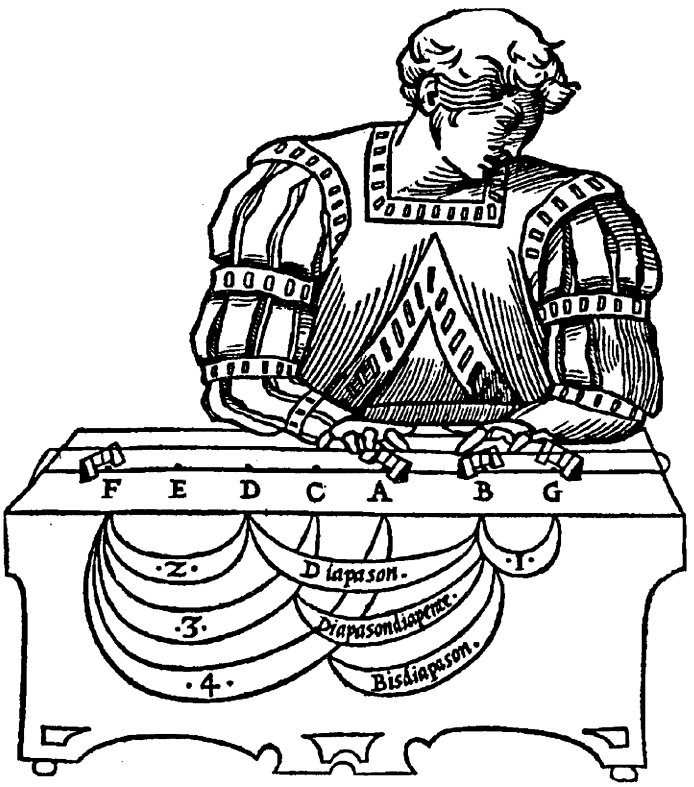
Vocal Intonation

An exposition of Pál Kardos's pedagogy of acoustically pure intonation for *a cappella* music

Judith Christine Pickering



A research project submitted in partial fulfilment of the requirements for the Masters of Music degree at the Australian National University

February 1995

B19263004

Preface

Good intonation is an essential condition of *a cappella* choral music, yet a systematic understanding of how to achieve it is rare among choral directors and aural trainers. Knowledge of theoretically based acoustical connections of notes and their relationships is a prerequisite for finding the best solution to the complex and often contradictory intonational problems inherent in a given choral work. Yet works on the pedagogy of vocal tuning are rare. Easley Blackwood in 1985 wrote that there are no writings giving practical instruction how to sing just intervals.[1] It is obvious Blackwood was unaware of the textbooks on choral tuning of the Hungarian scholar and choral trainer Pál Kardos, whose work is virtually unknown outside Hungary. The significance of Kardos's work in the area of vocal intonation lies in its practical orientation. In this it is a unique contribution to aural and choral training. He was concerned with the training of choral directors so that they could teach choristers to sing with acoustically pure intonation. Kardos saw pure intonation as a dying art, adversely effected by equal temperament, the indiscriminate use of piano accompaniment, and the lack of theoretical and practical knowledge of the acoustics, physics and mathematics of tuning.

The primary purpose of this work is to make available to the English-speaking world the valuable ideas of Kardos, and more generally provide an explanation of the theory and pedagogy of acoustically pure intonation in a cappella vocal music. I have drawn on all Kardos's writings, using a Hungarian interpreter to help translate from the Hungarian. Kardos's pedagogy was also orally transmitted to his students, and much of the material in this work is a result of three months spent in Hungary in 1991 studying with his colleague Eva Rozgonyi, his widow Mrs Pál Kardos and his son Gabor Kardos.

I amplify and elucidate his ideas with additional material and information from historical and scientific sources and other exponents of pure intonation. I draw on ancient Greek, medieval and Renaissance theories of intonation and harmonics and discuss the interplay between music theory and scientific theory as it applies to tuning and compositional practice.

I am indebted to the help of Eva Rozgonyi, Mrs Pál Kardos and Gabor Kardos, for their help and generous support. I thank also my supervisors Robyn Holmes and David Worrall. My thanks to Olle Palmqvist, Piroska Varga, Peter Campbell, Jonathon Beaumont and Deborah Crisp. Most of all, my gratitude goes to Michael Parkinson, without whose support this thesis would not have been written.

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1 Easley Blackwood, *The Structure of Recognisable Diatonic Tunings*, Princeton University Press, Princeton, 1985, p. 153.

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Abbreviations

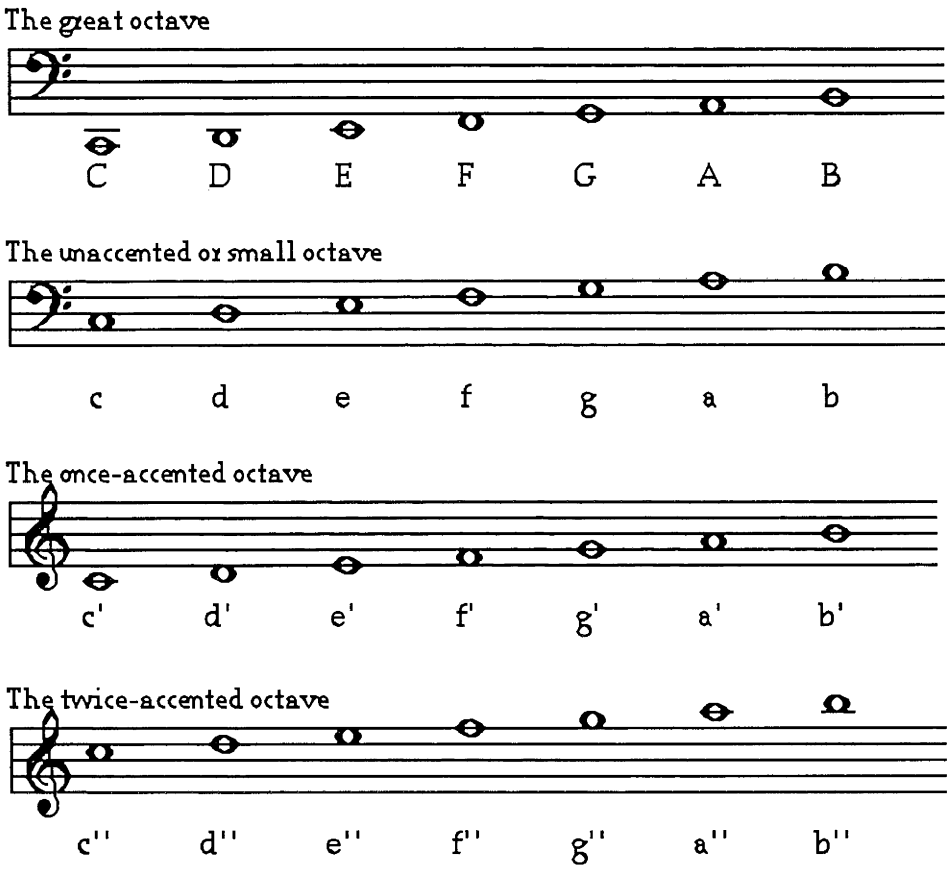
For ancient authors, text collections, periodicals, encyclopaedias, I have used those in the Oxford Classical Dictionary, 2nd ed., 1970.

Journal annotations follow Repertoire Internationale de la Literature Musicale (RILM).

Barker, *GMW* II	Andrew Barker, Greek *Musical Writings*, vol. 2, Cambridge University Press, Cambridge, 1989.
Kardos, *KK*	Pál Kardos, *Kórusnevelés - Kórushangzás* [*Choral Training, Choral Tone*], Zenemükiadó, Budapest, 1969, 1977.
Kardos, *Two Choral Works*	Pál Kardos, 'Bartók két kórusmüve a tiszta intonálásra nevelés tükrében (Ne hagyj itt!, Bolyongás) [Two Choral Works of Bartók in the Mirror of Education towards Pure Intonation]', *az ének-zene tanítása*, 1970, no. 4-5, szám.
Kardos, *A tiszta intonálásra*	Pál Kardos, *A tiszta intonálásra nevelés alapjai*, bolgár nyelvü kézirat, 1970.
Kardos, *KV*	Kardos, Pál, *Karvezetés III: Intonálás* [*Choral Direction III: Intonation*]', Tankönyvikiadó, Budapest, 1970.
Palisca, *Humanism*	Claude Palisca, *Humanism in Italian Renaissance Musical Thought*, Yale University Press, New Haven, 1985.
Palisca, *Studies*	Claude Palisca, *Studies in the History of Italian Music and Music Theory*, Clarendon Press, Oxford, 1994.
West, *AGM*	M. L. West, *Ancient Greek Music*, Clarendon Press, Oxford, 1992.

Nomenclature

Solfa (solfege) syllables are given in italics using modern spellings—*do, re, mi, fa, so, la, ti*. The solfa system used in this book is the relative solfa system where *do* is the tonic of a major key, *la* is the tonic of a minor key. Kardos does not usually refer to absolute pitch names of notes but rather the tonic solfa name. For extra clarity and ease I often include names of the notes in absolute pitch. I follow the German system: C D E F G A B c d e f g a b c c'd'e'f'g'a'b'c"d"e"f"g"a"b"c". Notes refered to in the text follow this nomenclature:



The language of ratios

Modern adherents of just tuning often argue that ratios are the only symbols that unambiguously represent just intervals.[2] Many discard the conventional names perfect fifth etc, in favour of 3:2 etc. Harry Partch is one, arguing that to think in terms of ratios opens 'new tonal vistas' and gets 'to the analysable root of music

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2 David B. Doty, 'Tutorial III', *1/1 Just Intonation*, vol. 1, no. 2, Spring 1986, p. 6.

and the core of the universe of tone'.[3] Eric Regener points out that the conventional names are now associated with equal temperament which cannot distinguish between different forms of tone, tritone, etc. and assumes that enharmonic intervals are the same when they are very different in non tempered systems.[4] A ratio term refers to only one size of interval. When clarity is required, only a ratio suffices. Regarding the use of ratios the following applies:

- ratios should be expressed in their least terms;
- the numerator should represent the higher tone, the denominator the lower tone;
- To add two ratios, multiply the two numerators and the two denominators;
- to subtract, divide by inverting the ratio to be subtracted then multiply.

Cents measurement

One way to quantify intervals is using cents measurement as devised by the English acoustic scientist, Alexander John Ellis' (1814-1890).[5] This system assumes a division of the octave into twelve equal parts. Each part is thus $\frac{1}{12}a$. This part is divided into one hundred equal parts or $\frac{1}{1200}a$ and this is the unit of measure, called a cent. The frequency ratio of $\frac{1}{1200}$ is $2^{1:1200}$. The octave consists of 1,200 cents. The tempered minor second equals 100 cents, a whole tone = 200 cents, a minor third = 300 cents and so forth.[6]

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3 Harry Partch, *Genesis of a Music*, Da Capo Press, New York, 1974, p. 76.

4 Eric Regener, 'The number seven in the theory of intonation', *JMTheory*, vol. 19, no. 1, 1975, p. 40.

5 Arthur Mendel, and Alexander J. Ellis, *Studies in the History of Musical Pitch*, Da Capo Press, New York, 1968.

6 The formula for converting ratios to cents is: $a/b = 10^{(c \div 3986.314)}$. The formula for converting cents to ratios is: a cent = $\frac{1200}{\log(2)}$ X log (a/b). Alternatively:

$$\frac{100}{\log(\sqrt[12]{2})} \text{ X } \log (a/b)$$

There is some debate about using cent measurement to quantify intervals since it assumes equal temperament as the standard. In the opposite camp is Murray Barbour who, believing that equal temperament is the pinnacle of tuning evolution, takes the equal tempered semitone of 100 cents as the ideal. Just or other forms of semitone are then calculated in terms of their deviation from the ideal. He argues that 'cents representation is the most convenient way of affording comparisons between systems.'[7] Cent values are easier to manipulate since, being logarithmic units, they are added and subtracted, rather than multiplied and divided, which is the case for calculation of ratios, or frequencies in terms of Hz. However converting cents into ratios and vice versa requires a scientific calculator, log tables or computing power.[8]

While it may be better in the long run to think in terms of ratios, cents measurement is useful for comparing acoustically pure intervals with their tempered versions. Kardos continually made comparison between the two forms since knowing how much bigger or smaller than the tempered an acoustic interval is allows the singer to adjust accordingly.

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where a/b is the ratio you wish to convert, and 'c' is its value in cents. 3986.314 is 1200 divided by log(2). Log (2) equals the power of ten which two is, ie $10^{\log(2)}=2$.

7 Murray Barbour, *Tuning and Temperament*, Da Capo Press, New York, 1972, pp. vi-vii.

8 The Macintosh program 'Just Intonation Calculator' enables fast manipulation of ratios. Forty eight tones may be displayed as whole number ratios, frequencies, cents, or synthesizer tuning units. Ratios may be displayed relative to a given fundamental or to successive notes in the scale. Robert Rich, 'The Just Intonation Calculator', *1/1 Just Intonation*, vol. 4, no. 4, 1988, pp. 4-7.

Introduction

Towards a definition of pure intonation

Terms such as pure intonation and just intonation are often used quite loosely to cover a range of concepts. The word 'just' had, in the English language, the sense of 'right', 'proper', 'perfect', 'exact'.[9] The earliest advocates of what was later called just intonation, were looking for alternatives to the prevailing Pythagorean tuning, with its impure thirds and sixths. Because some of their divisions of the scale resembled the second century A.D. theorist Claudius Ptolemy's syntonon diatonon the name 'syntonic diatonic' tuning was adopted by many, including Zarlino, to give their divisions the authority of antiquity. The terms Zarlino used when referring to just intonation include *sintono naturale* or 'natural syntonic',[10] *loro proportioni* or based on true proportions, and *perfetta forma loro*, in true and perfect ratios.[11] Walker credits Zarlino with the

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9 The Oxford English Dictionary (Oxford, 1933, vol. 5, p. 638) gives as the earliest reference General Perronet Thompson's *On the Principles and Practice of Just Intonation* (1850) which is very late, considering that Odington and other 13th century English theorists discuss the practice of singers basing their thirds on 5/4 and 6/5 ratios. See Hugo Riemann, *History of Music Theory*, Raymond Haggh, trans., Da Capo Press, New York, 1974, pp. 98-99, 280-288. John Harrison (1693-1776) refers in his book of 1775, *A Description concerning such Mechanism as will afford a nice, or true Mensuration of Time; together with Some Account of the Attempts for the Discovery of the Longitude by the Moon; and also An Account of the Discovery of the Scale of Musick* to a tuning system in which all the consonances are true in these terms 'as touching the Scale of Musick, As they have...said that it is confessed [meaning by all] that was there but only one Key to be used, all its 8 Notes, should then be tuned perfect Consonances, according to the Diatonick Scale, and as only wherein, a Tune to be play'd perfect; when as, in what they therein think, — that could not be: a foolish Imagination as of old, even that there was such a Scale at all, for no such Thing is in Nature to be found, but what will have Clashes in it, and they very great ones too!' Quotation kindly supplied by Charles Lucy via email correspondence, February, 1995. John Curwen mentions just intonation as being the form of tuning suitable for voices. John Curwen, *The art of teaching and the teaching of music*, Tonic Sol-fa Agency, London, 1875, p. 93.

10 Zarlino, *Sopplimenti*, pp. 140-49, cited in D. P. Walker, 'Some aspects of the musical theory of Vincenzo Galilei and Galileo Galilei', *Proceedings of the Royal Musical Association*, vol. 100, 1973-74, pp. 33-47. Zarlino here in fact distinguishes the artifical system of Ptolemy, the *sintono artifiziale* which has some impure intervals, and the system singers used, which although resembling this corrects the impure intervals because voices are natural, perfect and sang only true consonances on all degrees.

11 Zarlino, as represented by Galilei, *Discourse Concerning the Unison*, in Claude Palisca, *The Florentine Camerata*, Yale University Press, New Haven, 1989, p. 206.

systematization of just intonation, long before the observation of the harmonic series, and explains how

> a very simple mathematical scheme was widely accepted as the basis of a theory of consonance and of an ideal of musical practice, and this scheme was only considerably later, and very slowly, found to fit perfectly the empirically observable behaviour of vibrating strings and columns of air.[12]

The term 'natural' is often linked with 'pure' as attributes of just intonation. Zarlino contrasted the superiority of nature over art, of natural voices over artificial instruments, of *sintono naturale* or natural intonation over the artifice that merely imitated nature, hence was imperfect, such as the *sintono artifiziali* of Ptolemy. The association of 'natural' with just intonation was made even stronger with the 17th century discovery of the harmonic series which gave just intonation a basis in the natural world.[13] Definitions after the scientific revolution frequently derive the concept of the purity and naturalness of just intonation from the harmonic series.[14]

Acoustic purity often means intervals tuned so as to be beatless, possible to achieve only with consonant intervals greater than a minor third. Intervals in this set that are smaller than a minor third derive their validity presumably because they combine to form larger consonances that are beatless. Yet 'pure intonation' is rarely free of beats, since, as Helmholtz pointed out, notes other than unison, octave and, to all extensive purposes, the perfect fifth, will generate beats between different aural harmonics and also between the combination tones of the fundamental note. The simpler the ratio, the less interference, or beating,

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12 D. P. Walker, op. cit., pp. 33.

13 See Rudolf Haase, 'Music and natural law: What is "research into basic harmonic principles"?', *Morgen*, vol. 10, no. 46, 1986, pp. 100-103, for a discussion of this. It includes an analysis of the philosophical consequences of different views of universal harmony drawn from various natural sciences.

14 B. L. Green points out that theorists of the 18th century were looking to the physical sciences for the foundations of music and found a convincing basis in the newly discovered phenomenon of the harmonic series, which provided ratios in a 'fixed order that can be confirmed by concurrent phenomena, by natural successions of tones, and by the synthesis of tonal mixtures.' B.L. Green, *The Harmonic Series from Mersenne to Rameau*, PhD diss., Ohio State University, 1969, p. 19. Harry Partch resists any attempt to base pure intonation on the harmonic series and sticks to monochord-derived ratios. His definition is 'a system in which interval-and scale-building is based on the criterion of the ear and consequently a system and procedure limited to small-number ratios; the initial interval in Just Intonation is 2/1, and stemming from this are the wealth of musical intervals inherent in small-number tonal relationships. Just Intonation is the generic term describing this procedure.' op. cit., p. 71.

between upper harmonics and combination tones. The strength and audibility of these beats will determine how consonant a given 'purely tuned' interval is. Partials of vowels and other timbral qualities have an effect, as does amplitude and frequency.[15]

The term Kardos uses is 'pure intonation' (*tiszta intonálásra*). Although he does not define the term, my analysis of the assumptions inherent in his use of the term is as follows. Pure intonation refers to intonation that uses intervals with simple frequency ratios modelled on those found in the harmonic series in order to maximise acoustic purity. It also favours intervals with simple ratios which are regarded as more consonant, simple to sing, more 'natural' (*természetes*) or 'pure'. The set of intervals that fall into this category are, in order of consonance factor, 1:1, 2:1, 3:2, 4:3, 5:4, 6:5, 9:8, 10:9, 16:15. All the intervals in this set are superparticular (n+1:n), the epimoric proportion given special status by theorists from Pythagoras and Ptolemy through the Renaissance.[16]

What distinguishes Kardos's pure intonation from what was traditionally understood by the term 'just intonation', is that pure intonation is not a fixed system which defines in advance the precise ratio for every note in a scale. Even extended just intonation[17] is often a highly complicated set of rules used to

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15 For a discussion of the effects of different vowels see Hermann Helmholtz, *On the Sensations of Tone*, Dover Publications Inc, New York, 1954, pp. 103-119.

16 See Barker, *GMW II*, pp. 272 for an analysis of Ptolemy's use of epimorics (n+1:n) The Pythagoreans believed concords had to be multiple (mn:n) or epimoric as did most Greek theorists. For Boethius's view of the value of superparticular ratios see Boethius, *Fundamentals of Music*, Calvin M. Bower, trans., Claude Palisca, ed., Yale University Press, New Haven, 1989.

17 Extended just intonation is a system of just intonation extended by the addition of all such extra notes as required to achieve pure tuning on all degrees of the scale in major and minor keys and chromatic passages. Such tuning means that most of the chords are pure but often the melodic line is severely distorted and not all chromatic chords can be tuned with each component interval pure. The assumption is that such additions are still part of a fixed system which tries to 'set in concrete' the solution for every possible situation. See Easley Blackwood, op. cit., pp. 91-128. Another example of extending classic just intonations is that of David B. Doty 'Tutorial IV: The nine-tone diatonic scale', *1/1 Just Intonation*, vol. 3, no. 1, 1987, pp. 14-15. Doty argues that a just scale suitable for triadic harmony includes, in addition to the Syntonon Diatonic (1/1, 9/8, 5/4, 3/2, 5/3, 15/8), the pitches 10/9 and 21/16, which provide correct triads on the second and seventh degrees as well as a just dominant seventh chord. Many more examples emanating from contemporary experimentation with just intonation could be cited. The Just Intonation Network is a group of composers, instrument designers and theorists, mostly American, who are exploring the possibilities of just intonation, particularly in composition. Since the majority of these compositions are for instruments (the use of electronic music is strong) there is the problem of specifying in

predetermine the ratio of every interval used in advance. Kardos's pure intonation is a method for obtaining optimal tuning solutions by making flexible adjustments depending on the numerous, often complex factors in any given situation. Such solutions are applied to modal, pentatonic and infra-pentatonic scales as well as diatonic scales. He writes:

> The naturalness of a living intonation is expressed in the transitions of the colour changes where musicians immediately react to the finest nuances which are musical vibrations. This ideally occurs in the quality of a higher order of singing which is sensitive, flexible, and free. This does not require the singer to make the extraordinarily complex mathematical computations that would be required to analyse and produce a sound that is fixed, determined for a particular note once and for all. Rather, it requires a relatively simple grounding in basic principles and aural discrimination of a very high order. This can develop into an acuteness of hearing producing a super sensitivity to nuances of pitch. Certain necessary corrections are automatically resolved in the living, natural movements and flow of music. These corrections can be marked and instrumentally measured and exist both theoretically and practically. However there are often insoluble contradictions when the requirements for horizontal and vertical purity clash. Only by the compromise of one can optimal purity be reached. This shows the limits of mathematical purity of intonation.[18]

Singers are capable of greater flexibility than instruments with fixed pitch and can, with training as provided by Kardos, make instantaneous microtonal adjustments to achieve the optimal tuning solution in a given work. The complexities of reconciling linear and vertical factors requires an intonational analysis of every choral work. A knowledge of the broad principles of intonation is applied to form an individual solution in each case. Kardos outlines these basic principles with numerous exercises and examples. He applies the principles to the analysis of certain works as illustration or case study, but his contribution is

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advance the exact ratio for each scale degree or tone in a given work. Many experiment with using a particular set of notes much like modes, and generate different scales that exceed the limits of traditional just intonation as understood by Renaissance composers, but which are still bound by the principles of just intonation, eg. whole number ratios. Just intonation in America began with Harry Partch, (1901-1974). The goal of theorists using just intonation in previous centuries was to find the most ideal tuning for a culturally predominant scale. The goal of twentieth century composers was to discover or create a tuning that served their compositional goals, which often meant generating their own scales. Until the 1950s Partch was the only American composer devoted to just intonation. He was joined in the 1950s by Lou Harrison, and Ben Johnston. The list of composers has since grown, and the parameters of the definition of just intonation has also been stretched. For more information consult the network's journal, *1/1 Just Intonation.* I should add that Kardos's use of the term is closer to that of the Just Intonation Network, for whom Just Intonation is 'not the name of a particular scale, nor is it tied to any particular style. It is, rather, a set of principles which can be used to generate a virtually infinite variety of scales and chords which are applicable to any type of tonal music, regardless of style, or of historical or ethnic provenance.' David. B. Doty, 'Editorial', *1/1 Just Intonation* vol. 1, no. 1, 1985, p. 15. For Patch's use of just intonation see *Genesis of a Music,* 2nd ed., Da Capo Paperbacks, 1974.

18 Kardos, *Two Choral Works,*p. 167, my translation (abridged).

to give the tools necessary for choral or aural trainers to make their own analyses. In more complex works he shows that it is not always possible to reconcile the various problems and compromise solutions must be found. The melodic line may have to distort itself for the sake of the purity of the harmony, or where the melodic line is paramount, the harmonic purity may be sacrificed momentarily. In living intonation these become part of the interweaving of consonant and dissonant relationships that come to their point of resolution in a final cadence of acoustic purity. Kardos believed that

> intonation that is living, sensitive, and flexible is not primarily based on an exhaustive theoretical analysis of works but on musical education which is long lasting and conscious, from the most elemental to the highest degree. At the same time, when confronted with complexities in intonation, analysis provides an invaluable help. A conscious overview and knowledge of the structure of small elements of a work saves a lot of useless repetition.[19]

In summary, pure intonation uses ratios that are superparticular, whose numerators and denominators are whole numbers, and in which, ideally, all consonances are free of beats.

Pál Kardos—biographical note[20]



Pál Kardos was born on February 9, 1927 in Cegléd, Hungary. His first musical training was at the Teachers Training Institute in Nagykörös, where he received his first diploma in 1945. He studied in the Choral Conducting Faculty of the Liszt Academy, Budapest from 1947, receiving a diploma in 1952. From 1953 to 1961 he taught solfege in Miskolc at the Music Conservatory directing the Miskolc Conservatorium choir from 1955.

He taught choral conducting at the Teacher's Training High School, Szeged, between 1961 and 1972. There he formed his own mixed choir which became known as the *Béla Bartók Chamber Choir of Szeged*. In 1972 he took a

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19 ibid., p. 167.

20 From notes generously supplied by Kardos's son, Gabor Kardos, by mail in 1991.

position at the Zoltán Kodály Pedagogical Institute, Kecskemét where he remained until his death in 1978.

He gave numerous concerts, lectures, workshops in Hungary and overseas and was the author of over fourteen books and articles on the pedagogy of vocal intonation. These publications are classic texts used by music educators and students of choral conducting in Hungary. To review some of the more important, *Egyszólamúság az énekkari nevelésben - kézirat*[21] deals with intonation in unison work in choral education. 'Foundations of Education towards Pure Intonation'[22] (in English) outlines a step-by-step pedagogy of pure intonation in unison, pentatonic song material, beginning with the easiest interval to intone purely, the descending minor third, progressing to the more complex intervals encountered in pentatony. *Kórusnevelés - Kórushangzás*[23] is his classic text on the pedagogy of acoustically pure intonation as applied to *a cappella* choral music. It describes the harmonic series, ratios of intervals in the series, application of pure intonation to choral music and extensions of just intonation necessary in particular contexts. It contains numerous musical examples for practice and is accompanied by a practical handbook of musical excerpts. *Karvezetés III*[24] is a more general work for less erudite music teachers, covering the same areas but applied only to unison material such as is used in the classrooms in Hungary. It contains an analysis of Kodály's intonational two-part exercises, *Let Us Sing Correctly.*[25] 'Két kórusmüve a tiszta intonálásra nevelés tükrében'[26] analyses in detail the intonation of two choral works for children's chorus, *Ne hagyj itt*! and *Bolyongás* by Béla Barkók. *A Kromatika*

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21 Pál Kardos, *Egyszólamúság az énekkari nevelésben - kézirat.* [*Unison work in Choral Education*], Csongrád megyeés Szeged mj. város Népmüvelési Tanácsadója, 1968.

22 Pál Kardos, 'Foundations of Education towards Pure Intonation', Kodály Seminar, Kecskemét, 1972.

23 Pál Kardos, Pál, *Kórusnevelés - Kórushangzás* [*Choral Training, Choral Tone*], Zenemükiadó, 1969, 1977.

24 Kardos, Pál, *Karvezetés III: Intonálás* [*Choral Direction III: Intonation*], Tankönyvikiadó, Budapest, 1970.

25 Zoltán Kodály, *Énekeljünk Tisztán,* Editio Musica, Budapest, 1941.

26 Pál Kardos, 'Barkók két kórusmüve a tiszta intonálásra nevelés tükrében (Ne hagyj itt!, Bolyongás) [Two Choral Works of Bartók in the Mirror of Education towards Pure Intonation]' in *az ének-zene tanítása* 1970, vol, 4-5. szám. Unpublished translation and exposition of this article by Judith Pickering and Marta Moduna.

intonálásra[27] outlines the application of just intonation to chromatic work. These titles indicate the practical orientation of his work within the framework of a national music educational system which so heavily focuses on singing as the foundation of aural culture.

Outline of approach

Kardos structured his works with a pedagogically driven order, rather than a subject-logic approach. Thus he began with the easiest intervals to tune moving on to the more difficult, or from the simplest musical examples to the most complex. I have also used an order based on a hierarchy of pedagogical complexity but for the ease of the reader not versed in relative solfa I do so in categories of intervals rather than scale degrees. A system of music is, in part, an organisation of contextual configurations of pitch relationships. A tone in music is always a relation to one or more other tones, heard or implied. The tuning of every scale degree depends on its function in the melody or harmony, on its structural relationship to other intervals and notes. A tone such as c' has a different 'colour' when functioning as a tonic in C major than it does when functioning as the subdominant degree of G major. The same tones represent different degrees depending on tonal context, and in fact will change their intonation accordingly. The notes c', d', e', f' functioning as scale degrees I, II, III, IV in C major will be tuned differently from c', d', e', f' functioning as scale degrees III, IV, V, VI in the natural A minor or Aeolian mode. As all intervals, all scale degrees are interconnected, and their tuning varies depending on these relationships, it is not always advisable to follow a strict logical or linear order.

To demonstrate the validity and applicability of his theories I have selected new musical excerpts and made my own analyses of choral works based on his principles. I have added my own diagrams, charts and musical examples to clarify his ideas. Diagrams by Kardos or authors other than myself are so indicated with either a footnote or label beneath the diagram.

Chapter 1. Kardos's work comes from within a particular historical and nationally based system of aural training with its own set of assumptions and methods. Chapter 1 is a brief description of the Hungarian system of aural

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27 Pál Kardos, *A Kromatika intonálásra [Chromatic Intonation]*, Különlenyomat a Szegedi Tanárképzö Föiskola Tudományos Közleményeiböl, 1971.

training adequate to equip the reader with the necessary tools to understand his nomenclature, especially tonic solfa. A critique of the method is included.

Chapter 2 contains a historical overview of theories and debates on tuning and the interaction between scientific discoveries and musical theories of consonance and intonation. Such knowledge is necessary for well informed, judicious application of Kardos's ideas in terms of historical performance practice as well as for understanding the historical background to his ideas. The validity of his work needs to be tested against both historical development of theory and scientific understanding. The state of scientific knowledge at a given period of musical history has a crucial bearing on the relationship between contemporary music theory and compositional/performance practice. The harmonic series had no place in the theoretical writings, debates and practices of ancient Greek, medieval and Renaissance theorists of tuning, since it was not fully understood until the 17th century. Thus tuning based on the harmonic series could only date from that time. This is another fundamental difference between the conceptual basis of Kardos's acoustically pure intonation and the Renaissance adherents of just intonation. I give a brief history of the development of scientific knowledge about the harmonic series and the effect of this knowledge on music theory. I explain scientific concepts such as beats, combination tones and Helmholz's theories.

Chapter 3 introduces the theoretical principles of Kardos's system of tuning:

- **3.1** outlines how Kardos uses the harmonic series to derive a set of intervals that maximises acoustic purity in intonation;
- **3.2** describes how intervals generated from the harmonic series can be arranged to form major, minor, modal and pentatonic scales; and
- **3.3** shows the inherent contradictions of any tuning system based on pure ratios.

Chapter 4 deals with some basic principles of choral and vocal technique necessary for good intonation.

Chapter 5 deals with the characteristics of each interval in its relationship to other intervals and tonal systems:

- **5.1** compares acoustically pure and tempered intervals;

- **5.2** deals with the intonation of triads, on the basis of Kardos's assertion that the major triad forms the acoustic centre in pure intonation;

- **5.3** contains a lengthy essay on the intonation of the major second, since the flexible use of two forms of whole tone is the hallmark of Kardos's pure intonation;

- **5.4** deals with the semitone, and the scale degrees with which it is associated, particularly the 4th and 7th degrees in major;

- **5.5** covers the intonation of various forms of sixth, and more complex forms of third;

- **5.6** deals with the minor and major seventh, and the dominant seventh chord;

- **5.7** analyses the various forms of tritone and diminished fifth; and

- **5.8** surveys the intonation of chromatic passages.

Chapter 1 Kardos in context: the role of solfege in Hungarian aural culture

Aural training in Hungary is based on the use of tonic solfa. Kardos's work presupposes this system, thus the following account of Kardos's theories will utilise tonic solfa as the basis of the account. Its advantages are seen to be the way it cements tonal relationships and aids aural memory of the intonation of intervallic relationships.

> Solmisation works because it associates phonemes with various musical constructs that, with repetition, enable students to audiate sound from sight, and perceive sound to sight.[28]

The syllables convey meanings of musical structure, and thus the user is made aware of the structural function of a note. The way solfa is taught in Hungary, the solfa syllables also implicitly convey intonational relationships, based on the principles of pure intonation. Tonic solfa means that the tonic of a major scale is always *do* and the tonic of a minor scale is always *la*. Thus semitones in any mode are defined in relation to the Ionian mode. The notes *mi—fa* and *ti—do* are always semitones; they are located on different scale degrees in different modes. Historically this is a very similar approach to solmisation based on the Guidonian hexachord system, except that it has been extended to the octave of the diatonic scale. One difficulty in modal music is that a term such as *ti* refers in one mode (Ionian) to the leading tone, in another (eg Aeolian) to the 2nd degree. Thus *re* is the 2nd degree in Ionian; as the 4th in Aeolian it is in fact a comma lower, so it is not really the same note.

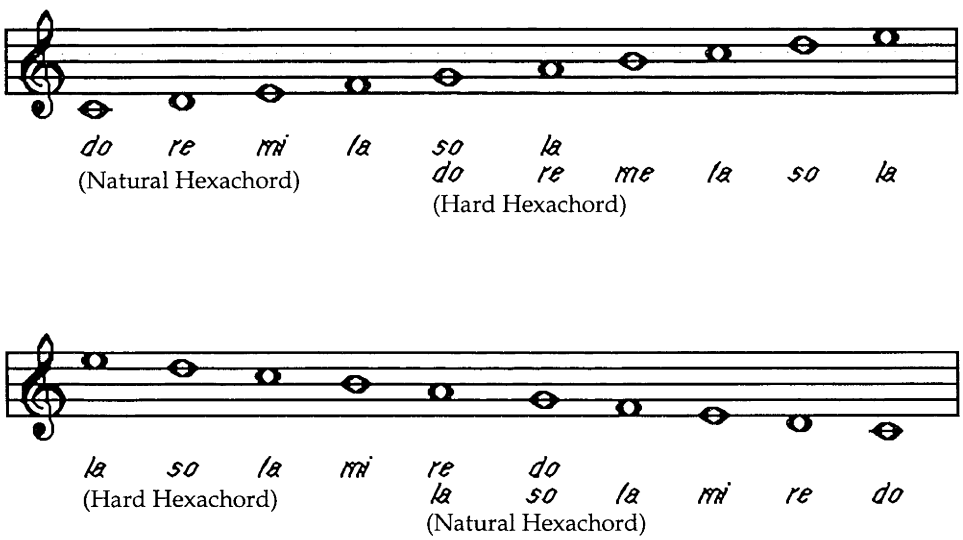
Is there a historical performance practice issue concerning the appropriate solmisation for medieval and renaissance music? Should we solmize with the solmisation in use in the period to really enter into the sound world and assumptions of the music? In other words should we use the Guidonian hand system for medieval music and the form advocated by Thomas Morley for English Renaissance music? The ability to determine the appropriate hexachord for solmization gives the necessary apparatus to make correct *musica ficta* decisions (which itself was defined as that which went beyond the hand—steps

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28 Timothy A. Smith, 'A Comparison of Pedagogical Resources in Solmization Systems', *Journal of Music Theory Pedagogy*, vol. 5, no. 1, 1991, pp. 3.

not found in the hand were 'feigned, false, coloured, acquired music').[29] I would hypothesise too, that the size of intervals in terms of ratio may have been encapsulated in the hexachord system.

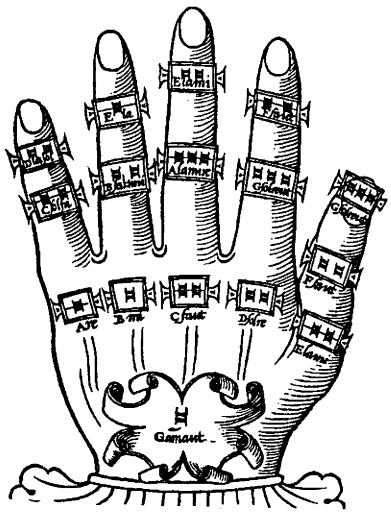
Hexachordal solfa is associated with the name of Guido d' Arezzo (995-1050) and was the precursor of 'movable *do*' systems. What Guidonian solfa has in common with the later forms such as are used in Hungary today is the relative nature of the system. It described a set of six tones and one semitone, whose internal relationships and placement of the semitone remained constant, but the starting pitch did not. The solfa syllables used were: *ut, re, mi, fa, sol, la.* The semitone was that between *mi* and *fa*. This set of six notes could begin on c' (natural hexachord), g' (hard hexachord) or f' (soft hexachord). When a melody went beyond the range of one hexachord the singer 'mutated' into the next hexachord:



A visual, mnemonic device used was the Guidonian hand: the fingers of the hand were used to map out the hexachordal relationships and mutations:

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29 *musica ficta, falsa, colorata, conjuncta, acquisita.* Thus Petrus frater dictus Palma ociosa, writes in 1336 'false music is the one which cannot be found in the gamut of the hand.' Prosdocimus de Beldemandis writing in 1425-8 says the steps contained within the hand are *musica vera,* all others *musica ficta.* Karol Berger, *Musica ficta,* Cambridge University Press, Cambridge, 1987, p. 12.



Solmization with the Guidonian hand was used for reading music for over 500 years. Tinctoris (1435—1511) devoted an entire treatise (*Expositio manus*, 1477) to its exposition. Other major exponents were Franchinus Gaffurius (1477-1522), and Thomas Morley (1557-1603). The 'hand' (*manus*) was the ancestor of the gamut or scale. It was a set of steps, referring to pitches whose definition was relative to other steps in the gamut representing the tonal material of music.[30] The ambitus of Gregorian chant meant a hexachord was adequate for the gamut of solfa syllables needed. The most important thing was to specify the location of the half-step. The hexachord was extended to the octachord by the addition of *si* which later became *ti* in some countries. Mersenne advocated extending the hexachord by the syllable *ci*.[31]

The form in use in Hungary was adapted from John Curwen's tonic solfa of the

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30 Berger, ibid., p. 3.

31 Albion Gruber, 'Mersenne and Evolving Tonal Theory', *Journal of Music Theory*, vol. 14, no. 1, 1970, p. 62.

19th century.[32] Curwen's syllables took just intonation into account.[33] Further, he was well aware of the two forms of supertonic required in just intonation. An illustration of the handsigns describing the individual qualities of each degree *ray* (*re*) is described as the 'roving or hopeful tone,—the Major supertonic, making with Fah and Lah the Chord R, in which case it is naturally sung a komma flatter, and may be distinguished as *Rah*.'[34] Thus there are two specified forms of supertonic, distinguished by two solfa syllables:

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32 John Curwen was a Congregationalist minister who wanted to find the best method of teaching Sunday school pupils to sing. He found what he was looking for in the work of Sarah Glover, whose scheme used a notation based on sol-fa syllables. He made his own modifications to the system and called it 'tonic sol-fa' to distinguish it from fixed *do* systems. He devoted his whole life to teaching this method, setting up his own printing press to publish teaching materials, and founding the Tonic Sol-fa College in 1869. It became the recognised method of teaching in English schools. Bernard Rainbow, 'Tonic Sol-fa', *Grove Dictionary of Music and Musicians*, vol. 19, Stanley Sadie, ed., Macmillan, London, 1980, pp. 61-65. See also his 'Rise of Popular Music Education in 19th century England', *Victorian Studies*, vol. 22, 1986, pp. 25-49.

33 Sydney Kleinman, *Moveable Solmization from Jean-Jacques Rousseau to John Curwen*, PhD diss., University of Paris-Sorbonne, 1971. Kleinman traces the development of moveable solfa from Rousseau to Curwen and shows how the *Das Tonwort* of C. Eitz had 53 syllables based on just intonation, and how Curwen took just intonation into account in his Tonic sol-fa.

34 John Curwen, 'Standard Course of Lessons and Exercises in the Tonic Sol-fa Method of Teaching Music', in Bernard Rainbow, 'Tonic Solfa', *New Grove Dictionary of Music and Musicians*, vol. 19, Macmillan Publishers, 1980, p. 65.

FIRST STEP



SOH

The GRAND or *bright* tone,—the Major DOMINANT, making with *Te* and *Ray* the Dominant Chord,—the Chord S, and with *Fa* also the chord ^{7}S.



ME

The STEADY or *calm* tone,—the Major MEDIANT, making with *Soh* and *Te* the rarely used chord M.



DOH

The STRONG or *firm* tone,—the Major TONIC, making with *Me* and *Soh* the Tonic Chord, the Chord D.

SECOND STEP



TE

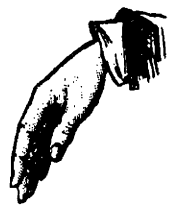
The PIERCING or *sensitive* tone,—the Major LEADING TONE, making with *Ray* and *Fah* the weak Chord T.



RAY

The ROUSING or *hopeful* tone,—the Major SUPERTONIC, making with *Fah* and *Lah* the Chord R,—in which case it is naturally sung a Komma flatter, and may be distinguished as *Rah*.

THIRD STEP



LAH

The SAD or *weeping* tone,—the Major SUBMEDIANT, making with *Doh* and *Me* the Chord L.



FAH

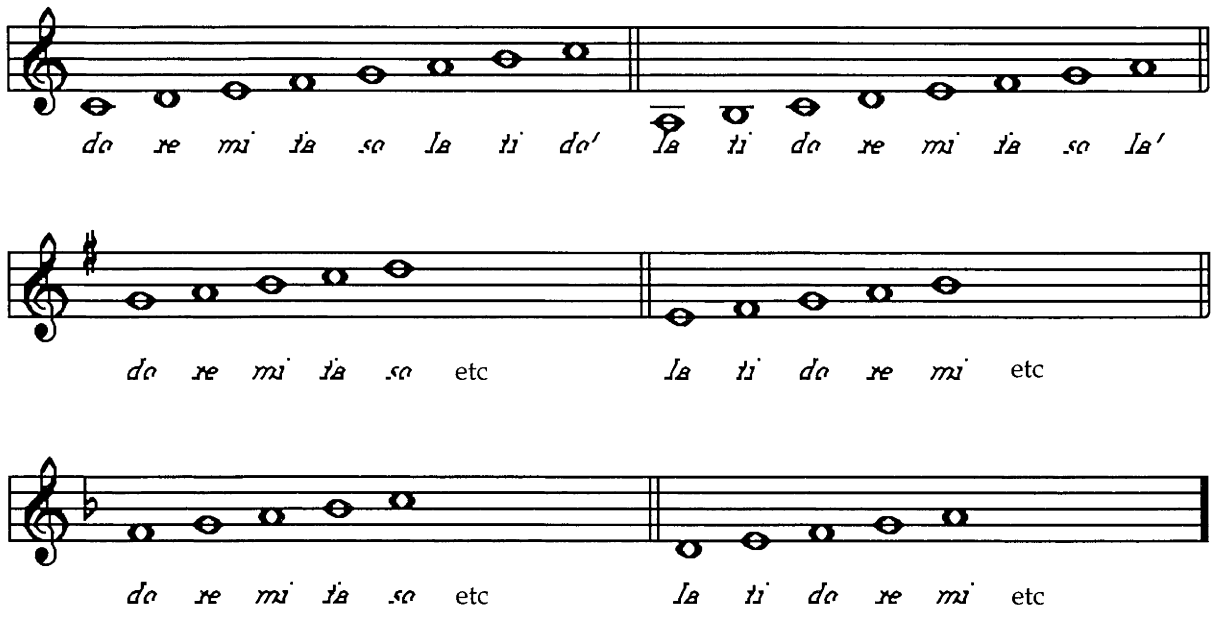
The DESOLATE or *awe-inspiring* tone,—the Major SUBDOMINANT, making with *Lah* and *Doh*, the Subdominant Chord,—the Chord F.

John Curwen, *Standard Course of Lessons and Exercises in the Tonic Sol-fa Method of Teaching Music*, 1880

The shape of the hand gestures provides a visual representation, not only of the seven notes of the octatonic scale, but also indicates their intonational character. The upward direction of *ray* indicates the height of the supertonic being a 9:8 tone from *doh*, likewise the pointing *te* points to its note of resolution, the tonic and shows it is a big whole tone from *lah*. *Fah* points down to *me* which is its note of resolution.

Kodály, wanting to develop a nation of musically literate people, decided that Curwen's method could help achieve that aim, and thus adopted Curwen's Tonic Sol-fa. In this system the tonic of a major key is *do*, of a minor key it is *la*. Thus it is sometimes called the 'la-minor' system. He also adopted the handsigns.

The following diagram outlines the solfa for C major and A minor, and indicates the transposition of solfa syllables to other keys:



Chromatic alterations are solmised as follows:



Concerning solfa syllables for the modes, the Hungarian system classifies the modes according to their Greek names as understood by Glarean, but they do not distinguish between authentic and plagal forms. Each mode uses a different syllable to represent the 'final' or 'tonic'. It thus allows sightreading in any mode based only on seven syllables. It is sometimes called 'key-signature moveable do' since all seven modes sharing the same key signature can be sung with the same solfa syllables representing the same absolute pitch. The solfa for the modes is thus:

Ionian	*do*	*re*	*mi*	*fa*	*so*	*la*	*ti*	*do*					
Dorian		*re*	*mi*	*fa*	*so*	*la*	*ti*	*do*	*re*				
Phrygian			*mi*	*fa*	*so*	*la*	*ti*	*do*	*re*	*mi*			
Lydian				*fa*	*so*	*la*	*ti*	*do*	*re*	*mi*	*fa*		
Mixolydian					*so*	*la*	*ti*	*do*	*re*	*mi*	*fa*	*so*	
Aeolian						*la*	*ti*	*do*	*re*	*mi*	*fa*	*so*	*la*
Locrian							*ti*	*do*	*re*	*mi*	*fa*	*so*	*la*	*ti*

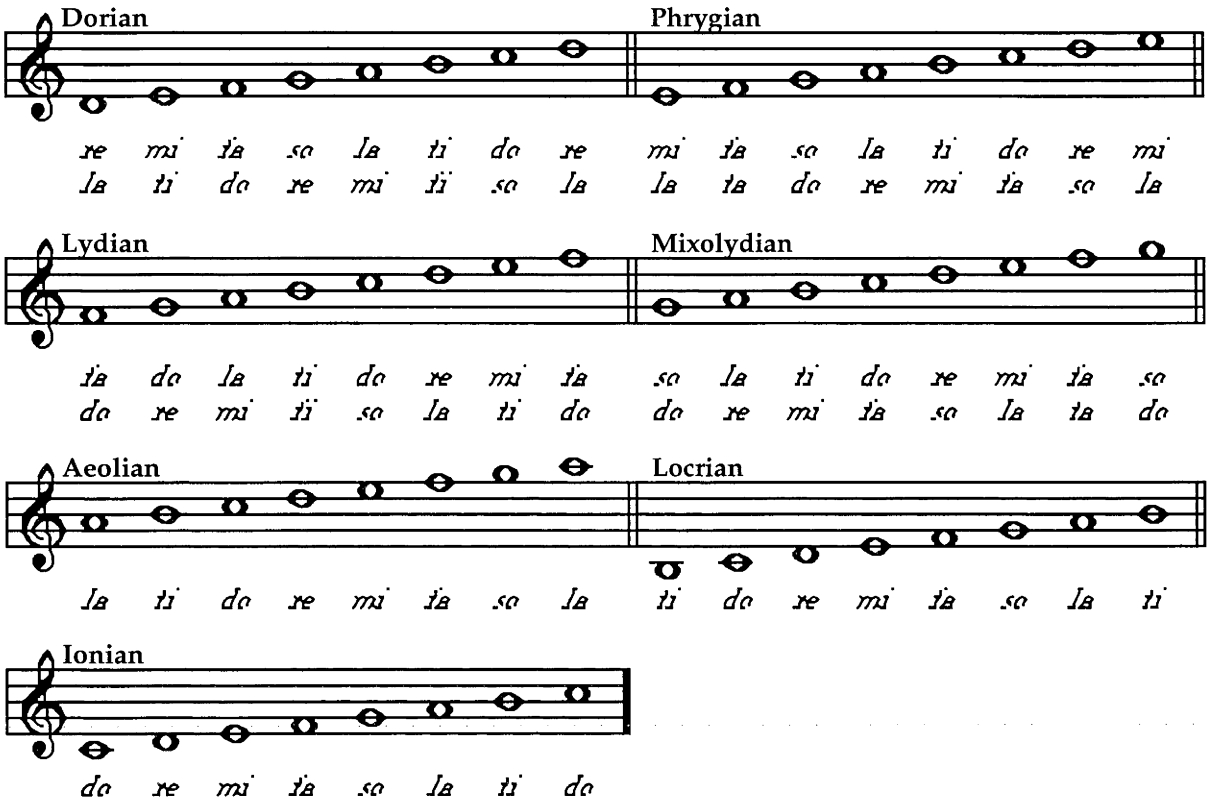
Since three modes begin with a major third, an alternative is to solfa these three with the solfa of the Ionian with a solfa alteration of the note that is different. Lydian is the same as Ionian except for the raised 4th degree so it can be solmised *do, re, mi, **fi**, so, la, ti, do.* Mixolydian is the same except for a flattened 7th, thus *do, re, mi, fa, so, la, **ta**, do* can be used. Likewise the three that begin with a minor third can be solmised according to the Aeolian mode. Thus Phrygian can be solmised *la, **ta**, do, re, mi, fa, so, la* and Dorian *la, ti, do, re, mi, **fi**, so, la.*[35] The Hungarians stress the transposing nature of modes; any mode can begin on any absolute pitch, eg Dorian in G, Phrygian in A etc.[36]

Modal scales as described in the Hungarian system, with the two alternative ways of solmising them, are as follows:

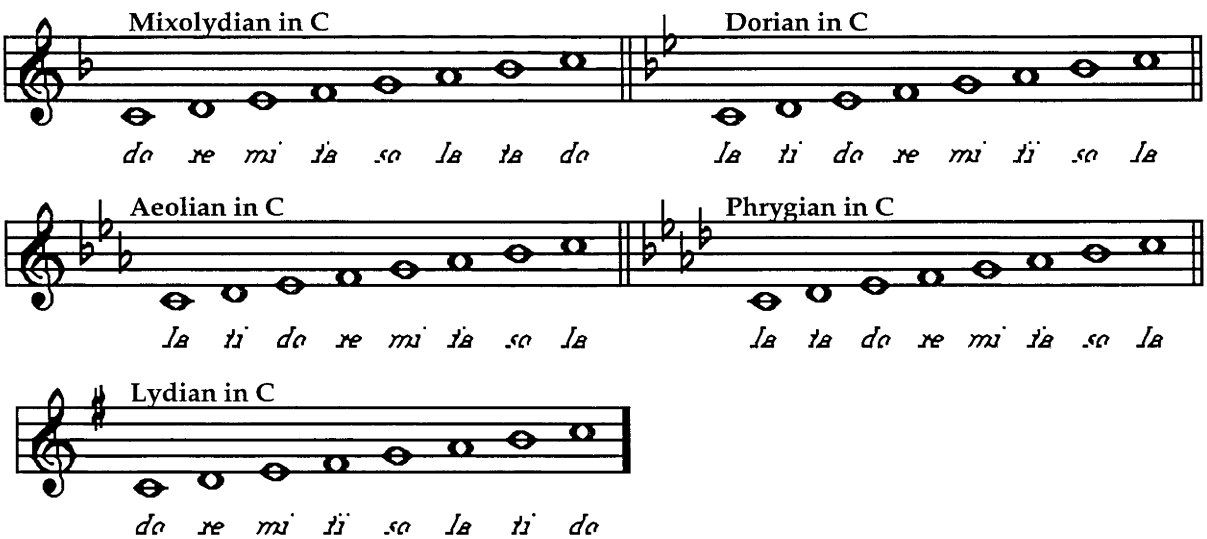
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35 Zarlino also stressed the major/minor quality of the modes. Zarlino, *On the Modes*, (Part IV of *Le istitutioni harmoniche* 1558), Vered Cohen, trans., Yale University Press, New Haven, 1983. See Wienpahl, 'Zarlino, the Senario, and Tonality', JAMS, vol. 12, 1959, p. 30.

36 Mersenne was among the first to stress that it is the pattern of tones and semitones that defines the mode, not the absolute pitch of the starting note, and that it does not matter where the modes are begun. Marin Mersenne, *Treatise on Consonances, dissonances, genera, Modes, and Composition, Harmonie Universelle contenant la théorie et la pratique de la musique*, Paris, 1636, cited by Albion Gruber, op. cit., p. 60.



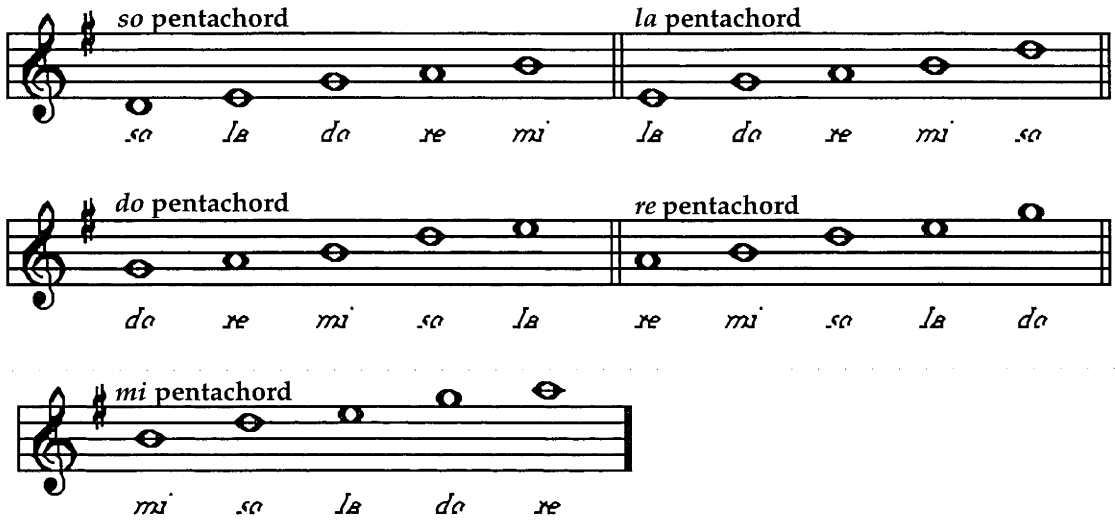
Some examples of modes in different 'keys':



Pentatony

Kodály and Bartók, based on folk music research in eastern European regions, concluded that pentatony had a strong influence on Hungarian musical culture

and exploited its perceived pedagogical virtues in aural training.[37] Kodály advocated avoiding music containing semitones in the early stages of training since 'correct intonation can better be achieved if semitones are postponed until whole tones are sung with sufficient assurance.'[38] Kardos applied principles of pure intonation to a great deal of pentatonic material.[39] Another interesting feature of pentatony is its absence of the feeling of a tonic. Each scale degree can function as the 'final'. There are thus five forms of anhemitonic pentatonic scale:



It is interesting to note that many cultures have used a form of solfa—in Indian music it is *Ni, Sa, Re, Ga, Ma, Pa, Dha.* The ancient Greeks used the syllables τε τω τη τα (pronounced *teh, toe, tay, tah*).[40]

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37 Not all folk music from the Hungarian region is founded on pentatony. See László Vikár and Gábor Bereczki, *Cheremis Folksongs,* Hungarian Academy of Sciences, Budapest, 1971; their *Chuvas Folksongs*, 1979, *Votyak Folksong*, 1989, and *Tatar Folksongs*, forthcoming.

38 Zoltán Kodály, *The Selected Writings of Zoltán Kodály*, Boosey & Hawkes, London, 1974, p. 217.

39 David Canright explores the virtues of pentatony for composers using just intonation in David Canright, 'Pentatonics I have known', *1/1 Just Intonation*, vol. 1/2, Spring 1985, pp. 1, 11-13.

40 Based on the four tetrachordal tones, they were sung in descending order to the notes of every genus and nuance (*chroai*). John Chalmers Jr., *Divisions of the Tetrachord*, Frog Peak Music, Hanover, 1993, p. 5.

Chapter 2 Historical, scientific background

2.1 Ancient Greece

A basic knowledge of ancient Greek harmonic theory is essential to any discussion of tuning, since later traditions of musical science and debates about tuning can be traced back to ancient Greece. Reliance on the authority of Greek treatises was one of the major ways Renaissance theorists justified their arguments. The subject is extraordinarily complicated, and it would be beyond the scope of this work to give more than a summary necessary for the understanding of the present topic.

Two major and opposing currents which flowed into Renaissance thinking and beyond were represented by Pythagorean numerology and Aristoxenean empiricism,[41] one based itself on the authority of reason (*logos*) the other on experience (*phainomenon*).[42] A third major force, as represented by Ptolemy and others,[43] argued that both were necessary for knowledge. This offered an

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41 Several authors have recently argued that these two traditions are not as opposed as they seem. Aristoxenus is stereotyped as the empiricist who based his theory on musical practice and argued that the human ear is the arbiter of correctness of pitches and harmonic functions. Litchfield argues that Aristoxenus's *Harmonics* is less based on musical practice than is supposed. He examines Aristoxenus's discussion of genera, his relationship to equal temperament, his proof that a fourth is comprised of two and one half tones, and the philosophical background to his theory. Malcolm Litchfield, 'Aristoxenus and Empiricism: A Reevaluation Based on his Theories', *JMTheory*, vol. 32, no. 1, 1988, pp. 51-75. Norman Cazden, 'Pythagoras and Aristoxenos Reconciled', *JAMS*, vol. 11, 1958, pp. 97-105.

42 Ptolemais in her *Pythagorean Elements of Music*, defined the Pythagoreans as *Kanonikos*, those who basing their science on reason and on measurement construct ratios in connection with attunement (*hērmosmenon*). The Aristoxenians were the harmonic theorists who begin from perception. Quoted in Porph. *Comm.* 22.22, 23-22, in Barker *GMW II*, pp. 239-240. This accords with Didymus's explanation of the difference between the Pythagoreans and Aristoxenians, the former who rely on reason alone, 'taking no further notice of perception' and even if 'perception bears witness' against their theorems, 'it is possible for them to be not in the least embarrassed by this sort of discord, but to pin their faith to reason and dismiss perception as going astray.' Didymus on schools of harmonic theory, first extract, Porph. *Comm.* 26.6-29, in Barker, *GMW II*, pp. 242-243.

43 Ptolemais argued that the Pythagorean mathematical approach and the Aristoxenian empiricist approach should go together, that a combination of reason and perception was required. Extracts preserved in Porph. in Ptol. *Harm.* pp. 22-6, West, *AGM*, p. 239. The date of Ptolemais is unknown, but was before the first century A.D.

alternative to Pythagorean tuning and became the basis of just tuning in the Renaissance.

The extant sources are too fragmentary to adequately reconstruct the history and theory of ancient Greek music. The science of Greek harmonics goes back to the fifth century B.C., yet no treatises survive from this period. Some information about the theories dating from this time are contained in reports of fourth century writers, such as in the quotations of Archytas, the writings of Plato and Aristotle, the Aristotelian *Problemata,*[44] Euclid's *Sectio Canonis* (attributed)[45] and Aristoxenus' *Elementa Harmonica.*[46] Later Aristoxenian writers include Cleonides,[47] Gaudentius, and Bacchius. The earliest complete text on Pythagorean metaphysics of music, that of Nicomachus, dates from the first century A.D.[48] From the second century A.D. there is the *Harmonics* of Claudius Ptolemy. Aristides Quintilianus' *De Musica*[49] dates from about the third or fourth century. Aristides' work aimed to create a compendium which embraced all aspects of music as it applied to human life, God and the cosmos. Both Ptolemy's and Aristides' works also give invaluable information about the works and ideas of earlier writers.

Greek Harmonic and Acoustic Theory

Greek harmonics was a major branch of music theory. The term 'harmonics'

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44 Translation of most of this work can be found in Barker, *GMW I*, pp. 190-204; *II*, pp. 85-97.

45 Thomas Mathiesen, 'An Annotated Translation of Euclid's Division of a Monochord,' *JMTheory*, vol. 19, 1975, pp. 236-259. Euclid was the first to systematically explain the Greater Perfect System, showing how all the notes of the Greater Perfect System can be found on the monochord.

46 English translations include: Aristoxenus, *The Harmonics of Aristoxenus*, Henry Macran, trans., Oxford University Press, Oxford, 1902; Aristoxenus, 'From the Harmonic Elements', *Source Readings in Music History*, Oliver Strunk, trans., Faber and Faber, London, 1952, pp. 25-34; Aristoxenus, *Elementa Rhythmica*, L. Pearson, trans., Clarendon Press, Oxford, 1990.

47 Cleonides's handbook is translated in Oliver Strunk's *Source Readings in Music History*, vol. 1, *Antiquity and the Middle Ages*, W. W. Norton & Co., New York, 1965, pp. 34-47.

48 Nicomachus of Gerasa, *Enchiridion* 6, Flora Levin, trans. Unpublished dissertation, Columbia University, 1967. See also her *The Harmonics of Nicomachus and the Pythagorean Tradition*, American Classical Studies, no. 1, The American Philological Association, 1975.

49 Aristides Quintilianus, *On Music: In Three Books*, Thomas Mathiesen, trans., Yale University Press, New Haven, 1983.

comes from the Greek *harmonia* from the verb *harmozein* meaning to bind or fasten together, to join, to adapt.[50] It is defined in different ways by different authors but concerns itself with tuning, the division of the octave and disposition of notes in a system; (*systēma*)[51] the division of musical space and with acoustic theory.[52] Aristotle called those who calculated interval ratios mathematically and judged them by ear *harmonikoi*. Other writers used the term to refer to those who judge by sense-perception (Aristoxenians).[53]

Division of the monochord

Music theory was based on divisions of a monochord.[54] Dividing the string in half gives a ratio of 2:1. This is the *diapason*, which means 'Through All'. Dividing the string one third from the end gives a ratio 3:2 or *dia pente* which means 'concord through five strings' (*dioxeia* by some Pythagoreans). Next is the interval derived by dividing the string in the ratio 4:3 or *dia tesserōn* or 'concord through four strings'[55] (*syllaba* by the Pythagoreans). The most fundamental intervals in musical harmony thus correspond to ratios of the whole numbers 1, 2, 3, 4.

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50 Solon Michaelides, *The Music of Ancient Greece: an Encyclopaedia*, Faber and Faber, London, 1978, p. 127.

51 According to Theon of Smyrna, a *systēma* is a specifically qualified container of intervals such as the tetrachord, pentachord and octachord. A *harmonia* is an ordering of *systēmata*. Theon Smyrn. 46.20-49.4, in Barker, *GMW II*, pp. 211-213. Compare with Adrastus, in Theon Smyrn. 49.6-62.4, Barker, *GMW II*, pp. 213-214.

52 This is how the term was understood by Aristoxenus, Plato, and Aristotle. See Aristoxenus, *Harmonic Elements*, II p. 36, Plato, *Republic*, II, 617B and Aristotle, *Politics* IV, 3, 1290A.

53 See Theophrastus, fr. 89, Plato, *Phaedro*, 268 de, both cited in West. p. 218.

54 An instrument with one string, although it often had more than one to compare intervals. Also called the canon (hence canonical) it was used to determine the mathematical relations of musical sounds. See Jan van Der Maas, 'Das Monochord, das Instrument des Harmonikers [The monochord, the instrument of the harmonist]', *Schriften über Harmonik* 9, Kreis der Freunde um Hans Kayser, Bern, 1985; Cecil Dale Adkins, *The Theory and Practice of the Monochord*, PhD diss., State University of Iowa, 1963; Cecil Dale Adkins, 'Monochord', *New Grove Dictionary of Music and Musicians*, vol. 12, pp. 495-96. Cecil Dale Adkins, 'The Technique of the Monochord', *Acta Musicologica*, vol. 34, 1967, pp. 34-43.

55 See Adrastus, in Theon Smyrn. 49.6-62.4, Barker, *GMW II*, p. 215.

Tetrachords

The tetrachord was the fundamental module or scalar unit from which more complex scalar and harmonic structures were built. A tetrachord was the interval of 4:3, a *diatessarōn* or perfect fourth subdivided into three smaller intervals. No sub-division of the fourth was considered a concord by any writer in the Greek period. They were melodic intervals. There were numerous theoretical divisions of the tetrachord, but three primary *genera* can be distinguished—the diatonic, chromatic, and enharmonic. Again the precise size in ratio (*chroa*, shade, nuance, hence 'chromatic') of the subdivisions varied from theorist to theorist, but in general the diatonic tetrachord contained in descending order two tones and the remainder (*leimma*, semitone).[56] The chromatic consisted of one and a half tones, semitone, semitone.[57] The enharmonic consisted of ditone, quarter-tone, quarter-tone.[58] By the Classical Period (late 5th century—4th B.C.), tetrachords had been extended by adding another tetrachord, and were joined either by disjunction (*diazeuxis*) with a note between, or by conjunction (*synēmmenōn*) with a common note to form a heptachord or octave. This too was extended in various ways to form the Greater Perfect System etc. Conjunct tetrachords were called *synemmenai*, disjunct tetrachords were called *diezeugmenai.*

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56 Adrastus's explanation of the diatonic is one of the most lucid. To paraphrase his argument: The tone is the difference between the fifth and fourth, the semitone a melodic interval smaller than a tone, but not half a tone since the tone cannot be divided into two equal parts. The voice sings, moving from lower note to next in sequence by making the semitone interval, then a tone, then another tone. The note that bounds the top of this sequence stands in the concord of a fourth with the bottom note. This *systēma* is called a tetrachord and is constituted of the three intervals, semitone, tone, tone. This genus is called 'diatonic' because it travels mostly through tones (*dia tōn tonōn*). In Theon Smyrn. 49.6-62.4, Barker, *GMW II*, p. 216.

57 Adrastus's explanation is that if the voice ascends by a semitone and then by another semitone then the third step has to be what is left of the tetrachord which is an incomposite triple semitone (*trihēmitoniaion*), ibid., p. 216. Barker points out that this is an Aristoxenian explanation, since the Pythagoreans would specify the ratio of the semitones as in Arist. Quint. *De Mus.* 95.19ff. *GMW II*, p. 217.

58 *Diesis tetartemorios*, or enharmonic diesis. An enharmonic diesis was approximately a quarter-tone, but the ratio varies from writer to writer. Diesis is used by some Pythagoreans such as Philolaus to refer to the semitone. Adrastus and others point out that the chromatic genus is very hard to sing, 'whereas the diatonic genus is rather simple and noble and more natural. That is why Plato prefers to adopt the diatonic', ibid., p. 217.

Pythagoras and the Pythagoreans

Pythagoras, legendary founder of philosophy, science and mathematics, was almost deified by his followers and later writers.[59] He was reputed to be the son of Apollo, who himself was the god of music and science. No writings survive, and Aristotle was the first to describe the beliefs of the Pythagoreans in any detail. His teachings were passed on by students, the most famous of whom was Philolaus. Archytas, a contemporary of Plato, was the last in the direct line of students. It was Archytas and Plato who welded Pythagorean numerology with harmonics in such a way as to create the tradition of Pythagorean musicology and acoustics.[60]

Pythagoras, it is thought, was born on the island of Samos in about 580 B.C.[61] He travelled throughout the centres of Greek civilisation, around the Mediterranean, and, according to one biographer, Iamblichus, spent twenty two years studying with priests in Egypt. In these places he was initiated into various religious mysteries and studied mathematical sciences. He eventually established a school at Crotona where he taught arithmetic, geometry, astronomy, and music.

It is thought that Pythagoras brought a diatonic scale based on tuning via a cycle of perfect fifths, fourths, and octaves from Egypt, and that this tuning was known to the Babylonians.[62] Legends abound concerning his discovery of the fundamental ratios based on divisions of a monochord.

No account of Pythagorean tuning can be complete without relating the legend of

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59 Sextus Empiricus talks of his followers deifying him. Sextus Empiricus *Adv. Math.* vii.94-5, *GMW II*, p. 30.

60 Barker, *GMW II*, p. 29.

61 570 B.C. according to David Fideler, in his introduction to Kenneth Sylvan Guthrie, *The Pythagorean Source book and Library: An Anthology of Ancient Writings Which Relate to Pythagoras and Pythagorean Philosophy*, Phanes Press, Michigan, 1987, p. 20. This work contains one of the more influential biographies of Pythagoras, that of Iamblichus, a third century A.D. neoplatonist. Iamblichus, *The Life of Pythagoras*, in Guthrie, pp. 57-156.

62 According to John Chalmers, Pythagorean tuning has been convincingly traced back to Mesopotamia and to at least the 2nd millennium B.C. Babylonian cuneiform tablets translated by Anne Draffkorn Kilmer and M. Duchesne-Guillem contained tuning orders for the Babylonian harp that used the cyclic technique for tuning all twelve chromatic tones. There is speculation that this scale and its tuning doctrines once brought to Greece by Pythagoras were then used to produce the numerous scale divisions and melody types. John Chalmers, *Divisions of the Tetrachord*, Frog Peak Music, Hanover, 1993, p. 5. Further information obtained from personal correspondence with John Chalmers.

Pythagoras' discovery of the ratios,[63] as disseminated by Nichomachus of Gerasa, a Pythagorean of the 2nd century A. D. This legend was transmitted to later periods by Boethius in his *De institutione musica* 1. 10-11.[64] Nichomachus' version is as follows:

> Once upon a time, while pondering with intense calculation whether it might be possible to devise some kind of instrumental aid for the ears which would be firm and unerring, such as vision obtains through the compass and the ruler or even the looking-glass, or the sense of touch obtains with the balance-beam or the invention of measurements; by a miraculous chance, he walked by a smithy and heard the hammers beating out iron on the anvil and giving off the sounds that are the most harmonious in combination with one another, except for one pair. He recognised among them the consonance of the octave, the fifth and the fourth. But he perceived that the interval between the fourth and the fifth was dissonant in itself, but was otherwise complementary to the greater of these two consonances. Delighted, therefore, since it was as if his purpose was being accomplished by a god, he ran into the smithy and found by various experiments that the difference of sound was consistent with the weight of the hammers, but not with the force of the blows, nor with the shapes of the hammers, nor the alteration of the iron being forged. Taking precise note of the weights of the hammers and their downward momentum, which was identical, he departed to his home.[65]

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63 Adrastus, 1st century A.D. also speaks of Pythagoras investigating the ratios through the lengths and thickness of strings, through tension 'arising from the turning of the kollaboi, or attachment of weights, through the holes of the bores or breath-tension in wind instruments, through solid bodies and weights such as discs and vessels. 'For whichever is taken according to one of the ratios mentioned, other factors being equal, it will produce the concord that corresponds to the ratio.' Adrastus, in Theon Smyrn. 49.6-62.4, Barker, *GMW II*, pp. 217-218. Gene Anderson traces the history of the legend in 'Pythagoras and the origin of music theory', *Indiana Theory Review*, vol. 6, no. 3, 1985, pp. 35-61. He shows that by 1492, the year of Gafori's Theorica Musicae, the story of the harmonious blacksmith' had acquired many accretions.

64 Anicius Manlius Severinus Boethius, *Fundamentals of Music*, Calvin M. Bower, trans., Claude Palisca, ed., Yale University Press, New Haven, 1989, pp. 18-19. The story as found in Boethius is essentially the same except that he specifies that Pythagoras examined the weights and found that those that weighted twice as much sounded a diapason, those in the ratio 3:2 produced a fifth, those in the ratio of 4:3 produced a fourth. He experimented by attaching weights to strings, blowing on pipes of different lengths, filling glasses with different amounts of water. He found the same ratios caused the same consonances.

65 Nicomachus of Gerasa, quoted in Flora Levin, *The Harmonics of Nicomachus and the Pythagorean Tradition*, op. cit., p. 70.

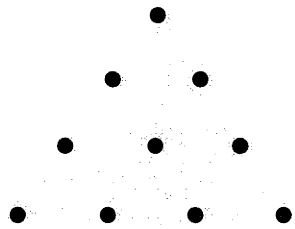


Gaffurio's woodcut[66]

66 Gaffurio's *Theorica musice*, I, 8 of 1492. Gaffurio attributes the blacksmith's story to Jubal before the flood. Jubal is illustrated overseeing six blacksmiths each swinging hammers weighing 4, 6, 8, 12, 16 pounds. Pythagoras demonstrates the tones of bells of different sizes, glasses filled with water, strings stretched with different weights, pipes of different lengths in harmonic ratios 4, 6, 8, 9, 12, 16. In fact the principles derived regarding ratios and sounds produced hold true only for the monochord. With hammers the pitch depends on the metal struck rather than the weight of the hammers. With weights attached to strings the frequency varies as the square of the weights. Bells and water glasses too have complex relationships. Cited in Palisca, *Humanism*, p. 228. James McKinnon traces the medieval provinance of the story. Medieval music treatises granted priority to Pythagoras rather than Jubal, after the 13th century this attribution was reversed. *Musical Quarterly*, vol. 64,

We shall see the inherent fallacies in the science of the above, but its authority akin to 'gospel truth' continued to the time of Vincenzo Galilei. It does furnish the ratios 2:1, 3:4, 4:3 and the epidoic ratio of 9:8 that is the difference between the hemiolic and epitritic proportions.

For the Pythagoreans the numbers 1-4 contained all harmony, all creation. 'What is the oracle at Delphi?—*Tetraktys*.'[67] *Tetraktys* means tetradizing or 'of ten', a name given to the arrangement of the first four integers in a triangular figure:



This contained the ratios 2:1, 3:2, 4:3, and was the numerical paradigm of the Pythagorean system.[68]

Tuning

Philolaus[69] was the Pythagorean who, in the late 5th C BC, established mathematical values for intervals other than those contained in the tetraktys. This allowed a mathematical description of a complete musical scale with all its possible internal divisions. The difference between the 3:2 and 4:3, the 9:8 epogdoon was used to subdivide the 4:3. A 4:3 so divided contained two epogdoa and a remainder of ratio 256:243. This he called *diesis*,[70] but it is more usually

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no. 1, 1978, pp. 1-28.

67 Iambl. *De vita Pythagorica* 82, in Guthrie, op. cit., p. 57. Also cited in West, *AGM*, p. 235.

68 It contained in addition the geometric principles of point, line, plane, solid. Being a triangle it incorporates odd and even, *peras* and *apeiron*. The Tetraktys was *Kosmos*, *Oranos* and *Pan*, universe, heaven, the All. Pythagoreans swore an oath by the Tetraktys calling it the 'fount of ever flowing nature'. See Guthrie, op. cit., p. 28-29; Walter Burkert, *Lore and Science in Ancient Pythagoreanism*, Harvard University Press, Cambridge, Massachusetts, 1972, p. 72.

69 A contemporary of Socrates, his *De Naturae*, analyses Pythagorean doctrines on music, and his account of Pythagorean ratios was the basis for Plato's *Timaeus*. Only a fragment of the former document survives and the only extant source for the following definitions of intervals by Philolaus is in Boethius, *Fundamentals of Music*, 3.8., Calvin Bower, trans., p. 97. See Michaelides, op. cit., p. 248; West, *AGM*, p. 235; Barker, *GMW II*, pp. 36-39.

70 Diesis, 'letting through' was more usually used to refer to the quarter-tone (*diesis tetartemorios*). Theon of Smyrna 'Diesis according to the school of Aristoxenus is the

called *leimma* or leftover.[71] Further divisions[72] were:

apotomē	offcut	a (9:8) tone minus a diesis/*leimma*
komma	chip	*apotomē* minus a diesis/*leimma*
diaschisma	splinter	half a diesis/*leimma*
schisma	crack	half a *comma*

Such divisions are necessary for the *pyknon*[73]of the enharmonic and chromatic genera. Archytas[74] was the first to work out the ratios for the tetrachord in the three genera.[75] There are innumerable variants of tetrachordal genera given by different theorists, but the one that is critical for later Pythagorean tuning used in the Middle ages and Renaissance is the diatonic as found in Eratosthenes and presupposed in Plato's *Timaeus*, 35-6.[76]

The Pythagorean diatonic genera

In the *Timaeus*, Plato described the divine Craftsman creating the world-soul by dividing the soul-stuff according to duple (2:1), hemiolic (3:2) epitritic (4:3) and epogdoic (9:8) proportions. The epitritic (4:3) were filled up with two epogdoic

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quarter-tone, while the Pythagoreans called diesis the semitone, 1/2 of a tone.' Quoted in Michaelides, op. cit., p.81. See Boethius, op. cit., p. 97.

71 *Leimma, limma*, from *leipein* 'to be wanting', ie remnant, remainder, left-over. Ptolemy defines the *leimma* as the interval by which the fourth exceeds the ditone, and is smaller than half a epidoic 9:8. Harmonics I, ch. 10, Barker, *GMW II*, p. 296.

72 Philolaus as cited in Boethius, op. cit., p. 97.

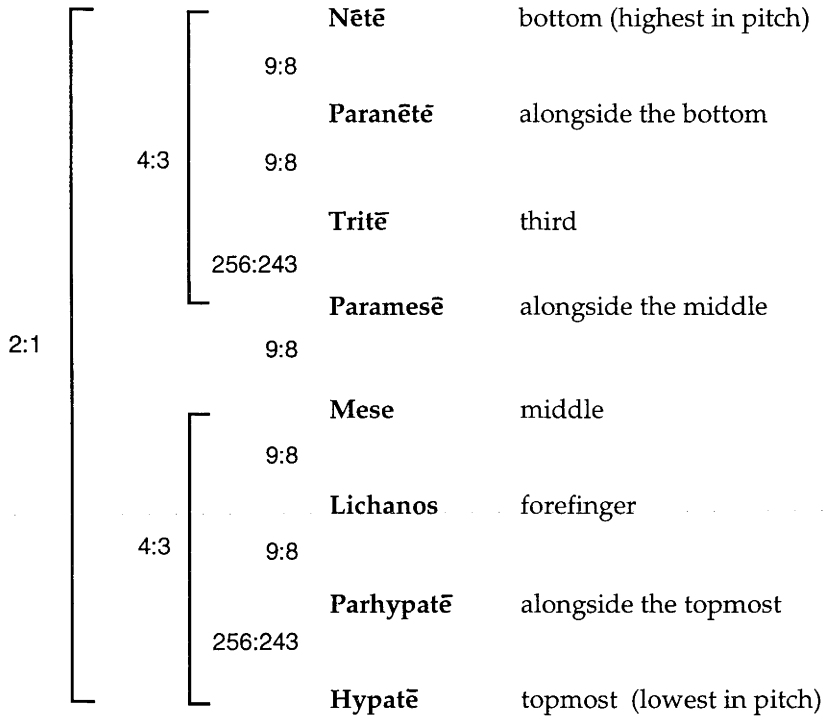
73 *Pyknon* meaning dense, compact, refers to the sum of the two small intervals that are less than the remainder of the tetrachord, which occurs in the enharmonic and chromatic genera. Michaelides, op. cit., p. 281.

74 Archytas lived in the half of the fourth century B.C., Tarentum. He was aware that sound is produced by vibrations of the air, its pitch depended on the rapidity of the pulsations, higher pitches being produced by more rapid pulsations and lower by slower pulsations. Porph, Comment., ed. I. Düring, pp. 56-57, cited in Michaelides, p. 27. According to Barker, his writings and Plato's adaptations of them were responsible for the whole tradition of musicology known as 'Pythagorean', Barker, *GMW, II*, p. 29.

75 According to Barker Archytas's divisions were hard to understand as they did not conform to the principles employed by other theorists. They were criticised by Ptolemy, *Harmonics*, book 1, ch. 14, as offending reason and the ear.

76 See Barker, *GMW II*, pp. 58-61.

intervals, leaving a remaining part, or *leimma* of 256:243.[77] Two such tetrachords with *leimma* at the bottom were joined by a tone of disjunction to form a diatonic octave. This is what is known as the Pythagorean diatonic scale. It can be represented with the Greek names (*onomasia*) for the notes[78] as follows:



This order containing the *leimma* at the bottom of the tetrachord was known as the diatonic Dorian[79] species. There are references to the 'method of concordance' for the generating of this scale from a given note via upwards and downwards movements through fourths and fifths, which was also the practical

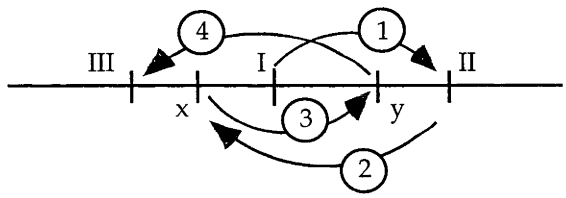
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77 *Timaeus*, 34b-36d, Barker, ibid., pp. 58-59. According to Ernest McClain the allegories contained in Plato's *Republic, Timaeus, Critias, Statesman* and *Laws* embody a systematic treatise on the science of tuning theory, foreshadowing its development in the West in the centuries that followed. Plato conceived four possible tuning systems as models of four different 'communities'. Tones, defined by numbers, were 'citizens' in a vast cosmological systems theory. Ernest McClain, *The Pythagorean Plato: Prelude to the Song Itself*, Nicolas Hays Ltd, New York, 1978.

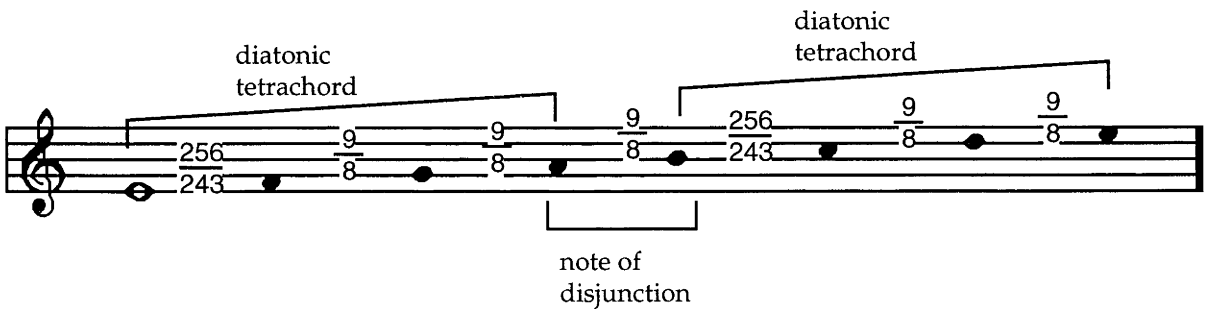
78 The names of the scale degrees derived from the names of the seven strings of the early Classical lyre. The names refereed to the position of the string in a lyre tilted away from the body. Therefore the highest in pitch was called 'bottom'. The strings nearest the body played the lower notes and were called 'highest'.

79 What the Renaissance (after Glarean) knew as Phrygian— semitone tone tone, tone, semitone tone tone, or e' to e" on a modern keyboard (although the tuning for the Greek would not, of course, be tempered like the latter).

musician's method for tuning. Although criticised by the Pythagoreans[80] for its reliance upon the ear rather than by ratios, Aristoxenus described a similar method[81] for generating a discordant interval via concords. He asserted that any interval smaller than a fourth can only be reliably generated aurally by going up and down by fifths and fourths to reach the desired interval. For example he showed how to generate a ditone downwards from a given note I by going up by a fourth to II, down by a fifth to point y, up by a fourth to point z then down by a fifth to III which is then a ditone from the starting note. Add ratios to this method and it was that used by the Pythagoreans:[82]



There is some evidence that the Pythagorean diatonic scale was generated using the method of concordance by going up or down from a given starting note by fourths and fifths, then bringing the seven notes into the range of an octave. Using modern notation based on a starting note e' this would be as follows:



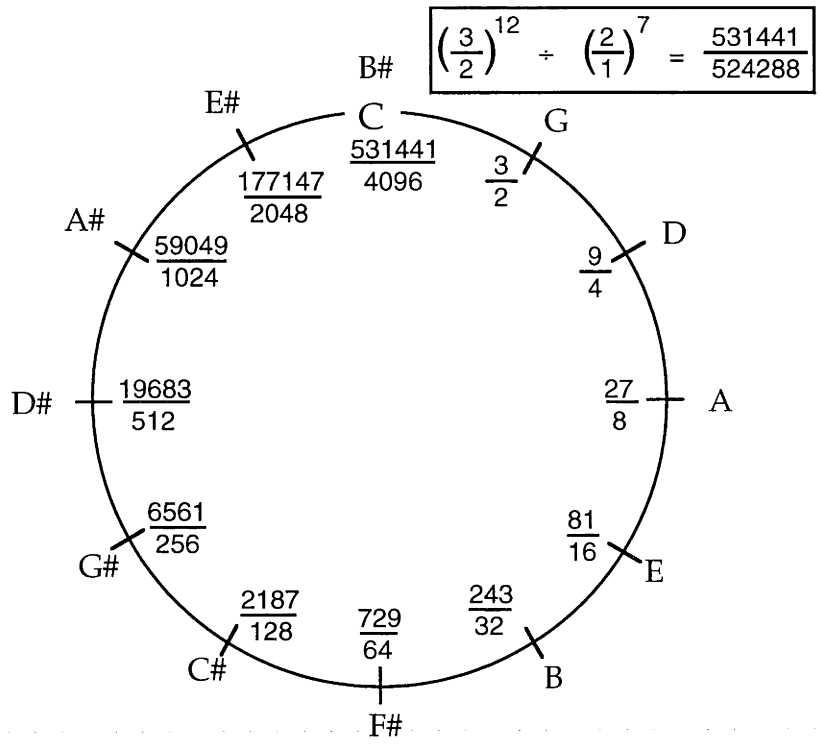
This is presumably the origin of modern textbook descriptions of Pythagorean tuning via a cycle of 12 fifths, the circle of fifths:

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80 See Ptolemy, *Harmonics*, I, 10, Barker *GMW II*, pp. 295-296.

81 See Aristoxenus, *Elementa Harmonica*, II, 55.3 ff, in Barker, *GMW II*, pp. 168-169.

82 See for example Euclid, *Sectio Canonis*, proposition 7, Barker, *GMW II*, pp. 203. Boethius also describes this method, see Anicius Manlius Severinus Boethius, *Fundamentals of Music*, Calvin M. Bower, trans., Claude Palisca, ed., Yale University Press, New Haven, 1989, pp. 98-103.



This circle makes clear the inherent imperfections of Pythagorean tuning. Twelve fifths differ from seven octaves by 531441:524288. Another traditional tale of Pythagoras' experiments was that he took a monochord and divided it in half seven times, thus producing seven perfect octaves. He took a second monochord and shortened it by two thirds making a perfect fifth. He repeated this process twelve times. The two series he found did not match up, but the series of fifths was sharper than the seven octaves by an interval with the ratio 531441:524288. This is known as the Pythagorean comma.

Although the values of ratios of the enharmonic and chromatic genera vary enormously from writer to writer, the diatonic of 256:243, 9:8, 9:8 was so common among Pythagorean writers it was considered canonic, particularly given its Platonic imprimatur. Erastosthenes, in the third century B.C., was the first to specify it. Eratosthenes' ratios for the three genera are, in descending order:[83]

Enharmonic	19:15	39:38	40:39
Chromatic	6:5	19:18	20:19
Diatonic	9:8	9:8	256:243

83 Ptol. *Harm.* in Barker, *GMW II*, p. 347.

Pythagoras was credited with the discovery of the fundamental intervals octave, fifth and fourth produced by the stopping of a string in the proportions 2:1, 3:2, 4:3. It was 2000 years until the discovery that corresponding vibration rates are inversely proportional to these ratios and the discovery of the relationship of frequency to pitch. The Pythagoreans disputed the reliability of the ear as an instrument for judging consonance. Reason is the instrument for judging intonational matters, reason which measures by means of number. Harmonics is based on numeric principles, hence it was a branch of mathematics. Aristoxenus, on the other hand, based his harmonics on experience.

Aristoxenian empiricism

Born around 350 B.C. at Tarentum, Southern Italy, and a pupil of Aristotle, Aristoxenus represented the empirical tendency in Greek harmonics. The notes of a scale were to be judged by the ear, not by mathematical ratios. Hearing allowed one to judge the size of intervals, intellect contemplated the functions of the notes. Accuracy of sense-perception was essential for a musician. Cleonides, representing the Aristoxenian tradition,[84] defined harmonics as:

> the speculative and practical science having to do with the nature of the harmonious. And the harmonious is what is made up of notes and intervals having a certain order. The parts of harmonics are seven: it has to do with notes, intervals, genera, systems (or scales), tones (or keys), modulations, and melodic composition.[85]

Aristoxenus divided the octave into twelve equal semitones. Each tetrachord was divided by 30, the tones consisting of 12 parts, the semitone of six. The intense diatonic octave consisted of two equal tetrachords, separated by the tone of disjunction. All these intervals were divided equally rather than by ratios of string lengths. The justification for this was that an interval is a movement of the voice through an unbroken pitch space. This space could be measured aurally rather than by ratios of string lengths. The same definition of consonance was derived using the ear as guide—only the fourth, fifth and octave could be judged accurately, and were hence consonant, the other intervals the ear found dissonant.

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84 Oliver Strunk, 'Aristoxenus—From the Harmonic Elements', *Source Readings in Music History*, Faber and Faber, London, 1952, pp. 25-34.

85 Cleonides, *Harmonic Introduction*, in Strunk, op. cit. pp. 34-35. See also Aristoxenus's less succinct definition in his *Harmonic Elements* also in Strunk, pp. 28-30. For a full translation of Aristoxenus see Henry S. Macran, trans., *The Harmonics of Aristoxenus*, Oxford, 1902.

Aristoxenian theory was controversial in the Renaissance in his insistence upon the ear over numerical ratios as authority for judging consonance, in contradiction to Pythagorean doctrine. His *Harmonic Elements* was not published until 1562.[86] The early experimental scientists such as Galilei and advocates of equal temperament used Aristoxenus to justify their views.[87]

Didymus[88]

The ratios of Didymus, as recorded by Ptolemy,[89] in descending order, are:

Enharmonic	5:4	31:30	32:31
Chromatic	6:5	25:24	16:15
Diatonic	9:8	10:9	16:15

Didymus was the first to suggest monochord division based on the super-particular ratios 16:15, 10:9 and 9:8. These values for the diatonic genera were valued by theorists of the Renaissance because a combination of any two created superparticular ratios for thirds, 5:4 and 6:5. The diatonic genus of Didymus contains all the ratios used in just intonation. However the order of tones with the 10:9 before the 9:8 was problematic for the just tuners, since 16:15 X 10:9 is 32:27. If the values are reversed then the tetrachord would contain within it a pure major and minor third, since 16:15 X 9:8 = 6:5 and 9:8 X 10:9 = 5:4. The Greek convention for the diatonic genus was to have the middle interval not exceed the highest in size, hence the 10:9 comes before the 9:8. Ptolemy, disregarding this convention, gave the ratios in the order required for the advocates of just intonation:

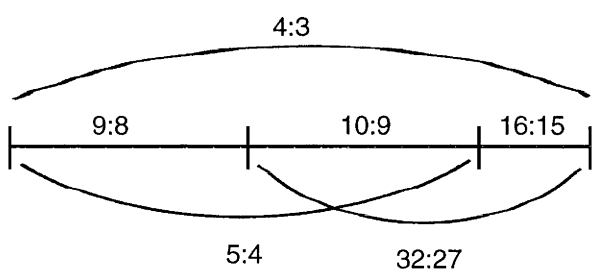
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86 Antonio Gogava, trans. Palisca points out in 'Scientific Empiricism in Musical Thought', *Studies*, p. 226, that ironically this translation was commissioned by Zarlino. See also Palisca, 'Aristoxenus Redeemed in the Renaissance', *Studies*, pp. 189-199.

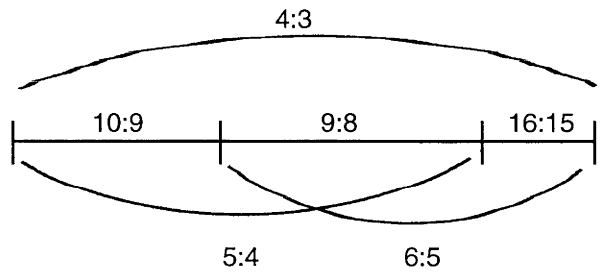
87 Palisca, *Studies*, p. 226. See also Ann Moyer, *Musica scientia: Musical Scholarship in the Italian Renaissance*, Cornell University Press, Ithaca, 1992, p. 280.

88 Barker believes this Didymus is not the same as Didymus of Alexandria, first century A. D. He suggests that Ptolemy deliberately called him Didymus the musician to make this clear. Didymus the musician was a grammarian and lived in the time of Nero. Barker, *GMW II*, p. 230.

89 Ptol, *Harm* II, 14, Barker, *GMW II*, pp. 349, 350.



Diatonic genera according to Didymus[90]



Syntonon diatonic according to Ptolemy

Ptolemy

The second century A. D. Alexandrian astronomer Claudius Ptolemy combined elements of the Pythagorean and Aristonexian approach, arguing that reason must be modified by the evidence of perception. Although mathematical reasoning was the means by which the laws underlying harmonic structures are adduced, and the intervals between notes were defined in terms of numerical ratios, these findings are tested against sense perception. Perception (*phainomenon)* gave the raw data and identified sonorous relations as concordant or discordant. Reason (*eulogon*) gave the underlying causes for consonance and dissonance which are mathematical relations between qualitative variables.[91] Experimental instruments such as monochords were required to in turn test the validity of the theorised numeric proportions against the judgement of perception. Ptolemy's treatise, *Harmonics,* contained details of Pythagorean theory, including how they constructed their scales from the numbers 1-4, and it also contained his tetrachordal divisions based on the numbers 1-5. What this allowed was scalar divisions based on superparticular or epimoric proportions (n+1:n). Ptolemy divided harmonic relations into three categories: homophones (octaves and their multiples), concords (fifths and fourths) and melodic intervals (elementary scalar divisions of the concords). The three classes grade from perfect to less perfect, from 1:1, 2:1, 3:2, 4:3, 5:4 etc. Further, each class combines to form the class above, the fifth and fourth combine to form the octave. They do so with ratios that most closely bisect the octave, ie 3:2 and 4:3. The melodic intervals also must have ratios that 'most

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90 Diagrams by author, based on values given in Ptol, *Harm* II, 14, Barker, *GMW II*, pp. 349, 350. The values for major and minor thirds are added.

91 Description of Ptolemy's theories based on Barker, *GMW* II, p. 273.

nearly make divisions into halves'. The smaller number ratios are more melodic. The tetrachord was then divided into two intervals of epimoric ratio. The remainder was divided in accordance with the law of near-equality, ie the simplest possible, and epimoric.[92]

Ptolemy's ratios, in descending order:[93]

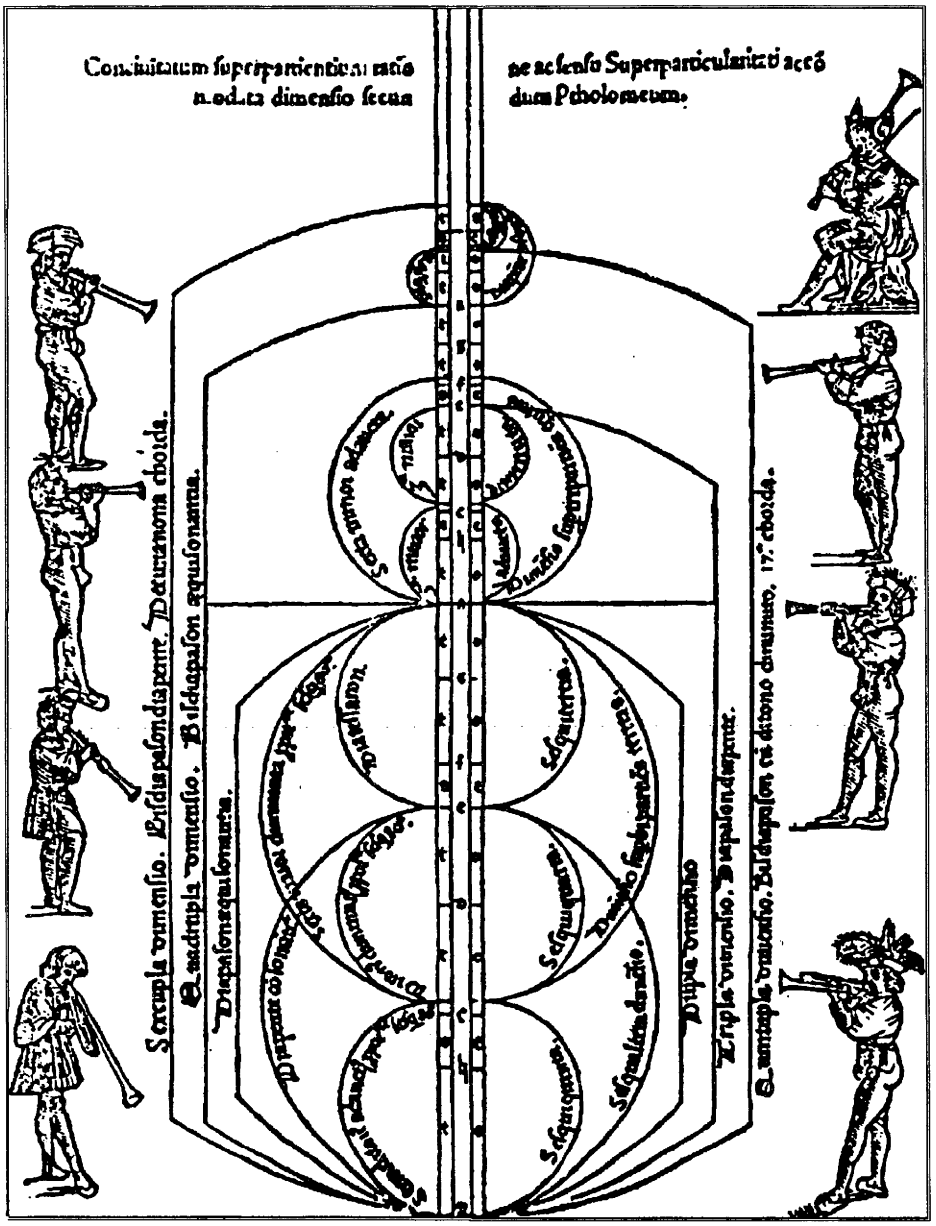
Enharmonic:	5:4	24:23	46:45	5:4 X 24:23 X46:45 = 4:3
Soft chromatic:	6:5	15:14	28:27	6:5 X 15:14 X 28:27 = 4:3
Tense chromatic:	7:6	12:11	22:21	7:6 X 12:11X 22:21= 4:3
Soft diatonic:	8:7	10:9	21:20	8:7 X 10:9 X 21:20 = 4:3
Tonic diatonic:	9:8	8:7	28:27	9:8 X 8:7 X 28:27 = 4:3
Ditonic diatonic:	9:8	9:8	256:243	9:8 X 9:8 X 256:243 = 4:3
Syntonic (tense) diatonic:	10:9	9:8	16:15	10:9 X 9:8 X 16:15 = 4:3
Even diatonic:	10:9	11:10	12:11	10:9 X 11:10 X 12:11 = 4:3

An examination of all genera shows that the superparticular, epimoric proportions are used except in the case of the ditonic diatonic, which corresponds to the Pythagorean ditonic diatonic. The syntonic diatonic tetrachord gives the ratios that came to be used in the just diatonic scale, which is also built only using epimoric ratios. However, Ptolemy did not classify as concordant the combinations of his melodic intervals to form 5:4 and 6:5 thirds.

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92 Barker, *GMW* II, pp. 270-275.

93 Ptolemy, *Harmonics*, Book 1:14, Barker, *GMW* II, pp. 347-350.



Dimensions of skilfully joined superpartiens numbers accomodated to the ratios and aural sense of Ptolemy, Gaffurius, *De Harmonia Musicorum Instrumentorum Opus*, 37

The ancient Greek theorists grappled with every issue related to tuning and delineated the parameters of later debates. They furnished a repertoire of tunings and approaches to tunings much of which was adopted by later theorists. The revival of ancient Greek theories in the Renaissance has long been recognised as a powerful force fuelling the development of musical theory in the Renaissance.[94] I would argue that the influence continues to the present, in furnishing ways of thinking about music and tuning. The romance of

94 Palisca, *Humanism*, pp. 1-50.

numerology still appeals to many twentieth century composers.[95]

2.2 Medieval and Renaissance debates on tuning

The status of the major and minor thirds (and, to a lesser degree, the major and minor sixth) as consonances and their mathematical definitions was at the heart of the historical debate on tuning between the Pythagoreans and advocates of just intonation. For the Pythagoreans,[96] no interval smaller than a fourth was consonant and Pythagorean tuning was the theoretically sanctioned system in the medieval period. Further, the Pythagorean ratios for major and minor third (81:64 and 32:27) could not be considered consonant by medieval definitions—they were not pleasing to the ear, nor based on superparticular ratios.[97] Franchino Gaffurio (1451-1522), despite his allegiance as a theoretician to Pythagoras, advised tempering the harshness of Pythagorean thirds and sixths in

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95 For example Schoenberg's *Variations on a recitative for organ,* op. 40, is suffused with number symbolism, according to Charles Ore, including expressions of the Pythagorean tetraktys of ratios based on integers 1,2,3,4 and their sum, 10. Charles William Ore, *Numbers and number correspondence in opus 40 by Arnold Schoenberg: Pythagoras and the Quadrivium revisited,* DMA diss., University of Nebraska, Lincoln, 1986. Colin Sterne discusses Schoenberg's numerology and Pythaorean influences in *Pierrot lunaire*: 'Pythagoras and Pierrot: an approach to Schoenber's use of numerology in the construction of Pierrot lunaire', *Perspectives of New Music,* vol. 21, nos. 1-2, 1983, pp. 506-34. Theodore van Huijstee discusses the Pythagorean origins of the mathematical basis of Xenakis's compositions. 'From Pythagoras to Xenakis', *Mens en melodie,* vol. 36, 1981, pp. 408-16.

96 As it was for all schools of Greek music theory. Aristoxenus observed that all intervals smaller than the fourth were dissonant, *Harmonic Elements* 1.20, see also Palisca, *The Florentine Camerata,* op. cit., p. 158.

97 The Pythagoreans also required consonances to be superparticular or multiple, but did not recognise the authority of sense perception. That a ratio sounded displeasing had no bearing on the argument. For Ptolemy, however, both mathematical reason and perception were necessary conditions. His argument was that mathematical definitions show the underlying principles that generate the experience of beauty and excellence that is evident to the ear. The ear tests the validity of the mathematical reasoning. If mathematical analysis cannot incorporate musical practice, if the findings of reason do not square with those of perception the reasoning must be wrong. And for Aristoxenus mathematical reasoning had no bearing on the subject, only the perception of the well-trained musical ear.

'Pleasingness' as a determinant of consonance is mentioned by Gaffurio in his *De Harmonia* of 1518 where is defines music as 'the arrangement of proportionately disjunct sounds in pleasing intervals, making a consonance by sense and reason. This consonance seems to arise in sound or tone through definite numbers and sizes.' Gaffurius, *De harmonia,* I. 1, Clement A. Miller, trans., Musicological Studies and Documents, American Institute of Musicology, 1977, p. 35. We can see from this passage that both sense and reason are both arbiters of consonance.

practice.[98] There was a pressure from musical practice as early as the 13th century to accept thirds as consonant and to recast them in less dissonant ratios.[99] Johannes de Garlandia reported that practising musicians accepted major and minor thirds as imperfect consonances.[100] Anonymous IV noted that thirds were deemed consonant in 'the west of England' while Walter Odington was the among the first to associate major and minor thirds with the just ratios of 5:4 and 6:5.[101]

Various theorists from the 15th century experimented with monochord divisions that would allow pure major thirds. Bartolome Ramos de Pareja[102] (1440—after 1491) was the first to advocate in a *musica pratica* treatise an alternative to Pythagorean tuning, that more closely resembled practice[103] while

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98 This procedure he called *communicata* or *participata*, in which intervals were altered slightly to be more harmonious. *Practica Musicae*, III, 2. See Palisca, *Humanism*, p. 234.

99 See Mark Lindley, 'Pythagorean intonation and the rise of the triad', *Royal Musical Association Research Chronicle*, vol. 16, pp. 4-61.

100 Several theorists at this time began to define thirds as imperfect consonances. Garlandia in his treatise *De musica mensurabili*, printed in Coussemaker, *Script.* I:97 ff defines the major and minor third as imperfect consonances. Likewise the author of *Ars cantus mensurabilis* states that 'those consonances are imperfect when the ear distinguishes two sounds which are quite different, yet are not discordant. Of these there are two: ditone and semiditone.' Quoted in Hugo Reimann, *A History of Music Theory*, Books I and II, Raymond H. Haggh, trans., Da Capo Press, New York, 1974, p. 96. A treatise regarded as contemporaneous with Odington dated around end of 13th century, *De musica libellus*, preserved in Coussemaker, Anonymous VII, Script. I:378 states: 'It is further to be noted that the most necessary intervals are the unison, the minor, and the major third, the fourth, the fifth, and the octave, as the discant always keeps its relationship with the tenor by one of these intervals. the unison and the octave are perfect consonances; the minor and major third imperfect, the fourth and fifth intermediate ones.' Quoted in Reimann, p. 94.

101 He concluded that the fifth 3:2 should be the result of the combination of 5:4 and 6:5 since 5:4 X 6:5 = 3:2 and noted that the Pythagorean ditonus consisting of two 9:8 tones would exceed the value of the 5:4 third by 81:80 and the Pythagorean semiditonus was smaller by 81:80 than the 6:5 minor third. Because the 5:4 and 6:5 united make a diapente they are considered consonant by 'many' and are blended pleasingly by human voices. See Riemann, op. cit., pp. 98-99.

102 Ramis's *Musica Practica*. See Johannes Wolf, ed., *Musica Practica Bartolomei Ramide Pareia*, Leipzig, 1901; Edward E. Lowinsky, 'Music of the Renaissance as Viewed by Renaissance Musicians', *The Renaissance Image of Man and the World*, Columbus, Ohio, 1966; J. M. Barbour, *Tuning and Temperament*, East Lansing, 1951, pp. 88-92, Palisca, *Humanism*, pp. 232-235.

103 Ramos's pupil Giovanni Spataro maintained that singers sang pure thirds and tempered (*participatio*) the rational proportions of Pythagoraean intervals. This showed that Pythagorean doctrine was unsuited to musical practice. Giovanni Spartaro, *errori*, Error 16, fol. 21v, quoted in Palisca, *Humanism*, p. 235.

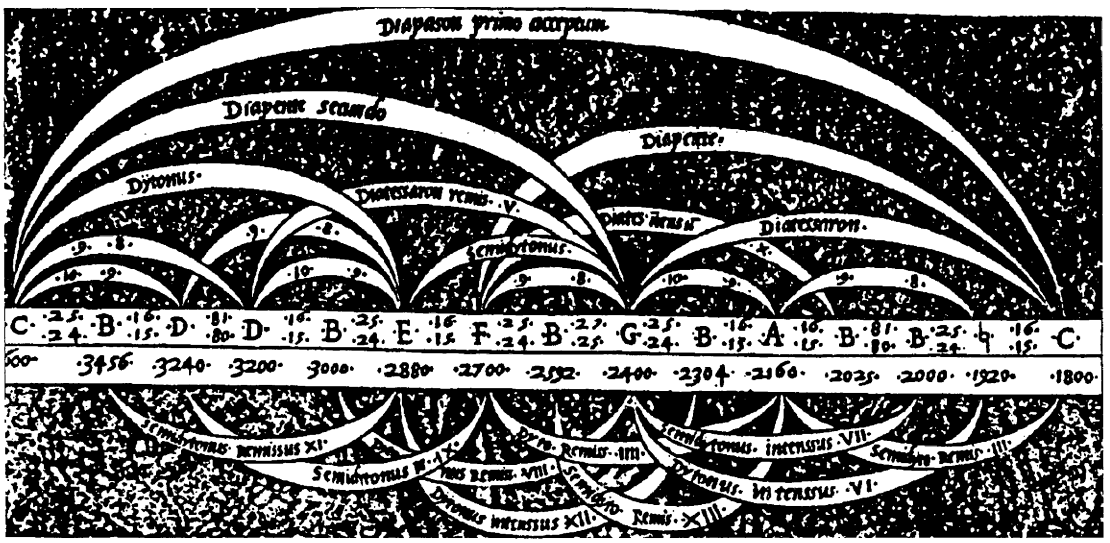
Fogliano, in 1529, was the first to do so in a *Musica theorica* one.[104] The monochordal divisions of the two compared:

Monochord division of Ramos de Pareja:[105]

a	b	c	d	e	f	g	a
	9:8	16:15	10:9	9:8	16:15	9:8	10:9
		Didymus' diatonic			Ptolemy's syntonon diatonic		

Monochord division of Fogliano:[106]

c	d	e	f	g	a	b	c'
	10:9	9:8	16:15	9:8	10:9	9:8	16:15
			Ptolemy's syntonon diatonic				



Divisions of the monochord subject to the ratio of pure numbers, from Fogliano, *Musica theorica*, III, 1, fol 34v[107]

Ramo's pupil Spataro[108] pointed out that the syntonic diatonic tetrachord of the

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104 Fogliano, *Musica theorica*, II, 5, quoted in Palisca, *Humanism*, p. 241.

105 Bartolemé Ramos de Pareja, *Musica practica*, Bologna, 1482, quoted in Palisca, *Humanism*, p. 233.

106 Fogliano, *Musica theorica*, II, 5, quoted in Palisca, *Humanism*, p. 241.

107 Quoted in Palisca, *Humanism*, p. 242.

108 The syntonic diatonic tuning of Ptolemy 'which divides the tetrachord by the ratios 16:15 at the bottom, then 9:8 and 10:9...is that practiced in active music today.' Giovanni Spartaro, *Errori de Franchino Gafurio*, Benedictus Hectoris, Bologna, 1521, Error 16, fol. 21v, quoted in Palisca, *Humanism*, p. 235. See also p. iv, Error 26, pt. v, Error 16, cited in Palisca, *Studies*, p. 219, n. 58. Frieder Rempp points out that the battle between Franchino Gafori and

second century astronomer, Claudius Ptolemy, removed the faults of Pythagorean tuning, and that this was the tuning used by practical musicians. Notes e to a in both Ramos and Fogliano's systems were the same as Ptolemy's syntonon.[109] His work had been described by Gaffurio and his *Harmonics* translated and published in 1562.[110] The new 'just' or 'syntonic' tunings thus had the authority of antiquity.

In both the above monochordal divisions the major thirds are all 5:4. The minor thirds are 6:5 except b to d which is Pythagorean, ratio 32:27, smaller than 6:5 by a syntonic comma:

$$\frac{16}{15} \times \frac{10}{9} = \frac{32}{27} \qquad \frac{6}{5} \div \frac{32}{27} = \frac{81}{80}$$

The fourth from d to g is an ad comma fourth, ratio 27:20:

$$\frac{9}{8} \times \frac{9}{8} \times \frac{16}{15} = \frac{27}{20} \qquad \frac{4}{3} \div \frac{27}{20} = \frac{81}{80}$$

The fifth from g to d is de comma 40:27:

$$\frac{10}{9} \times \frac{9}{8} \times \frac{16}{15} = \frac{4}{3} \qquad \frac{4}{3} \times \frac{10}{9} = \frac{40}{27} \qquad \frac{3}{2} \div \frac{40}{27} = \frac{81}{80}$$

All these intervals deviate from pure by the syntonic comma, the Achilles' heel of just intonation systems. The impossibility of having all intervals pure in an octatonic scale based on either tetrachord fuelled the new empiricist theories that rejected numerological definitions of consonance as *numerus sonorus* and called for acceptance of temperament.

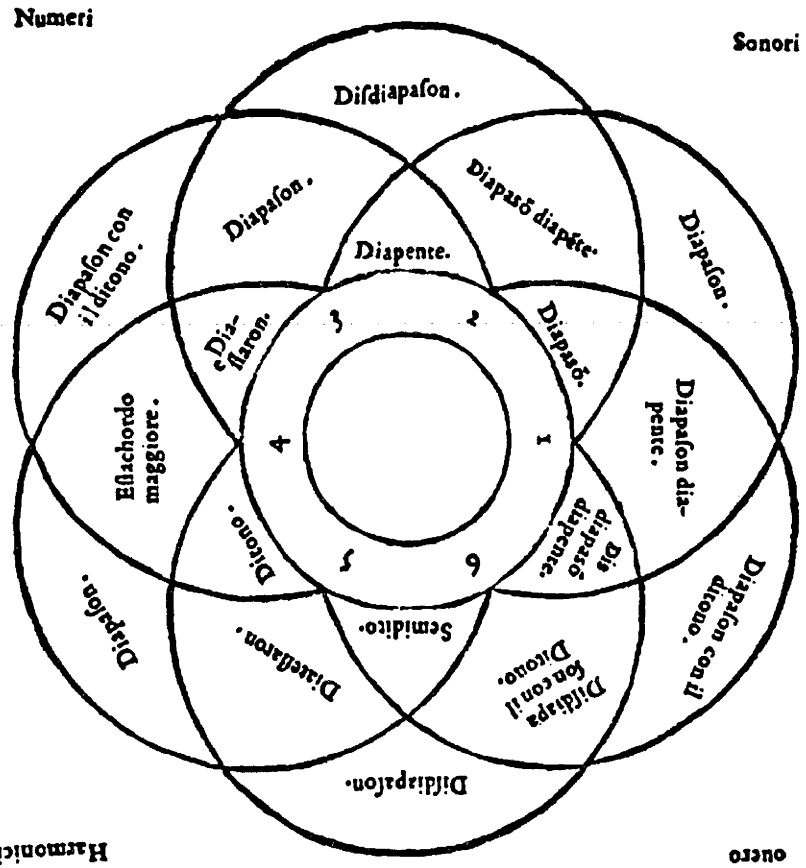
Giovanni Spataro symbolised a historic change through which a new sense of musical reality based on practice and on emotional effects asserted itself against a speculative theory remote from contemporary musical practice. It also signalled the demise of the view of music as a sounding metaphysic. Frieder Rempp, 'Bemerkungen zumselbstverständnis der italienischen Musiktheorie im 16. Jahrhundert [Observations on the way Italian music theory of the 16th century viewed itself]', *Musiktheorie*, vol. 4, no. 2, pp. 100-112.

109 According to Spataro, Ramos's divisions were independent of Ptolemy and Didymus. Palisca, *Humanism*, p. 232.

110 The first published Latin translation of Ptolemy's *Harmonics* was not until 1562 by Antonio Gogava. Leoniceno had translated it in 1499 on behalf of Gaffurio and Giovanni Battista Augio completed one in 1545. See Palisca, *Humanism*, pp. 117-128.

Zarlino, and the debate with Vincenzo Galilei, Benedetti and Girolamo Mei

Gioseffo Zarlino (1517-1590) replaced the Pythagorean *tetraktys* and its consequent impure thirds with a *senario* (1:2:3:4:5:6:) as mathematical basis of perfect art. He asserted that Ptolemy's syntonic diatonic tuning produced all the perfect and imperfect consonances in their purest form and was thus the system used for all vocal music.[111] All consonances are contained in the *senario*, including the 5:4, 6:5, and 5:3. (The minor sixth 8:5 was also allowed by a sleight of hand).[112]



Sonorous or Harmonic Numbers, from Zarlino, *Institutioni*, I, 15, p. 25[113]

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111 In response to Galilei's attack regarding the 32:27 third between D and F, Zarlino made a distinction between Ptolemy's syntonic diatonic and that used in practice. Ptolemy's was a man-made imitation of nature, hence *sintono artifiziale*; voices were natural and used *sintono naturale*. *Sopplimenti*, pp. 140-49, cited in Walker, op. cit., p. 35.

112 Zarlino writes 'Although it is not found in actuality among the parts of the senario, it is found in potential, for it takes its form from the parts of which it is a composite, that is, from the diatessaron and semiditone, because it is composed of these two consonances. For within the first cubic number, 8, its form attains actuality.' *Instutioni*, I, 16, p. 27, quoted in Palisca, *Humanism*, pp. 248-9.

113 Quoted in Palisca, *Humanism*, p. 249.

However a number of intellectual and scientific currents were to challenge Zarlino's numerology and he became caught up in a raging debate centering around the Florentine Camerata, the experiments of Vincenzo Galilei, Mei, Benedetti and others. Vincenzo Galilei, with missionary zeal, set about proving that the sonorous or harmonic number as material cause of consonant sound was a myth. Musical numerologists confused the corporeal, the vibrating body, with the incorporeal, the abstract numbers measuring the divisions of the body. Galilei revealed the inaccuracies of the famous legend of Pythagoras's discovery of the ratios, showing that the ratios 2:1, 3:2, 4:3 only produce the intervals associated with them when referring to string-lengths whose weight, thickness and material are equal. Weights attached to strings must be in the ratio 9:4 to produce the fifth, 16:9 for the fourth. This meant numbers outside the *senario* produced the perfect fifth and fourth, furthermore these numbers are also not superparticular.[114] Galilei argued vigorously that the superiority of a tuning system could not be proven on mathematical grounds, as the ear had no regard for systems. It operated on a subjective level devoid of quantitative measurement. The ear tolerated small deviations from true ratios and could not detect smaller differences than one-ninth of a tone.[115]

Vincenzo Galilei's debate with Zarlino centred on definitions of tuning sung and played in actual practice. If syntonic diatonic monochordal tuning is used, too many consonances would be out of tune.[116] Practising musicians did not follow a

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114 Vincenzo Galilei, *Discorso intorno all'opere di messer Gioseffo Zarlino da Chioggia*, G. Marescotti, Florence, 1589. Argument paraphrased in Palisca, *Studies*, pp. 227-231. See also *Discorso particolare intorno alla diversita delle forme del Diapason*, Florence, ed. and trans. in Claude Palisca, *The Florentine Camerata*, pp. 18-91, paraphrased in Palisca, *Studies*, p. 230 where he shows the octave may be obtained through ratios 4:1 when weights are attached to strings and 8:1 in terms of volumes of organ pipes. In his 'A Particular Discourse concerning the Unison' he showed that to produce a unison two strings must be made of the same material, be of the same thickness, length, quality and tension. The duple proportion does not only apply to the octave, it can be found also in dissonant intervals such as a tritone produced by 2 coins in duple proportion. Translated by Claude Palisca in *The Florentine Camerata*, op. cit., pp. 198-208.

115 Palisca, *Studies*, p. 231.

116 Zarlino based his syntonic diatonic on C. A glaringly obvious impurity is the D—E third. As with Ramos and Fogliano's tunings, the D—F minor third is 32:27, smaller than the 6:5 minor third by a syntonic comma:

C	D	E	F	G	A	B	C
9:8	10:9	16:15	9:8	10:9	9:8	16:15	

single diatonic species but mixed them according to their need, Galilei argued. Singers adjusted to their situation using a mixture of the diatonic ditoniaion and diatonic syntonon.[117] Zarlino replied:

> The forms of the consonances and other intervals that we use in our times in vocal and natural compositions are not products of art nor inventions of man but primarily of nature itself, collocated and registered among many things and especially among the parts of the perfect number, which is the senario, as I declared in the *Instutoni*, in which they find their true forms. They are then ordered and rediscovered by art in the species that I call and shall always call natural, named syntonic diatonic by Ptolemy.[118]

The imperfections that occurred in the syntonic diatonic were avoided by voices which were natural and completely flexible.[119] They could steer the harmony to a good consonance where a preordained tuning system would lead to impure one.[120] Musical science had as its aim the defence of the natural canon or monochord.

Galilei pointed out that if the singers deviated from the intervals of the syntonic in order to avoid its pitfalls, then they no longer were following the syntonic. Against Zarlino's assertion that some intervals are natural, some artificial, he argued that all intervals were equally natural, whether or not they fell within the senario. The 81:64 third is as natural as the 5:4, and 'whether we sing the fifth in the 3:2 ratio or not, is of no more importance than that a crow or a raven lives three hundred or four hundred years...'[121]

Zarlino, *Institutioni*, Bk. ii, ch. 39. This can be deduced also in Bk iii, Chapter 18, *The Art of Counterpoint*, Guy Marco and Claude Palisca, trans., Yale University Press, New Haven, 1968, pp. 36-37. Galilei points out this discrepancy in *Dialogo della musica antica et della moderna*, G. Marescotti, Florence, 1581, p. 10, cited in Palisca, *Studies*, p. 224.

117 The viola d'arco, lute and fretted lyra used Aristoxenus's intense diatonic with its equal semitones. Organs, harpsichords and harps use two unequal semitones. Transverse flutes, cornettei etc adjust to various species according to the situation. Galilei, *Dialogo*, op. cit., pp. 30-31, cited in Palisca, *Humanism*, p. 271.

118 Zarlino, *Sopplimenti musicali*, I, 1, p. 8., passage translated by Palisca, *Humanism*, p. 272. See also Jane Piper Clendinning, 'Zarlino and the helicon of Ptolemy: A translation with commentary of book III, chapter 3 of Gioseffo Zarlino's Sopplimenti musicali', *Theoria*, vol. 2, 1987, pp. 39-58.

119 See also Maria Maniates, *Mannerism in Italian Music and Culture, 1530 -1630*, University of North Carolina Press, Chapel Hill, 1979, pp. 133-157.

120 Citied in Palisca, *Humanism*, p. 272.

121 Galilei, *Discorso intorno all'opere di messer Gioseffo Zarlino da Chioggia*, Marescotti,

Galilei advocated the equal tuning of Aristoxenus' intense diatonic. A variety of different semitones and whole tones requiring such things as a different note for D# and E^b etc was unsatisfactory. He favoured Aristoxenus' solution which divided the octave into twelve equal parts and built up the intervals from these. A semitone was one part, a tone two parts, a minor third three parts etc.[122] Galilei argued that equal temperament was a necessary compromise for all instrumental music. He did not claim that this was what singers used because a well trained ear always sought the purist interval possible. However it was not feasible to describe numerically which interval they did choose. Galilei upheld the primacy of sense observation over reliance on authority:

> Men who profess an art or a science do not in writing about it go off half cocked as does Zarlino. But when they find a writer who cites the authority of another more ancient than he, they seek to get to the bottom of the thing, and even more when writing about matters heard from friends. When someone has seen to the bottom of the things of any author who treats of matters that are subject to sensation, he examines whether they are true or not, and after reporting their opinions and recognising that they do not truly pass muster he adds his own opinion with all due modesty.[123]

It had been correspondence with Girolamo Mei (1519—1594) that fuelled Galilei's arguments. Mei, as a result of exhaustive study of Greek treatises, concluded that musical science and musical practice were separate entities in ancient Greece. Mei pointed out that, since the imperfect consonances were not recognised in Greek theory, Pythagorean tuning was adequate. Further, the fact that the ancients were so concerned with theories of consonance based on numerical proportions yet sang in unison also showed that theory and practice were separate arenas. Mei criticised Zarlino's *senario* as being not derived from any ancient Greek author.

Giovanni Battista Benedetti (1530-90)

Benedettti believed theory must be supported by practice, mathematics by sensory perception. He showed how in justly tuned vocal music discrepancies in starting and finishing notes occured; the composition would not end on the same pitch it began. If singers were able to continually adjust their pitch to each other maintaining the consonance of simple ratios, the absolute pitch of the work

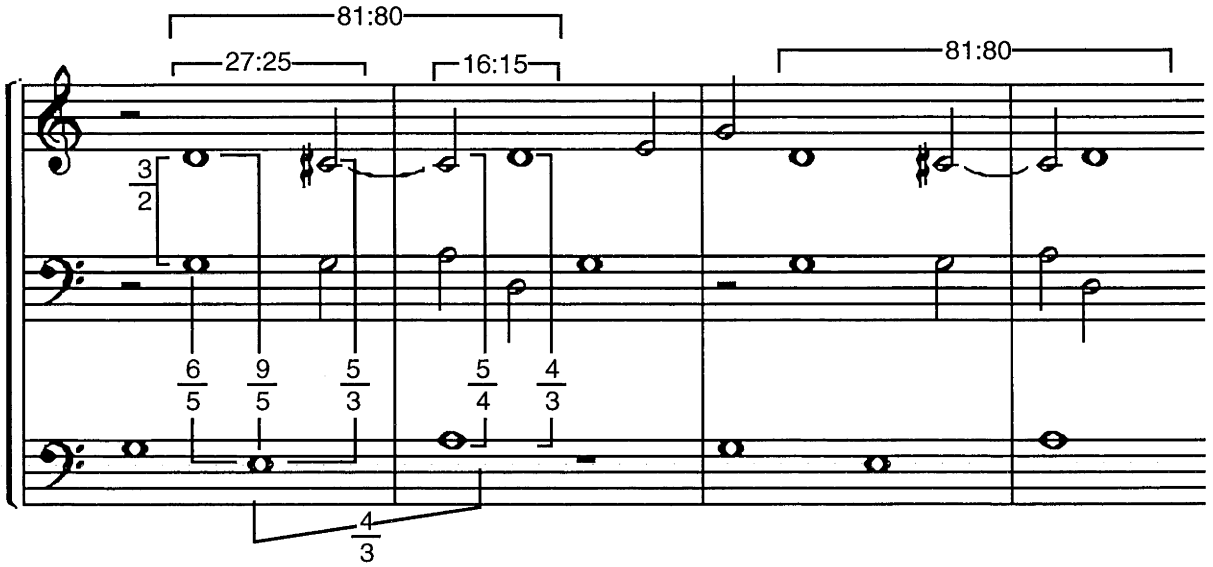
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Florence, 1589, quoted in Palisca, *Studies*, p. 224.

122 For the reception of Aristoxenus's tuning and reliance on the ear over numerical ratios in the Renaissance, see Palisca, 'Aristoxenus Redeemed in the Renaissance', *Studies*, pp. 189-199.

123 Galilei, unpublished response to Zarlino's *Sopplimenti* of 1588, MS Galilei 5, fol. 42r, Florence, Bibliotheca Nazionale Centrale, translated by Palisca, *Humanism*, p. 270.

would not remain the same. He gave several examples demonstrating this phenomenon. The following example demonstrates a descent in pitch from original to finishing note by a comma in each repetition of a progression where the consonance between superius and tenor necessitated a large descending semitone (27:25) and a small ascending semitone (16:15):



Benedetti, *Diversarum*[124]

In the above example involving c#' the superius descends by a maximum semitone (27:25) and ascends by a natural semitone 16:15. The maximum semitone occurs because the intervals between superius and tenor are a perfect fifth (g—d') and minor third (e—g) a total of a small minor seventh (9:5). When the superius descends to c#' it forms a 5:3 major sixth with the tenor (e—c#'), and the difference between the seventh and sixth is 9:5 ÷ 5:3 = 27:25, thus it descends by 27:25. The tenor ascends to a, creating a 5:4 major third with c#'. When the superius ascends against the tenor a to d' creating a 4:3 fourth with it, it does so by a 16:15, since 4:3 ÷ 5:4 = 16:15. The difference between the two forms of semitone being 27:25 ÷ 16:15 = 81:80, the piece has dropped a comma, and will continue to drop by 81:80 each time the pattern is repeated. The requirement that a composition end on the same tone in which it began being impossible in syntonic tuning, the only solution was to make judicious adjustments to the pure intervals of the *senario*. Temperament was a necessity in vocal as well as instrumental music, Benedetti concluded.

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124 Benedetti, *Diversarum speculationum mathematicarum & physicorum liber*, Turin, 1585, p. 280, quoted in Palisca, *Humanism*, p. 264.

Zarlino had an able supporter of just intonation in Johannes Kepler (1571-1630), who described Vincenzo's attempts at tempering 'ruinous'[125] and claimed that the revolution of the planets formed a just scale.[126]

The Dutch Calvinist scientist and music theorist, Isaac Beeckman (1588-1637), also upheld the virtues of just intonation. He rejected the idea of making flexible adjustments as a solution to the inherent contradictions between vertical sonority and stable pitch based on flexible adjustments, since 'singing is to repeat often and alternately the same higher and exactly the same lower pitch.'[127] To sing two different 'd's was to introduce a foreign note into the melody. His solution was to form 18 submodes (modes of the modes) to each mode which fixed the position of tones and semitones according to the intonational requirements of a given mode in a given melodic context.[128] This suited the diatonic, simple melodies of the Calvinist Psalter, with which he was chiefly concerned, but did not solve the tuning problems of more complex polyphonic writing.[129]

Another Dutchman, Simon Stevin, rejected rational proportions as a way of describing musical intervals, asserting that music was a continuum and the proper division of the octave was equal semitones divided using the twelfth root of two.[130]

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125 Owen Gingerich, 'Kepler, Galilei, and the Harmony of the World', *Music and Science in the Age of Galileo*, Victor Coelho, Kluwer Academic Publishers, Dordrecht, pp. 45-67. Although supporting just intonation, Kepler's view of the cause of consonance was geometric rather than mathematical. He disagreed with the tendency to use number symbolism as a discrete, abstract quantity, but saw geometry as the underlying principle. Kepler, J. *Gesammelte Werke*, 6, 100, cited in H. F. Cohen, *Quantifying Music*, D. Reidel Publishing Co, Dordrecht, 1984, p. 17.

126 See Gingerich, op. cit., pp. 48-50, 54-57, Barbour, op. cit., p. 97.

127 Cohen, op. cit., p. 152.

128 Described in Cohen, ibid., pp. 152-156.

129 Rudolf A. Rasch, 'Six Seventeenth-Century Dutch Scientists', *Music and Science in the Age of Galileo*, Victor Coelho, Kluwer Academic Publishers, Dordrecht, pp. 185-211. Beeckman was concerned with pedagogical issues regarding congregational singing of the repertory of Genevan psalms, and his journal abounds with references to the interpretation of psalms in just intonation, fitting them into Glareanus or Zarlino's classification of modes, and attempts at hexachordal solmisation of them. See p. 190.

130 *The Principle Works of Simon Stevin*, vol. 5, *Music*, A. D. Fokker, ed., Amsterdam, 1966, pp. 431-33., see Stillman Drake, 'Renaissance Music and Experimental Science', *Journal of the History of Ideas*, vol. 31, 1970, pp. 491.

2.3 Baroque empiricism, the relationship between scientific discovery and music theory

From the time of Vincenzo's son, Galilei Galileo, scientific experimentation changed thinking about tuning. Numerological theories relating to the ratios of string lengths were replaced by definitions based on the vibrations of those same lengths, and the relationship of speed of vibration to string length.[131] In fact the same proportions were shown to apply and thus frequency ratios replaced string-length ratios. As Sigalia Dostrovsky puts it: 'a periodic vibration can be heard as a tone, and two tones make a consonant interval when the ratio of their frequencies is a ratio of small integers.'[132] Galileo identified pitch with frequency, with the consequent relocation of the cause of consonance:

> I say that the length of strings is not the direct and immediate reason behind the forms [ratios] of musical intervals, nor is it their tension, nor their thickness, but rather, the ratio of the numbers of vibrations and impacts of air waves that go to strike our eardrum, which likewise vibrates according to the same measure of times. This point established, we may perhaps assign a very congruous reason why it comes about that among sounds differing in pitch, some pairs are received in our sensorium with great delight, others with less, and some strike us with great irritation; we may thus arrive at the reason behind perfect consonances, and imperfect, and dissonances.[133]

Benedetti reaffirmed the Aristotelian understanding that vibration frequency varies inversely with string length.[134] If a string is divided so that one side is twice as long as the other side, and the two parts are simultaneously plucked, then the octave will be heard. The longer part of the string will complete one period of vibration in the time that the shorter completes two such periods (*intervallum tremoris*).[135]

The vibrations of a plucked string were first measured at the time of Galileo (1564—1642) when improvements in mechanical clocks enabled accurate

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131 See Sigalia Dostrovsky, 'The origins of vibration theory: the Scientific Revolution and the nature of Music', *Archive for History of Exact Sciences*, 14, 1975, pp. 169-218.

132 ibid., p. 170. Mersenne was one of the first to redescribe just tuning in frequency ratios. See Gruber, 'Mersenne and Evolving Tonal Theory', op. cit., p. 38.

133 Galileo Galilei, *Two New Sciences*, Stillman Drake, trans., University of Wisconsin Press, 1974, p. 104.

134 Galileo attempted to prove the inverse proportionality between string length and frequency in 1638, but did not do so convincingly. In 1636 that Mersenne published a geometrical proof, based on correspondence from Beeckman in 1629. Cohen, op. cit., p. 123.

135 Benedetti, *Diversarum*, passage translated in Palisca, *Studies*, pp. 215-216.

measurement of seconds, hence the capacity to measure how many vibrations per second (the first pendulum clock was produced by Christiaan Huygens in 1652). Marin Mersenne, in 1648, was the first to record an attempt to measure a musical frequency by taking a string long enough that its vibrations could be counted, then shortening it until it was in unison with the note whose frequency he wanted to count. The latter was calculated by multiplying the counted frequency by the ratio of original to final lengths.[136] Robert Hooke in 1672 used a cogwheel with a fixed number of teeth and rotated it at a rate he could count and held a card against it. The pitch so produced was associated with the known frequency. The advances of Heinrich Hertz (1857-1894) in 1888 with radio waves facilitated the measurement of sound vibrations. The unit for measuring frequency is named 'hertz' or Hz after him. Something that makes one complete vibration every second (or cycle per second, cps) has a frequency of 1 Hz.[137]

The discovery of the Harmonic Series

The discovery of the harmonic series led to fundamental changes in thinking about consonance, tuning, and the nature of musical sound. Kardos' entire system is founded on his understanding of the role of the harmonic series in tuning, which in turn is derived from a line of theorists from Rameau to Hindemith who have attempted to find a natural basis for music in the harmonic series.

It had been observed by musicians and theorists who used the monochord to derive musical ratios as far back as Aristotle[138] that a vibrating string vibrates not only at the fundamental frequency as determined by its length and thickness, but it also vibrates with different frequencies at the same time. During the age of enlightenment many mathematicians and scientists in different countries worked on understanding this phenomenon. Marin Mersenne (1588-1648) had

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136 Ian Johnston, *Measured Tones*, Institute of Physics Publishing, Bristol, 1989, p. 35.

137 ibid., p. 35.

138 Aristotle's *Problems* on physics argues that the tone produced by a vibrating string produces also its upper octave and that if the high string of a lyre is stopped the string of the lower octave sounds. *Problemata*, xix. $918^{a}8$, $921^{b}42$, $919^{b}24$. Girolamo Fracastoro in 1546 demonstrated the sympathetic vibration of unison strings. *De sympathia et antipathia rerum*, Venice, 1546, cited in Claude Palisca, *Studies*, p. 208. This discovery was applied by Mersenne and by Descartes to the sympathetic vibration of strings which were in simple ratios to each other. Palisca, *Studies*, p. 208.

already noted that a single plucked string produced other tones beside the pitch of the string, and identified them aurally as follows:

> these tones follow the ratio of the numbers 1,2,3,4,5 because one hears four tones different from the natural one, of which the first is at the octave above[2:1], the second at the twelfth [3:1], the third at the fifteenth [4:1] and the fourth at the major seventeenth [5:1].[139]

The cause for this had been discovered in 1677 by John Wallis in England, who showed that overtones occurred as a result of the fact that there are points of no vibration in the string (nodes): in other words, partial vibration.[140] Francis Roberts was the first to give an accurate statement of the harmonic series in 1692, but only in reference to the trumpet.[141] Joseph Sauveur (1653-1716) demonstrated in 1704 how a musical note from any natural vibrator produced overtones and further the overtones from strings and most air columns formed a perfect harmonic series. He also showed that a string sounding an overtone vibrated with nodes and antinodes.[142] Sauveur was influential in the understanding of beats.[143] Sauveur's work on overtones was like a revelation to Rameau, for it grounded his theory of harmony in nature.[144] The development of Rameau's theories paralleled the development of scientific understanding of the harmonic series. His first theories were based on the first six harmonics of a vibrating system, which he referred to as the *senario*.[145] He then became aware of the resonant body which produced the same overtones as the harmonic divisions of

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139 Marin Mersenne, *Harmonie Universelle, Fourth Book on Instruments*, prop. 9., R. E. Chapman, trans., Martinus Nijhoff, The Hague, 1957, pp. 208-211.

140 John Wallis, *Opera Mathematica*, Oxford, 1695-99, see Dostrovsky, op. cit., pp. 204-206.

141 Francis Roberts, 'A discourse concerning the Musical Notes of the Trumpet', *Philosophical Transactions*, XVI, 1692, p. 563, cited in B.L. Green, op. cit., p. 14.

142 Dostrovsky, op. cit., pp. 206-209.

143 See Sigalia Dostrovsky, op. cit., p. 202. Carlton Maley op. cit., see Richard Semmens, 'The early eighteenth-century discussion of musical acoustics by Etienne Loulié', *Canadian university music review*, vol. II, 1981, pp. 177-206, for the researchers of Sauveur's contemporary, Loulié. Musical intervals are explained in terms of patterns of sound vibrations (as opposed to the traditional comparison of string lengths), lucid descriptions of overtones and harmonics, and the fixing of musical pitches through the phenomenon of beating. A fuller discussion of the phenomenon of beats is contained in Appendix B.

144 Rameau, *Nouveau systémede musique théroique*, 1976. For the influence of Sauveur see Johnston,op. cit., pp. 108-113, Palisca, *Studies*, pp. 207-208. For the theories of Rameau, see Joan Ferris, 'The Evolution of Rameau's Harmonic Theories', *Journal of Music Theory*, 3, 1959, pp. 231-56.

145 Rameau, *Traité de l'Harmonie Reduite à ses Principes naturels*, 1722.

a string. Marin Mersenne was the first to realise the correlation between these overtones and the divisions of a monochord based on superparticular ratios. Combination tones were first mentioned in print in 1744 by George Andreas Sorge.[146] Ohm, in 1843, formulated what is known as Ohm's law, which stated that musical tones can be analysed into a sum of simple vibrations, to each such single vibration corresponds a simple tone which is sensible to the ear, and which has a pitch determined by the periodic time of the corresponding motion of the air.[147]

Jean-Baptiste-Joseph Fourier (1768-1830) devised a method of analysing the components of the harmonic series. The 'Fourier theorem', states that any periodic oscillation curve, with frequency *f* can be analysed into a set of simple sine curves with frequencies *f*1, *f*2, *f*3, *f*4, *f*5, etc, each with its own amplitude. This set of sine curves is called a set of harmonics. If the fundament is truly periodic the overtones are harmonic, and whole number multiples of the fundamental. For example, if the fundamental is 100 Hz. then the other component frequencies will be simple multiples of it in a series: 200, 300, 400, 500, 600 Hz. etc. Thus harmonics are not the same as overtones.[148] A didjeridu, for example, has as its first overtone a note a tenth higher than its fundament. The overtones of a piano string are sharp and thus not harmonic, a cylinder closed at one end has as its first overtone the third harmonic.[149] The term 'spectrum' refers to the exact set of frequencies present in a complex waveform. If all the components are harmonic the term 'harmonic spectrum of frequencies' is used.

The audibility of harmonics is an important consideration in evaluating their influence. Marin Mersenne claimed he could hear the first seven harmonics of the harmonic series. Plomp's experiments have more recently verified the legitimacy of such a claim, demonstrating that the limit of distinguishability of harmonic overtones for most listeners is the eighth harmonic.[150]

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146 See Appendix B for a fuller discussion of these.

147 Helmholtz, op. cit., p. 33.

148 Alexander Ellis objected to the use of the word 'overtone' as a translation of Helmholtz's term *oberton*, or 'upper tone'. Ellis in Helmholtz, op. cit., p. 25 n.

149 Johnson, op. cit., pp. 105-107.

150 Plomp, 1964, cited in Plomp, *Aspects of Tone Sensation*, Academic Press, London, 1976, pp. 3-6. It depends also on the frequency of the fundamental, the best results obtaining for the region around 98 Hz or G.

Hermann von Helmholtz 1821-1894

Helmholtz, in his work on the anatomy of the ear and processes of hearing, discovered the physiological basis for combination tones, beats and the role of harmonics, leading to his psycho-physical theory of consonance. He hypothesised that the organ of Corti acts as a resonating system, and is able to conduct a harmonic analysis. The basilar membrane acts as if it were made up of numerous short strings with short, taut 'strings' near the oval window, and longer, slacker ones further down. If the fluid around it vibrates at a pure frequency then one string will resonate with it. This resonant vibration will cause the nerve cell below to fire, thus the brain can recognise the frequency by knowing which string resonated. Helmholtz saw the way the ear worked as a row of graded resonators. The process of recognising frequency entailed locating where on the row the resonance occurred. Since a complex vibration is equivalent to the summation of pure tones of harmonically related frequencies, more than one part of the basilar membrane will respond simultaneously, depending on which harmonics are present. The cochlea thus sends to the brain a number of electrical signals along different fibres of the auditory nerve, one for each harmonic.

The psycho-acoustical theory of consonance[151]

For Helmholtz, consonance is the absence of dissonance, and dissonance is caused by rapid beating. A certain number of beats[152] per second can be tolerated, but once a critical threshold is reached the sensation is of roughness. The critical amount varies with pitch but it is worst at the distance of a semitone and disappears at the distance of a minor third. However the ear also responds to the overtones of complex sounds. When two notes with overtones are sounded there is overlap in the harmonics leading to a considerable degree of 'roughness' caused by the relationship between the harmonics. Certain intervals do not have this overlap. The resonant peaks of intervals an octave apart exactly coincide with every second peak of the lower. Adding the two together produces no roughness at all. The theory states that this is why they are seen to be so

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151 For explanations of Helmholtz's pycho-physical theory of consonance see Johnston, op. cit., pp. 252-257; John Rigden, *Physics and the Sound of Music*, University of Missouri, John Wiley & Sons, New York. 1977, pp. 236-240; Murray Campbell, *The Musician's Guide to Acoustics*, Schirmer Books, New York, 1988, pp. 165-170; Juan G. Roederer, *Introduction to the Physics and Psychophysics of Music*, Springer-Verlag, New York, 1979, pp. 140-145.

152 See Appendix B.

consonant. Perfect consonance means the absence of extra roughness. Similarly notes of frequencies in ratio 3:2 create an overlap where every second peak of the higher coincides with every third one of the lower. The ratio of fundamental frequencies mean some of the peaks are superimposed, leading to very small overlap roughness. Other combinations of notes whose frequencies are in the small whole number ratios behave similarly. Thus the perfect consonances have the lowest degree of dissonance. Helmholtz also analysed combination tones, showing that if the fundamental frequencies are not in simple ratios their combination tones will be dissonant with the primary tones.

Conclusion

Helmholtz felt he had discovered the answer to the riddle of the Sphinx—why do small integer ratios produce the consonances?

> The enigma which, about 2500 years ago, Pythagoras proposed to science, which investigates the reasons of things, 'Why is consonance determined by the ratios of small whole numbers?' has been solved by the discovery that the ear resolves all complex sounds into pendular oscillations, according to the laws of sympathetic vibration, and that it regards as harmonious only such excitements of the nerves as continue without disturbance. The resolution into partial tones, mathematically expressed, is effected by Fourier's law, which shews how any periodically variable magnitude, whatever be its nature, can be expressed by a sum of the simplest periodic magnitudes. The length of the periods of the simply periodic terms of this sum must be exactly such, that either one or two or three or four, and so on, of their periods are equal to the period of the given magnitude. This, reduced to tones, means that the pitch numbers of the partial tones must be exactly once, twice, three times, four times an so on, respectively, as great as that of the prime tone. These are the whole numbers which determine the ratios of the consonances.[153]

The scientific revolution, Palisca argued, created a climate of experimentation favourable to the acceptance of the new directions already taken by early Baroque music, and 'the new acoustics replaced that elaborate conglomeration of myth, scholastic dogma, mysticism and numerology that had been the foundation of the older musical theory.'[154] However, from my analysis of theorists such as Rameau, Helmholtz, Hindemith and Kardos, who derive their theories of harmony, consonance, and tuning from the harmonic series, number symbolism still prevailed. It is simply that numerology was based on the harmonic series instead of the monochord as the acoustical foundation of music. Consider the view of Levarie and Levy:

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153 Helmholtz, op. cit., pp. 229.

154 Palisca, *Studies*, p. 235.

> We notice an exact coincidence of the overtone series with the results of our divisions on the monochord. This is not an accident but a symptom of the truth that the norms of our psyche correspond with those of the physical world. Man is part of, and not outside, nature. The physicist usually explains the overtone series as a natural fact, and he is, of course, correct in his point of view...The musician explains the overtone series as a psychological reality which exists not only in nature but which necessarily agrees with a particular norm of our soul...The overtone series is a law of objective nature as well as of our subjective apperception. The two must not be in conflict.[155]

Kardos was the inheritor of two and a half thousand years of experimentation, debate, discussion, and discovery regarding vocal tuning. Theorists and scientists have struggled over that entire period to find answers to what was accepted experientially; the sonority and pleasingness of intonation based on intervals which can be expressed mathematically as small whole number ratios, and acoustically as derived from lower overtones of the harmonic series. Most admitted that a system based on these principles had inherent limitations, but against that was the argument that singers do not have to follow a pre-determined, fixed system. Even Galilei admitted that 'well-trained singers do indeed sing all intervals in just intonation' but that 'it is very complicated and difficult to describe'. He would need another book to do so.[156] Kardos wrote that book, as we shall see.

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155 Siegmund Levarie and Ernst Levy, *Tone, a Study in Musical Acoustics*, Kent State University Press, 1968, pp. 47-8.

156 Walker, op. cit., p. 17.

Chapter 3 Introduction to principles of pure intonation in *a cappella* works

3.1 Kardos's acoustically pure tuning based on the ratios of the harmonic series[157]

Kardos used the harmonic series as the conceptual model on which he bases his theory and pedagogy of pure intonation. The same mathematical relationships as found in the harmonic series can be found by divisions of a monochord. One difference between the two ways of measuring intervals is that the monochord provides a linear model, the harmonic series a vertical one. In some ways, for a musical audience reared on modern staff notation, Kardos's reliance on the vertical relationships of the harmonic series makes sense.[158] It is conceptually easier to grasp the relationship of ratios to intervals by means of a notated harmonic series, than to linear diagrams of numerical and geometrical proportions. Harry Partch is an example of the opposite approach, a modern exponent of the monochord as conceptual tool.[159]

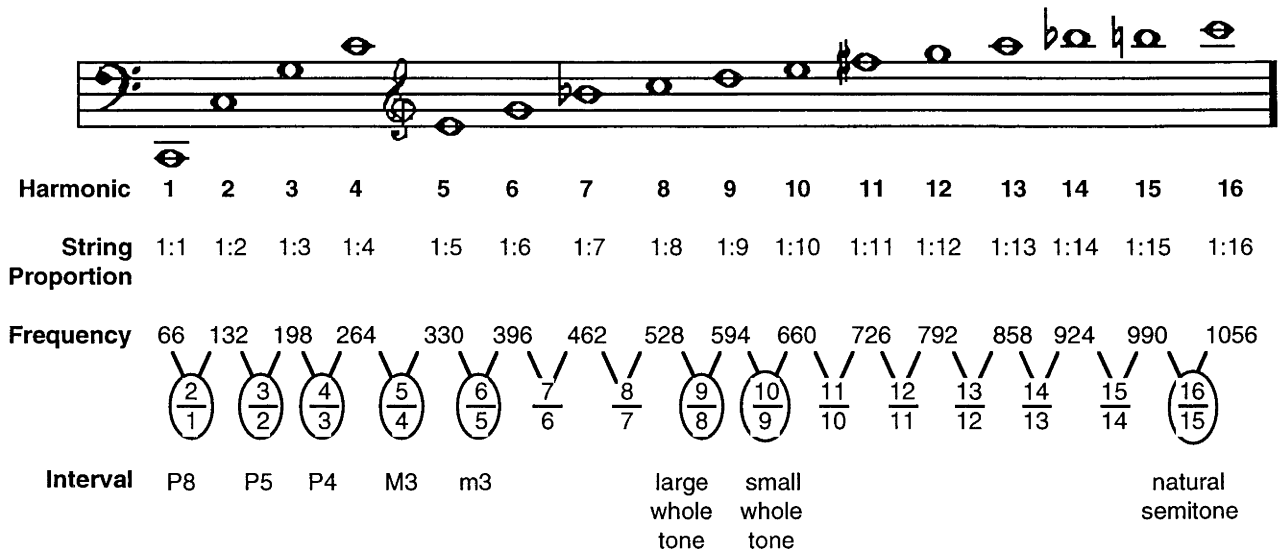
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157 Kardos adapted a Harmonium to replicate the sounds of the harmonic series.

158 He gives a brief account of how Pythagoras derived his scale from division of a monochord, as well as Didymus, and compares the two. He makes the point that the Pythagorean diatonic scale was used for a Greek musical culture that was unison, and that in certain unison settings Pythagorean intonation is appropriate. Kardos, *KK*, pp. 218-231.

159 Partch, *Genesis of a Music*, op. cit., p. 87.

The harmonic series:[160]



A vocal tone always contains a harmonic series in which the frequencies of the harmonics are related to each other as 1:2:3:4:5:6:7:8:9:10 etc. The original tone is the fundamental or first harmonic. Acoustically pure intonation is a system of

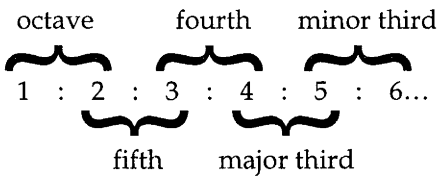
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160 Table adapted from Kardos, *KK*, p. 22. Kardos's Hz values differ from those found in other texts. C in Helmholtz is 66 Hz, not 64. Helmholtz, op. cit., p. 17. The Hz values given in Alldahl's harmonic series corresponds with Helmholtz. Per-Gunnar Alldahl, *Choral Intonation*, Gehrmans Musikförlag, Stockholm, 1990, p. 1. I have accordingly change Kardos's values to match Helmholtz. The Hertz values correspond to the number of vibrations per second. One Hz = 1 oscillation per second, the Hz values between two adjacent harmonics is always the same although the interval gets smaller the higher it is in the series. A difference of 66 Hz is an octave difference between harmonics 1 and 2, but between the eighth and ninth it only represents a tone. However the proportional relationship reflects its intervallic size and corresponds exactly to the defining ratio relationship. 128 Hz is twice the frequency of 64, ratio 2:1 and is an octave. The third harmonic is three times the frequency of the first, and the relationship between the third and second is that of three to two or 3:2, ie a perfect fifth and so on up the series. See Wilford W. Berard, 'The Eleventh and Thirteenth Partials', *Journal of Music Theory*, vol. 5, 1961, pp. 95-109 for a simple mathematical formula for determining the frequencies of a given partial. The frequency values and notation of the overtone series as given by Kardos concur with this article, whereas Berard points out the mistakes of most theorists, including Helmholtz, in notating the 11th and 13th partial as F and A, when they are closer to F# and A♭. Hindemith is the only one to notate them 'correctly', according to Berard. Kardos's notation is identical. Kardos lists Hindemith and Helmholtz in his Bibliography, almost the only non-Hungarians listed. Hindemith's description for the generation of a scale, chords, and harmonic relations from the harmonic series may have been a model for Kardos. From my reading of Hindemith, his description seems very similar to that of Kardos. Paul Hindemith, *Unterweisung im Tonsatz*, Schott's Söhne, Mainz, 1940. English version, Paul Hindemith, *The craft of musical composition*, Associated Music Publishers, New York, 1945, pp. 15-50.

tuning which seeks to replicate the ratios of notes in the harmonic series, since these are seen to have the greatest sonority, and the least dissonance caused by beating between harmonics and combination tones. Intervals that can be derived from neighbour notes in the series, their frequency ratio and conventional name are, in order of primacy:

harmonic no.	**frequency**	**ratio**	**name of interval**
Fundamental	*f*		
1st—2nd	2*f*	2:1	octave
2nd—3rd	3*f*	3:2	perfect fifth
3rd—4th	4*f*	4:3	perfect fourth
4th—5th	5*f*	5:4	major third
5th—6th	6*f*	6:5	minor third

The first six harmonics of the series contain a *senario* of consonant intervals:

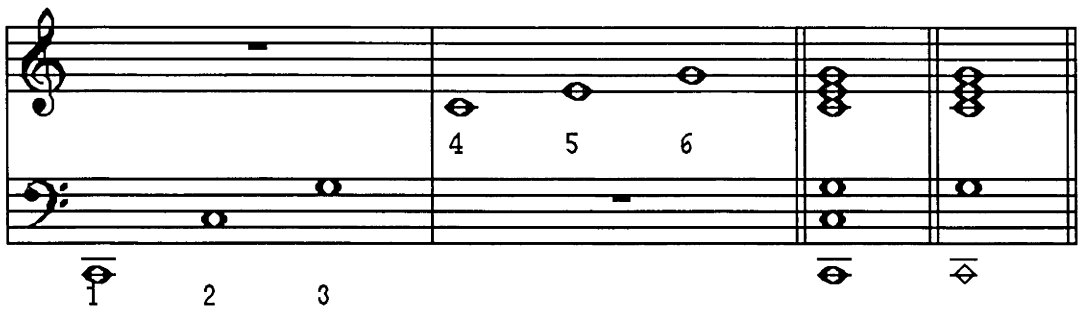


According to Kardos these intervals are the most fundamental as they are the first and most strongly heard of the harmonics. It is thus seen to be no accident that, in different periods of music history, they form the basis of harmony, providing the building blocks for a harmonic system consisting of triads of major and minor thirds, in various combinations, encased within the strong resonances of perfect fifths. The major triad—the foundation of classic harmony—is so primal precisely because it makes explicit what is implicit in the harmonic spectrum of the fundamental.[161]

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161 Hindemith concurs with this view and its derivation in the harmonic series. He writes:

'Tones 1-6 of the overtone series...outline the extended major triad, which is to the trained and the naive listener alike one of the most impressive phenomena of nature, simple and elemental as rain, snow, and wind. Music, as long as it exists, will always take its departure from the major triad and return to it...The feeling for the purity, the harmonic completeness, and the satisfying effect of the triad...is accordingly just as natural to us as the body's sense



Intervals in the harmonic series after the sixth harmonic reveal vicissitudes, complexities and are not so consonant:

harmonic no.	frequency	ratio	name of interval
6th—7th	7*f*	7:6	small minor third
7th—8th	8*f*	7:8	maximum whole tone
8th—9th	9*f*	8:9	big whole tone
9th—10th	10 *f*	10:9	small whole tone
10th—11th	11*f*	11:10	
11th—12th	12*f*	12:11	
12th—13th	13*f*	13:12	
13th—14th	14*f*	14:13	
14th—15th	15*f*	15:14	
15th—16th	16*f*	16:15	natural semitone

6th—7th harmonic

The ratio 7:6 between the sixth and seventh harmonic could be described as a

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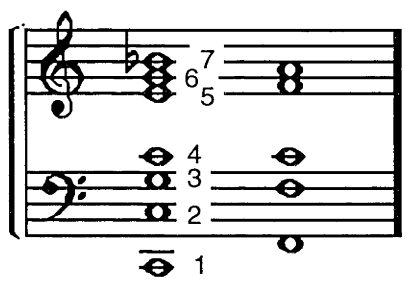
of space. It is based on the nature of the ear itself.' Hindemith, op. cit., pp. 22-23.

Schoenberg also wrote: 'To explain the constitution of the major triads we may cite the prototype in the overtone series.' Arnold Schoenberg, *Theory of Harmony*, Roy E. Carter, trans., University of California Press, Berkeley-Los Angeles, 1978, p. 312.

minor third but it is significantly narrower than the 6:5 minor third. $\frac{7:6}{6:5} \times 100 =$ 97.5% of 6:5. The theory postulates that the lower in the harmonic series the interval, the more fundamental to our hearing, the more sonorous. The minor third that occurs first in the series—between the fifth and sixth harmonic—has more primacy than the one that follows. However the 7:6 third has an important function in the dominant seventh chord.

The first seven harmonics: the dominant seventh triad

The first seven notes in the series 1:2:3:4:5:6:7 as a chord generate what is known as the dominant seventh chord. The dominant seventh is regarded by Karods, as by Rameau, as fundamental because we hear it in the harmonic series. The seventh is flat, forming a smaller minor third than the one directly below. The tension of the 7th degree in a dominant seventh, which displays a downward pull towards its resolution on the 3rd degree of the tonic, helps create the flatness that matches its acoustic prototype:



The seventh harmonic thus reveals the first tuning ambiguity—not all minor thirds are equal. A minor third, when in the context of the third third in the dominant seventh chord, should be smaller than the minor third directly below it.

7th—8th harmonic

The interval between the seventh and eighth harmonic, ratio 8:7, is ignored by most theorists including Kardos. It is rarely used in acoustically pure intonation or other tuning systems.[162] If it is classed as a very large whole tone, then logical consistency would demand it be regarded as more primal than the 9:8 and 10:9

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162 Archytas (c. 400 B.C.) uses intervals with ratio 8:7 in the diatonic tetrachord which went 9:8, 7:8, 28: 27. See Barker, *GMW II*, pp. 46-52.

versions that follow, but it is not.[163]

8th—9th harmonic

The ratio of interval between the eighth and ninth harmonic is the same as the Pythagorean tone, 9:8. Kardos called it the big whole step—*nagy egészhang*. It has been called the Major Tone, Tonus Major, the large just whole tone, major second. I will call it the big whole tone or 9:8 tone.

9th—10th harmonic

The interval between the ninth and tenth note in the series is also called a major second but it is slightly smaller having a ratio of 10:9 instead of 9:8. Kardos called it the small whole step (*kis egészhang*). Other terms include tonus minor, small just whole tone. I will call it the small whole tone. The two whole tones as successive steps form the first three notes of a just major scale. The ratios 9:8 and 10:9 multiplied equals 5:4 or a pure major third: $\frac{9}{8} \times \frac{10}{9} = \frac{5}{4}$.

Kardos ignores the intervals between the eleventh and fifteenth harmonics. This is presumably because intervals with these ratios are not used in the repertoire of western choral music familiar to Kardos.

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163 Several recent writers comment on the logical inconsistency of this disregard for the ratios of 7. For example Burdette Lamar Green, op. cit., p. 13 points out that 'it is curious that the ratios 7:6, and 8:7 are rejected while 9:8, 10:9 and 16:15 are accepted. In light of this last ratio (16:15), how can ratios involving 11, 13, and 14 be rejected? All of them should be valid. The same dilemma is presented by the proportions, or sets of ratios, that are available in the series. Taken in order: 1:2:3, 2:3:4, 3:4:5, and 4:5:6 are accepted; 5:6:7, 7:8:9 are rejected; 8:9:10 is vital...

15th—16th harmonic

The interval between the fifteenth and sixteenth notes of the harmonic series has a ratio of 16:15. Kardos called it the natural semitone (*természetes félhang*). According to theories based on numerological principles there is nothing particularly natural about this interval in itself, apart from being superparticular. Its value lies in the fact that it allows pure tuning of other consonant intervals, such as the minor third in a just scale. Other authors call it the diatonic semitone.[164]

We now have a set of intervals taken from adjacent notes in the harmonic series suitable for use in an intonation system: 1:1 perfect unison, 2:1 perfect octave, 3:2 perfect fifth, 4:3 perfect fourth, 5:4 major third, 6:5 minor third, 9:8 big whole tone, 10:9 small whole tone, 16:15 natural semitone. Other intervals are derived from interrelationships between the above set of intervals and from non-adjacent harmonics:

- 8:5 minor sixth, being the interval between the fifth and eight harmonic;
- 5:3 major sixth, being the interval between the third and fifth harmonic;
- 5:9 minor seventh, being the interval found between the fifth and ninth harmonic;
- 15:8 major seventh, being the interval between the eighth and fifteenth harmonic.

The use of each of these intervals in an acoustically pure tuning system is justified by the following factors:

- Its primary position in the harmonic series. The further down the scale it occurs the more primal it is, the more audible it is, and the greater its

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164 Blackwood, op. cit., p. 70. He points out that the semitone should only be used to refer to the tempered semitone since it is the only interval that is exactly half a tone. However Zarlino calls it a semitone and defines 'semi' as from *semus* meaning 'shy' or 'imperfect, rather than meaning 'half', the 'whole tone is never diminished or made imperfect by exactly one half its size, since no [superparticular] proportion...can be divided into two equal parts.' Gioseffo Zarlino, *The Art of Counterpoint*, op. cit., p. 38. Panaetius, quoted in Porphyry's (c. 233-304) Commentary, 65.21-67. 10., argued that the designation 'half-tone' was a misuse of language, just as one could not say the interval between white and black or between hot and cold is divided in half. The tone could no more be divided in half and thus be a semitone than a mule was a semi-ass. Barker, *GMW II*, pp. 237-239.

postulated consonance factor, thus giving rise to a 'hierarchy of consonances'. The most consonant interval is seen to be perfect unison, followed by 2:1, 3:2, 4:3, 5:4, 6:5...9:8, 10:9...16:15.[165]

- The simplicity of its ratios—small, simple ratios are seen to be easier to sing than ones with complex ratios.[166]

- The mathematical relationship of one interval to another. For example it is not possible to use two identical major seconds and have a pure major third. A natural semitone is required to maintain the purity of a perfect fourth that consists of a pure major third plus a semitone:

  $$\frac{4}{3} \div \frac{5}{4} = \frac{4}{3} \times \frac{4}{5} = \frac{16}{15}$$

  This would explain Kardos's choice of the 16:15 semiton over the 15:14 one, since given that 15:14 is prior to 16:15 in the harmonic series it would be preferable according to the first point. But, according to this point, the 16:15 is the interval required to create pure minor thirds.

A discussion of the implied theory of consonance contained in Kardos's system, and its historical precedents can be found in Appendix C.

3.2 Scales in pure intonation

Tunings for different scales can be generated by using a hierarchy of pure intervals derived from the harmonic series. Kardos only specified the tunings for the pure major and minor,[167] pointing out that *a cappella* vocal music is a flexible, living art, wherein the best tuning needs to be determined for each interval, not predetermined like the temperament of a fixed tuning instrument.

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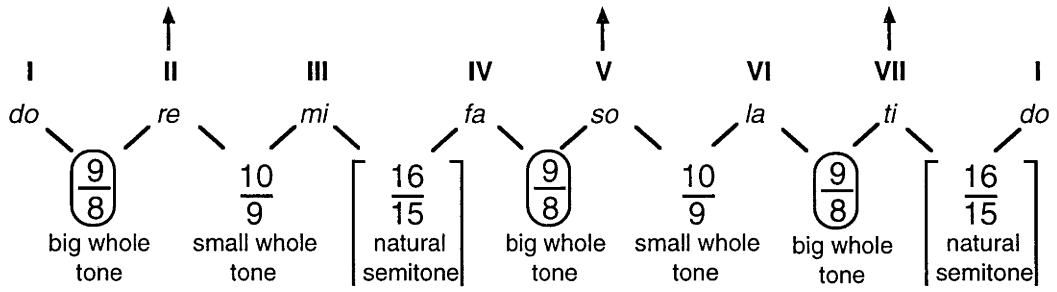
165 Kardos, *KK*, p. 22.

166 This principle is implicit in any comparison Kardos makes between a simple and complex ratio, whether the latter is the result of Pythagorean, tempered or other tunings. See for example Kardos, *Two Choral Works*, p. 166. See also Pál Kardos, 'Foundations of Education towards Pure Intonation', Kodály Seminar, Kecskemét, 1972, p. 39-40. Kardos's argument that intervals with small ratios are easier to sing than those with complex ones is similar to Partch's view of consonance being a hierarchy based on simplicity of ratio, but Kardos is talking about the singer's ability to intone an interval, not its consonance factor. For the latter see Partch, op. cit., p. 87.

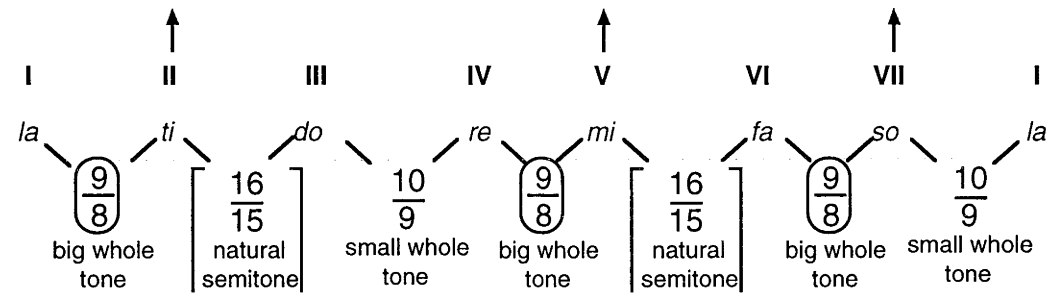
167 Kardos, *KK*, p. 34.

The pure diatonic major and minor scales

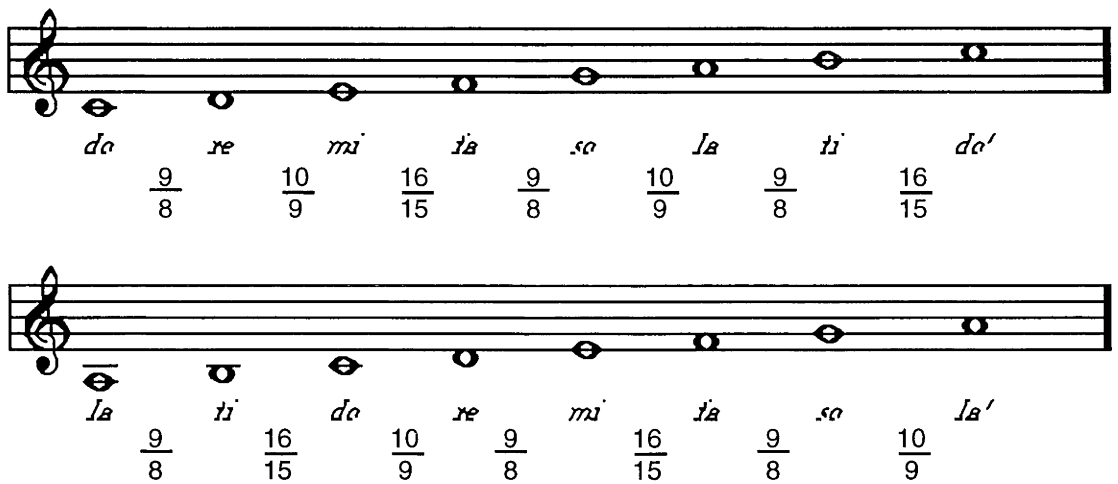
Major scale degrees



Minor scale degrees:



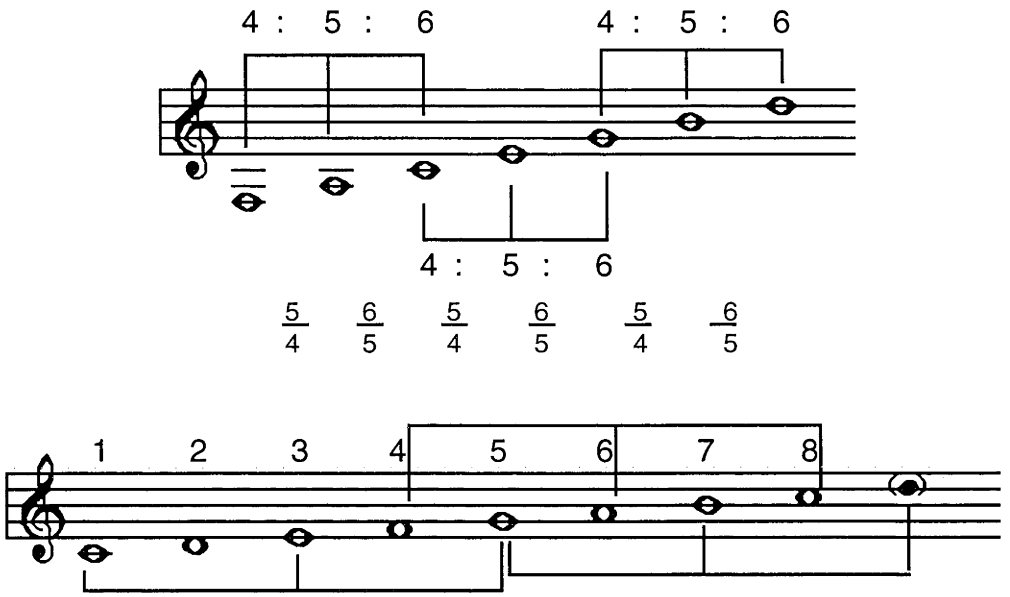
Harmonic and melodic factors will cause redistribution of the big and small whole tones to maintain optimum purity of tuning. Remembering the relativity of tonic solfa, this translated into a C major scale and an A minor scale would be as follows:



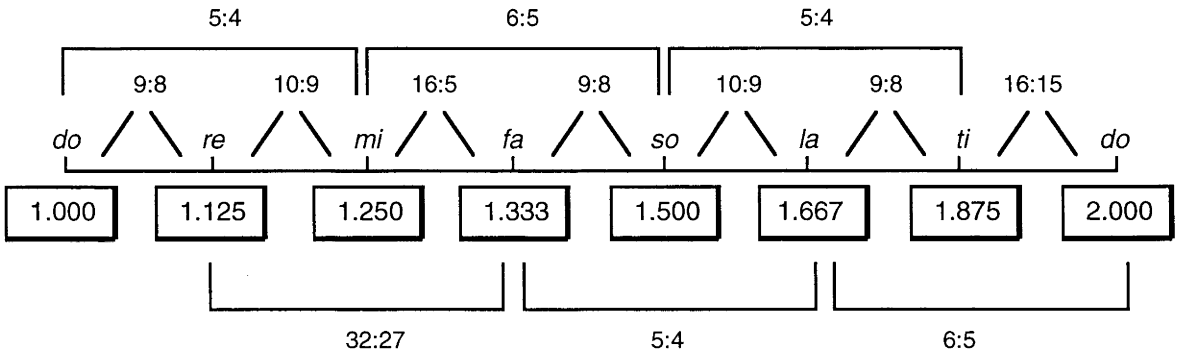
Two natural semitones, two small whole tones and three big whole tones add to exactly an octave:

$$\frac{16}{15}\times\frac{16}{15}\times\frac{10}{9}\times\frac{10}{9}\times\frac{9}{8}\times\frac{9}{8}\times\frac{9}{8}=2$$

Kardos did not spell out how to generate the above scale from the harmonic series, but that it comes from the harmonic series is implicit in all his writings. It can be derived from the harmonic series by taking the 4th, 5th and 6th harmonics which form a pure major triad. Three such triads joined together, then placed in the same octave, create a major scale with the same ratios as above:



The following diagram converts the ratios of each note's relationship to the first note into decimals. The decimal numbers act like string lengths. To find the interval between any two notes, divide the higher by the lower:



$\frac{9}{8} \times \frac{10}{9} = \frac{90}{72} = \frac{5}{4}$ Pure major third

$\frac{16}{15} \times \frac{9}{8} = \frac{144}{120} = \frac{6}{5}$ Pure minor third

$\frac{10}{9} \times \frac{16}{15} = \frac{160}{135} = \frac{32}{27}$ Pythagorean minor third

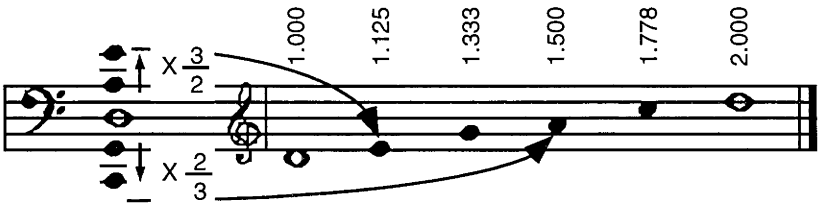
$\frac{9}{8} \times \frac{256}{243} = \frac{2304}{1944} = \frac{32}{27}$ Pythagorean derivation of the Pythagorean minor third

This procedure is the same as used to generate a just scale, although the just scale was not originally taken from the harmonic series but generated by using a monochord subdivided into 9:8 and 10:9 tones and 16:15 semitones to enable pure 5:4 and 6:5 thirds and hence triads (see above).

Pentatonic scales

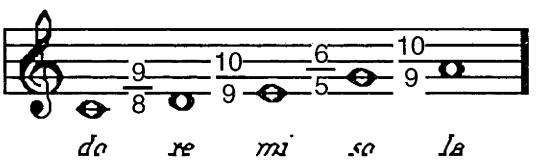
Anhemitonic pentatonic scales (without semitones) are found to be free of some of the more ambiguous aspects of tuning. That is one reason why Hungarian music educators advocate extensive use of pentatonic children's song material in the early stages.[168] An anhemitonic pentatonic scale can be generated using Pythagorean tuning procedures with multiplications of 3:2. Two fifths in a circle of fifths above the starting note and two fifths below, brought down or up into the range of an octave, generates an anhemitonic pentatonic scale:

168 Kardos, 'The Foundations of Education towards Pure Intonation', bases his pedagogy of pure intonation on the pentatonic scale.

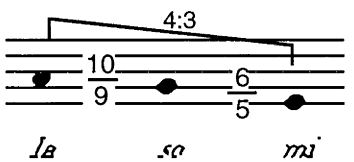


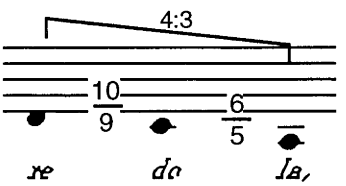
Pythagorean pentatonic scale, Ian Johnston, *Measured Tones*, p. 8

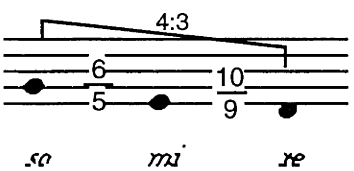
The ratios of intervals between each degree of this scale are 9:8 32:27 9:8 32:27 9:8. Another simple way is to use the 'method of concordance' based on descending fourths: c'—g—d—A—E, brought back into the range of an octave c'—d'—e'—g'—a'. However a scale generated in this manner uses only the Pythagorean 9:8 whole tones and minor thirds, which are not the tunings used in Kardos's pentatony. The thirds are acoustically impure, hence would be rejected out of hand by Kardos. A just tuning for a pentatonic scale using pure minor thirds and a combination of 9:8 and 10:9 whole tones could be:

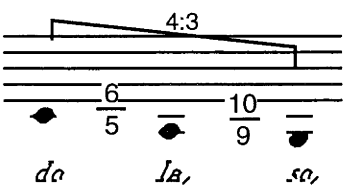


Vocal music based on pentatonic tonalities often consists of infra-pentatonic modules consisting of a 4:3 perfect fourth subdivided into a 6:5 minor third and 10:9 small whole tone. The two units when conjoined to form a full pentachord reveal various tuning ambiguities associated with the two forms of whole tone. The size of the note of disjunction between the two modules will usually be 9:8, but depends upon the particular melodic pattern. Fuller discussion of these patterns can be found in section 5.3.









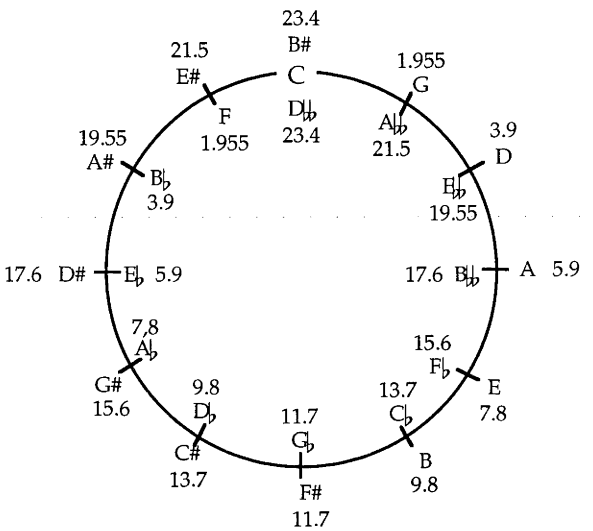
Modal scales

The placement of big and small whole tones depends on harmonic and melodic

context and will be discussed in section 5.4.

3.3 The inherent contradictions of pure tuning

Tuning systems based on pure ratios contain inherent limitations. No combination of 2:1 octaves and 3:2 perfect fifths will produce any other pure interval.[169] There is no such things as enharmony in either Pythagorean or just tuning. The familiar cycle of fifths displays not only the anomaly of the Pythagorean comma but also the syntonic comma associated with just intonation:



Kardos, *Two Choral Works*, p. 163

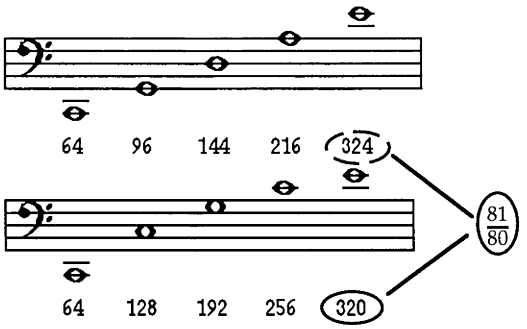
The Pythagorean comma is the difference between seven 9:8 tones and an octave, or seven pure 2:1 octaves and twelve pure 3:2 fifths: $\frac{3}{2}^{12} \times \frac{2}{1}^{7} = \frac{531441}{524288}$ or 23.4 cents. The 'enharmonic' notes on the outer and inner circle of fifths are all non identical by $\frac{531441}{524288}$ or 23.4 cents. Equal temperament evenly distributes the comma so that the first fifth above C is 1.955 smaller than the pure 3:2 fifth, the second fifth d above C is 3.9 cents lower than the pure d and so on.

Not only the fifths but the thirds also hide contradictions:

- the 4th fifth in the circle of fifths starting with C is an e', it corresponds with the first major third in the harmonic series based on C which is also e', but the e' in the circle of fifths does not equate with the fifth harmonic, in fact it is

169 See Blackwood, op. cit., p. 48, for a mathematical proof of this statement.

higher by 81:80:



Kardos, *Two Choral Works*, p. 164

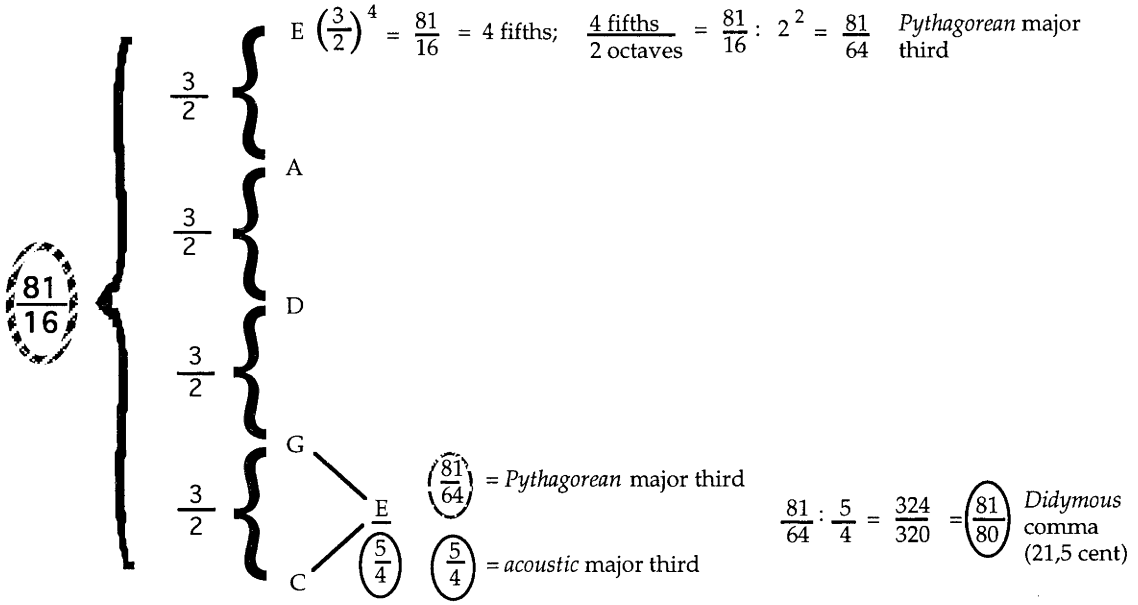
This difference of 81:80 is called the syntonic comma or comma of Didymus after the Greek theorist credited with discovering it. Without recourse to overtones four perfect fifths exceed two octaves plus a major third by 81:80—

$$\frac{2}{1}^{2} \times \frac{5}{4} = \frac{5}{1} \qquad \frac{3}{2}^{4} = \frac{81}{16} \qquad \frac{81}{16} \div \frac{5}{4} = \frac{81}{80}$$

The syntonic comma is also the difference between the 9:8 and the 10:9 forms of whole tone and if an interval is composed of the wrong form of whole tone it will deviate from its pure form by a syntonic comma (see below).

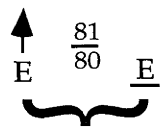
$$\frac{9}{8} \div \frac{10}{9} = \frac{81}{80}$$

The interval generated by four perfect fifths brought down two octaves is the Pythagorean ditone. This can be expressed mathematically in the following:



Kardos, *Two Choral Works*, p. 164

The Pythagorean major third e' is a syntonic comma larger than the acoustic major third e:



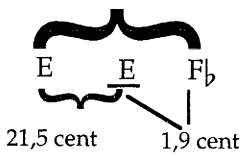
Kardos, *Two Choral Works*, p. 164

Three notes considered equal e, e and f♭ can be compared as follows: In the enharmonic 5th rows there is a Pythagorean comma difference between each of the sharpened and flattened fifth rows. The difference between the enharmonic e and f♭ is a Pythagorean comma of 23.4 cents:

+12	B#	~	C	0
+11	E#	~	F	- 1
+10	A#	~	B♭	- 2
+ 9	D#	~	E♭	- 3
+ 8	G#	~	A♭	- 4
+ 7	C#	~	D♭	- 5
+ 6	F#	~	G♭	- 6
+ 5	B	~	C♭	- 7
+ 4	E	~	F♭	- 8
+ 3	A	~	B𝄫	- 9
+ 2	D	~	E𝄫	-10
+ 1	G	~	A𝄫	-11
0	C	~	D𝄫	-12

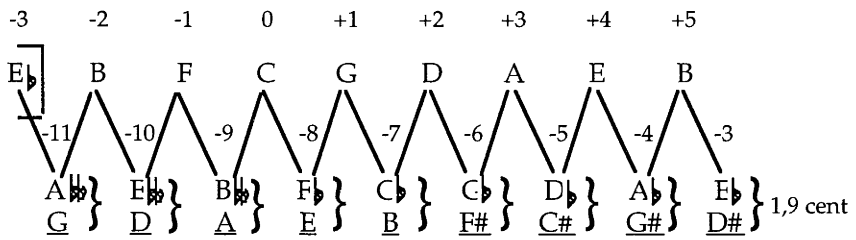
Kardos, *Two Choral Works*, p. 165

The difference between e and f♭ is the difference between the Pythagorean and syntonic commas: $\frac{531441}{524288} \div \frac{81}{80} = \frac{32805}{32768}$. In cents this is 23.4 - 21.5 cents = 1.9 cents, and is called the schisma, or crack.



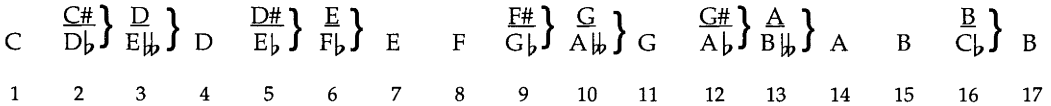
Kardos, *Two Choral Works*, p. 165

Kardos points out that there was a possibility of a schismatic exchange or substitution already in the Arabian-Persian system of 17 grades:[170]



Adapted from Kardos, *Two Choral Works*, p. 166

In a tone row:



Adapted from Kardos, *Two Choral Works*, p. 166

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170 Kardos, *Two Choral Works*, p. 166.

C# is 1.9 higher or a schisma higher than D♭ which is the lower 5th fifth. D is equally higher by the same amount than E♭ etc.

Further anomalies are associated with just intonation. Three acoustic major thirds are narrower, while four acoustic minor thirds are broader than the octave:

$\left(\frac{5}{4}\right)^3 = \frac{125}{64}$ while $\frac{128}{64} = \frac{2}{1}$ (= octave) instead of $\left(\frac{6}{5}\right)^4 = \frac{1296}{625}$ where $\frac{1250}{625} = \frac{2}{1}$

Expressed in cents: three acoustic major thirds have the value of 386.3 x 3 = 1158.9 cents instead of a 1200 cent octave. Four minor thirds equal 315.6 x 4= 1262.4 cents instead of 1200. This data shows the blatant contradictions of the most consonant and simple intervals.

Translating these principles into practice, I return to the scale in just ratios outlined above. The advantage of the just scale is the use of two forms of whole tone which allow all three major thirds and two of the three minor thirds to be pure. The disadvantage is the syntonic comma which causes certain intervallic relationships to be impure. These can be delineated as follows:

- An interval that contains a 9:8 whole tone when its acoustic form requires a 10:9 tone will be greater than acoustic form by 81:80 (ad comma);
- An interval that contains a 10:9 when the acoustic requires a 9:8 whole tone will be smaller than its pure form by 81:80 (de comma);
- A 6:5 minor third requires a 9:8 whole tone and 16:15 semitone (in any order). A minor third containing a 10:9 whole tone will be smaller than 6:5 by 81:80—

  $\frac{10}{9} \times \frac{16}{15} = \frac{32}{27}$.

  In fact this is the same as the Pythagorean minor third although the derivation is different. The Pythagorean third is the consequence of having a 9:8 whole tone and a 256:243 semitone (*leimma*)—

  $\frac{256}{243} \times \frac{9}{8} = \frac{32}{27}$.

- A 5:4 major third requires a 9:8 and a 10:9 whole tone (in any order). If it consists of two 9:8 whole tones it will be 81:64, the Pythagorean third which is bigger than 5:4 by 81:80. This would not normally occur in just intonation,

since all major thirds are pure. Likewise if a major third consists of two 10:9 whole tones it will be smaller than 5:4 by 81:80, a de comma third—

$\frac{9}{8}X\frac{9}{8}=\frac{81}{64}$ $\frac{81}{64}\div\frac{5}{4}=\frac{81}{80}$ $\frac{10}{9}X\frac{10}{9}=\frac{100}{81}$ $\frac{5}{4}\div\frac{100}{81}=\frac{81}{80}$;

- The 5:3 pure major sixth should consist of a 3:2 and a 10:9. If it consists instead of a 3:2 and 9:8 it will be ad comma, the Pythagorean major sixth. This occurs in the just major scale between degrees II and VII— $\frac{3}{2}X\frac{9}{8}=\frac{32}{37}$;

- The inversion of the Pythagorean major third is the Pythagorean minor sixth, $\frac{128}{81}$ less a comma than the pure 8:5 version. Again since all the major thirds in the just scale are pure it would not normally occur in just intonation;

- A 4:3 fourth in just intonation should contain a 9:8, 10:9 and 16:15 (in any order), or alternatively a 5:4 major third and a 16:15. If it consists of two 9:8 whole tones and a 16:15 it will be an ad comma fourth. If it consists of two 10:9 whole tones and a 16:15 it will be a de comma fourth—

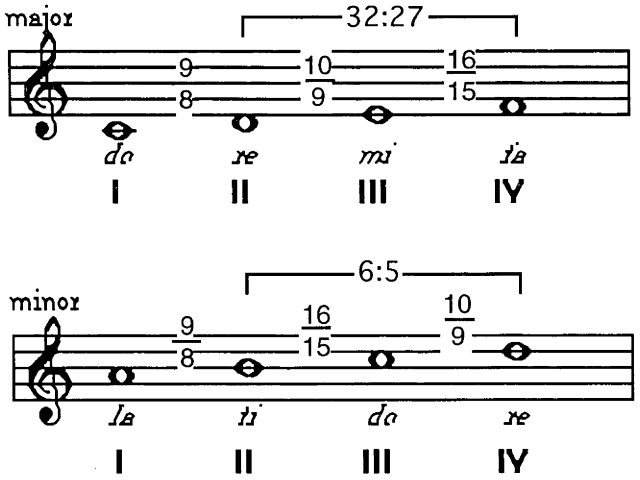
$\frac{9}{8}X\frac{9}{8}X\frac{16}{15}=\frac{27}{20}$ $\frac{27}{20}\div\frac{4}{3}=\frac{81}{80}$ $\frac{10}{9}X\frac{10}{9}X\frac{16}{15}=\frac{320}{243}$ $\frac{4}{3}\div\frac{320}{243}=\frac{81}{80}$;

- The difference between the 3:2 perfect fifth and 4:3 perfect fourth is 9:8. If the interval between the fourth and fifth is 10:9 the fifth will be de comma—

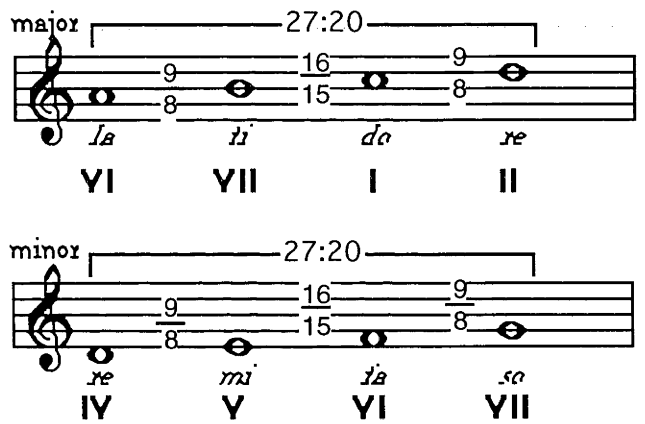
$\frac{4}{3}X\frac{10}{9}=\frac{40}{27}$;

The syntonic comma frequently adheres to certain intervals associated with the 2nd degree in the major. Using the C major (and A natural minor) scale above the following intervals are larger or smaller than pure by 81:80:

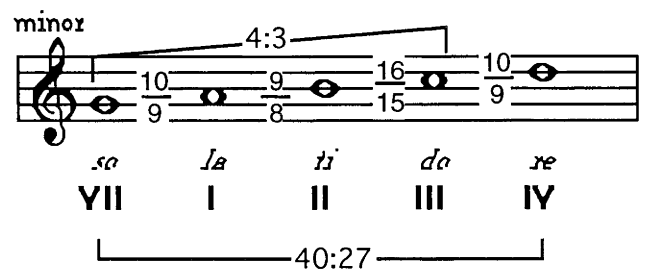
- The minor third between degrees II and IV in both major and minor is 32:27, which is smaller than the pure 6:5 by 81:80. This is because the 2nd degree is 9:8 from the tonic, leaving between it and the 4th a 10:9 and a 16:15—



- The fourth between degrees VI and II of major is an ad comma fourth, 27:20, larger than a pure 4:3 by 81:80. It is also found between degrees IV and VII of minor:

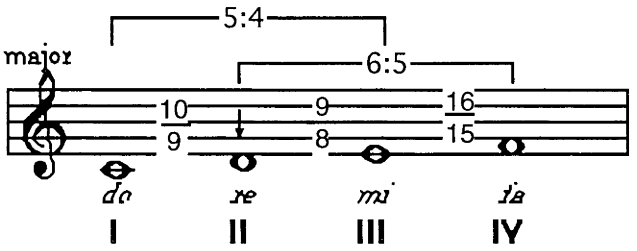


- The fifth between degrees II and VI in major, and between degrees VII and IV in minor is a de comma fifth:

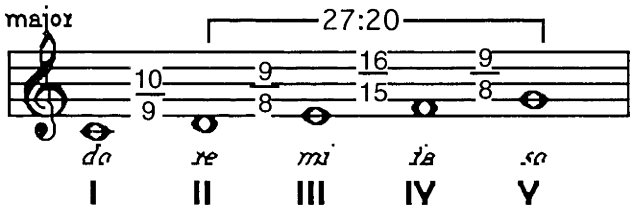


Equal temperament distributes the comma evenly between all notes, yet this means that all notes are equally out of tune. Kardos's solution is flexible intonation, the exchanging of big and small whole tones to allow pure tuning on

all degrees.[171] For example, if in the case of the diagram above the 2nd degree is lowered by 81:80 by reversing the order of big and small whole tones between degrees I, II and III, there will be a 10:9 tone between degrees I and II and a 9:8 tone between degrees II and III. This would mean that the major third I—III, *do—mi* will still be pure, but the minor third between II—IV *re* and *fa* will also be pure:



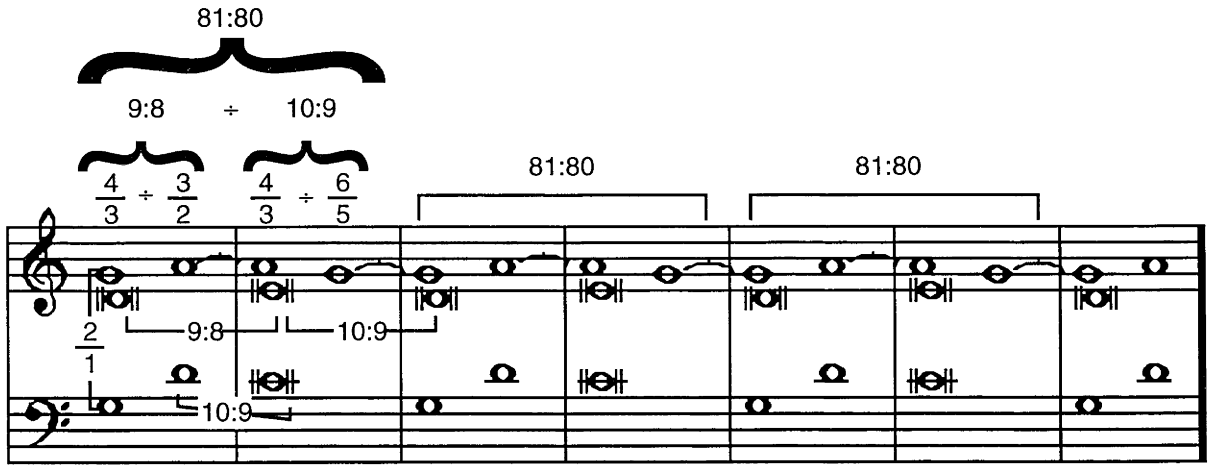
Such adjustments require a careful analysis of the entire work to avoid creating impurities in another intervallic relationship or a rise or drop in pitch in the entire work. The solution outlined above would create an ad comma fourth between degrees II and V:



If this was corrected by lowering the 5th, the entire work would have modulated downwards by 81:80, a phenomenon well known in the Renaissance and often called 'wandering tonics'. Benedetti, in 1563, demonstrated that a rise in pitch could just as easily occur:

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171 In support of this proposed solution, Carola Anderson argues that the Didymus comma, supposed difficulties in modulation and the progressive lowering of pitch when pure triads are used, can be mitigated by feasible pitch adjustments in performance. Carola B. Anderson, 'The Trouble with Didymus', *1/1 Just Intonation*, vol. 1, no. 3, Summer 1985, pp. 1, 11-13.



Benedetti, *Diversarum speculationum mathematicarum & physicorum liber*, 1585[172]

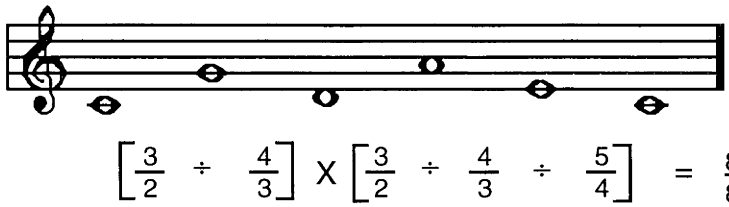
My analysis of this example is as follows. Although it is not possible to conclusively state the final or the mode from the above, let us assume it is based on a final of g'. In mode it is one of the modes beginning with a two whole tones (in the modal understanding of the period it would most probably be Mixolydian according to Glarean terminology or mode 7 according to Zarlino's 1573 nomenclature).[173] In any case the same adjustments would be necessary. The alto d' is a common tone and therefore cannot be the part to adjust. The progression g' to a' then has to be a 9:8 to retain the purity of fourth followed by fifth, since 4:3 ÷ 3:2 = 9:8, (if it is in a mode on g' then this would in any case be normal). When the alto part moves from d' to e' it has to do so with a 9:8 step to create a pure 4:3 with the high a', since again a 3:2 ÷ 4:3 is 9:8, (the normal progression between degrees V and VI would be 10:9). When the soprano a' descends to g' it has to do so by a 10:9 tone. The g' in bar 2 is 81:80 higher than the g' in bar 1. The e' descending to d' will now do so by a 10:9 to create a pure 4:3 with the new g'. The tenor's descent to c' will also be by a 10:9 tone. The whole work thus rises in pitch a 81:80 comma every repetition of this pattern, in 8 bars a rise of 4 syntonic commas has occurred. Benedetti gives several other such examples in the *Diversarum* which demonstrate how certain combinations of purely tuned intervals will lead to a rise or fall in pitch.

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172 Turin, p. 279, reprinted in Palisca, *Humanism*, p. 263. Alto ratios added by the author.

173 *Istitutioni Harmoniche*, IV, 1, 'On the Modes', trans. Vered Cohen, ed., New Haven, 1983. In the 1558 version Zarlino used Glarean's names and order.In the 1573 version he renumbered them to begin the authentic and plagal modes on C as modes 1 and 2, the rationale being that he based his tuning on the just intonation of the C major scale, based on Ptolemy's syntonic diatonic, and as a more rationale way out of the confusion of names. Owing to the 'diversity of their ordering, the variety of number, and the difference in names found in all these authors, on cannot draw anything but confusion of mind.' p. 10.

Christiaan Huygens (1629-1695) showed the same phenomenon in the most simple example:



$$\left[\frac{3}{2} \div \frac{4}{3}\right] \times \left[\frac{3}{2} \div \frac{4}{3} \div \frac{5}{4}\right] = \frac{81}{80}$$

Huygens, *Oeuvres completés*, 20, 77[174]

A progression that involves an ascent by a fifth, followed by a descent by a fourth, another ascent by a fifth, another descent by a fourth, followed by a descent by a major third, will lead to a rise in pitch of a comma. Huygens, like Kardos, observed that singers solved the problem of the comma by either keeping internal intervals pure and accepting the consequent modulation in pitch, or by unconsiously adjusting (tempering) certain intervals. The first alternative tended to occur in slow, moving passages, the second in fast passages where they still remembered the absolute pitch of the piece.[175]

In conclusion it is apparent that acoustically pure intonation is much more complex than the fine tuning of nuances in the pitch of single tones or of individual intervals. As Kardos points out, a tone which is crystal clear in relation with one tone may be false in relation to another tone. There are times when in order to make one interval pure the absolute pitch of a note sung earlier has to be disregarded, and a new interpretation of its tuning made. At other times to retain the absolute pitch of the piece and avoid comma modulations, it may be necessary to sacrifice the purity of the interval. '"Pure" and "false" are, in musical terms, involved and contradictory concepts, and yet purity is such a simple, easily understood thing in that it is a natural phenomenon.'[176] The following analyses will show how to make these adjustments in various harmonic or melodic contexts.

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174 Cited in Cohen, op. cit., p. 40.

175 ibid., p. 215.

176 Kardos, Pál, *Two Choral Works*, p. 163.

Chapter 4 Choral technique

Basic principles of intonation

The subject of vocal production and choral training is vast, and although highly relevant to vocal intonation, is beyond the scope of the current work.[177] Nevertheless, some basic principles of choral technique as outlined by Kardos[178] that are important in achieving acoustically pure intonation are:

- **Balance**, which in turn depends upon such factors as:

  homogeneity of dynamics;

  blending of tone colour;

  uniformity of vowel production;

  breathing technique;

  aural training in acoustically pure intonation.

- **Linear factors include:**

  counteracting the effects of gravitational pull;

  maintaining the pitch level of sustained notes, repeated notes, and notes that return to the same pitch.

- **Vertical factors include:**

  tonal balance of harmonic intervals, appropriate 'weighting' of component notes;

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177 Consult the bibliography for further information on this topic. A good starting place is the chapter on the acoustics of the singing voice in Johnston, op. cit., pp. 331-352; also Johan Sundberg, 'Acoustics of the Singing Voice', (Readings from) *Scientific American*, pp. 16-24, W. H. Freeman and Co, San Franscisco, 1978.

178 Kardos, *KK*, p. 43.

voice-crossing;

dissonance-friction.[179]

Balance

Kardos defines choral balance as a homogeneous sound from the whole choir as well as from each part, with equal emphasis on tone-colour, intonation and dynamics. It is a perfect blending, a whole which is greater than the sum of its parts, a new tonal quality, a rejuvenation of tonal brightness and beauty of tone of individual voices through their unification in the choir.[180] The technical factors required to achieve this end include uniformly shaped vowels, refinement of tone colour, attention to maintaining a consciously controlled balance of sound by each member to the whole, a highly refined aural awareness of intonational principles and the ability to make minute adjustments in line with this awareness.[181]

Blend

The importance of blend for intonation has been born out by experiments since Kardos. Allen Goodwin has investigated the phenomenon of choral blend by identifying spectral differences between vocal sounds produced in solo and in ensemble singing. Those produced in ensemble conditions had stronger fundamental frequencies in combination with fewer and weaker upper partials than those singing solo.[182]

Good choral tone, Kardos advises, needs singing which is relaxed, effortless, without tightening or straining of the voice. There is also a certain dynamic level which is comfortable, and has a feeling of looseness. A first step towards tonal

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179 Kardos, *KK*, p. 31. Dissonance-friction is caused by the pull of dissonance notes to their consonant resolution. A major seventh pulls towards the octave, an harmonic semitone pulls towards unison.

180 Kardos, *KK*, p. 8.

181 Kardos, *KK*, pp. 8-9.

182 Goodwin, Allen Wayne, 'An acoustical study of individual voices in choral blend', *JResearchMEducation*, vol. 38, no. 2, 1980, pp. 119-28. Singers had to blend with the sustained tones of a prerecorded ensemble heard in earphones along with a singer's own voice. These individual voices were recorded in isolation, then compared.

balance is a single note sung with this dynamic level, in the most comfortable middle range of each voice type. Choristers need to focus on blending with those around them, on sustaining a steady tone, on matching their dynamic level as well as their tone colour or timbre to each other. Kardos did not recommend vibrato. Singers can tune to the resonance and overtones of each other if they stand close and listen to the resonance. No choir can produce a beautiful tone if they cannot achieve a balanced, homogeneous, pure unison. Kardos uses the terms *arsis* and *thesis* to refer respectively to the build up and release of intensity. Arrows slanting slightly up and down indicate respectively *arsis* and *thesis*. Breathing technique is essential for balance, dynamics, sustaining tones, phrasing and maintaining the correct pitch.[183]

Uniformly produced vowels are an absolute requirement for intonation, as every vowel creates its own formants, and if a certain vowel is produced in different ways the partials will create as much dissonance as singing out of tune.[184] Rogers Covey-Crump emphasises the need for pure, uniformly produced vowels as an essential requirement for *a cappella* singing. A cruel test is to ask the singers to sing the word 'Alleluia' slowly on one note without any vibrato, dividing it into its five syllables. 'Al' and 'le' will probably sound flat, 'lu' will be in tune, 'i' will be sharp, and the final 'a' flat. Thus one word, sung in unison on a single note will have four changes in pitch. Covey-Crump does not explain why, but a simple explanation is the different formants of the vowels involved. He goes on to say that a trained singer can find the right vowel placement to correct these pitch deviations, but a single badly placed vowel will ruin the purity of a justly tuned triad.[185] Sten Ternstrom has shown experimentally the effect of change of vowel on purity of intonation.[186] Amateur male choir singers attempted to sing just intervals from a synthetic vowel tone with varied spectrum and vibrato

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183 A set of vocal exercises putting these factors into practice is to be found in Kardos, *KK*, Exercise 1—25.

184 See Helmholtz, op. cit., pp. 103-119. Elsewhere, (p. 206) he explains how certain vowels reinforce the partials that will conflict, giving the example of the vowel *a* as in 'father' which would, in the case of a major third, create dissonance between the 4th partial of the lower note and the 3rd partial of the higher note, creating a semitone dissonance. If the vowel is changed to *o* as in 'no' the dissonance disappears.

185 Rogers Covey Crump, op. cit., p. 159.

186 Sten Ternstrom, Johan Sundberg, 'Acoustical factors related to pitch precision in choir singing', *Speech Transmission Laboratory* q. progress and status report, 1982/3, Oct 1982, pp. 76-90.

content. Change of vowel caused significant changes in intonation, whereas absence of vibrato and presence of common and high partials facilitated intonation.

Tessitura and harmonics

Pitch has an influence on the audibility of harmonics. The lower the voice, the richer in audible harmonics. Intervals smaller than a fourth can sound 'grumbly' in the thick timbre of the bass part.[187] This fact was recognised by Christiaan Huygens (1629-1695) who pointed out the effects of pitch upon degree of consonance. Thirds in the bass were not agreeable consonances, according to Huygens.[188]

The principal of gravitation in singing:

There is a common tendency towards flatness in singing. Kardos calls this tendency the principle of gravitation. When an ascending progression is being sung there is a tendency to under reach the note in question, whereas there is a tendency to over reach a descending interval. This 'gravitational pull' needs to be actively resisted.

Good vocal technique is a necessary but not sufficient condition for pure intonation. Detailed knowledge of the principles of pure intonation is also necessary. The next section gives a step by step outline of these principles, based on Kardos's work in this area.

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187 Johan Sundberg, 'Singing and Timbre', *Music, room and acoustics* (Publications of the Royal Swedish Academy of Music 17), pp. 57-81, Kungl. Musikaliske Akademien, Stockholm, 1977. Sundberg compares the formant technique in a soprano singing high pitches as well as the background of a male's singing formant. He discusses the differences between bass, baritone and tenor timbres and between male and female voices.

188 Cited in Cohen, op. cit., p. 211.

Chapter 5 Basic characteristics of intervals and their inter-relationships

5.1 Equal temperament versus acoustically pure intonation

Even Zarlino, history's most eloquent advocate of just intonation in vocal music, conceded that it is impractical to attempt to fix every nuance of ideal purity on a musical instrument. The only solution for musical instruments of fixed tuning was the tempered system.[189] Yet for Zarlino, Kardos and others, this was not the solution for singers of unaccompanied music. 'Just as in nature there is no enharmony, in vocal singing there is no temperament,' Kardos asserted.[190] Choral singing can be either pure or false but never tempered. Zarlino argued that singers always instinctively favoured pure tuning based on the simple ratios of the syntonic tuning (just intonation).[191] Against this Vincenzo Galilei argued that in polyphonic music singers must continually adjust to each other and thus only the octave remained pure. Further, he continued, the ear can be trained to hear harmony in tempered, ie imperfect intervals, and constant exposure to the new tempered intervals had corrupted human ears, making people actually prefer the 'narrow' fifth over the pure interval.[192] 'I tell you, besides, that the fifth is generally heard with greater pleasure in the size that Aristoxenus gives it, than in its true sesquialter ratio.'[193] Kardos believed that the widespread use of

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189 Zarlino, *Le Instutioni*, II., 45, quoted in Palisca, *Humanism*, p. 250. Zarlino devised 'an equally tempered diatonic monochord' that combined the Pythagorean diatonic and Ptolemaic diatonic syntonon. The comma was divided into seven and evenly distrubuted across the major and minor thirds.

190 Kardos, *Two Choral Works*, p. 166.

191 See *Sopplimenti musicali*, chaps. 33-37.

192 Vincenzo Galilei, *Dialogo della musica antica e della moderna*, 1581, paraphrased by Jamie James, *The Music of the Spheres*, Abacus, London, 1994, p. 91. Partial translation in Oliver Strunk, *Source Readings in Music History*, W. W. Norton & Co, New York, 1965, pp. 112-133. See also Vincenzo Galilei, *Discorso intorno all'opere di messer Gioseffo Zarlino da Chioggia*, G. Marescotti, Florence, 1589. Argument paraphrased in Palisca, *Studies*, pp. 227-231. See also *Discorso particolare intorno alla diversita delle forme del Diapason*, Florence, ed. and trans. in Claude Palisca, *The Florentine Camerata*, op. cit., pp. 18-91.

193 Galilei, *Dialogo*, 53, quoted in Palisca, *Studies*, p. 194. Palisca goes on to say that to Galilei the 3:2 fifth sounded and he found the Aristoxenian fifth had a 'grace and certain softness

equal tuning has had adverse effects on choral and aural culture, yet he would maintain, as did Zarlino, that choirs with finely tuned hearing singing *a cappella* will strive for pure intervals.[194] There is some historical evidence for this view. Bruce Haynes discussing the intonational performance practices of the 17th and 18th century argues that flexibility in tuning, not temperament, was the practice among singers and non-fixed pitch instrumentalists. Even when performing with keyboard instruments they did not conform strictly to the temperament of the keyboard.[195] Lindsay Norden asserted in 1936 that 'no singer can sing *a cappella* in any temperament...*A cappella* music, therefore, is always sung in just or untempered intonation.'[196] Timothy Smith in 1991 writes:

> In truth, most musicians have never performed, and never will perform, within the constraints of equal temperament. Accomplished wind players, string players, and singers routinely give colour to pitches by raising or lowering them. This process is governed in tonal music by the mental and physical attributes of tonality, and in atonal music by the pitches that immediately surround the pitch in question.[197]

Other contemporary writers are selective in the application of tuning systems: the Swedish aural trainer, Per-Gunnar Alldahl, asserts that just intonation is appropriate for older music, but music with modulation, chromaticism, alteration requires some tempering, while free-tonal music, even in *a cappella* singing, assumes equal temperament. Related to this debate is the use of equal tempered keyboards as a rehearsal prop in choral practice or aural classes. Both Kodály and Kardos strongly disagreed with the use of pianos in aural music education.[198] Alldahl advocates a very small, judicious use of the piano in

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and languidity that suited the current taste.

194 See for example Kardos's 'Barkók két kórusmüve a tiszta intonálásra nevelés tükrében (Ne hagyj itt!, Bolyongás) [Two Choral Works of Bartók in the Mirror of Education towards Pure Intonation]', *az ének-zene tanítása* 1970/4-5, szám, p. 166. Pál Kardos, 'Foundations of Education towards Pure Intonation', Kodály Seminar, Kecskemét, 1972, p. 39-40.

195 Bruce Haynes, 'Beyond temperament: non-keyboard intonation in the 17th and 18th centuries', *Early Music*, vol. 24, no. 3, August 1991, pp. 357-381.

196 Lindsay Norden, 'A New Theory of Untempered Music', *Musical Quarterly*, XXII, 1936, p. 218. See Rogers Covey-Crump, 'Pythagoras at the Forge: tuning in music before 1600', *Companion to Medieval and Renaissance Music*, Tess Knighton and David Fellows, eds., J. M. Dent and Sons, Ltd, London, 1992, p. 159.

197 Timothy Smith, 'A comparison of Pedagogical Resources in Solmization Systems', *Journal of Music Theory Pedagogy*, vol. 5, no. 1, 1991, pp. 1-25.

198 See Zoltán Kodály, *Énekeljünk Tisztán*, Editio Musica Budapest, 1941, p. 1. English translation, *Let Us Sing Correctly*, Boosey and Hawkes, 1954, p. 2. Lindsay Norden also decries the use of pianos: 'One of the worst fallacies in present-day music is the endeavor to

rehearsal and aural training situations.[199] However one major problem for training in pure or just intonation is a reliable aural model. Pianos with their tempered tuning do not provide this. Developing electronic keyboards linked to computer programs capable of flexible tuning would provide a major breakthrough. There are encouraging signs that computer-generated sound sources with flexible intonation will prove a useful sound model in the future.[200]

Acoustically pure intervals that differ from the tempered forms by a significant amount need to be intoned with special care. The table below maps out the difference between acoustically pure and tempered consonant intervals in terms of cents:

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teach *a cappella* music with the assistance of a piano in equal temperament...The singers have been taught false intervals, but are supposed to sing pure ones, for all *a cappella* music must be sung in just intonation.' Lindsay Norden, op. cit., p. 218.

199 Per-Gunnar Alldahl, *Choral Intonation*, op. cit., p. 39, p. 35.

200 The 1/1 Just Intonation Network in California, and the Alternative Tuning Network abounds with reports of various experimental instruments using alternative tunings. The Just Intonation Network can be contacted at 535 Stevenson Street, San Francisco, CA 94103.

Comparison between acoustic and tempered intervals

Interval name (conventional)	Ratio (A)	Acoustically pure form (A)	Tempered form (T)	Difference (A-T)
		in cents		
Perfect unison	$\frac{1}{1}$	0	0	0
Semitone	$\frac{16}{15}$	111.7	100	+11.7
Major second (small)	$\frac{9}{8}$	182.4	200	- 17.6
Major second (large)	$\frac{10}{9}$	203.9	200	+ 3.9
Minor third	$\frac{6}{5}$	315.6	300	- 15.6
Major third	$\frac{5}{4}$	386.3	400	- 13.7
Perfect fourth	$\frac{4}{3}$	498.04	500	- 1.96
Perfect fifth	$\frac{3}{2}$	701.96	700	+ 1.96
Minor sixth	$\frac{8}{5}$	813.7	800	+ 13.7
Major sixth	$\frac{5}{3}$	884.4	900	- 15.6
Minor seventh (small)[201]	$\frac{16}{9}$	996	1000	- 4
Minor seventh (large)[202]	$\frac{9}{5}$	1018	1000	+ 18
Major seventh[203]	$\frac{15}{8}$	1088	1100	- 12
Perfect octave	$\frac{2}{1}$	1200	1200	0

201 The minor seventh which is the inversion of the big whole tone—perfect octave minus big whole tone, $2 \div \frac{9}{8} = \frac{16}{9}$.

202 The minor seventh is the inversion of the small whole tone, a perfect octave minus small whole tone, $2 \div \frac{10}{9} = \frac{9}{5}$.

203 Perfect octave minus a semitone, $2 \div \frac{16}{15} = \frac{15}{8}$.

More complex intervals such as augmented fourths and diminished fifths will be tabulated separately as there are so many different versions which depend heavily on context.

Two factors adversely affect intonation—gravitational pull and difference from the tempered form. When these two aspects combine, extra care must be taken to achieve pure intonation. The following is a set of guidelines devised by Kardos for achieving the pure rather than equal-tempered form.[204]

- If an interval is larger than the tempered, care is required when it occurs in an ascending direction because the higher of the two notes needs to be sharper than the tempered;
- Intervals that are smaller than the tempered require extra care when they occur in a descending direction because the lower note needs to be sharper than the tempered;
- Intervals larger than the tempered form in a descending direction do not cause problems because of the tendency to over-reach the lower interval;
- Intervals smaller than the tempered form in an ascending direction do not create difficulty because of the tendency to under reach the higher note.

Upward arrows are placed above the second note of ascending intervals that are wider than the tempered forms, and the second note of descending intervals that are narrower than the tempered to indicate they need to be tuned high.

These principles will now be analysed in terms of the following interval classes: perfect unison and octave; perfect fourths and fifths; major and minor intervals.

The perfect consonances

Perfect unison and perfect octave are the only intervals that are identical in the acoustic and tempered forms. In practice this provides one stable reference point. A starting note can be given from the piano. A single note or octave drone[205] can

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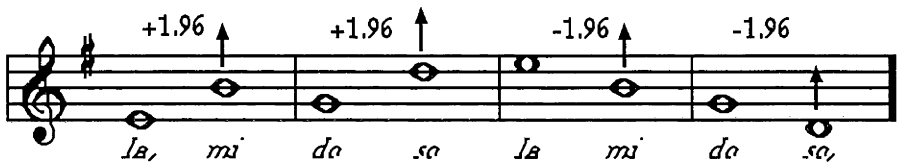
204 Pál Kardos, 'Foundations of Education towards Pure Intonation', pp. 6-14.

205 Except with pianos with stretched octaves. The piano's overtones depart considerably from a harmonic series owing to the stiffness of the piano strings. Further it is impossible to keep the three strings per note in the treble and two in the bass tuned in perfect unison for any

provide a useful reference point for intonation exercises.

The perfect fourth and perfect fifth deviate from the tempered form by only 1.96 cents, the fifth being 1.96 cents larger, the fourth being 1.96 cents smaller. However, because these intervals have such acoustic primacy in the harmonic series, it is particularly important that these intervals are intoned purely, thus even a small degree of deviation from acoustic purity is significant:[206]

- the higher note of a melodic ascending fifth and the lower note of a descending fourth should be intoned sharper than the piano would indicate;
- in harmonic contexts the higher note of the fifth should be slightly sharper, that of the fourth slightly flatter:



Thus care is required whenever a melodic passage calls for an ascending fifth or descending fourth. I have selected the following examples to illustrate cases where care in intonation is necessary to achieve pure upward fifths and pure downward fourths:



The Hungarian folk song begins with an ascending perfect fifth in a minor

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time. This is what gives the piano its distinctive timbre. Because its overtones and those of the human voice do not match, it is difficult to tune to a piano drone. Remember too that the absolute pitch of a piano note and a justly tuned note will differ by varying amounts depending on the reference note. For example if the reference note is A = *do* 440Hz then B on the piano will be 493.16, while the justly tuned B will be 495 Hz and so on. Kardos elsewhere (*KK* p. 30) says that sometimes the pitch instability of singers may be greater than the danger of using a tempered instrument and that a single pitch from a piano can be used for giving starting notes, for intonational support or checking the pitch.

206 Huygens noted in the 17th century that the higher up the scale of consonance the less the ear can tolerate tempering. Cohen, op. cit., p. 233. Helmholtz showed how the greater the degree of coincidence of the harmonics and combination tones present in a given dyad, the more disastrous the consequences for deviation from pure ratio. Intervals with fewer coinciding notes will have less additional beating from mistuning, this being the reason why mistuning the major third is more tolerable than mistuning the octave, fifth, fourth. Helmholtz, op. cit., p. 185.

tonality, *la—mi*. The *mi* or b' needs to be tuned from above.

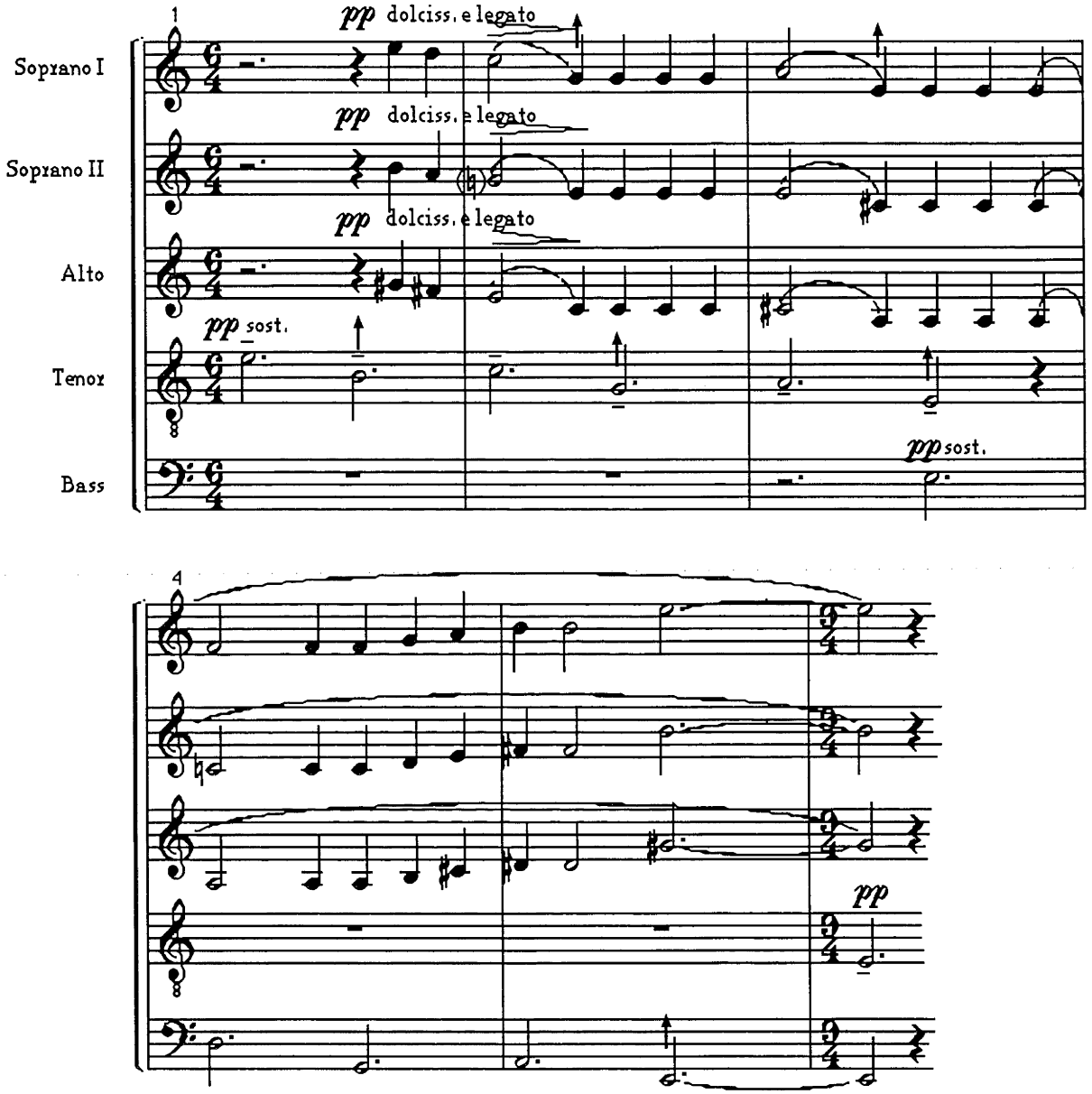


This children's song in major tonality contains two difficult melodic intervals in a row, an ascending fifth followed by a descending fourth. The opening ascending *do—so* requires the *so* or d" to be tuned high, which for children will be particularly difficult since a leap of a fifth is a large distance to cover for young voices. The *re* or d' of the following descending fourth needs to be tuned high, the pull of gravity resisted.



This seemingly simple children's song also hides difficulties—the descending fourth. The notes marked with an arrow need to be tuned high.

The following example, an excerpt from Benjamin Britten's *Hymn to St. Cecilia*, could almost have been written as an exercise in descending fourths:



Benjamin Britten, *Hymn to St. Cecilia*, Op. 27, bars 1-6[207]

Minor and major intervals

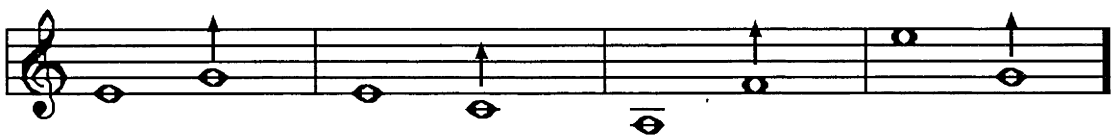
The pure forms of the minor and major intervals deviate from the tempered forms by a significant amount. Minor intervals are generally larger than their tempered form while major intervals tend to be smaller. Thus, Kardos advised:

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207 Benjamin Britten, *Hymn to St. Cecilia*, Op. 27, bars 1-6, Boosey and Hawkes, London, 1942.

- minor intervals, being larger than their tempered equivalents, must be intoned with special care when in an upward direction;

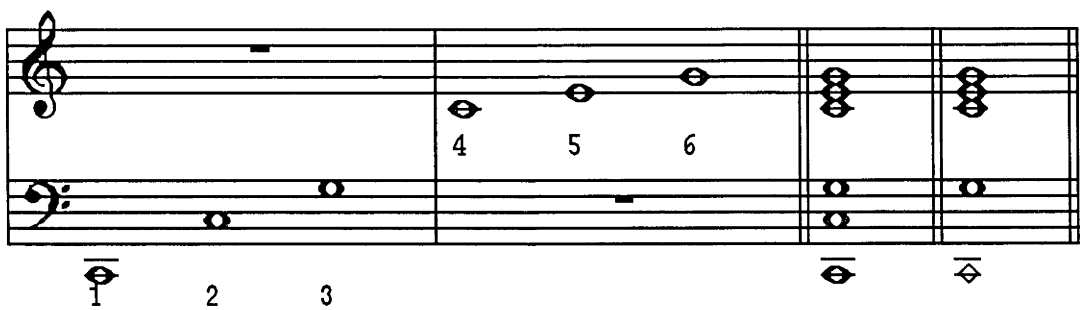
- major intervals, being smaller than the tempered version must be intoned very carefully in a downward direction. Ascending the minor intervals should be broad and wide; descending, the major intervals should be tightened and narrow:



5.2 Triads and thirds

The Intonation of the major triad

The major triad forms the acoustic centre of pure intonation and is thus the starting place for training in pure intonation, Kardos asserts.[208] Its centrality lies in the fact that the first six harmonics of the harmonic spectrum of a sung note form a major triad. A choir singing a major triad is in fact replicating the first six notes of the harmonic spectrum:



This view echoes Jean-Phillipe Rameau,[209] (1722-1764) who hypothesised that the major triad is satisfying because of the number of overlaps in the harmonics of

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208 Kardos, *KK*, p. 23.

209 Rameau's *Theory of Harmony* was written without knowledge of overtones. He revised his science of music in the later years of his life following acquaintance with the work of Sauveur on overtones, seeing in overtones the scientific proof of his theories of the fundamental bass, and the primacy of the major triad. These are contained in his 1726 treatise *Nouveau systèmede musique thérorique*. Johnston, op. cit., pp. 108-113, Palisca, *Studies*, pp. 207-208.

the three component notes. Rameau also asserted that the three positions of the chord, (root, 1st, 2nd inversion) should be regarded as the same chord, since so much of the pattern of overlapping harmonics is common to all three. The note which is the lowest in the 'root' position, should thus be called the fundamental bass. Because the three chords, tonic, dominant and subdominant have the similar pattern of harmonics even if at different frequencies, the ear will detect this similarity when one of these chords is played after another. This is the reason why tonic, dominant, subdominant, tonic patterns are so fundamental in functional harmony. For Zarlino, without the benefit of knowledge of the harmonic series, the major triad had semi-mystical overtones.[210]

I would argue that the major chord as final functions not only as the resolution of all deliberate dissonant relationships in a choral composition of both Renaissance polyphony or functional harmony style, it also serves as the resolution of inevitable tuning vicissitudes. Numerous complex factors related to the vertical and horizontal relationships of notes, vocal technique, vowel colour etc, mean faultless tuning of every interval is an ideal impossible to achieve in practice. Yet a perfectly tuned chord in the final cadence functions as the resolution of tuning imperfections and dissonances. Even if choristers have no in-depth knowledge of the principles of pure intonation and do not analyse each interval for its optimum tuning, if they have good aural sensitivity and listen to each other they should sing the final chord in tune. Tuning a chord is like focusing a camera lens. When it is perfectly in tune it will have a special lustre and resonance.

To tune a major triad in acoustic conformity with the harmonic series the following principles, as outlined by Kardos, should be applied:

- The root of the chord, which occurs three times in the harmonic series (harmonics 1, 2, and 4) should be slightly louder than the fifth. If another part duplicates the root an octave or more higher it should blend into the lower, as would the second or fourth harmonic into the fundament;[211]

- The fifth, which occurs twice in the series (harmonics 3 and 5) should be slightly louder than third;

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210 See Wienpahl, op. cit., pp. 27-41.

211 Kardos, *KK*, p. 26. *Karvezetés III*, p. 131.

- The third, which only occurs once (fifth harmonic) is the softest, most sensitive and adjustable note. The justly tuned major third, being 13.7 cents smaller than the tempered, is less sharp than one might think. All too often, conductors, hearing that the third is out of tune in a chord, ask for it to be sharper, when in fact the reverse is the remedy.

Thus choristers should analyse which part of the triad they are singing. The part that has the fundamental is the part that other parts tune to. The conductor's catch-cry 'tune to the basses' applies only when the basses have the root of the chord. A chord will sound particularly pure if the distribution of its parts is the same as the distribution of overtones of a given note. In the following example the sung chord replicates the overtones of the bass note:



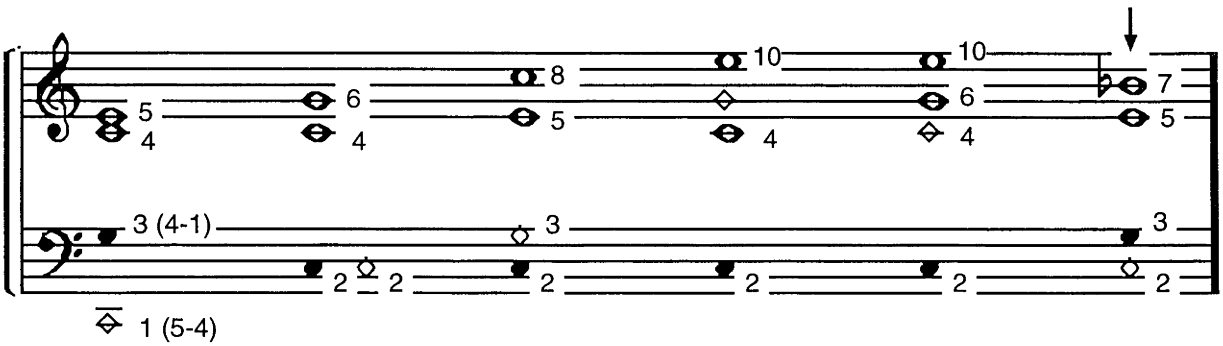
Triads and difference tones[212]

Kardos advocated using the phenomenon of difference tones as an aid to tuning. Difference tones are more audible if the chord is perfectly in tune acoustically.[213] The diagram below shows the first difference tone produced by various combinations of the triadic relationship. If the notes notated as semibreves are sung as a sustained chord the first difference tone should be faintly heard:

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212 See Appendix B.

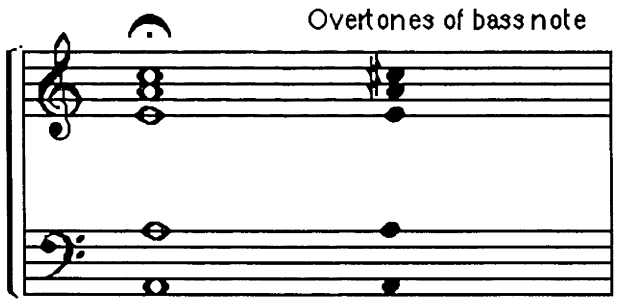
213 There is a story about Zoltán Kodály overhearing a children's choir in rehearsal. He came in and asked, in some surprise, where the men were hiding. He had heard deep bass undertones or difference tones resounding below the children's voices because their singing was so in tune.



Adapted from Kardos, *KK*, p. 24

Minor triads

Minor triads are recognised by many theorists as harder to sing in tune than major triads. This is because the minor third produces beating among its higher harmonics and difference tones. Also it will conflict with the major third that is the fifth harmonic. In churches with long reverberation times the building itself will seem to sound a major third.[214]



Owing to its unstable and inconclusive character the minor triad was avoided as a final until the Baroque period. The manner of avoidance was to omit the third altogether in the final close.[215] From the sixteenth century major thirds were increasingly used in final cadences having been elevated to the status of

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214 Johnston, op. cit., p. 21.

215 Since, in theory at least, no thirds were consonant before Fogliano's treatise of 1529. From the time of Zarlino, thirds were predominant in final chords. Wienpahl examined 5,179 pieces of music from the period 1500-1700 and found that the final third was used most heavily in the period 1580-1620, when 93.8% of final chords included the third. Wienpahl, op. cit., p. 28. Wienpahl does not specify if the relative distribution of the major and minor thirds. I would surmise that Zarlino prefered the 4:5:6 major triad on numerologicl grounds. Zarlino describes the major third as happy, the minor third as sad. Wienpahl, p. 28.

imperfect consonance once just intonation had changed their size to 5:4. It is debatable whether theory preceded practice.

Even once minor thirds were admitted as imperfect consonances, if the prevailing modality would normally call for a minor triad as final, the third was either omitted or this was changed to major in the final chord, whether by means of an accidental or not. This is what is referred to as the *Tierce de Picardie* cadence.[216] The *Tierce de Picardie* continued to be used throughout the seventeenth and even eighteenth century.[217]



Antonio Lotti,*Vere languores nostros*

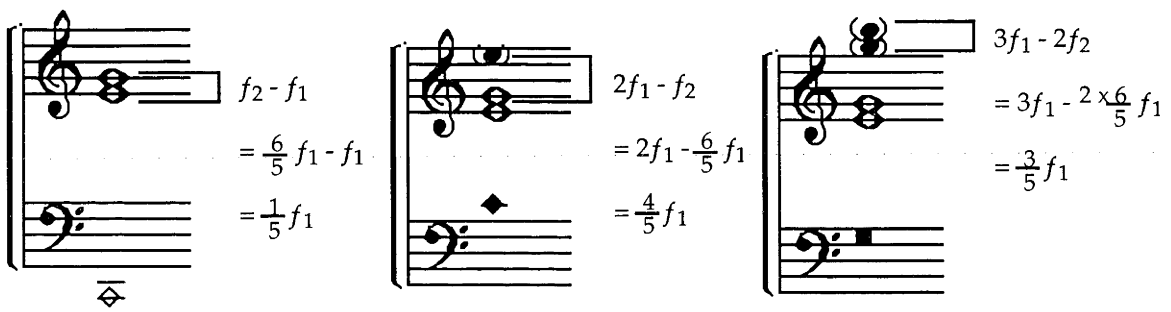
The minor third will only sound unstable and dissonant in relation to overtones if it is in the context of a minor triad, where it conflicts with the fifth harmonic of the bass note. If the minor third (*so—mi*) is sung as a harmonic interval without a bass note it will generate as its second combination tone a note a major third below, (*do*) thus making it the upper of the two thirds in a major triad. A minor third that is purely intoned will generate combination tones. The first combination tone is the difference between the two tones. A minor third has the ratio 6:5, the first difference tone is thus $f_2\text{-}f_1 = \frac{6}{5} f_1 - f_1 = \frac{6}{5} f_1 - \frac{5}{5} f_1 = \frac{1}{5} f_1$. If for

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216 There seems to be differences of opinion as to when it came to be fashionable. Ian Johnston asserts it is associated with the Flemish school of composers in the 15th Century including Josquin des Prez. Johnston, op. cit., p. 21. Donnington puts it as the 16th century.

217 Thomas Morley, *Plaine and Easie Introduction*, London, 1597, pp. 155-6: 'Morley's Master Gnorimus corrects...his pupil's final close in the minor by writing in a Picardy Third'. Francesco Bianciardi, *Breve regola*, Siena, 1607, rule 9: 'In the final closes, one always ends with the major third.' All these examples are quoted in Robert Donnington, *The Interpretation of Early Music*, revised ed., Faber and Faber, London, 1990, pp. 140-1.

the purposes of demonstration the minor third was e′—g′ the first difference tone will be C. The second difference tone is that found between the first harmonic of the lower note ($2f_1$) and the upper note (f_2). $2f_1 - f_2 = 2f_1 - \frac{6}{5}f_1 = \frac{4}{5}f_1$. The first harmonic of the lower note is e″ and the upper note is g′, the combination tone is c′, a major third below e′. The third combination tone is the difference between the second harmonic of the lower note ($3f_1$) and the first harmonic of the higher note ($2f_2$). $3f_1 - 2f_2 = 3f_1 - 2 \times \frac{6}{5}f_1 = \frac{3}{5}f_1$. Thus the minor third with its combination tones replicates the first six harmonics of the harmonic series, and the sound of an acoustically pure major triad. Thus the minor third has very high consonance value:



Thus a minor third generates the sound of a major triad. The difference tones are only faintly perceptible to an acute ear. For the well trained chorister, however, they are an excellent way to achieve pure intonation. Thus the minor third on its own is anything but dissonant or unstable!

In summary, the intonation of the third in minor triads is extremely sensitive. Kardos directed the part creating the minor note to adjust to the other parts. In just intonation the major thirds are smaller and the minor thirds larger than the tempered, thus the difference between the major and minor thirds is comparatively small—386-316 = 70 cents. Alternating between major and minor triad as indicated in the diagram below shows the size of the alteration is quite small.

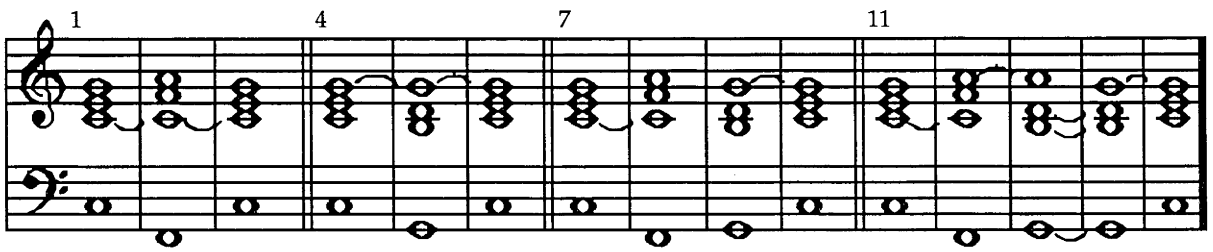


Voice leading

Complications arise in moving from one note in a chord to another, for example

from tonic to the dominant below the tonic which entails descending a fourth. Care must be taken descending a fourth to keep the second note high. Owing to the use of two different forms of the 2nd degree (in the dominant chord it is tuned high, in subdominant or tonic relatives, eg II and VI the supertonic is tuned a syntonic comma lower), progressing from subdominant relatives to dominant, (II—V) or from tonic relatives to dominant (VI—V) present special difficulty. The 2nd degree may not be in a position to adjust if it is the common tone, and other tones may need to adjust instead. This can lead to 'wandering tonics'.

The harmonic progression below shows that the different intonational requirements of the 2nd degree create difficulty. I—IV—I is straight forward, as is I—V—I and I—IV—V—I. However the suspended soprano a' in bar 12 sounding simultaneously with G in the bass will create conflicting influences on the d' in the alto. It needs to be low to create a perfect fifth with the soprano, high to create an octave and a fifth with the bass. If the altos tune to the bass, the a' will be de comma fifth, if the sopranos therefore adjust the a' up there will be two forms of a', one a syntonic comma higher. Because the d' is sustained over bars 13-14 the descent from a' to g' will be by a 9:8 whole tone instead of a 10:9 whole tone. If the altos form their d' against the soprano a' then the g' in bar 14 will be low, a 81:80 drop in pitch will have occurred. Here is a case where 'tune to the basses' is good advice:



Triads in inversion

Different intervallic relationships are involved in triads in inversion. Kardos did not go into this in detail. The part which has the fundamental is the part other parts must tune to. In a 6:4 chord the fifth is in the bass and the first interval is a perfect fourth. The acoustic fourth at 498 cents is 2 cents smaller than the tempered at 500 cents.

Thirds as diads and melodic intervals

We have dealt with thirds as components of a triad, this section deals with the intonation of melodic thirds as found in infra pentatonic melodic material.

The 6:5 minor third

Music educators in Hungary since Zoltán Kodály have advocated the descending minor third as a starting point for learning to sing in tune[218] and assert its primal nature as forming the basis of melodic culture throughout the world. The descending minor third is easier to tune than the ascending minor third as we have seen, being 315.6 cents larger than the tempered minor third of 300 cents. Surveying children's folk literature[219] for the presence of the upward minor third one finds it is far less common than the descending minor third. The ubiquitous *so–mi* or *la–so–mi* children's chants are generally built on the descending minor third. For example in such chants or songs as *I'm the King of the Castle and You're the Dirty Rascal, Rain Rain Go Away* or *Ring a Ring a Rosy* the descending minor third predominates. It seems that ancient culture sensed which intervals are easier to sing in tune.

Kardos analyses the role of the second minor third a fifth lower, a common feature in Hungarian folk literature, and known as the lower fifth answer, *do—la*:[220]

$$\frac{6}{5}$$
$$\frac{3}{2}\left\{\begin{matrix} so\text{----}mi \\ do\text{----}la \end{matrix}\right\}\frac{3}{2}$$
$$\frac{6}{5}$$

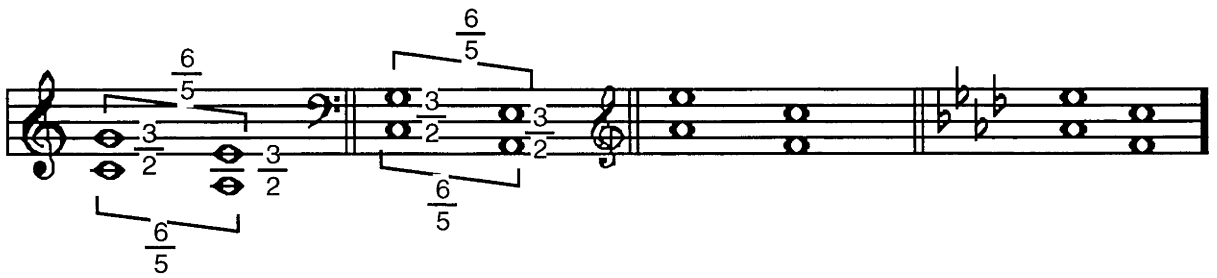
These two minor thirds are united by the strong consonance of the perfect fifth:

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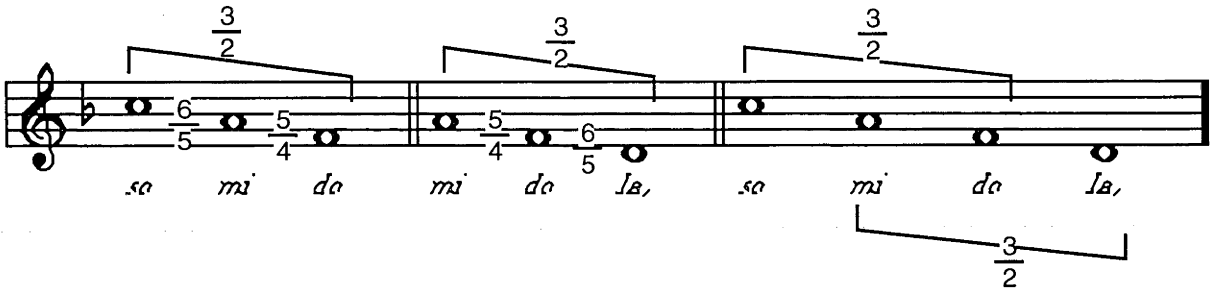
218 Kardos, Pál, *A tiszta intonálásra*, p. 2. See also Lajos Bárdos, *Természetes hangrendszerek* [natural tonal systems], Zenemükiadó, Budapest, 1969.

219 In European or European derived folk culture.

220 Kardos, *A tiszta intonálásra*, p. 3.



In a chain form they extend to include the major and minor triads in melodic (broken) form:[221]



Kardos argues that there are numerous melodic examples from primal cultures throughout the world that show an ancient form of anhemitonic infra-pentatony that omits the 2nd degree (*re*) and hence the tuning complexities it brings in its wake. The following is one of the many example he cites:[222]



The interlocking minor and major triads *la do mi* and *do mi so* form a tetratonic system which contains no tuning ambiguities.[223] The following diagram shows

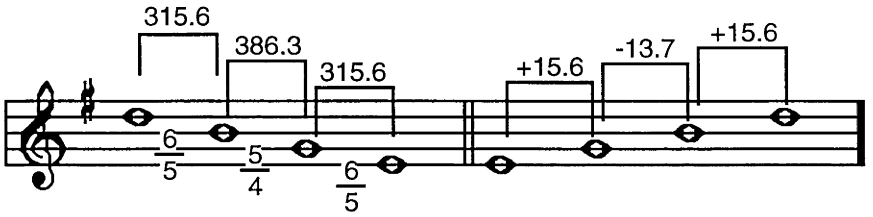
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221 Kardos, *A tiszta intonálásra*, p. 3.

222 Kardos, *A tiszta intonálásra*, p. 4.

223 These triads are pure or non-beating in the justly tuned major and minor scales. It is the 2nd degree triad that creates the difficulties. Here, however, Kardos is talking of the *la do mi* and *do mi so* triads as a form of pre-pentatony, and not in their diatonic contexts. Pál Kardos, *A tiszta intonálásra*, op. cit., p. 5.

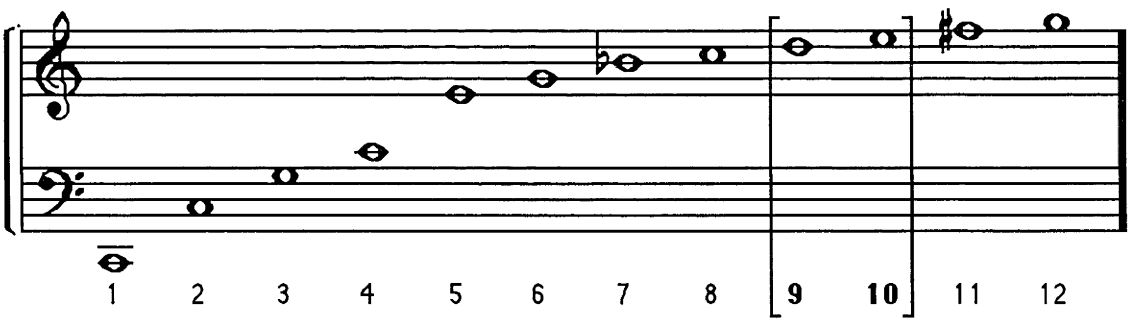
the difference between the tempered and pure forms of this tetratonic system:[224]



The tuning of the minor third is heavily influenced by its tonal context and in the diatonic context by the size of whole tone it contains or borders. Further discussion of such influences is outlined after the section analysing the complexities of tuning whole tones.

5.3 The major second

One of the most critical aspects of Kardos's pure intonation is its use of two forms of major second—the 9:8 big whole tone and the 10:9 small whole tone. In the harmonic series:



In fact there are several types of major second derivable from the harmonic series. The interval found between the 7th and 8th notes in the harmonic series is larger than the big whole tone, and the interval between the 10th and 11th note of the harmonic series is smaller than the small whole tone. These are rarely used. Outside the series there is the tempered second which in size falls in between the big and small whole tones. The following diagram compares the different forms of major second found in the harmonic series with the tempered major second:

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224 Adapted from Kardos, *A tiszta intonálásra*, p. 9.

Major Seconds

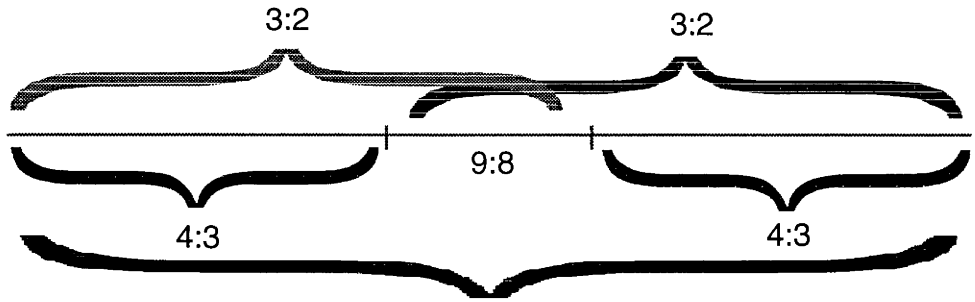
	Big Whole Tone	Tempered	Small Whole Tone	Interval between 10th and 11th harmonics	Interval between 7th and 8th harmonics
Ratio	9:8	$\sqrt[6]{2}$	10:9	11:10	8:7
Cents	203.9	200	182.4	165	231.174
Difference	+ 3.9		17.6	35	31.174

Just intonation also has big and small whole tones, but their order tends to be fixed much like the tetrachordal divisions of ancient Greece. A flexible use of the 9:8 and 10:9 tones allows various combinations of melodic and harmonic intervals to be tuned acoustically purely. What is necessary is knowing whether to sing a big or small whole tone in a given musical example. This is complex, and Kardos explained at length how to determine whether a given tone is a 9:8 or 10:9.

The 9:8 big whole tone

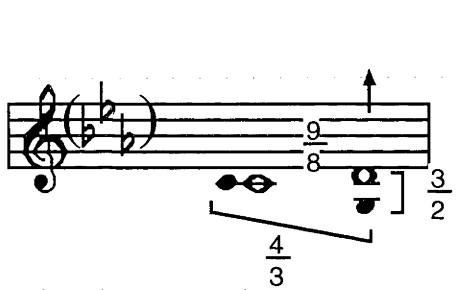
The big whole tone is often called the Pythagorean major second since it is the interval Pythagoras[225] reputedly derived from the difference between the ratios of fifth and fourth—$\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$—and because it is the only form of whole tone used in Pythagorean tuning. This 9:8 interval was the basic building block of any scale and was called *epogdoon*. In medieval periods it was called *tonus* the tone:

225 It was actually Philolaus who derived the values for intervals smaller than the octave, fifth and fourth.

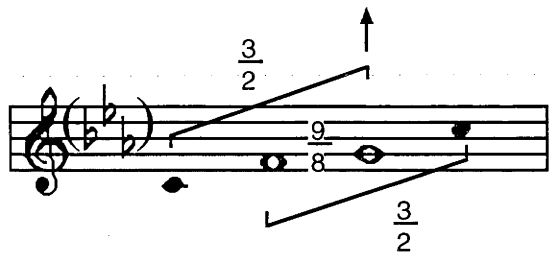


2:1 octave
3:2 fifth
4:3 fourth
9:8 big whole tone

Harmonic factors influence whether a major second is the larger or smaller form. The following diagrams by Kardos outline the more common harmonic influences giving rise to the big whole tone.



In major	*do*	*re*
In minor	*la*	*ti*
Scale degrees	1	2



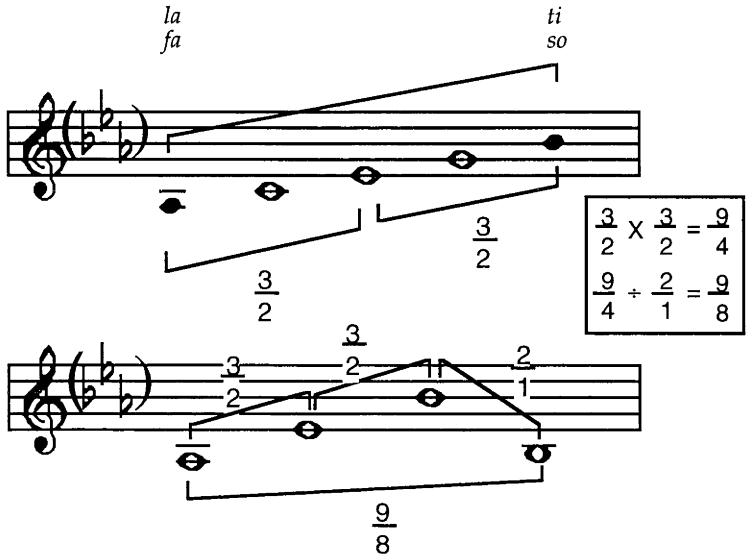
In major	*fa*	*so*
In minor	*re*	*mi*
Scale degrees	4	5



In major	*la*	*ti*
In minor	*fa*	*so*
Scale degrees	6	7

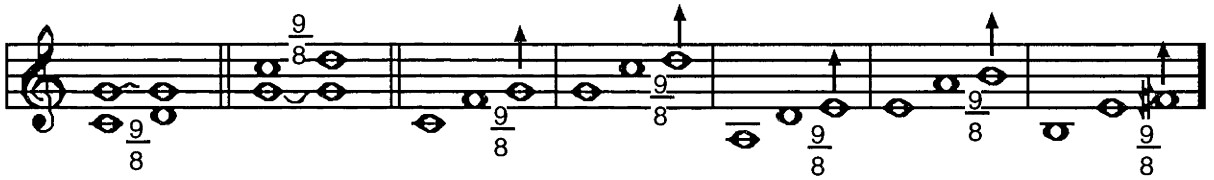
Location of The Big Whole Tone, Adapted from diagrams 69-71, Kardos, *KK*

The big whole tone can be mathematically derived from various combinations of intervals. It is also the projection of two perfect fifths within one octave. Perfect fifth + perfect fifth = major ninth. Take the major ninth down an octave and it is a big whole tone from the original starting note:

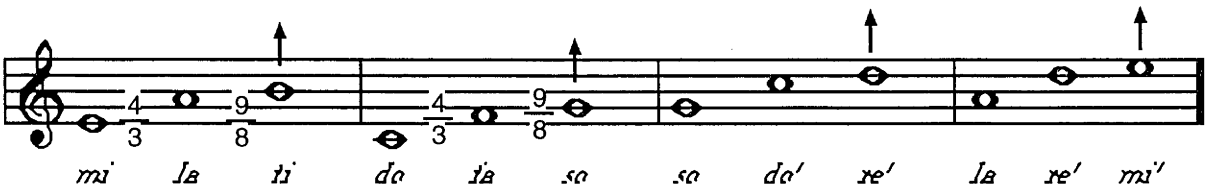


Adapted from diagram 72, Kardos, *KK*, 2nd diagram by author

The big whole tone as a major second interval bridging a perfect fourth and perfect fifth $\left(\frac{3}{2} \div \frac{4}{3} = \frac{3}{2} \times \frac{3}{4} = \frac{9}{8}\right)$ is found in contexts involving these intervals:

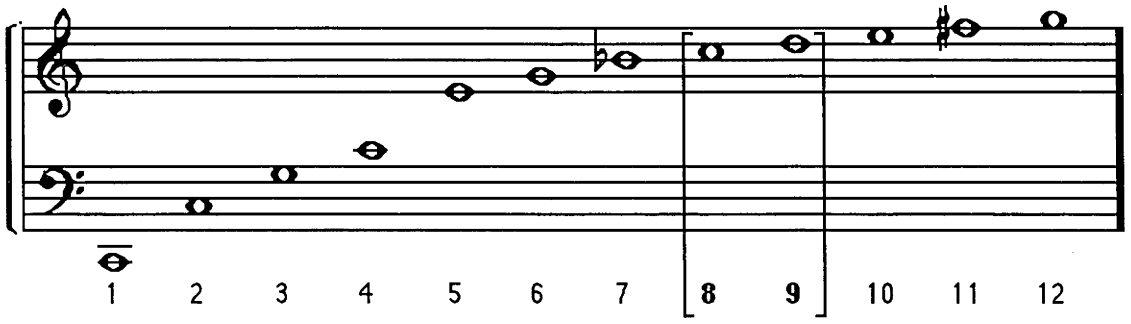


The following example shows various three note systems containing the big whole tone in contexts where the gravitational pull makes care in intonation especially important:

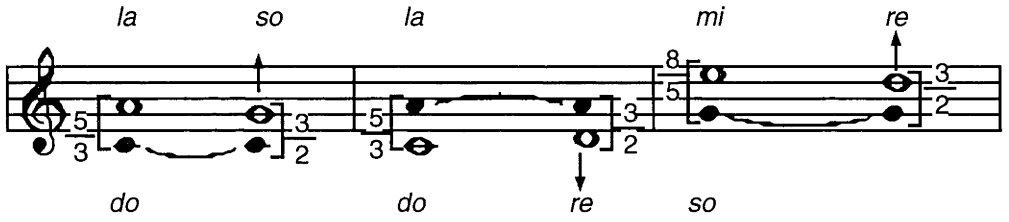


The 10:9 small whole tone

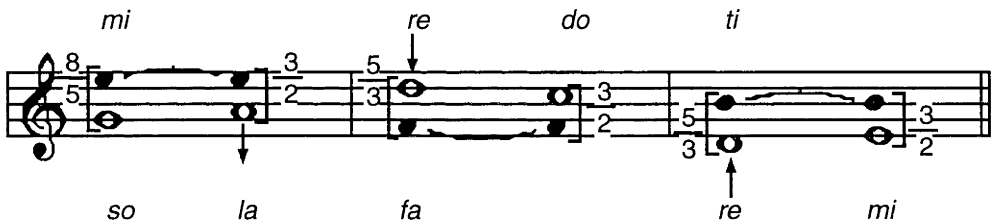
The 10:9 small whole tone as found between the 9th and 10th notes of the harmonic series at 182.4 cents is 17.6 cents smaller than the tempered major second, and 21.5 cents smaller than the big whole tone. Kardos derived it from the harmonic series:



The small whole tone occurs in typical contexts such as the following:[226]



Scale degrees						
In major	6	5	(1	2)	3	2
In minor	8	7	3	4	(5	4)



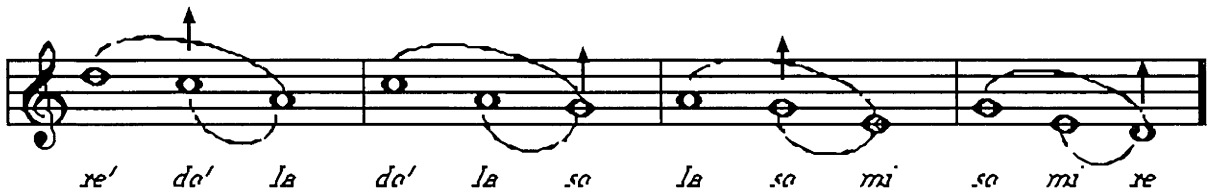
Scale degrees						
In major	5	6	(2	1)	2	3
In minor	7	8	4	3	(4	5)

Scale degrees in brackets indicate that this tuning is not normal in that mode but does occur. The small whole tone can be mathematically derived as the interval remaining when a minor third is subtracted from a perfect fourth:

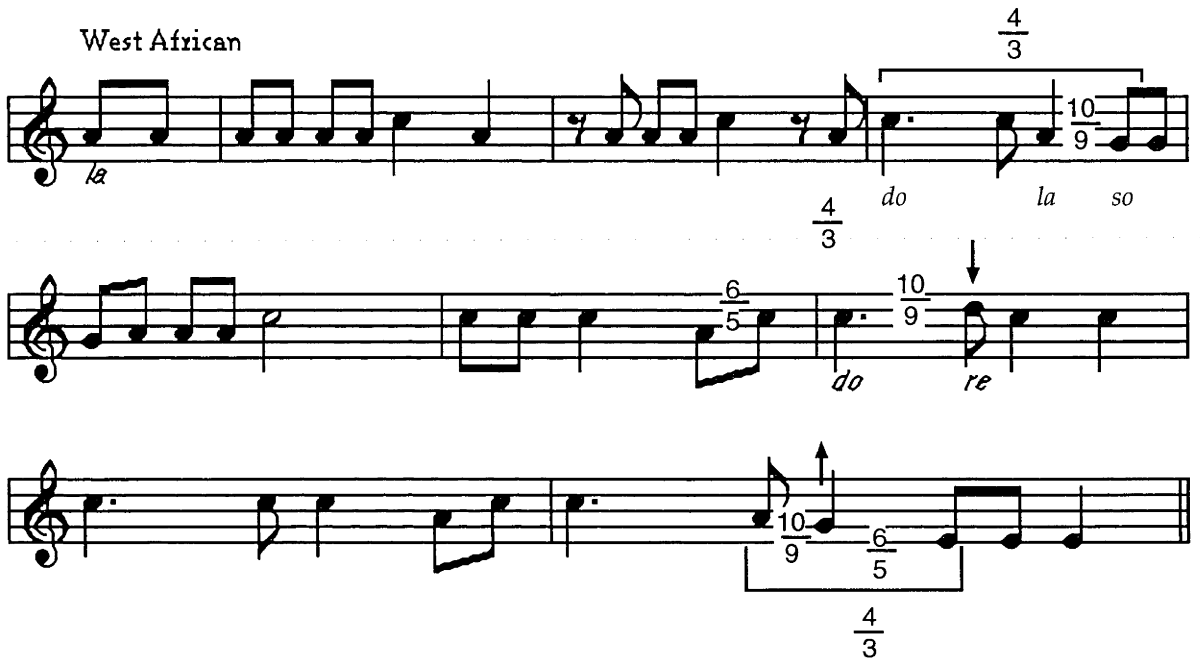
$$\frac{4}{3} \div \frac{6}{5} = \frac{9}{10}$$

226 See Kardos, *KK*, pp. 32, 33.

It thus occurs in contexts involving a perfect fourth, divided into a minor third and small whole tone:

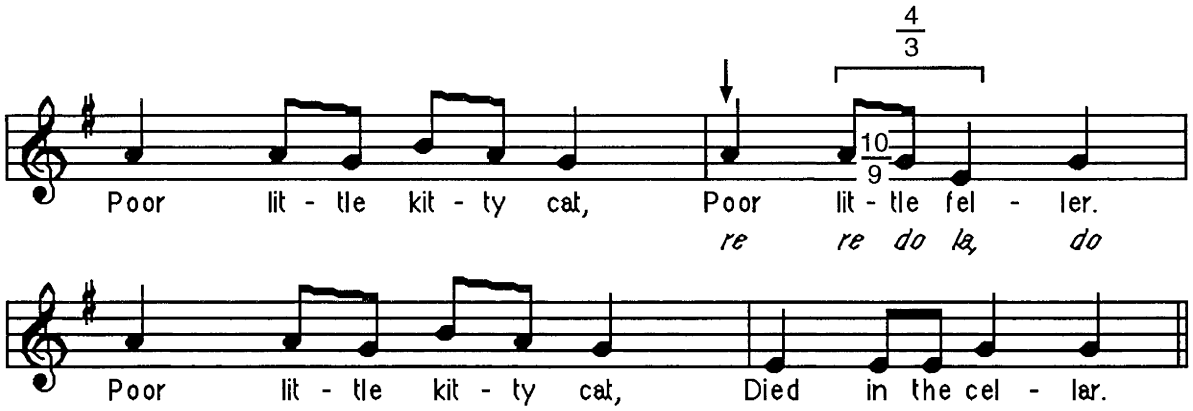


The following examples from folk literature show contexts where the small whole tone is found:

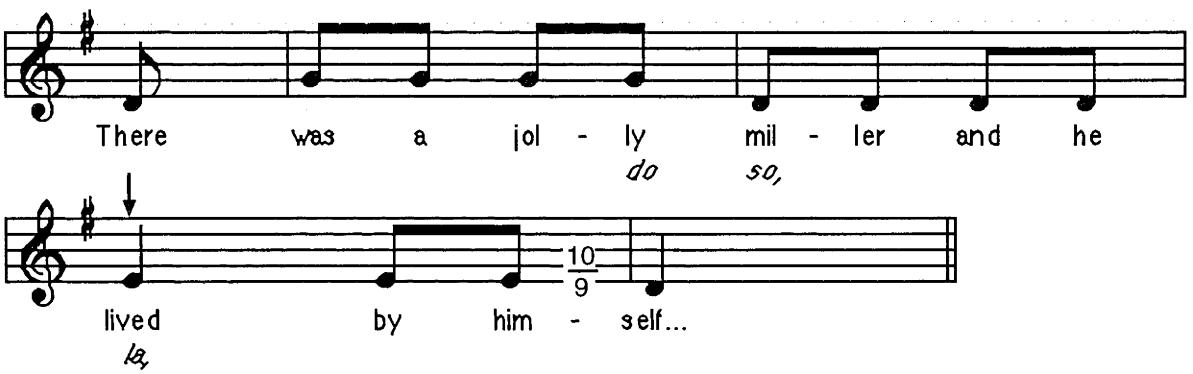


Kardos, *A tiszta intonálásra*, p. 19. Solfa and ratios added by author

The following children's song is dominated by *la* feeling leading to a low *re*, and small whole tone between *re* and *do*:[227]



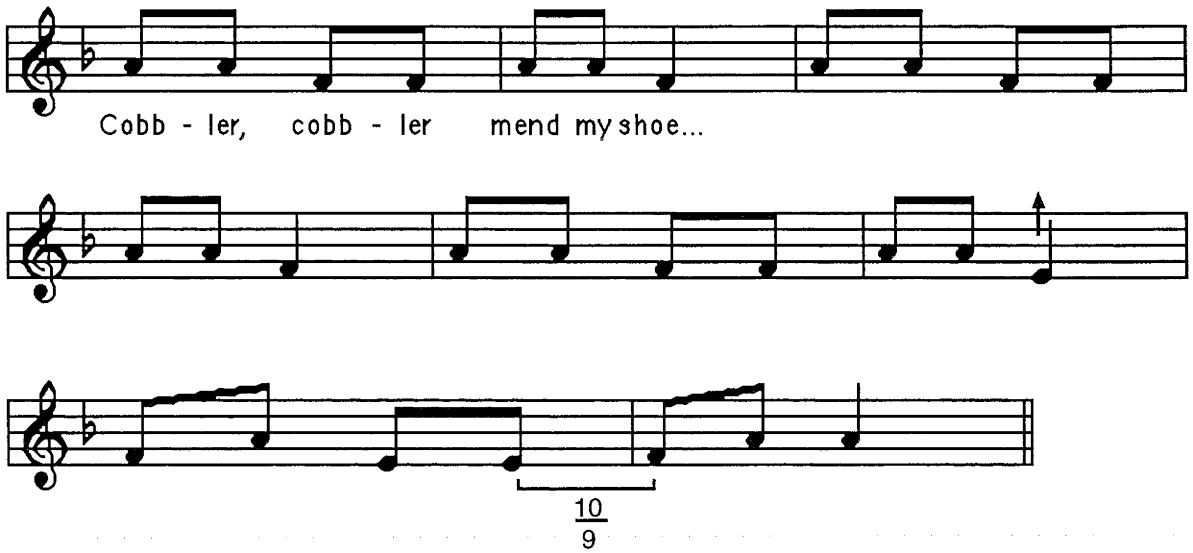
The next example, again a simple children's song, illustrates the small whole tone that is usual between the 5th and 6th scale degrees, *so* and *la*:[228]



227 Selected by the author.

228 Selected by the author.

The familiar song *Cobbler, cobbler, mend my shoe* is based on the descending minor third. When it descends by a fourth the three note module of 4:3, 6:5 and 10:9 occurs, perfect fourth, minor third, small whole tone, *re—mi—so*:[229]

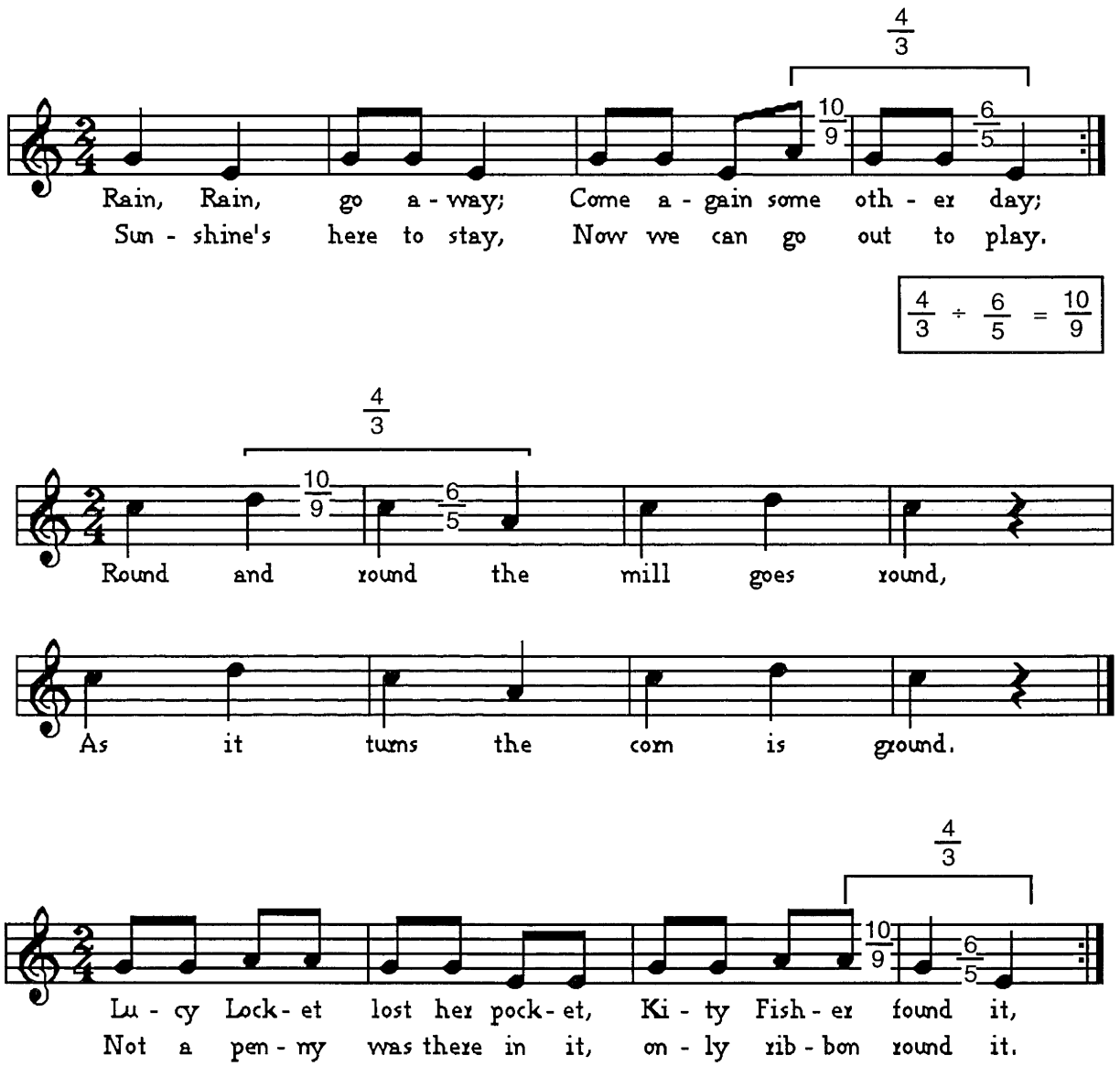


Here the 4:3 divided into 10:9 and 6:5 is of the *la—so—mi* system:

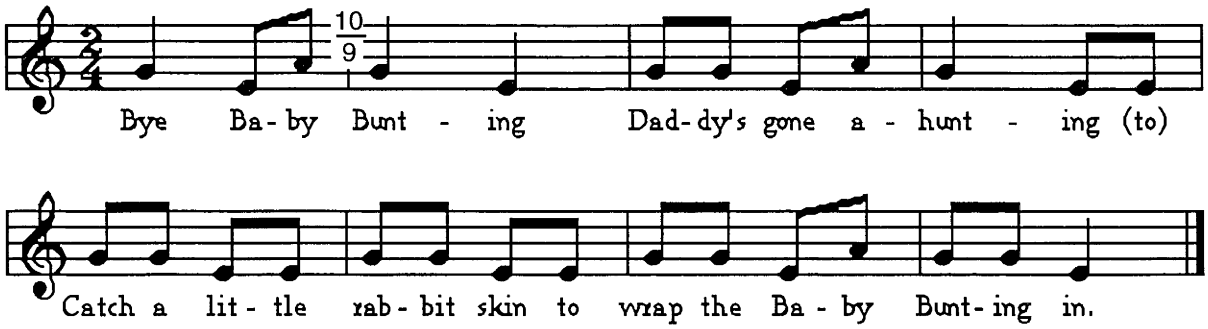


229 Selected by the author.

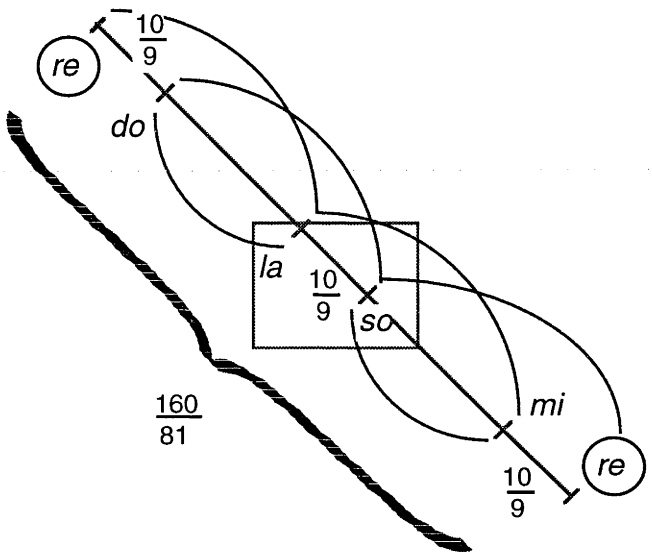
The ubiquitous *la so mi* chants[230] as found in *I'm the King of the Castle* or *Rain, Rain, Go Away* are common examples consisting of the small whole tone and minor third:



230 Selected by the author.



Kardos pointed out that the three note system containing a minor third plus a small whole tone making up a perfect fourth is often extended in chain form thus:[231]



He pointed out[232] that two outer poles of this chain are not a perfect octave apart but in fact about a tenth of a tone narrower: $(\frac{4}{3})^2 \times \frac{10}{9} = \frac{160}{81} = \frac{80}{81}$.

231 Kardos, *A tiszta intonálásra*, p. 19.

232 Kardos, *A tiszta intonálásra*, p. 19.

Flexible intonation: adjustable *re*

As was said before, the advantage of having two forms of whole tone is that it allows a pure major third since:

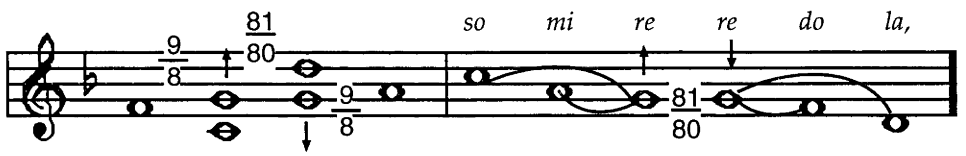
$$\frac{9}{8} X \frac{10}{9} = \frac{5}{4}$$

Pythagorean tuning uses only the big whole tone, thus the major third is impure since:

$$\frac{9}{8} X \frac{9}{8} = \frac{81}{64}$$

Yet even in just intonation not all thirds are pure: the triad on the 2nd degree is not. This impurity can be solved by having two forms of the 2nd degree.[233] The addition of a second 2nd degree which is lower by a syntonic comma ($\frac{81}{80}$ or 21.5 cents)[234] allows pure tuning in all six major and minor triads. This still does not solve all tuning problems[235] since there are occasions when, for melodic reasons, it is preferable for another note to adjust.

The reason why there are two forms of 2nd degree is because it cannot be reconciled with the fifth and sixth degrees simultaneously:[236]



The high 2nd degree: *re* connected with *so*

The 2nd degree is a big whole tone from the 1st degree when it is in melodic or harmonic relationship with the 5th degree. If the 5th degree is below the 2nd

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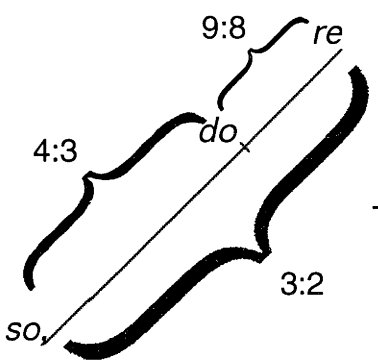
233 See Easley Blackwood, op. cit., pp. 66-128.

234 The difference between the big and small whole tones.

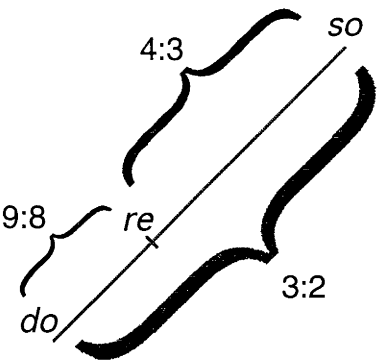
235 For example, in some progressions it will require one part to alternate between the two forms of *re* which is melodically unsatisfactory. This occurs in progressions IV-II-V-I, or III-VI-II-V.

236 See Charles Shackford, 'Some Aspects of Perception: Sizes of Harmonic Intervals in Performance', *Journal of Music Theory*, vol. 5, 1961, pp. 162- 202.

degree they are a perfect fifth apart. The 5th is a perfect 4th below the 1st degree. A perfect fifth minus a perfect fourth is a big whole tone. If the 5th degree is above, it forms a perfect fifth with the 1st degree and a perfect fourth with the 2nd. Again, a perfect fifth minus a perfect fourth is a big whole tone:



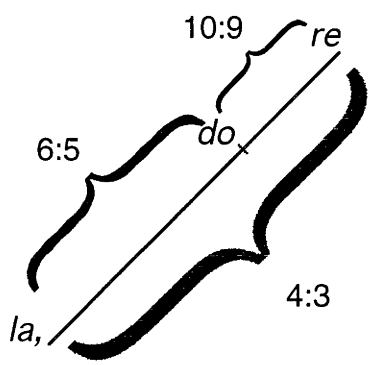
$$\frac{3}{2} \div \frac{4}{3} = \frac{9}{8}$$



$$\frac{3}{2} \div \frac{4}{3} = \frac{9}{8}$$

The low 2nd degree: *re* connected with *la*

The 2nd degree is low when in melodic or harmonic relationship with the sixth degree. It is a small whole tone distance from the 1st degree. This occurs when it forms a perfect fourth with the sixth degree below (*la—re*) since a perfect fourth minus a minor third is a small whole tone. It also occurs in the context of the 6th degree above, since a major sixth minus a perfect fifth is also a small whole tone:



$$\frac{4}{3} \div \frac{6}{5} = \frac{20}{18} = \frac{10}{9}$$

In summary, the intonation of the 2nd degree in a major scale, or *re*, fluctuates depending on whether the 5th or 6th degree is asserting its influence more strongly. This influence can be harmonic, eg a *la* or a *so* in another part (II chord versus a V chord) or melodic—a *la* or *so* prior or following a *re* in the melodic

line will effect its intonation. The big whole tone is found wherever there is a perfect fifth minus a perfect fourth. In pentatonic contexts it appears between *re* and *do* when *re* and *so* form the outer poles (ie. a perfect fifth minus a perfect fourth), and between *mi* and *re* when *mi* and *la* form the perfect fifth. Thus *re* is ambiguous. It is either closer to *do* or to *mi* depending on its context. When it is connected with the *so* below it is a big whole tone away from *do*. When it is connected with the *la* below it, it is a big whole tone away from *mi*.

re in minor, *la* -pentatonic and modal scales

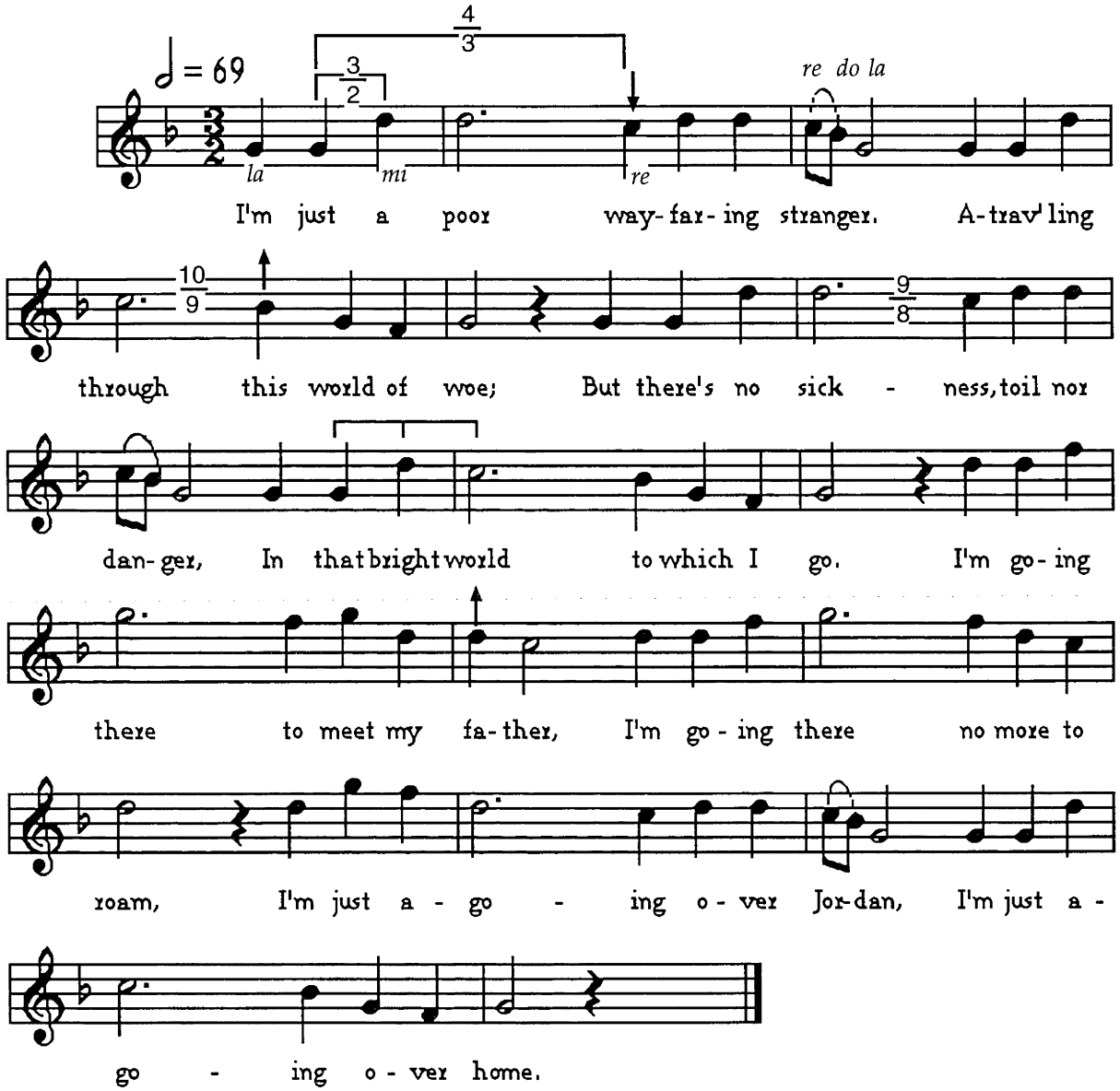
In the minor scale *re* is the 4th degree and is usually low, because of its relationship of perfect fourth to the tonic *la*. Similarly *re* will more commonly take the low form in *la* pentatonic scales. The following example shows that the same *do—re—mi* trichord will require two tunings of *re* depending upon its harmonic context:



Diagram 98, Kardos, *KK*

Even in unison literature implied tonal relationships still influence the intonation. A discreet drone, even a mental one, can help. In the *la* pentatonic folk song *I'm Just a Wayfaring Stranger* a *la* drone hummed, played or imagined shows the strong *la* feeling of the piece that influences the intonation of *re*. *Re* in this piece is closer to *do* than to *mi*:

I'm just a wayfaring stranger



The placement of big and small whole tones in modal scales

Modal scales differ in the location of the big and small whole tones. Since context is so important in defining the location of big and small whole tones, Kardos does not spell out the order of big and small whole tones in modal scales, but rather gives some pointers. Modes that begin with a major second between degrees I and II are the Ionian, Dorian, Lydian, Mixolydian, and Aeolian. Degree II must be tuned high when it is in intervallic relationship with degee V, since 3:2 ÷ 4:3 = 9:8:

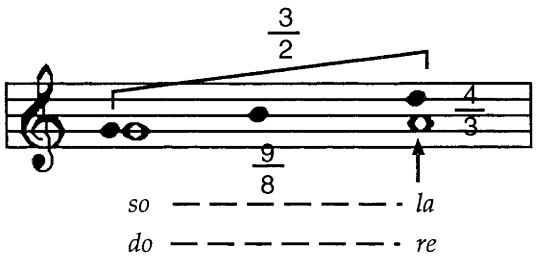


Diagram 82, Kardos, *KK*

As with the major, it will be low when in connection with degree VI. If in a Mixolydian scale a $\frac{6}{4}$ harmonic structure predominates, the distance between degrees I and II is a small whole tone.[237] Although Kardos did not enlarge on this point this is because the harmonic note above the g' is a sixth, above the a' it is a fifth, and $\frac{5}{3} \div \frac{3}{2} = \frac{10}{9}$:

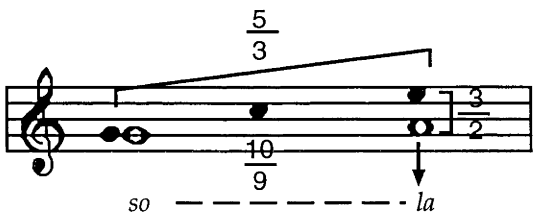
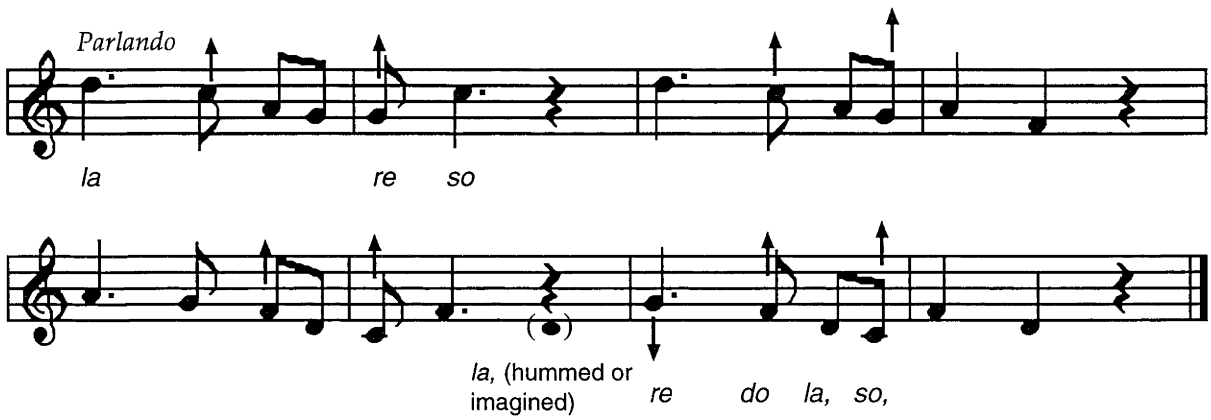


Diagram 83, Kardos, *KK*

237 Kardos, *KK*, p. 35.

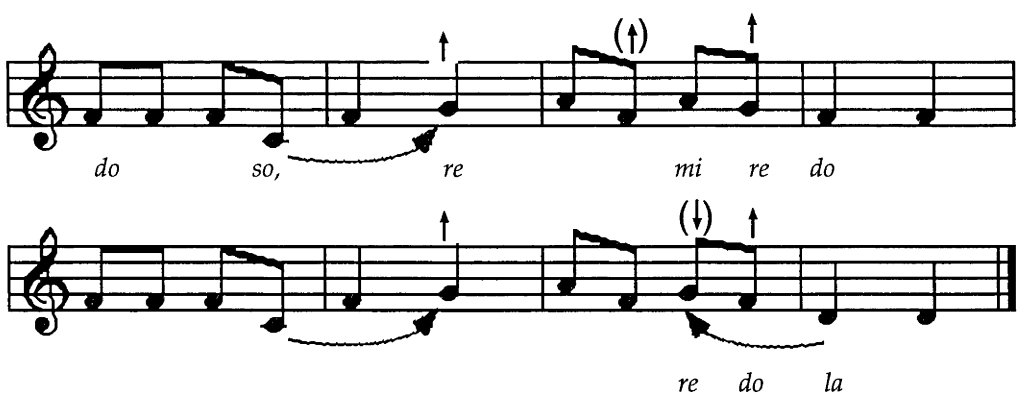
Two forms of *re* in the same piece

Often there are competing influences of *la* and *so* even in unison melodies. The Transdanubian 'Peacock' melody is one such example. The first part is under the influence of *so*, which is replaced by the tonal influence of *la* in the second part.[238] Two interpretations of *re* are necessary:



Peacock melody

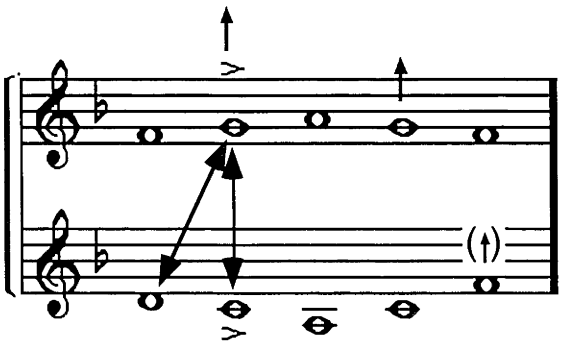
The following simple children's song[239] also has two forms of *re*. In the first part of the song, the tonal influence is major, with the dominant or *so* causing a high *re*. But the *la* of the last bar requires the *re* in the second last bar to be low:



238 Eva Roszgonyi, a student of Kardos, explained to me how she heard peasants sing this song: as if unconsciously to prepare for the change to a lower *re* under the influence of *la* they hummed a *la* just prior to it.

239 Selected by the author.

The next example from Kodaly's *Let Us Sing Correctly,* shows how a preceding *la* may influence the intonation of *re* even when it occurs in connection with *so* and thus should be high:



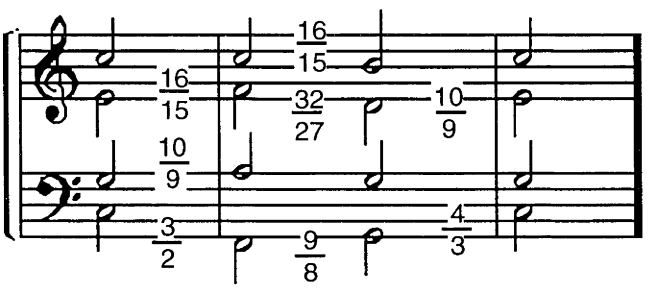
Kodaly, *Let Us Sing Correctly* no. 46

The influence of functional harmonic relationships on the intonation of *re*

The subdominant and dominant role of *re* create different intonational requirements. In the context of a dominant chord *re* tends to be high, in the context of a subdominant (relative, eg a II chord) *re* tends to be low:

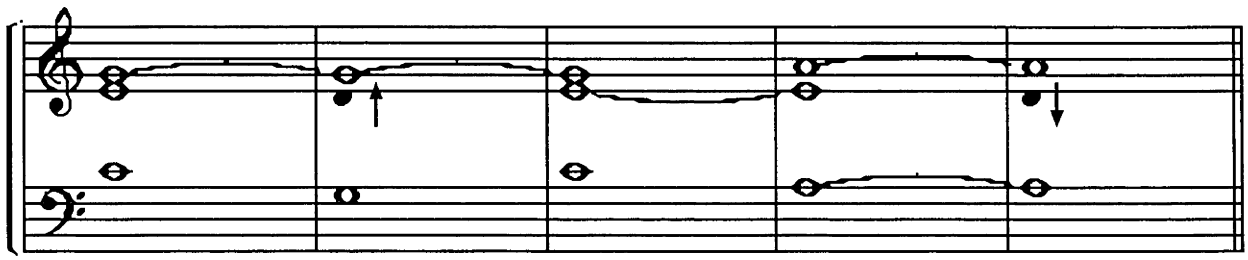


Even in a straight-forward I—IV—V—I progression where *re* only occurs in its high form with *so,* the influence of *fa* preceding it may make singers sing it too low, alternatively the *fa—re* melodic interval will be a 32:27 Pythagorean, smaller minor third:

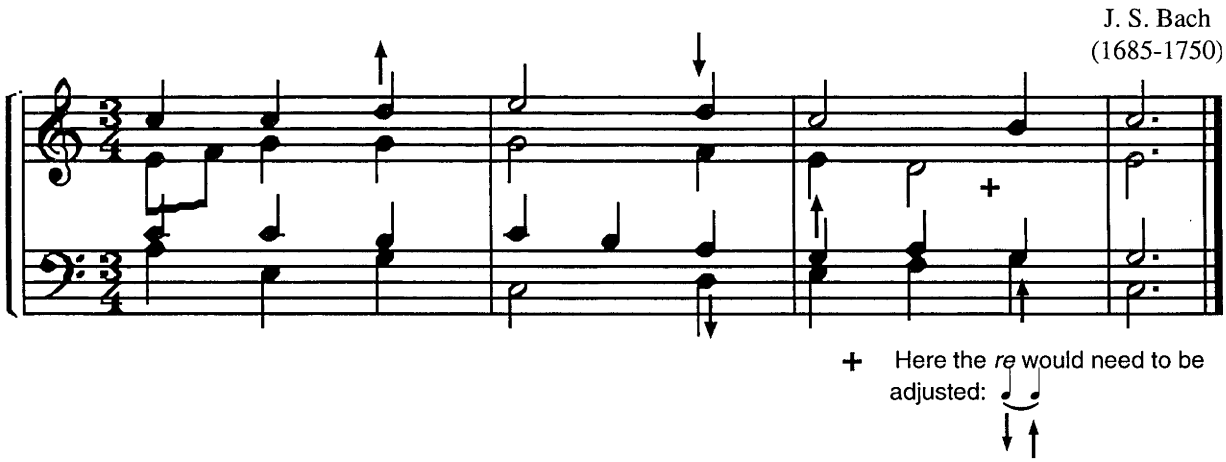


I—IV—V—I progression involving a Pythagorean minor third in the alto line

The 2nd degree will frequently alternate between its higher and lower form in the same piece depending on the influence of the 5th or 6th degree in other parts:



The following straightforward Bach excerpt[240] shows how a 2nd degree functioning in a dominant (V) chord is high, but later in the same work when it functions as the root in a subdominant relative chord (II) it should be tuned low. Although the alternation between the 5th and 6th degree would normally require the *re* or supertonic to adjust up and down a comma, if *re* is the common note, as it is in the penultimate bar, it cannot change so other notes must adjust or else a momentary intonational dissonance must be tolerated:



It is not always merely the *so* or the *la* that exerts influence on *re*. The third of a dominant chord (*ti*) demands a high *re*. A low *re* occurs in certain subdominant connections with *fa*. Since subdominant and dominant follow on from each other so frequently, it is necessary to be very aware of these differences.

Often, as when *re* is exposed, it cannot be the tone to adjust, as the adjustments would be too noticeable: in such cases the other parts need to adjust. This occurs when the *re* is part of a *cantus firmus*, is a sustained, common tone, or is in a prominent melodic line:

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240 Cited in Kardos, *KK*, p. 39.

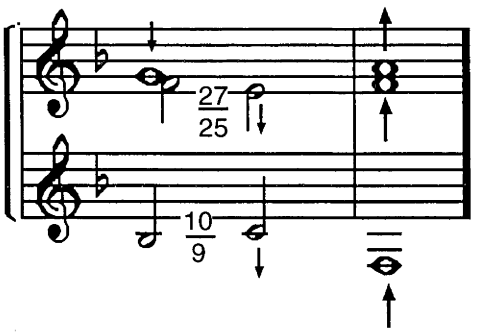


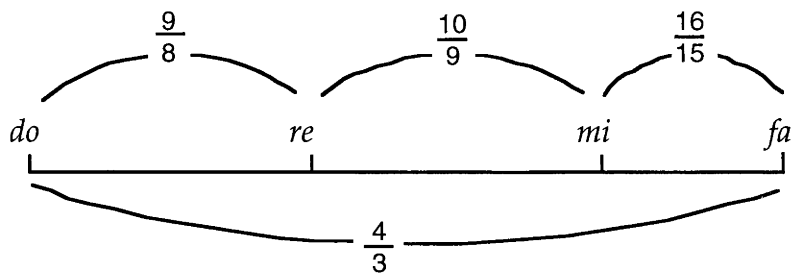
Diagram 204, Kardos, *KK*

In this example the 2nd degree *re* (g'), as part of a subdominant chord with the 4th degree *fa* (b♭), needs to be low. When the lower part ascends from *fa* to *so*, (b♭—c') normally a 9:8 whole tone, the *so* needs to be lower than normal to create a 3:2 with the *re*. Thus it moves by a 10:9 tone. The *so* being a comma lower than normal then necessitates a low *ti*, which will mean the mezzo moves by a 27:25 semitone, a comma larger than the normal 16:15, between the tonic and 7th degree.

5.4 The semitone

The derivation of the semitone in ancient Greece, as we have seen, was the remainder when a 4:3 was subdivided into tones. Two tones plus a semitone made up the diatonic tetrachord. In just intonation the 16:15 semitone is what is left over when the fourth is divided up into tones when the two tones are 9:8 and 10:9:

$$\frac{4}{3} \div \left(\frac{9}{8} \times \frac{10}{9} \right) = \frac{16}{15}$$



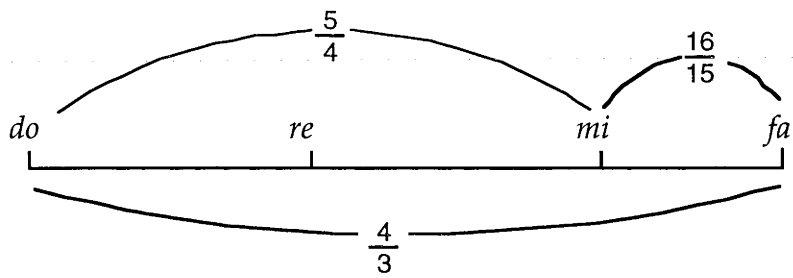
$\frac{10}{9}$ $\frac{9}{8}$ $\frac{16}{15}$

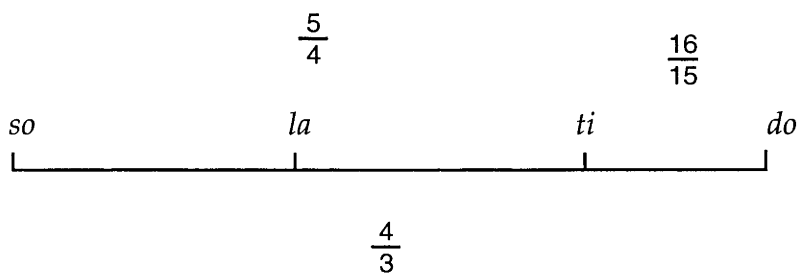
so *la* *ti* *do*

$\frac{4}{3}$

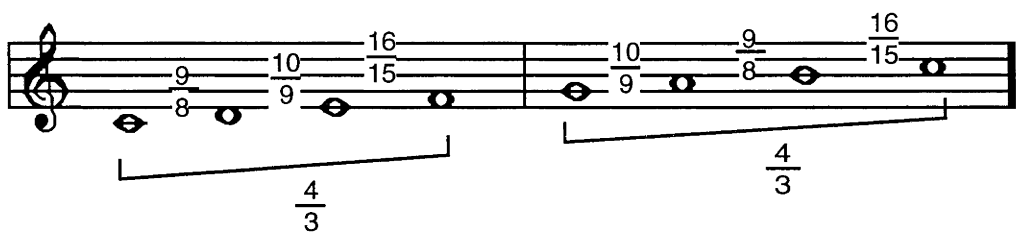
The tetrachord *do—fa* is the same as Didymus' diatonic, the tetrachord *so—do* is the same as Ptolemy's syntonic diatonic. The interval between *mi—fa* and *ti—do* is a 16:15 semitone. The 16:15 semitone is also the difference between the pure 5:4 major third and the pure 4:3 fourth.

$$\frac{4}{3} \div \frac{5}{4} = \frac{16}{15}$$

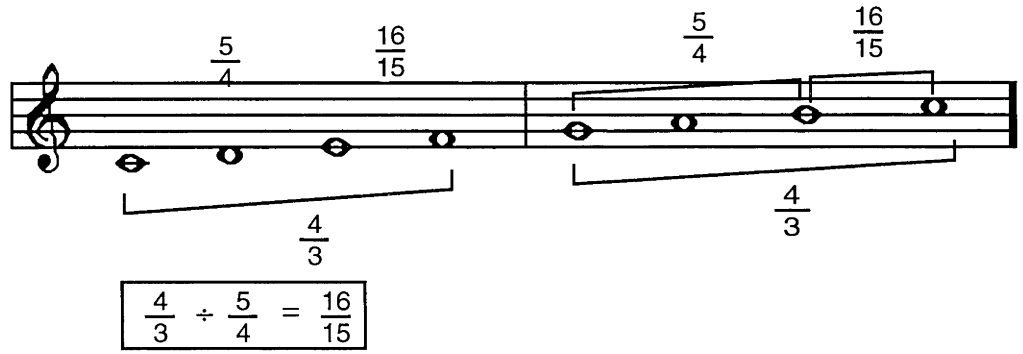




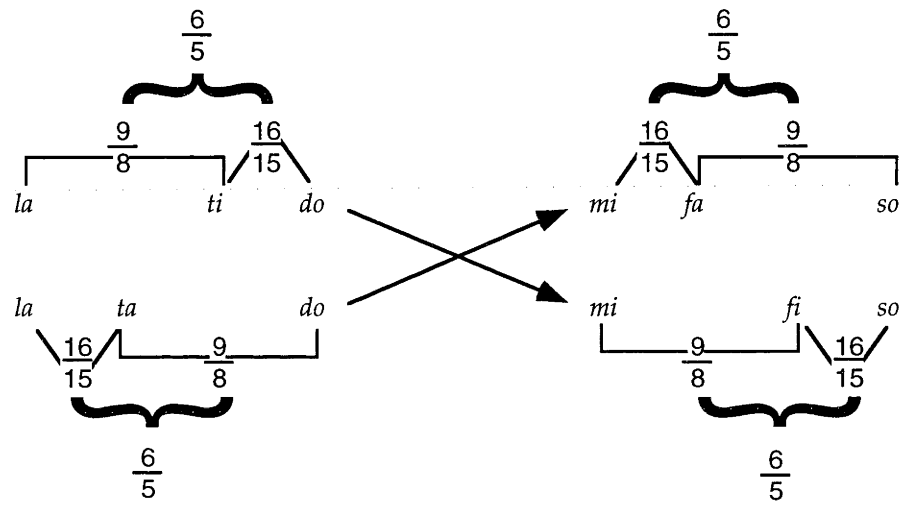
On a stave assuming a final of C major:



$$\frac{4}{3} \div \left(\frac{9}{8} \times \frac{10}{9}\right) = \frac{16}{15}$$



A pure 6:5 minor consists of a 9:8 whole tone and a 16:15 semitone: $\frac{9}{8} X \frac{16}{15} = \frac{6}{5}$. The following diagram[241] maps out different locations of the 16:15 semitone:



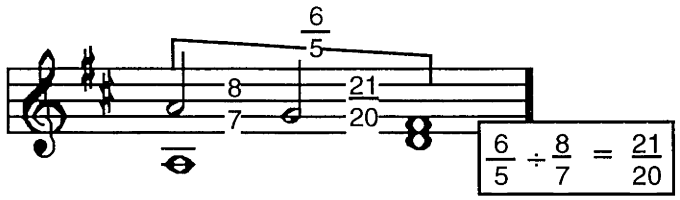
Intonation of the 4th and 7th degree in a major scale (*fa* and *ti*)

Anhemitonic pentatonic scales are free of the tuning complexities of the 4th and 7th degrees of a major scale (*fa* and *ti*). These two scale degrees bring into play many of the intervals and relationships that cause difficulties in tuning—the semitone, tritone, diminished fifth, and complex intervals such as the Pythagorean minor third, Pythagorean major sixth etc. That is one reason why Hungarian music educators advocate first using song material which does not contain *fa* and *mi* (ie pentatonic and infra-pentatonic melodies) in the developmentally early stages of music education.

The intonation of *fa* is difficult owing to the influence of *fa* in the context of the dominant seventh. As is analysed in more detail in section 5.6, the low

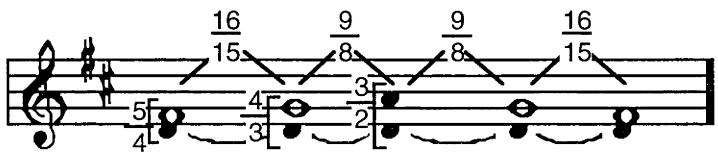
241 Based on Kardos *KV*, p. 71.

subdominant *fa* resolving to mediant does so by a 21:20 semitone:



Small semitone in the context of the resolution of the dominant 7th, based on diagram 122, Kardos, *KK*

The influence of this progression is so pervasive it tends to make singers want to make the *fa—mi* step small, even when the tonality or harmony calls for a high *fa* and a natural semitone between *mi* and *fa*. Singers need to distinguish when to sing *fa* high or low. In the following context *fa* is high:



Based on diagram 123, *Kardos, KK*

As a general principle, Kardos advises, the *mi—fa* step upwards should be felt as wide, the *fa—mi* downwards step as narrow.

The best way to introduce *fa* is as a passing note in an otherwise pentatonic frame. This is frequent in children's song material as cadence:



Thus *fa* fits into the intervallic frame of so and *mi*. Kardos advises singers to begin with the descending pure minor third, then fill it in:

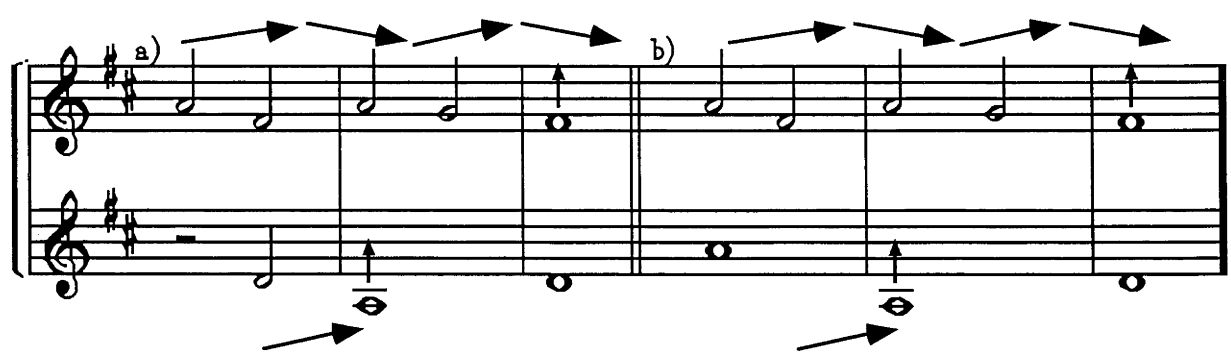


Diagram 120, *Kardos, KK*

Further advice given by Kardos on this exercise is to make sure that the upper note of the alto's octave from a' to a in 120b melts into the texture of the lower note. This octave must be in tune before the tuning of the *fa* passing note (g') can be achieved. A descending octave leap is difficult to sing accurately as it is easy to over-reach the lower note. One further exercise by Kardos for practicing the intonation of *fa* is given below:

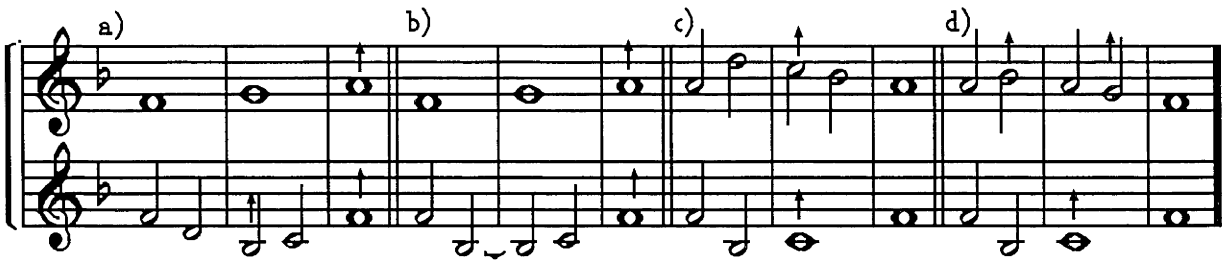
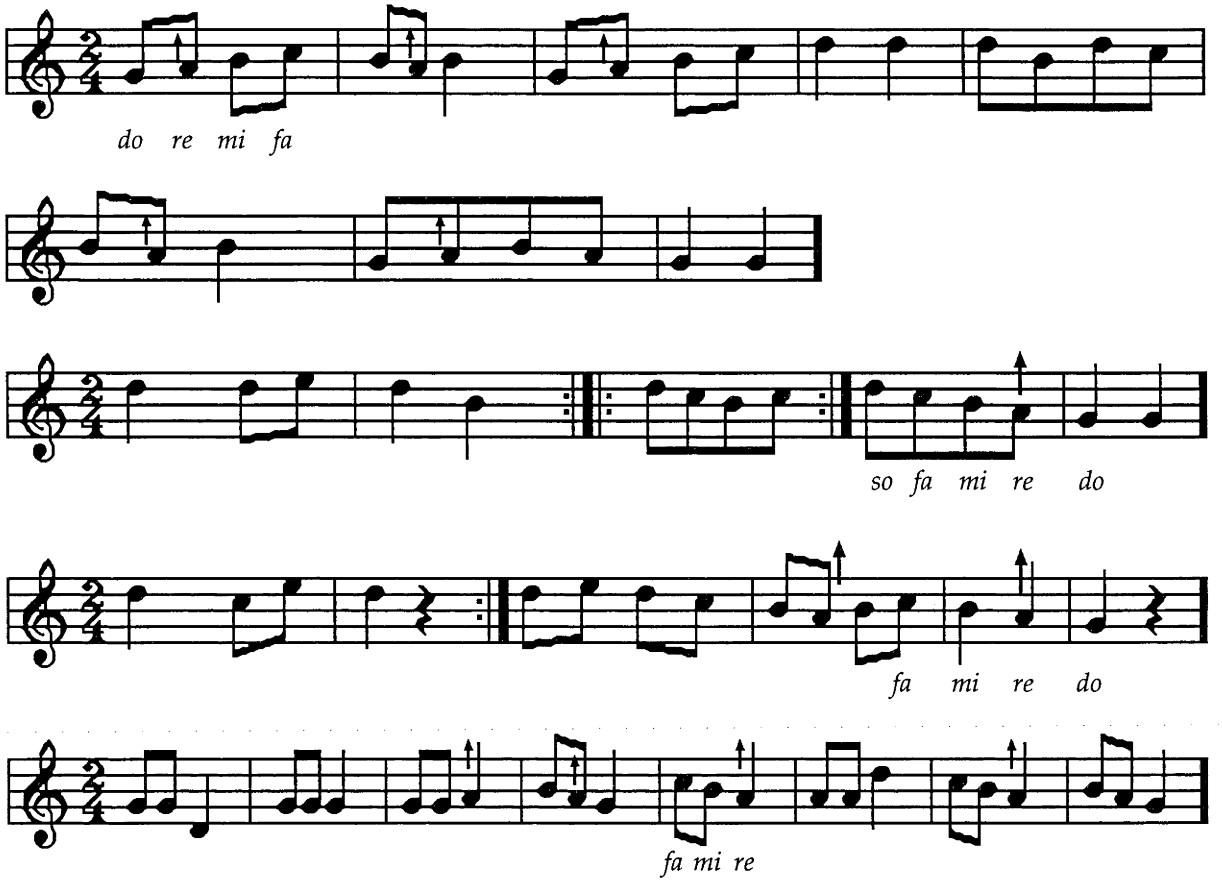


Diagram 124, *Kardos, KK*

In a justly tuned major (and minor) scale the interval between *fa* and *so* is usually a big whole tone. In diagram 124, Kardos explains, it is a small whole tone, because the *re* (g') is low owing to the influence of the *do la fa* (f'-d'- b♭) broken chord in the alto. The low *re* will cause a low *so*, thus the distance between the *la* and the *so* will be a small whole tone. This is an example of cases where the note that would normally adjust cannot because it is either a common or sustained note. Thus the other parts have to adjust to it. In descending a fifth from the tonic to subdominant, care is required to avoid over reaching the subdominant. In 124b the jump from *do* to *fa* below needs care that the *fa* is not too low.

Below are some simple folk songs chosen by Kardos for practice of *fa*:





Songs for practicing *fa*, Kardos, *KV*, p.75

The leading tone

Melodic and harmonic factors effect the intonation of the leading tone. Kardos did not advocate simple conformity to the just tuned scale where the distance between the leading tone and tonic is always a natural semitone, 16:15 regardless of other influences.[242] The function of the leading tone is to lead to the tonic, thus it is often natural to sing the leading tone high with a smaller semitone than 16:15. Further, the tempered semitone which is so much smaller (11.7 cents) permeates our hearing making a high leading tone seem more natural.[243] The Pythagorean semitone at 19.5 cents is smaller still: this gives a very high leading tone. However, a step by a 16:15 to tonic is the more acoustically correct. The V—I

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242 *Kardos, KK,* p. 40.

243 Suzanne Clercx shows references to high leading tones in the 15th century. She translates a passage from Ciconia's 'Nova Musica', which shows a fifteenth century reference to a diesis of a one-fifth tone or a 'chromatic' semitone resulting in a high leading tone. Suzanne Clercx, *Johannes Ciconia: Un musicien liegeois et son temps,* Liége, 1954, quoted in J. W. Herlinger, 'Marchetto's Division of the Whole Tone', JAMS, vol. 14, 1981, p. 206.

progression calls for a 16:15 semitone:

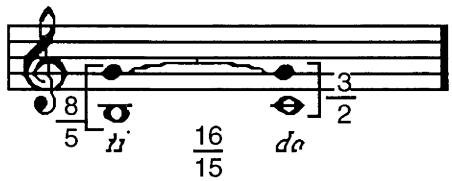


Diagram 102, Kardos, *KK*

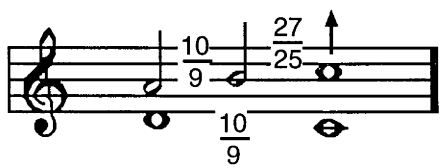
Some general guidelines, provided by Kardos, for intonation of the leading tone are as follows:[244]

- Generally the leading tone should be felt to be high;
- In a diatonic or modal melodic passage the acoustic semitone is usually the more natural progression from leading tone to tonic;
- The leading tone as part of a dominant chord should form a pure 5:4 third with the dominant, thus should not be sharp. The *ti* in a dominant chord in root position is the third which should be softer and blends into the harmony;



Diagram 109, Kardos, *KK*

- The leading tone is higher if it is approached from below by a big whole tone 9:8 than if it is approached by a small whole tone, 10:9. In the just tuned major scale *la—ti* is a big whole tone and the distance between *la* and *do* is a natural minor third 6:5. A natural minor third minus a big whole tone leaves a natural semitone: $\frac{6}{5} \div \frac{9}{8} = \frac{16}{15}$ However a pure minor third minus a small whole tone leaves what is known as the maximum semitone: $\frac{6}{5} \div \frac{10}{9} = \frac{50}{54} = \frac{27}{25}$. At 133.238 cents it is 21.507 cents bigger than the natural at 111.731, thus the distance between the leading tone and tonic would be extremely large;



244 Kardos, *KK*, pp. 40-41.

- A leading tone *ti* which is approached from above from a low *re* forms a Pythagorean minor third with the low *re*. This interval is smaller than the acoustic minor third. *re—do* = $\frac{10}{9}$; *do—ti*= $\frac{16}{15}$. $\frac{10}{9}$ X $\frac{16}{15}$ = $\frac{160}{135}$ = $\frac{32}{27}$

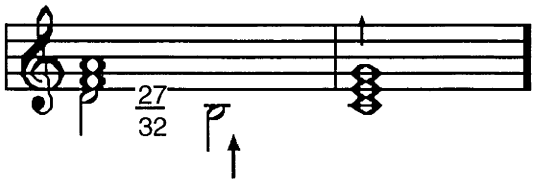


Diagram 107, *Kardos, KK*

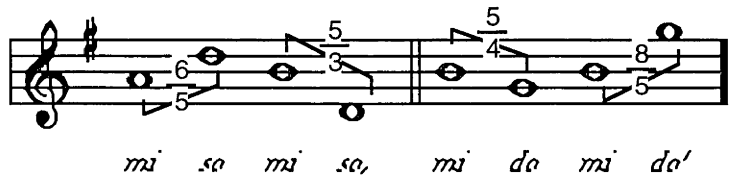
The diatonic diminished fifth between *fa* and *ti* (in this case between f' and b) has a ratio of $\frac{64}{45}$. This diminished fifth is not therefore divisible into two acoustic minor thirds. It consists of two asymmetric minor thirds, an acoustic and a Pythagorean minor third: $\frac{64}{45} \div \frac{6}{5} = \frac{32}{27}$. The following example by Fr. Marpurg contains a diminished fifth from c' to f#, *fa—ti*. It is also excellent for the practice of both *fa* and *ti*:



Diagram 197, Kardos, *KK*

5.5 The major and minor sixth

The acoustic 5:3 major sixth is the inversion of the acoustic 6:5 minor third: $\frac{2}{1} \div \frac{6}{5} = \frac{5}{3}$. The acoustic 8:5 minor sixth is the inversion of the 4:3 major third: $\frac{2}{1} \div \frac{5}{4} = \frac{8}{5}$.



Acoustic major and minor sixths as inversions of minor and major thirds

Three kinds of major sixth are as follows:

	Acoustic Major sixth	**Tempered Major sixth**	**Pythagorean Major sixth**
Ratio	$\frac{5}{3}$	$\left(\sqrt[12]{2}\right)^9$	$\frac{27}{16}$
Cents	884.3	900	905.9
Difference from tempered	- 5.7 cents		+5.9 cents

The acoustic major sixth

The inversion of the pure minor third is an acoustic major sixth. It consists of an acoustic perfect fifth and a small whole tone:

$$\frac{3}{2} X \frac{10}{9} = \frac{30}{18} = \frac{5}{3}$$

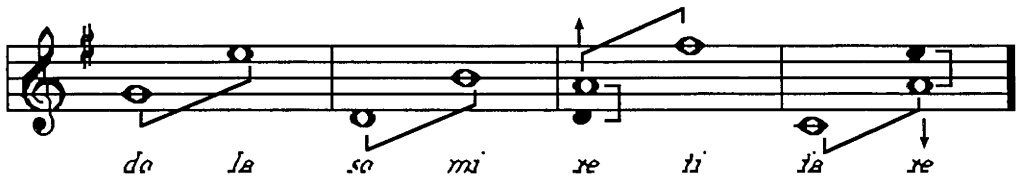
Pythagorean and acoustic major sixths compared

The inversion of the 32:27 Pythagorean minor third is the 27:16 Pythagorean major sixth. The Pythagorean major sixth consists of an acoustic perfect fifth and a big whole tone: $\frac{3}{2} X \frac{9}{8} = \frac{32}{37}$. The acoustic major sixth contains a 3:2 fifth and a 10:9 small whole tone. The difference between the Pythagorean and acoustic major sixths is a syntonic comma, $\frac{81}{80}$.



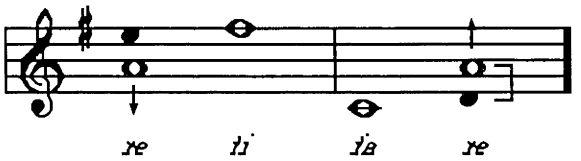
Pythagorean and acoustic major sixths. Diagrams 137, 138, Kardos, *KK*

Acoustic major sixths are found in the following contexts:



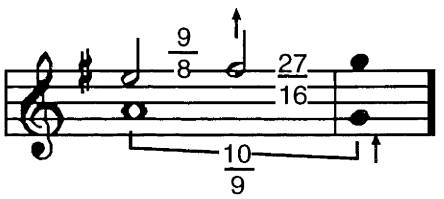
Acoustic major sixths. Diagram 144, Kardos, *KK*

Pythagorean major sixths will occur where a 3:2 fifth is extended by a 9:8 whole tone:



Pythagorean major sixths. Diagram 146, Kardos, *KK*

The Pythagorean major sixth that occurs between a low super-tonic and the leading tone (low *re*— *ti*) is difficult to intonate, (as is a Pythagorean sixth between a subdominant *fa* and high supertonic *re*):



Pythagorean major sixth arising from a low 2nd degree. Diagram 147, Kardos, *KK*

An acoustic major sixth as occurs between a high supertonic (*re*) and leading tone (*ti*) does not create special difficulty. The consonance of the major sixth approached from a natural semitone makes the intonation of the leading tone clear:

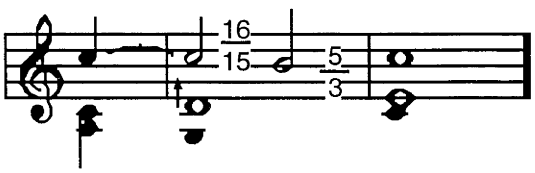


Diagram 151, Kardos, *KK*

The minor sixth

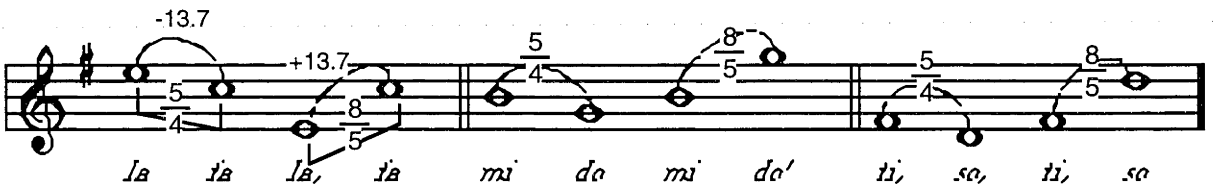
The pure minor sixth, ratio 8:5 is the inversion of the pure major third, 5:4:

$$\frac{2}{1} \div \frac{5}{4} = \frac{8}{5}$$

Minor sixths

Interval name	Pythagorean minor sixth	Tempered minor sixth	Just minor sixth
Ratio	$\frac{128}{81}$	$\left(\sqrt[12]{2}\right)^8$	$\frac{8}{5}$
Cents	792.18	800	813.686
Difference from tempered	- 7.82		+13.686

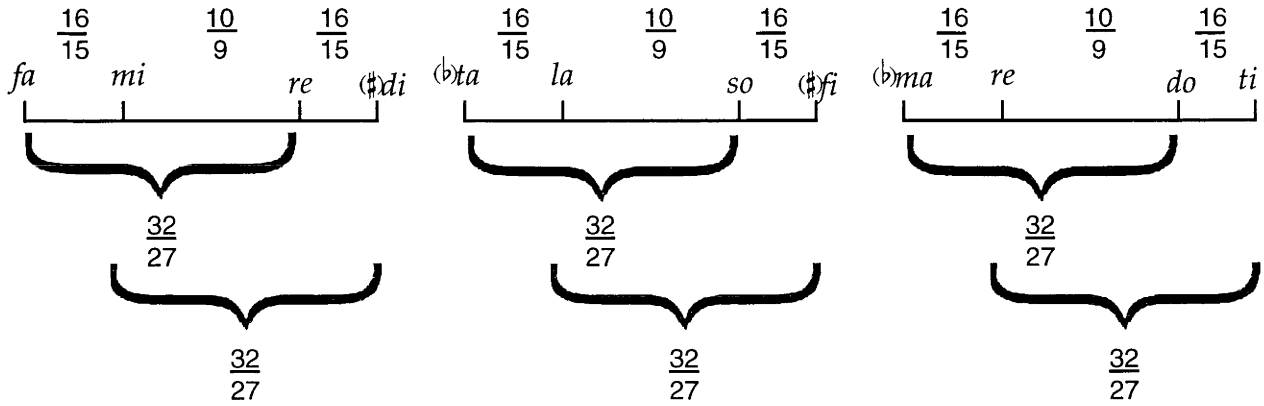
The acoustic minor sixth is 13.7 cents larger than the tempered. Thus it requires care in an ascending direction. The second note needs to be particularly high, which is difficult since it is a large leap:



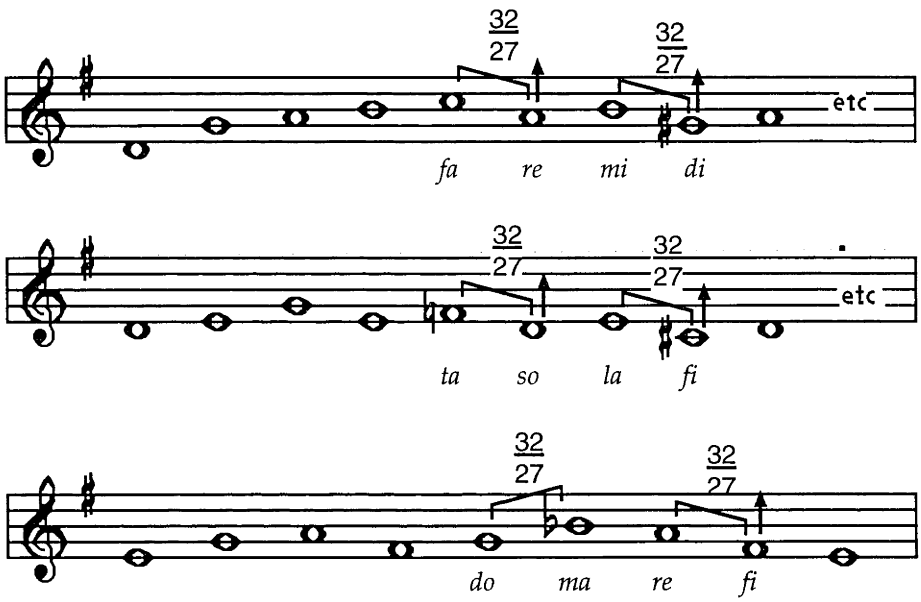
The Pythagorean minor third

Kardos conceded that some harmonic or melodic contexts necessitate a Pythagorean minor third even in natural or just intonation systems. The Pythagorean minor third occurs when a 9:8 big whole tone is subtracted from a 4:3 perfect fourth since $\frac{4}{3} \div \frac{9}{8} = \frac{32}{27}$. Alternatively it occurs as the sum of a small whole tone and a natural semitone: $\frac{10}{9} \times \frac{16}{15} = \frac{160}{135} = \frac{32}{27}$.

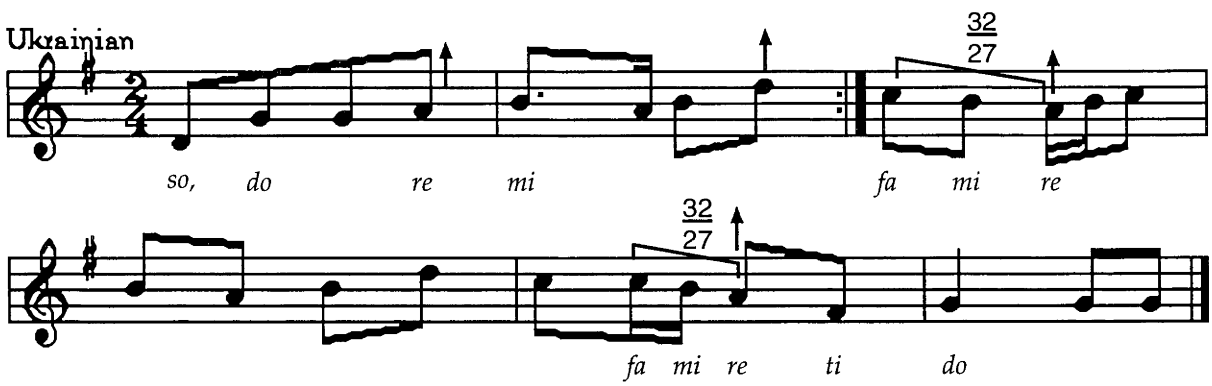
The following diagrams illustrate the various contexts in which the Pythagorean minor third occurs:



Pythagorean minor thirds[245]

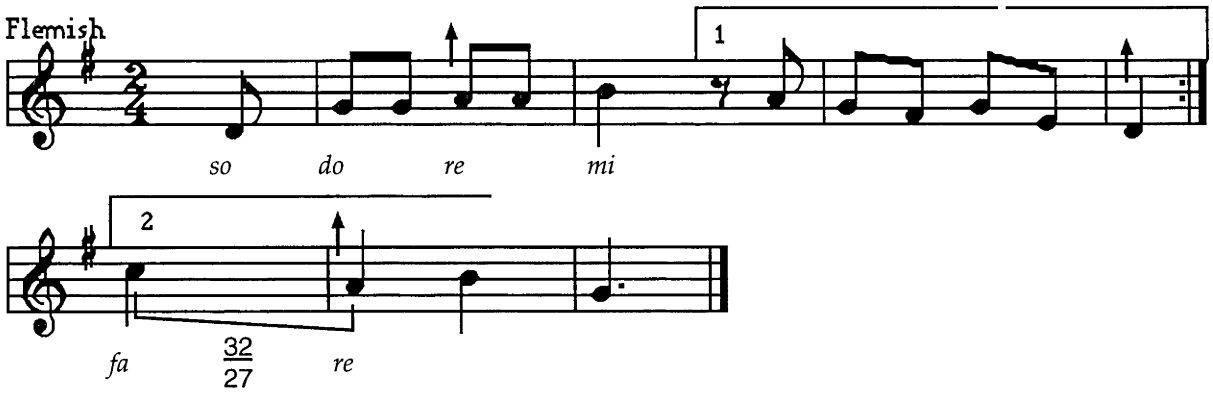


In selected musical examples:[246]

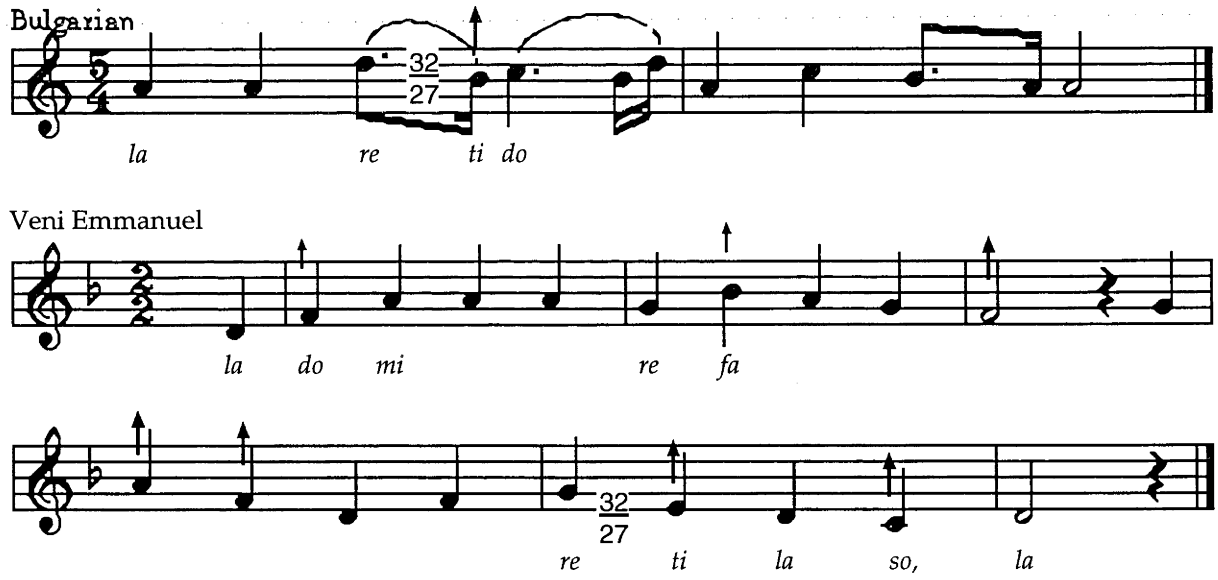


245 Based on Kardos's selection in his *A tiszta intonálásra*, p. 36.

246 Based on Kardos, *A tiszta intonálásra*, p. 35.



In both the Ukrainian and Flemish examples above, the *re* functioning as the 2nd degree is high in relation to the 5th degree *so*. The minor third between degrees IV and II, *fa* and *re* is therefore Pythagorean. The next two examples are in *la* tonality (Aeolian). The *re* is low and thus forms a Pythagorean minor third with *ti*:



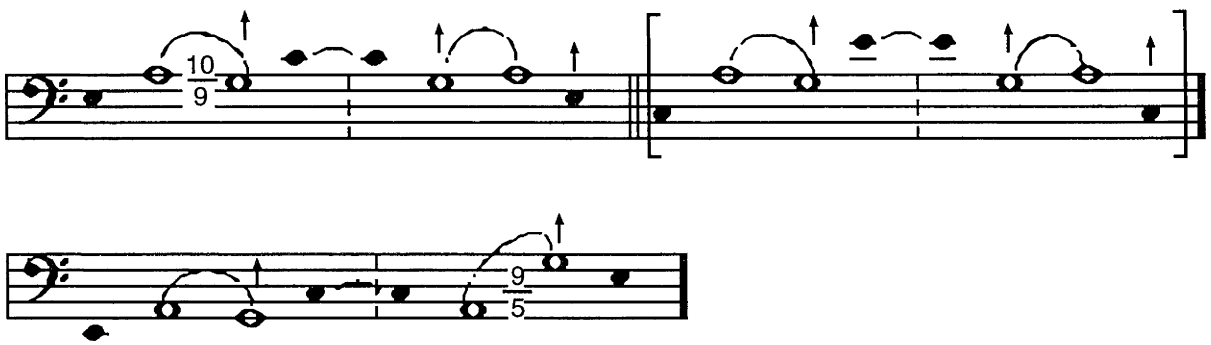
This shows the influence of the high versus low *re*. A high *re* connected to *so* causes the minor third between *fa* and *re* to be Pythagorean, a low *re* causes the minor third between *re* and *ti* to be Pythagorean.

5.6 The minor and major seventh

The minor seventh

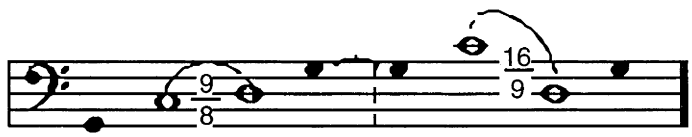
		small (Pythagorean) minor seventh	tempered minor seventh	big minor seventh
Ratio	$\frac{7}{4}$	$\frac{16}{9}$	$\left(\sqrt[12]{2}\right)^{10}$	$\frac{9}{5}$
Cents	968.826	996	1000	1017.596
Difference from tempered	- 31.174	- 4		+ 17.596

The minor seventh is the inversion of the major second, and since in pure intonation there are two forms of whole tone, there are correspondently two forms of minor seventh. The inversion of the small whole tone 10:9 is the large minor seventh 9:5, 1017.596 cents. The following diagrams show the placement of the minor seventh:[247]



The 9:5 large minor seventh, inversion of 10:9 tone, based on Kardos, *KV*, p. 178

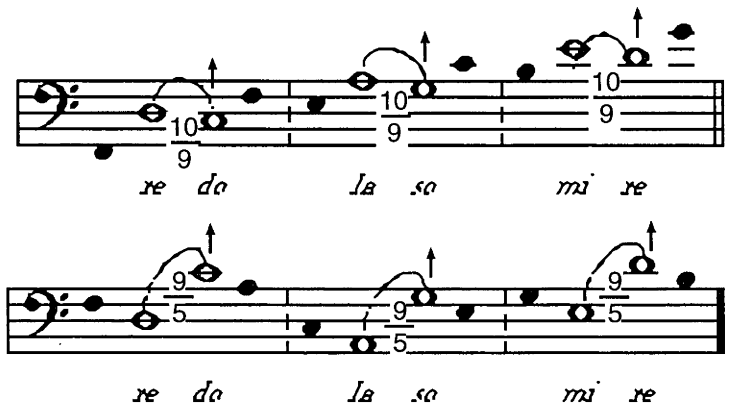
The inversion of the 9:8 big whole tone is the 16:9 small minor seventh, 996 cents:



The small minor seventh, inversion of 9:8 tone, based on Kardos, *KV*, p. 178

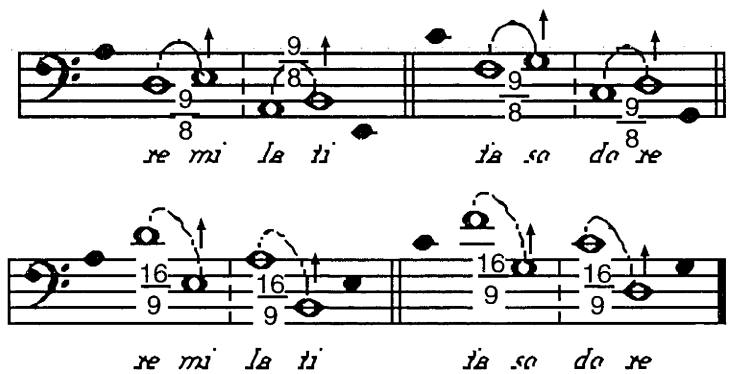
247 The following diagrams illustrating the minor seventh are adapted from *KV*, p. 178.

The following diagram shows contexts where the small whole tone and consequent large minor seventh can be found:



9:5 large minor seventh, inversion of the small whole tone, basd on Kardos, *KV*, p. 178

The next diagram shows the big whole tone and small minor seventh:



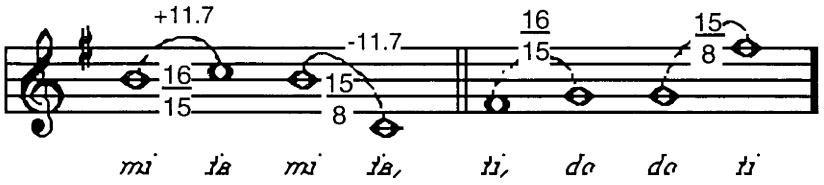
16:9 small minor seventh, inversion of the big whole tone, based on Kardos, *KV*, p. **178**

The major seventh

	just major seventh	Pythagorean major seventh	tempered major seventh
Ratio	$\frac{15}{8}$	$\frac{243}{128}$	$\left(\sqrt[12]{2}\right)^{11}$
Cents	1088.269	1109.775	1100
Difference from tempered	- 11.731	+ 1109.775	

The major seventh is the inversion of the semitone, a perfect octave minus a semitone is a major seventh, $2 \div \frac{16}{15} = \frac{15}{8}$. Since in just intonation there is only one form of semitone, all major sevenths have the proportion 15:8, or 1088.269 cents.

Being 11.731 cents smaller than the tempered, the seventh requires care in a descending direction, to ensure that the second note is not too low:



The seventh harmonic, ratios of 7 and dominant seventh chords

In his article 'The number seven in the theory of intonation', Eric Regener points out that the number 7 has been 'the stepchild of intonation theory' since Zarlino's senario.[248] Long before this it was excluded from the Pythagorean quaternary. From the Baroque, the dominant seventh chord was admitted into harmonic theory, justified by its presence in the harmonic series as harmonics 4:5:6:7. For Rameau seventh chords had great importance. He derived all chords from either the perfect major triad or the (dominant) seventh chord.[249] Helmholtz defines chords of the seventh as being constructed by uniting two consonant triads containing two common tones that are consonant to each other, hence reducing the proportion of perceived dissonance. Thus c—e—g joined to e—g—b becomes c—e—g—b (with a major seventh), the notes c—b being a dissonant relationship.[250] The intervals between seventh chords based on consonant triads of the scale are alternating major and minor thirds. Kardos only mentioned the dominant seventh in detail.

The dominant seventh chord

The first seven harmonics, 1:2:3:4:5:6:7 generate the sound of what is called the dominant seventh triad. In fact harmonics 4:5:6:7 alone suffice, but according to Kardos the importance of the dominant seventh triad in functional harmonic writing lies in the fact that it derives from the first seven harmonics. To replicate the harmonic series as closely as possible, the seventh note should be intoned slightly flat, since the 7:6 interval is smaller than the 6:5 interval (just minor

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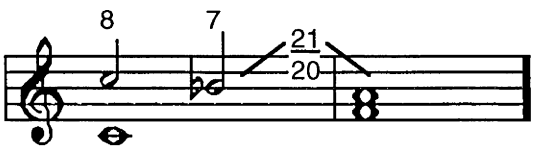
248 Regener, op. cit., p. 140.

249 Rameau, *Traité de l'Harmonie Reduite à ses Principes naturels*, 1722, p. 45, cited in Ferris, op. cit., p. 239.

250 Helmholtz, op. cit., pp. 343-350.

third) by $\frac{36}{35}$ since $\frac{6}{5} \div \frac{7}{6} = \frac{36}{35}$. The tension of the 7th degree in a dominant seventh, which displays a downward pull towards its resolution on the 3rd degree of the tonic, (subdominant to mediant) helps create the flatness required in acoustic terms. There are further acoustic reasons for so doing. Easley Blackwood points out that when the dominant seventh is tuned so that its frequencies are in the ratios 4:5:6:7, beats in lower-order harmonics are nulled and difference tones reinforce a pitch two octaves below the root of the chord.[251]

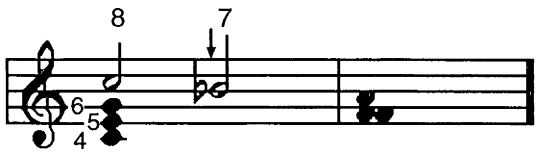
The 7th has a very strong pull to resolution. The distance between the subdominant acting as 7th and the mediant acting as third in the triad of resolution (*fa—mi*) is a very small semitone, ratio 21:20, 84.467 cents. This difference can be computed from a number of angles. Firstly, the outer poles of the 4:5:6:7 chord has a ratio of 7:4, compared with the more standard just minor seventh of $\frac{16}{9}$ between the dominant and subdominant (the interval between subdominant and dominant, or 4th and 5th degrees in the just major scale is 9:8, the inversion of this is $2 \div \frac{9}{8} = \frac{16}{15}$). The difference then between the two minor sevenths is $\frac{16}{9} \div \frac{7}{4} = \frac{64}{63}$. The subdominant, or 4th degree is flatter by $\frac{64}{63}$ when functioning as in a dominant seventh context than the subdominant or 4th degree in other contexts. The semitone of resolution between *fa* and *mi* or between the 7th of the dominant seventh chord and the third of the tonic chord, (or subdominant to mediant) is thus smaller than the natural 16:15 semitone that is usually between the 3rd and 4th degrees of a major scale by $\frac{64}{63} \div \frac{16}{15} = \frac{21}{20}$. Its cents value is 84,467, ie is 27.264 cents smaller than the more usual just semitone of 111.7 cents:



Illustrated below are two different contexts creating two forms of b♭. In the first the b♭' is lower by $\frac{64}{63}$ than the b♭' in the second, non dominant seventh context, 64:63 being the difference between 8:7 and 9:8: $\frac{7}{8} \div \frac{9}{8} = \frac{64}{63}$.

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251 Blackwood, op. cit., pp. 81-83.

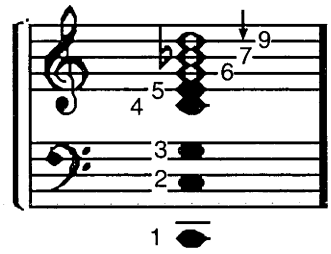




The dominant 9th

The dominant 9th, which Kardos does not analyse, is also based on the harmonic series, 4:5:6:7:9. The interval between the ninth and the seventh is 9:7, which at 435 cents is 48.686 cents larger than the pure 5:4 major third at 386.314 cents. It consists of a 7:8 tone and a 9:8 tone:





5.7 The tritone, augmented fourth, diminished fifth.

The tritone as *diabolus musicae* was prohibited as a melodic progression in the medieval and early Renaissance period (1300-1560). Musica ficta rules were designed in large part to correct the tritone that occurred between f' and b'[252] or these modal degrees in transposition.[253] In fact, not only was an ascending or descending melodic progression between f' and b' banned, but so was an indirect tritone, when filled in by some of the intervening steps.[254] It was allowed in some cases when it resolved to the consonance of the perfect fifth, ie when it continues from f' through b' to c". Tinctoris asserted it was either difficult or

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252 b' then being known as ♮ or 'hard' (*durum*) in contrast to its 'softened' form ♭ (*molle*).

253 Jacques de Liége, in around 1321-5 writes of how 'ancient writers' (the practitioners of *ars antiqua*) found ways of avoiding the tetrachord which went b'-c"d"e" so as to avoid the ♮, replacing it with tetrachords with a soft b (or b♭), so that the 'fifth and sixth modes, may proceed through the [perfect] fourth above its proper final which is F.' Quoted, with numerous other writers of the same period reiterating such principles in Berger,op.cit., p. 70.

254 Some theorists allowed indirect melodic tritones when there was more than one intervening step.

impossible to sing a melodic tritone as a direct leap.[255] Zarlino wrote that 'the tritone, semidiapente, and others like them must not be used, however, despite the fact that certain moderns write them and justify the progressions as chromatic.'[256] The diminished fifth was also prohibited. Ramos stated that the difference between these two intervals mattered more to theorists but for practical musicians both were equally dissonant and thus equally to be avoided.[257]

In the Renaissance, the tritone along with other dissonant intervals began to be used for dramatic effect by composers of the *seconda practica*.[258]Artusi's attack on Monteverdi's innovations centred upon his breaking such rules for dramatic effect.

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255 Tinctoris, quoted in Berger, op. cit., p. 72.

256 Gioseffo Zarlino, *Le Istitutioni harmoniche*, G. A. Marco and C.V. Palisca, trans., Yale University Press, New Haven, 1968, p. 110.

257 Ramos de Pereia, *Musica practica*, p. 50., quoted in Berger, op. cit. p. 79.

258 Vincentino, in 1555, writes of how thetritone occasionally occurs, despite its difficulty for singers, when setting words with 'astonishing effect'. The ascending tritone is lively and powerful, the descending one is funereal and sad. Cited in Berger, ibid., p. 75.

Below are seven different forms of tritone or diminished fifth:

tritones, diminished fifths, augmented fourths

	Acoustic diminished fifth	Pythagorean diminished fifth	Diatonic augmented fourth	Tempered tritone	Diatonic diminished fifth	Pythagorean augmented fourth	Acoustic augmented fourth
Ratio	$\frac{7}{5}$	$\frac{1024}{729}$	$\frac{45}{32}$	$\left(\sqrt[12]{2}\right)^6 = \sqrt{2}$	$\frac{64}{65}$	$\frac{729}{512}$	$\frac{10}{7}$
Ratio in decimals	1.4	1.4046	1.4062	1.4142	1.4222	1.4237	1.4285
Cents	582.5	588.2	590.2	600	609.8	611.7	617.5
Difference from tempered	- 17.5	- 11.8	- 9.8		+ 9.8	11.7	17.5

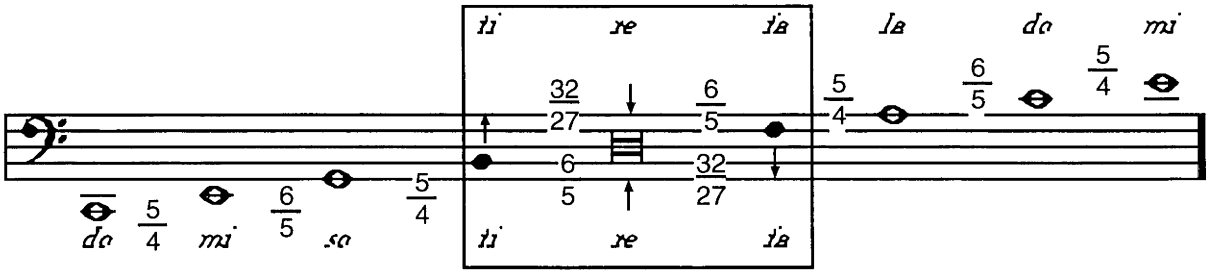
The diminished triad

The diminished triad based on degree VII will contain one 32:27 Pythagorean and one 6:5 acoustic minor third,[259] the two forming a diminished triad of $\frac{64}{45}$. The placement of the Pythagorean minor third depends on the height of the middle note. If *re* is high, then the *ti—re* will be acoustic, the *re— fa* Pythagorean. If *re* is low then this is reversed:[260]

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259 This ratio can be derived a number of ways. If there is a perfect fourth from tonic to subdominant and a natural semitone between leading tone and tonic then $\frac{4}{3} \times \frac{16}{15} = \frac{64}{65}$. Alternatively $\frac{2^2}{1} \div \frac{3^2}{2} \div \frac{4}{5} = \frac{64}{65}$.

260 Kardos, *KK*, pp. 51-52.



Based on diagram 170, Kardos, *KK*

In the first example below, *re* is high in relation to *so*. Thus the Pythagorean minor third occurs between the high *re* and*fa*:

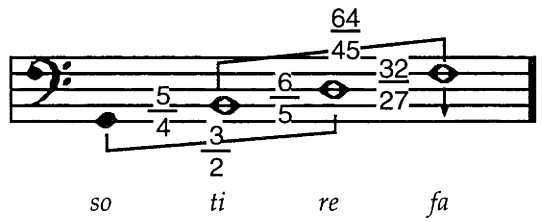


Diagram 168, Kardos, *KK*

In the next example, *re* is low in relation to *la*, and the Pythagorean minor third occurs between low *re* and *ti*:

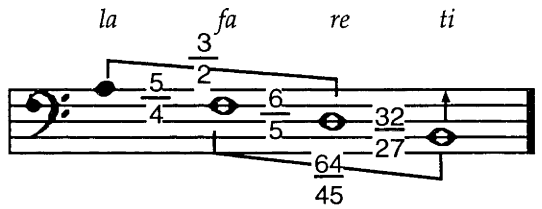
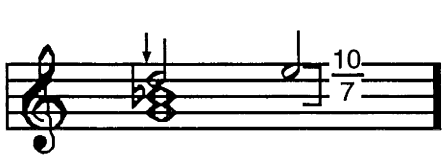
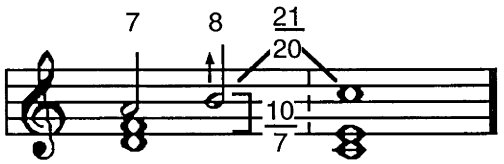


Diagram 169, Kardos, *KK*

The acoustic 10:7 augmented fourth

The acoustic augmented fourth or tritone is the interval between the 7th and 10th harmonic. It particularly occurs in the context of the progression subdominant relative (II) to dominant (VII, V^7) to tonic. The melodic progression from degree VI to VII (*la*—*ti*) is via a 7:8 whole tone, creating a high leading tone which then resolves to tonic by a small 21:20 semitone:

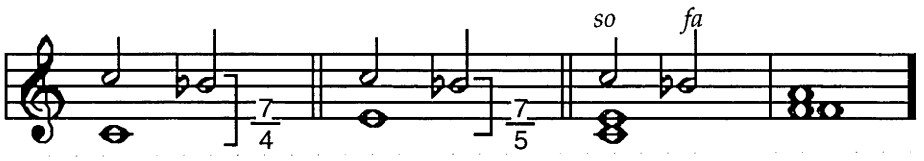




The acoustic 7:5 diminished fifth

The acoustic diminished fifth is 35 cents smaller than the augmented fourth, 582.5 cents compared with 617.5 cents. The acoustic diminished fifth is 17.5 cents smaller, the augmented fourth 17.5 cents bigger than the tempered. The acoustic diminished fifth has a downward tension, the augmented fourth an upward tension. Both occur in the context of the dominant seventh, and derive from the 7th harmonic.

The example below compares *fa* as a minor seventh against a *so* below as occurs in the dominant seventh (or between the 4th and 7th harmonic), and *fa* against *ti* (between harmonics 5 and 7):



The tritone as a harmonic interval

Despite its instability and complexity the tritone often occurs in homophonic writing when the voice leading of a part progresses from subdominant to dominant to tonic. The most common form of harmonic tritone is in cadential formulas containing the 4th scale degree (*fa*) and the leading tone (*ti*). It is easier in examples where only one part moves. Kardos advised that the note that enters first in a consonant setting is the referential tone. The note that creates the interval of a tritone must adjust to that referential tone. If two notes a tritone apart enter simultaneously then the note that resolves most closely to the tonic is the referential tone. In the case of degrees the subdominant and leading tone, this is the leading tone. These principles are illustrated in the following two exercises. Kardos's direction for the singing of these exercises is as follows: The *fa* in the second bar of the following exercise should tune to the *ti* (or the b♭' to e'). The sustained e' must remain steady in pitch. The reverse is the case in 222b where *ti* tunes to *fa* (or the e' to b♭'):

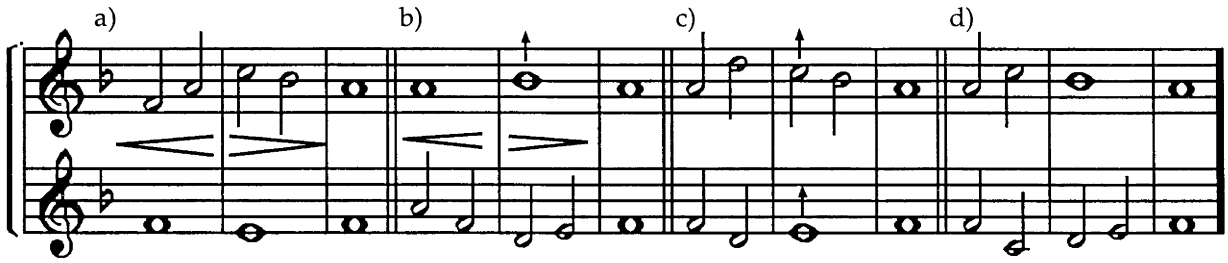


Diagram 222 Kardos, *KK*

5.8 Chromatic intonation, division of the whole tone, further definitions of semitone

Medieval and early Renaissance theorists, in addition to the prohibition of the tritone and diminished fifth, also banned the diminished octave and the chromatic semitone. Chromatic alteration came to be used as an effect, *maniera*,[261] from 1560 on and became the hallmark of mannerist writing (of which Orlando di Lasso and Gesualdo were masters). But even before that, the chromatic semitone was used by some musicians, notable among them Marchetto da Padova.[262] The whole tone does not divide evenly, and theoretical writing on the various divisions of the tone abounds with a confusing array of terminology. In the medieval period the Pythagorean 9:8 tone was divided into a minor and a major semitone. The minor semitone (originally called leimma or

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261 See Claude Palisca, 'Towards an intrinsically Musical Definition of Mannerism in the Sixteenth Century', in *Studies in the History of Italian Music and Music Theory*, op. cit., pp. 312-346. He cites the 16th century theorist Ghiselin Danckerts who criticised those composers writing in the new manner (*nuova maniera*) who used accidentals not merely to avoid false relations or prohibited relations but to achieve new melodic effects. Quoted on p. 314. Taisnier in 1559 criticised composers who introduced the chromatic diesis. Aristoxenus (4th century B.C.) had proposed a fractional division of the whole tone in a system transmitted by Boethius. Medieval theorists, inclined towards the Pythagorean view, typically rejected Aristoxenus's system. Richard C. Crocker, 'Aristoxenus and Greek Mathematics', *Aspects of Medieval and Renaissance Music: A Birthday Offering to Gustave Reese*, Jan LaRue, ed., New York, 1966, pp. 96-110.

262 For Marchetto's division of the whole tone, his tuning system, and use of chromatic intervals see Berger, op. cit.,pp. 22-29; Jan Herlinger, *Marchetto's Influence: the manuscript evidence*, University of Notre Dame, South Bend, 1990; Jan Herlinger, *The Lucidarium of Marchetto of Padua: A Critical Edition*, Translation and Commentry, PhD diss., University of Chicago, 1978; Jan Herlinger, 'Marchetto's Division of the Whole Tone', *JAMS*, 34, 1981 pp. 193-216. Herlinger, Jan W., *The Lucidarium of Marchetto of Padua: A Critical Edition*, Translation and Commentry, PhD diss., University of Chicago, 1978. Marchetto's *Lucidarium in arte musicae planae* of 1317-18 was the first to present music showing chromaticism, the term 'permutation' being used for such chromatic progressions. Marchetto also divided the whole tone into fifths, thus abandoning Pythagorean tuning. The minor and major semitones were two fifths and three fifths respectively.

'left-over')[263] was what was left over when a fourth, ratio 4:3, was divided into two whole tones: $\frac{4}{3} \div \frac{9}{8} \times \frac{9}{8} = \frac{256}{243}$. This is the semitone used in a Pythagorean scale. The major semitone, ratio 256:243 originally called *apotome* was the remainder when a *leimma* was subtracted from a 9:8 since $\frac{9}{8} \div \frac{256}{243} = \frac{2187}{2048}$.[264] Similarly in pure intonation the whole tone does not evenly divide into two acoustically pure semitones. A big whole tone minus a natural semitone produces the major chroma: $\frac{9}{8} \div \frac{16}{15} = \frac{135}{128}$. A small whole tone minus a natural semitone produces a minor chroma: $\frac{10}{9} \div \frac{16}{15} = \frac{150}{144} = \frac{25}{24}$. Thus the whole tone, whether the 9:8 or the 10:9 version, cannot divide equally into two natural semitones. Below is a table showing eight types of semitone.

minor seconds, semitones and chromas

interval name:	**minor chroma**		**Pythagorean semitone, *leimma*, limma,**	**major chroma**	**tempered semitone**	**diatonic semitone natural semitone**	**apotome**	**maximum semitone**
ratio:	$\frac{25}{24}$	$\frac{21}{20}$	$\frac{256}{243}$	$\frac{135}{128}$	$\sqrt[12]{2}$	$\frac{16}{15}$	$\frac{2187}{2048}$	$\frac{27}{25}$
cents:	70.672	84.467	90.225	92.179	100	111.731	113.685	133.238
Difference	-29.328	-15.533	- 9.775	- 7.821		+11.731	+13.685	+33.238

In dealing with chromatic passages, the size of the semitones will depend upon the size of the whole tone that is divided, what tuning system is appropriate for the period of the piece, and what other vertical and horizontal factors influence the interval in question. Obviously, singers are not likely to get overly anxious about whether their Pythagorean semitone is correctly sung with a ratio of $\frac{256}{243}$ or whether it should be a major or minor chroma. A basic principle is to keep

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263 To confuse matters further, some Pythagoreans, such as Philolaus, called this 256:243 the *diesis*.

264 For example, Gaudentius, a first century A.D. Aristonexian, writes in his *Harmonic Introduction*', 'The remainder of the leimma to complete the tone is called *apotome;* commonly this is also a semitone. Therefore one of the semitones will be the major and the other one the minor.' Quoted in Solon Michaelides, *The Music of Ancient Greece: An Encyclopaedia*, Faber and Faber, London, 1978, p. 26.

semitones very small in Pythagorean contexts, to listen to other parts in relation to the given chromatic step, to remember the pitch of notes before and after, know if it is a big or small whole tone that is being divided, etc. Basic knowledge of the theoretically correct solution will aid the intonation of chromatic steps.

The size of the chromatic semitone depends on the size of whole tone being divided:

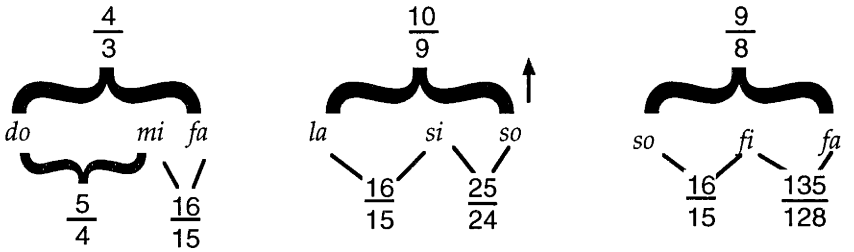


Diagram from Kardos, *Chromatic Intonation*, p. 383.

A chromatic passage can take on the following proportions:

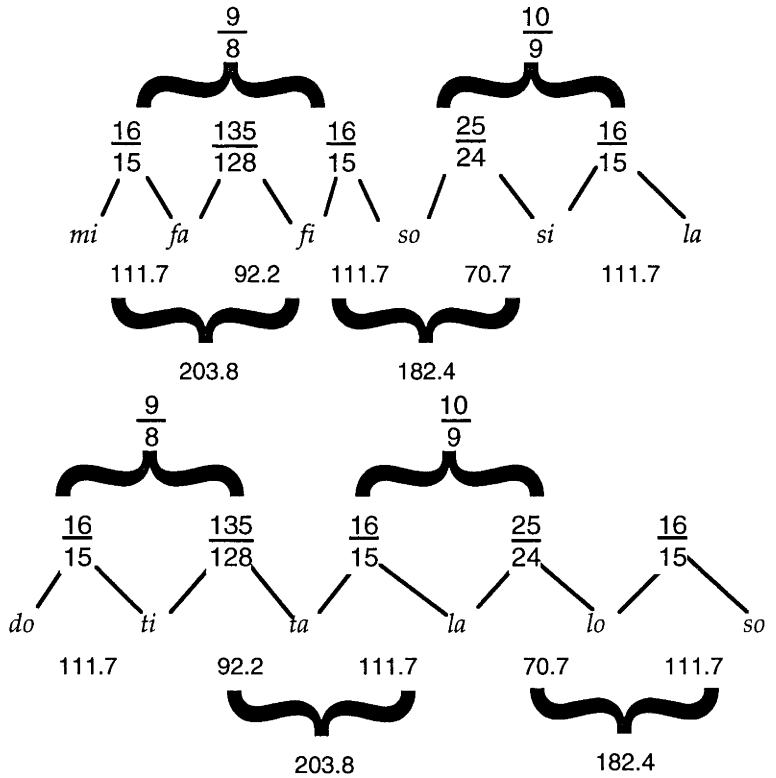
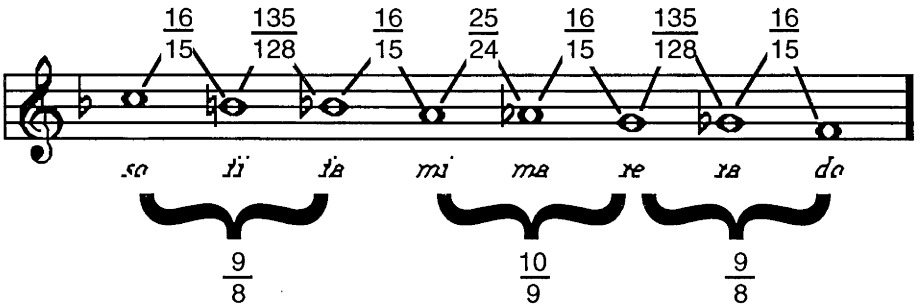


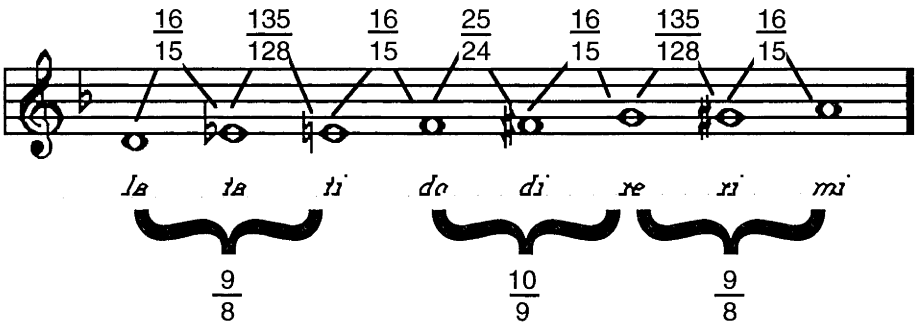
Diagram from Kardos, *Chromatic Intonation*, p. 383

The following descending chromatic passage shows the different forms of chromatic step as determined by the size of the whole tone:



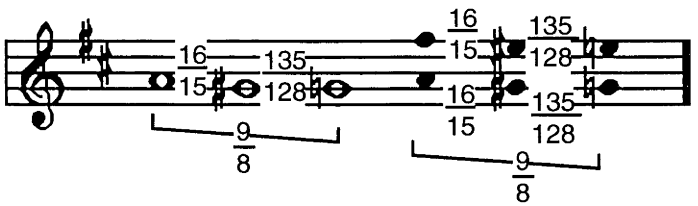
Kardos, *Chromatic intonation*, p. 103

In an ascending chromatic passage:



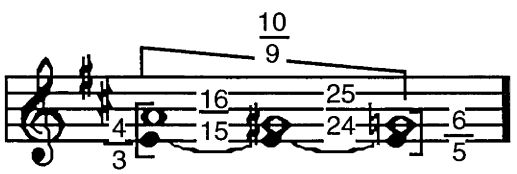
Kardos, *Chromatic Intonation*, p. 103

The following examples show chromatic semitones in the context of pure intonation.



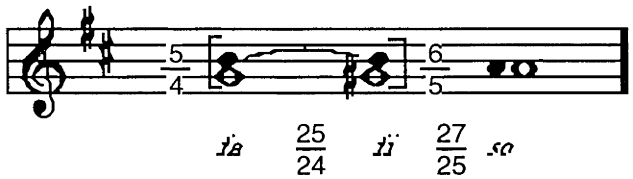
Based on diagram 226, Kardos, *KK*

In the above example, the distance between a' and g' (*so* and *fa*) is a big whole tone. The chromatic descent a'—g#—g♮ (*so—fi—fa)*will consist of a natural semitone followed by a major chroma. But in the next passage involving a chromatic descent *so—fi—fa* above a sustained *re* the need for the last interval to be a pure minor third requires the minor chroma between *fa* and *fa* rather than the major chroma:



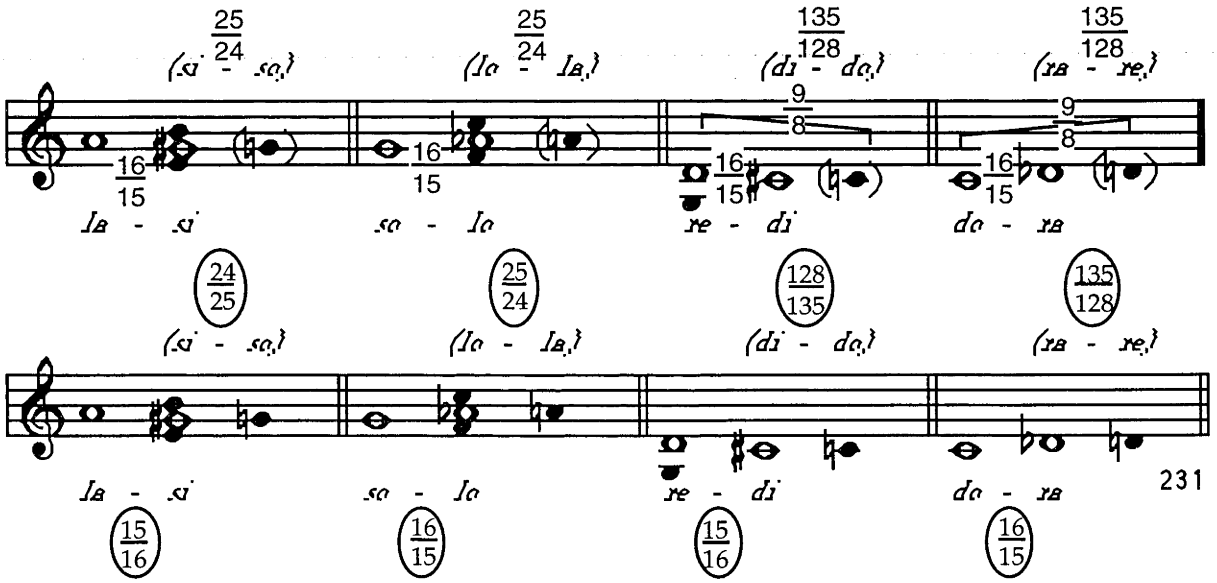
Based on diagram 229, Kardos, *KK*

Thus harmonic considerations have as much influence as melodic. In the following example a major third becomes a minor third by the alteration of the *fa* to *fi*, below a sustained *la*. The interval resulting from the difference between an acoustic major and minor third is $\frac{5}{4} \div \frac{6}{5} = \frac{25}{24}$. Thus the step *fa—fi* is a minor chroma:



Based on diagram 227, Kardos, *KK*

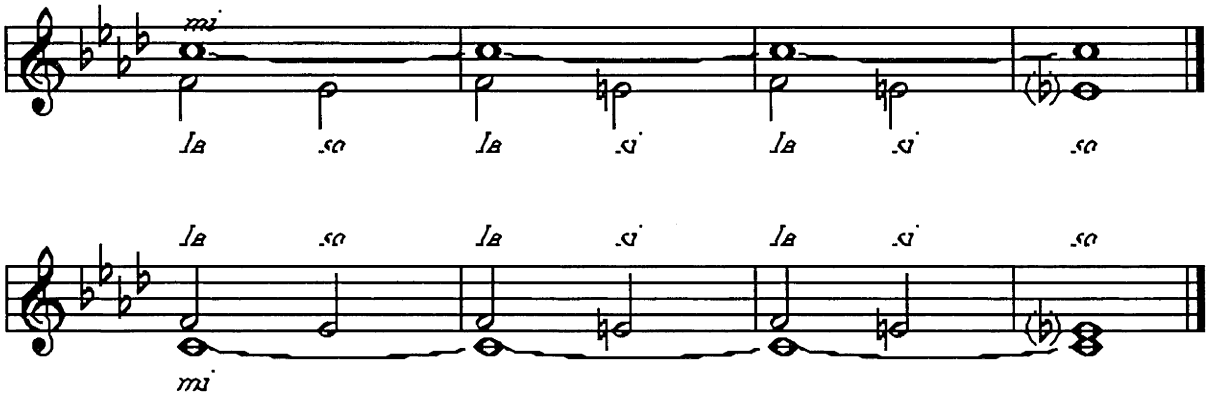
Augmented unison steps are generally smaller than semitones:



Based on diagram 231, Kardos, *KK*

The following are two simple exercises for practicing chromatic steps, the first involving the division of a 9:8 tone (*so—fi—fa*) the second involving the division of a 10:9 tone (*la—si—so*):





Below are some of the more common chromatic alterations:

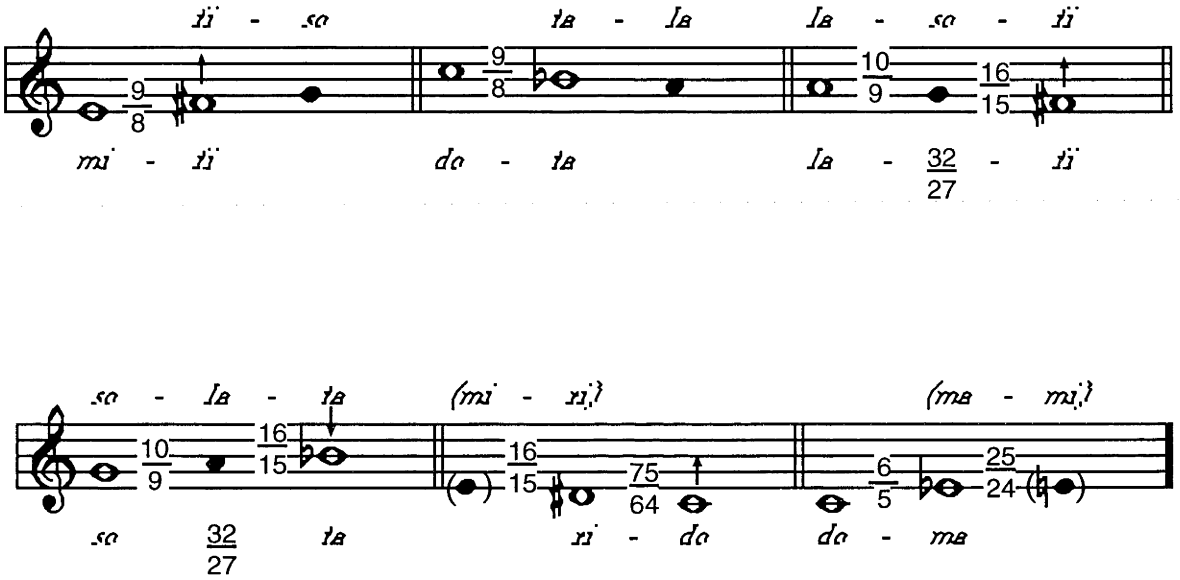


Diagram 233/234, Kardos, *KK*

The following diagram is an exercise by Kardos for practice of chromatic alteration. His directions for its correct interpretation are as follows:[265]

265 Kardos, *KK*, p. 70.

The black f' notes of 235 can be played on the piano as a reference point. Bar one of 235a has a diminished fifth (*do — fi*) between the alto tonic f' and soprano b♮. Bar two has an augmented fourth (*ti—fa*) between the soprano b♭' and the alto e'. There is a diminished fifth in the fifth bar of 235b between the altered note c#'(*si*) in the alto and the g' (*re*) in the soprano. There is a diminished fifth between the altered 2nd and the 6th scale degree (*ri — la*) in 235c, fourth bar. This last exercise should be sung first without, and then with, the bracketed alterations.



Diagram 235, Kardos, *KK*

Augmented seconds, diminished thirds

Other forms of chromatic alteration include the augmented second and diminished third. The table below compares these with the forms of minor third:

augmented seconds, diminished and minor thirds

Augmented second	Diminished third	Pythagorean minor third	tempered minor third	Acoustic minor third
$\frac{75}{64}$	$\frac{16}{15} \times \frac{16}{15} = \frac{256}{225}$	$\frac{32}{27}$	$\left(\sqrt[12]{2}\right)^3$	$\frac{6}{5}$
274.582 cents	223.5 cents	294.135 cents	300 cents	315.641

The diminished third

An example of a diminished third between *fa-* and *ri* is below. Kardos points out that although it is larger than a chromatic division of either the 9:8 or 10:9 whole tones, it has a strong tension towards its centre on *mi*:[266]

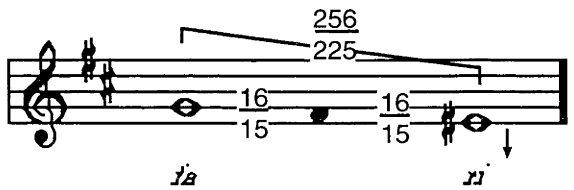


Diagram 239, Kardos, *KK*

The following exercise by Kardos includes a melodic diminished third between the alto f♮ and d#'. The distance between e' and f♮' is a 16:15 semitone, and the distance between d#' and e' is also a 16:15 semtone. 16:15 X 16:15 = 256:225. There is a harmonic augmented fifth, ratio 45:32, between the f♮' and the b' in the soprano, ratio 45:32. Kardos advises singers to prehear the *si* or d#' while singing the *ta* or f♮':

266 Kardos, *KK*, p. 74.



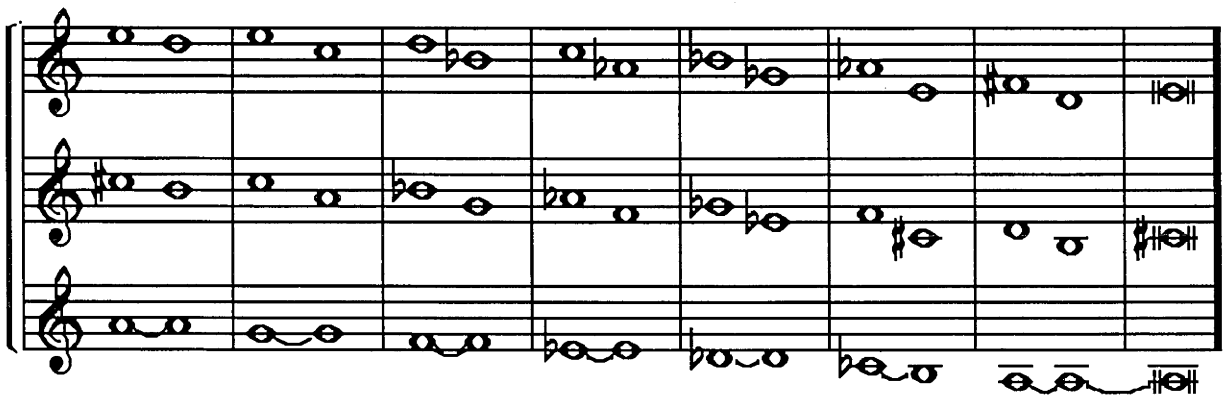
Diagram 242., Kardos, *KK*

The augmented second

The augmented second is frequent in the harmonic minor. The ratio of the augmented second between *fa* and *si* is 75:64 because $\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)^2\left(\frac{1}{2}\right)=\frac{75}{64}$. In cents it is 274.582.

The whole tone scale

The following exercise involves the intonation of a whole tone scale:



Based on diagram 267, Kardos, *KK*

Summary of tuning problems

Problems with intonation can be caused by many factors, and when these occur systematic analysis is necessary to diagnose the cause and remedy it. The

following is a check list of possible causes of tuning problems:[267]

1. Vocal technique:

- poor breathing, lack of support, straining, tension, inertia with descending intervals, insufficient momentum to reach the higher note of ascending intervals, etc.

2. Horizontal versus vertical factors:

- It is important to ascertain whether the problem is caused by vertical or horizontal factors, and to identify those notes whose incorrect tuning becomes the source of subsequent deviations.

3. Can horizontal and vertical factors be reconciled?

- Often tuning a harmonic interval acoustically will require the melodic interval to be compromised and vice versa. A drop in pitch in the entire work can be the result of having each harmonic interval pure! Choices often need to be made as to whether it is preferable for the harmony to be pure at the expense of the melodic line, or whether certain chords need to be 'tempered' in order to avoid a drop in pitch of the entire work.

Generally, a well trained choir with acute hearing will naturally produce the simplest, most natural solutions instinctively. It is only when this does not occur that intervention is necessary.

A piano can be used to give the starting note and to check the pitch at the end, as a single note on a piano does not imply tempered relationships.

The complications of achieving purity of intonation are impractical with instruments with fixed pitch. However a singer with a well-trained ear can make the necessary minute adjustments instantaneously, and thus achieve acoustically pure intonation.

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267 taken from Kardos, *KK*, p. 116.

Conclusion

The inherent limitations of any tuning system have occupied music theorists for two and a half thousand years. One of the solutions posed to the problem of the syntonic comma in just intonation was temperament. Equal temperament is seen by many to be the evolutionary pinnacle of tuning, and has been the unchallenged standard in many parts of the world.[268] Other forms of tuning or temperament are regarded as historically interesting relics. The early music and authentic performance movement has revived interest in historical tunings for original instruments in some circles. Yet little work has been done on the appropriate tuning for early unaccompanied vocal music. This is ironic, since temperament was a compromise necessitated by the inflexibility of fixed-pitch instruments. Even in periods where instruments were tuned with various forms of mean tone temperament the evidence suggests[269] that singers tuned justly wherever possible.

One application of Kardos's work could be in the historical performance practice arena. His work gives the necessary information to apply the principles of just intonation to the tuning of Renaissance vocal works. Kardos, from my reading of his works, seems to have taken a largely a-historic approach, globally applying the principles of pure intonation to all Western repertoire, including folk-music styles. His motivation was not that of the authentic performance purist, but driven by the conviction that acoustically pure intonation is the most suitable form of intonation for almost all unaccompanied vocal music whether choral or unison song. This has two implications. On the positive side, it means that he has extended its application to a range of music outside the Renaissance polyphonic repertoire. He has showed that folk-music, children's songs, Baroque, classical, and even many twentieth century works can be interpreted with the principles of pure intonation. This could have further extension into the area of

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268 David B. Doty points out that what some composers call 'the tyranny of equal temperament' coincided with the advent of the industrial revolution, and the 'consequence was the development of an orchestra of standarised instruments designed to play a standardised scale.' David. B. Doty, 'Editorial', *1/1 Just Intonation*, vol. 1, no. 1, 1985, p. 15.

269 Even Galilei observed that singers who are well trained sing all intervals justly with '*il loro udito perfetto*', *Discorso*, p. 31, quoted in Walker, *Studies in Musical Science in the Late Renaissance*, The Warburg Institute, London, 1978, p. 21.

ethno-musicology, and the consideration of what form of tuning was used in non-Western cultures. Evidence suggests that classical Indian music, for example, was based on just intonation.[270] On the negative side, some may argue that he has not been selective enough. There may be historical or stylistic limitations to Kardos's ideas. Medieval vocal music was composed with the inherent limitations and properties of Pythagorean tuning in mind. Thirds were not used in places calling for a consonance, since their Pythagorean proportions were acknowledged to sound dissonant. Medieval music, then, should, from the authentic performance point of view, be tuned with Pythagorean tuning. Yet, as should be clear from the historical section 2.2, this is not incontrovertibly the case. There is evidence that singers did not, in practice, sing the Pythagorean version of thirds and sixths, but instinctively 'tempered' or 'sweetened' these intervals to their just proportions.[271] If this was the case, then just tuning applies also to medieval vocal music. Similarly there is evidence that many 20th century works assume equal temperament. Against this, there are new movements in compositional practice exploiting the possibilities of harmonic chanting. The Australian composer, Sarah Hopkins, is one example. There is the 'new sonority' of the Estonian Arvo Pärt, Alfred Schnittke, John Tavernor, and Henryk Górecki. In the instrumental and electronic domains Harry Partch, Warren Burt, David Canright, Ben Johnston, Lou Harrison, and Larry Polansky are some of composers making extensive use of just intonation. Lou Harrison also applied

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270 The frequencies of notes in an octave in Classical Indian music are basically identical with just ratios. The frequencies of the octave divisions are as follows:

Krishta	Prathama	Dvitiya	Tritiya	Chaturtha	Panchama	Atiswarya	Krishta
	9:8	10:9	16:15	9 :8	9:8	10:9	16:15

The 9:8 interval is called *kakali* (sweet) the 10:9 interval is called *sadharana* (ordinary) and the 16:15 interval is called *antara* (intermediate). There is much debate as regards the division of the octave into 22 shrutis. Most now agree that shrutis are used to derive just ratios. The 9:8 consists of 4 shrutis, the 10:9 of 3, the 16:15 of 2. See Rai Bahadur Bishan Swarup, *Theory of Indian Music*, Mittal Publications, Delhi, India, 1987; Lok Virsa Jairazbhoy, *The Raags of North Indian Music - Their Structure and Evolution*, Nazir J Islamabad, Pakistan, 1977. Carrie Starr Clark shows how the Kirana tradition the texture of raga is a monophonic line against a drone, with no modulation. Microtonal adjustments based on just intonation principles affect both the pitch placement of scale degrees and variation in their intonation within a raga. 'Just intonation in the Kirana vocal tradition of North India, ' *1/1 Just Intonation*, vol. 5, no. 3, pp. 1, 11-12.

271 See the historical outline above, section 2.2, and Reimann, op. cit., pp. 92-119.

principles of just intonation in his choral music.[272] This last group of composers, who have affiliated themselves by forming the Just Intonation Network, consciously use just intonation because:

> we care, and because virtually every shred of perceptual, acoustical, societal, and in a less supportable manner, aesthetic evidence tells us conclusively that the way we perceive frequency relations is an integral and complex system inexorably linked to our other ratiometric perceptions of the universe. To not use rational tunings is much more of a conscious decision of avoidance, a choice to leave a particular parameter undetermined.[273]

Kardos's practical work on how to perform music in just intonation, and the compositions of the members of the Just Intonation Network, overturn the following assertions of Easley Backwood, made in 1984:

> the quest for perfection in tuning of a given musical fragment is the pursuit of an *ignis fatuus*. The author has been unable to discover any major composer whose style conforms to the inherent limitations and properties of just tuning. Nor are there theoretical writings giving practical instruction as to how to sing or play just intervals, or how to compose with them. It must be concluded that at present, just tuning is of no practical use with regard to the existing Western repertoire.[274]

Kardos's work forms a practical set of instructions as to how to sing with acoustically pure intonation, and demonstrates that most choral music lends itself to being tuned purely to greater or lesser degree. He was acutely aware of the impossibility of reconciling vertical and horizontal factors in the final analysis, but argued that, to aim for the greatest degree of purity possible, was an ideal worth striving for. Some other writers would argue it is not practicable to expect the average chorister or choirmaster to master the complex mathematical and analytical factors necessary, but against this argument is the debate about whether choristers instinctively search for simple ratios and pure tuning. There is evidence to suggest that acoustically pure intonation is much easier and more natural than tempered intonation. Kardos's work makes this intuitive knowledge more conscious and systematic.

Kardos's pedagogy of just intonation could also prove a useful resource for performers of contemporary compositions based on pure tuning, since the resistance on the part or the inability of performers to tune using just intonation

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272 David Lee Brunner, 'The Choral Music of Lou Harrison', PhD diss., 519, 1989.

273 Larry Polansky, 'Confessions of a Lousy Carpenter: Some thoughts on Composing for Standard Instruments in Just Intonation', *1/1 Just Intonation*, vol. 1, no. 1, 1985, p. 1.

274 Blackwood, op. cit., p.153.

is frequently cited as an impediment to composers using just intonation.

I believe that a systematic examination of the theoretical tuning associated with each composition will inform decisions about appropriate tuning. Renaissance works from around the time of Zarlino should be tuned with just. Some atonal compositions should be tuned with tempered intonation, many twentieth century works suit just intonation. Children's folk songs and traditional folk music probably suit acoustic intonation. Sensitive, aesthetically and historically aware artistic decisions should be made on a case-by-case basis.

A logical continuation of Kardos's work is how to make his principles and practices translatable into a sound source for use as a model. Kardos did not show how to make pure intonation audible on an instrument. His work predates the possibilities generated by computer music, although he did adapt a harmonium to play a just scale with additional notes to provide the necessary alternative tunings, but this was for his own use only.

Partch outlined how to adapt readily available instruments, realising that the human ear needs an instrument capable of sounding the theoretical possibilities in order to test, prove, and correct ideas about tuning. It makes an enormous difference to hear the various scales discussed by theorists from ancient Greece to the present. He showed how to construct a harmonic canon, how to use the cello, or the harmonium to this end, and gives detailed descriptions of the notations and tunings of various instruments he devised or adapted himself, including adapted violas, adapted guitars, chromelodeons, kitharas, and kotos.[275]

George Kirck argues that the use of software synthesis programming to realise compositions written in extended just intonation, offers an aural window into the millennia of musical compositions based on ratio tuning systems.[276] The use of electronic instruments with user-programmable tuning capabilities are currently being explored.[277]

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275 Partch, op. cit., pp. 95-105, 195-258.

276 George T. Kirck, 'Computer Realization of Extended Just Intonation Compositions', *Computer Music Journal*, vol. 11, no. 1, Spring 1987, pp. 69-75. George Kirck discusses the digital sound synthesis of microtonal music composed in extended just intonation. He examines the theoretical aspects of just intonation, the extended just intonation notational system developed by Ben Johnston, and computer programming issues.

277 See Robert W. Jones, 'Tuning the Macintosh', *1/1 Just Intonation*, vol. 2, no. 1, 1986, pp. 6-7; Harold M. Waage, 'The Intelligent Keyboard', *1/1 Just Intonation*, vol. 1, no. 4, 1985, pp. 1,

Several contemporary composers believe that equal temperament is a very self-limiting system that has dried up the well-springs of sound resources available to composers. Such composers are now championing the merits of just intonation. In conclusion I quote David Doty, speaking of the work of Harry Partch and Lou Harrison:

> These farsighted musicians recognised that in the acoustically pure intervals of Just Intonation and in the diverse traditions of World music was to be found enough material to fruitfully occupy generations of composers. I will go one step further and assert that, for a skilled composer equipped with an understanding of the subtleties of Just Intonation, almost any musical style, except, perhaps, the excesses of the various 'avant-gardes' of the present century, may serve as a point of departure for a powerful new musical expression.[278]

12-13; Robert Rich, 'Tuning and Transposition of the Prophet Five', *1/1 Just Intonation*, vol.1, no. 4, 1985, pp. 4-6.

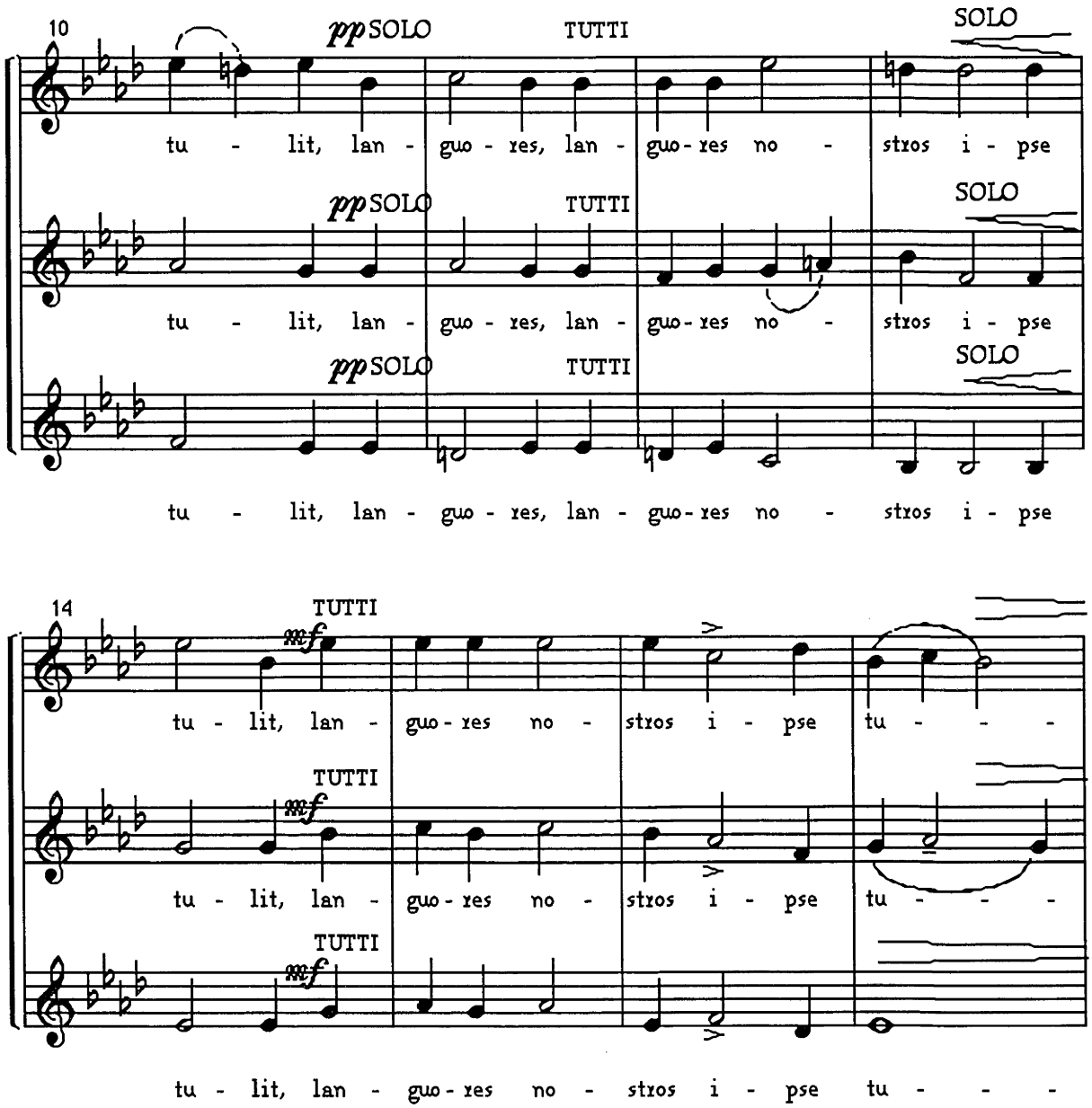
278 David. B. Doty, 'Editorial', op. cit., p. 15. He goes on to say that 'while Just Intonation provides a more subtle and varied palette of consonances than does any of the equal temperaments, it also vastly expands the range of musically meaningful dissonances available to the composer.'

Appendix A: analysis of musical works

Vere Languores Nostros: Antonio Lotti 1667—1740[279]

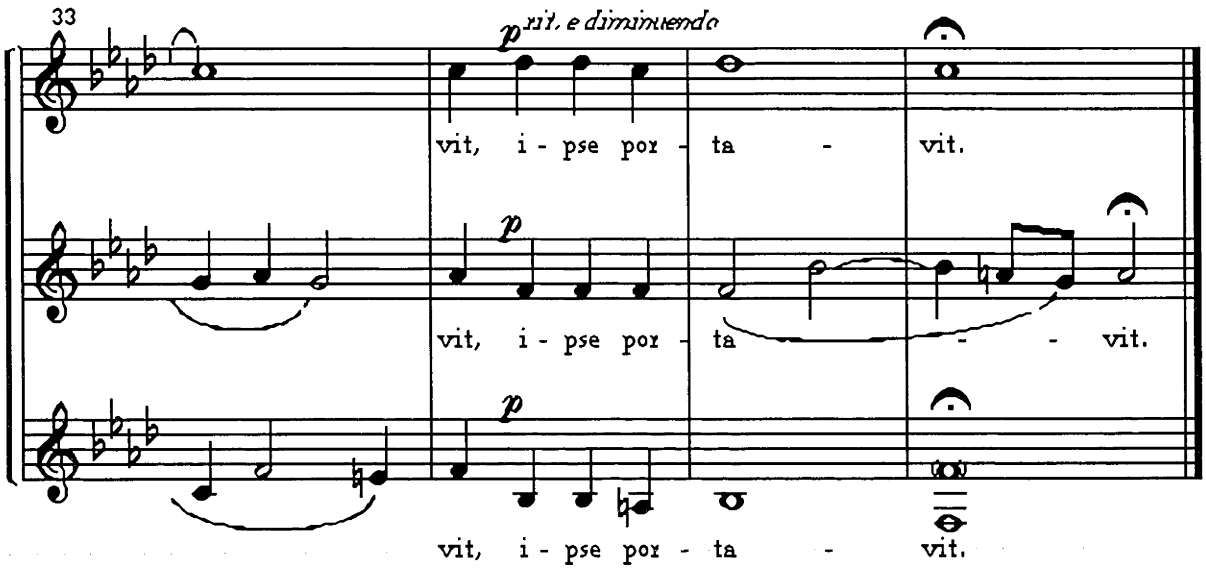


279 This work was not analysed by Kardos. The following analysis is an application of the principles discussed to a choral work made by the author. I was able to check the validity of my analysis with Eva Roszgony, his colleague. The edition used is from *Ezer év Kórusa* Forrai Miklós, ed., Editio Musica Budapest, pp. 256-257.


10
pp SOLO
TUTTI
SOLO
tu - lit, lan - guo - res, lan - guo - res no - stros i - pse
pp SOLO
TUTTI
SOLO
tu - lit, lan - guo - res, lan - guo - res no - stros i - pse
pp SOLO
TUTTI
SOLO
tu - lit, lan - guo - res, lan - guo - res no - stros i - pse
14
TUTTI
mf
tu - lit, lan - guo - res no - stros i - pse tu - - -
TUTTI
mf
tu - lit, lan - guo - res no - stros i - pse tu - - -
TUTTI
mf
tu - lit, lan - guo - res no - stros i - pse tu - - -


18
SOLO
lit, et do - lo - res no - stros, do - lo - res
SOLO
lit, et do - lo - res no - stros, do - lo - res
SOLO
lit, do - lo - res no - stros, do - lo - res
22
f TUTTI
no - stros i - pse por - ta - vit, i - pse por -
f TUTTI
no - stros i - pse por - ta - vit, i - pse por -
f TUTTI
no - stros i - pse por - ta - vit, i - pse por -


26
fff SOLO
ta - vit, et do - lo - res no - - -
fff SOLO
ta - vit, et do - lo - res no - - -
fff SOLO
ta - vit, et do - lo - res no - - -
30
fff TUTTI
stros i - pse por - ta - - - - -
fff TUTTI
stros i - pse por - ta - - - - -
fff TUTTI
stros i - pse por - ta - - - - -



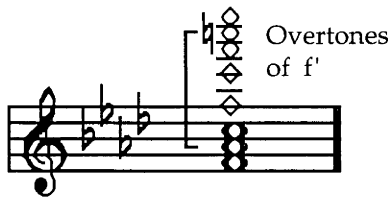
This piece illustrates many of the principles outlined. Space does not allow an exhaustive bar by bar analysis. I have included selected elements that are intonationally interesting. *Vere Languores* illustrates just intonation in vocal compositions which modulate. Modulation is often associated with tempered systems, but Kardos argued that voices will choose untempered intervals where ever possible and can instantaneously adjust to the new tonal centre. In the following piece, each time the composition modulates a new tuning interpretation is required. The same notes have a markedly different intonational 'colour'.

Bars 1—4



The first chord, held for a full bar, establishes the tonality of F minor. The minor triad creates intonational instability, since the minor third interferes with the major third that is inherent in the overtone series of the root of the chord. There

will be acoustic dissonance between the fifth overtone of the alto's f' (a♮''') and the mezzo's a♭'. This dissonance gives the minor chord its special character:



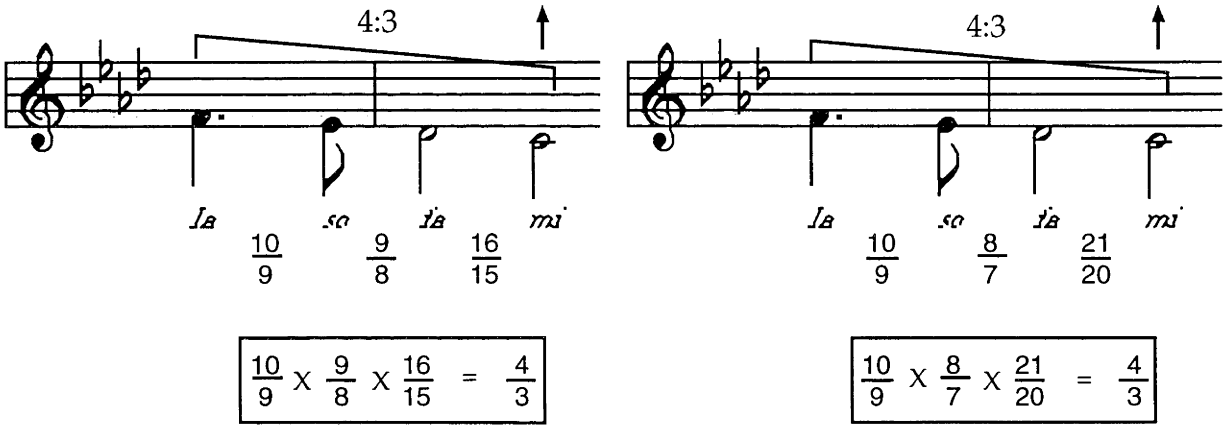
Clash of a♭' and a♮'''

Bars 1—4 are in F minor. The first four bars consist of alternating tonic and dominant (V^6) in a harmonic pattern of I—V^6—V—I—V^6—I—IV^{7-6}—V. The soprano line holds a c'' as a common tone with the harmony changing underneath, thus it alternates between being the fifth of the tonic or the root of the dominant except in the fourth bar, where it is the seventh, and the fifth bar, where it is the third in the new tonality of relative major. This illustrates the principal that, where there is a sustained note, the other parts need to adjust to that note. It is melodically unpleasant to have microtonal adjustments of a sustained note. This is especially the case when the common tone is the root of a triad. The root is the most solid note, the fifth is next in stability, and the third is the most sensitive, adjustable note. After a break in phrasing, such as a breath and a new cadence, as occurs in bar 5, it is not as noticeable to adjust the c' common-tone to its new tonal surroundings. When the dynamics are soft it is also less noticeable if the common tone has to adjust.

The first chord of bar 2 has the c'' common tone change its function from fifth of a minor triad to the root of a major triad in first inversion. The alto descends from tonic to leading tone; the distance between the f' (*la*) to the e♮' (*si*) is a natural semitone, ratio 16:15. The next note in the alto is middle c' and the distance from between the e♮' and c' (*si* to *mi*) is an acoustic major third, ratio 5:4. The acoustic major third is smaller than the tempered or Pythagorean major third, and so the altos need to take care they do not overreach the lower note.

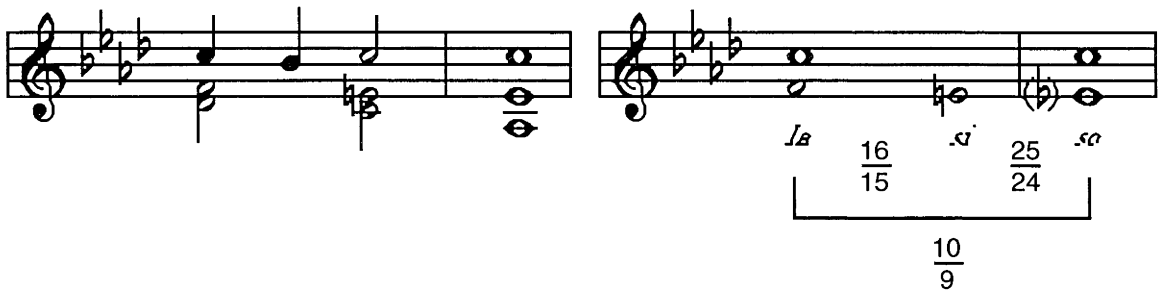
The alto's melodic line in bars 3 and 4 is *la—si —la —so—fa—mi*. The *si* in *la—si—la* is functioning as a leading tone and is a 16:15 semitone from *la*. The pattern *la so fa mi* has a strong downward pull to the *mi* which functions as the root of the dominant cadential chord. The first problem is the stepwise descent

la—so—fa (f'—e♭'—d') covering a distance of a major third, because the acoustic major third is smaller than the tempered. In descending, the singers need to be careful they do not make the steps too big. From *la* to *so* is a 10:9 small whole tone; from *so* to *fa* a 9:8 big whole tone. The second problem is caused by the dissonance of the suspended seventh that will make the IV^{7-6} chord sound like a seventh chord: the singers may instinctively tune the *so—fa* step as an 8:7, tuning the *fa* like the 7th harmonic. If this is the case then the *fa—mi* step will need to be very small, 21:20, otherwise a drop in pitch will occur. The third problem is that there is a dramatic, textural climax at this point in the phrase. Thus there are three reasons for care: melodic, harmonic and textural. Care must be taken with the *fa* to keep it high, and even more so the *mi*. Two alternative tunings are:



Bars 5—8

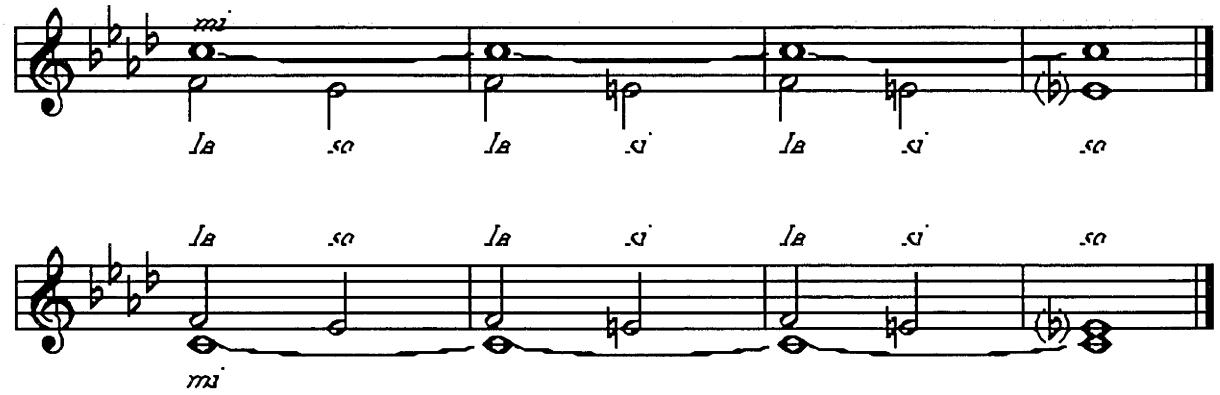
Bar 5 modulates to A♭ major. The transition from the cadence in the dominant of F minor in bar 4 and the A♭ major tonic chord needs care. The soprano common tone c" has a very different colour here as the third in an A♭ major tonic chord. The new phrase allows a new intonational interpretation of the same note. The third being the most sensitive, adjustable note should be very quiet and, for textual reasons, intense. The temptation to sing it louder than the other parts due to its high tessitura should be resisted. The mezzo transition from having the third of the dominant chord to being the fifth of the tonic in a new key between bars 4 and 5 requires a chromatic progression *la—si—so*, f'—e♮'—e♭:



The interval *la*—*so* is a small whole tone, 10:9. Between *la* and *si* is a natural semitone, ratio 16:15, between *si* and *so* a very small semitone of 25:24.

$$\frac{10}{9} \div \frac{16}{15} = \frac{25}{24}$$

The step from *si* to *so* is therefore a very small one. The following exercise can be used to prepare the intonation of this step:



The difficulty of the chromatic step from *si* to *so* is lessened by the phrasing in this context. The *si* occurs at the end of the first phrase and the *so* at the beginning of the next, thus there is opportunity to take a breath between the two notes. The soft, intense dynamics of the A♭ major chord of bar 5 also helps, as does the unexpectedness of A♭ major as a new tonic after the V of F minor. The chord in bar 4 functioning as V in *la* minor becomes I of *do* major.

The sopranos in bar 6 need to be careful with the ascending minor third which is bigger than the tempered. Tiredness may compound the problem—the dotted minim that precedes it will tend to sink, and loose impetus. The following exercise illustrates two situations containing an ascending minor third. The second version is much easier to intone correctly because the impetus of an upbeat acts like a springboard impelling the singer upwards. The first version is

harder because sustaining the first note diminishes energy:

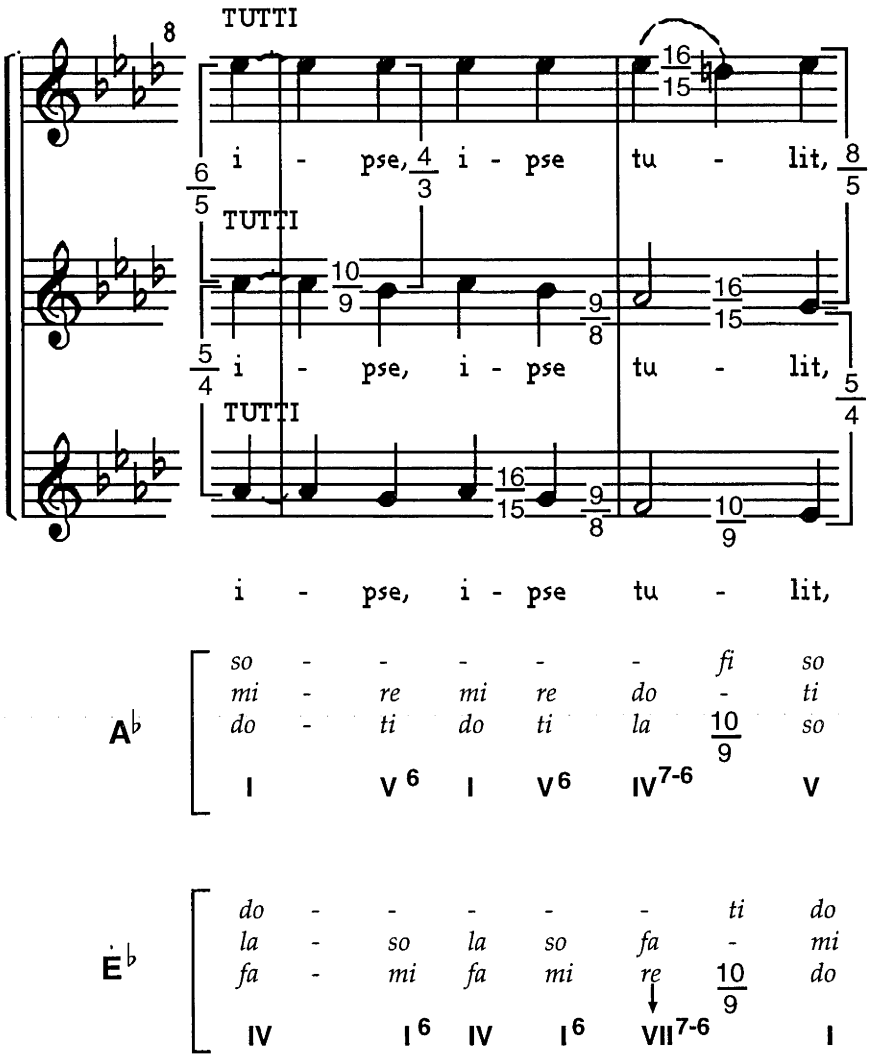


The soprano line in bars 7 and 8 has a similar situation to that of the altos in bars 3 and 4. Here it occurs in the context of a dominant seventh resolving to the tonic. The distance from the low d♭" resolving to c" (or from *fa—mi*) is a small semitone, 21:20. The *mi* needs to be high.

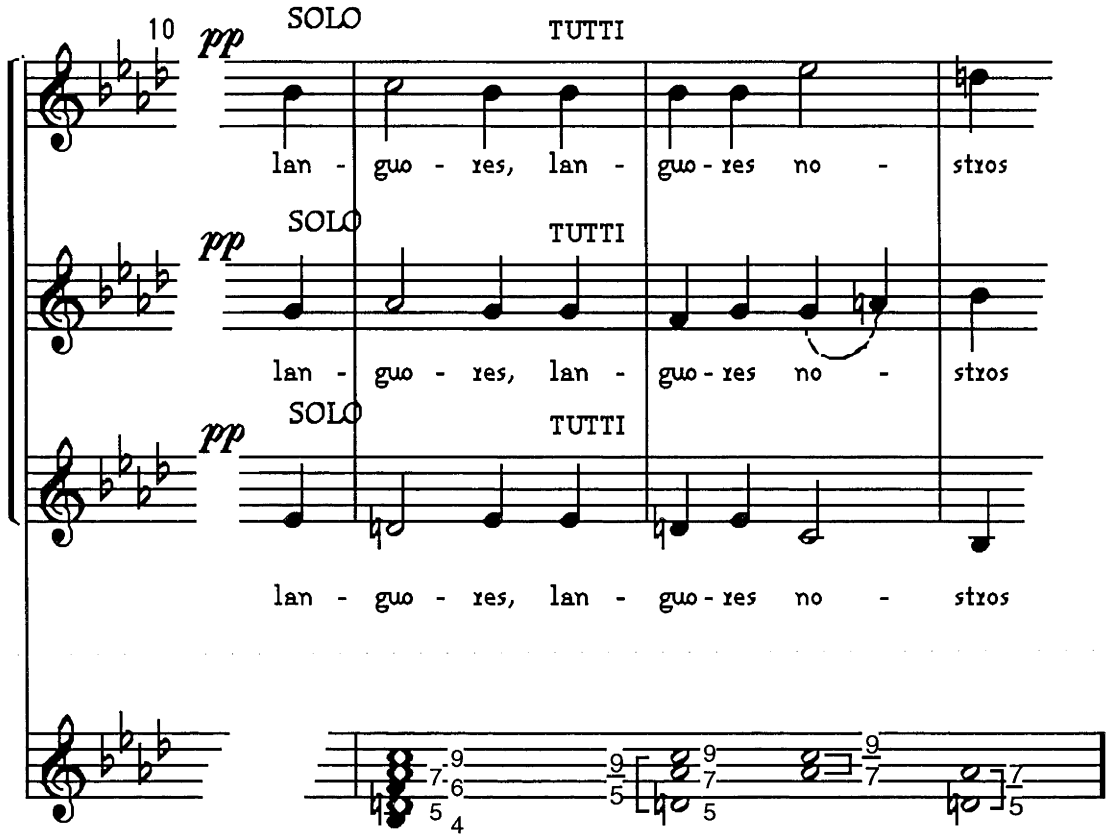
Bars 8—10

Bars 8—10 can be interpreted in two ways, either as continuing in A♭ major, or as having modulated to the dominant, E♭ major. In A♭ these bars repeat the I—V^6—V—I—V^6—IV^{7-6}—V pattern of bars 1—4 in the relative major of A♭. The common tone e♭" of the soprano changes its function from being the fifth of the tonic chord to the root of the dominant. Because it is a common tone the other parts need to adjust to it. Both the *ti* (g') in the alto and the *re* (b♭') of the mezzo in the V^6 chord are high, and need to adjust to the soprano e♭".

The high tessitura of these bars has a weakening effect upon the A♭ major feel, and makes it operate more as if in E♭ major. A comparison of the internal relationships in either key shows that in this particular case the same ratios apply. The only note in question is degree II (*re*) in the E♭ interpretation, which would normally be high—a 9:8 from the tonic *do*. However in a II chord it would in any case be low— a 10:9 from *do*.


TUTTI
TUTTI
TUTTI
i - pse, i - pse tu - lit,
i - pse, i - pse tu - lit,
A♭
so - - - - - fi so
mi - re mi re do - ti
do - ti do ti la 10/9 so
I V 6 I V6 IV7-6 V
E♭
do - - - - - ti do
la - so la so fa - mi
fa - mi fa mi re 10/9 do
IV I 6 IV I6 VII7-6 I

Bars 10—12



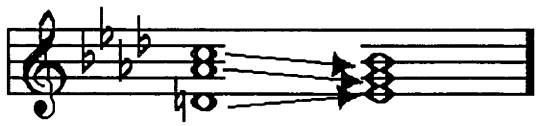
Bars 10—12 are in E♭ major. The pattern is I—VII^{ø7}—I—VII^{ø7}—I. The VII^{ø7} chord could be considered as a dominant ninth without the root (this chord also omits the fifth *re*), one of the most interesting chords in acoustically pure intonation. If we model it on the harmonic series its components will form the ratios 4:5:6:7:9, or in this case 5:7:9. It spans a 9:5 minor seventh[280] between the outer poles of the

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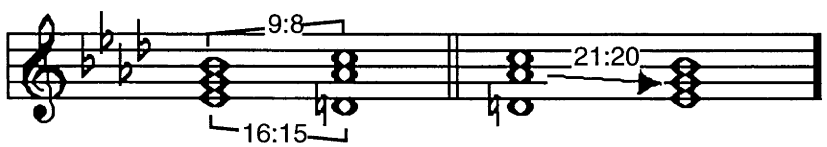
280 The small minor seventh is the inversion of the big whole tone, $\frac{2}{1} \times \frac{9}{8} = \frac{16}{9}$. The large minor seventh is the inversion of the small whole tone, $\frac{2}{1} \times \frac{10}{9} = \frac{9}{5}$. Normally degrees VII—VI, *ti—la* would be the small 16:9 version, since VI—VII or *la—ti* is a 9:8 whole tone. Here, however, I interpret the chord's function as that of the dominant ninth, which when modelled on the harmonic series would lead to a 9:5 ratio between these two components. If the 16:9 span is chosen, then the following ratios would apply. The diminished fifth between d♮' and a♭' would be 64:45, the major third between a♭' and c" would be 6:5. The progression from the 3:2 chord to the 16:9 would be via a 10:9 small whole tone from b♭'to c", *so* to *la*, and a 16:15 semitone from e♭"to d♮', *do* to *ti*.

alto d♮' and soprano c". It contains a diminished fifth between mezzo a♭' and alto d♮', ratio 7:5, and a large minor third, 9:7.

The dominant ninth functions as a leading chord, each component displaying a strong pull to the note it resolves to, here the *ti* resolves to *do* (d♮' to e♭'), *fa* to *mi* (a♭' to g') *la* to *so* (c" to b♭). Thus as a chord it has a very contracted tension, and will feel tight and narrow:

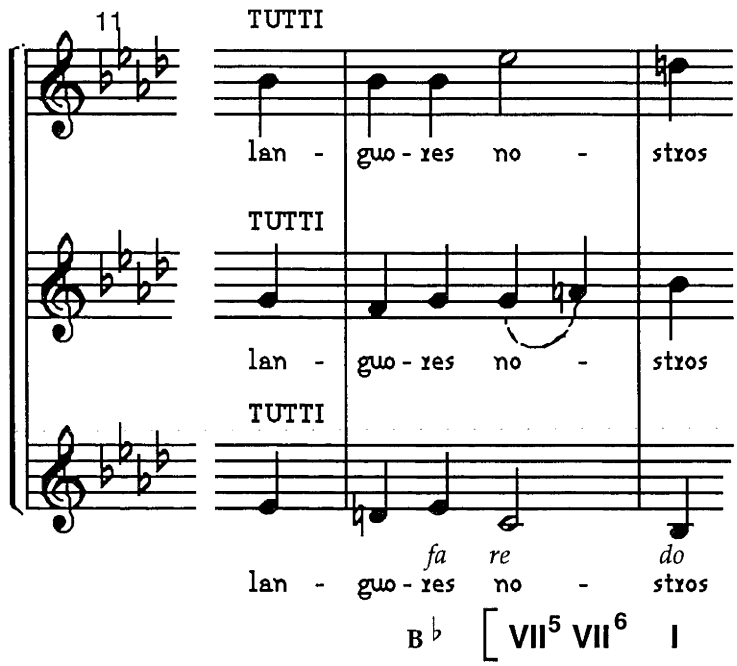


The I and the VII$^{\varnothing 7}$ chords have the ratios 3:2 and 9:5: the difference between them is 6:5. The progression from the 3:2 to the 9:5 would therefore have to be by subdivisions of 6:5, the simplest being a 9:8 whole tone from b♭'to c", *so* to *la*, (normally a 10:9), and a 16:15 from e♭'to d♮', *do* to *ti*. 9:8 X 16:15 = 6:5. The interval between *fa* and *mi* needs to be considered. In dominant seventh contexts owing to the flatness of the seventh (7:4 from the root to seventh instead of 16:9, a difference of 64:63) the semitone of resolution will be correspondingly small, $\frac{16}{15} \div \frac{64}{63} = \frac{21}{20}$. Here the root is omitted, the relationship in question is between the third and the seventh (*ti—fa*, d♮'—a♭") ratio 7:5. Comparing this acoustic diminished fifth as found in the dominant seventh chord, ratio 7:5 with the diatonic diminished fifth 64:45, the difference is also 64:63 since $\frac{7}{5} \div \frac{64}{45} = \frac{64}{63}$ hence the semitone will still be $\frac{21}{20}$:

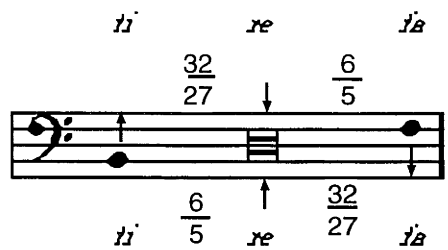


Having resolved to the tonic in bar 11, the composition immediately begins to

modulate to the dominant of E♭ major, B♭ major. The tutti repetition of *languores* substitutes the dominant 9th chord with a V^6, a B♭ in first inversion which is weaker in effect. This is followed by a II—VII^6—I cadence in B♭:



The II—VII^6 progression requires analysis. The height of the 2nd degree in a II is often lowered to create a pure 6:5:4 triad in relation to *fa* and *la* Here this seems natural melodically and harmonically: the alto's line *fa—re—do* (e♭'—c'—b♭) has a strong leading effect to the new b♭ *do*. However the 2nd degree needs to be high in relation to the 7th. In the context of a VII chord *re* confronts simultaneous competing demands in relation to *ti* and to *fa*. One of the minor thirds has to be Pythagorean 32:27, the other 6:5:



Because the *re* or c' is a common tone, and is already low, the *re—ti* (c'—a♮) would be Pythagorean. The VII^6 being in first inversion is a Pythagorean major

sixth, ratio 27:16. At 905.865 cents, it is 21.5 cents bigger than the acoustic 5:3 at 884.359 or an 81:80 syntonic comma bigger. The VII^{6} could of course be interpreted as a dominant seventh without the root, and the same tuning principles would apply as for the VII$^{\varnothing 7}$ of bar 11 analysed above. The chord has a strong pull to the tonic, e♭'' to d♮'', a♮' to b♭', c' to b♭':

Bars 13—18



The composition continues to pass through various related keys. Even if these do not represent full modulation, it is important for a chorister to be aware of the tonal centre and nexus of relationships in order to correctly interpret the best tuning. For example the B♭ tonality having been established on the first chord of bar 13 the solo section in bars 13-14 reverts the tonality to E♭. The second beat in

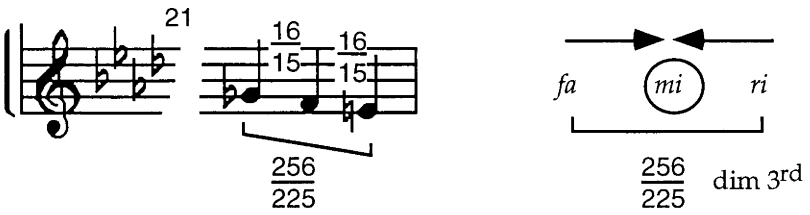
bar 16 changes the tonality to A♭ major with an unexpected tonic-relative VI:

Bars 18—22



Bars 18—22 are in B♭ minor. Bar 21 in the alto features a diminished third which has a strong tension towards the *mi* in the centre because of the melodic pattern *fa—mi—ri—mi*. In fact this chromatic pattern should be compared to the *la—si—so* pattern of bars 4—5 of the mezzo, which spans a small whole tone, 10:9 and contains a 16:15 and a 25:24 semitone. The diminished third is significantly

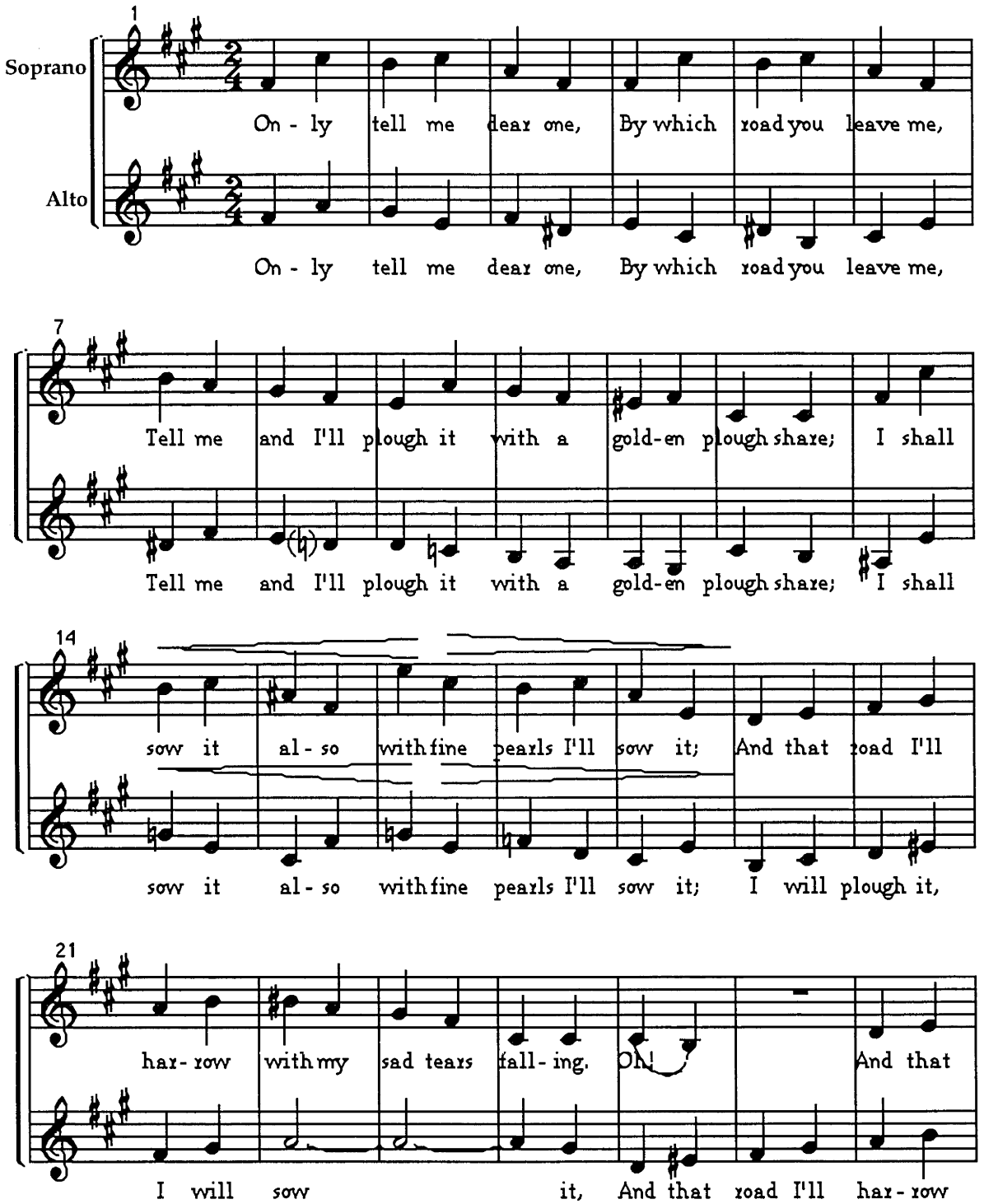
bigger, 223.5 cents compared with 182.4 cents:



Bars 23—30

Bar 23—24 has a V—I in B♭ minor followed by a V—I in A♭ major. The altos need to sing the descending octave leaps with care that they do not descend too far, and the ascending fifth from a♭' to e♭' with care that they do not under-reach it. The chromatic steps *so—si—la* are the same ratios as discussed above. The cadence on C major acts as the dominant of F minor, and bars 31 to the end are in the F minor that began the piece. The final chord features the *Tierce de Picardie* which avoids the acoustic dissonance of the minor triad by making it conform to 4:5:6 of the harmonic series:

Ne hagyi itt!: Béla Bartók (1881—1945)[281]



281 Based on my translation of Pál Kardos, 'Bartók két kórusmüve a tiszta intonálásra nevelés tükrében (Ne hagyj itt!, Bolyongás)' [Two Choral Works of Bartók in the Mirror of Education towards Pure Intonation], *Az ének-zene tanítása* 1970/4-5, szám, p. 163. I am indebted to the help of Hungarian interpreter, Marta Moduna, for help with the initial translation. A translation of the entire article is available from the author. Edition used is Béla Bartók, *Kórusmüvei*, Zenemükiadó, Budapest, 1942, pp. 12-14.


28
road I'll har-row with my sad tears fall - ing.
with my sad tears fall - - - - ing.
35
On - ly tell me dear one, By which road you leave me, Tell me
Tell me by which road
On - ly tell me dear one, By which road you leave me, Tell me
42
and I'll plough it with a gold-en plough share; I shall sow it
you leave me. I shall sow it
and I'll plough it with a gold-en plough share; I'll plough


49
al - so with fine pearls I'll sow it; And that road I'll har - row
al - so with fine pearls I'll sow it; And that road I'll har - row
it I'll sow it; I will
56
with my sad tears fall - ing. Oh! And that road I'll
with my tears And that road I'll har - row
plough it, And that road I'll har - row with
63
har-row with my sad tears fall - ing.
with my ma - my fall - ing tears.
my sad tears fall - - - ing.

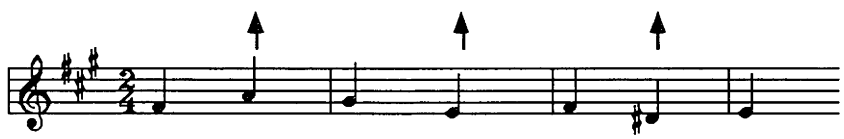
I have selected this piece because it illustrates the application of acoustically pure intonation to works outside the period and style commonly associated with just intonation (namely the Renaissance). Béla Bartók derived much of the tonal language of his children's choruses from Hungarian folk music and its modality. The tonal centre of *Ne Hagyj Itt!* is what Kardos called *la* minor. It is imbued with a minor modal tonality with F# being *la*.

A second reason for choosing this work it that it shows the impossibility of simultaneously achieving vertical and horizontal purity, and that choices need to be made between various possible compromise solutions. There is no fixed 'right answer' and conductors should compare the possible solutions and make their own choices as to which sounds most musical.

The interpretation that follows is entirely Kardos'. It is a paraphrase of my translation of his 'Two Choral Works of Bartók in the Mirror of Education towards Pure Intonation'. I have made minor additions of terminology that help clarify his ideas for readers not versed in the Hungarian method of aural education.

Alto bars 1-3

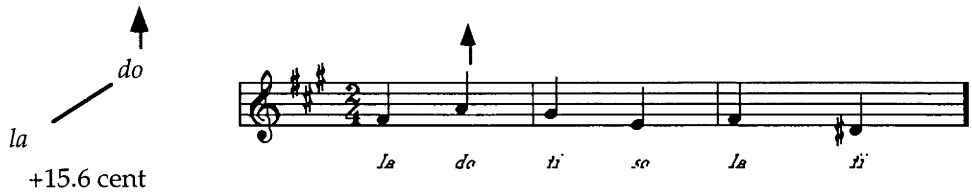
The opening bars of the alto contain a sequence of thirds that is hard to tune:



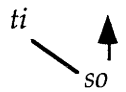
Kardos, *Two Choral Works*, p. 168

The first interval from *la* to *do* (f#'—a') being a 6:5 acoustic minor third at 315.6 cents is 15.6 cents wider the tempered minor third of 300 cents. An ascending acoustic minor third is a very broad, wide movement. The *do* should be intoned high:[282]

282 ibid., p. 167.



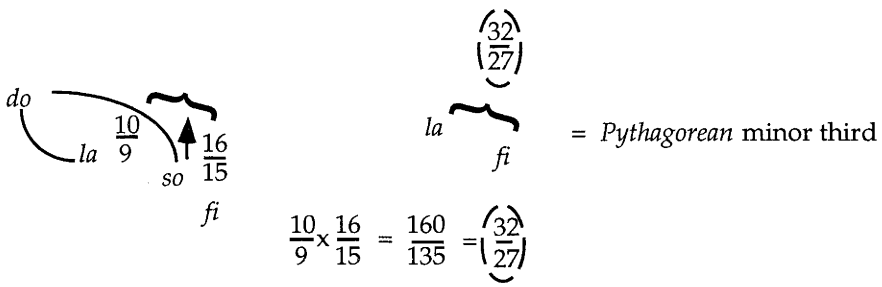
The second interval *ti—so* (g#' to e) has the opposite problem. Being a 5:4 descending acoustic major third at 386.3 cents it is narrower by 13.7 cents than the tempered major third of 400 cents. The gravitational pull needs to be resisted and the lower note needs to be tuned high:



-13.7 cent

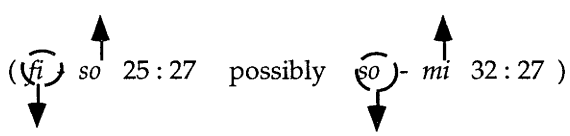
Kardos, *Two Choral Works*, p. 168

The minor third in the third bar from *la* to *fi* (f#' to d#') is a Pythagorean minor third which is 21.5 cents narrower than the acoustic minor third. It consists of a 10:9 small whole tone plus a 16:15 semitone:

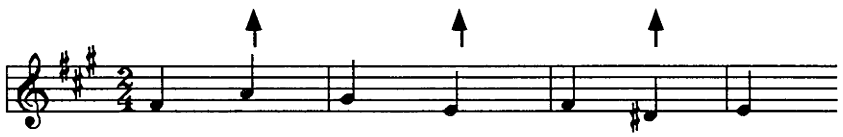


Kardos, *Two Choral Works*, p. 168

At 294.1 cents it is 5.9 cents narrower than the tempered, and 21.5 cents narrower than the acoustic minor third. Again, as it occurs in the direction of gravity, it is very difficult not to over-reach the f#' and intone the second interval too low. The f#' should be intoned very high. If the *la-fi* (f#' - d#') minor third was intoned as an acoustic minor third, followed by the natural semitone from *fi* to *so* (d#' to e') (the second note of bar 3 to the first note of bar 4) the *so* (e') would be lower by one comma, (81:80) than the previous *so*, and a drop in pitch of the whole piece would be set in motion. One of the following notes would then have to be adjusted. For example the interval between *so* (e') and the following *mi* (c#') would have to be a Pythagorean minor third:



Kardos, *Two Choral Works*, p. 168



Kardos, *Two Choral Works*, p. 168

The first three bars therefore contain a series of thirds in which the second note must be intoned high, and with great care.

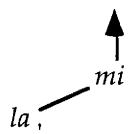
Soprano bars 1-3

The tuning of the first three bars of the soprano part is less problematic:



Kardos, *Two Choral Works*, p. 169

The opening 2:3 perfect fifth from *la* to *mi* (f#' to c#") at 701.955 cents is wider than the tempered by only 1.955 cents. An upward arrow is a reminder to intone the c#" slightly sharp:

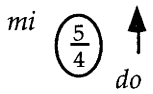


Kardos, *Two Choral Works*, p. 169

Singers should be directed to sing the top note of the fifth from above, not from below. It should be sung a little softer than the first note *la* (f#'), which is contrary to the instinctive tendency to sing higher notes louder. If the *mi* (c#") is stronger than the *la*, then the melody will be very angular. It should be sung legato rather than as disjunct notes.

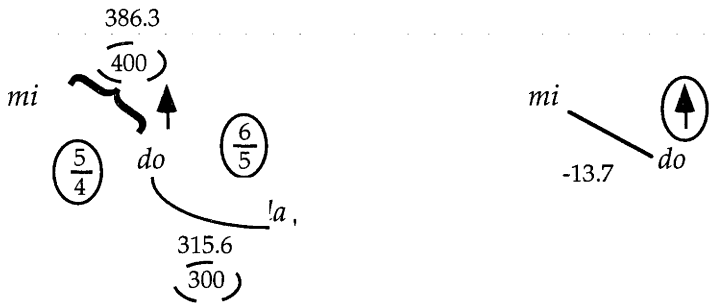
The first note of the second bar (b' or re) is a very sensitive note and serves as an illustration of situations where vertical and horizontal purity of tuning cannot be reconciled. The melody of the soprano taken in isolation would dictate a low *re* in relation to the *la* tonality of the piece, a 9:8 big whole tone from the *mi* (c#"). But this would create a dissonant Pythagorean minor third as the harmonic interval between the alto *ti* (f#")and a low *re* (b'). Consequently the sopranos will usually automatically adjust to the alto f#" and intone a high *re* (b') as the acoustic minor third is more consonant. But this high *re* as a melodic note is out of place in the tonal environment of *la*. The *re* will fluctuate because of this

contradiction. If the influence of the altos is weak it will stay low, and for a second, *re–ti* (b'-g#') will be a Pythagorean minor third. But if the influence of the alto is strongly felt, if the acoustic environment provides optimal listening, then the *ti* (f#') will force the *re* (b') higher and the soprano will intone a high *re* (b'). The beauty of the two notes sounding together will make the choir forget the *la* tonal centre for a moment. The *do* (a') in the third bar of the soprano is a 5:4 acoustic major third from the previous *mi* (c#"). Thus the *do* (a') has to be high:



Kardos, *Two Choral Works*, p. 169

This *do* (a') is critical to the intonation of the whole piece, occurring as part of a descending minor triad. It should be intonated with a fermata to establish how high it needs to be in this harmonic and tonal environment:



Kardos, *Two Choral Works*, p. 169

The following diagram indicates with upward arrows the sensitive notes of the soprano melody of the first three bars (the next three bars repeat this pattern). These few bars provide the key to the intonation of the whole piece:

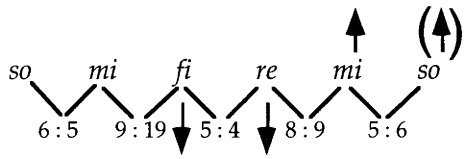


Kardos, *Two Choral Works*, p. 169

Bars 4-6

The soprano melody is repeated in bars 4-6. The contradictions caused by the *re* (b') in the second bar of the soprano occur again in the fifth bar against a *fi* (d#') in the alto. Again there are two alternative tunings, the first as dictated by the tonality (a low *re*), the second as dictated by the harmonic context (high *re*). If the sopranos sing a low *re* (b') then in order for the harmonic interval *re—fi* (b'—d#') to be an acoustic minor sixth it will push down the natural height of the *fi*

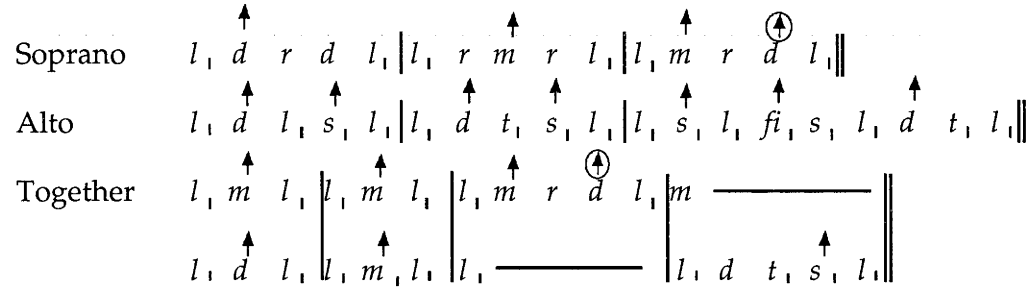
(d#'). If the *fi* (d#') is low then the following note, *re* (b'), which should be an acoustic 5:4 major third, will be low and then the next note *mi* (c#') will be a 9:8 big whole tone:



Kardos, *Two Choral Works*, p. 170

If the soprano sings a high *re* (b') then this situation will not occur.

In bars 1—6 the stable, referential notes are *so, la, ti, do, mi.* The sensitive, ambiguous, and adjustable notes are the *re* and the *fi.* The choir should prepare this work by finding the position of the constant notes with the following exercise:



Kardos, *Two Choral Works*, p. 170

The ability to remember the absolute pitch of one note is important in this work because the adjustable notes should be tuned in conformity with the memory of the absolute pitch of the constant notes.

Bar 7

The interval in the seventh bar should be analysed vertically:



Kardos, *Two Choral Works*, p. 171

The leap of a 4:3 perfect fourth, from f#' to b', or from *la* to *re* in the soprano

produces a low *re*. Thus the following *do* (a') has to be high, (*re—do* = 10:9). The simultaneous e'—d#' (*so—fi*) step in the alto should be a 16:15 semitone, but the *fi* (d#') will only be a consonant 8:5 acoustic minor sixth with the *re* (b') if it is high. This results in the *re—do* interval being 9:8. The alternatives are:

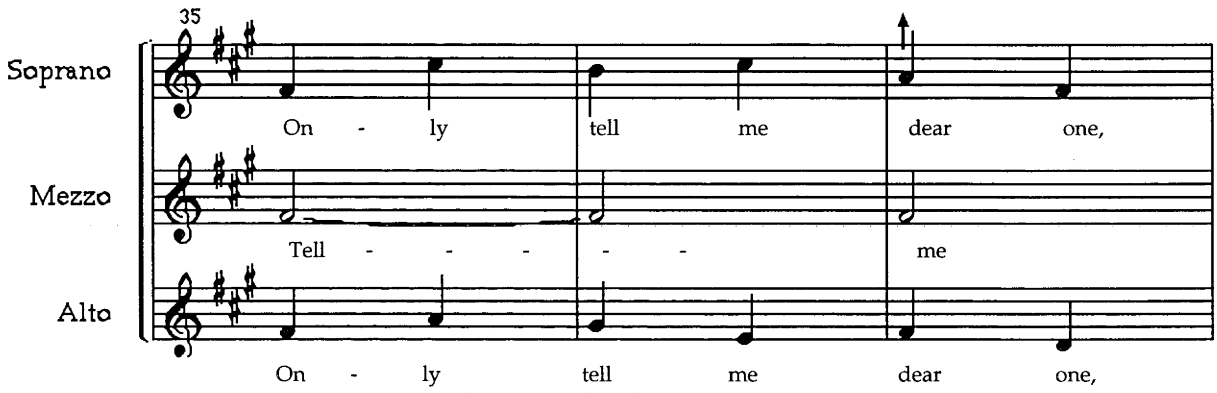
- if the parts are horizontally in tune then the *fi—re* interval will be a Pythagorean minor sixth (ratio 128:81) which is narrower by a comma (81:80) than the acoustic 8:5 minor sixth;

- if the soprano adjusts to the *fi* (d#') in the alto and thus sings a high *re* (b') this will require an 27:20 ad comma fourth jump from f#' to b' or *la—re;*

- if the altos adjust to the melodically dictated low *re* of the soprano, the *so—fi* step (e'—d#') will be wider than the 16:15 semitone by a comma, a 27:25 large semitone (133.2 cents).

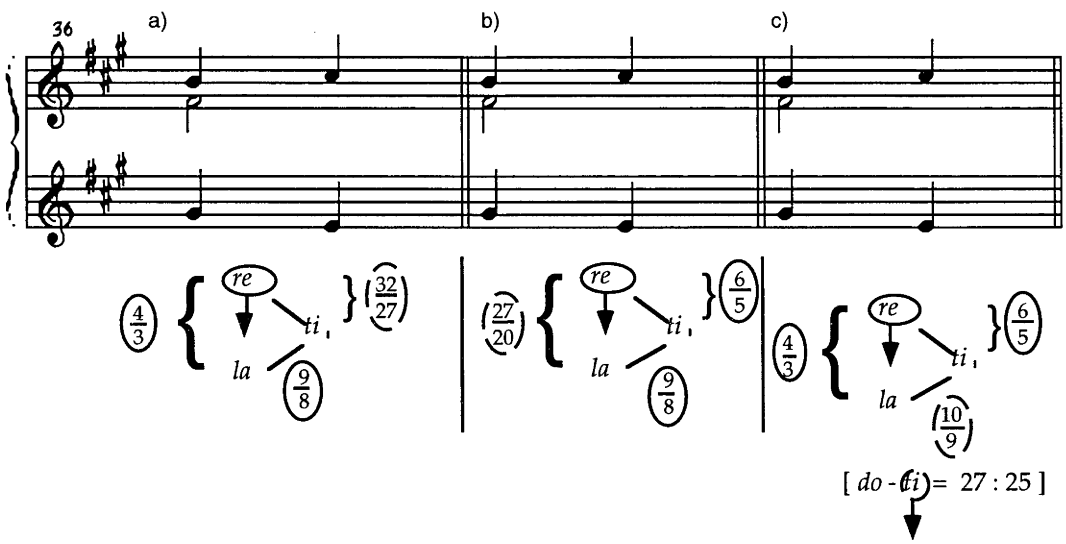
Even the purest performances will require the use of one of the above compromises. Any measurement of intonational purity would show a deviation from mathematical purity in this circumstance. Which alternative is used will depend on the vicissitudes of the live circumstances such as the acoustics of the room, the sensitivity of the choir, the directions of the conductor.

Theoretically the same problem occurs in bar 36 between the three parts, but in practice, after proper preparation of the parts, the choir's ability to make instantaneous adjustments should resolve it in a moment. With fixed-pitch instruments we have to decide in advance which of several intonational possibilities to choose, but this is not necessary with a choir. Kardos advised that it is not wise to over-complicate musical performance with too much theoretical analysis, or to try to set in concrete the changeable nature of living intonation. It is necessary to be aware of the irresolvable intervallic relationships in a piece and use this knowledge in pedagogical work, but not to be so caught up in theoretical ratios that the living intonation is stifled.



Kardos, *Two Choral Works*, p. 171

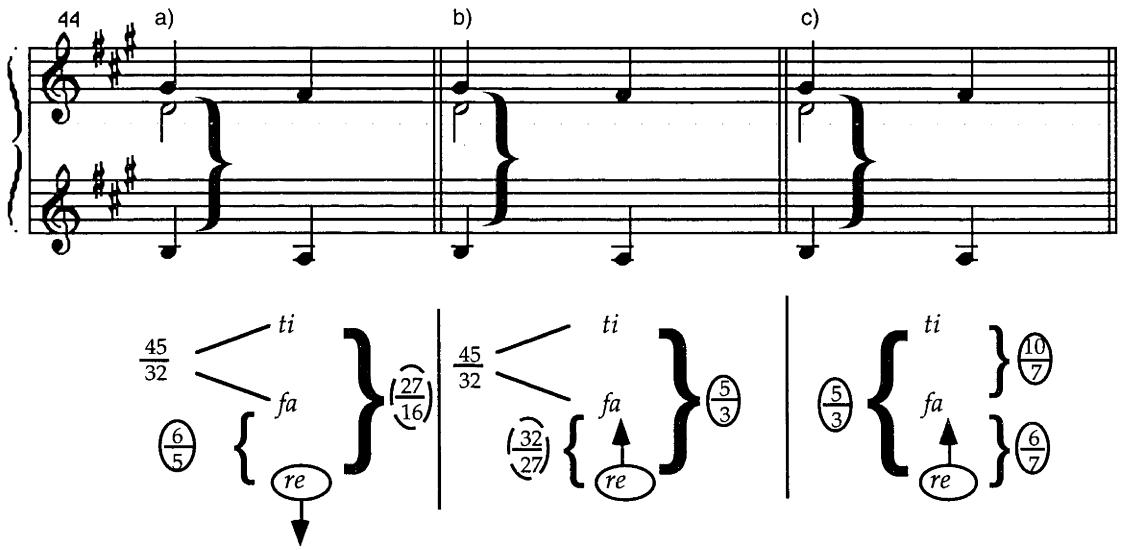
As a general theoretical principle perfect intervals are less tolerant of the disturbing presence of a comma, than the minor and major intervals. The further apart the major or minor interval the less sensitive it is to the presence of an ad comma deviation. There are three alternative solutions to the contradictions of bars 36 and 44:



Kardos, *Two Choral Works*, p. 172

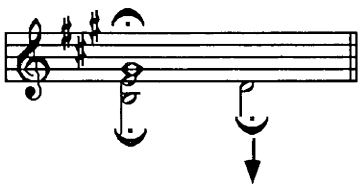
The first solution (a) of bar 36 is with the low *re* (b). The Pythagorean minor third as a harmonic interval is not too conspicuous in this context. The second solution (b) is worse with its ad comma fourth (27:20) due to the high *re.* An ad comma fourth harmonic interval would be too dissonant. The third solution (c) is the most beautiful, but the step between *do—ti* (a'—g#') in the alto would have to be a large semitone (27:25) which is very unnatural and only a choir with excellent intonational training would be capable of intoning it. The descent from the *ti* to the *so* in the alto would be 100:8, smaller than a major third by a comma—$\frac{5}{4} \div \frac{81}{80} = \frac{81}{100}$.

Bar 44



Kardos, *Two Choral Works*, p. 172

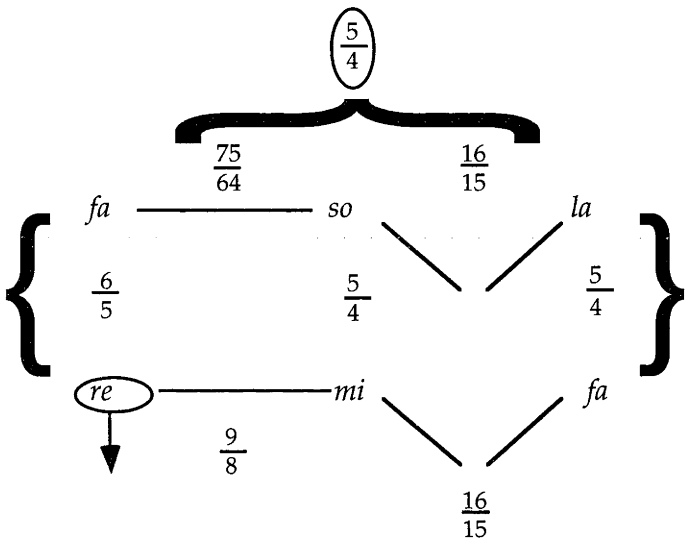
If, as in version a) above, we have a low *re* (b) in the alto, the minor third *re—fa* (b—d') with the mezzo will be acoustic, the minor sixth *re—ti* (b—g') with the soprano will be Pythagorean. This sounds better than version b) where the minor sixth is acoustic and the minor third is Pythagorean. Harmonically the c) version would be the most beautiful because it is found naturally in the harmonic series in over tones 6:7:10. In situations where the harmony is sustained this would be the way to intone this chord. Yet the harmonic beauty of one chord would occur at the expense of the horizontal purity—the 7th overtone is out of tune, and this would be too great a price to pay because the whole melodic line would vacillate. The three alternative tunings of bar 44 outlined in the diagram above can be sung and compared. The pure intervals (circled ratios) should be sung first, then the non-pure (ratios with dotted circles) The (c) version can be sung as follows:



Kardos, *Two Choral Works*, p. 173

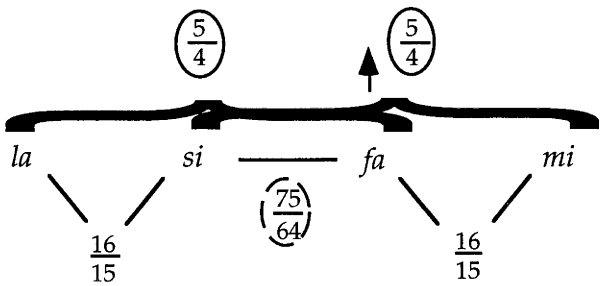
Sing *ti—so—re,* (b—e'—g#') as a chord, building up from the lowest to highest note then sing *ti—fa—re* (b—d'—g#').

Bar 19 contains an augmented second:



Kardos, *Two Choral Works*, p. 173

The ratio of the augmented second is $\frac{75}{64}$, or 274.6 cents, larger by 25.4 cents than the enharmonic tempered minor third at 300 cents. Thus it is particularly difficult in a descending direction, the second note is marked with an upward arrow to remind singers to intone it high:



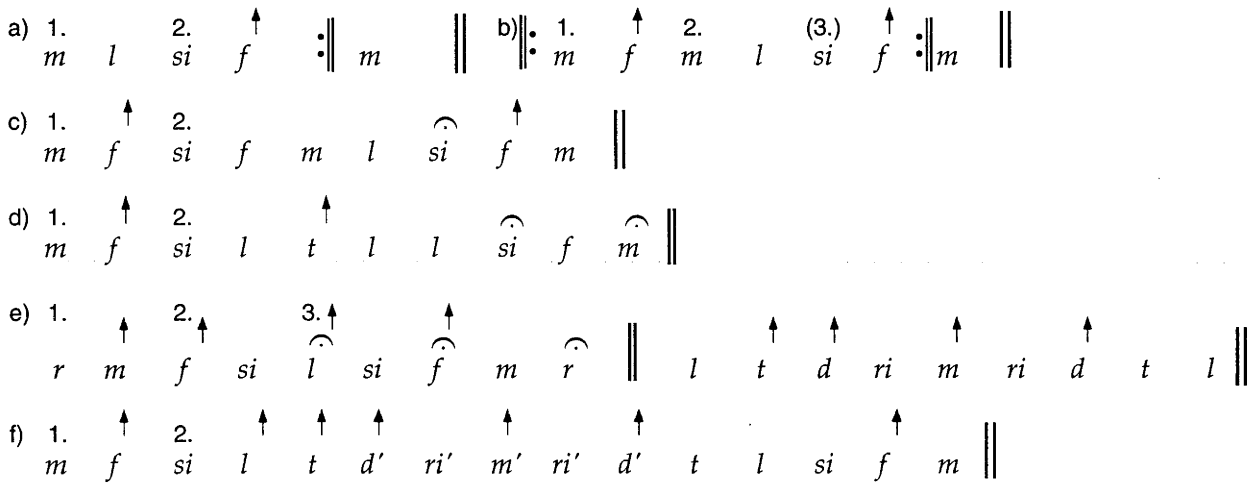
Kardos, *Two Choral Works*, p. 73

An acoustic 5:4 major third minus a 16:15 semitone is an augmented second: $\frac{5}{4} \div \frac{16}{15} = \frac{75}{64}$

$\frac{5}{4}$	Acoustic major third	386.3 cent
$\frac{16}{15}$	Natural semitone	111.7 cent
$\frac{75}{64}$	Augmented second	274.6 cent

Kardos, *Two Choral Works*, p173

The augmented second can be practiced by singing an acoustic major third between fa and *la*, and then inserting the *si*:



Kardos, *Two Choral Works*, p. 174

In canon the *si* should always be softer than the *mi*, and the *la* also softer than the *fa*.

From bar 35 to 46 it is best to practice the parts in separate pairs: mezzo/soprano, mezzo/alto, alto/soprano, then all together. The harmonic structure should be dissected before singing the whole work. Correcting the tuning in one and two parts will allow the intonated intervals to adjust until they reach their right character. It is very important after every run through to check the pitch. Practice all the parts separately, so that each line can be heard on its own. Dissonance between the parts can distort an individual melodic line when singing tutti. Many discrepancies that are unclear when practicing the whole choir can come to light. The harmony can be correct but unbalanced.

Appendix B: beats, combination tones and tuning

Beats have been used for tuning instruments long before their physiological cause was understood. Arnolt Schlick in a work on tuning for organ builders and organists dated 1511 describes the rate of beating necessary for tempering thirds and fifths.[283] Isaac Beeckman in 1619 states that the criteria for judging the justness of fifths is that they do not 'wow-wow'[284] and hypothesised that dissonance is caused by beats.[285] Christiaan Huygens in 1661 defines two criteria for consonance, being agreeable to the ear and the absence of beats.[286] Joseph Sauveur was the first to mention them in print in 1701. Research into this phenomenon was continued by Robert Smith, Giuseppe Tartini, Jean-Baptiste Rameau,[287] Jean le Rond, d' Alembert, Joseph-Louis Langrane, Matthew Young, Ernst Chladni, and George Simon Ohm[288]culminating in the findings of Hermann von Helmholtz.

Beats are a regular pulsation in loudness which occur when two notes, whose frequencies are very close but not the same, are sounded together. The beats cease when the two frequencies are equal. The number of pulsations heard in one second is equal to the difference in frequency of the two notes. When the interval between the two notes is widened the pulsations become too fast for individual beats to be distinguished; instead a tone whose frequency is equal to the difference in the frequencies of the original notes is heard. Notes separated by a minor third or more are heard as two distinct pitches. In the area between 20 and 66 Hz the sensation is described as 'roughness'.[289] Helmholtz showed that the physiological

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283 Arnolt Schlick, *Spiegel der Orgelmacher und Organisten*, 1511, cited in Cohen, op. cit., p. 248.

284 Cited in Cohen, ibid., p. 144.

285 ibid., p. 242.

286 ibid., p. 226

287 Rameau also saw a correlation between dissonance and the rapidity and frequency of beats, and consonance and the cessation of beats. Rameau, *Generation*, 17, cited in Joan Ferris, 'The Evolution of Rameau's Harmonic Theories, *Journal of Music Theory*, vol. 3, 1959, pp. 231-256.

288 Carlton Maley, V. *The theory of beats and combination tones, 1700-1863*, Garland, New York, 1990.

289 This description is derived from the following texts: Helmholtz, op. cit., pp. 159-173;

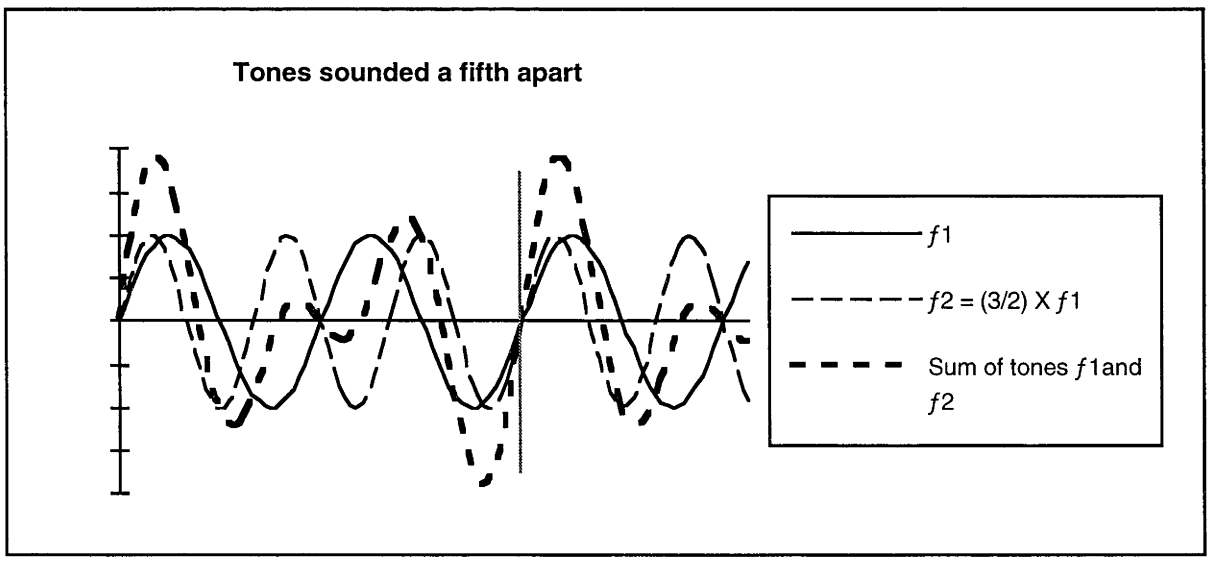
basis for these is in the non linearity of the hearing apparatus.[290]

Beats have an important role in tuning instruments, since two notes that should be unison can be tuned until the beats cease, which occurs when $f1 = f2$, or when the difference between $f1$ and $f2 = 0$.

Combination tones

Combination tones are frequencies generated by the addition and subtraction of the overtones of two sounded notes. Several combination tones are generated by the sounding of two pure tones. The first and most audible combination tone is the difference between the first, lower tone ($f1$) and the second, higher tone ($f2$), or $f2 - f1$. This is sometimes called the simple difference tone or a Tartini tone.[291] If two notes are related to each other by a simple ratio, then the difference between them is a simple ratio. If the two notes were a' (440 Hz) and e" (660 Hz) then the difference tone will be an a (220 Hz).

To explain in terms of wave forms, two notes in ratio 3:2 each produce a periodic oscillation curve. The two curves combined generate a third curve. The combination curve repeats itself exactly, and its frequency is the difference in frequency of the first and second curve.



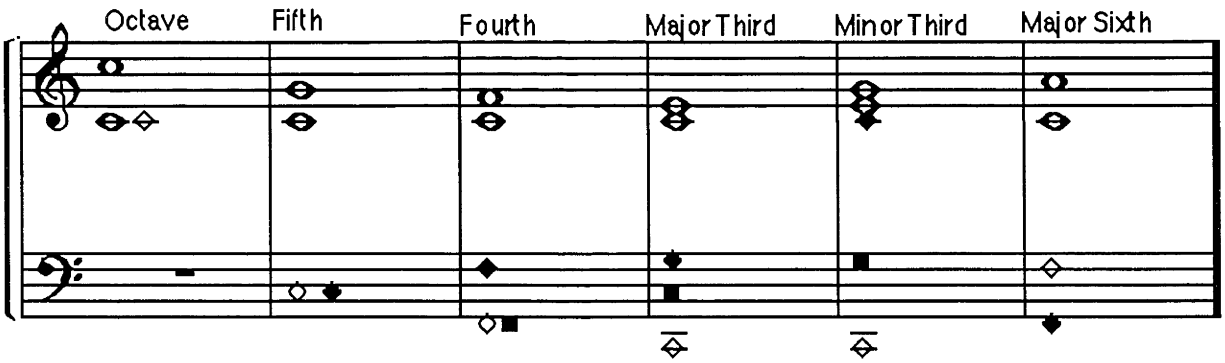
Johnston,op. cit., pp. 247-248; Rigden, op. cit.,pp. 80-85; Juan G. Roederer, *Introduction to the Physics and Psychophysics of Music*, Springer-Verlag, New York, 1979, pp. 27-32.

290 Helmholtz, op. cit., pp. 129-151. See also Johnston, op. cit., pp. 248-250.

291 So named after Tartini who was credited with discovering them in 1754.

Consider the example of sounding two pure tones a fifth apart. The diagram above illustrates that in the time interval marked by the vertical dotted line, the lower tone (f_1) goes through two complete cycles, the upper tone (f_2) goes through three complete cycles (ie they are in the ratio 3:2) and the combined tone goes through one complete cycle (it is in the ratio 1:2 to the lower note, ie it is an octave lower). The combination tone has the frequency $f_2 - f_1 = \frac{3}{2}f_1 - f_1 = \frac{1}{2}f_1$.

The second combination tone is the difference between the first harmonic of the lower note $2f1$ and the upper note $f1$. The third combination tone is the difference between the second harmonic of the lower note ($3f1$) and the first harmonic of the higher note ($2f2$). Helmholtz also investigated the phenomenon of sum tones which are the sum of the two frequencies ($f1+f2$), but as it is highly debatable that these are audible, they have little relevance to tuning. Combination tones are audible if the two fundamental tones are loud enough (the sound intensity level at the ear needs to be above 75db) and not much more than a 3:2 apart. The two fundamental tones need to be fairly high as the difference tones may be several octaves lower. Below are some combination tones generated by simple consonant intervals:



The first difference tone is indicated by the white diamond, the second by a black diamond, the third by a black square. Notice that the first difference tone is the lowest, subsequent difference tones are higher. Plomp[292] observed in 1976 that the difference tones subsequent to the first are also audible when the two original tones are at a low level of loudness. Difference tones can be used to aid tuning because when the choir is singing the original interval acoustically purely at the right sound-intensity level and the choristers have acute hearing, they will hear the corresponding difference tone. Kardos, after Hindemith,[293] asserts that the

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292 Plomp, op. cit.,pp. 28-30.

293 Hindemith, op. cit. p. 58.

first difference tone can be used as a 'trigonometrical point' enabling measurement of the size of the sounded interval (and its degree of acoustic purity) by triangulation.[294] Combination tones also play a part in theories of consonance and dissonance (see below).

294 Kardos *KK*, p. 25. Tartini used difference tones as a guide to intonation for his students of violin. Walker, op. cit. p. 137.

Appendix C: theories of consonance

The implied hierarchical theory of consonance contained in Kardos's work is based on both acoustic and numerological factors. Acoustically, consonance is a function of position in the harmonic series: the lower in the harmonic series the more consonant. Mathematically, consonance is a function of simplicity of ratio: the simpler the ratio, the more consonant. Both ways of defining consonance have a venerable history,[295] as is evident in the historical outline above.

Numerological definitions of consonance go back to the Pythagoreans, for whom the concords (*symphonia*) included only intervals based on ratios based on the numbers 1-4. No interval smaller than a 4:3 was considered consonant by any Greek theorist.[296] In the medieval period, Boethius' transmission of the Pythagorean definition of consonance was ratified by papal degree: Pope John XXII promulgated an edict in 1324 permitting only 'concords such as the octave, fifth, and fourth, that enrich the melody and may be sung above the simple ecclesiastical chant'.[297] We have seen the process by which the sacred sanctum of consonances was extended by Zarlino's senario to include intervals in the ratios 1:2:3:4:5:6.[298]

From Pythagoras to Zarlino, consonance was defined in terms of ratios of string lengths. Benedetti was the first to establish a connection with the vibration cycles of strings, stating in 1585 that the concordance of intervals depends on the coincidence of the terminations of vibration-cycles.[299] The intervals with the

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295 See Leonard Meyer, *Emotion and Meaning in Music*, University of Chicago Press, Chicago, 1956, p. 230.

296 Archytas in the 4th century had recognised the four ratios of 5—5:4, 8:5, 6:5, 5:3 as scale degrees and Ptolemy described a diatonic tuning that allowed 5:4 and 6:5 tunings on most degrees of the scale, yet these were not defined as consonant.

297 Boethius, *Fundamentals of Music*, op. cit.,Book I, 6., p. 15. 'all musical consonances consist of a duple, triple, quadruple, sesquialter, or sesquitertian ratio.' Papal edict cited in Johnston, op. cit., p. 19. Claude V. Palisca, *Humanism*, pp. 36-42.

298 The minor sixth (8:5) too was admitted, explained as being composed of a fourth and a minor third.

299 'The order of agreement of the terminations of the percussions of the air waves', *Diversarum speculationum mathematicarum & physicorum liber*, Turin, 1585, quoted in Palisca, *Studies*, p. 214.

most simple ratios have the greatest concurrences of vibrations and are therefore most consonant. Benedetti has, accordingly, a scale of consonance with unison being the most consonant, followed respectively by octave, fifth, fourth and so on.[300] Conversely, he showed that 'consonance decreases in proportion to the size of the product of the ratios.'[301] The coincidence theory of consonance, which holds that the degree of consonance is a function of the frequency of the meeting-points of vibrations of two pitches, was further refined in the wake of the scientific revolution and promulgated by Galileo Galilei, Descartes, and Mersenne.[302]

The relative weighting of reason versus experience as determinants of consonance occupied the minds of Renaissance and early Baroque theorists. The Pythagorean denial of experiential factors meant that intervals that sounded harsh could still be considered consonant if the mathematics were right.[303] Much debate centered on the status of the fourth, which by numerological definitions had greater consonance than the third, but by current aesthetic judgement had come to be regarded as less consonant.[304] 'Pleasingness' as a determinant of consonance is linked to the concept of the validity of information gained via the

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300 ibid, p. 216.

301 Palisca, *Studies*, p. 204.

302 See H. F. Cohen, *Quantifying Music: The Science of Music at he First stage of the Scientific Revolution, 1580-1650*, D. Reidel, Dordrecht, 1984, and Palisca, *Studies*, pp. 202-235. Galileo's well known theory of harmony was based on his experiments with pendulums. A simple explanation is contained in Johnston, op. cit., pp. 27-35. In fact the coincidence theory is a view of consonance based on numerological definitions, numbers representing frequency ratios instead of string length.Cohen, op. cit., pp. 232-236, lists the problems with the coincidence theory in detail.

303 Kepler, in his *Harmonice Mundi*, 1619, wrote:

> 'The Pythagoreans were so addited to... philosophising in numbers, that they failed to keep to the judgement of their ears, though it was by means of this that they had initially been brought to this philosophy; they defined solely by their numbers what is a melodic interval and what is not, what is consonant and what is dissonant, thus doing violence to the naturl instinctive judgement of the ear.
>
> Quoted in D.P. Walker, 'Kepler's Celestial Music', *Journal of the Warburg and Courtauld Institutes*, vol. 30, 1967, p. 235.

304 Tinctoris rejected the fourth in 1477, because it sounds to the ear like a discord, and cited Aristoxenus's reliance on the ear by way of justification for this stance. Johannes Tinctoris, *The Art of Counterpoint*, Musicological Studies and Documents, vol. 5, American Institute of Musicology, Rome, 1961, pp. 18-19.

senses. Descartes added pleasingness to simplicity of ratio as determinants of consonance.[305] Mersenne made a distinction between scientific and aesthetic explanations. 'Sweetness' (*doux*) was determined by coincidence of vibration, 'agreeability' (*agreable*) represented auditory pleasure. In terms of sweetness the fourth ranked in its usual place after the fifth. In terms of agreeability it ranked last in the list of consonances.[306]

Later hierarchical gradations of intervals include the grades of suavity of Leonhard Euler in 1739,[307] Rameau's theory of decreasing perfection from octave onwards, and Paul Hindemith's classes of chord intensity.[308] Other theories of consonance based on vibration speeds also include that of Diderot.[309]

Harry Partch is a twentieth century inheritor of the numerological approach. Denying the role of the harmonics in theories of consonance he bases his hierarchy on simplicity of ratio, beginning with 'absolute consonance' and progressing into 'an infinitude of dissonance'.[310] The consonance of the intervals decreases as the odd numbers of their ratios increase—1:1, 2:1, 3:2, 4:3, 5:4, 5:3, 6:5, 7:6 etc.

Schoenberg, in 1911, also theorised that degrees of consonance depend on

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305 Palisca, *Studies*, p. 218.

306 Mersenne, Marin, *Harmonie Universelle, Fourth Book on Instruments*, prop. 9., R. E. Chapman, trans., Martinus Nijhoff, The Hague, 1957, prop. 32, coroll. 2;82. Reimann cites many examples from 13th century manuscripts showing that already in composition the third had usurped the fourth and that the fourth was 'in itself consonant, but becomes a dissonance in progression.' op. cit., pp. 97-98. This introduces another determinant of consonance, that of context. Salinas's explanation for this dilemma is worthy of quoting: 'For just as the vine loves to bind itself to the elm, and wife to husband, and neither could exist well if separated from the other; just so the fourth longs to bind itself to the fifth, and is hardly able, if separated from it, to sustain itself. However in this union the fourth requires the upper rather than the bottom position.' Salinas, 1577, 56, quoted in Cohen, op. cit., p. 64.

307 Leonhard Euler, *Tentamen novae theoriae musicae*, Typographia Academiae scientiarum, St Petersburg, 1739, cited in Palisca, *Studies*, p. 216.

308 Hindemith, op. cit., pp. 106-108.

309 Paul Chanier, discusses Diderot's theory of the acoustics of a vibrating string, where he regards the qualities of major and minor modes as inherent, derived from heaven. Chanier argues that this is a concept that pervades 18th century music. *L' opposition majeur-mineur: est-elle dans la nature?* [The major-minor opposition: does it exist in nature?], V. de Provence, Aix-en-Provence, 1986.

310 Partch argues that the overtone series has nothing to do with this hierarchy, it has to do with the ear's preference for small-number intervals. Partch, op. cit., p. 87.

position in the harmonic series.[311]

Although Kardos did not explain the physiological basis for a hierarchy of consonance based on the harmonic series, it is consistent with Helmholtz's theory that degrees of consonance are based on the ear's response to harmonics. Consonance is the absence of dissonance.[312] Sensory theories of consonance argue that consonance is a function of absence of beats in the cochlea, or to the coincidence of ensuing neural discharges. A more recent version hypothesises that brain circuits tuned to each frequency can be shared where the patterns of neural firing for two frequencies coincide.[313] Plomp's definition incorporates Helmholtz's' definition which relies on both the ratios of fundamentals and congruence among the lower order harmonics, but adds to this the physical distance between the tone's frequencies on the ear's basilar membrane.[314] Plomp uses the terms 'tonal consonance' to refer to the acoustic component of consonance devoid of musical context.[315]

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311 See Johnny Reinhard, 'Arnold Schoenberg's conception of tonality', *1/1 Just Intonation*, vol. 5, no. 3, 1989, pp. 4-7. Reinhard points out that despite his reputation as a proponent of atonal music, Schoenberg considered his serial music tonal. He stressed the derivation of the major triad from the harmonic series, a viewpoint closer in spirit to the tradition of just intonation than to the Pythagorean tuning via the cycle of fifths, from which 12-tone equal temperament is derived.

312 Helmholtz, op. cit., p. 204, writes:

> 'when two or more simple tones are sounded at the same time, they cannot go on sounding without mutual disturbance, unless they form with each other certain perfectly definite intervals. Such an undisturbed flow of simultaneous tones is called a *consonance*. When these intervals do not exist, beats arise, that is, the whole compound tones, or individual partial and combinational tones contained in them or resulting from them, alternately reinforce and enfeeble each other. The tones then do not coexist undisturbed in the ear. They mutually check each other's uniform flow. This process is called *dissonance*.

William Hutchinson, in 'The acoustic component of Western Music', *Interface*, Netherlands, vol. 7, no. 1, 1978, pp. 1-29 has devised a means for measuring dissonance in a superposition of complex tones, based on an extension of the Helmholtz-Plomp and Levelt model of beats as the cause of dissonance. This measure of dissonance was found to correlate with psychological rank orderings of dissonance and its absence (consonance), and with orderings of consonance and dissonance found in Western common practice and pedagogy.

313 Paul Boomsliter and Warren Creel, 'The Long Pattern Hypothesis in Harmony and Hearing' *Journal of Music Theory*, vol. 1, 1961, pp. 2-31.

314 Plomp, op. cit., pp. 72-74.

315 ibid., p. 68. Some authors distinguish 'acoustic consonance/dissonance' from 'syntactic consonance/dissonance'. The distinction between perfect and imperfect consonances is a measure of acoustic consonance, while the perfect fourth is considered dissonant in many

Modern definitions tend to weld together elements from all the above approaches and emphasise context. Norman Cazden, in his 'The definition of consonance and dissonance' includes the attributes of agreement, accord, and tunability, as well as gradations of sonorous euphony. He believes that consonance and dissonance refer to the interaction of sonorous aggregates, as well as the perceptual expectations aroused. They are dynamic events which function thoughout harmonic progressions. Consonance reflects the pole of stability, dissonance of instability. They do not have their origin in nature but in a 'second nature' which is constituted by a music system that is culturally determined.[316] Cazden argues that extra-musical philosophic priorities posit an idealised, static view of consonance, for example the fabled law of simple ratios divorced from practice and context. Such definitions of consonance apply, he argues, to pre-musical properties of isolated auditory signals, which he terms euphony. These signals attain musical roles only by entering into configurations dependent on expectation, memory. These aspects are phenomenally, not actually present, arising from acculturation. Consonance and dissonance are not reflections of universal law immanent in psychoacoustic response to sound-signals but a part of a culturally determined music system restricted to European culture.[317]

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cases because it is a syntactic dissonance (ie., in the syntax of tonal heirarchy, fourths are often considered dissonant and resolve to thirds).

316 Norman Cazden, 'The definition of consonance and dissonance', *International review of the aesthetics and sociology of music*, vol. 11, no. 2, 1980, pp. 123-68.

317 Norman Cazden, 'The systematic reference of musical consonance response', *International review of the aesthetics and sociology of music*, vol. 3, no. 2, 1972, pp. 217-45.

Appendix D: glossary of terms

Acoustically Pure Intonation
a form of tuning intervals so that they are free of beats in the lower order harmonics.

Ad comma an interval that exceeds its acoustic enharmonic by a syntonic comma.

Anhemitonic without semitones.

Beats a regular pulsation in loudness which occurs when two notes, whose frequencies are very close but not the same, are sounded together. The number of pulsations heard in one second is equal to the difference in frequency of the two tones. Beats cease when the two frequencies are equal. When the interval is widened the pulsations become too fast to distinguish. Instead a tone whose frequency is equal to the difference in the frequencies of the orginal tones is heard (a difference tone). Notes separated by a minor third or greater are heard as two distinct pitches. In the area around 20 and 66 Hz the sensation is described as 'roughness'.

Apotomē major semitone, ratio $\frac{2187}{2048}$. See *leimma*, q.v.

Arsis a building of musical tension. *Thesis* is the release of tension.

Canon a means of determining the ratios of the intervals of music, often used interchangeably with monochord (q.v.)

Cent 1:1200 of an equally tempered octave, one hundredth of an equally tempered semitone, ratio $\sqrt[1200]{2}$

Chroa, chroea Shade, nuance, refers to the particular size of subdivision of the tetrachord in each genus, defines the variety of intervals composing a particular genus.

Chroma—Major Interval between a large whole tone and a natural semitone:

$$\frac{9}{8} \div \frac{16}{15} = \frac{135}{128}$$

Chroma—Minor Interval between a major and a minor third. $\frac{5}{4} \div \frac{6}{5} = \frac{25}{24}$

Combination tones frequencies generated by the addition and subtraction of the overtones of two sounded notes. The 'simple difference tone' is the difference between the lower and higher tone—$f2$-$f1$. The second combination tone is the difference between the first harmonic of the lower note and the upper note—$2f2$-$ff1$.

Comma—syntonic or Ptolemaic
Interval between a just major third, ratio 5:4 and Pythagorean ditone or major third, ratio 81:64 (two large whole tones 9:8 X 9:8 = 81:64). 5:4 ÷ 81:64 = 5:4 x 64:81 = 324:320 = 81:80.

Comma—ditonic or Pythagorean
Interval between six Pythagorean whole tones, ratio 53441:524288, 74:73.

De comma an interval that that is smaller than its acoustic enharmonic by a syntonic comma.

Diapason Octave—*Diapason*, which means 'Through All' is the interval that arises from dividing string in half = ratio 2:1 = octave.

Diapente Fifth—the interval that arises from dividing the string of a monochord (q.v.) one third from the end = ratio 3:2 commonly known as the perfect fifth.

Diatessaron Fourth— the same division if you pluck the other side of the monochord = ratio 4:3, (perfect fourth). Perfect fourth and fifth thus two sides of string division with 2 thirds on one side (*dipente*) and one third on other side (*diatesseron*).

Didymus comma = Syntonic comma. Called by Kardos and certain other writers the comma of Didymus because he was credited with defining it. It is the difference between the big and small whole tone (9:8 ÷ 10:9 = 81:80).

Ditone Major third formed by two equal tones, eg 9:8 X 9:8 = 81:64, Pythagorean major third. Used by Zarlino to mean a justly tuned major third (5:4).

Equal Temperament The division of the octave into equal parts. Twelve tone equal temperament is the one found in piano tuning today, where the octave is divided into 12 semitones each worth 100 cents or ratio $\sqrt[12]{2}$.

Epogdoon Greek word meaning that which consists of a whole and one eighth of the whole, in music it denotes the ratio 9:8 or whole tone, which is the interval by which the 5th

exceeds the 4th, hence a tone.

Equal temperament The division of the octave into an equal number of parts, ie twelve equal semitones, each being $\sqrt[12]{2}$.

Genera Three ways of subdividing the tetrachord. The outer poles maintain their fixed proportion, the middle two are moved to create a set of melodic intervals. *Diatonic* genera—semitone, tone, tone; *chromatic*—semitone, semitone, tone and a half; *enharmonic*—quarter tone, quarter tone, ditone.

Just Intonation Just—pure. A system of tuning that uses ratios that are superparticular, whose numerators and denominatos are whole numbers, and in which, theoretically, all consonances are true (1:2:3:4:5:6).

Leimma Greek word for the Pythagorean semitone, from *leipein*, meaning to be wanting. It is the remnant or remainder when the fourth 4:3 was divided into two whole tones: $\frac{4}{3} \div \frac{9}{8} \times \frac{9}{8} = \frac{256}{243}$. This is the ratio of the semitone used in Pythagorean tuning. A Pythagorean whole tone, ratio 9:8 divides unevenly into the leimma and the apotome which is the larger semitone, ratio $\frac{2187}{2048}$.

Monochord An instrument with a single (mono) string. A string stretched over a wooden base which marks the string-lengths for a tuning system, a diagram showing these lengths, directions for constructing such a diagram. The monochord was used to determine the mathematical relations of musical sounds. It was called a 'canon' or *kanon* because it canonised these intervals.

Pentatonic scale Five note scale.

Ptolemaic Sequence A sequence of intervals derived from Ptolemy's syntonon diatonic genera which form a diatonic scale in the ratios 9:8, 10:9, 16:15, 9:8, 10:9, 9:8, 16:15.

Ratio A mathematical means of defining an interval between two tones in terms of their proportional relationship. Since the 18th century this has been in terms of frequency or vibrations per second, reduced to its lowest possible terms.

Schisma Difference between the syntonic and ditonic commas, ratio 32805:32768 (approx. 2 cents).

Superparticular Ratio A ratio where the antecedent exceeds the consequent by 1, eg 5:4, 4:3, 3:2, 2:1.

Temperament To temper means to vary the pitch from its true ratio slightly. Temperament refers to a system of tempering all intervals to a predetermined amount.

Thesis The release of musical tension.

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