{index}_{\mathrm{APS}}(D) = \frac{1}{2}\mathrm{index}(\tilde{D}) - \frac{1}{2}\eta(D_N).$$

If D is a Spin^c-Dirac operator twisted by a vector bundle V, then, by the Atiyah–Singer index theorem, this becomes a special case of Corollary 2.16:

$$\mathrm{index}_{\mathrm{APS}}(D) = \int_M \hat{A}(M)e^{c_1(L)/2}e^{-R_V/2\pi i} - \frac{1}{2}\eta(D_N).$$

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