Underdetermination of Imprecise Probabilities

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Declaration

This thesis is solely the work of its author. No part of it has previously been submitted for any degree, or is currently being submitted for any other degree. To the best of my knowledge, any help received in preparing this thesis, and all sources used, have been duly acknowledged.

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Abstract

In a fair finite lottery with n tickets, the probability assigned to each ticket winning is $\frac{1}{n}$ and no other answer. That is, $\frac{1}{n}$ is *unique*. Now, consider a fair lottery over $\mathbb{N}$. What probability is assigned to each ticket winning in this lottery? Well, this probability value must be smaller than $\frac{1}{n}$ for all $n \in \mathbb{N}$. If probabilities are real-valued, then there is only one answer: 0, as 0 is the only real and non-negative value that is smaller than $\frac{1}{n}$ for all $n \in \mathbb{N}$.

However, this answer seems counter-intuitive, as it violates *Regularity*: for all possible events A that are assigned a probability, A is assigned a positive probability. It is possible for ticket i to win in the second lottery. So, the set $\{i\}$ should be assigned a positive probability and not 0. Consequently, in order to satisfy *Regularity*, probabilities must be allowed to take on 'non-real' values, e.g. a positive infinitesimal ε that is smaller than $\frac{1}{n}$ for all $n \in \mathbb{N}$.

So, what regular probability is assigned to $\{i\}$ in the second lottery? This probability value must be positive but smaller than $\frac{1}{n}$ for all $n \in \mathbb{N}$. Given the definition of ε, ε is a correct answer. But ε *isn't* unique, because every other positive infinitesimal is a correct answer as well and there are uncountably many positive infinitesimals. After all, every positive infinitesimal is strictly between 0 and $\frac{1}{n}$ for all $n \in \mathbb{N}$. Nothing about a fair lottery over $\mathbb{N}$ uniquely determines ε as the correct answer and nothing about the lottery rules out other positive infinitesimals as a correct answer as well. What probability to assign to $\{i\}$ is *underdetermined*.

In this thesis, I explore this difference between fair finite lotteries and fair infinite lotteries. The constraints governing the former uniquely determine the probabilities assigned to events in those lotteries, but this isn't true for the latter lotteries, when *Regularity* is a constraint on probabilities. Now, although there are uncountably many correct probability values that can be assigned to $\{i\}$ in the second lottery, there are some applications of probability theory, for which at least one correct probability value must be chosen in order to accomplish that application, e.g. calculation of expected utilities. In this thesis, I spell out sufficient conditions for when a choice like that is problematic. These conditions apply to precise probabilities, imprecise probabilities, and qualitative probabilities. So, I will consider Benci et al. [2018]'s and Pruss [2021]'s suggestion of allowing probabilities to be imprecise, as well as resorting to qualitative probabilities, to overcome the choice problem. I will argue that both don't solve the problem. Furthermore, there's a hitherto unnoticed difficulty in deriving imprecise probabilities by supervaluating over precise 'non-real' probability values. To avoid this difficulty, precise probability values should be real-valued. Thus, *Regularity* cannot be a constraint on probabilities. But as it turns out, dropping *Regularity* as a constraint and requiring precise probabilities to be real-valued solve the choice problem.

Contents

1 Introduction

Consider a fair finite lottery with n tickets. What is the probability assigned to each ticket winning in this lottery? Well, the answer is simple: $\frac{1}{n}$. In particular, let P be a probability function[1] correctly representing this lottery, i.e. for all events A in this lottery, $P(A) = \frac{\#(A)}{n}$ where $\#(A)$ is the cardinality of A. Here's the next question: is P *unique*? That is, is P the only probability function that correctly represents this lottery? Well, yes! There are no other probability functions that correctly represents this lottery.

Now, consider another fair lottery but instead of finitely many tickets, there are *infinitely* many tickets. Perhaps a number in $[0, 1]$ will be randomly chosen. What probability is assigned to each number getting chosen in this lottery? Let P' denote a probability function correctly representing a fair lottery over $[0, 1]$. What is $P'(\{\omega\})$ where $\omega \in [0, 1]$? We know that $P'(\{\omega\}) < \frac{1}{n}$ for all $n \in \mathbb{N}$, because there are more than n tickets in this second lottery. In fact, there are uncountably many. If $P'(\{\omega\}) \geq \frac{1}{n}$ for some $n \in \mathbb{N}$, then $P'(\{\omega_1, \ldots, \omega_{n+1}\}) > 1$ where $\omega_1, \ldots, \omega_{n+1} \in [0, 1]$. The probability axioms are violated.[2] So, $P'(\{\omega\}) < \frac{1}{n}$ for all $n \in \mathbb{N}$. If P' is real-valued, then there is only one answer: 0, as 0 is the only real and non-negative value that is smaller than $\frac{1}{n}$ for all $n \in \mathbb{N}$.

However, this answer seems counter-intuitive. The probability of an impossible event is 0, i.e. $P'(\emptyset) = 0$. Impossible events seem less probable than possible events. After all, the former, by definition, can never happen, while the latter can still happen. And it seems intuitive to assign a lower probability to less probable events. So, since impossible events seem less probable than possible events, it seems impossible events should be assigned a lower probability than possible events. Because it is still *possible* for a particular number to be chosen in the second lottery, $P'(\{\omega\})$ should be positive and not 0. In accordance with this intuition, the assignment of 0 probability should be reserved only for impossible events.

Call the following constraint **Regularity**: for all possible events A that are assigned a probability, A is assigned a positive probability. **Regularity** seems like an intuitive constraint on probabilities. Indeed, a number of authors have advocated various versions of **Regularity**, e.g. Shimony [1955], Kemeny [1955], Jeffreys [1961], Bernstein & Wattenberg [1969], Stalnaker [1970], Lewis [1980], Bascelli et al. [2014], Hofweber [2014], Benci et al. [2018], Bottazzi et al. [2019], Bottazzi & Katz [2021a,b], etc. Call a probability function P *regular* just in case for all nonempty events A in the domain of P, $P(A) > 0$. While there are many advocates of **Regularity** and despite its intuitive appeal, **Regularity** also has many opponents, e.g. Hájek [MS], Williamson [2007], Hájek [2012], Easwaran [2014a], Pruss [2014], Parker [2019], Pruss [2021], etc.

[1]A function P is a *probability function* just in case (i) $P(A) \geq 0$ for all events A that P is defined for; (ii) $P(\Omega) = 1$; (iii) If $A \cap B = \emptyset$ and $P(A)$ and $P(B)$ are defined, then $P(A \cup B) = P(A) + P(B)$.

[2]Hereafter, whenever I mention the 'probability axioms', I mean those axioms spelled out in footnote 1.

The issues surrounding **Regularity** are highly complex because of the alternative number systems needed to ensure that probability functions are regular. Indeed, no real-valued probability functions correctly representing fair infinite lotteries are regular. To see why, first note that a fair infinite lottery has more than n tickets for all $n \in \mathbb{N}$. Second, note that the real numbers, denoted by $\mathbb{R}$, satisfy the *Archimedean property*: for any two positive real numbers x and y, there is an $n \in \mathbb{N}$ such that $x < ny$. If there were a regular and real-valued probability function P correctly representing a fair infinite lottery, then $P(\{\omega\}) = y > 0$ where y is a positive real number and ω is a ticket in the lottery. Since 1 and y are positive real numbers, there is an $n \in \mathbb{N}$ such that $1 < ny$. Re-arranging, $y > \frac{1}{n}$. So, $P(\{\omega\}) = y > \frac{1}{n}$. Re-arranging again, $nP(\{\omega\}) > 1$. But $nP(\{\omega\}) = P(\{\omega_1, \ldots, \omega_n\}) > 1$ where $\omega_1, \ldots, \omega_n$ are all tickets in the given fair infinite lottery. Recall that a fair infinite lottery has more than n tickets for all $n \in \mathbb{N}$. Thus, the probability axioms are violated. Therefore, no real-valued probability functions correctly representing fair infinite lotteries are regular.

1.1 Non-Archimedean Number Systems

To ensure that probability functions correctly representing fair infinite lotteries are regular, alternative number systems that enrich $\mathbb{R}$ are needed. And there are many such alternative number systems, e.g. the hyperreals, the surreals, the Levi-Civita field, etc.[3] For the sake of convenience, call all these alternative number systems *non-Archimedean*. Why? It is because there are numbers in these listed fields[4] that do not satisfy the Archimedean property, i.e. there exist positive numbers x and y in each respective field such that there does not exist an $n \in \mathbb{N}$ such that $x < ny$.

Now, what numbers are added to $\mathbb{R}$ in these alternative systems? There are 3 kinds.

[3]For more details on the hyperreals, see Wenmackers [2016]. Although the surreals and the Levi-Civita field are listed here as potential co-domains of regular probability functions correctly representing fair infinite lotteries, note that Bottazzi & Katz [2021a,b] have argued that these fields are in fact *not* applicable to probabilities.

[4]A *field* is a set F equipped with two binary operations called *addition*, denoted by $+ : F \times F \to F$, and *multiplication*, denoted by $\cdot : F \times F \to F$, satisfying the *field axioms*:

Commutativity. *For all* $x, y \in F$, $x + y = y + x$ *and* $x \cdot y = y \cdot x$.

Associativity. *For all* $x, y, z \in F$, $(x + y) + z = x + (y + z)$ *and* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Identity. *There exists an element* $0 \in F$ *such that* $x + 0 = 0 + x = x$ *for all* $x \in F$. *Also, there exists an element* $1 \in F$ *such that* $x \cdot 1 = 1 \cdot x = x$ *for all* $x \in F$.

Inverses. *For any* $x \in F$, *there exists a* $y \in F$ *such that* $x + y = y + x = 0$. *Also, for any* $0 \neq x \in F$, *there exists a* $y \in F$ *such that* $x \cdot y = y \cdot x = 1$.

Distributivity. *For all* $x, y, z \in F$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

Distinct Identities. $1 \neq 0$.

Hereafter, let $(F, +, \cdot)$ denote a field.

To illustrate, let x be a number in a non-Archimedean field and let $|x|$ denote the absolute value of x. Here are the 3 kinds of numbers added to $\mathbb{R}$:

1. Infinite numbers: x is *infinite* just in case $|x| > n$ for all $n \in \mathbb{N}$.

2. Infinitesimals, the reciprocals of infinite numbers: x is an *infinitesimal* just in case $|x| < \frac{1}{n}$ for all $n \in \mathbb{N}$. Hereafter, let ε denote a positive infinitesimal, i.e. $0 < \varepsilon < \frac{1}{n}$ for all $n \in \mathbb{N}$.[5]

3. Real (and non-zero) numbers plus or minus a positive infinitesimal, e.g. $\frac{1}{2} \pm \varepsilon$, $-\frac{\sqrt{2}}{3} \pm 2\varepsilon$, $\frac{3}{4} \pm \pi\varepsilon$, $-100 \pm \varepsilon^2$, etc.

Say that x is *finite* just in case it is not infinite. Also, note that 0 is the only real-valued infinitesimal, though it is not a positive infinitesimal. To see how these non-Archimedean fields violate the Archimedean property, consider 1 and ε: there is no $n \in \mathbb{N}$ such that $1 < n\varepsilon$.

However, because I'm primarily interested in probability functions, I'm only interested in those numbers in these non-Archimedean fields that fall between 0 and 1, inclusive of both 0 and 1. That is, the relevant numbers of interest to me are the following:

1. All the real numbers r such that $0 \leq r \leq 1$.

2. All the positive infinitesimals.

3. All the positive real numbers plus or minus a positive infinitesimal strictly between 0 and 1, e.g. $\frac{1}{2} \pm \varepsilon$, $\frac{3}{4} \pm \pi\varepsilon$, $\frac{\sqrt{2}}{2} \pm 100\varepsilon$, etc.

Hereafter, call a probability function P *non-Archimedean* just in case there are non-real values in its image that are in some non-Archimedean field.[6] Though there are many issues surrounding **Regularity**, all of which highly complex, in this thesis, I shall limit my attention to a feature of regular probability functions correctly representing fair infinite lotteries: underdetermination.

[5] Hájek [2003] has argued that there is actually no successful denotation or reference of a particular positive infinitesimal here. That is, ε fails to refer to or denote any particular positive infinitesimal. In fact, ε is just a *placeholder*. Consider this analogy: let x denote a positive real number. x does not 'pick out' a particular positive real number; there is no successful denotation or reference of a particular positive real number here, as x is just a placeholder for such a number, whatever it may be. Just as x is merely a placeholder for a positive real number, ε is merely a placeholder for a positive infinitesimal. In either case, there is no successful denotation or reference of a particular number. For the purposes of this thesis, I will put aside this problem. However, the reader is free to refer to Pruss [2021] for a reply to Hájek [2003].

[6] Since I'm allowing probability functions to be non-Archimedean, probability functions cannot be required to be *countably additive*, i.e. $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ if all the A_i's are pairwise disjoint and $P(A_i)$ is defined for all A_i's. This explains why probability functions, as defined in footnote 1, are only *finitely additive*, i.e. if $A \cap B = \emptyset$ and $P(A)$ and $P(B)$ are defined, then $P(A \cup B) = P(A) + P(B)$. More details will be provided soon on why probability functions cannot be required to be countably additive if they are allowed to be non-Archimedean.

1.2 Infinitesimal Underdetermination

In order to explain what underdetermination is, let's revisit the fair lottery over $[0, 1]$. What *regular* probability is assigned to each number getting chosen in this lottery? We know that this value has to be strictly between 0 and $\frac{1}{n}$ for all $n \in \mathbb{N}$. Given the definition of a positive infinitesimal above, ε is one answer. Here's the next question: is ε the only answer? That is, is it *unique*? Well, no. Because 2ε, ε^2, $\varepsilon^2 + 2\varepsilon$, and for that matter, every other positive infinitesimal is an answer too. After all, every positive infinitesimal is strictly between 0 and $\frac{1}{n}$ for all $n \in \mathbb{N}$. And because there are uncountably many positive infinitesimals, there are also uncountably many answers. Nothing about a fair lottery over $[0, 1]$ uniquely determines ε as the answer and nothing about the lottery rules out other positive infinitesimals as an answer as well. That is, what value to assign to each number getting chosen in a fair lottery over $[0, 1]$ is *underdetermined* by the constraints governing a probability function correctly representing this lottery, in the sense that the constraints do not uniquely determine a probability value to be assigned to each number getting chosen.

Consider again a fair finite lottery with n tickets. The probability function that correctly represents this lottery is *unique*, in the sense that there are no other probability functions that correctly represent this lottery. The unique probability assigned to each ticket winning in this lottery is *fully* determined by the constraints governing a probability function correctly representing this lottery; there is no underdetermination. Indeed, the probability assigned to each ticket winning in a fair finite lottery with n tickets is unequivocally $\frac{1}{n}$ and no other answer. Contrast this with fair infinite lotteries. Although only a fair lottery over $[0, 1]$ was considered above, the general point holds: what *regular* probability to assign to each ticket winning in a fair infinite lottery is not uniquely determined by the constraints governing a regular probability function correctly representing this lottery; there is underdetermination. To sharpen the contrast, a regular probability function correctly representing a fair finite lottery is real-valued and unique, whereas a regular probability function correctly representing a fair infinite lottery is non-Archimedean and not unique.

So, the probability assigned to each ticket winning in a fair finite lottery is fully determined by the constraints governing a regular probability function correctly representing this lottery. But this doesn't carry over to fair infinite lotteries, i.e. what regular probability to assign to each ticket winning in a fair infinite lottery is underdetermined by the constraints governing a regular probability function correctly representing this lottery. This may come as a surprise. After all, since fair lotteries are the main object of interest in this thesis, at least one of the constraints governing a probability function correctly representing a fair lottery over Ω (the sample space) is the following:

Fairness. *For all* $\omega, \omega' \in \Omega$, $\{\omega\}$ *is equiprobable to* $\{\omega'\}$.

Fairness and the probability axioms seem strong enough to uniquely determine the probability assigned to each ticket winning in fair lotteries. Indeed, in fair finite lotteries, **Fairness** and the probability axioms are strong enough to uniquely determine the probability assigned to each ticket winning. And by additivity, the probabilities assigned to other events in the same lottery will be uniquely determined too.

Furthermore, if probabilities are real-valued, then **Fairness** and the probability axioms are strong enough to uniquely determine the real-valued (albeit irregular) probability assigned to each ticket winning in a fair *infinite* lottery. Of course, if probabilities are real-valued, then **Regularity** must be dropped as a constraint on probabilities. Only when **Regularity** is a constraint on probabilities, **Fairness** and the probability axioms cannot uniquely determine what *regular* probability to assign to each ticket winning in fair infinite lotteries, which may come as a surprise.

Are there other constraints that can be added to **Fairness**, **Regularity**, and the probability axioms such that the probability assigned to each ticket winning in a fair lottery will always be uniquely determined? This is a complex question. At least for a fair lottery over $\mathbb{N}$, my answer is in the negative. But let's not get ahead of ourselves here. In order to even begin answering this question, a precise characterisation of the kind of underdetermination we're aiming to rule out needs to be given first. To do so, the notion of a 'probabilistic scenario' has to be defined. Following DiBella [MS], define a *probabilistic scenario* to be a triple $\langle \Omega, \mathcal{F}, C \rangle$ where Ω is the sample space (the set of possibilities), $\mathcal{F}$ is an algebra[7] on Ω, and C is the set of constraints on events in $\mathcal{F}$. To specify a probabilistic scenario is then to specify Ω, the algebra of events $\mathcal{F}$, and the set of constraints C on events in $\mathcal{F}$. Since I'm primarily interested in fair infinite lotteries in this thesis, I'm only interested in probabilistic scenarios $\langle \Omega, \mathcal{F}, C \rangle$ where Ω is infinite and **Fairness** is one of the constraints, though not necessarily the only constraint, in C. Note that because **Fairness** quantifies over all singleton sets of Ω, $\mathcal{F}$ must at least contain all the singleton sets in order for **Fairness** to be satisfied. Hereafter, a probability function P *correctly represents* a probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ iff P is defined on $\mathcal{F}$ and satisfies all the constraints in C.[8]

I will now state the kind of underdetermination I'm interested in without explanation first.

Infinitesimal Underdetermination. *With respect to a probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ where Ω is infinite and **Fairness** is one of the constraints, though not necessarily the only constraint, in C, infinitesimal underdetermination arises if and only if there exist at least two probability functions P_1 and P_2 such that all of the following hold:*

1. *P_1 and P_2 are both defined on $\mathcal{F}$ and both satisfy all the constraints in C.*

[7] That is, $\Omega \in \mathcal{F}$; if $A \in \mathcal{F}$, then $\neg A \in \mathcal{F}$; if $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

[8] See footnote 1 for the definition of a probability function.

2. *$P_1(A) \leq P_1(B)$ iff $P_2(A) \leq P_2(B)$ for all events A and B in $\mathcal{F}$.*

3. *There exists an event C in $\mathcal{F}$ where $P_1(C) < P_2(C)$ or $P_1(C) > P_2(C)$.*

4. *For all events A in $\mathcal{F}$, $|P_1(A) - P_2(A)| < \frac{1}{n}$ for all $n \in \mathbb{N}$.*

Explication is in order. First, note that **Infinitesimal Underdetermination** only applies to pairs of probability functions, at least one of which is non-Archimedean.

Theorem 1. *There are no real-valued probability functions P_1 and P_2 where $P_1 \neq P_2$ that satisfy all four conditions of **Infinitesimal Underdetermination**.*[9]

This does not mean that real-valued probability functions do not exhibit underdetermination whatsoever. Instead, **Theorem 1** just implies that real-valued probability functions do not exhibit **Infinitesimal Underdetermination**. For all we know, real-valued probability functions may exhibit some other kind of underdetermination distinct from **Infinitesimal Underdetermination**.

Why focus on **Infinitesimal Underdetermination** and not the more general phenomenon of underdetermination that may be exhibited even by real-valued probabilities? It's because I think that **Infinitesimal Underdetermination** is worthy of philosophical attention, as it is a *problematic* kind of underdetermination. More specifically, **Infinitesimal Underdetermination** consists of a set of conditions, the fulfillment of which by two probability functions P_1 and P_2 satisfies *another* condition required for my argument that our choosing P_1 over P_2 to represent a probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ where Ω is infinite and **Fairness** is one of the constraints in C is problematic. That is, if P_1 and P_2 fulfill **Infinitesimal Underdetermination**, then a further proposition q about P_1 and P_2 is true. My argument that there is something problematic about our choosing P_1 over P_2 to represent a probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ where Ω is infinite and **Fairness** is one of the constraints in C in turn requires proposition q to be true. I acknowledge that there is a lot to unpack here. But the details will be provided progressively in this introduction. Also, since it is such a mouthful to explain what I mean when I said that **Infinitesimal Underdetermination** is a problematic kind of underdetermination, I will just stick to saying that **Infinitesimal Underdetermination** is a problematic kind of underdetermination, instead of the clarification I've provided above.

Since **Infinitesimal Underdetermination** is problematic, we should find ways to rule it out. In earlier drafts of this thesis, instead of the current version of **Infinitesimal Underdetermination**, the earlier versions only had two conditions: conditions 2 and 3. Plus, the earlier versions didn't have the qualifications that Ω is infinite and **Fairness** is one of the constraints, though not necessarily the only constraint, in C. As a result, the earlier versions admitted real-valued probability functions representing all

[9]All proofs can be found in the Appendix.

kinds of probabilistic scenarios into my characterisation of underdetermination then. My aim in those earlier drafts was the same as my current aim: to argue that the kind of underdetermination I've characterised is problematic. However, I've received a litany of counterexamples, all of which fulfill my earlier characterisation of underdetermination but are intuitively *unproblematic*.[10] The striking similarity among all these counterexamples is that they are exclusively framed in terms of real-valued probabilities representing probabilistic scenarios that are *not* fair infinite lotteries. Because I had to expend a big part of the discussion arguing that there is something philosophically different between these counterexamples and the main object of interest in this thesis, namely non-Archimedean probabilities correctly representing fair infinite lotteries, the main object of interest was thus obscured by all these counterexamples.

So, I've decided to strengthen my characterisation of underdetermination once and for all in order to rule out all these counterexamples framed exclusively in terms of real-valued probabilities representing probabilistic scenarios that are not fair infinite lotteries. **Infinitesimal Underdetermination** is the result. Furthermore, as we will see soon, my argument for why **Infinitesimal Underdetermination** is problematic will not work for a characterisation of underdetermination that consists only of conditions 2 and 3 and lacks the qualifications that Ω is infinite and **Fairness** is one of the constraints in C.

Let's examine the conditions of **Infinitesimal Underdetermination** one by one, starting with the easiest: condition 3. This condition states that P_1 and P_2 assign different values to C, which implies that P_1 and P_2 are *different* probability functions.[11] For underdetermination to arise, there must be at least two different probability functions. If there were only one probability function defined on $\mathcal{F}$ that satisfied all the constraints in C, then there would be *no* underdetermination, as the constraints in C would fully determine the probability function.

Now, consider condition 1. This condition states that both P_1 and P_2 are defined on the same algebra of events $\mathcal{F}$. That is, they assign the *same* events probabilities, although it is not the case that they assign the same events the *same* probabilities, in virtue of

[10]Thanks to Alan Hájek, Michael Nielsen, Nicholas DiBella, and the anonymous referees for one of my papers for providing the counterexamples.

[11]Strictly speaking, there is no need to state that $P_1(C) < P_2(C)$ or $P_1(C) > P_2(C)$, as the former implies that $P_1(\neg C) > P_2(\neg C)$. As long as there is an event C in $\mathcal{F}$ such that $P_1(C) < P_2(C)$, there is always another event where the ordering is reversed, e.g. $P_1(\neg C) > P_2(\neg C)$. So, the second disjunct can be dropped. Also, note that condition 3 *cannot* be simplified to 'There exists an event C in $\mathcal{F}$ where $P_1(C) \neq P_2(C)$'. For if condition 3 were simplified this way, then **Infinitesimal Underdetermination** would admit probability functions P_1 and P_2 that are *not* comparable, i.e. there exists an event C in $\mathcal{F}$ such that $P_1(C) \nleq P_2(C)$ and $P_1(C) \ngeq P_2(C)$. When non-Archimedean probabilities are concerned, there will be incomparable probability functions, as there are many non-Archimedean fields. Say P_1 assigns to C a hyperreal value and P_2 assigns to C a value set in the Levi-Civita field. Then, $P_1(C) \nleq P_2(C)$ and $P_1(C) \ngeq P_2(C)$. For the purposes of this thesis, I'm not interested in probability functions like that. Does any part of my argument hang on this antecedent ruling out of incomparable probability functions? That is, is it true that if incomparable probability functions were not antecedently ruled out, my argument would not work? No, my argument would still work. However, for the sake of not complicating my argument unnecessarily, incomparable probability functions will be excluded from the discussion.

condition 3. Condition 1 also states that P_1 and P_2 satisfy all the constraints in C. That is, the constraints in C do not uniquely determine a probability function, which then gives rise to underdetermination.

Consider condition 2 now. This condition states that P_1 and P_2 order the same events the same way. Why is this condition needed? It is because the constraints in C may not fully specify the order of events in $\mathcal{F}$. That is, there may be some *order*-underdetermination, in that the constraints in C do not uniquely determine the order of events in $\mathcal{F}$. Order-underdetermination is itself philosophically interesting. After all, in some cases, the constraints in C do not uniquely determine a probability function exactly because the order of events is underdetermined. Though philosophically interesting, order-underdetermination will not be the focus of this thesis, as all the cases I will examine are such that the constraints in C *fully* determine the order of events in $\mathcal{F}$. Yet, the constraints in C are still not strong enough to uniquely determine a probability function. This may be surprising. After all, in a fair finite lottery, the constraints of this lottery fully determine the order of events, which in turn uniquely determines the probability function correctly representing this lottery. But this doesn't carry over to fair infinite lotteries. That is, we can specify a set of constraints C that fully determines the order of events in an algebra $\mathcal{F}$ on an infinite Ω. Yet, C is still not strong enough to uniquely determine a probability function. That said, although C fully determines the order of events in $\mathcal{F}$ in all the cases I will examine, condition 2 will still be included in **Infinitesimal Underdetermination** for the sake of precisely characterising the kind of underdetermination I'm interested in.

Lastly, consider condition 4. This condition states that P_1 and P_2 are *infinitely close* to each other, in that for all events A in $\mathcal{F}$, $|P_1(A) - P_2(A)| < \frac{1}{n}$ for all $n \in \mathbb{N}$. There are two reasons why condition 4 is included in **Infinitesimal Underdetermination**. I've already stated the first reason, which is to rule out all the counterexamples framed exclusively in terms of real-valued probabilities representing probabilistic scenarios that are not fair infinite lotteries. In earlier drafts of this thesis, I wanted to argue that my characterisation of underdetermination then (consisting only of conditions 2 and 3 and lacking the qualifications that Ω is infinite and **Fairness** is one of the constraints in C) is problematic. But because of all these counterexamples, though fulfilling my earlier characterisation of underdetermination but are still intuitively unproblematic, my argument then was not successful. So, I've added condition 4, as well as the qualifications that Ω is infinite and **Fairness** is one of the constraints in C, in hopes of ruling out all these counterexamples.

The reader may better appreciate my second reason for why I added condition 4 if I briefly introduce a notion essential to my argument for why **Infinitesimal Underdetermination** is problematic. An essential part of my argument is spelling out when two probability functions P_1 and P_2 where $P_1 \neq P_2$ have the same *representational power* in representing a given probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$. That is, I'm interested in the following question: exactly when are P_1 and P_2 *equally good* (or *bad*) models or representations of a given probabilistic scenario $\langle\Omega, \mathcal{F}, C\rangle$? Note that I'm not interested in spelling out what consists in the representational power of a *single* probability function. Instead, I'm interested in spelling out a set of conditions, which when fulfilled together, is sufficient for two different probability functions to have the same representational power in representing a given probabilistic scenario $\langle\Omega, \mathcal{F}, C\rangle$. In other words, I'm more interested in the comparison rather than the absolute. And I think that it is possible to answer when two different probability functions have the same representational power, without having to answer what consists in the representational power of a single probability function.[12]

So, what is this set of conditions, which when fulfilled together, is sufficient for two different probability functions to have the same representational power in representing a given probabilistic scenario $\langle\Omega, \mathcal{F}, C\rangle$? Here's my answer:

Same Representational Power. *With respect to a probabilistic scenario $\langle\Omega, \mathcal{F}, C\rangle$, if probability functions P_1 and P_2 satisfy conditions* 1, 2, *and* 4 *of* ***Infinitesimal Underdetermination****, then P_1 and P_2 have the same representational power in representing $\langle\Omega, \mathcal{F}, C\rangle$.*

Note that $\langle\Omega, \mathcal{F}, C\rangle$ need not be a fair infinite lottery. Also, note that it is possible for $P_1 = P_2$ in **Same Representational Power** since condition 3 of **Infinitesimal Underdetermination** is not included. In virtue of condition 1 of **Infinitesimal Underdetermination**, both P_1 and P_2 are defined on the same algebra of events $\mathcal{F}$ and both satisfy all the constraints in C. In virtue of condition 2, both order the same events the same way. And in virtue of condition 4, P_1 and P_2 are not just close to each other, but *infinitely* close to each other.[13]

As we will see soon, **Same Representational Power** plays an important role in my argument for why **Infinitesimal Underdetermination** is problematic. In fact, recall that I was alluding to a proposition q above such that if P_1 and P_2 fulfill **Infinitesimal Underdetermination**, then proposition q about P_1 and P_2 is true. Now, I can state what proposition q is. It is simply the proposition that P_1 and P_2 have the same representational power in representing a probabilistic scenario $\langle\Omega, \mathcal{F}, C\rangle$. Importantly, note that **Same Representational Power** cannot be used to argue that my earlier characterisation of

[12] Think of the weights of objects as an analogy. To find out whether two objects A and B have the same weight, we don't have to know what their individual weights are exactly. Just put A and B on a weighing *balance*. If the balance doesn't tip towards either object, then they have the same weight. Importantly, note that a weighing balance itself does not tell us how heavy each individual object is.

[13] Nicholas DiBella has commented that the satisfaction of condition 1 of **Infinitesimal Underdetermination** alone suffices for two different probability functions to have the same representational power. After all, both probability functions are defined on the same algebra of events $\mathcal{F}$ and both satisfy all the constraints in C. Intuitively, that alone seems enough for two different probability functions to have the same representational power. Nicholas may be right, but I'll keep **Same Representational Power** the way it is. If the satisfaction of condition 1 suffices for two different probability functions to have the same representational power, then, by entailment, **Same Representational Power** is right.

underdetermination, which consists only of conditions 2 and 3 and lacks the qualifications that Ω is infinite and **Fairness** is one of the constraints in C, is problematic. After all, conditions 1 and 4 feature in the statement of **Same Representational Power**. So, **Same Representational Power** only applies to a strict subset of all the cases fulfilling my earlier characterisation of underdetermination. However, in order to argue that my earlier characterisation of underdetermination is problematic, *all* cases need to be considered. This is why in earlier drafts of this thesis, I wasn't successful in arguing that my characterisation of underdetermination then was problematic. Therefore, for the sake of using **Same Representational Power** to argue that my characterisation of underdetermination is problematic, conditions 1 and 4 are added. And for the sake of ruling out all the counterexamples framed exclusively in terms of real-valued probabilities representing probabilistic scenarios that are not fair infinite lotteries, the qualifications that Ω is infinite and **Fairness** is one of the constraints in C are added, resulting in **Infinitesimal Underdetermination**.[14]

Say probability functions P_1 and P_2 satisfy all four conditions of **Infinitesimal Underdetermination** with respect to a given fair infinite lottery, i.e. $\langle \Omega, \mathcal{F}, C \rangle$ where Ω is infinite and **Fairness** is one of the constraints in C. In virtue of **Same Representational Power**, P_1 and P_2 represent equally well $\langle \Omega, \mathcal{F}, C \rangle$. Yet, in virtue of condition 3 of **Infinitesimal Underdetermination**, $P_1 \neq P_2$. This means that the constraints in C underdetermine a probability function. Contrast this with a fair *finite* lottery. With respect to a fair finite lottery, i.e. $\langle \Omega, \mathcal{P}\Omega, C \rangle$ where Ω is finite, $\mathcal{P}\Omega$ is the powerset (set of all subsets) of Ω, and **Fairness** and the probability axioms are the *only* constraints in C, C uniquely determines *the* probability function correctly representing $\langle \Omega, \mathcal{P}\Omega, C \rangle$; there is no underdetermination. After all, this probability function is real-valued and in virtue of **Theorem 1**, there are no real-valued probability functions P_1 and P_2 where $P_1 \neq P_2$ that satisfy all four conditions of **Infinitesimal Underdetermination**.

[14] To be sure, in earlier drafts of this thesis, I employed a stronger version of **Same Representational Power**, which stated that with respect to a probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$, if probability functions P_1 and P_2 satisfy condition 2 of **Infinitesimal Underdetermination**, then P_1 and P_2 have the same representational power in representing $\langle \Omega, \mathcal{F}, C \rangle$. However, Michael Nielsen gave a counterexample to this stronger version of **Same Representational Power**. Consider tossing a coin that's slightly biased towards tails. Let P_1 assign $\{H\}$ a value of 0.49 and P_2 assign $\{H\}$ a value of 0.1. Between these two probability functions, the order of events is preserved: $\emptyset$, $\{H\}$, $\{T\}$, $\{H, T\}$ in increasing probability. So, P_1 and P_2 satisfy the stronger version of **Same Representational Power**. But intuitively, P_1 is a better representation of the probabilistic scenario than P_2; they don't have the same representational power. In fact, it is plausible that P_1 is a *correct* representation of the scenario, while P_2 is a *misrepresentation*. Thus, the stronger version of **Same Representational Power** is wrong. Here's my diagnosis of what went wrong: P_1 and P_2 are too 'far apart'. After all, $P_1(\{H\}) - P_2(\{H\}) = 0.39$. If this diagnosis is correct, then adding the requirement that the two relevant probability functions must be *infinitely* close to each other will rule out all analogous counterexamples. This explains the current statement of **Same Representational Power**.

1.3 Correct Numerical Models

Up to now, we've been talking about when probability functions exhibit **Infinitesimal Underdetermination**. But are there even probability functions that satisfy all four conditions of **Infinitesimal Underdetermination**? That is, do these probability functions *exist* in the first place? After all, if these probability functions do not exist at all, then there is no point in talking about **Infinitesimal Underdetermination**. Well, there *are* probability functions that satisfy all four conditions of **Infinitesimal Underdetermination**. First, note that Bernstein & Wattenberg [1969] have proven that there exist regular and non-Archimedean probability functions correctly representing a fair lottery over $[0, 1]$.[15] Second, Pruss [2021] has proven that if there exists a non-Archimedean probability function P_1, then there exist uncountably many other non-Archimedean probability functions P_2 set in the same non-Archimedean field as P_1 such that P_1 and P_2 satisfy all four conditions of **Infinitesimal Underdetermination**.[16] Note that P_1 and P_2 in Pruss [2021]'s result can correctly represent other probabilistic scenarios beside fair infinite lotteries and both need not be regular. Combine the results by Bernstein & Wattenberg [1969] and Pruss [2021] to see that there exist regular and non-Archimedean probability functions P_1 and P_2, both of which correctly represent a fair lottery over $[0, 1]$, that satisfy all four conditions of **Infinitesimal Underdetermination**. Nothing about a fair lottery over $[0, 1]$ uniquely determines P_1 as *the* probability function correctly representing it and nothing about the lottery rules out P_2 as a probability function correctly representing it. That is, both P_1 and P_2 correctly represent a fair lottery over $[0, 1]$. After all, in virtue of **Same Representational Power**, P_1 and P_2 represent equally well a fair lottery over $[0, 1]$.

Now, define a *numerical model* $\mathcal{M}$ of a probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ to be a quadruple $\langle \Omega, \mathcal{F}, C, \Sigma \rangle$ where Σ is the set of all ordered pairs (A, B) for every event $A \in \mathcal{F}$ and B is the probability assigned to A. Here's an example. Consider a fair coin toss. We have $\Omega = \{H, T\}$ and let $\mathcal{P}\{H, T\}$ denote the powerset of $\{H, T\}$. In this example, C will contain only **Fairness** and the probability axioms. For this probabilistic scenario $\langle \{H, T\}, \mathcal{P}\{H, T\}, C \rangle$, $\Sigma = \{(\emptyset, 0), (\{H\}, 0.5), (\{T\}, 0.5), (\{H, T\}, 1)\}$. The quadruple $\langle \{H, T\}, \mathcal{P}\{H, T\}, C, \Sigma \rangle$ is then a numerical model for this fair coin toss scenario, i.e. $\mathcal{M}_1 = \langle \{H, T\}, \mathcal{P}\{H, T\}, C, \Sigma \rangle$. Note that the probability assigned to A

[15] In addition, Wenmackers & Horsten [2013] have proven that there exist regular and non-Archimedean probability functions correctly representing a fair lottery over $\mathbb{N}$. Also, Benci et al. [2013] have proven that there exist regular and non-Archimedean probability functions correctly representing each of the following scenarios: (i) a fair lottery over $\mathbb{N}$, (ii) a fair lottery over $\mathbb{Q}$, (iii) a fair lottery over $\mathbb{R}$, and (iv) fair infinite coin tosses.

[16] To be sure, Pruss [2021] did not prove the above statement directly. Instead, he proved a theorem that implies the above statement. Also, note that the theorem Pruss [2021] proved is only true in totally ordered non-Archimedean fields such that every finite number in the field is *infinitely close* to a real number, i.e. letting x be a finite number in the field, there is always a real number r such that $|x - r| < \frac{1}{n}$ for all $n \in \mathbb{N}$. The reader can refer to Pruss [2021] for more details.

need not be a precise probability under this definition of a numerical model. That is, an imprecise probability can be assigned to A instead. Here's an example. Consider another coin toss, but the coin is biased towards tails this time. Say a probability of $(0, \frac{1}{2})$ is assigned to $\{H\}$ and a probability of $(\frac{1}{2}, 1)$ is assigned to $\{T\}$. Let $\Sigma' = \{(\emptyset, 0), (\{H\}, (0, \frac{1}{2})), (\{T\}, (\frac{1}{2}, 1)), (\{H, T\}, 1)\}$ and let C' contain the appropriate constraints for this probabilistic scenario. The quadruple $\langle \{H, T\}, \mathcal{P}\{H, T\}, C', \Sigma' \rangle$ is then a numerical model for this second coin toss scenario, i.e. $\mathcal{M}_2 = \langle \{H, T\}, \mathcal{P}\{H, T\}, C', \Sigma' \rangle$.

Next, call a numerical model $\mathcal{M} = \langle \Omega, \mathcal{F}, C, \Sigma \rangle$ of a probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ *correct* just in case the probability assignment to the events in $\mathcal{F}$ satisfies all the constraints in C. So, the numerical model $\mathcal{M}_1 = \langle \{H, T\}, \mathcal{P}\{H, T\}, C, \Sigma \rangle$, as defined in the previous paragraph, is a correct numerical model for the fair coin toss scenario. Here's an example of an *incorrect* numerical model of the same scenario. Let $\Sigma^* = \{(\emptyset, 0), (\{H\}, 0.1), (\{T\}, 0.9), (\{H, T\}, 1)\}$. The numerical model $\mathcal{M}_3 = \langle \{H, T\}, \mathcal{P}\{H, T\}, C, \Sigma^* \rangle$ is an incorrect numerical model for the fair coin toss scenario, as the probability assignment to the events in $\mathcal{P}\{H, T\}$ does not satisfy **Fairness**. In fact, for a fair finite lottery with n tickets, there is only one correct numerical model for the scenario.

Back to a fair lottery over $[0, 1]$. Since P_1 and P_2 both correctly represent this lottery, there are multiple—in fact uncountably many—correct numerical models of this lottery. Without getting too much into the technical details, let the probabilistic scenario $\langle [0, 1], \mathcal{F}, C \rangle$ be a fair lottery over $[0, 1]$. The reader can refer to Bernstein & Wattenberg [1969] for the exact specification of $\mathcal{F}$ and for the constraints that are in C. P_1 and P_2 are both defined on $\mathcal{F}$ and both satisfy all the constraints in C. Define $\Sigma_1 = \{(A, P_1(A)) : A \in \mathcal{F}\}$ and $\Sigma_2 = \{(A, P_2(A)) : A \in \mathcal{F}\}$. Then, $\langle [0, 1], \mathcal{F}, C, \Sigma_1 \rangle$ and $\langle [0, 1], \mathcal{F}, C, \Sigma_2 \rangle$ are two different correct numerical models—among uncountably many others—of $\langle [0, 1], \mathcal{F}, C \rangle$, as P_1 and P_2 both correctly represent $\langle [0, 1], \mathcal{F}, C \rangle$. Importantly, note that among all the correct numerical models of $\langle [0, 1], \mathcal{F}, C \rangle$, there may be some numerical models that assign *imprecise* probabilities to some events in $\mathcal{F}$.

1.4 The *Chosen* Correct Numerical Model

Here's where things start to get tricky. Though no particular numerical model is *the* correct numerical model of a fair lottery over $[0, 1]$, in order to perform calculations involving probabilities (precise or imprecise) assigned to events in this lottery, a particular correct numerical model has to be *chosen* first. Here's an example of such a calculation. Consider the following bet:

> Bet A: If the number chosen in a fair lottery over $[0, 1]$ is 0.5, then you'll be awarded $\frac{1}{\varepsilon}$ units of utility. Otherwise, no units of utility will be awarded or lost.

In order to calculate bet A's expected utility, a particular correct numerical model of a fair lottery over $[0,1]$ has to be chosen first. For if no correct numerical model were chosen, then how could we possibly calculate bet A's expected utility?

Note that the calculation of expected utilities is just an illustration. After all, probability theory has widespread application, e.g. calculation of the expected value, probabilistic logic, predictive modelling, risk management, reliability theory, etc. My general point is that in at least some applications of probability theory, a particular correct numerical model has to be chosen first in order to perform that application.

For a moment, think of world maps as an analogy. Say, for a given world map, a colour must be chosen to represent Australia in order to complete the map. We can choose either red or green to represent Australia. Both these colours are good representations of Australia. But for the sake of completing the map, a particular colour has to be chosen. If no colour is chosen, then the map will never be completed. In fact, for a given map, we cannot choose *both* colours to represent Australia. That is, for a given map, once we've chosen red to represent Australia, then we cannot choose green to represent Australia and vice versa. Only one colour can be chosen to represent Australia in a given map.[17] Now, back to probabilities. For some applications of probability theory, e.g. calculation of expected utilities, a particular correct numerical model has to be chosen first in order to perform that application, just like how a particular colour has to be chosen first to represent Australia in order to complete that given map.

So, although there is no such thing as *the* correct numerical model of a fair lottery over $[0,1]$, the phrase '*the chosen correct numerical model*' is appropriate when referring to the particular correct numerical model that has been chosen. Since there are uncountably many correct numerical models of a fair lottery over $[0,1]$, a natural follow-up question arises: given that a choice has to be made, exactly which correct numerical model do we choose then? After all, there is a considerable degree of freedom in making this choice, as there are uncountably many correct numerical models to choose from. The freedom we have in choosing a particular correct numerical model among uncountably many such models translates to a freedom in assigning a probability to an event C, to which different correct numerical models assign different probabilities. For example, let $C = \{\omega\}$ where $\omega \in [0,1]$. Suppose $P_1(\{\omega\}) = \varepsilon$ while $P_2(\{\omega\}) = 2\varepsilon$. If the chosen correct numerical model is $\langle [0,1], \mathcal{F}, C, \Sigma_1 \rangle$, as defined a few paragraphs ago, then we have chosen to assign a probability value of ε to $\{\omega\}$. If the chosen correct numerical model is $\langle [0,1], \mathcal{F}, C, \Sigma_2 \rangle$ instead, as defined a few paragraphs ago, then we have chosen to assign a probability value of 2ε to $\{\omega\}$. What probability value to assign to $\{\omega\}$ is, in a sense, up to us.

Importantly, note that the question *isn't* the following: given that a choice has to be made, exactly which correct numerical model *did* we choose? This question suggests that a choice has already been made. And answering it is straightforward: just ask ourselves which model we chose to arrive at the answer. Instead, the question relevant

[17] Thanks to Alan Hájek for providing this analogy.

to this thesis concerns the deliberative process *before* making a choice. Since there are uncountably many correct numerical models of a fair lottery over $[0, 1]$ and a choice of a particular correct numerical model has to be made, exactly which correct numerical model *do* we choose? Now, answering this latter question is not straightforward.

Contrast this with fair finite lotteries. For a fair finite lottery with n tickets, there is only one correct numerical model for the scenario. There is no need to *deliberate* on which correct numerical model to choose, as there is only one. If we have to choose a correct numerical model, then the choice is obvious. Put in another way, there is *no* freedom in deciding which correct numerical model to choose. If a choice has to be made, we must choose the only existing correct numerical model. What probability values to assign to events in this lottery is *not* up to us, as they are fully determined by **Fairness** and the probability axioms.

Consider again the correct numerical models $\langle[0, 1], \mathcal{F}, C, \Sigma_1\rangle$ and $\langle[0, 1], \mathcal{F}, C, \Sigma_2\rangle$ of a fair lottery over $[0, 1]$, as defined a few paragraphs ago. Say we've decided after deliberation to choose $\langle[0, 1], \mathcal{F}, C, \Sigma_1\rangle$ and not $\langle[0, 1], \mathcal{F}, C, \Sigma_2\rangle$. Is doing so problematic? More generally, say we've decided after deliberation to choose $\langle[0, 1], \mathcal{F}, C, \Sigma_1\rangle$ and not uncountably many other correct numerical models. Is doing so problematic? Why or why not? Well, if a particular correct numerical model 'stands out' above uncountably many other correct numerical models, then there is no problem in choosing that model over others. Perhaps we have good reason to choose that model over others. That's why that model 'stands out' above the rest in the first place. In fact, if we have good reason to choose that particular correct numerical model over others, then we may even be rationally *required* to choose that model; there is no freedom in our choice.

However, it isn't obvious that there is always a particular correct numerical model that 'stands out' above the rest. Sometimes, we may not have good reason to single out a particular correct numerical model among uncountably many others. In cases like that, it isn't obvious exactly which correct numerical model to choose. But this doesn't mean that we should therefore choose *none*. For the sake of performing some application of probability theory, a choice of a particular correct numerical model has to be made regardless. And depending on exactly which correct numerical model is chosen for that application of probability theory, we may end up with some results that wouldn't be true if some *other* correct numerical model were chosen instead.

For example, recall bet A. Now, consider bet B.

> Bet B: If the result of a fair coin toss is heads, then you'll be awarded 2 units of utility. Otherwise, no units of utility will be awarded or lost.

Bet B's expected utility is $\frac{1}{2} \times 2 = 1$ unit. Depending on exactly which correct numerical model of a fair lottery over $[0, 1]$ is chosen, bet A's expected utility changes. If the chosen correct numerical model is $\langle[0, 1], \mathcal{F}, C, \Sigma_1\rangle$, as defined a few paragraphs ago, then bet A's expected utility is $\varepsilon \times \frac{1}{\varepsilon} = 1$ unit. As prescribed by expected utility reasoning, we should be indifferent between choosing bet A and choosing bet B. After all, the expected utilities of both bets are equal: 1 unit each. However, if the chosen correct numerical model is $\langle [0,1], \mathcal{F}, C, \Sigma_2 \rangle$ instead, as defined a few paragraphs ago, then bet A's expected utility is $2\varepsilon \times \frac{1}{\varepsilon} = 2$ units. As prescribed by expected utility reasoning, we should strictly prefer choosing bet A to choosing bet B. After all, bet A's expected utility is higher than bet B's. So, depending on exactly which correct numerical model of a fair lottery over $[0,1]$ is chosen, the prescribed course of action by expected utility reasoning changes.

It is in these instances like the example given above that I argue that our choosing a *particular* correct numerical model over others becomes problematic. To be clear, I'm talking about cases where it seems no particular correct numerical model 'stands out' above the rest, yet a choice of a particular correct numerical model has to be made regardless. And depending on exactly which correct numerical model is chosen, we end up with some results that wouldn't be true if some *other* correct numerical model were chosen instead. In the example detailed above, there is an *observable* change in betting behaviour, depending on exactly which correct numerical model is chosen. More details and nuances will be provided soon in the next section.

1.5 Aims

One of the main aims of this thesis is to provide sufficient conditions for when our choosing a *particular* correct numerical model over other correct numerical models becomes problematic.[18] Note that these sufficient conditions concern only fair infinite lotteries and not probabilistic scenarios in general. After I've listed the sufficient conditions, I will limit my attention in particular to the specific probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C \rangle$ where **Fairness** and **Regularity**[19] are two of the constraints in C (though not the only two), $\mathcal{F}$ is the *smallest* algebra required to satisfy all the constraints in C, and the constraints in C fully specify the order of events in $\mathcal{F}$. Why focus on this specific probabilistic scenario? More details will be given soon in the next section. But after specifying $\langle \mathbb{N}, \mathcal{F}, C \rangle$, I aim to show that there are two probability functions P_1 and P_2 correctly representing $\langle \mathbb{N}, \mathcal{F}, C \rangle$ that satisfy all four conditions of **Infinitesimal Underdetermination**. From these two probability functions P_1 and P_2, two different correct numerical models expressed in terms of precise probabilities of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ can be defined. I will then show that choosing one correct numerical model of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ over the other satisfies the sufficient conditions for when a choice like this is problematic. Therefore, choosing one model over the other is

[18] Why have I italicized the word 'particular'? It is to emphasize that the sufficient conditions concern when our choosing model A over model B becomes problematic. These sufficient conditions do not concern when it becomes problematic to be faced with an inevitable choice of choosing *at least one* model. That is, I'm more interested in how that choice unfolds rather than the fact that we are faced with such a choice.

[19] Strictly speaking, I'm not including **Regularity** in C but a qualitative version of it: for all nonempty events $A \in \mathcal{F}$, A is more probable than $\emptyset$.

problematic. Next, I explore potential constraints that we can add to C such that the new collection of constraints uniquely determines *the* probability function defined on $\mathcal{F}$ that satisfies all the constraints in C', which contains the old constraints in C and also the new added ones, i.e. $C \subseteq C'$. However, I argue that there is no such constraint that can be added to C for which an intuitive justification can be provided.

Note that strictly speaking, the probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C' \rangle$ is a different probabilistic scenario from $\langle \mathbb{N}, \mathcal{F}, C \rangle$. So, it may reasonably be wondered whether adding a constraint to C even *addresses* the problem concerning the particular probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C \rangle$. After all, once a constraint is added to C, we are talking about a wholly different probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C' \rangle$. Here's my reply. This worry isn't too serious, because probabilistic scenarios like $\langle \mathbb{N}, \mathcal{F}, C' \rangle$ and $\langle \mathbb{N}, \mathcal{F}, C \rangle$ can be understood as *tokens* of the more general *type* of probabilistic scenarios, namely a fair lottery over $\mathbb{N}$ in this case. In particular, call $\langle \Omega, \mathcal{F}, C \rangle$ a *token probabilistic scenario* hereafter and define a *type probabilistic scenario* as a set of token probabilistic scenarios $\langle \Omega, \mathcal{F}, C \rangle$ that satisfy a common specification. For example, a fair lottery over $\mathbb{N}$ is a set of $\langle \Omega, \mathcal{F}, C \rangle$ where $\Omega = \mathbb{N}$ and **Fairness** is one of the constraints in C. In this framework, there can be a hierarchy of different types of probabilistic scenarios. Think of a fair lottery over $\mathbb{N}$ again. The most general type of a fair lottery over $\mathbb{N}$ is the set of all $\langle \Omega, \mathcal{F}, C \rangle$ where $\Omega = \mathbb{N}$ and **Fairness** is one of the constraints in C. But there may be token probabilistic scenarios with different algebras in this set. A less general type of a fair lottery over $\mathbb{N}$ is the set of all $\langle \Omega, \mathcal{F}, C \rangle$ where $\Omega = \mathbb{N}$, $\mathcal{F}$ is fixed for all $\langle \Omega, \mathcal{F}, C \rangle$ in the set, and **Fairness** is one of the constraints in C. That is, the latter type is a subset of the former type.

Once we recognise the distinction between token probabilistic scenarios and type probabilistic scenarios, it isn't worrying that we switch from talking about token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C \rangle$ to talking about another token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C' \rangle$, upon adding a constraint to C. This is because both token probabilistic scenarios still fall under a certain type probabilistic scenario, an example of which is the set of all $\langle \Omega, \mathcal{F}, C \rangle$ where $\Omega = \mathbb{N}$, $\mathcal{F}$ is fixed for all $\langle \Omega, \mathcal{F}, C \rangle$ in the set, and **Fairness** is one of the constraints in C. Call a type probabilistic scenario *basic with respect to* $\langle \Omega, \mathcal{F}, C \rangle$ just in case all other token probabilistic scenarios $\langle \Omega', \mathcal{F}', C' \rangle$ in the set are such that $\Omega = \Omega'$, $\mathcal{F} = \mathcal{F}'$, and $C \subseteq C'$. Now, here's a rephrasing of my claim a paragraph ago in this framework. I claim that in the type probabilistic scenario basic with respect to $\langle \mathbb{N}, \mathcal{F}, C \rangle$, there are no token probabilistic scenarios $\langle \mathbb{N}, \mathcal{F}, C' \rangle$ where $C \subsetneq C'$, the constraints of which an intuitive justification can be provided for.

Choosing a particular correct numerical model of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ over another, both of which are expressed in terms of precise probabilities, is problematic. How can we then avoid this problem? Well, one solution is to choose *neither* model. But hold on, isn't refraining to choose at all when a choice has to be made itself a *bigger* problem? If no correct numerical model were chosen, then how could we possibly apply probability theory to—say, calculate expected utilities? My reply is simple. Not all correct numerical models of $\langle \mathbb{N}, \mathcal{F}, C\rangle$ are expressed exclusively in terms of precise probabilities; there is at least one correct numerical model that is expressed in terms of precise *and* imprecise probabilities. We can always choose the latter model. After all, as the argument currently stands, it seems to only be problematic when a particular correct numerical model expressed exclusively in terms of precise probabilities is chosen over another model like that.

Indeed, some authors, e.g. Benci et al. [2018] and Pruss [2021], have suggested that the chosen correct numerical model need not be exclusively in terms of precise probabilities. That is, *imprecise* probabilities can be assigned to events in $\mathcal{F}$ too. So, it should be allowed that the chosen correct numerical model can assign some events imprecise probabilities. Moreover, there seems to be only one correct numerical model of $\langle \mathbb{N}, \mathcal{F}, C\rangle$ expressed in terms of precise and imprecise probabilities, whereas there are uncountably many correct numerical models of the same scenario expressed exclusively in terms of precise probabilities. In this sense, *the* former correct numerical model of $\langle \mathbb{N}, \mathcal{F}, C\rangle$ 'stands out' from the rest of the models. After all, there seems to be only one correct numerical model of $\langle \mathbb{N}, \mathcal{F}, C\rangle$ expressed in terms of precise and imprecise probabilities. This apparent fact, by itself, may give us good enough reason to choose the correct numerical model expressed in terms of precise and imprecise probabilities to represent $\langle \mathbb{N}, \mathcal{F}, C\rangle$.

However, I will show that there are in fact multiple correct numerical models of $\langle \mathbb{N}, \mathcal{F}, C\rangle$ expressed in terms of precise and imprecise probabilities. So, the assertion that there is only one such model is *false*. Here's the main reason why there are multiple—in fact, uncountably many—such correct numerical models. Recall that for a fair infinite lottery, no real-valued probability functions correctly representing this scenario are regular. Therefore, a regular probability function correctly representing a fair infinite lottery must be non-Archimedean. However, there is a property satisfied by the reals that isn't satisfied by any non-Archimedean field. Here's that property:

Least Upper Bound Property. *Let $(X, \leq)$ be a partially-ordered set.*[20] *For all nonempty subsets $A \subseteq X$ that have an upper bound in X, A has a least upper bound in X. That is, if there exists an $x \in X$ such that $y \leq x$ for all $y \in A$ (A has x as an upper bound), then there exists a $\ell \in X$ such that:*

- *$y \leq \ell$ for all $y \in A$. (ℓ is an upper bound for A.)*
- *For all $x \in X$ such that $y \leq x$ for all $y \in A$, $\ell \leq x$. (ℓ is the least upper bound for A.)*

Before showing why non-Archimedean fields do not satisfy the **Least Upper Bound Property**, the concept of an ordered field needs to be defined first. An *ordered field* is a

[20]That is, X is the ground set and $\leq$ is a binary relation between members of X such that for all $a, b, c \in X$, (i) $a \leq a$, (ii) if $a \leq b$ and $b \leq c$, then $a \leq c$, and (iii) if $a \leq b$ and $b \leq a$, then $a = b$. Hereafter, call $\leq$ a *partial order* iff $\leq$ satisfies (i)-(iii).

field[21] $(F, +, \cdot)$ equipped with a total order[22] $\leq$ defined on F such that:

- **Addition Preserves Order**: For all $x, y, z \in F$, if $x \leq y$, then $x + z \leq y + z$.
- **Multiplication Preserves Order**: For all $x, y \in F$, if $0 \leq x$ and $0 \leq y$, then $0 \leq x \cdot y$.

Hereafter, let $(F, +, \cdot, \leq)$ denote an ordered field where $\leq$ is a total order. The real numbers form an ordered field, as well as all the listed non-Archimedean fields from before, e.g. the hyperreals, the surreals, the Levi-Civita field, etc. Only ordered fields are relevant to this thesis. Also, it is important to note that all ordered fields are partially-ordered sets. After all, a total order is a partial order, but not vice versa.

Call an ordered field $(F, +, \cdot, \leq)$ *complete* just in case $(F, +, \cdot, \leq)$ satisfies the **Least Upper Bound Property**.[23] And call an ordered field $(F, +, \cdot, \leq)$ *Archimedean* just in case $(F, +, \cdot, \leq)$ satisfies the Archimedean property: for any two $x, y \in F$ such that $0 < x$ (i.e. $0 \leq x$ and $x \nleq 0$) and $0 < y$ (i.e. $0 \leq y$ and $y \nleq 0$), there exists an $n \in \mathbb{N}$ such that $x < n \cdot y$ (i.e. $x \leq n \cdot y$ and $n \cdot y \nleq x$).[24]

Theorem 2. *A complete ordered field $(F, +, \cdot, \leq)$ is Archimedean.*

Theorem 2 implies that all non-Archimedean ordered fields are *not* complete, in the sense that the **Least Upper Bound Property** isn't satisfied. In fact, for any complete ordered field $(F, +, \cdot, \leq)$, $(F, +, \cdot, \leq)$ is isomorphic to the reals.[25]

[21] See footnote 4 for the definition of a field.

[22] A binary relation $\leq$ on a set X is a *total order* iff $\leq$ is a partial order such that for all $a, b \in X$, either $a \leq b$ or $b \leq a$. See footnote 20 for the definition of a partial order.

[23] Note that $(F, +, \cdot, \leq)$ satisfies the **Least Upper Bound Property** iff it satisfies the following:

Greatest Lower Bound Property. *Let $(X, \leq)$ be a partially-ordered set. For all nonempty subsets $A \subseteq X$ that have a lower bound in X, A has a greatest lower bound in X. That is, if there exists an $x \in X$ such that $x \leq y$ for all $y \in A$ (A has x as a lower bound), then there exists a $g \in X$ such that:*

- *$g \leq y$ for all $y \in A$. (g is a lower bound for A.)*
- *For all $x \in X$ such that $x \leq y$ for all $y \in A$, $x \leq g$. (g is the greatest lower bound for A.)*

The proof of this claim is well-understood and therefore will not be reproduced here.

[24] Strictly speaking, the statement of the Archimedean property above implies that $\mathbb{N} \subseteq F$. But it is possible that $\mathbb{N} \nsubseteq F$. Regardless, for any ordered field $(F, +, \cdot, \leq)$, there is always a subset $A \subseteq F$ that 'imitates' $\mathbb{N}$. By **Identity**, there are distinguished elements 0_F and 1_F in F. Let $A = \{1_F, 1_F + 1_F, 1_F + 1_F + 1_F, \ldots\}$. Define $2_F = 1_F + 1_F$, $3_F = 1_F + 1_F + 1_F$, etc. So, $A = \{1_F, 2_F, 3_F, \ldots\}$. Note that there is a bijective function $f : A \to \mathbb{N}$ such that $f(n_F) = n$. Now that we've established that there is always a subset of F that 'imitates' $\mathbb{N}$, here's a more general statement of the Archimedean property: for any two $x, y \in F$ such that $0_F < x$ and $0_F < y$, there exists a $n_F \in A$ such that $x < n_F \cdot y$.

[25] Let $(\mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, \leq_{\mathbb{R}})$ denote the real field where $\leq_{\mathbb{R}}$ is a total order. The claim is that there exists a bijection $f : F \to \mathbb{R}$ such that:

- $f(x + y) = f(x) +_{\mathbb{R}} f(y)$.
- $f(x \cdot y) = f(x) \cdot_{\mathbb{R}} f(y)$.
- If $x \leq y$, then $f(x) \leq_{\mathbb{R}} f(y)$.

A bijection f between two ordered fields that satisfies the three conditions above is called an *isomorphism*. The proof of this claim is well-understood but tedious. Therefore, it will not be reproduced here.

It is this failure of the **Least Upper Bound Property** in non-Archimedean ordered fields that gives rise to multiple—in fact, uncountably many—correct numerical models of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ expressed in terms of precise and imprecise probabilities. More details will be given soon. Now, among all such models, it isn't obvious that there is always a particular correct numerical model that 'stands out' above the rest. Sometimes, we may not have good reason to single out a particular correct numerical model among uncountably many others. In cases like that, it isn't obvious exactly which correct numerical model (expressed in terms of precise and imprecise probabilities) to choose. So, it seems we are still stuck in the same predicament as before. Previously, when only correct numerical models expressed exclusively in terms of precise probabilities are considered, it was reasonable to ask the following question: given that a choice has to be made, exactly which correct numerical model do we choose? Now, say we restrict our choice to only those correct numerical models expressed in terms of precise and imprecise probabilities. Since there are multiple—in fact, uncountably many—models like that, the same question can reasonably be asked: given that a choice has to be made, exactly which correct numerical model do we choose?

Say we've decided after deliberation to choose *this* particular correct numerical model expressed in terms of precise and imprecise probabilities over uncountably many other models like that. Here's the next question: is doing so problematic? That is, does our choosing this particular correct numerical model expressed in terms of precise and imprecise probabilities over other models like that satisfy the sufficient conditions for when a choice like this is problematic? I will argue that the answer is *yes*. However, the technical groundwork needed to reach this answer is by no means straightforward. The main technical complication is that **Infinitesimal Underdetermination** and **Same Representational Power** need to be generalised in a manner that's applicable to imprecise probabilities too. Doing so is the other main aim of this thesis. This is no easy task and I fully recognise that my generalisations of **Infinitesimal Underdetermination** and **Same Representational Power** may not be convincing as generalisations of these principles.

Regardless, there is no reason not to try. As we will see soon, my generalisations of **Infinitesimal Underdetermination** and **Same Representational Power** are in terms of *qualitative* probabilities. In fact, in the technical framework of qualitative probabilities I'm working in, two different correct *qualitative* models of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ can be defined. The concept of a correct qualitative model will be introduced soon. Now, it isn't obvious that there is always a particular correct qualitative model that 'stands out' above the rest. Sometimes, we may not have good reason to single out a particular correct qualitative model among uncountably many others. In cases like that, it isn't obvious exactly which correct qualitative model to choose. So, again, it seems we are still stuck in the same predicament as before. And the same questions from before can reasonably be asked here too. Given that a choice has to be made, exactly which correct qualitative model do we choose? Say we've decided after deliberation to choose *this* particular correct qualitative model over uncountably many other models like that. Is doing so problematic? That is, does our choosing this particular correct qualitative model over other models like that satisfy the sufficient conditions for when a choice like this is problematic? Again, I will argue that the answer is *yes*.

The discussion above suggests that there are multiple correct models of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ within each *kind* of model. That is, there are multiple correct numerical models of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ expressed exclusively in terms of precise probabilities; there are also multiple correct numerical models of the same token probabilistic scenario expressed in terms of precise and imprecise probabilities; and there are multiple correct qualitative models of the same token probabilistic scenario. Though there are different kinds of models, the predicament we are stuck with in each case is the same. Since there are multiple correct models of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ within each kind of model, no particular model of which 'stands out' above the rest in the same kind, exactly which correct model within each kind do we choose, given that a choice has to be made? Say we've decided after deliberation to choose this *particular* correct model over other correct models within a certain kind of model. Is doing so problematic? I will argue that it is. So, it seems inevitable that we will run into a problem. A choice must be made, but no matter how that choice unfolds, there is a problem. And it is no good either to refrain from choosing, as refraining from choosing when a choice must be made is itself a bigger problem. We seem to be caught between a rock and a hard place. This is the predicament: there is a problem no matter what we choose and there is a problem if we don't choose.[26]

[26]Alan Hájek has argued that there is no real predicament here, by attempting to parody the situation. Here's his parody. Consider Buridan's Ass. There is a donkey equally hungry and thirsty. Equal distance, albeit in opposite directions, from the donkey are a stack of hay and a pail of water. Now, the donkey can decide to go to the stack of hay first or the pail of water first. If it does either, it survives. If it does neither, it dies. But because the donkey is equally hungry and thirsty and the stack of hay is no closer to the donkey than the pail of water and vice versa, neither option 'stands out' from the other. So, the donkey has no good reason to go to either first. Struggling to make a choice, the donkey consequently dies before it manages to make up its mind. What a dumb ass! It could've just chosen one and survived. Even though neither option 'stands out' from the other, choosing one regardless is intuitively not problematic, as the donkey gets to live. In fact, choosing one option may even be rationally *required*. But failing to choose is problematic, as this failure will cause the donkey to die.

According to Hájek, the same lesson applies to choosing a correct model for $\langle \mathbb{N}, \mathcal{F}, C \rangle$ over other correct models. There may be no correct model that 'stands out' above other correct models within a certain kind of model, just like how the stack of hay doesn't 'stand out' above the pail of water and vice versa. But that doesn't mean that choosing this *particular* correct model over other correct models is problematic, when a choice has to be made. This is just like how it isn't problematic for the donkey to choose the stack of hay over the pail of water or vice versa, even though neither 'stands out' above the other. After all, all these models are *correct* models and there is no problem in choosing a correct model. Just *choose* one! Don't be like the donkey in Buridan's Ass. In fact, not choosing when a choice has to be made is then a problem. So, the predicament I've spelled out above is not a real one, or so it is argued.

Here's my reply. Buridan's Ass is not analogous to choosing a correct model for $\langle \mathbb{N}, \mathcal{F}, C \rangle$ over other correct models. There is an important difference. Recall that I'm interested in cases where it seems no particular correct model 'stands out' above the rest, yet a choice of a particular correct model has to be

How can we then get out of this predicament? Well, perhaps *not* simply by allowing probabilities to be imprecise, as have been suggested by both Benci et al. [2018] and Pruss [2021]. For even if probabilities are allowed to be imprecise, we are still stuck in the same predicament as when only precise probabilities are considered. Here's a potentially better solution. First, recognise the fact that no correct model of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ 'stands out' above the rest is what gives rise to the predicament in each kind of model. If, within each kind of model, there were a correct model that 'stands out', then there would be no problem in choosing that model. So, in order to avoid the predicament, we just have to find that correct model that 'stands out'. But, as I've said above, there may not always be such a model. This leaves us with one option. Identify the exact constraints in C of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ that give rise to the multiplicity of correct models within a certain kind of model. Then, rule out those constraints as legitimate constraints on the most general type probabilistic scenario that $\langle \mathbb{N}, \mathcal{F}, C \rangle$ is a token of, on pain of producing an inevitable problem if those constraints were treated as legitimate. In our case, the relevant type probabilistic scenario is a fair lottery over $\mathbb{N}$, which is the set of all $\langle \Omega, \mathcal{F}, C \rangle$ where $\Omega = \mathbb{N}$ and **Fairness** is one of the constraints in C.

To be clear, say statement A in C of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ is what gives rise to the multiplicity of correct models within a certain kind of model. I'm suggesting that in order to avoid the predicament, statement A should not be treated as a legitimate constraint on a fair lottery over $\mathbb{N}$. That is, statement A should not be included in the C's of any $\langle \Omega, \mathcal{F}, C \rangle$ where $\Omega = \mathbb{N}$ and **Fairness** is one of the constraints in C. For if statement A were included, then we would run the risk of getting stuck in the same predicament all over again. But we want to avoid the predicament. That can be accomplished by ruling out statement A as a legitimate constraint on a fair lottery over $\mathbb{N}$. In other words, we *allow* for violations of statement A in a fair lottery over $\mathbb{N}$.

However, ruling out statement A as a legitimate constraint on a fair lottery over $\mathbb{N}$ is only half the job done. The reason is simple. Let C' be the set of constraints in C, except statement A, i.e. $C' \subseteq C$. All the correct models of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ will also be correct models of $\langle \mathbb{N}, \mathcal{F}, C' \rangle$, in virtue of $C' \subseteq C$. That is, if a correct model satisfies all the constraints in C, then that model will definitely satisfy all the constraints in C' too. This implies that

made regardless. But this characterisation is not all. The cases I'm interested in must fulfill a further characterisation. Depending on exactly which correct model is chosen, we end up with some results that wouldn't be true if some *other* correct model were chosen instead. That is, our choice of a particular correct model ultimately affects a result, which wouldn't be true if we were to choose differently. In the case of choosing a correct model for $\langle \mathbb{N}, \mathcal{F}, C \rangle$, an example of such a result may be our actions or preferences. After all, depending on exactly which correct model is chosen, the expected utilities of some of our actions change. Buridan's Ass seems importantly different from choosing a correct model for $\langle \mathbb{N}, \mathcal{F}, C \rangle$, in that this further characterisation isn't satisfied by Buridan's Ass. If the donkey chooses to go to the stack of hay first, it survives. But if the donkey instead chooses to go to the pail of water first, it still survives. No matter what the donkey's choice is, it survives. The same cannot be said of choosing a correct model for $\langle \mathbb{N}, \mathcal{F}, C \rangle$. Depending on exactly which correct model is chosen, our actions and preferences change. Therefore, it isn't obvious that Buridan's Ass is analogous to choosing a correct model for $\langle \mathbb{N}, \mathcal{F}, C \rangle$ over other correct models. More discussion to follow soon.

that model is also a correct model of $\langle \mathbb{N}, \mathcal{F}, C' \rangle$. In a sense, the token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C' \rangle$ 'inherits' all the correct models of $\langle \mathbb{N}, \mathcal{F}, C \rangle$. But if all the correct models of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ are 'inherited' by $\langle \mathbb{N}, \mathcal{F}, C' \rangle$, then $\langle \mathbb{N}, \mathcal{F}, C' \rangle$ will also 'inherit' the predicament. This is why ruling out statement A as a legitimate constraint on a fair lottery over $\mathbb{N}$ is only half the job done in avoiding the predicament. Here's the other half. We need to introduce a new constraint—call this constraint statement B—to C'—call this new set C^* containing the constraints in C' and statement B—such that all the correct models of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ are *not* correct models of $\langle \mathbb{N}, \mathcal{F}, C^* \rangle$. Note that $C' \subseteq C^*$ and $C \cap C^* = C'$. Also, note that $\langle \mathbb{N}, \mathcal{F}, C \rangle$, $\langle \mathbb{N}, \mathcal{F}, C' \rangle$, and $\langle \mathbb{N}, \mathcal{F}, C^* \rangle$ are all tokens of the general type probabilistic scenario of a fair lottery over $\mathbb{N}$, i.e. **Fairness** is a constraint in C, C', and C^*. Since $\langle \mathbb{N}, \mathcal{F}, C^* \rangle$ doesn't 'inherit' the correct models of $\langle \mathbb{N}, \mathcal{F}, C \rangle$, the predicament is avoided. On the one hand, while statement A is ruled out as a legitimate constraint on a fair lottery over $\mathbb{N}$, I will argue that statement B, on the other hand, should be a legitimate constraint on a fair lottery over $\mathbb{N}$. That is, statement B should be included in the C's of any $\langle \Omega, \mathcal{F}, C \rangle$ where $\Omega = \mathbb{N}$ and **Fairness** is one of the constraints in C. Independent reasons for thinking so will be provided later in this thesis.

As we will see soon, statement A is **Regularity** and its qualitative counterpart.[27] In fact, there are plenty of independent[28] reasons for rejecting **Regularity** and its qualitative counterpart as constraints on probabilities, e.g. see Hájek [MS], Williamson [2007], Hájek [2012], Easwaran [2014a], Pruss [2014], Parker [2019], Pruss [2021], etc. While these reasons will not be reproduced in this thesis, I will offer a potentially original and independent reason for why precise probabilities correctly representing a fair lottery over $\mathbb{N}$ should be set in a complete ordered field. My argument essentially relies on how the failure of the **Least Upper Bound Property** in non-Archimedean ordered fields implies that imprecise probabilities, derived from supervaluating over precise non-Archimedean probabilities correctly representing fair infinite lotteries, fail to satisfy a property I call *tightness*. I will then argue that imprecise probabilities should intuitively satisfy the property of tightness. More details to follow soon. As it turns out, requiring imprecise probabilities, derived from supervaluating over precise probabilities correctly representing fair infinite lotteries, to satisfy the property of tightness suggests that those precise probabilities are set in a complete ordered field. But since any complete ordered field is Archimedean (**Theorem 2**), probability functions correctly representing fair infinite lotteries will inevitably admit violations of **Regularity**, if they are set in these fields. After all, a regular probability function correctly representing a fair infinite lottery must be non-Archimedean. If precise probabilities correctly representing fair infinite lotteries are to be set in a complete ordered field, then **Regularity** cannot be a constraint on precise probabilities. Importantly, it cannot be a constraint on a fair lottery over $\mathbb{N}$.

[27] See footnote 19 for the statement of the qualitative counterpart of **Regularity**.

[28] Independent of the predicament spelled out above.

To be sure, while statement A is **Regularity** and its qualitative counterpart, statement B is the following, as well as its qualitative counterpart:

Archimedean Probability. *For all events $A \in \mathcal{F}$ such that $P(A) > 0$, there exists an $n \in \mathbb{N}$ such that $P(A) > \frac{1}{n}$.*[29]

The reason for rejecting statement A but accepting statement B as a legitimate constraint on a fair lottery over $\mathbb{N}$—or more generally, as a legitimate constraint on fair infinite lotteries—is the same: probabilities correctly representing fair infinite lotteries should be set in a complete ordered field, for the sake of ensuring that the imprecise probabilities, derived by supervaluating over those precise probabilities, satisfy the property of tightness. Since there is independent reason, apart from avoiding the predicament spelled out above, to reject statement A but accept statement B, no C's of any $\langle \Omega, \mathcal{F}, C \rangle$ where $\Omega = \mathbb{N}$ and **Fairness** is one of the constraints in C should contain statement A but instead, every such C should contain statement B. As explained above, doing so will avoid the predicament.

In fact, for the particular token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C^* \rangle$ I'm interested in where $\mathcal{F}$ is the *smallest* algebra required to satisfy all the constraints in C^*, C^* contains all the constraints in C, except for statement A which is replaced by statement B, and C^* fully determines the order of events in $\mathcal{F}$, there is only one correct numerical model of $\langle \mathbb{N}, \mathcal{F}, C^* \rangle$. Similarly, once qualitative probabilities are required to satisfy a qualitative analogue of **Archimedean Probability**, there is only one correct qualitative model of the same token probabilistic scenario. Since there is only one correct model of $\langle \mathbb{N}, \mathcal{F}, C^* \rangle$ within each kind of model, we will not be caught in the predicament mentioned above.

It is important to note that **Regularity** not being a constraint on precise probabilities doesn't imply that there must be a failure of **Regularity** in every correct numerical model of a token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$. It is easy to see why this is false. Just consider a token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ where Ω is *finite* and **Fairness** is one of the constraints in C, i.e. a fair finite lottery. Instead, **Regularity** not being a constraint on precise probabilities just implies that there *must* be some token probabilistic scenarios $\langle \Omega, \mathcal{F}, C \rangle$ which admit of correct numerical models with violations of **Regularity**.[30]

As far as I know, in the current literature, there are plenty of proofs showing that regular non-Archimedean probability functions exist, e.g. see Bernstein & Wattenberg [1969], Wenmackers & Horsten [2013], Benci et al. [2013], etc.[31] But I don't know of any rigorous attempts to derive imprecise probabilities by supervaluating over regular non-Archimedean probability functions. To be sure, Benci et al. [2018] and Pruss [2021] have

[29]The qualitative analogue of **Archimedean Probability** will be given later in this thesis.

[30]Similarly, **Regularity** being a constraint on precise probabilities implies that there *must not* be any token probabilistic scenarios $\langle \Omega, \mathcal{F}, C \rangle$ which admit of correct numerical models with violations of **Regularity**.

[31]It is interesting to note that all these papers only provide correct numerical models for fair infinite lotteries. It can reasonably be wondered whether there are regular non-Archimedean probability functions modelling other probabilistic distributions—say, a normal distribution, etc.

suggested supervaluating over a family of regular non-Archimedean probability functions to derive imprecise probabilities. However, I find no rigorous attempts to do so in either paper. Because of the lack of such an attempt, the challenges surrounding the derivation of imprecise probabilities by supervaluating over regular non-Archimedean probability functions are unknown, unexplored, and underappreciated. In fact, I suspect that this lacuna in the literature on regular non-Archimedean probabilities is the reason why the philosophical significance of the **Least Upper Bound Property** isn't fully appreciated. Regardless, in this thesis, I will show why this principle is important in the derivation of imprecise probabilities by supervaluating over precise probabilities. This demonstration of the importance of the **Least Upper Bound Property** may be original, given the lacuna in the literature discussed above. And since the argument I'm going to present against **Regularity** relies on this importance, my argument may similarly be original. While I don't intend my demonstration of the importance of the **Least Upper Bound Property** to be a knock-down argument against regular non-Archimedean probabilities, I do hope that it will at least prompt more research on how to derive imprecise probabilities by supervaluating over regular non-Archimedean probability functions.

The discussion above suggests that *contra* Benci et al. [2018] and Pruss [2021], simply allowing probabilities to be imprecise will not single out a particular correct numerical model of $\langle \mathbb{N}, \mathcal{F}, C \rangle$. After all, there isn't just one correct numerical model expressed in terms of precise and imprecise probabilities, but uncountably many. Since there isn't a particular correct numerical model that 'stands out' among all these models, we are still stuck with the predicament spelled out above, even though probabilities are already allowed to be imprecise. Instead, ruling out **Regularity** and its qualitative counterpart as legitimate constraints on a fair lottery over $\mathbb{N}$ and requiring probabilities to satisfy **Archimedean Probability** and its qualitative counterpart got us out of the predicament. In other words, allowing for violations of **Regularity** and its qualitative counterpart in a fair lottery over $\mathbb{N}$ got us out of the predicament.

However, the suggestion by Benci et al. [2018] and Pruss [2021] still has its merits. This is because even if probabilities are required to satisfy **Archimedean Probability** and its qualitative counterpart, there may still be other token probabilistic scenarios $\langle \Omega, \mathcal{F}, C \rangle$, for which there are multiple correct numerical models expressed exclusively in terms of precise probabilities set in a complete ordered field. As is the case with non-Archimedean probabilities, there isn't always a particular correct numerical model that 'stands out' among all these models. The predicament from before looms in the background again. However, this time, it can be easily avoided. I will show that it is possible to define a correct numerical model of $\langle \Omega, \mathcal{F}, C \rangle$ expressed in terms of precise and imprecise probabilities by supervaluating over all the correct numerical models expressed exclusively in terms of precise probabilities set in a complete ordered field. Furthermore, I will show that this correct numerical model I've defined of $\langle \Omega, \mathcal{F}, C \rangle$ is the *only* model that satisfies the property of tightness among all the correct numerical models expressed in terms of precise and imprecise probabilities. So, although there are many correct numerical models of $\langle \Omega, \mathcal{F}, C \rangle$ expressed exclusively in terms of precise probabilities set in a complete ordered field, there is only one correct numerical model expressed in terms of precise and imprecise probabilities that satisfies the property of tightness. In this sense, the latter model 'stands out' from the rest of the models. This fact, by itself, gives us good enough reason to choose this model, especially when imprecise probabilities should satisfy the property of tightness. Thus, we have avoided the predicament.

1.6 Structure of Thesis

Here's how the thesis will be structured. In §2, I will provide sufficient conditions for when our choosing a *particular* correct model over other correct models is problematic. In §3, I specify the exact token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C \rangle$ I'm interested in for the purposes of this thesis. That is, I specify $\mathcal{F}$ and the constraints in C. But because the constraints in C are in terms of *qualitative probabilities*, I will also introduce the notion of qualitative probabilities in this section. After specifying $\langle \mathbb{N}, \mathcal{F}, C \rangle$, I will show that there are two probability functions P_1 and P_2 correctly representing $\langle \mathbb{N}, \mathcal{F}, C \rangle$ that satisfy all four conditions of **Infinitesimal Underdetermination**. From these two probability functions P_1 and P_2, two different correct numerical models expressed in terms of precise probabilities of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ can be defined. I will then show that choosing one correct numerical model of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ over the other satisfies the sufficient conditions for when a choice like this is problematic. Therefore, choosing one model over the other is problematic. Next, I explore potential constraints that we can add to C such that the new collection of constraints uniquely determines *the* probability function defined on $\mathcal{F}$ that satisfies all the constraints in C', which contains the old constraints in C and also the new added ones, i.e. $C \subseteq C'$. However, I argue that there is no such constraint that can be added to C for which an intuitive justification can be provided.

In §4, Benci et al. [2018]'s and Pruss [2021]'s approach of allowing probabilities to be imprecise will be explored. In this section, I will show that there are in fact multiple correct numerical models of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ expressed in terms of precise and imprecise probabilities. So, the assertion that there is only one such model is false. The property of tightness will also be introduced in this section. After introducing the property of tightness, I will show that if we require imprecise probabilities to satisfy the property of tightness, then there are *no* correct numerical models of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ expressed in terms of precise and imprecise probabilities. So, with the property of tightness, Benci et al. [2018]'s and Pruss [2021]'s approach of allowing probabilities to be imprecise will not even get off the ground for $\langle \mathbb{N}, \mathcal{F}, C \rangle$.

In §5, I will show that our choosing a particular correct numerical model expressed in terms of precise and imprecise probabilities over other models like that satisfies the sufficient conditions for when a choice like this is problematic. But to do this, **Infinitesimal Underdetermination** and **Same Representational Power** need to be generalised first using the framework of qualitative probabilities. Then, in the framework of qualitative probabilities I'm working with, I will show that two different correct qualitative models of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ can be defined that satisfy all four conditions of the generalised version of **Infinitesimal Underdetermination**. Again, choosing one correct qualitative model of $\langle \mathbb{N}, \mathcal{F}, C \rangle$ over the other satisfies the sufficient conditions for when a choice like this is problematic. Now, recall that C^* contains all the constraints in C, except for statement A which is replaced by statement B. At this point, I will show that unlike $\langle \mathbb{N}, \mathcal{F}, C \rangle$, there is only one correct numerical model and only one correct qualitative model of $\langle \mathbb{N}, \mathcal{F}, C^* \rangle$. In fact, I will also show that there is a representation-theoretic principle 'linking' these two models. But most importantly, since there is only one correct model within each kind of model, the predicament spelled out above is avoided.

The focus of §6 will be on token probabilistic scenarios $\langle \Omega, \mathcal{F}, C \rangle$ in general that admit multiple correct numerical models expressed exclusively in terms of precise probabilities set in a complete ordered field. I will show that it is possible to define a correct numerical model of $\langle \Omega, \mathcal{F}, C \rangle$ expressed in terms of precise and imprecise probabilities by supervaluating over all the correct numerical models expressed exclusively in terms of precise probabilities set in a complete ordered field. Furthermore, I will show that this correct numerical model I've defined of $\langle \Omega, \mathcal{F}, C \rangle$ is the *only* model that satisfies the property of tightness among all the correct numerical models expressed in terms of precise and imprecise probabilities. So, although there are many correct numerical models of $\langle \Omega, \mathcal{F}, C \rangle$ expressed exclusively in terms of precise probabilities set in a complete ordered field, there is only one correct numerical model expressed in terms of precise and imprecise probabilities that satisfies the property of tightness. In this sense, the latter model 'stands out' from the rest of the models. This fact, by itself, gives us good enough reason to choose this model, especially when imprecise probabilities should satisfy the property of tightness. Thus, we have avoided the predicament.

Note that §1 through §6 are the main parts of this thesis. §7 is mainly a response to a comment given by Nicholas DiBella. In §7, I will discuss why the usual conception of the nature of a qualitative probability in the literature is mistaken. In §8, I conclude the thesis. The proofs of all the theorems are in the Appendix.

2 A Problematic Choice

In this section, I will provide sufficient conditions for when our choosing a particular correct numerical model over other correct numerical models is problematic. Let $\langle \Omega, \mathcal{F}, C \rangle$ be any token probabilistic scenario and let $\mathcal{M}_1 = \langle \Omega, \mathcal{F}, C, \Sigma_1 \rangle$ and $\mathcal{M}_2 = \langle \Omega, \mathcal{F}, C, \Sigma_2 \rangle$ be two correct numerical models of $\langle \Omega, \mathcal{F}, C \rangle$. Here, I may refer to $\mathcal{M}_1$ and $\mathcal{M}_2$ as *probabilistic models* as well. Call a token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ *fully specified* just in case (i) Ω, $\mathcal{F}$, and C are all specified and (ii) the constraints in C fully determine the order of events in $\mathcal{F}$. In other words, (ii) states that for any two events A and B in $\mathcal{F}$, according to the constraints in C, either A is at most as probable as B or A is at least as probable as B. Call a token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ *under-specified* just in case it is not fully specified.

In the introduction, I said that choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ is *problematic* if doing so fulfills all the sufficient conditions for when a choice like this is problematic. This phrasing suggests that our choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ is either problematic or not, *simpliciter*. However, strictly speaking, I do *not* assume this. Instead, the set of conditions I'm about to spell out concerns when our choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ is problematic *with respect to* (w.r.t) an end result R. Why introduce this nuance? More details will be given soon. While R can be any end result influenced by our choice of $\mathcal{M}_1$ over $\mathcal{M}_2$, in this section, R will always be our preferences or a normative prescription on which course of action to perform between a pair of actions.

Before stating the sufficient conditions, an intuition pump will be provided detailing a case study in which choosing a particular model over another is intuitively problematic w.r.t an end result R. This case study involves a phenomenon for which there are multiple models. Though this phenomenon is not a probabilistic scenario, token or type, there are two reasons why I'm interested in this case study. The first reason is that this case study shares several important similarities with the main object of interest in this section, which is the multiplicity of correct numerical models for a given fair infinite lottery.[32] First, both involve a multiplicity of models. Second, in both cases, no particular model seems to 'stand out' above the rest, yet a choice of a particular model has to be made regardless. Last, in both cases, depending on exactly which model is chosen, we end up with some results that wouldn't be true if some *other* model were chosen instead. Because of the similarities above, examining this case study will prove useful and informative when formulating the sufficient conditions for when our choosing probabilistic model $\mathcal{M}_1$ over probabilistic model $\mathcal{M}_2$ is problematic w.r.t an end result R.

The second reason is that I'm trying to correct an apparent imbalance in the literature and in the comments I've received between problematic and unproblematic cases of choosing a particular model over other competing models. As far as I know, of all the

[32]Of course, the multiplicity of correct *qualitative* models will be discussed in a later section.

provided examples, almost all, if not all, belong in the latter category, with almost none, if not none at all, in the former. And this imbalance, I suspect, is the root cause for a trend I've noticed in the literature and in the comments I've received. Philosophers, who are inclined to think that choosing a particular model over others is never problematic, tend to liken a potentially problematic case of choosing a particular model over others to paradigmatically unproblematic cases. Then, because of the alleged analogy between the potentially problematic case and the paradigmatically unproblematic cases, the former case is thus dismissed as another unproblematic case. Since we are lacking an intuitively problematic case of choosing a particular model over others, these philosophers face no resistance at all in likening the potentially problematic case to paradigmatically unproblematic cases. For if an intuitively problematic case of choosing a particular model over others were provided, then these philosophers would have to come up with further argument on why the potentially problematic case resembles the paradigmatically unproblematic cases more than the intuitively problematic case. This is why I see value in providing a case study in which choosing a particular model over another is intuitively problematic (w.r.t an end result R).

So, what are some paradigmatically unproblematic cases of choosing a particular model over other competing models? Pruss [2021] has provided some examples.

> "Models have semantically irrelevant features. The reader offered the example of a subway map that could indicate the same route with different colors. Perhaps even more tellingly, a mathematical model of some aspect of reality could use any of a host of different set-theoretic constructions of numbers—the natural number 2 could be identified with the set $\{\emptyset, \{\emptyset\}\}$, or the set $\{\{\emptyset\}\}$, etc.—without this making any difference to how well the model represents what is being modeled. Indeed, in this way, even classical real-valued probabilities are subject to underdetermination, as it is underdetermined which of the equivalent constructions of real numbers (say, Dedekind cuts, lower parts of Dedekind cuts, upper parts of Dedekind cuts, equivalence classes of Cauchy sequences, infinitely long decimals, infinitely long binary numbers, etc., which in turn can be layered on different constructions of natural numbers, sequences, etc.) is to be used for the values of the probabilities. The typical scientist does not care about such questions: as long as the maps or constructions are isomorphic, the specific choice *makes no difference*." (Pruss, 2021, p. 14, emphasis mine)

Why did I italicise the phrase 'makes no difference'? An explanation will be given soon. To be clear, there are two examples in the quote above. The first example is a subway map for which different colours can be used to indicate the same route. Intuitively, there is no problem in choosing red over green or vice versa to indicate a subway route. After all, a passenger normally doesn't decide which subway route to take based on the colour used to indicate it on the map. Instead, a passenger decides which subway route to take based on her starting point and destination. That is, if a passenger's starting point and destination remain the same, then changing the colour used on the map to indicate a subway route should intuitively not change which routes she has decided to take. So, when it comes to deciding which colour to use to indicate a subway route, it "makes no difference" to the route the passenger has decided to embark on if red is chosen over green or vice versa. And because the specific choice of colour "makes no difference", choosing red over green to indicate a subway route is intuitively unproblematic.

The second example is a mathematical model, for which there are different set-theoretic constructions of numbers. As stated in the quote above, there are many ways to construct the real numbers. But intuitively, there is no problem in choosing one construction over another for the sake of modelling a real-life phenomenon. After all, an arithmetic operation on the real numbers will deliver the same result across all constructions, i.e. dividing, multiplying, adding, or subtracting two given real numbers will deliver the same result across all constructions. So, when it comes to modelling a real-life phenomenon, it "makes no difference" to modelling the given phenomenon which construction of the real numbers is adopted. And because the specific choice of construction "makes no difference", choosing one construction over another is intuitively unproblematic.

Adding to the two examples listed above, in the comments I've received on previous drafts of this thesis, there is a litany of examples of choosing a particular model over other competing models, all of which are intuitively unproblematic. Here are some of them: using different colours to represent countries on a world map, Buridan's Ass[33], choosing which action to perform among actions of equal expected utility, choosing between the Celsius and Fahrenheit scales to represent temperature, choosing which utility function to represent an agent's utilities among all the utility functions derived from the agent's preferences, etc.[34] Yet, as far as I know, no one has provided a single example of choosing a particular model over other competing models that is intuitively problematic w.r.t an end result R.

Here's the structure of this section. In §2.1, I will provide an intuition pump detailing an example of choosing a particular model over other competing models that is intuitively problematic w.r.t an end result R. After that, I will argue why this example is problematic w.r.t an end result R. I will then compare this example to the paradigmatically unproblematic cases spelled out above in §2.2, noting the most philosophically salient differences between the cases. These differences will give us a clue what some of the sufficient conditions are for when our choosing probabilistic model $\mathcal{M}_1$ over probabilistic model $\mathcal{M}_2$ is problematic w.r.t an end result R. Finally, in §2.3, I will formulate

[33]See footnote 26.

[34]Thanks to Alan Hájek, Michael Nielsen, Nicholas DiBella, Katie Steele, and the anonymous referees for one of my papers for providing these examples.

the sufficient conditions for when our choosing probabilistic model $\mathcal{M}_1$ over probabilistic model $\mathcal{M}_2$ is problematic w.r.t an end result R.

2.1 An Intuition Pump

Consider now the following intuition pump:

> **Tornado**: In a tornado-prone area, there are two towns A and B. These towns are situated far enough from each other such that if a tornado hits one town, there will be minimal damage to the other town. Currently, a group of experts has determined that at noon tomorrow, there will be a tornado heading towards the general direction of both towns. They know that it will only hit one town as that is the case in every computer simulation they've run of the tornado. But here's what they cannot determine precisely: will the tornado hit town A or town B? This is because there are two models $\mathcal{M}_1$ and $\mathcal{M}_2$ of the internal conditions of the tornado.[35] Upon using $\mathcal{M}_1$ in the computer simulation of the tornado, it predicts that the tornado will hit town A. But upon using $\mathcal{M}_2$ in the computer simulation, it predicts that the tornado will hit town B instead. As far as the experts know, both $\mathcal{M}_1$ and $\mathcal{M}_2$ are equally well-supported by all the available data they have on the tornado. Time is running out and the experts need to decide which town to evacuate. Furthermore, due to limited resources, only one town can be evacuated. Unable to determine which model is better and because time is running out, the experts decide to just choose $\mathcal{M}_1$ and therefore evacuate town A, while hoping that they made the right choice by luck. When questioned on why they chose $\mathcal{M}_1$ over $\mathcal{M}_2$, the experts could not give any reason, other than the fact that a choice *has to* be made in the interest of preventing the sure deaths of the citizens in one town if they fail to choose a model before the time is up.[36]

I hope the reader shares my intuition that there is something unsettling and problematic about the situation in **Tornado**. At least for me, what's unsettling and problematic is how the experts decided to choose $\mathcal{M}_1$ over $\mathcal{M}_2$ w.r.t the town to be evacuated. That is, the thought process behind choosing $\mathcal{M}_1$ is bothering me. Here's a reason why. Arguably, the goal of the situation in **Tornado** is to minimise the loss of human lives. To achieve this goal, the internal conditions of the tornado need to be modelled as accurately as possible, so as to predict with relative accuracy where the tornado will head. Then,

[35] Note that $\mathcal{M}_1$ and $\mathcal{M}_2$ need *not* be probabilistic models here, in the sense defined in the introduction.

[36] I acknowledge that **Tornado** may not be realistic. Rarely in science are there two models of the same phenomenon equally well-supported by the *same* data, yet each model still delivers a radically different result about the phenomenon as compared to the other model. However, note that the purpose of this intuition pump is to elicit intuitions; it need not be realistic. That is, if **Tornado** *were* to happen, what intuitions would you have about it? That's the purpose of stating **Tornado** as an intuition pump here.

using this information, we evacuate the town that the tornado is predicted to hit, thus minimising the loss of human lives. For the sake of modelling the internal conditions of the tornado as accurately as possible, a model should be chosen based on a careful comparison between the internal conditions of the tornado as detailed *in the model* and all the available data we have on those internal conditions. The higher the 'match' between a model and all the available data, the more accurate that model is. For example, say, according to the data we have, the tornado has an average wind speed of $300\,\mathrm{km\,h^{-1}}$ with a diameter of $2\,\mathrm{km}$, etc. We want to choose a model with values closest to these values. In other words, a model should be chosen based on its proximity to the data.

Generally speaking, when choosing a model for a scientific phenomenon, we want to follow where the data leads us. Ideally, the data leads us to a particular model which we will then choose as the most accurate representation of the given scientific phenomenon. This is not the case in **Tornado** unfortunately. In **Tornado**, all the available data led the experts to two models instead. Here, data will not provide any guidance. That is, it seems $\mathcal{M}_1$ is chosen not because it 'matches' the data the best, but because the experts *have to* choose at least one model in the interest of preventing the sure deaths of the citizens in one town if they fail to choose a model before the time is up. After all, since $\mathcal{M}_2$ 'matches' the data equally well as $\mathcal{M}_1$ does, it cannot be that $\mathcal{M}_1$ is chosen because of its proximity to the data. So, it seems that choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ is an *arbitrary* choice, not guided by data, between the two available models. But as explained above, at least in **Tornado**, a model should be chosen based on its proximity to the data and not arbitrarily. Since how the experts decided to choose $\mathcal{M}_1$ over $\mathcal{M}_2$ in **Tornado** falls short of ideality, I find that there is something unsettling and problematic about how the experts arrived at their decision w.r.t the town to be evacuated.

Here's another way of seeing why there is something unsettling and problematic about how the experts arrived at their decision w.r.t the town to be evacuated. Put yourself in the shoes of a citizen living in town B. Say you've just received news that the experts have decided to evacuate town A, even though both models $\mathcal{M}_1$ and $\mathcal{M}_2$ are equally well-supported by all the available data they have on the tornado. You'll probably feel a little puzzled at how the experts decided to evacuate town A. After all, from their epistemic standpoint, the tornado is equally probable to hit town A as compared to town B. So, why choose to evacuate town A but abandon town B? Perhaps you'll even feel angry that the experts are effectively gambling with the lives in town B. Moreover, it's a gamble with odds that are *not* in favour of the lives in town B. Above all, you may feel angry at the experts' lack of sufficient research beforehand on tornadoes to ensure that a situation like **Tornado** will never arise. If you have these emotions detailed above, then it suggests that you think that there is something unsettling and problematic about how the experts arrived at their decision w.r.t the town to be evacuated. After all, if there were nothing problematic whatsoever, then you wouldn't feel puzzled or angry at all.

Another reason why arbitrarily choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ is problematic w.r.t the town to be evacuated is that making a choice this way doesn't seem to be a trustworthy or reliable method to minimise the loss of human lives. Note that the only way to minimise the loss of human lives in **Tornado** is for the experts to make a *lucky* guess. That is, they guess that the tornado will hit town A and as luck will have it, the tornado hits town A. But what if the experts weren't lucky? The consequences would then be disastrous. And even if the experts were lucky this time round, they may not have the same stroke of luck the next time a similar situation arises; the experts won't be lucky every time. As an analogy, consider the following method of forming true beliefs. For any proposition q, we toss a fair coin to decide whether we believe that q. If the coin lands heads, we believe that q; otherwise, we do not believe that q. Intuitively, this method is not a trustworthy or reliable method to form true beliefs. After all, why think that the results of fair coin tosses are truth-tracking? Since forming beliefs based on the results of fair coin tosses isn't a trustworthy or reliable method to form true beliefs, there's something problematic about using this method to form true beliefs. Similarly, why think that the experts' arbitrary choice of $\mathcal{M}_1$ over $\mathcal{M}_2$ tracks where the tornado will hit? For this reason, how the experts decided to choose $\mathcal{M}_1$ doesn't seem to be a trustworthy or reliable method to minimise the loss of human lives. This means that there's something problematic about arbitrarily choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ in order to minimise the loss of human lives, just like how there is something problematic about forming beliefs based on the results of fair coin tosses in order to form true beliefs; doing so isn't a trustworthy or reliable method to form true beliefs. We should not play dice with people's lives but in **Tornado**, it seems that the experts are doing exactly that.

Don't get me wrong. There's nothing problematic about the *existence* of multiple models that 'match' the data equally well. In fact, for such a complex phenomenon like a tornado, it may even come as no surprise that there are multiple such models. And if the computer simulations predict that the tornado will hit—say town A—*no matter which model is chosen*, then there is no problem in arbitrarily choosing $\mathcal{M}_1$ for the computer simulations too. But a problem arises when the prediction of which town the tornado will hit depends on exactly which model is chosen for the computer simulation, where all the models under discussion here are equally well-supported by the available data. That is, the town the tornado is predicted to hit changes if a different model, equally well-supported by the data, is used for the computer simulation. Cases like that are always tricky, as it isn't clear which model to use for the computer simulation, yet a model still has to be chosen regardless. Because this computer simulation is used to decide which town to evacuate, this choice effectively amounts to a life-or-death situation for the citizens in both towns. And it seems problematic to leave the fate of the citizens up to an arbitrary choice. Note that I'm *not* saying that how the experts decided to choose $\mathcal{M}_1$ over $\mathcal{M}_2$ is problematic w.r.t the town to be evacuated because they chose arbitrarily. Arbitrary choices are not always problematic. Instead, I'm saying that how the experts arrived at their decision is problematic w.r.t the town to be evacuated because they chose arbitrarily in a situation where there is reason *against* choosing arbitrarily.

As with all intuition pumps, objections are expected, to which I turn now. *Objection* 1. The first objection goes something like the following:

> 'For a moment, let's assume that I'm right that there is something problematic about how the experts decided to choose $\mathcal{M}_1$ over $\mathcal{M}_2$ w.r.t the town to be evacuated. Now, the same can be said if the experts were to choose $\mathcal{M}_2$ over $\mathcal{M}_1$. This means that no matter which model the experts choose, how they arrive at their decision is always problematic. So, by my lights, in order to avoid making a choice in a problematic way, the experts must not choose any model at all. But surely the worst thing to do here is failing to choose any model and thus not evacuating any town. After all, in any situation, we always strive to achieve the best case scenario and avoid the worst case scenario. In **Tornado**, the best case scenario is that the tornado hits the town that has already been evacuated, while the worst case scenario is that the tornado hits the town that isn't evacuated. If no town is evacuated, then we will certainly get the worst case scenario. But if some town is evacuated, then there is still at least some chance that we will get the best case scenario. So, overall, evacuating a town is better than not evacuating any town at all. Since my argument above implies that the experts should not choose any model at all and thus not evacuate any town, I'm wrong in that there is something problematic about how the experts decided to choose $\mathcal{M}_1$ over $\mathcal{M}_2$ w.r.t the town to be evacuated.'

Here's my reply. Consider a moral trilemma, in which there are two options A and B such that performing A and performing B are both moral evils of equal intensity, but performing *neither* is a greater evil than individually performing A or B. Furthermore, I take it that when confronted with only moral evils, we should always perform the lesser evil, or at least one of the lesser evils. So, in the moral trilemma detailed above, we should either perform A or perform B, even though performing either is a moral evil; performing neither should be avoided as doing so is the greater evil. Importantly, especially in moral trilemmas like the one stated above, it is *false* that we should not perform A or B. Instead, given the situation in the moral trilemma stated above, we should still either perform A or perform B.

The reader can treat **Tornado** as an *epistemic* trilemma. As in the moral case, when confronted with only epistemic evils, we should always perform the lesser evil, or at least one of the lesser evils. Choosing $\mathcal{M}_1$ may be epistemically problematic, because from the experts' epistemic standpoint, both models $\mathcal{M}_1$ and $\mathcal{M}_2$ are equally good representations of the internal conditions of the tornado. So, why choose $\mathcal{M}_1$ but not $\mathcal{M}_2$? The same goes for choosing $\mathcal{M}_2$. But this doesn't mean that the experts should therefore not choose any model at all, for the greater epistemic evil here is failing to choose any model. When confronted with only epistemic evils, we should always avoid performing the greater evil. Instead, we should perform the lesser evil, or at least one of the lesser evils. And in **Tornado**, the least of all the epistemic evils is to either choose $\mathcal{M}_1$ or choose $\mathcal{M}_2$. So, even though choosing either model is an epistemic evil, the experts should still either choose $\mathcal{M}_1$ or choose $\mathcal{M}_2$; choosing neither should be avoided as doing so is the greater epistemic evil. To be clear, the point here is that in epistemic trilemmas like **Tornado**, it is *false* that the experts should not choose $\mathcal{M}_1$ or $\mathcal{M}_2$. Instead, given the situation in **Tornado**, the experts should still either choose $\mathcal{M}_1$ or choose $\mathcal{M}_2$. With that, the above objection fails.

Objection 2. The second objection goes something like the following:

> 'Sure, my argument doesn't imply that the experts should not choose any model at all and thus not evacuate any town. Instead, the experts should either choose $\mathcal{M}_1$ or choose $\mathcal{M}_2$, though it seems that this choice can only be made arbitrarily. Choosing neither should be avoided at all costs. But doesn't the fact that the experts are forced to choose a model arbitrarily imply that an arbitrary choice between $\mathcal{M}_1$ and $\mathcal{M}_2$ isn't problematic at all? After all, if we are forced to do something, then it cannot be problematic to do that thing.'

Here's my reply. Let's say that it is morally evil to lie. One day, someone comes up to me and points a gun at my head saying 'Tell a lie right now or else I'll shoot you in the head!'. Wanting to hang on to my dear old life, I hurriedly told a lie without a second thought. Does being forced to tell a lie imply that it isn't morally evil to tell a lie at that moment? Intuitively, no. Even though I am forced to tell a lie, doing so is still considered a moral evil. Being forced to do so doesn't change that. And I suspect that the same goes for epistemic cases. To drive the point home, consider the following epistemic case. A fair coin is about to be tossed. Say you're in a situation where you're forced to either believe that the coin will land heads or believe that the coin will land tails. Further assume that a credence of 50% in a proposition q is *too low* for you to believe in q. Now, in this situation, come what may, you believe in a proposition q to which only 50% credence is assigned. Does being forced to believe in such a proposition q imply that it isn't epistemically evil to believe in q at that moment? Intuitively, no. Being forced to believe in q doesn't make doing so any less epistemically problematic than if you weren't forced to do so. After all, a credence of 50% in a proposition q is still too low for you to believe in q, regardless of whether you were forced to do so or not. So, it's *false* that if we are forced to do something, then it cannot be problematic to do that thing. With that, the above objection fails.

Objection 3. The third objection goes something like the following:

> 'I should put myself in the shoes of the experts in **Tornado**. If I were to call the shots in **Tornado**, what would *I* have done differently from the experts? Would I not have chosen a model arbitrarily like them? Surely not, as by my lights, the experts should either choose $\mathcal{M}_1$ or choose $\mathcal{M}_2$, even though this choice has to be made arbitrarily. So, it seems that there is nothing that I would have done differently. But if there's nothing that I would have done differently, then that suggests that there's nothing wrong with what the experts did.'

Here's my reply. If we find ourselves in a situation like **Tornado**, then it is already 'too late'. By that time, there's only one thing that should be done: choose a model arbitrarily. As I've argued above, even though choosing a model arbitrarily in **Tornado** is epistemically problematic, the experts should do so regardless, for failing to choose any model at all is a greater epistemic evil here. In particular, if we find ourselves already in an epistemic trilemma like **Tornado**, then the only way out is to impale ourselves on one of the horns in the trilemma, no matter how unattractive that may seem. The best we can do in such a trilemma is to at least ensure that we get impaled by the smaller horn. In the case of **Tornado**, the smaller horn is choosing a model arbitrarily.

Furthermore, though there is an epistemic reason against choosing a model arbitrarily in **Tornado**, there is still a *practical* reason in favour of doing so. By choosing a model arbitrarily, we stand a chance in avoiding the worst case scenario in **Tornado**, as detailed in objection 1. But if we don't choose any model at all, then the worse case scenario is guaranteed to happen. Some chance is still better than no chance at all. So, while there is an epistemic reason against choosing a model arbitrarily in **Tornado**, there is also a practical reason in favour of doing so. And sometimes, practical reasons to do something trump epistemic reasons *not* to do that thing. Recall the example of someone pointing a gun to my head saying 'Tell a lie right now or else I'll shoot you in the head!'. I have a *moral* reason not to lie, as doing so is morally evil. But in the gun situation, I also have a practical reason to lie: to stay alive. I daresay that the practical reason to lie trumps the moral reason not to lie in this case. Therefore, all things considered, I should lie in the gun situation, despite the fact that lying is a moral evil. The same goes for the practical and epistemic reasons for and against arbitrarily choosing a model in **Tornado** respectively. Considering the situation in **Tornado**, a model should still be chosen arbitrarily, *all things considered*. However, note that the presence of a practical reason for choosing a model arbitrarily does not imply that doing so is no longer epistemically problematic. It is still epistemically problematic, just like how lying is still morally evil even though I have a practical reason to lie in the gun situation above.

Although there's nothing we should be doing other than choosing a model arbitrarily once we find ourselves in a situation like **Tornado**, there are some things that we can (and should) do after we get out of the predicament in **Tornado**. For instance, we should do everything in our power to prevent a situation like **Tornado** from happening again. This includes investing more money in the research of tornadoes and in the procuring of more sensitive instruments to measure the internal conditions of tornadoes, gathering more accurate and complete data on the tornado, analysing why models $\mathcal{M}_1$ and $\mathcal{M}_2$ delivered different predictions for where the tornado will hit, etc. All these are done in hope of there being a model that 'stands out' above the rest in 'matching' the available data on a given tornado, so that a situation like **Tornado** can be prevented in the future. Put simply, if we don't want to get impaled again by one of the horns in an epistemic trilemma, then we should do everything in our power to prevent ending up in another epistemic trilemma like **Tornado**. If we prevent ourselves from ending up in epistemic trilemmas, then we won't be forced to do anything that's epistemically problematic. Also, note that if the reader agrees with me that the situation in **Tornado** should at least prompt measures to prevent its happening again in the future, then this suggests that the reader finds the situation in **Tornado** problematic. For if there were nothing problematic whatsoever, then why put in place measures to prevent similar situations from happening again? With the considerations above, this objection fails.

Objection 4. The fourth objection goes something like the following:

> 'There is definitely something unsettling and problematic about the situation in **Tornado**. But it doesn't have anything to do with how the experts decided to choose $\mathcal{M}_1$ over $\mathcal{M}_2$ w.r.t the town to be evacuated. That is, it isn't the thought process behind choosing $\mathcal{M}_1$ that's unsettling and problematic. Instead, what's unsettling and problematic is the *set-up* of **Tornado**. How can there be two models $\mathcal{M}_1$ and $\mathcal{M}_2$ that are equally well-supported by all the available data on the tornado, yet they still deliver different predictions for where the tornado will hit? Surely, if $\mathcal{M}_1$ and $\mathcal{M}_2$ are equally well-supported by all the available data on the tornado, then they *must* deliver the same prediction for where the tornado will hit. Conversely, if they deliver different predictions, then $\mathcal{M}_1$ and $\mathcal{M}_2$ cannot be equally well-supported by all the available data. Put simply, **Tornado** is an *impossible* situation. So, any intuitions we have about it cannot be trusted.'

Here's my reply. This objection under-appreciates how difficult it is to model such a complex phenomenon like a tornado and consequently, how difficult it is to predict where it is going. Moreover, since the advent of using artificial intelligence and machine learning to advance science, it has been observed that some scientific results produced from machine learning are neither repeatable, replicable, nor reproducible.[37] That is, the *same* data can

[37] *Repeatability* is defined as "obtaining the exact same results of an experiment with the same precision by the same team in the same location, under the same experimental setup such as hardware and software settings on multiple trials". *Replicability* is defined as "obtaining the exact same results of an experiment with the same stated precision by a different team in the same or different location, under the same experimental setup such as hardware and software settings on multiple trials". *Reproducibility* is defined

produce multiple machine-learning models, with each model delivering a different result regarding the phenomenon the data is of.[38] So, it is *false* that if two models $\mathcal{M}_1$ and $\mathcal{M}_2$ are equally well-supported by all the available data on a given phenomenon, then they must deliver the same result regarding that phenomenon. With that, this objection fails.

Objection 5. The fifth objection goes something like the following:

> 'I simply don't have the intuition that there is something unsettling and problematic about the situation in **Tornado** at all. In fact, **Tornado** is exactly like one of the paradigmatically unproblematic cases of choosing a particular model over other competing models provided by Pruss [2021]. Recall the example, provided by Pruss [2021], of the subway map for which different colours can be used to indicate the same route. Choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ in **Tornado** is exactly like choosing red over green to indicate a subway route on the map. Since the latter is unproblematic, the former is too.'

Here's my reply. I simply disagree with this objection: there is one philosophically significant difference between **Tornado** and the subway map, which I will explain in the next section. But for now, if the reader is inclined to object this way, I urge her to reconsider. If the reader still finds this objection compelling even after reconsideration, then, unfortunately, all I can give her is an incredulous stare of disbelief.

2.2 Comparing Tornado and Paradigmatically Unproblematic Cases

Recall the example of a subway map for which different colours can be used to indicate the same route. Let R_{map} be the route a passenger decides to embark on to get to her destination. Stipulate that red and green represent equally well a certain route on the map and say that red is chosen over green to indicate that route. Now, since both colours represent equally well that route on the map, red is chosen not because it represents that route better than green does. But instead, red is chosen *arbitrarily*, just like how the experts chose a model in **Tornado**. And intuitively, R_{map} doesn't change if green were chosen instead of red. After all, a passenger normally doesn't decide which subway route to take based on the colour used to indicate it on the map. Instead, a passenger decides which subway route to take based on her starting point and destination. That is, if a passenger's starting point and destination remain the same, then changing the colour used on the map to indicate a subway route should intuitively not change which routes she has decided to take. Changing the colour used to indicate a subway route makes no difference to R_{map}. For this reason, arbitrarily choosing red over green to indicate a route on the map is intuitively unproblematic. This shows that arbitrary choices are not always

as "obtaining the exact same results of an experiment with the same stated precision by a different team in a different location, under a different experimental setup such as hardware and software settings over multiple trials". (Alahmari et al., 2020, p. 211861)

[38] See Gundersen & Kjensmo [2018], Alahmari et al. [2020], and Beam et al. [2020] for more details.

problematic, as making a choice this way is problematic in one case (**Tornado**) but is unproblematic in another (the example of the subway map).

So, what's the difference between **Tornado** and the example of the subway map? Let $R_{tornado}$ be the town to be evacuated. Recall that model $\mathcal{M}_1$ is chosen over model $\mathcal{M}_2$ in **Tornado**. As detailed in **Tornado**, $R_{tornado}$ changes if model $\mathcal{M}_2$ were chosen instead of $\mathcal{M}_1$. That is, if model $\mathcal{M}_1$ is chosen, then $R_{tornado}$ is town A; if model $\mathcal{M}_2$ is chosen, then $R_{tornado}$ is town B. So, $R_{tornado}$ is *sensitive* to exactly which model is chosen in **Tornado**, whereas R_{map} isn't sensitive at all to exactly which colour is chosen to indicate a subway route on the map. *This* is the philosophically salient difference between **Tornado** and the subway map. And I conjecture that it is this difference that explains why, on the one hand, there is no problem w.r.t R_{map} in choosing red over green or vice versa to indicate a subway route, but, on the other hand, choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ and vice versa in **Tornado** are problematic w.r.t $R_{tornado}$.

To drive the point home, suppose for a moment that no matter which model is chosen, every computer simulation run of the tornado predicts that the tornado will hit town A. Then, intuitively, there is no problem w.r.t $R_{tornado}$ in choosing $\mathcal{M}_1$ for the computer simulation. The same goes if $\mathcal{M}_2$ were chosen instead. After all, choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ or vice versa *makes no difference* at all to $R_{tornado}$ if **Tornado** were modified this way. In either case, the tornado is predicted to hit town A and therefore, town A should be evacuated. Choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ in this modified case of **Tornado** is then exactly like choosing red over green to indicate a subway route on the map. Since the latter is unproblematic, the former is too. However, if we keep **Tornado** the way it is originally, then this analogy with the subway map dissipates, contrary to what objection 5 says in the previous section.

Generally speaking, for any cases that involve choosing a particular model over other competing models, doing so isn't problematic w.r.t an end result R if R isn't sensitive to the particular choice of model. That is, R remains the same even if some other model were chosen instead. In cases like that, it *makes no difference* at all to R exactly which model is chosen. This is the reason why, in §2, I italicised the phrase 'makes no difference' in the quote by Pruss [2021] when he provided the two paradigmatically unproblematic cases of choosing a particular model over other competing models. For any cases that involve choosing a particular model over other competing models, doing so is problematic w.r.t an end result R *only if* R is sensitive to the particular choice of model. That is, R changes if some other model were chosen instead. In these latter cases, it *makes a difference* to R exactly which model is chosen.

Now, I'll explain why I introduced the nuance 'w.r.t an end result R' to my analysis. The reason is simple. Recall the example of a subway map and the definition of R_{map} again. We've established above that R_{map} doesn't change if red is chosen over green or vice versa to indicate a subway route. However, there may be another end result R'_{map}

that is sensitive to such a choice. For example, R'_{map} may be the aesthetics of the map, i.e. choosing red over green may make the map more pleasing to the eye. The point here is that for any cases that involve choosing a particular model over other competing models, even though an end result R isn't sensitive to the particular choice of model, for all I know, there may be another end result R' that *is* sensitive to the particular choice of model. Put simply, there may be many end results, some of which are sensitive to the particular choice of model, some of which are not at all. If there is such a mixture of end results, then it is difficult or even impossible to tell whether choosing a model over another in this case is problematic or not, *simpliciter*. It is this possible mixture of end results that prompted me to introduce the nuance 'w.r.t an end result R' to my analysis.

2.3 The Sufficient Conditions

With everything in place, I'm all set to state the sufficient conditions for when our choosing probabilistic model $\mathcal{M}_1$ over probabilistic model $\mathcal{M}_2$ is problematic w.r.t an end result R. Here they are, without explication first:

Problematic Choice. *For a token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ where Ω is infinite and **Fairness** is one of the constraints, though not necessarily the only constraint, in C, choosing $\mathcal{M}_1$ over $\mathcal{M}_2$, both of which are correct numerical models of $\langle \Omega, \mathcal{F}, C \rangle$, is problematic w.r.t an end result R if all of the following hold:*

1. *$\langle \Omega, \mathcal{F}, C \rangle$ is fully specified.*
2. *$\mathcal{M}_1$ and $\mathcal{M}_2$ have the same representational power in modelling $\langle \Omega, \mathcal{F}, C \rangle$.*
3. *R changes if $\mathcal{M}_2$ were chosen instead of $\mathcal{M}_1$.*

Explication is in order. Let's start with the easiest condition to explain: condition 3. If condition 3 weren't included in **Problematic Choice**, then **Problematic Choice** would also apply to cases where R remains the same no matter which model is chosen. But as explained in the previous section, if R remains the same no matter which model is chosen, then choosing a particular model over other competing models isn't problematic w.r.t R. After all, in these cases, it makes no difference whatsoever to R exactly which model is chosen. This means that **Problematic Choice** without condition 3 delivers the wrong verdict for some cases. Therefore, for the sake of delivering only correct verdicts, condition 3 must be included in **Problematic Choice**.

Consider condition 2 now. For a moment, suppose that $\mathcal{M}_1$ is a better model of $\langle \Omega, \mathcal{F}, C \rangle$ than $\mathcal{M}_2$ is. Then, choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ to represent $\langle \Omega, \mathcal{F}, C \rangle$ is not problematic at all. In fact, doing so may even be rationally required. After all, we want (and are also rationally obliged) to choose a model that *best* represents $\langle \Omega, \mathcal{F}, C \rangle$, if such a model exists. Generally speaking, for any cases that involve choosing a particular model over other competing models, doing so isn't problematic w.r.t an end result R if the chosen model is a better model of the given phenomenon than any other models, i.e. the chosen model has more representational power in modelling the given phenomenon than any other models. So, for there to be a problem w.r.t R when choosing $\mathcal{M}_1$ over $\mathcal{M}_2$, $\mathcal{M}_1$ and $\mathcal{M}_2$ must have the *same* representational power in modelling $\langle \Omega, \mathcal{F}, C \rangle$. Here's a natural follow-up question: when do $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same representational power in modelling $\langle \Omega, \mathcal{F}, C \rangle$? This is a loaded question. But for now, let's just consider probabilistic models expressed exclusively in terms of precise probabilities. That is, let $\mathcal{M}_1 = \langle \Omega, \mathcal{F}, C, \Sigma_1 \rangle$ where $\Sigma_1 = \{(A, P_1(A)) : A \in \mathcal{F}\}$ and $\mathcal{M}_2 = \langle \Omega, \mathcal{F}, C, \Sigma_2 \rangle$ where $\Sigma_2 = \{(A, P_2(A)) : A \in \mathcal{F}\}$. $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same representational power in modelling $\langle \Omega, \mathcal{F}, C \rangle$ just in case P_1 and P_2 satisfy **Same Representational Power**. How about probabilistic models expressed in terms of precise and imprecise probabilities? An answer will be provided in §4 when we discuss such models.

Also, I hope that the reader has a clearer idea now of what I said earlier in §1.2. There, I said that **Infinitesimal Underdetermination** is a problematic kind of underdetermination. By this, I mean that **Infinitesimal Underdetermination** consists of a set of conditions, the fulfillment of which by two probability functions P_1 and P_2 implies that P_1 and P_2 have the same representational power in representing a fully specified $\langle \Omega, \mathcal{F}, C \rangle$ where Ω is infinite and **Fairness** is one of the constraints in C. Then, by using the method specified above, two models $\mathcal{M}_1$ and $\mathcal{M}_2$ can be defined from P_1 and P_2 respectively. Since P_1 and P_2 have the same representational power in representing $\langle \Omega, \mathcal{F}, C \rangle$, $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same representational power in modelling $\langle \Omega, \mathcal{F}, C \rangle$ too, which is just condition 2 in **Problematic Choice**. This is what I meant when I said that **Infinitesimal Underdetermination** is a problematic kind of underdetermination.

Finally, consider condition 1. In order to explain why I've included condition 1 in **Problematic Choice**, consider the Thomson's Lamp paradox.

> "The Thomson's Lamp paradox supposes a lamp that starts (say) off, but its switch is toggled infinitely often during a finite interval of time. The paradox is supposed to be that we have no good answer to the question of whether after the interval of time the lamp is on or off (Thomson, 1954). (Benacerraf, 1962) argues that there is no real paradox, but simply an insufficiency of information. The setup tells us what happens after a finite number of switch togglings, but says nothing about what happens after an infinite number. Both the answer that the lamp is on and that the lamp is off at the end of the interval are logically compatible with the story, but to know which answer is true, we would need more information, say about some law of nature specifying what happens in infinite switching situations. This is a plausible response to the paradox." (Pruss, 2021, p. 16)

Due to insufficient information, guessing that the lamp is on after the finite interval of time is as good as guessing that it is off instead. If we are forced to give a definite answer, saying that the lamp is on instead of off is intuitively unproblematic. After all, as explained in the quote above, both answers "are logically compatible with the story". The same goes for under-specified token probabilistic scenarios $\langle \Omega, \mathcal{F}, C \rangle$ where Ω is infinite and **Fairness** is one of the constraints, though not necessarily the only constraint, in C. Consider this example of such a token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ where (i) Ω and $\mathcal{F}$ are not specified and (ii) C contains the probability axioms, **Regularity**, and **Fairness**. Put simply, this scenario is a regular fair infinite lottery over an unspecified sample space. It is obvious that there are uncountably many correct numerical models for this scenario, with some models even having a different $\mathcal{F}$ from others. After all, there are uncountably many infinite Ω's, and consequently uncountably many $\mathcal{F}$'s too. Let's call two of these models $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. Intuitively, choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ to represent this scenario is not problematic at all, just like how saying that the lamp is on instead of off is not problematic. Just like Thomson's Lamp, there is insufficient information about the token probabilistic scenario stated above. In fact, we can just treat $\mathcal{M}_1$ and $\mathcal{M}_2$ as various *precisifications* of the scenario. And for under-specified token probabilistic scenarios like the one stated above, there is intuitively no problem in choosing one precisification over another. So, for there to be a problem w.r.t R when choosing $\mathcal{M}_1$ over $\mathcal{M}_2$, $\langle \Omega, \mathcal{F}, C \rangle$ has to be *fully specified* in the sense spelled out in §2. That is, (i) Ω, $\mathcal{F}$, and C are all specified and (ii) the constraints in C fully determine the order of events in $\mathcal{F}$.

There may be no problem in choosing a particular model over other competing models to represent an under-specified token probabilistic scenario like the one stated above. But the situation changes for a fully specified token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ where Ω is infinite and **Fairness** is one of the constraints, though not necessarily the only constraint, in C. Furthermore, this is a token probabilistic scenario for which there are two correct numerical models $\mathcal{M}_1$ and $\mathcal{M}_2$ of the same representational power, choosing between which ultimately affects an end result R. For token probabilistic scenarios like these, any information that can be supplied *has been* supplied, except perhaps the tedious and exhaustive specification of *all* the exact probabilities (precise and imprecise) to be assigned to *every* event in $\mathcal{F}$.[39] Yet, there still exist two correct numerical models $\mathcal{M}_1$ and $\mathcal{M}_2$ of the same representational power, choosing between which ultimately affects an end result R. Moreover, in order to apply probability theory to—say, calculate the expected utility of performing an action, at least one probabilistic model must be chosen, i.e. if no probabilistic model were chosen, how could we then calculate the expected utility of performing an action? At least to me, there's something unsettling and problematic about this situation. Arguably, we want to choose a probabilistic model based on how well it represents the given token probabilistic scenario. But because $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same representational power in modelling the given token probabilistic scenario, choosing

[39] I will say more about this exception in the next section of this thesis.

one over the other cannot be due to the fact that the chosen model represents the given token probabilistic scenario better than the other. Instead, it must be chosen *arbitrarily*, just like how a model is chosen in **Tornado**.

I daresay that our choosing probabilistic model $\mathcal{M}_1$ over probabilistic model $\mathcal{M}_2$ resembles **Tornado** more than the example of the subway map for which different colours can be used to indicate the same route. After all, as I've mentioned above in §2, **Tornado** shares three important similarities with our choosing probabilistic model $\mathcal{M}_1$ over probabilistic model $\mathcal{M}_2$, with the third similarity driving the crucial wedge between the example of the subway map and our choosing probabilistic model $\mathcal{M}_1$ over probabilistic model $\mathcal{M}_2$. First, both **Tornado** and our choosing probabilistic model $\mathcal{M}_1$ over probabilistic model $\mathcal{M}_2$ involve a multiplicity of models. Second, in both cases, no particular model seems to 'stand out' above the rest, yet a choice of a particular model has to be made regardless. Last and most importantly, in both cases, depending on exactly which model is chosen, we end up with some results that wouldn't be true if some *other* model were chosen instead.

In **Tornado**, such a result is $R_{tornado}$. In our choosing probabilistic model $\mathcal{M}_1$ over probabilistic model $\mathcal{M}_2$, such a result is R. Both $R_{tornado}$ and R change if some other model were chosen instead in each respective case, whereas R_{map} doesn't change even if some other colour were chosen instead to indicate the same route on the map. This is the crucial difference between the example of the subway map and our choosing probabilistic model $\mathcal{M}_1$ over probabilistic model $\mathcal{M}_2$. To be sure, there may be another end result R'_{map} in the example of the subway map that *changes* upon choosing some other colour instead, e.g. R'_{map} may be the aesthetics of the map, i.e. choosing red over green may make the map more pleasing to the eye. But recall from the discussion at the end of §2.2 that this possible mixture of end results, some of which are sensitive to the particular choice of model, some of which are not, is the reason why I've included the nuance 'w.r.t an end result R' in my analysis. That is, w.r.t R, our choosing probabilistic model $\mathcal{M}_1$ over probabilistic model $\mathcal{M}_2$ resembles the experts' choosing of a particular model in **Tornado** w.r.t $R_{tornado}$ more than choosing red over green in the map example w.r.t R_{map}. In fact, I'll even go further to say that our choosing probabilistic model $\mathcal{M}_1$ over probabilistic model $\mathcal{M}_2$ w.r.t R is analogous in all philosophically salient aspects to the experts' choosing of a particular model in **Tornado** w.r.t $R_{tornado}$. Therefore, since the experts' choosing of a particular model in **Tornado** is problematic w.r.t $R_{tornado}$, our choosing probabilistic model $\mathcal{M}_1$ over probabilistic model $\mathcal{M}_2$ is problematic w.r.t R.

Here's another way to bring out the problem. Consider this scenario: you're ill and you have to take pills to recover. But there are many kinds of pills. So, which pill should you take? Surely, the pill that you should take *isn't* determined by arbitrarily choosing a pill out of all the available pills. That is, the pill you should take isn't determined, or for that matter, isn't ever determined, by your arbitrary choice. Instead, you go to a doctor who will first examine your condition and then determine which pill you should take. The point here is that medical prescriptions are *not* determined by your arbitrary choices. And I suspect that the same goes for normative prescriptions.

Say R is the action that you should perform between actions A and B. Further stipulate that if $\mathcal{M}_1$ is chosen to represent the given token probabilistic scenario, then $R = A$; if $\mathcal{M}_2$ is chosen instead, then $R = B$. Then, choosing a particular model over the other effectively amounts to choosing your own normative prescription for which action to perform. That is, choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ effectively amounts to choosing the normative prescription that you should perform A and not B. Similarly, choosing $\mathcal{M}_2$ over $\mathcal{M}_1$ effectively amounts to choosing the normative prescription that you should perform B and not A. Now, recall that because $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same representational power in modelling the given token probabilistic scenario, choosing one over the other must be done *arbitrarily*, just like how a model is chosen in **Tornado**. This means that the normative prescription for which action to perform is *determined* by your arbitrary choice. But, as I've argued above, like medical prescriptions, normative prescriptions are *not* determined by your arbitrary choices. Therefore, arbitrarily choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ is problematic w.r.t R. To be sure, I'm not saying that there cannot be any *coincidental* or *lucky* 'matches' between the action you chose arbitrarily and the action you should perform. There may be such 'matches'; the pill you chose arbitrarily may very well coincide with the pill you should take. Instead, I'm saying that the action you should perform is *not determined* by your arbitrary choice of a model, just like how the pill you should take is not determined by your arbitrary choice of a pill.

Don't get me wrong. There may be some arbitrary choices made in the past that led to the current situation that the normative prescription is tailored for, but this doesn't mean that those arbitrary choices *determined* the normative prescription. For example, say you're walking home and there are two paths you can take: the right path and the left path. Let's say you chose arbitrarily to take the left path. But unbeknownst to you, there is an assailant further down the left path. When you approach the assailant, he runs up to you and points a gun to your head saying 'Tell a lie right now or else I'll shoot you in the head!'. As discussed in §2.1, the normative prescription here is that you should lie in this situation, all things considered. So, your arbitrary choice to take the left path led to this gun situation you're currently in that the normative prescription just stated is tailored for. But this doesn't mean that your arbitrary choice to take the left path *determined* the normative prescription just stated. After all, intuitively, in the gun situation, what determines the normative prescription to lie is the outweighing of the moral reason not to lie by the practical reason to lie. Your arbitrary choice to take the left path does not enter the equation at all. Therefore, even though your arbitrary choice to take the left path led to the gun situation, the normative prescription to lie still isn't determined by that choice.

Now, I'll address two objections to **Problematic Choice**. *Objection* 1. The first objection goes something like the following:

> 'Say R is the action that you should perform between actions A and B. Further stipulate that if $\mathcal{M}_1$ is chosen to represent the given token probabilistic scenario, then $R = A$; if $\mathcal{M}_2$ is chosen instead, then $R = B$. Since R is sensitive to our choice between $\mathcal{M}_1$ and $\mathcal{M}_2$, any preference between actions A and B can be treated as a 'tie-breaker' between the two models. That is, if A is preferred to B, then choose $\mathcal{M}_1$ to represent the given token probabilistic scenario; if B is preferred to A, then choose $\mathcal{M}_2$ instead. No need for any arbitrary choices here.'[40]

I have two replies. First, this objection only works if we *have* a preference between actions A and B. What about those cases where we *don't* have any such preference? Then, it seems that we're back to square one. Here's the second and more important reply. At least for fair lotteries, choosing which numerical model to represent such token probabilistic scenarios should intuitively depend on the specification and, more importantly, the symmetry of the scenario. Our preferences between actions A and B, if we have any at all, should have no part to play in our choice of such a model. After all, our preferences may not even be *rational* at all.[41] With these two considerations above, this objection fails.

Objection 2. The second objection goes something like the following:

> 'Consider the following token probabilistic scenario $\langle\{H, T\}, \mathcal{P}\{H, T\}, C\rangle$ where $\mathcal{P}\{H, T\}$ is the powerset of $\{H, T\}$ and C contains the probability axioms, **Regularity**, and the statement 'The coin is biased towards tails'. Put simply, this scenario is about tossing a coin that's biased towards tails. It is easy to verify that $\langle\{H, T\}, \mathcal{P}\{H, T\}, C\rangle$ is fully specified.[42]
>
> Yet, there are still uncountably many correct numerical models for this scenario. Let P_1 be a probability function that assigns 0.4 probability to $\{H\}$ and 0.6 probability to $\{T\}$. Let P_2 be a probability function that assigns $0.4 + \varepsilon$ probability to $\{H\}$ and $0.6 - \varepsilon$ probability to $\{T\}$ where ε is a positive infinitesimal. Define $\mathcal{M}_1 = \langle\{H, T\}, \mathcal{P}\{H, T\}, C, \Sigma_1\rangle$ where $\Sigma_1 = \{(A, P_1(A)) : A \in \mathcal{P}\{H, T\}\}$ and $\mathcal{M}_2 = \langle\{H, T\}, \mathcal{P}\{H, T\}, C, \Sigma_2\rangle$ where $\Sigma_2 = \{(A, P_2(A)) : A \in \mathcal{P}\{H, T\}\}$. Clearly, $\mathcal{M}_1$ and $\mathcal{M}_2$ are correct numerical models of $\langle\{H, T\}, \mathcal{P}\{H, T\}, C\rangle$. Furthermore, $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same representational power in modelling $\langle\{H, T\}, \mathcal{P}\{H, T\}, C\rangle$ too.

[40] Thanks to an anonymous referee for one of my papers for suggesting this.

[41] For examples of irrational preferences, see Ellsberg [1961]. To be sure, Ellsberg [1961] intended for these preferences to be intuitively *rational*. But I disagree; I think the preferences he gave are *irrational*.

[42] Thanks to Alan Hájek for providing this example. But to be sure, Alan didn't intend for **Regularity** to be included in C. I was the one who included it.

Also, it is easy to find actions A and B such that if $\mathcal{M}_1$ is chosen to represent $\langle\{H,T\},\mathcal{P}\{H,T\},C\rangle$, then we should perform A over B, whereas if $\mathcal{M}_2$ is chosen instead, then we should perform B over A. Let R be the action that we should perform between actions A and B. Given the stipulations above, if $\mathcal{M}_1$ is chosen to represent $\langle\{H,T\},\mathcal{P}\{H,T\},C\rangle$, then $R = A$; if $\mathcal{M}_2$ is chosen instead, then $R = B$. So, R is sensitive to the particular choice of model.

Intuitively, it is *unproblematic* w.r.t R to choose $\mathcal{M}_1$ over $\mathcal{M}_2$ to represent $\langle\{H,T\},\mathcal{P}\{H,T\},C\rangle$. After all, all we know about the coin is that it is biased towards tails. In fact, in some sense, this situation is like Thomson's Lamp all over again. There is just not enough information provided of $\langle\{H,T\},\mathcal{P}\{H,T\},C\rangle$ to determine the exact probabilities to be assigned to $\{H\}$ and $\{T\}$, just like how there is not enough information provided of the set-up of Thomson's Lamp to determine whether the lamp is on or off after the finite interval of time. And just as it is unproblematic to say that the lamp is on after the finite interval of time, it is also unproblematic to choose $\mathcal{M}_1$ over $\mathcal{M}_2$ to represent $\langle\{H,T\},\mathcal{P}\{H,T\},C\rangle$.

Why think that choosing a particular model over other competing models to represent fair infinite lotteries is any different from choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ to represent $\langle\{H,T\},\mathcal{P}\{H,T\},C\rangle$?'

Here's my reply. There may be no difference between choosing a particular model over other competing models to represent a fair infinite lottery over an unspecified sample space and choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ to represent $\langle\{H,T\},\mathcal{P}\{H,T\},C\rangle$. But if the former case is swapped out for a *fully specified* token probabilistic scenario $\langle\Omega,\mathcal{F},C\rangle$ where Ω is infinite and **Fairness** is one of the constraints, though not necessarily the only constraint, in C, then there is a difference.

On the one hand, for $\langle\{H,T\},\mathcal{P}\{H,T\},C\rangle$, there is not enough information provided to determine the exact probabilities to be assigned to $\{H\}$ and $\{T\}$. But on the other hand, for a *fully specified* token probabilistic scenario $\langle\Omega,\mathcal{F},C\rangle$ where Ω is infinite and **Fairness** is one of the constraints, though not necessarily the only constraint, in C, it seems that the information provided should be enough to determine the exact probabilities to be assigned to each event in $\mathcal{F}$. To drive the point home, consider for a moment fair *finite* lotteries, i.e. token probabilistic scenarios $\langle\Omega,\mathcal{P}\Omega,C\rangle$ where Ω is finite, $\mathcal{P}\Omega$ is the powerset of Ω, and C contains the probability axioms and **Fairness**. In these scenarios, C completely determines the probabilities to be assigned to every event. And this is true even for *under-specified* token probabilistic scenarios $\langle\Omega,\mathcal{P}\Omega,C\rangle$ where Ω is an unspecified finite set and C contains the probability axioms and **Fairness**. Even in these latter scenarios, C still completely determines the probabilities to be assigned to every event. So, it is surprising if this doesn't carry over to *fully specified* token probabilistic scenarios

$\langle \Omega, \mathcal{F}, C \rangle$ where Ω is infinite and C contains more than just the probability axioms and **Fairness**. Therefore, there is a difference between choosing a particular model over others to represent a *fully specified* token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$, where Ω is infinite and C contains more than just the probability axioms and **Fairness**, and choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ to represent $\langle \{H, T\}, \mathcal{P}\{H, T\}, C \rangle$. With that, this objection fails.[43]

[43]Adding to the point I've made above, I find it puzzling why $\mathcal{M}_1$ is chosen to represent $\langle \{H, T\}, \mathcal{P}\{H, T\}, C \rangle$. After all, there is not enough information provided to determine the exact *precise* probabilities to be assigned to $\{H\}$ and $\{T\}$. But there *is* enough information provided to determine the *imprecise* probabilities to be assigned to $\{H\}$ and $\{T\}$. If, indeed, all we know about the coin is that it is biased towards tails, then why not assign $\{H\}$ an imprecise probability of $(0, \frac{1}{2})$ and $\{T\}$ an imprecise probability of $(\frac{1}{2}, 1)$? Define $\mathcal{M}_3 = \langle \{H, T\}, \mathcal{P}\{H, T\}, C, \Sigma_3 \rangle$ where $\Sigma_3 = \{(\emptyset, 0), (\{H\}, (0, \frac{1}{2})), (\{T\}, (\frac{1}{2}, 1)), (\{H, T\}, 1)\}$. Why not choose $\mathcal{M}_3$ instead to represent $\langle \{H, T\}, \mathcal{P}\{H, T\}, C \rangle$? $\mathcal{M}_3$ does seem to reflect the available information better than $\mathcal{M}_1$ or $\mathcal{M}_2$. But of course, if the statement 'All probabilities are precise' is added to C in the scenario spelled out in the objection above, then this point does not apply to the newly modified token probabilistic scenario.

3 A Fair Lottery Over the Natural Numbers

In this section, I will specify the exact token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C \rangle$ I'm interested in for the purposes of this thesis. That is, I will specify $\mathcal{F}$ and the constraints in C. But because the constraints in C are in terms of *qualitative probabilities*, I have to first introduce the notion of qualitative probabilities. This will be done in §3.1. Then, in §3.2, I will list the constraints in C, after which I will define the *smallest* algebra $\mathcal{F}$ required to satisfy all the constraints in C. Also, in this subsection, I will show that $\langle \mathbb{N}, \mathcal{F}, C \rangle$ is fully specified. Having already specified each item in the triple, all that's left to do in showing that $\langle \mathbb{N}, \mathcal{F}, C \rangle$ is fully specified is prove that C fully determines the order of events in $\mathcal{F}$.

After showing that $\langle \mathbb{N}, \mathcal{F}, C \rangle$ is fully specified, in §3.3, I will show that there are at least two probability functions P_1 and P_2 correctly representing $\langle \mathbb{N}, \mathcal{F}, C \rangle$ that satisfy all four conditions of **Infinitesimal Underdetermination**. From these two probability functions P_1 and P_2, two different correct numerical models $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\langle \mathbb{N}, \mathcal{F}, C \rangle$, both of which are expressed in terms of precise probabilities, can be defined respectively. Hereafter, for the sake of convenience, I will call numerical models expressed exclusively in terms of precise probabilities *precise models* and numerical models expressed in terms of precise and imprecise probabilities *imprecise models*. After defining $\mathcal{M}_1$ and $\mathcal{M}_2$, I will show that choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ satisfies the conditions of **Problematic Choice** w.r.t an end result R, which will be specified soon. Therefore, choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ is problematic w.r.t R.

In §3.4, I explore potential constraints that we can add to C such that the new collection of constraints uniquely determines *the* probability function defined on $\mathcal{F}$ that satisfies all the constraints in C', which contains the old constraints in C and also the new added ones, i.e. $C \subseteq C'$. However, I will argue that there is no such constraint that can be added to C for which an intuitive justification can be provided.

3.1 Qualitative Probabilities

As usual in the literature, let Ω be the sample space and $\mathcal{F}$ be an algebra on Ω. By *qualitative probability*, I mean the binary relation of an event A being at most as probable as an event B; it is possible that $A = B$. For any two events A and B, let $A \preceq B$ represent the statement 'A is at most as probable as B'. From this relation, other relations can be defined.

Definition 1. *Let A and B be events.*

1. $A \sim B := A \preceq B \wedge B \preceq A$. *This represents the statement 'A is equiprobable to B'.*
2. $A \prec B := A \preceq B \wedge B \npreceq A$. *This represents the statement 'A is less probable than B'.*

In most cases, I will ensure that the relations $\preceq$ and $\prec$ point in the same direction. Occasionally, however, it will be more convenient to use the notations $\succeq$ and $\succ$ instead. Hereafter, whenever I say '$A \succeq B$' and '$A \succ B$', they are equivalent to $B \preceq A$ and $B \prec A$ respectively.

Note that $\mathcal{F}$ is the domain of a qualitative probability $\preceq$, i.e. if $A \preceq B$ or $A \succeq B$, then both A and B are in $\mathcal{F}$, although the converse isn't necessarily true. That is, even if both A and B are in $\mathcal{F}$, that doesn't imply that either $A \preceq B$ or $A \succeq B$; it is still possible that $A \npreceq B$ and $A \nsucceq B$. More about this soon. Now, for all qualitative probabilities $\preceq$, $\preceq$ obeys the following constraints. For $A, B, C \in \mathcal{F}$,

Non-Negativity. $\emptyset \preceq A$.

Non-Triviality. $\emptyset \prec \Omega$.

Transitivity. *If $A \preceq B$ and $B \preceq C$, then $A \preceq C$.*

Additivity. *Let $A \cap C = B \cap C = \emptyset$. $A \preceq B$ iff $A \cup C \preceq B \cup C$.*

Next, call a qualitative probability $\preceq$ *regular* just in case $\preceq$ satisfies the following constraint:

Qualitative Regularity. *For all $\emptyset \neq A \in \mathcal{F}$, $\emptyset \prec A$.*

Qualitative Regularity is the qualitative analogue of **Regularity**, as stated in the introduction. Note that I'm not assuming that **Qualitative Regularity** is a constraint on *all* qualitative probabilities $\preceq$. That is, while there exist regular qualitative probabilities, *irregular* qualitative probabilities may exist too, in that these latter qualitative probabilities admit violations of **Qualitative Regularity**. The only constraints I'm assuming that hold for all qualitative probabilities $\preceq$ are **Non-Negativity**, **Non-Triviality**, **Transitivity**, and **Additivity**. Indeed, these four constraints are common across all major axiomatizations of qualitative probability, e.g. Keynes [1921], de Finetti [1937], Koopman [1940], Luce [1968], Domotor [1969], Krantz et al. [1971], Fine [1973], etc.

Just like **Qualitative Regularity**, note that I'm not assuming that the following is a constraint on all qualitative probabilities $\preceq$.

Totality. *For all A and B in $\mathcal{F}$, either $A \preceq B$ or $A \succeq B$.*

That is, while there exist qualitative probabilities that satisfy **Totality**, qualitative probabilities that don't may exist too. To be clear, there may be events A and B such that (i) $\emptyset \preceq A$ and $\emptyset \preceq B$ but (ii) $A \npreceq B$ and $A \nsucceq B$. From (i), both A and B are in $\mathcal{F}$. Yet, according to (ii), $A \npreceq B$ and $A \nsucceq B$, violating **Totality**. This is why I said that while it is true that if $A \preceq B$ or $A \succeq B$, then both A and B are in $\mathcal{F}$, the converse isn't necessarily true, i.e. even if both A and B are in $\mathcal{F}$, that doesn't imply that either $A \preceq B$ or $A \succeq B$.[44]

And just like **Qualitative Regularity** and **Totality**, note that I'm not assuming that the following is a constraint on all qualitative probabilities $\preceq$.

Monotone Continuity. *Let $A, B \in \mathcal{F}$ and $A_i \in \mathcal{F}$ for all $i \in \mathbb{N}$. Suppose:*

1. $A_i \subseteq A_{i+1}$ *for all* $i \in \mathbb{N}$.
2. $A = \bigcup_{i=1}^{\infty} A_i$.
3. $A_i \preceq B$ *for all* $i \in \mathbb{N}$.

Then, $A \preceq B$.

That is, while there exist qualitative probabilities that satisfy **Monotone Continuity**, qualitative probabilities that don't may exist too. Before explaining why I'm not assuming that **Monotone Continuity** is a constraint on all qualitative probabilities $\preceq$, some explication on what this principle is saying is required. **Monotone Continuity** is formulated by Villegas [1964] as a qualitative analogue of **Countable Additivity**, which is the statement that $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ if all the A_i's are pairwise disjoint and $P(A_i)$ is defined for all A_i's. Indeed, **Countable Additivity** implies the following statement[45]:

Upper Bound Continuity. *Let $A, B \in \mathcal{F}$ and $A_i \in \mathcal{F}$ for all $i \in \mathbb{N}$. Suppose:*

1. $A_i \subseteq A_{i+1}$ *for all* $i \in \mathbb{N}$.
2. $A = \bigcup_{i=1}^{\infty} A_i$.
3. $P(A_i) \leq P(B)$ *for all* $i \in \mathbb{N}$.

Then, $P(A) \leq P(B)$.

Notice that **Monotone Continuity** is just the qualitative analogue of **Upper Bound Continuity**.[46] In fact, Villegas [1964] has proven that if a qualitative probability $\preceq$ satisfies **Monotone Continuity**, then there does not exist a countable partition $\{A_i : i \in \mathbb{N}\}$ of Ω such that $A_i \sim A_j$. Put simply, if a qualitative probability $\preceq$ satisfies **Monotone Continuity**, then there are no fair countably infinite lotteries. But since the main object of interest in this section is a fair lottery over $\mathbb{N}$, it cannot be assumed that **Monotone Continuity** is a constraint on all qualitative probabilities $\preceq$. Furthermore, since we are going to talk about non-Archimedean probability functions, we cannot assume that all numerical probabilities satisfy **Countable Additivity**, as non-Archimedean ordered

[44] For examples of such events A and B, see Keynes [1921] and Fishburn [1986].

[45] The proof of this claim is straightforward and thus will not be reproduced here.

[46] This has already been noticed by DiBella [2021].

fields are not complete (**Theorem 2**). Therefore, it seems only fitting that we also don't assume that all qualitative probabilities $\preceq$ satisfy **Monotone Continuity**, which is the qualitative analogue of **Countable Additivity**.

Now, we're almost all set to specify the exact token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C \rangle$ I'm interested in for the purposes of this thesis. All that's left to do is to restate **Fairness**, but in terms of qualitative probabilities this time.

Fairness. *For all* $\omega, \omega' \in \Omega$, $\{\omega\} \sim \{\omega'\}$.

3.2 Specifying the Constraints and the Smallest Algebra

3.2.1 The Constraints

Before stating the constraints, note that a fair lottery over $\mathbb{N}$ is a *type probabilistic scenario*, i.e. a set of token probabilistic scenarios $\langle \Omega, \mathcal{F}, C \rangle$ where $\Omega = \mathbb{N}$ and **Fairness** is one of the constraints in C. This means that there are many *different* token probabilistic scenarios that fall under the same type probabilistic scenario of a fair lottery over $\mathbb{N}$. To be clear, under this type probabilistic scenario of a fair lottery over $\mathbb{N}$, there may be token probabilistic scenarios with different $\mathcal{F}$'s and different C's, though the sample space is the same across all token probabilistic scenarios in this set: $\mathbb{N}$, and **Fairness** must be at least one of the constraints in all such C's. In this subsection, I'm only going to specify *one* token probabilistic scenario under this type probabilistic scenario of a fair lottery over $\mathbb{N}$. The reader should therefore not misinterpret me as saying that the token probabilistic scenario I'm going to specify is *paradigmatic* of a fair lottery over $\mathbb{N}$. That's not what I'm saying at all. Instead, the token probabilistic scenario I'm going to specify is just one example of a fair lottery over $\mathbb{N}$. After all, there may very well be *infinitely* many other token probabilistic scenarios that fall under the same type probabilistic scenario of a fair lottery over $\mathbb{N}$. However, I'm not interested in these other token probabilistic scenarios.

Having clarified the aim of this subsection, what are some constraints that are intuitively satisfied by a fair lottery over $\mathbb{N}$? Or rather, what are some constraints that appear in *every* C of all the token probabilistic scenarios that fall under the same type probabilistic scenario of a fair lottery over $\mathbb{N}$? Here are some intuitive ones, without explanation first:

1. **Non-Negativity**.
2. **Non-Triviality**.
3. **Transitivity**.
4. **Additivity**.
5. **Fairness**.

6. **Comparing Finite and Infinite Sets**: For all finite $A \in \mathcal{F}$ and infinite $B \in \mathcal{F}$, $A \preceq B$.

Since **Non-Negativity**, **Non-Triviality**, **Transitivity**, and **Additivity** are constraints on all qualitative probabilities $\preceq$, they have to appear in every C of all the token probabilistic scenarios that fall under the same type probabilistic scenario of a fair lottery over $\mathbb{N}$. And since we are talking about a *fair* lottery over $\mathbb{N}$, the same goes for **Fairness**. However, a little more explication is required for **Comparing Finite and Infinite Sets**. Intuitively, in any fair infinite lottery, a finite set is always at most as probable as an infinite set in containing the winning ticket. After all, the former contains *fewer* members than the latter. Since a fair lottery over $\mathbb{N}$ is a fair infinite lottery, **Comparing Finite and Infinite Sets** has to appear in every C of all the token probabilistic scenarios that fall under the same type probabilistic scenario of a fair lottery over $\mathbb{N}$. Hereafter, let $\#(A)$ denote the cardinality of A.

Theorem 3. *Let A and B be any two finite subsets of $\mathbb{N}$ such that $\#(A) \leq \#(B)$. Assume* ***Non-Negativity, Non-Triviality, Transitivity, Additivity, Fairness****, and* ***Comparing Finite and Infinite Sets****. Then, $A \preceq B$.*

Theorem 3 just states that in a fair lottery over $\mathbb{N}$, for any two finite subsets A and B of $\mathbb{N}$, if A contains at most as many members as B, then $A \preceq B$, which is intuitively true.

Now, what are the constraints that appear in the C of the particular token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C \rangle$ I'm interested in for the purposes of this thesis? There are only two. Note that these two constraints need *not* appear in every C of all the token probabilistic scenarios that fall under the same type probabilistic scenario of a fair lottery over $\mathbb{N}$.

1. **Qualitative Regularity**.

2. **Partition**: Consider the following partitions of $\mathbb{N}$:

$$A_1 = \{1, 1+n, 1+2n, \ldots\}.$$

$$A_2 = \{2, 2+n, 2+2n, \ldots\}.$$

$$\ldots$$

$$A_n = \{n, 2n, 3n, \ldots\}.$$

and

$$B_1 = \{1, 1+m, 1+2m, \ldots\}.$$

$$B_2 = \{2, 2+m, 2+2m, \ldots\}.$$

$$\ldots$$

$$B_m = \{m, 2m, 3m, \ldots\}.$$

Let $A = \bigcup_{k=1}^{p} A_{i_k}$ be the union of any p A_i's and $B = \bigcup_{j=1}^{q} B_{i_j}$ be the union of any q B_i's. If $p = 0$, then $A = \emptyset$. Similarly, if $q = 0$, then $B = \emptyset$. Now, let $A' \subseteq A$ and $A^* \cap A = \emptyset$ where $A' \cup A^*$ is finite. Similarly, let $B' \subseteq B$ and $B^* \cap B = \emptyset$ where $B' \cup B^*$ is finite. Then, $(A - A') \cup A^* \preceq (B - B') \cup B^*$ iff *either* of the following holds:

(a) $\frac{p}{n} < \frac{q}{m}$.

(b) $\frac{p}{n} = \frac{q}{m}$ and $\#(A^*) - \#(A') \leq \#(B^*) - \#(B')$.[47,48]

Explication is in order. Why is **Qualitative Regularity** included as a constraint in the C of the particular token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C \rangle$ I'm interested in? There is only one reason for doing so: for the sole purpose of arguing that choosing probabilistic model $\mathcal{M}_1$ over probabilistic model $\mathcal{M}_2$, both of which are correct precise models of $\langle \mathbb{N}, \mathcal{F}, C \rangle$, is problematic w.r.t an end result R.

Let's consider **Partition** now. Intuitively, $\frac{p}{n}$ is the *asymptotic density*[49] of $A = \bigcup_{k=1}^{p} A_{i_k}$, as defined in **Partition**. For example, consider the following partition of $\mathbb{N}$:

$$A_1 = \{1, 4, 7, \ldots\}.$$

$$A_2 = \{2, 5, 8, \ldots\}.$$

$$A_3 = \{3, 6, 9, \ldots\}.$$

Say $A = A_1 \cup A_2$. Then, $p = 2$ and $n = 3$. So, $\frac{p}{n} = \frac{2}{3}$ and that is the asymptotic density of A. The same goes for $\frac{q}{m}$, i.e. $\frac{q}{m}$ is the asymptotic density of $B = \bigcup_{j=1}^{q} B_{i_j}$, as defined in **Partition**. And in a fair lottery over $\mathbb{N}$, for any two sets A and B such that the asymptotic density of A is less than that of B, $A \preceq B$. This explains condition (a) in **Partition**. However, when we are considering two sets A and B of *equal* asymptotic densities, we must tread more carefully. Notice that it is *inconsistent* with the rest of the constraints I've listed to say that if A and B have equal asymptotic densities, then $A \preceq B$. This statement cannot hold for all such A's and B's, on pain of contradiction. For instance, consider $A = \mathbb{N}$ and $B = \mathbb{N} - \{1\}$. Both sets have equal asymptotic densities: 1. But it cannot be that $\mathbb{N} \preceq \mathbb{N} - \{1\}$. For if it were the case, then, by **Additivity**, $\{1\} \preceq \emptyset$. Contradiction, as by **Qualitative Regularity**, $\{1\} \succ \emptyset$. Therefore, sets A and B having equal asymptotic densities doesn't imply that $A \preceq B$; a finer-grained comparison is required.

Here's how to deliver the correct verdict regarding $\mathbb{N}$ and $\mathbb{N} - \{1\}$, i.e. $\mathbb{N} - \{1\} \prec \mathbb{N}$. Notice first that $\mathbb{N} - \{1\}$ has a 'deficit' of 1 member. After all, $\mathbb{N} - \{1\} \subsetneq \mathbb{N}$. In the

[47] It is possible that $n = m$. If $n = m$, then the two partitions above are identical.

[48] This constraint is inspired by DiBella [2021].

[49] The asymptotic density of a set $A \subseteq \mathbb{N}$ is defined as $\lim_{n\to\infty} \frac{\#(A \cap \{1,\ldots,n\})}{n}$.

parlance of **Partition**, let $A = \mathbb{N}$, $A^{'} = \{1\}$, and $A^* = \emptyset$.[50] Then, $\mathbb{N} - \{1\} = (A - A^{'}) \cup A^*$ and $\#(A^*) - \#(A^{'}) = 0 - 1 = -1$, which reflects the 'deficit' of 1 member. As for $\mathbb{N}$, let $B = \mathbb{N}$, $B^{'} = \emptyset$, and $B^* = \emptyset$. Then, $\mathbb{N} = (B - B^{'}) \cup B^*$ and $\#(B^*) - \#(B^{'}) = 0 - 0 = 0$, i.e. $\mathbb{N}$ has no 'deficit' or 'surplus' of members. Since $\mathbb{N}$ and $\mathbb{N} - \{1\}$ have equal asymptotic densities and $\mathbb{N} - \{1\}$ has a 'deficit' of 1 member but $\mathbb{N}$ has none, $\mathbb{N} - \{1\} \prec \mathbb{N}$, in accordance with **Qualitative Regularity**.

In order to further demonstrate the method above on comparing sets of equal asymptotic densities, here's another example. Consider the sets $\{1\} \cup \{4, 6, 8, \ldots\}$ and $\{2\} \cup \{1, 3, 5, 7, \ldots\}$, both of which have equal asymptotic densities of $\frac{1}{2}$. In the parlance of **Partition**, let $A = \{2, 4, 6, 8, \ldots\}$, $A^{'} = \{2\}$, $A^* = \{1\}$, $B = \{1, 3, 5, 7, \ldots\}$, $B^{'} = \emptyset$, and $B^* = \{2\}$.[51] Then, we have $\{1\} \cup \{4, 6, 8, \ldots\} = (A - A^{'}) \cup A^*$ and $\{2\} \cup \{1, 3, 5, 7, \ldots\} = (B - B^{'}) \cup B^*$. Also, $\#(A^*) - \#(A^{'}) = 1 - 1 = 0$ and $\#(B^*) - \#(B^{'}) = 1 - 0 = 1$. That is, even though one member (i.e. 2) was, in some sense, 'removed' from $\{1\} \cup \{4, 6, 8, \ldots\}$, it was 'replaced' by another member (i.e. 1). Therefore, $\{1\} \cup \{4, 6, 8, \ldots\}$, like $\mathbb{N}$, has no 'deficit' or 'surplus' of members. But for $\{2\} \cup \{1, 3, 5, 7, \ldots\}$, there is a 'surplus' of 1 member. After all, 2 was added. And for that reason, $\{1\} \cup \{4, 6, 8, \ldots\} \prec \{2\} \cup \{1, 3, 5, 7, \ldots\}$, despite the fact that both sets have equal asymptotic densities of $\frac{1}{2}$. And indeed, $\{1\} \cup \{4, 6, 8, \ldots\} \prec \{2\} \cup \{1, 3, 5, 7, \ldots\}$ is the correct verdict for the comparison between $\{1\} \cup \{4, 6, 8, \ldots\}$ and $\{2\} \cup \{1, 3, 5, 7, \ldots\}$, if we assume that $\{2, 4, 6, 8, \ldots\} \sim \{1, 3, 5, 7, \ldots\}$. To see why, note that since $\{1, 3, 5, 7, \ldots\} \subsetneq \{2\} \cup \{1, 3, 5, 7, \ldots\}$, $\{1, 3, 5, 7, \ldots\} \prec \{2\} \cup \{1, 3, 5, 7, \ldots\}$.[52] Apply **Transitivity** to $\{2, 4, 6, 8, \ldots\} \sim \{1, 3, 5, 7, \ldots\}$ and $\{1, 3, 5, 7, \ldots\} \prec \{2\} \cup \{1, 3, 5, 7, \ldots\}$ to get $\{2, 4, 6, 8, \ldots\} \prec \{2\} \cup \{1, 3, 5, 7, \ldots\}$. By **Fairness**, $\{1\} \sim \{2\}$. Then, apply **Additivity** to get $\{1\} \cup \{4, 6, 8, \ldots\} \sim \{2\} \cup \{4, 6, 8, \ldots\} = \{2, 4, 6, 8, \ldots\}$. Finally, apply **Transitivity** to $\{1\} \cup \{4, 6, 8, \ldots\} \sim \{2, 4, 6, 8, \ldots\}$ and $\{2, 4, 6, 8, \ldots\} \prec \{2\} \cup \{1, 3, 5, 7, \ldots\}$ to get $\{1\} \cup \{4, 6, 8, \ldots\} \prec \{2\} \cup \{1, 3, 5, 7, \ldots\}$. This explains condition (b) in **Partition**.

So, why is **Partition** included as a constraint in the C of the particular token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C \rangle$ I'm interested in? There are two reasons. Firstly, I want to compare not just finite sets, but infinite sets as well. For example, I want to say that the even numbers $\{2, 4, 6, \ldots\}$ is equiprobable to the odd numbers $\{1, 3, 5, \ldots\}$ in containing the winning ticket[53], as well as comparisons like the set $\{2, 5, 8, \ldots\}$ is less probable in containing the winning ticket than the set $\{1, 3, 5, \ldots\}$, the set $\{1\} \cup \{10, 20, 30, \ldots\}$ is

[50] The relevant partition here consists of only one member: $\mathbb{N}$ itself.

[51] The relevant partition for both cases consists of the sets $\{1, 3, 5, 7, \ldots\}$ and $\{2, 4, 6, 8, \ldots\}$.

[52] See **Lemma 2** for the proof.

[53] Wenmackers & Horsten [2013] have shown that there is a regular probability function P that represents a fair lottery over $\mathbb{N}$ which assigns higher probability to the odd numbers $\{1, 3, 5, \ldots\}$ than to the even numbers $\{2, 4, 6, \ldots\}$. According to this P, $\{2, 4, 6, \ldots\}$ *isn't* equiprobable to $\{1, 3, 5, \ldots\}$ in containing the winning ticket. Instead, $\{2, 4, 6, \ldots\}$ is *less* probable than $\{1, 3, 5, \ldots\}$ in containing the winning ticket. This is why **Partition** does not appear in every C of all the token probabilistic scenarios that fall under the same type probabilistic scenario of a fair lottery over $\mathbb{N}$, but only in the C of the particular token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C \rangle$ I'm interested in.

less probable in containing the winning ticket than the set $\{4, 8, 12, \ldots\}$, etc. In order to make comparisons like that, **Partition** is needed. Secondly, **Partition** is needed to ensure that $\langle \mathbb{N}, \mathcal{F}, C \rangle$ is fully specified. More about this soon.

These are the constraints in the C of the particular token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C \rangle$ I'm interested in. Namely, **Non-Negativity**, **Non-Triviality**, **Transitivity**, **Additivity**, **Fairness**, **Comparing Finite and Infinite Sets**, **Qualitative Regularity**, and **Partition**. For the sake of differentiating this collection of the eight constraints just listed from the other collections of constraints in other token probabilistic scenarios, let $C^{\#}$ denote the former collection. That is, $C^{\#}$ contains as members the eight constraints just listed.

3.2.2 The Smallest Algebra

What are the various subsets of $\mathbb{N}$ that must be contained in an algebra $\mathcal{F}$ such that a probability function P defined on $\mathcal{F}$ satisfies all the constraints in $C^{\#}$? Here is a helpful list:

1. All finite sets. (Because **Fairness** quantifies over all the singleton sets of $\mathbb{N}$ and all algebras $\mathcal{F}$ contain $\emptyset$ and are closed under finite unions.)

2. All cofinite sets.[54] (Because the algebra I'm interested in contains all the finite sets and all algebras $\mathcal{F}$ are closed under complements.)

3. Let $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ be a partition of $\mathbb{N}$ such that:

$$G_1 = \{1, 1+n, 1+2n, \ldots\}.$$

$$G_2 = \{2, 2+n, 2+2n, \ldots\}.$$

$$\ldots$$

$$G_n = \{n, 2n, 3n, \ldots\}.$$

   Every set G such that G is a nonempty union of any number of G_i's. (Because **Partition** is a constraint in $C^{\#}$.)

4. Let $G' \subseteq G$ and $G^* \cap G = \emptyset$ such that $G' \cup G^*$ is finite. Every set H such that $H = (G - G') \cup G^*$. (Because **Partition** is a constraint in $C^{\#}$ and all algebras $\mathcal{F}$ are closed under complements and finite unions.)

For the sake of differentiating the smallest algebra containing the four kinds of subsets listed above from the other algebras in other token probabilistic scenarios, let $\mathcal{F}^{\#}$ denote the former algebra. I will now show that the collection of all and only those subsets listed

[54]A set $A \subseteq \Omega$ is *cofinite* just in case $\Omega - A$ is finite.

above is an algebra. And since this algebra contains no more and no fewer subsets than the ones listed above, it is the smallest algebra containing the four kinds of subsets listed above.

Let's start with some definitions. Call a subset $S \subseteq \mathbb{N}$ a *maximal arithmetic progression* just in case S is infinite and its members can be arranged in an increasing order $a_1, \ldots, a_n, \ldots$ where $n \in \mathbb{N}$ such that:

1. $a_n = a_1 + (n-1)m$ for some common difference $m \in \mathbb{N}$. (It is an arithmetic progression.)

2. $a_1 \leq m$. (It is maximal.)

For instance, the set of even numbers $\{2, 4, 6, \ldots\}$ is a maximal arithmetic progression while the set $\{4, 6, 8, \ldots\}$ isn't as $4 \not\leq 2$.

Next, call a subset $S = S_1 \cup \ldots \cup S_k \subseteq \mathbb{N}$ a *group of maximal arithmetic progressions* (or 'a group[55] of maximal arithmetic progression' if $k = 1$) sharing the same common difference just in case:

1. $S \neq \emptyset$.

2. S_i is a maximal arithmetic progression where $1 \leq i \leq k$.

3. $S_i \cap S_j = \emptyset$ if $i \neq j$.

4. Let $S_i \neq S_j$. Arrange the elements of S_i in increasing order $a_1, \ldots, a_n, \ldots$. Similarly, arrange the elements of S_j in increasing order $b_1, \ldots, b_n, \ldots$. Then, $a_{n+1} - a_n = b_{n+1} - b_n = m$, i.e. all maximal arithmetic progressions in S share the same common difference m.

5. $k \leq m$.

For instance, $\mathbb{N} = \{1, 2, 3, \ldots\}$ is a group of maximal arithmetic progression as it consists of a single ($k = 1$) maximal arithmetic progression $\{1, 2, 3, \ldots\}$ of common difference 1 ($m = 1$) where $k \leq m$. The set $\{2, 3, 5, 6, 8, 9, 11, \ldots\}$ is a group of maximal arithmetic progressions too. This is because this set is the union of $\{2, 5, 8, 11, \ldots\}$ and $\{3, 6, 9, \ldots\}$, both of which are mutually disjoint maximal arithmetic progressions sharing the same common difference 3 ($m = 3$). Since there are two maximal arithmetic progressions in this set, $k = 2$. So, $k \leq m$. Importantly, notice that the set G, as defined in the third kind of subsets in the list of four above, is a group of maximal arithmetic progressions.

Now, define $\mathcal{H}$ to be the set of all subsets $A \subseteq \mathbb{N}$ fulfilling the following form:

$$A = (A_1 - A_2) \cup A_3 \tag{1}$$

[55] The usage of the word 'group' here has nothing to do with group theory.

where A_1 is either $\emptyset$ or a group of maximal arithmetic progressions (or a group of maximal arithmetic progression), A_2 is a finite set such that $A_2 \subseteq A_1$, and A_3 is also a finite set such that $A_1 \cap A_3 = \emptyset$.[56] Note that it is possible for A_1, A_2, or A_3 to be empty. Call the ordered triple $\{A_1, A_2, A_3\}$ the *form* A fulfills just in case (i) A_1, A_2, and A_3 fulfill the conditions spelled out above and (ii) $A = (A_1 - A_2) \cup A_3$.

Crucially, notice that $\mathcal{H}$ contains all and only those subsets that are listed above. Let's go through each of the four kinds of subsets listed above to verify. By setting $A_1 = \emptyset$, all finite subsets are in $\mathcal{H}$, in virtue of A_3 being finite. Also, by setting $A_1 = \mathbb{N}$ and $A_3 = \emptyset$, all cofinite subsets are in $\mathcal{H}$, in virtue of A_2 being finite. Now, consider the set G, as defined in the third kind of subsets in the list of four above. To see why $G \in \mathcal{H}$, set $A_1 = G$ and $A_2 = A_3 = \emptyset$. After all, as I've mentioned above, G is a group of maximal arithmetic progressions. Last, it is easy to see that subsets that fall under the fourth kind of subsets are also in $\mathcal{H}$. Just set $A_1 = G$, $A_2 = G^{'}$, and $A_3 = G^*$. So, $\mathcal{H}$ contains all those subsets that are listed. Does $\mathcal{H}$ contain only those subsets that are listed above? For reductio, assume that $\mathcal{H}$ contains a subset A that doesn't fall under any of the kinds of subsets listed above. This means that A is an infinite subset that is not cofinite and not a group of maximal arithmetic progressions. It is also not $(G - G^{'}) \cup G^*$ for all G, $G^{'}$, and G^*, as defined in the last kind of subsets in the list of four above. Let the ordered triple $\{A_1, A_2, A_3\}$ be the form of A, i.e. $A = (A_1 - A_2) \cup A_3$. Since A is infinite, $A_1 \neq \emptyset$. By the definition above, this means that A_1 is a group of maximal arithmetic progressions (or a group of maximal arithmetic progression). But then, this implies that there is some G, as defined in the third kind of subsets in the list of four above, such that $A_1 = G$. Similarly, there is some $G^{'}$ such that $A_2 = G^{'}$ and there is some G^* such that $A_3 = G^*$. So, $A = (G - G^{'}) \cup G^*$ for some G, $G^{'}$, and G^*. Contradiction. Therefore, $\mathcal{H}$ contains all and only those subsets that are listed above.

Note that the maximal arithmetic progressions that make up A_1 may vary for a particular $A \in \mathcal{H}$. For instance, let $A = \mathbb{N}$. It is possible for A_1 to be made up of a single maximal arithmetic progression, namely $\{1, 2, 3, \ldots\}$, or multiple maximal arithmetic progressions, e.g. $\{1, 3, 5, \ldots\}$ and $\{2, 4, 6, \ldots\}$. Regardless, for all $A \in \mathcal{H}$, the form A fulfills is unique.

Theorem 4. *For all $A \in \mathcal{H}$, the form A fulfills is unique, i.e. if both ordered triples $\{A_1, A_2, A_3\}$ and $\{A_1^{'}, A_2^{'}, A_3^{'}\}$ are forms that A fulfills, then $\{A_1, A_2, A_3\} = \{A_1^{'}, A_2^{'}, A_3^{'}\}$.*

Theorem 5. *$\mathcal{H}$ is an algebra.*

Since $\mathcal{H}$ is an algebra (**Theorem 5**) and $\mathcal{H}$ contains all and only those subsets that are listed above, $\mathcal{H} = \mathcal{F}^{\#}$, i.e. $\mathcal{H}$ is the smallest algebra containing the four kinds of subsets listed above. At this point, I've specified every item in the token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ I'm interested in.

[56] This definition is inspired by DiBella [2021].

All that's left to do in this subsection is to prove that $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$ is fully specified. And to do that, all that remains is to prove that $C^\#$ fully determines the order of events in $\mathcal{F}^\#$. First, note that all sets $A \in \mathcal{F}^\#$ have defined asymptotic densities. Let $d(A)$ denote the asymptotic density of $A = (A_1 - A_2) \cup A_3 \in \mathcal{F}^\#$, i.e. $d(A) = \frac{k}{m}$ if $A_1 \neq \emptyset$ where k is the number of maximal arithmetic progressions in A_1 and m is the shared common difference among all such arithmetic progressions. If $A_1 = \emptyset$, then $d(A) = 0$. Next, define $N(A) = \#(A_3) - \#(A_2)$.

Theorem 6. *For any two sets $A \in \mathcal{F}^\#$ and $B \in \mathcal{F}^\#$, if $A \cap B = \emptyset$, then $N(A \cup B) = N(A) + N(B)$.*

Consider the following way of ordering the events in $\mathcal{F}^\#$:

Regular Ordering. *For any two sets $A \in \mathcal{F}^\#$ and $B \in \mathcal{F}^\#$, $A \preceq B$ just in case one of the following holds:*

1. *$d(A) < d(B)$.*
2. *$d(A) = d(B)$ and $N(A) \leq N(B)$.*

I hope the reader finds the first condition of **Regular Ordering** intuitive. As I've explained above in §3.2.1, in a fair lottery over $\mathbb{N}$, if $d(A) < d(B)$, then $A \preceq B$. But when $d(A) = d(B)$, we must tread more carefully, in order to satisfy **Qualitative Regularity**. For the sake of satisfying **Qualitative Regularity**, it cannot be such that if $d(A) = d(B)$, then $A \preceq B$. For example, consider $\{i\}$ and $\{i, j\}$. $d(\{i\}) = d(\{i, j\}) = 0$. If it were the case that $d(A) = d(B)$ implies that $A \preceq B$, then $\{i\} \preceq \{i, j\}$ and $\{i, j\} \preceq \{i\}$. By **Definition 1**, $\{i\} \sim \{i, j\}$. Apply **Additivity** to get $\emptyset \sim \{j\}$, which violates **Qualitative Regularity**. In fact, if it were the case that $d(A) = d(B)$ implies that $A \preceq B$, then all finite sets would be equiprobable to $\emptyset$, in violation of **Qualitative Regularity**. So, for sets with equal asymptotic densities, a finer-grained comparison is needed. Condition 2 of **Regular Ordering** is one such finer-grained comparison. To see why, note that $N(\{i\}) = 1$ while $N(\{i, j\}) = 2$. So, $N(\{i\}) < N(\{i, j\})$. By condition 2 of **Regular Ordering**, $\{i\} \prec \{i, j\}$, which is the correct verdict, if we want to satisfy **Qualitative Regularity**.

To drive the point home, here's another example. Consider the sets $\{2, 4, 6, \ldots\}$ and $\{4, 6, 8, \ldots\}$. $d(\{2, 4, 6, \ldots\}) = d(\{4, 6, 8, \ldots\}) = \frac{1}{2}$. If it were the case that $d(A) = d(B)$ implies that $A \preceq B$, then $\{2, 4, 6, \ldots\} \preceq \{4, 6, 8, \ldots\}$ and $\{4, 6, 8, \ldots\} \preceq \{2, 4, 6, \ldots\}$. By **Definition 1**, $\{4, 6, 8, \ldots\} \sim \{2, 4, 6, \ldots\}$. Apply **Additivity** to get $\emptyset \sim \{2\}$, which violates **Qualitative Regularity**. So, a finer-grained comparison is needed. Note that $N(\{2, 4, 6, \ldots\}) = 0$ while $N(\{4, 6, 8, \ldots\}) = -1$. Hence, $N(\{4, 6, 8, \ldots\}) < N(\{2, 4, 6, \ldots\})$. By condition 2 of **Regular Ordering**, $\{4, 6, 8, \ldots\} \prec \{2, 4, 6, \ldots\}$, in accordance with **Qualitative Regularity**. This is the rationale behind condition 2 of **Regular Ordering**.

Theorem 7. *The constraints in $C^{\#}$ imply* ***Regular Ordering****.*

Theorem 8. ***Regular Ordering*** *implies that for any two sets $A \in \mathcal{F}^{\#}$ and $B \in \mathcal{F}^{\#}$, either $A \preceq B$ or $A \succeq B$.*

Corollary 1. *The constraints in $C^{\#}$ imply that for any two sets $A \in \mathcal{F}^{\#}$ and $B \in \mathcal{F}^{\#}$, either $A \preceq B$ or $A \succeq B$.*

By **Corollary 1**, $C^{\#}$ fully determines the order of events in $\mathcal{F}^{\#}$.

Corollary 2. *$\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is fully specified.*

Why am I interested in this particular token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$? My reasons are simple. First, **Fairness** is one of the constraints in $C^{\#}$. Second, the first item in the triple $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is infinite, i.e. $\mathbb{N}$ is infinite. And third, $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is fully specified (**Corollary 2**). Put simply, $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ satisfies all the conditions listed in **Problematic Choice** that are applicable to token probabilistic scenarios. To be sure, I'm interested in any token probabilistic scenarios $\langle \Omega, \mathcal{F}, C \rangle$ such that (i) **Fairness** is one of the constraints in C, (ii) Ω is infinite, and (iii) $\langle \Omega, \mathcal{F}, C \rangle$ is fully specified. But from previous drafts of this thesis, it just seems easier, technically speaking, to specify $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ than to specify a token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ that satisfies (i)-(iii) and intuitively describes—say, a fair lottery over $[0, 1]$.

The reader may also wonder why I'm interested in this particular algebra $\mathcal{F}^{\#}$ for a fair lottery over $\mathbb{N}$. After all, there are many other algebras on $\mathbb{N}$. Why $\mathcal{F}^{\#}$ in particular? Moreover, $\mathcal{F}^{\#}$ doesn't even contain all the subsets of $\mathbb{N}$ with defined asymptotic densities. For example, the set $\mathbb{P}$ of all prime numbers has a defined asymptotic density of 0, but $\mathbb{P} \notin \mathcal{F}^{\#}$. Why am I not interested in an algebra $\mathcal{F}$ such that $\mathcal{X} \subseteq \mathcal{F}$ where $\mathcal{X}$ is the set of all subsets of $\mathbb{N}$ with defined asymptotic densities? I'll respond to the second question first. Note that $\mathcal{X}$ is *not* an algebra.[57] But according to the definition of a token probabilistic scenario, the second item in the triple $\langle \Omega, \mathcal{F}, C \rangle$ must be an algebra. So, in order for $\mathcal{X} \subseteq \mathcal{F}$, $\mathcal{F}$ must contain subsets of $\mathbb{N}$ that have *no* defined asymptotic densities. The trouble with working with such an algebra $\mathcal{F}$ is that it is technically difficult to specify a set of constraints that fully determines the order of events in $\mathcal{F}$. While it is easy to specify constraints that compare sets with defined asymptotic densities, in order to fully determine the order of events in a $\mathcal{F}$ such that $\mathcal{X} \subseteq \mathcal{F}$, further constraints must be specified that govern the comparisons between (i) sets A and B without defined asymptotic densities and (ii) sets A and B, where A has a defined asymptotic density, but B doesn't. Based on my previous attempts to specify the latter constraints, doing so isn't straightforward at all. That's why, for the sake of saving myself some trouble, instead of adding more subsets to $\mathcal{X}$ to make it an algebra, I decided to remove subsets from $\mathcal{X}$ to

[57]The proof of this claim is well-understood and therefore will not be reproduced here.

make it an algebra; $\mathcal{F}^{\#}$ is the result. And as we've seen in §3.2.1, I've managed to specify a set of constraints $C^{\#}$ that fully determines the order of events in $\mathcal{F}^{\#}$ (**Corollary 1**). In short, I'm interested in $\mathcal{F}^{\#}$ because it is relatively easier, technically speaking, to specify constraints that fully determine the order of events in $\mathcal{F}^{\#}$. This answers the first question above as well.

3.3 Two Different Correct Precise Models

Let $\mathcal{F}^{\#}$ be the domain of probability function P_1 such that $P_1(A) = d(A) + \big(\varepsilon \cdot N(A)\big)$ where ε is a positive infinitesimal. In this subsection, I will show that P_1 satisfies all the constraints in $C^{\#}$. To do so, we need a 'linking' principle linking qualitative probabilities and precise numerical probabilities, since the constraints in $C^{\#}$ are in terms of qualitative probabilities. Here's the 'linking' principle I'm assuming:

Precise Representation. *For any two sets $A \in \mathcal{F}^{\#}$ and $B \in \mathcal{F}^{\#}$, $A \preceq B$ iff $P(A) \leq P(B)$.*

Note that the 'P' in **Precise Representation** need not only denote P_1. It can be any probability function defined on $\mathcal{F}^{\#}$.

Theorem 9. *Under **Precise Representation**, P_1 satisfies all the constraints in $C^{\#}$.*

Since P_1 is defined on $\mathcal{F}^{\#}$ and satisfies all the constraints in $C^{\#}$ (under **Precise Representation**), P_1 correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ (under **Precise Representation**).

Now, consider the following probability function P' defined on $\mathcal{F}^{\#}$ such that $P'(A) = d(A) + \big(r \cdot \varepsilon \cdot N(A)\big)$ for any particular $r \in \mathbb{R}^{+} - \{1\}$.[58,59]

Theorem 10. *Under **Precise Representation**, P_1 and P' satisfy the four conditions of **Infinitesimal Underdetermination**.*

Theorem 10 shows that the probability function P' correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ as well (under **Precise Representation**). This suggests that a probability function correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is not unique. After all, there are multiple—in fact, uncountably many—probability functions that correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. As a matter of fact, for every $r \in \mathbb{R}^{+}$, there is a probability function P_r correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ such that $P_r(A) = d(A) + \big(r \cdot \varepsilon \cdot N(A)\big)$. In other words, $C^{\#}$ does not uniquely determine a probability function that correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$; there is underdetermination. This may be surprising because, after all, $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is already *fully specified* and **Fairness** is one of the constraints in $C^{\#}$. To drive the point home, consider for a moment fair *finite* lotteries, i.e. token probabilistic scenarios $\langle \Omega, \mathcal{P}\Omega, C \rangle$ where Ω

[58] $\mathbb{R}^{+}$ is the set of all positive real numbers.

[59] This definition of P' is inspired by Pruss [2021].

is finite, $\mathcal{P}\Omega$ is the powerset of Ω, and C contains **Non-Negativity**, **Non-Triviality**, **Transitivity**, **Additivity**, and **Fairness**. In these scenarios, C completely determines the probabilities to be assigned to every event under **Precise Representation**. And this is true even for *under-specified* token probabilistic scenarios $\langle \Omega, \mathcal{P}\Omega, C \rangle$ where Ω is an unspecified finite set and C contains **Non-Negativity**, **Non-Triviality**, **Transitivity**, **Additivity**, and **Fairness**. That is, even in these latter scenarios, C still completely determines the probabilities to be assigned to every event under **Precise Representation**. So, it is surprising that this doesn't carry over to $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, which is not only fully specified but $C^{\#}$ contains more than just **Non-Negativity**, **Non-Triviality**, **Transitivity**, **Additivity**, and **Fairness**.

At this point, the reader may notice that the results above only hold if we assume **Precise Representation**. But why think that **Precise Representation** is true in the first place? After all, Easwaran [2014a] has proposed several other 'linking' principles. I will state these other 'linking' principles in due time. But before doing that, note that Easwaran [2014a] adopted an alternative notion of conditional probability. Traditionally in Kolmogorov [1950], the conditional probability of A, given B, is defined as $\frac{P(A \cap B)}{P(B)}$, whenever $P(B) > 0$. Under this definition, the conditional probability of A, given B, is undefined if $P(B) = 0$, as division by 0 is undefined. However, as Hájek [2003] has argued, it seems that the conditional probability of A, given B, has defined value even when $P(B) = 0$ for some A's and for some B's. For example, the conditional probability of B, given B, is intuitively 1 even when $P(B) = 0$. So, the traditional definition of conditional probability has to be revised.

Alternative theories of conditional probabilities can be found in Popper [1955] and Rényi [1970].[60] In these two theories, conditional probabilities are the primitive two-place notion not defined as a ratio of two unconditional probabilities. Let $P(A, B)$ represent a function in either of these theories denoting the conditional probability of A, given B. In both theories, for all algebras $\mathcal{F}$, $P(A, A) = 1$ for all nonempty $A \in \mathcal{F}$ and $P(A \cap B, C) = P(A, B \cap C) \cdot P(B, C)$. While Easwaran [2014a] did not specify which theory he adopted for his proposals, it can be assumed that his proposals are flexible between the two theories.

Here are Easwaran [2014a]'s proposals:

Proper Containment. *$A \prec B$ iff $P(A) < P(B)$ or $A \subsetneq B$.*

Conditionality. *$A \prec B$ iff $P(A, A \cup B) < P(B, A \cup B)$.*

Both **Proper Containment** and **Conditionality** imply that $\emptyset \prec A$ for all nonempty sets $A \in \mathcal{F}$.[61] However, there are 3 distinct reasons why I haven't considered either principle

[60] It is important to note that for my purposes, I'm more interested in the set-theoretic formulation of Popper [1955]'s theory, as formulated by van Fraassen [1976].

[61] Refer to Easwaran [2014a] for the proof.

as a plausible 'linking' principle for the purposes of this section. First, the probabilities in either principle are *real-valued*, whereas the probabilities in **Precise Representation** are allowed to be non-Archimedean. Indeed, Popper [1955]'s and Rényi [1970]'s theories are theories of real-valued probabilities. But in this section, I'm interested in *non-Archimedean* probabilities. So, **Precise Representation** seems to be a better fit for the purposes of this section. Second, while the 'link' between the relation $\prec$ and precise numerical probabilities is defined in each principle, neither principle defines the 'link' between the relation $\sim$ and precise numerical probabilities. But in order to show that P_1 satisfies all the constraints in $C^{\#}$ (**Theorem 9**), such a 'link' has to be defined. In contrast, the 'link' between the relation $\preceq$ and precise numerical probabilities is defined in **Precise Representation**. So, **Precise Representation** seems to be a better fit for the purposes of this section.

The third reason comes in two parts, one for each principle. Here's the first part for **Proper Containment**. **Proper Containment** implies some results that conflict with what is implied by the constraints in $C^{\#}$. According to the constraints in $C^{\#}$, $\{i\} \prec \{j,k\}$ where $\{i\} \cap \{j,k\} = \emptyset$. Furthermore, notice that for a real-valued probability function P, $P(\{i\}) = P(\{j,k\}) = 0$. So, both conditions of **Proper Containment** fail. Therefore, according to **Proper Containment**, $\{i\} \not\prec \{j,k\}$, in conflict with what is implied by the constraints in $C^{\#}$.[62]

Here's the second part of the third reason for **Conditionality**. The same point goes for **Conditionality**, i.e. **Conditionality** implies some results that conflict with what is implied by the constraints in $C^{\#}$. According to the constraints in $C^{\#}$, $\mathbb{N} - \{1\} \prec \mathbb{N}$. For a real-valued probability function P, $P(\mathbb{N} - \{1\}) = P(\mathbb{N}) = 1$ as $P(\{1\}) = 0$. But this means that $P(\mathbb{N} - \{1\}, (\mathbb{N} - \{1\}) \cup \mathbb{N}) = P(\mathbb{N} - \{1\}, \mathbb{N}) = 1$ and $P(\mathbb{N}, (\mathbb{N} - \{1\}) \cup \mathbb{N}) = P(\mathbb{N}, \mathbb{N}) = 1$. So, $P(\mathbb{N} - \{1\}, (\mathbb{N} - \{1\}) \cup \mathbb{N}) = P(\mathbb{N}, (\mathbb{N} - \{1\}) \cup \mathbb{N})$. Therefore, according to **Conditionality**, $\mathbb{N} - \{1\} \not\prec \mathbb{N}$, in conflict with what is implied by the constraints in $C^{\#}$.[63] Because of the four reasons listed above, **Precise Representation** seems to be a better fit for the purposes of this section.

I will now define two different correct precise models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Then, I will show that choosing one model over the other is problematic w.r.t an end result R. Let probability function P_2 be defined on $\mathcal{F}^{\#}$ such that $P_2(A) = d(A) + \big(2 \cdot \varepsilon \cdot N(A)\big)$. Define precise model $\mathcal{M}_1 = \langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}, \Sigma_1 \rangle$ where $\Sigma_1 = \{(A, P_1(A)) : A \in \mathcal{F}^{\#}\}$. Similarly, define precise model $\mathcal{M}_2 = \langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}, \Sigma_2 \rangle$ where $\Sigma_2 = \{(A, P_2(A)) : A \in \mathcal{F}^{\#}\}$. Since both P_1 and P_2 correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ (under **Precise Representation**), $\mathcal{M}_1$ and $\mathcal{M}_2$ are both correct precise models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Furthermore, since P_1 and P_2 both satisfy conditions 1, 2, and 4 of **Infinitesimal Underdetermination** (**Theorem 10**), by **Same Representational Power**, $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same representational

[62]This has been pointed out in Thong [MS].
[63]This has been pointed out in Thong [MS].

power in modelling $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Yet, they are still different models. According to $\mathcal{M}_1$, a probability of ε is assigned to $\{1\}$. But according to $\mathcal{M}_2$, a probability of 2ε is instead assigned to $\{1\}$.

Consider the two following bets:

> Bet A: If the number chosen in a fair lottery over $\mathbb{N}$ is 1, then you'll be awarded $\frac{2}{3\varepsilon}$ units of utility. Otherwise, no units of utility will be awarded or lost.
>
> Bet B: If the result of a fair coin toss is heads, then you'll be awarded 2 units of utility. Otherwise, no units of utility will be awarded or lost.

Bet B's expected utility is $\frac{1}{2} \cdot 2 = 1$ unit. Depending on exactly which correct precise model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is chosen, bet A's expected utility changes. If the chosen correct precise model is $\mathcal{M}_1$, then bet A's expected utility is $\varepsilon \cdot \frac{2}{3\varepsilon} = \frac{2}{3}$ unit. As prescribed by expected utility reasoning, we should choose bet B over bet A. After all, bet B's expected utility is higher than bet A's. However, if the chosen correct precise model is $\mathcal{M}_2$ instead, then bet A's expected utility is $2\varepsilon \cdot \frac{2}{3\varepsilon} = \frac{4}{3}$ units. As prescribed by expected utility reasoning, we should choose bet A over bet B. After all, bet A's expected utility is higher than bet B's. So, depending on exactly which correct precise model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is chosen, the prescribed course of action by expected utility reasoning changes.

Let R be the bet that we should take between bets A and B, as detailed above. Following the explanation above, if $\mathcal{M}_1$ is chosen, then $R = B$. However, if $\mathcal{M}_2$ is chosen instead, then $R = A$. So, R is sensitive to our choice between $\mathcal{M}_1$ and $\mathcal{M}_2$. Is our choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ to model $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ problematic w.r.t R? Let's check whether all the conditions of **Problematic Choice** are fulfilled. First, $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is fully specified (**Corollary 2**) with an infinite set as the first item in the triple, and **Fairness** is one of the constraints in $C^{\#}$. Second, $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same representational power in modelling $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Third, R is sensitive to our choice between $\mathcal{M}_1$ and $\mathcal{M}_2$. So, all the conditions of **Problematic Choice** are fulfilled. By **Problematic Choice**, our choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ to model $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is therefore problematic w.r.t R.

Here's a natural follow-up question: exactly which constraints in $C^{\#}$ give rise to the problem? To answer this question, note first that it is the multiplicity of correct precise models, none of which 'stands out' from the rest, that led to the problem. After all, if there were only one correct precise model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, then there would be no problem. So, which constraints in $C^{\#}$ give rise to the problem? Well, they are simply those constraints that give rise to the multiplicity of correct precise models in the first place. But how can we in turn determine what these constraints are? After all, $C^{\#}$ contains 8 constraints. Here's a potential way. Consider a token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$ that admits only one correct precise model. Since $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$ admits only one correct precise model but $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ admits multiple correct precise models, the difference between C^{*} and $C^{\#}$ must be what led to the multiplicity of correct precise models. I will now specify $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$.

Let C^* contain only the following statement:

Irregular Ordering. *For any two sets $A \in \mathcal{F}^\#$ and $B \in \mathcal{F}^\#$, $A \preceq' B$ iff $d(A) \leq d(B)$.*

Theorem 11. *Under **Precise Representation**, for all probability functions P that correctly represent $\langle \mathbb{N}, \mathcal{F}^\#, C^* \rangle$, $P(A) = d(A)$.*

Theorem 11 shows that the probability function P correctly representing $\langle \mathbb{N}, \mathcal{F}^\#, C^* \rangle$ is unique, i.e. if both probability functions P and P^* correctly represent $\langle \mathbb{N}, \mathcal{F}^\#, C^* \rangle$, then $P = P^*$. Accordingly, there is only one correct precise model of $\langle \mathbb{N}, \mathcal{F}^\#, C^* \rangle$, i.e. $\mathcal{M}_3 = \langle \mathbb{N}, \mathcal{F}^\#, C^*, \Sigma_3 \rangle$ where $\Sigma_3 = \{(A, P(A)) : A \in \mathcal{F}^\#\}$ and P is defined on $\mathcal{F}^\#$ such that $P(A) = d(A)$.

Having defined $\langle \mathbb{N}, \mathcal{F}^\#, C^* \rangle$ and shown that there is only one correct precise model $\mathcal{M}_3$ modelling it, let's determine the differences between $C^\#$ and C^*.

Theorem 12. *$\preceq'$ satisfies all of the following: **Non-Negativity**, **Non-Triviality**, **Transitivity**, **Additivity**, **Fairness**, and **Comparing Finite and Infinite Sets**.*

Theorem 13. *$\preceq'$ does not satisfy **Qualitative Regularity** and **Partition**.*

Theorems 12 and **13** show that the salient differences between $C^\#$ and C^* are the satisfaction of **Qualitative Regularity** and **Partition**. To be clear, $\preceq'$ satisfies the first condition of **Partition**, i.e. if $d(A) < d(B)$, then $A \preceq' B$. It is the second condition of **Partition** that is violated by $\preceq'$. And as I've explained above in §3.2.1, the second condition of **Partition** is included, on pain of violating **Qualitative Regularity**. In other words, if **Qualitative Regularity** were dropped from $C^\#$, then there wouldn't be any need to include the second condition of **Partition**. So, strictly speaking, there is really only one salient difference between $C^\#$ and C^*: the satisfaction of **Qualitative Regularity**. That is, the satisfaction of **Qualitative Regularity** is what gives rise to the multiplicity of correct precise models of $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$, and the violation of the same principle is what gives rise to the uniqueness of $\mathcal{M}_3$ in modelling $\langle \mathbb{N}, \mathcal{F}^\#, C^* \rangle$.

To drive the point home, consider the definitions of P and P'. For all sets $A \in \mathcal{F}^\#$, $P(A) = d(A)$ (**Theorem 11**) and $P'(A) = d(A) + \big(r \cdot \varepsilon \cdot N(A)\big)$ for any particular $r \in \mathbb{R}^+ - \{1\}$. The *sole* purpose of including the term '$r \cdot \varepsilon \cdot N(A)$' in the definition of P' is to satisfy **Qualitative Regularity** under **Precise Representation**. Indeed, the inclusion of this term is responsible for introducing the multiplicity of correct precise models of $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$. After all, there are uncountably many members in $\mathbb{R}^+ - \{1\}$ and for each member in this set, there is a probability function P_r such that $P_r(A) = d(A) + \big(r \cdot \varepsilon \cdot N(A)\big)$. Then, different correct precise models can be defined from P_r and $P_{r'}$ where $r \neq r'$. Since there are uncountably many $r \in \mathbb{R}^+ - \{1\}$, there are also uncountably many correct precise models of $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$. If **Qualitative Regularity** were dropped from $C^\#$ (and therefore there wouldn't be any need to include the second condition of **Partition**), then there wouldn't be any need to include the term '$r \cdot \varepsilon \cdot N(A)$' in the definition of P'; P will do just fine.

3.4 Potential Constraints

Are there any potential constraints that we can add to $C^{\#}$ such that the new collection of constraints uniquely determines *the* probability function defined on $\mathcal{F}^{\#}$ that satisfies all the constraints in $C_2^{\#}$, which contains the old constraints in $C^{\#}$ and also the new added ones, i.e. $C^{\#} \subseteq C_2^{\#}$, under **Precise Representation**? In this subsection, I will argue that there are no such constraints for which an intuitive justification can be provided.

Broadly speaking, there are two *kinds* of constraints. The first kind is *relational*. That is, a relational constraint specifies the relations between events or between probabilities. For example, **Fairness** is a relational constraint. It specifies the relation between any two singleton sets, i.e. any two singleton sets are equiprobable. Importantly, relational constraints need not exclusively be in terms of qualitative probabilities; they can be in terms of numerical probabilities too. Consider the numerical version of **Fairness**: for all $\omega, \omega' \in \Omega$, $P(\{\omega\}) = P(\{\omega'\})$. This version is a relational constraint too. After all, it still specifies the relation between any two singleton sets, i.e. any two singleton sets are assigned equal probability. Here's another example of a relational constraint: **Finite Additivity**, which states that if $A \cap B = \emptyset$ and both $P(A)$ and $P(B)$ are defined, then $P(A \cup B) = P(A) + P(B)$. **Finite Additivity** specifies the relation between the probabilities $P(A \cup B)$ and $P(A) + P(B)$, i.e. $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$ and both $P(A)$ and $P(B)$ are defined.

The other kind of constraints is *monadic*. That is, a monadic constraint specifies some information about events or probabilities, considered *on their own*. Here's an example: **Normalization**, which states that $P(\Omega) = 1$. **Normalization** specifies some information about the probability of Ω, considered *on its own*, i.e. $P(\Omega) = 1$. Here's another example of a monadic constraint: **Non-Negativity (Numerical)**, which states that for all events A in the domain of P, $P(A) \geq 0$. **Non-Negativity (Numerical)** specifies some information about the probability of an event A, considered on its own, i.e. $P(A) \geq 0$. Here's yet another example of a monadic constraint: **Regularity**, which specifies some information about the probability of a nonempty set A, considered on its own, i.e. $P(A) > 0$. However, note that **Qualitative Regularity** isn't a monadic constraint. Instead, it is a relational constraint. After all, it specifies the relation between two events: $\emptyset$ and a nonempty set A, i.e. $\emptyset \prec A$. Importantly, monadic constraints are expressed exclusively in terms of numerical probabilities. Qualitative probabilities, as explained in §3.1, are *binary* relations between 2 events. So, monadic constraints cannot be expressed in terms of qualitative probabilities, for that is a contradiction in terms. Put simply, monadic constraints are constraints that specify some information about the numerical probability assigned to *an* event, considered on its own.

Now, consider for a moment fair *finite* lotteries. There are two ways to specify these scenarios. First, we can include as constraints in the specification of these scenarios the following monadic constraints: **Normalization**, **Non-Negativity (Numerical)**, and perhaps **Regularity**, as well as the following relational constraints: **Finite Additivity** and the numerical version of **Fairness**. Then, from these constraints, we *calculate* the probabilities to be assigned to each event, except Ω, whose probability is *specified* by **Normalization**. That's one way of specifying a fair finite lottery. Here's the other way. Keep all the monadic constraints as listed above in the specification of the lottery, as well as the relational constraint of **Finite Additivity**. But replace **Fairness** with another *monadic* constraint stating that for every singleton set $\{i\}$ in the lottery, $P(\{i\}) = \frac{1}{n}$, where n is the number of tickets in the lottery. In this way, the probability to be assigned to a singleton set *isn't* calculated from the constraints. Instead, it is *specified*, although the probabilities to be assigned to the other non-singleton events, except Ω, are still calculated from the constraints. Practically speaking, there is no significant difference between these two ways of specifying a fair finite lottery. After all, the constraints in one specification imply the constraints in the other specification. So, there is good intuitive justification for including in the specification of a fair finite lottery with n tickets the monadic constraint that for every singleton set $\{i\}$ in the lottery, $P(\{i\}) = \frac{1}{n}$.

With this insight, let's consider $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$. Is there a *relational* constraint that can be added to $C^\#$ to form $C_2^\#$ such that $C_2^\#$ uniquely determines *the* probability function defined on $\mathcal{F}^\#$ that satisfies all the constraints in $C_2^\#$, under **Precise Representation**? The answer is in the negative, as $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$ is already fully specified (**Corollary 2**). In particular, according to the constraints in $C^\#$, for any two sets $A \in \mathcal{F}^\#$ and $B \in \mathcal{F}^\#$, either $A \preceq B$ or $A \succeq B$ (**Corollary 1**). In other words, there do not exist sets A and B in $\mathcal{F}^\#$ such that $A \npreceq B$ and $A \nsucceq B$, if all the constraints in $C^\#$ are satisfied. So, adding a further relational constraint to $C^\#$ will not introduce any new comparisons, as all possible comparisons between the events in $\mathcal{F}^\#$ have already been specified according to $C^\#$. Also, adding a further relational constraint to $C^\#$ cannot change the existing comparisons between any two sets A and B in $\mathcal{F}^\#$. For if it did, then the new collection of constraints $C_2^\#$ would be inconsistent. This is because according to $C^\#$, $A \preceq B$ (or $A \succeq B$), but according to $C_2^\# - C^\#$, $A \npreceq B$ (or $A \nsucceq B$). That is, for the sake of consistency, all existing comparisons between any two sets A and B in $\mathcal{F}^\#$ must remain. Put simply, adding a further relational constraint will neither change any existing comparison nor add any new ones. But if all the existing comparisons remain, then all the correct precise models of $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$ will also remain, of which there are uncountably many. Therefore, no relational constraint is a member of $C_2^\# - C^\#$ where $C_2^\#$ uniquely determines *the* probability function defined on $\mathcal{F}^\#$ that satisfies all the constraints in $C_2^\#$, under **Precise Representation**.

If $C_2^\# - C^\#$ were nonempty, then it must contain *monadic* constraints. But what monadic constraint is in $C_2^\# - C^\#$ such that $C_2^\#$ uniquely determines *the* probability function defined on $\mathcal{F}^\#$ that satisfies all the constraints in $C_2^\#$, under **Precise Repre-**

sentation? It is easy to verify that if $C_2^{\#} - C^{\#}$ contains only **Normalization**, **Non-Negativity (Numerical)**, and **Regularity**, then there will still be uncountably many correct precise models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C_2^{\#} \rangle$. So, there must be a further constraint in $C_2^{\#} - C^{\#}$. Here's an example.

The Probability of a Singleton Set. *For all $i \in \mathbb{N}$, $P(\{i\}) = \varepsilon$.*

If **The Probability of a Singleton Set** is added to $C^{\#}$ to form $C_2^{\#}$, then there is only one probability function that correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C_2^{\#} \rangle$: P_1, the definition of which is $P_1(A) = d(A) + \big(\varepsilon \cdot N(A)\big)$. Consequently, there is only one correct precise model modelling $\langle \mathbb{N}, \mathcal{F}^{\#}, C_2^{\#} \rangle$: $\mathcal{M}_4 = \langle \mathbb{N}, \mathcal{F}^{\#}, C_2^{\#}, \Sigma_4 \rangle$ where $\Sigma_4 = \{(A, P_1(A)) : A \in \mathcal{F}^{\#}\}$. However, is there good intuitive justification for including **The Probability of a Singleton Set** in $C_2^{\#}$? I don't think so. Why is the probability to be assigned to a singleton set ε, and not 2ε, ε^2, $\varepsilon^2 + 2\varepsilon$, or for that matter, any other positive infinitesimal? After all, nothing about a fair lottery over $\mathbb{N}$ uniquely determines ε as the probability to be assigned to a singleton set and nothing about the lottery rules out every other positive infinitesimal as a probability as well. So, it's difficult to justify why **The Probability of a Singleton Set** should be included in $C_2^{\#}$. Unless the reader can provide me an intuitive justification for including **The Probability of a Singleton Set** in $C_2^{\#}$, I will not be convinced of using this manoeuvre to get out of the problem spelled out in §3.3.

Note also that including **The Probability of a Singleton Set** in $C_2^{\#}$ is different from including in the specification of a fair finite lottery with n tickets the constraint that for every singleton set $\{i\}$ in the lottery, $P(\{i\}) = \frac{1}{n}$. For the latter case, there *is* good intuitive justification for including such a constraint. Recall from above the two different ways of specifying a fair finite lottery. In the first way, the constraints specified are **Normalization**, **Non-Negativity (Numerical)**, **Regularity**, **Finite Additivity** and the numerical version of **Fairness**. In the second way, the constraints specified are the first four constraints just listed and for every singleton set $\{i\}$ in the lottery, $P(\{i\}) = \frac{1}{n}$. Notice that the constraints in the first set imply the constraints in the second, and vice versa. So, in the fair finite lottery case, there is good intuitive justification for including the constraint that for every singleton set $\{i\}$ in the lottery, $P(\{i\}) = \frac{1}{n}$. But the same cannot be said of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$: while the constraints in $C_2^{\#}$ imply those in $C^{\#}$, the other way round isn't true. Therefore, there seems to be no intuitive justification for including **The Probability of a Singleton Set** in $C_2^{\#}$.

4 Imprecise Probabilities and Tightness

In §3.2, I defined the particular token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ I'm interested in for the purposes of this thesis and also showed in the same subsection that $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is fully specified (**Corollary 2**). Then, in §3.3, I defined two correct precise models $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ such that our choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ is problematic w.r.t an end result R. With this problem at hand, I began exploring in §3.4 potential constraints that we can add to $C^{\#}$ such that the new collection of constraints uniquely determines *the* probability function defined on $\mathcal{F}^{\#}$ that satisfies all the constraints in $C_2^{\#}$, which contains the old constraints in $C^{\#}$ and also the new added ones, under **Precise Representation**. But I argued that there are no such constraints for which an intuitive justification can be provided, ending §3 with a somewhat pessimistic conclusion.

In this section, I'll begin with an optimistic note. First, notice that $\mathcal{M}_1$ and $\mathcal{M}_2$, as defined in §3.3, are *precise* models, i.e. both are numerical models expressed exclusively in terms of precise probabilities. But why limit ourselves to such models? Why not consider *imprecise* models too, i.e. numerical models expressed in terms of precise *and* imprecise probabilities? Indeed, some authors, e.g. Benci et al. [2018] and Pruss [2021], have suggested that the chosen correct numerical model need not be exclusively in terms of precise probabilities. That is, *imprecise* probabilities can be assigned to events in $\mathcal{F}^{\#}$ too.

> "[I]t has to be borne in mind that one can always consider the entire family of [non-Archimedean] functions modelling a given situation, rather than an—arbitrary—re-presentative of it ... Such a family is the set of all [non-Archimedean] functions that meet a common specification, such as 'a fair lottery on [the natural numbers]', ... As a whole, the family shows us how much the probabilities of a given event, and the order of probabilities of multiple events can vary ...
>
> To put it differently, there may be multiple, equally good ways to model the same situation ... What matters is what is true (or false) on all ways of making these arbitrary choices—what is supertrue (or superfalse)—as well as the spread of possible assignments. We need not project these arbitrary choices onto what is being modelled." (Benci et al., 2018, p. 544)

> "An approach in terms of imprecise probabilities, where a *family* of [non-Archimede-an] probability functions is assigned ... may well be able to escape the underdetermination worries." (Pruss, 2021, p. 4, his emphasis)

So, it should be allowed that the chosen correct numerical model is an imprecise model. Moreover, there seems to be only one correct imprecise model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, whereas there are uncountably many correct precise models of the same scenario. As a matter of fact, as proven in §3.3, there is a correct precise model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ for every $r \in \mathbb{R}^{+}$.

In this sense, *the* former correct imprecise model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ 'stands out' from the rest of the models. After all, there seems to be only one correct imprecise model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. This apparent fact, by itself, may give us good enough reason to choose the correct imprecise model to represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$.

However, here's where the optimistic note starts to turn sour. The main goal of this section is to show that there are in fact multiple correct imprecise models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. So, the assertion that there is only one such model is false. Here's how this section will be structured. Definitions and notations will first be introduced in §4.1. Then, in §4.2, I show that there are in fact multiple correct imprecise models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Last, in §4.3, the property of *tightness* will be introduced, after which I will show that if we require imprecise probabilities to satisfy this property of tightness, then there are *no* correct imprecise models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. So, with the property of tightness, Benci et al. [2018]'s and Pruss [2021]'s approach of allowing probabilities to be imprecise will not even get off the ground for $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$.

4.1 Definitions and Notations

Consider any fully specified token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ such that for every $A \in \mathcal{F}$, if there exist two probability functions P and $P^{'}$ correctly representing $\langle \Omega, \mathcal{F}, C \rangle$ such that $P(A) = a$ and $P^{'}(A) = a^{'}$ where $a < a^{'}$, then for every $a \leq a^{*} \leq a^{'}$, there exists a probability function P^{*} correctly representing $\langle \Omega, \mathcal{F}, C \rangle$ such that $P^{*}(A) = a^{*}$. There is a lot to unpack here. First, recall that a probability function P correctly represents $\langle \Omega, \mathcal{F}, C \rangle$ just in case P is defined on $\mathcal{F}$ and satisfies all the constraints in C. Now, consider the following token probabilistic scenario $\langle \{H, T\}, \mathcal{P}\{H, T\}, C \rangle$ where $\mathcal{P}\{H, T\}$ is the powerset of $\{H, T\}$ and C contains the probability axioms, **Regularity**, and the statement 'The coin is biased towards tails'. Put simply, this scenario is about tossing a coin that's biased towards tails. It is easy to verify that $\langle \{H, T\}, \mathcal{P}\{H, T\}, C \rangle$ is fully specified.

Let P be a probability function that assigns $\{H\}$ a probability of 0.1 and $P^{'}$ be a probability function that assigns $\{H\}$ a probability of 0.4. Obviously, both P and $P^{'}$ correctly represent $\langle \{H, T\}, \mathcal{P}\{H, T\}, C \rangle$. Most importantly, notice that for every number $0.1 \leq x \leq 0.4$—in fact, for every number $0 < x < 0.5$, there is a probability function P^{*} that correctly represents $\langle \{H, T\}, \mathcal{P}\{H, T\}, C \rangle$ such that $P^{*}(\{H\}) = x$. After all, for any such P^{*}, $0 < P^{*}(\{H\}) < P^{*}(\{T\})$, in accordance with the constraints in C. In other words, there are no 'jumps' or 'discontinuations' in the range of possible probability values to be assigned to $\{H\}$. In this sense, the range of possible probability values to be assigned to $\{H\}$ is *continuous*. For the purposes of this section, I'm only interested in fully specified token probabilistic scenarios that admit continuous ranges of possible probability values to be assigned to their events. Hereafter, call such token probabilistic scenarios *relevant*, i.e. a token probabilistic scenario $\langle\Omega,\mathcal{F},C\rangle$ is *relevant* just in case (i) $\langle\Omega,\mathcal{F},C\rangle$ is fully specified and (ii) for every $A\in\mathcal{F}$, if there exist two probability functions P and $P^{'}$ correctly representing $\langle\Omega,\mathcal{F},C\rangle$ such that $P(A)=a$ and $P^{'}(A)=a^{'}$ where $a<a^{'}$, then for every $a\leq a^{*}\leq a^{'}$, there exists a probability function P^{*} correctly representing $\langle\Omega,\mathcal{F},C\rangle$ such that $P^{*}(A)=a^{*}$.

To drive the point home, say a further constraint is added to C to form $C^{'}$ such that (i) every probability function P that assigns $\{H\}$ a value x where $0.2\leq x\leq 0.3$ *no longer* correctly represents $\langle\{H,T\},\mathcal{P}\{H,T\},C^{'}\rangle$ but (ii) every probability function P that assigns $\{H\}$ a value x where either $0.1\leq x<0.2$ or $0.3<x\leq 0.4$ still correctly represents $\langle\{H,T\},\mathcal{P}\{H,T\},C^{'}\rangle$. Because there is a 'jump' or 'discontinuation' in the range of possible probability values to be assigned to $\{H\}$, i.e. from 0.2 to 0.3, this new token probabilistic scenario $\langle\{H,T\},\mathcal{P}\{H,T\},C^{'}\rangle$ is *irrelevant*. And for that reason, I'm not interested in it for the purposes of this section.

Back to relevant token probabilistic scenarios $\langle\Omega,\mathcal{F},C\rangle$. Let $\mathcal{S}$ be the set of all precise probability functions P set in the same ordered field $\mathbb{F}$ that correctly represent $\langle\Omega,\mathcal{F},C\rangle$. Next, for each event $A\in\mathcal{F}$, define $\mathcal{S}_A=\{x\in\mathbb{F}:\exists P\in\mathcal{S}(P(A)=x)\}$, i.e. $\mathcal{S}_A$ contains all the various outputs of every $P\in\mathcal{S}$ at A. Now, call a function P_{im} with $\mathcal{F}$ as its domain an *imprecise probability function* just in case:

1. If $\mathcal{S}_A=\{a\}$, then $P_{im}(A)=a$.
2. If $\#(\mathcal{S}_A)>1$, then $P_{im}(A)$ is an *interval* with $0\leq a_1\leq 1$ as the lower bound and $0\leq a_2\leq 1$ as the upper bound of the interval, satisfying all of the following:
   (a) a_1 is a lower bound and a_2 is an upper bound for $\mathcal{S}_A$. That is, for all $x\in\mathcal{S}_A$, $a_1\leq x$ and $a_2\geq x$.
   (b) $P_{im}(A)=[a_1,a_2]$ only if $a_1\in\mathcal{S}_A$ and $a_2\in\mathcal{S}_A$.
   (c) $P_{im}(A)=[a_1,a_2)$ only if $a_1\in\mathcal{S}_A$ and $a_2\notin\mathcal{S}_A$.
   (d) $P_{im}(A)=(a_1,a_2]$ only if $a_1\notin\mathcal{S}_A$ and $a_2\in\mathcal{S}_A$.
   (e) $P_{im}(A)=(a_1,a_2)$ only if $a_1\notin\mathcal{S}_A$ and $a_2\notin\mathcal{S}_A$.[64]

Note that if $\#(\mathcal{S}_A)>1$, then, in virtue of $\langle\Omega,\mathcal{F},C\rangle$ being relevant, $\mathcal{S}_A$ is an infinite set. For if $\mathcal{S}_A$ were a finite set, then there would be 'jumps' and 'discontinuations' in the possible probability values to be assigned to A. To see why, let $a_1\in\mathcal{S}_A$ and $a_2\in\mathcal{S}_A$ such that $a_1<a_2$. If $\mathcal{S}_A$ were a finite set, then $\mathcal{S}_A$ cannot contain all the numbers x such that $a_1\leq x\leq a_2$, in violation of $\langle\Omega,\mathcal{F},C\rangle$ being relevant. Therefore, if $\#(\mathcal{S}_A)>1$, then $\mathcal{S}_A$ is an infinite set.

[64]The co-domain of P_{im} is the union of the given ordered field $\mathbb{F}$ and the set of all intervals (i.e. closed intervals, half-open intervals, and open intervals) of $\mathbb{F}$.

Here are some examples of imprecise probability functions P_{im}. Recall the token probabilistic scenario $\langle\{H,T\},\mathcal{P}\{H,T\},C\rangle$ from before describing tossing a coin that's biased towards tails. In this token probabilistic scenario, $\mathcal{S}_{\{H\}} = \{x \in \mathbb{F} : 0 < x < \frac{1}{2}\}$ and $\mathcal{S}_{\{T\}} = \{x \in \mathbb{F} : \frac{1}{2} < x < 1\}$ for some ordered field $\mathbb{F}$. Define a function $P_{im(1)}$ with $\mathcal{P}\{H,T\}$ as its domain such that $P_{im(1)}(\emptyset) = 0$, $P_{im(1)}(\{H,T\}) = 1$, $P_{im(1)}(\{H\}) = (0, \frac{1}{2})$, and $P_{im(1)}(\{T\}) = (\frac{1}{2}, 1)$. It is easy to verify that $P_{im(1)}$ is an imprecise probability function. Here's another example. Define a function $P_{im(2)}$ with $\mathcal{P}\{H,T\}$ as its domain such that $P_{im(2)}(\emptyset) = 0$, $P_{im(2)}(\{H,T\}) = 1$, $P_{im(2)}(\{H\}) = (0,1)$, and $P_{im(2)}(\{T\}) = (0,1)$. $P_{im(2)}$ is also an imprecise probability function. After all, 0 is still a lower bound for both $\mathcal{S}_{\{H\}}$ and $\mathcal{S}_{\{T\}}$, while 1 is still an upper bound for both $\mathcal{S}_{\{H\}}$ and $\mathcal{S}_{\{T\}}$. Now, here's an example that's *not* an imprecise probability function. Define a function $P_{im(3)}$ with $\mathcal{P}\{H,T\}$ as its domain such that $P_{im(3)}(\emptyset) = 0$, $P_{im(3)}(\{H,T\}) = 1$, $P_{im(3)}(\{H\}) = (0, \frac{1}{4})$, and $P_{im(3)}(\{T\}) = (\frac{1}{2}, 1)$. $P_{im(3)}$ isn't an imprecise probability function because $\frac{1}{4}$ isn't an upper bound for $\mathcal{S}_{\{H\}}$, i.e. there exists an $x \in \mathcal{S}_{\{H\}}$ such that $x > \frac{1}{4}$.

At this point, the reader may be wondering why I didn't specify in the definition of an imprecise probability function that a_1 and a_2 are the *greatest lower bound* (G.L.B)[65] and the *least upper bound* (L.U.B)[66] for $\mathcal{S}_A$ respectively, where a_1 is the lower bound and a_2 is the upper bound of the interval $P_{im}(A)$. After all, there may be multiple lower bounds and upper bounds for $\mathcal{S}_A$, but *the* G.L.B and *the* L.U.B for $\mathcal{S}_A$ are unique, i.e. there is only one G.L.B and only one L.U.B. And in virtue of the possibility of there being multiple lower bounds and upper bounds for $\mathcal{S}_A$, there may exist multiple imprecise probability functions P_{im} for the given relevant token probabilistic scenario $\langle\Omega,\mathcal{F},C\rangle$. Take $\langle\{H,T\},\mathcal{P}\{H,T\},C\rangle$ for example. In the previous paragraph, I've defined two imprecise probability functions for this particular token probabilistic scenario: $P_{im(1)}$ and $P_{im(2)}$. However, if I were to specify that a_1 and a_2 must be the G.L.B and the L.U.B respectively, then there would be only one imprecise probability functions for $\langle\{H,T\},\mathcal{P}\{H,T\},C\rangle$: $P_{im(1)}$. $P_{im(2)}$ would no longer be an imprecise probability function because though 0 is the G.L.B for $\mathcal{S}_{\{H\}}$, it isn't the G.L.B for $\mathcal{S}_{\{T\}}$. Instead, the G.L.B for $\mathcal{S}_{\{T\}}$ is $\frac{1}{2}$. Similarly, though 1 is the L.U.B for $\mathcal{S}_{\{T\}}$, it isn't the L.U.B for $\mathcal{S}_{\{H\}}$. Instead, the L.U.B for $\mathcal{S}_{\{H\}}$ is $\frac{1}{2}$. In fact, for any relevant token probabilistic scenarios $\langle\Omega,\mathcal{F},C\rangle$, if I added the specification that a_1 and a_2 must be the G.L.B and the L.U.B respectively, then there is at most one imprecise probability function P_{im} for $\langle\Omega,\mathcal{F},C\rangle$ set in the given ordered field, in virtue of the G.L.B and the L.U.B for each $\mathcal{S}_A$ being unique.

My reason for not including the specification that a_1 is the G.L.B and a_2 is the L.U.B for $\mathcal{S}_A$ is simple. It's because we're working with non-Archimedean probabilities, i.e. probabilities that take on values in some non-Archimedean field, which is *not* complete (**Theorem 2**). That is, there may exist an event $A \in \mathcal{F}$ such that the G.L.B or the

[65]The G.L.B for a set is a lower bound that is greater than or equal to every lower bound for that set.
[66]The L.U.B for a set is an upper bound that is less than or equal to every upper bound for that set.

L.U.B *doesn't* exist for $\mathcal{S}_A$. If I were to include the specification that a_1 is the G.L.B and a_2 is the L.U.B for $\mathcal{S}_A$, then $P_{im}(A)$ wouldn't be defined for such an event A. This in turn implies that P_{im} may not even be defined on $\mathcal{F}$, as there may exist an $A \in \mathcal{F}$ such that P_{im} assigns no value to. So, P_{im} may not correctly represent $\langle \Omega, \mathcal{F}, C \rangle$ at all. In fact, if there indeed exists an event $A \in \mathcal{F}$ such that the G.L.B or the L.U.B doesn't exist for $\mathcal{S}_A$, then no P_{im} (with the added specification that a_1 is the G.L.B and a_2 is the L.U.B for $\mathcal{S}_A$) is defined on $\mathcal{F}$. Consequently, no P_{im} correctly represents $\langle \Omega, \mathcal{F}, C \rangle$. Put simply, in order not to prematurely rule out Benci et al. [2018]'s and Pruss [2021]'s proposals of allowing probabilities to be imprecise, I cannot include the specification that a_1 is the G.L.B and a_2 is the L.U.B for $\mathcal{S}_A$ in the definition of an imprecise probability function. Note that the exclusion of this specification will result in more 'oddities', as we will see soon. I request that the reader take note of these 'oddities', which will ultimately cumulate in my argument against utilising precise non-Archimedean probabilities.

Having defined an imprecise probability function, I will now define when $P_{im}(A) \leq P_{im}(B)$.

Comparing Imprecise Probabilities. *For any two sets $A \in \mathcal{F}$ and $B \in \mathcal{F}$, $P_{im}(A) \leq P_{im}(B)$ iff for every $P \in \mathcal{S}$, $P(A) \leq P(B)$.*

Indeed, **Comparing Imprecise Probabilities** agrees with what Benci et al. [2018] said in the quote cited in §4. "[T]here may be multiple, equally good ways to model the same situation", with each $P \in \mathcal{S}$ being a way. But "[w]hat matters is what is true (or false) on all ways of making these arbitrary choices—what is supertrue (or superfalse)—as well as the spread of possible assignments", with P_{im} detailing the spread of possible assignments. Since every probability function $P \in \mathcal{S}$ assigns to A a probability less than or equal to that of B, it is *supertrue* that A is assigned a probability less than or equal to that of B. And because P_{im} is, in a sense, derived from all these P's, it seems only fitting that P_{im} also reflects this supertrue fact, which is what matters according to Benci et al. [2018]. So, $P_{im}(A) \leq P_{im}(B)$. This is the rationale behind **Comparing Imprecise Probabilities**.

With **Comparing Imprecise Probabilities**, I can now state the analogue of **Precise Representation**, but for imprecise probabilities this time. Recall that **Precise Representation** quantifies over sets in $\mathcal{F}^{\#}$. So, the same goes for its analogue.

Imprecise Representation. *For any two sets $A \in \mathcal{F}^{\#}$ and $B \in \mathcal{F}^{\#}$, $A \preceq B$ iff $P_{im}(A) \leq P_{im}(B)$.*

As is the case with **Precise Representation**, note that the 'P_{im}' in **Imprecise Representation** need not only denote a particular imprecise probability function. It can be any imprecise probability function defined on $\mathcal{F}^{\#}$.

Here's an oddity implied by the definitions just provided. Consider the token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, as defined in §3.2. It is easy to verify that $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is relevant. Define a function $P_{im(4)}$ with $\mathcal{F}^{\#}$ as its domain such that all of the following hold:

1. $P_{im(4)}(\emptyset) = 0$.

2. $P_{im(4)}(\mathbb{N}) = 1$.

3. For all sets $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) = 0$, $P_{im(4)}(A) = d(A)$.

4. For all sets $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) \neq 0$, $P_{im(4)}(A) = (0, 1)$.

Again, it is easy to verify that $P_{im(4)}$ is an imprecise probability function. And not only is $P_{im(4)}$ an imprecise probability function, the following is true as well:

Theorem 14. *Under* ***Comparing Imprecise Probabilities*** *and* ***Imprecise Representation****,* $P_{im(4)}$ *correctly represents* $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$*, i.e.* $P_{im(4)}$ *is defined on* $\mathcal{F}^{\#}$ *and satisfies all the constraints in* $C^{\#}$*.*

But intuitively, $P_{im(4)}$ does *not* correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. After all, for every $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) \neq 0$, $P_{im(4)}(A) = (0, 1)$. How can such a probability function correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$?

Here's my reply. I share the intuition stated above. That is, I find it uncomfortable that $P_{im(4)}$ correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, according to **Comparing Imprecise Probabilities** and **Imprecise Representation**. But notice that if I added the specification that for every $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) \neq 0$, an imprecise probability assigned to A, which is an interval, must have as the lower bound the G.L.B and as the upper bound the L.U.B of $\mathcal{S}_A$, then $P_{im(4)}$ will *no longer* correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. After all, with the specification above, $P_{im(4)}$ isn't even an imprecise probability function. This is because 0 isn't the G.L.B for $\mathcal{S}_A$ for all $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) \neq 0$. Similarly, 1 isn't the L.U.B for $\mathcal{S}_A$ for all $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) \neq 0$. In fact, if this specification (albeit, in a more general form) is added to the definition of an imprecise probability function, then no imprecise probability function correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. The reason is simple: there exists a set $A \in \mathcal{F}^{\#}$ such that the G.L.B or the L.U.B doesn't exist for $\mathcal{S}_A$, e.g. $A = \{i\}$. More details about this in §4.2.

For the sake of not prematurely ruling out Benci et al. [2018]'s and Pruss [2021]'s proposals of allowing probabilities to be imprecise, I didn't include the specification above (albeit, in a more general form) in the definition of an imprecise probability function. But by excluding this specification, we ran into the oddity spelled out above. So, if we want to avoid the oddity, that specification has to be included in the definition of an imprecise probability function, though this means that Benci et al. [2018]'s and Pruss [2021]'s proposals are *wrong*, as no imprecise probability function then correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Of course, Benci et al. [2018] and Pruss [2021] can always reject **Comparing**

Imprecise Probabilities or **Imprecise Representation**. I would just want to know their reasons for doing so. After all, as I've explained above, **Comparing Imprecise Probabilities** seems to agree with what Benci et al. [2018] said in the quote cited in §4. And **Imprecise Representation** is just the analogue of **Precise Representation**, which, as I've argued in §3.3, seems to be a better fit for the purposes of this thesis than **Proper Containment** and **Conditionality**.

Here's another reply of a different tack. The intuition that $P_{im(4)}$ does not correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ may be reasonably questioned. It isn't implausible that $P_{im(4)}$ correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. After all, $P_{im(4)}$ isn't *trivial*; it still does tell us certain relations between some events. For example, consider $\{2, 4, 6, \ldots\}$ and $\{1, 3, 5, \ldots\}$. By the definition of $P_{im(4)}$, $P_{im(4)}(\{2, 4, 6, \ldots\}) = P_{im(4)}(\{1, 3, 5, \ldots\}) = \frac{1}{2}$. And by **Imprecise Representation**, $\{2, 4, 6, \ldots\} \sim \{1, 3, 5, \ldots\}$, in accordance with the constraints in $C^{\#}$. Here's another example. Consider $\{10, 20, 30, \ldots\}$ and $\{4, 8, 12, \ldots\}$. By the definition of $P_{im(4)}$, $P_{im(4)}(\{10, 20, 30, \ldots\}) = \frac{1}{10}$ and $P_{im(4)}(\{4, 8, 12, \ldots\}) = \frac{1}{4}$. By **Imprecise Representation**, $\{10, 20, 30, \ldots\} \prec \{4, 8, 12, \ldots\}$, in accordance with the constraints in $C^{\#}$. So, we can still glean from $P_{im(4)}$ some information about $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Furthermore, this information that we can glean from $P_{im(4)}$ is *correct*. This suggests that it isn't implausible that $P_{im(4)}$ correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$.

As for the discomfort that we have (or, at least that I have) about $P_{im(4)}$ correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, this feeling can be explained in some other ways besides $P_{im(4)}$ not correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. For example, notice that there are many other—in fact, uncountably many other—imprecise probability functions that assign events $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) \neq 0$ *tighter* intervals for imprecise probabilities. The reason why we feel uncomfortable about $P_{im(4)}$ correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is because we think that these other imprecise probability functions are *better* representations of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ than $P_{im(4)}$ is, i.e. these other imprecise probability functions have more representational power than $P_{im(4)}$. More details about when two imprecise probability functions have the same representational power in representing the given token probabilistic scenario will be provided in §5. But for now, the important point is that $P_{im(4)}$ having less representational power doesn't mean that $P_{im(4)}$ is therefore an incorrect representation of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Instead, $P_{im(4)}$ still correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. It's just that there are imprecise probability functions that do so better.

To conclude this subsection, let $\mathcal{M}_{im}$ denote an imprecise model hereafter. Call an imprecise model $\mathcal{M}_{im} = \langle \Omega, \mathcal{F}, C, \Sigma \rangle$ *interval-based* iff $\langle \Omega, \mathcal{F}, C \rangle$ is relevant and $\Sigma = \{(A, P_{im}(A)) : A \in \mathcal{F}\}$ for some imprecise probability function P_{im} defined on $\mathcal{F}$. Next, call an interval-based model $\mathcal{M}_{im}$ *correct* just in case P_{im} correctly represents the given token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$. In this section, I'm only interested in correct interval-based models $\mathcal{M}_{im}$.

4.2 Different Correct Imprecise Models for a Fair Lottery Over the Natural Numbers

The main goal of this subsection is to prove that there are multiple correct interval-based models $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Define a precise probability function P_x with $\mathcal{F}^{\#}$ as its domain such that $P_x(A) = d(A) + \big(x \cdot N(A)\big)$ for any particular x such that $0 < x < \frac{1}{n}$ for all $n \in \mathbb{N}$.

Theorem 15. *Under* ***Precise Representation****, P_x satisfies all the constraints in $C^{\#}$.*

Theorem 15 implies two things. Here's the first. For all x such that $0 < x < \frac{1}{n}$ for all $n \in \mathbb{N}$, P_x correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Now, fix a non-Archimedean ordered field $\mathbb{F}$ and let $\mathcal{S}$ be the set of all P_x's set in $\mathbb{F}$, i.e. $\mathcal{S} = \{P_x : x \in \mathbb{F}\}$. Note that by the definition of P_x, $P_x(\{i\}) = x$. After all, $d(\{i\}) = 0$ and $N(\{i\}) = 1$. This means that $\mathcal{S}_{\{i\}} = \{x \in \mathbb{F} : \forall n \in \mathbb{N}(0 < x < \frac{1}{n})\}$, i.e. $\mathcal{S}_{\{i\}}$ is the set of all positive infinitesimals in $\mathbb{F}$, which is the second thing that **Theorem 15** implies. In other words, for every positive infinitesimal $x \in \mathbb{F}$, there exists a correct precise model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ that assigns x to $\{i\}$.

Theorem 16. *$\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is relevant.*

Theorem 17. *For every $u \in \mathbb{F}$ such that $0 \leq u \leq 1$, u is an upper bound for $\mathcal{S}_{\{i\}}$ iff $u > \frac{1}{n}$ for some $n \in \mathbb{N}$.*

Like **Theorem 15**, **Theorem 17** implies two things. Here's the first. There are uncountably many upper bounds $0 \leq u \leq 1$ for $\mathcal{S}_{\{i\}}$. After all, there are uncountably many u's such that $u > \frac{1}{n}$ for some $n \in \mathbb{N}$. Here's the second and more important thing. The L.U.B for $\mathcal{S}_{\{i\}}$ *does not exist.* For every upper bound $0 \leq u \leq 1$ for $\mathcal{S}_{\{i\}}$, $u > \frac{1}{n}$ for some $n \in \mathbb{N}$, by **Theorem 17**. But there is always a $0 \leq u' \leq 1$ such that $\frac{1}{n+1} \leq u' \leq \frac{1}{n}$, which implies that $u' < u$ since $u' \leq \frac{1}{n} < u$, i.e. $u' = \frac{1}{n+1}$. And by **Theorem 17**, u' is also an upper bound for $\mathcal{S}_{\{i\}}$. In other words, for every upper bound $0 \leq u \leq 1$ for $\mathcal{S}_{\{i\}}$, there is always a smaller upper bound u' for the same set, implying that the L.U.B for $\mathcal{S}_{\{i\}}$ does not exist. However, note that the G.L.B for $\mathcal{S}_{\{i\}}$ exists: 0.

This is the reason why I cannot include in the definition of an imprecise probability function the specification that for every set $A \in \mathcal{F}$, if P_{im} assigns A an interval, then the upper bound of that interval must be the L.U.B and the lower bound of that interval must be the G.L.B for $\mathcal{S}_A$. If such a specification were added, then there wouldn't be any imprecise probability functions correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. For reductio, assume that there is. Let such an imprecise probability function be P_{im}. In virtue of P_{im} correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, P_{im} is defined on $\mathcal{F}^{\#}$. This means that P_{im} assigns every event in $\mathcal{F}^{\#}$ a probability (precise or imprecise). However, there exists an $A \in \mathcal{F}^{\#}$, i.e. $A = \{i\}$, such that (i) $\#(\mathcal{S}_A) > 1$ and (ii) the G.L.B or the L.U.B for $\mathcal{S}_A$

doesn't exist. So, by the specification stated above, P_{im} doesn't assign A any probability (precise or imprecise) at all. Contradiction. Therefore, there are no imprecise probability functions correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, if the specification above is added to the definition of an imprecise probability function. Here's another way to see why P_{im} (with the added specification stated above) doesn't correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. In a fair lottery, equal probabilities (precise or imprecise) must be assigned to *every* singleton set. Since P_{im} doesn't assign a probability (precise or imprecise) to *any* singleton set, P_{im} doesn't correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, which intuitively describes a fair lottery over $\mathbb{N}$.

However, by excluding the specification stated above from the definition of an imprecise probability function, we run into another problem. Since there are uncountably many upper bounds $0 \leq u \leq 1$ for $\mathcal{S}_{\{i\}}$, there are also uncountably many imprecise probability functions P_{im} (*without* the specification stated above) that correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Define a function $P_{im(u)}$ with $\mathcal{F}^{\#}$ as its domain for any particular $u \in \mathbb{F}$ such that all of the following hold:

1. $0 \leq u \leq 1$.
2. u is an upper bound for $\mathcal{S}_{\{i\}}$.
3. $P_{im(u)}(\emptyset) = 0$.
4. $P_{im(u)}(\mathbb{N}) = 1$.
5. For all sets $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) = 0$, $P_{im(u)}(A) = d(A)$.
6. For all sets $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) > 0$, if $d(A) + u \leq 1$, then $P_{im(u)}(A) = (d(A), d(A) + u)$. If $d(A) + u > 1$, then $P_{im(u)}(A) = (d(A), 1)$.
7. For all sets $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) < 0$, if $d(A) - u \geq 0$, then $P_{im(u)}(A) = (d(A) - u, d(A))$. If $d(A) - u < 0$, then $P_{im(u)}(A) = (0, d(A))$.

Theorem 18. *$P_{im(u)}$ is an imprecise probability function.*

Theorem 19. *Under* ***Comparing Imprecise Probabilities*** *and* ***Imprecise Representation****, $P_{im(u)}$ correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, i.e. $P_{im(u)}$ is defined on $\mathcal{F}^{\#}$ and satisfies all the constraints in $C^{\#}$.*

Theorems 18 and **19** show that there are uncountably many imprecise probability functions $P_{im(u)}$ that correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. After all, there are uncountably many upper bounds $0 \leq u \leq 1$ for $\mathcal{S}_{\{i\}}$.

Now, consider these two imprecise probability functions $P_{im(u)}$ and $P_{im(u')}$ where $u < u'$. Define $\mathcal{M}_{im(u)} = \langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}, \Sigma_{im(u)} \rangle$ where $\Sigma_{im(u)} = \{(A, P_{im(u)}(A)) : A \in \mathcal{F}^{\#}\}$. Similarly, define $\mathcal{M}_{im(u')} = \langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}, \Sigma_{im(u')} \rangle$ where $\Sigma_{im(u')} = \{(A, P_{im(u')}(A)) : A \in \mathcal{F}^{\#}\}$. It is easy to verify that $\mathcal{M}_{im(u)}$ and $\mathcal{M}_{im(u')}$ are two different correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. In fact, $\mathcal{M}_{im(u)}$ and $\mathcal{M}_{im(u')}$ are just two out of an uncountable infinity of correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$; there is one such model for each $P_{im(u)}$, as defined above.

Among all the existing correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, it isn't obvious that there is always a particular model that 'stands out' above the rest. Sometimes, we may not have good reason to single out a particular model among uncountably many others. In cases like that, it isn't obvious exactly which model to choose. But for the sake of applying probability theory to—say, calculate the expected utility of performing an action, at least one model must be chosen. For if none were chosen, how could we then calculate the expected utility of performing an action? So, it seems that we are still stuck in the same predicament as before when only precise models are considered. Given that a choice has to be made, exactly which model do we choose? And say we've decided after deliberation to choose $\mathcal{M}_{im(u)}$ over uncountably many other correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Is doing so problematic w.r.t an end result R? I will answer this question in the affirmative in §5.

Taking a step back, it seems that we are trapped between a rock and a hard place. If I include in the definition of an imprecise probability function the specification that for every set $A \in \mathcal{F}$, if P_{im} assigns A an interval, then the upper bound of that interval must be the L.U.B and the lower bound of that interval must be the G.L.B for $\mathcal{S}_A$, then there are no correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. This is because there are no imprecise probability functions that correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. And if we limit ourselves to choosing only interval-based models, any actions associated with $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ will thus have *no* expected utilities, which seems like an unattractive result. But if I exclude such a specification from the definition of an imprecise probability function, then there are too many—in fact, uncountably many—correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, choosing between which is problematic w.r.t an end result R. Either there are no correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ or there are too many. Both are problematic and there seems to be no sweet spot here.

4.3 The Property of Tightness

Recall the token probabilistic scenario $\langle \{H, T\}, \mathcal{P}\{H, T\}, C \rangle$ from before describing tossing a coin that's biased towards tails. In this token probabilistic scenario, $\mathcal{S}_{\{H\}} = \{x \in \mathbb{F} : 0 < x < \frac{1}{2}\}$ and $\mathcal{S}_{\{T\}} = \{x \in \mathbb{F} : \frac{1}{2} < x < 1\}$ for some ordered field $\mathbb{F}$. Recall also that the function $P_{im(1)}$ with $\mathcal{P}\{H, T\}$ as its domain is such that $P_{im(1)}(\emptyset) = 0$, $P_{im(1)}(\{H, T\}) = 1$, $P_{im(1)}(\{H\}) = (0, \frac{1}{2})$, and $P_{im(1)}(\{T\}) = (\frac{1}{2}, 1)$. And the function $P_{im(2)}$ with $\mathcal{P}\{H, T\}$ as its domain is such that $P_{im(2)}(\emptyset) = 0$, $P_{im(2)}(\{H, T\}) = 1$, $P_{im(2)}(\{H\}) = (0, 1)$, and $P_{im(2)}(\{T\}) = (0, 1)$. Both functions $P_{im(1)}$ and $P_{im(2)}$ are imprecise probability functions, as proven above. However, I hope the reader shares my intuition that $P_{im(1)}$ represents $\langle\{H,T\},\mathcal{P}\{H,T\},C\rangle$ *better* than $P_{im(2)}$. After all, the comparison between $\{H\}$ and $\{T\}$, i.e. $\{H\} \prec \{T\}$, is 'reflected' in the imprecise probabilities assigned by $P_{im(1)}$, but not $P_{im(2)}$. Here's what I mean. Hereafter, let $\underline{P}_{im}(A)$ denote the lower bound and $\overline{P}_{im}(A)$ denote the upper bound of $P_{im}(A)$. Then, $P_{im}(A)$ is an interval iff $\underline{P}_{im}(A) < \overline{P}_{im}(A)$; $P_{im}(A)$ is a precise value iff $\underline{P}_{im}(A) = \overline{P}_{im}(A)$. The comparison between $\{H\}$ and $\{T\}$, i.e. $\{H\} \prec \{T\}$, is 'reflected' in the imprecise probabilities assigned by $P_{im(1)}$, in the sense that $\frac{1}{2} = \overline{P}_{im(1)}(\{H\}) \leq \underline{P}_{im(1)}(\{T\}) = \frac{1}{2}$. But for $P_{im(2)}$, $1 = \overline{P}_{im(2)}(\{H\}) > \underline{P}_{im(2)}(\{T\}) = 0$. So, the comparison between $\{H\}$ and $\{T\}$ isn't 'reflected' in the imprecise probabilities assigned by $P_{im(2)}$. This is the reason why I think that $P_{im(1)}$ represents $\langle\{H,T\},\mathcal{P}\{H,T\},C\rangle$ better than $P_{im(2)}$.

Indeed, I think that all imprecise probability functions P_{im} *should* satisfy the following statement.[67]

Imprecise Reflection. *Let $\langle\Omega,\mathcal{F},C\rangle$ be any relevant token probabilistic scenario. For any two sets A and B in $\mathcal{F}$, suppose:*

1. *For all $P \in \mathcal{S}$, $P(A) \leq P(B)$.*

2. *$\mathcal{S}_A \cap \mathcal{S}_B = \emptyset$.*

Then, $\overline{P}_{im}(A) \leq \underline{P}_{im}(B)$.

Here's what **Imprecise Reflection** says intuitively. First, note that the following is true:

Theorem 20. *Let $\langle\Omega,\mathcal{F},C\rangle$ be any relevant token probabilistic scenario. For some $A \in \mathcal{F}$ and $B \in \mathcal{F}$, if A and B fulfill the two conditions stated in **Imprecise Reflection**, then, for every $x \in \mathcal{S}_A$, $x < y$ for any $y \in \mathcal{S}_B$.*

Put simply, every probability value that can be assigned to A is strictly less than any probability value that can be assigned to B. Here's an example. Let $\mathcal{S}_A = \{x \in \mathbb{F} : 0 < x < \frac{1}{4}\}$ and $\mathcal{S}_B = \{x \in \mathbb{F} : \frac{3}{4} < x < 1\}$ for some ordered field $\mathbb{F}$. Notice that every $x \in \mathcal{S}_A$ is less than any member $y \in \mathcal{S}_B$. Given $\mathcal{S}_A$ and $\mathcal{S}_B$, as defined above, it seems only fitting to require as a *minimal* constraint that the upper bound of an imprecise probability assigned to A is less than or equal to the lower bound of an imprecise probability assigned to B. After all, the highest probability that can be assigned to A is $\frac{1}{4}$, which is less than the lowest probability that can be assigned to B, which in turn is $\frac{3}{4}$. Now, back to $P_{im(1)}$ and $P_{im(2)}$. Notice that while $P_{im(1)}$ satisfies **Imprecise Reflection**, $P_{im(2)}$ does not, which explains why $P_{im(1)}$ represents $\langle\{H,T\},\mathcal{P}\{H,T\},C\rangle$ better than $P_{im(2)}$.

Importantly, recall the function $P_{im(u)}$, as defined towards the end of §4.2.

[67]Why 'should' and not 'do'? It's because according to my definition of an imprecise probability function P_{im} in §4.1, there may exist a P_{im} that doesn't satisfy **Imprecise Reflection**, although I think that there's something suspicious about such a P_{im}. And recall that I adopted the current definition of an imprecise probability function for the sake of not prematurely ruling out Benci et al. [2018]'s and Pruss [2021]'s proposals of allowing probabilities to be imprecise.

Theorem 21. *$P_{im(u)}$ does not satisfy* ***Imprecise Reflection****.*

Violating **Imprecise Reflection** doesn't mean that $P_{im(u)}$ is *trivial*. Indeed, $P_{im(u)}$ *isn't* trivial at all; it still tells us certain relations between some events. For example, consider $\{2, 4, 6, \ldots\}$ and $\{1, 3, 5, \ldots\}$. Note that $d(\{2, 4, 6, \ldots\}) = d(\{1, 3, 5, \ldots\}) = \frac{1}{2}$ and $N(\{2, 4, 6, \ldots\}) = N(\{1, 3, 5, \ldots\}) = 0$. By the definition of $P_{im(u)}$, $P_{im(u)}(\{2, 4, 6, \ldots\}) = P_{im(u)}(\{1, 3, 5, \ldots\}) = \frac{1}{2}$. And by **Imprecise Representation**, $\{2, 4, 6, \ldots\} \sim \{1, 3, 5, \ldots\}$, in accordance with the constraints in $C^{\#}$. Here's another example. Consider $\{10, 20, 30, \ldots\}$ and $\{4, 8, 12, \ldots\}$. Note that $d(\{10, 20, 30, \ldots\}) = \frac{1}{10}$, $N(\{10, 20, 30, \ldots\}) = 0$, $d(\{4, 8, 12, \ldots\}) = \frac{1}{4}$, and $N(\{4, 8, 12, \ldots\}) = 0$. By the definition of $P_{im(u)}$, $P_{im(u)}(\{10, 20, 30, \ldots\}) = \frac{1}{10}$ and $P_{im(u)}(\{4, 8, 12, \ldots\}) = \frac{1}{4}$. By **Imprecise Representation**, $\{10, 20, 30, \ldots\} \prec \{4, 8, 12, \ldots\}$, in accordance with the constraints in $C^{\#}$. Here's yet another example. Consider $\{10, 15, 20, \ldots\}$ and $\{3, 6, 9, \ldots\}$. Note that $d(\{10, 15, 20, \ldots\}) = \frac{1}{5}$, $N(\{10, 15, 20, \ldots\}) = -1$, $d(\{3, 6, 9, \ldots\}) = \frac{1}{3}$, and $N(\{3, 6, 9, \ldots\}) = 0$. By the definition of $P_{im(u)}$, if $\frac{1}{5} - u \geq 0$, then $P_{im(u)}(\{10, 15, 20, \ldots\}) = (\frac{1}{5} - u, \frac{1}{5})$ for some $u \in \mathbb{F}$ where $\mathbb{F}$ is a non-Archimedean ordered field, $0 \leq u \leq 1$, and $u > \frac{1}{n}$ for some $n \in \mathbb{N}$. If $\frac{1}{5} - u < 0$, then $P_{im(u)}(\{10, 15, 20, \ldots\}) = (0, \frac{1}{5})$. Similarly, by the definition of $P_{im(u)}$, $P_{im(u)}(\{3, 6, 9, \ldots\}) = \frac{1}{3}$. In virtue of $\frac{1}{5} < \frac{1}{3}$, $\{10, 15, 20, \ldots\} \prec \{3, 6, 9, \ldots\}$, by **Imprecise Representation**, in accordance with the constraints in $C^{\#}$. So, we can still glean from $P_{im(u)}$ some information about $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Furthermore, this information that we can glean from $P_{im(u)}$ is *correct*.

But even though $P_{im(u)}$ isn't trivial, $P_{im(u)}$ still does not 'reflect' the *full* order of events, as implied by the constraints in $C^{\#}$, in the sense that $P_{im(u)}$ does not satisfy **Imprecise Representation**. For example, consider the set $\{i\}$. Note that $d(\{i\}) = 0$ and $\mathcal{S}_{\{i\}} = \{x \in \mathbb{F} : \forall n \in \mathbb{N}(0 < x < \frac{1}{n})\}$. Now, consider a set $A \in \mathcal{F}^{\#}$ such that $d(A) = \frac{1}{n}$ and $N(A) = 0$. Note that $\mathcal{S}_A = \{\frac{1}{n}\}$. So, $\mathcal{S}_{\{i\}} \cap \mathcal{S}_A = \emptyset$. Further note that for all $P_x \in \mathcal{S}$, $P_x(\{i\}) \leq P_x(A)$.[68] All the conditions of **Imprecise Reflection** are thus satisfied, but the consequent of **Imprecise Reflection** isn't. By the definition of $P_{im(u)}$, $P_{im(u)}(\{i\}) = (0, u)$. This means that $\overline{P}_{im(u)}(\{i\}) = u$. And by the definition of $P_{im(u)}$, $P_{im(u)}(A) = \frac{1}{n}$. This means that $\underline{P}_{im(u)}(A) = \frac{1}{n}$. Therefore, $\overline{P}_{im(u)}(\{i\}) = u > \frac{1}{n} = \underline{P}_{im(u)}(A)$, in violation of **Imprecise Reflection**. But this assignment of imprecise probabilities seems counter-intuitive. After all, according to every $P_x \in \mathcal{S}$, $\{i\}$ is *infinitely* less probable than A, i.e. $P_x(\{i\})$ is a positive infinitesimal while $P_x(A) = \frac{1}{n}$. So, it seems only fitting that the upper bound of an imprecise probability assigned to $\{i\}$ should be less than or equal to $\frac{1}{n}$, in accordance with **Imprecise Reflection**. But curiously and surprisingly, $\overline{P}_{im(u)}(\{i\}) = u > \frac{1}{n}$ instead, as shown above. Therefore, though $P_{im(u)}$ isn't trivial and correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ under **Comparing Imprecise Probabilities** and **Imprecise Representation** (**Theorem 19**), there is still some reason to be suspicious of

[68] See the top of §4.2 for the definition of P_x.

$P_{im(u)}$. Namely, it doesn't satisfy **Imprecise Reflection**, which intuitively is a *minimal* constraint on imprecise probabilities.

Are there more constraints on all imprecise probability functions P_{im} other than **Imprecise Reflection**? It seems that there are. After all, it isn't enough to merely 'reflect' the order of events in the imprecise probabilities. 'Reflecting' how *much* less (or more) probable an event is as compared to another event is important too. To illustrate, consider this token probabilistic scenario $\langle\{H,T\},\mathcal{P}\{H,T\},C'\rangle$ where C' contains the probability axioms, **Regularity**, the statement 'The coin is biased towards tails', and some other statement to the effect that $\mathcal{S}'_{\{H\}} = \{x \in \mathbb{F} : 0 < x < \frac{1}{4}\}$ and $\mathcal{S}'_{\{T\}} = \{x \in \mathbb{F} : \frac{3}{4} < x < 1\}$ for some ordered field $\mathbb{F}$. Define a function $P_{im(5)}$ with $\mathcal{P}\{H,T\}$ as its domain such that $P_{im(5)}(\emptyset) = 0$, $P_{im(5)}(\{H,T\}) = 1$, $P_{im(5)}(\{H\}) = (0, \frac{1}{4})$, and $P_{im(5)}(\{T\}) = (\frac{3}{4}, 1)$. It is easy to verify that $P_{im(5)}$ is an imprecise probability function. But I hope the reader shares my intuition that $P_{im(5)}$ represents $\langle\{H,T\},\mathcal{P}\{H,T\},C'\rangle$ *better* than $P_{im(1)}$, even though both functions satisfy **Imprecise Reflection** w.r.t $\langle\{H,T\},\mathcal{P}\{H,T\},C'\rangle$. This is because in $\langle\{H,T\},\mathcal{P}\{H,T\},C'\rangle$, $\{T\}$ is at least 3 times as probable as $\{H\}$. So, it seems only fitting that the imprecise probabilities assigned to these events should also 'reflect' this fact, in the sense that the lower bound of an imprecise probability assigned to $\{T\}$ should be at least 3 times as great as the upper bound of an imprecise probability assigned to $\{H\}$. According to the definition of $P_{im(5)}$, $\overline{P}_{im(5)}(\{H\}) = \frac{1}{4}$ while $\underline{P}_{im(5)}(\{T\}) = \frac{3}{4}$. So, $\underline{P}_{im(5)}(\{T\}) \geq 3 \cdot \overline{P}_{im(5)}(\{H\})$. But according to the definition of $P_{im(1)}$, $\overline{P}_{im(1)}(\{H\}) = \frac{1}{2} = \underline{P}_{im(1)}(\{T\})$. So, $\underline{P}_{im(1)}(\{T\}) \ngeq 3 \cdot \overline{P}_{im(1)}(\{H\})$. Therefore, while $P_{im(5)}$ 'reflects' the fact that $\{T\}$ is at least 3 times as probable as $\{H\}$, $P_{im(1)}$ doesn't. This explains why $P_{im(5)}$ represents $\langle\{H,T\},\mathcal{P}\{H,T\},C'\rangle$ better than $P_{im(1)}$, even though both functions satisfy **Imprecise Reflection** w.r.t $\langle\{H,T\},\mathcal{P}\{H,T\},C'\rangle$.

The above discussion suggests that there must be more constraints on all imprecise probability functions P_{im} other than just **Imprecise Reflection**. After all, **Imprecise Reflection** merely requires the order of events to be 'reflected' in the imprecise probabilities, but it says nothing about 'reflecting' how much less (or more) probable an event is as compared to another event. Here's a statement that can distinguish between $P_{im(1)}$ and $P_{im(5)}$.

Maximal Distance. *Let $\langle\Omega,\mathcal{F},C\rangle$ be any relevant token probabilistic scenario. For any two sets A and B in $\mathcal{F}$, suppose:*

1. *For all $P \in \mathcal{S}$, $P(A) \leq P(B)$.*
2. *$\mathcal{S}_A \cap \mathcal{S}_B = \emptyset$.*

Now, define $\mathcal{X}$ to be the set of all x's in an ordered field $\mathbb{F}$ such that x is an upper bound for $\mathcal{S}_A$ and a lower bound for $\mathcal{S}_B$. Then, $|\overline{P}_{im}(A) - \underline{P}_{im}(B)| \geq |x - x'|$ for any two x and

x' in $\mathcal{X}$.[69,70]

Intuitively, **Maximal Distance** says that the 'distance' between $\overline{P}_{im}(A)$ and $\underline{P}_{im}(B)$, i.e. $|\overline{P}_{im}(A) - \underline{P}_{im}(B)|$, must be the *widest* 'distance' among all the 'distances' between any two x and x' in $\mathcal{X}$, i.e. $|x - x'|$. To see how **Maximal Distance** distinguishes $P_{im(1)}$ from $P_{im(5)}$, first note that both functions satisfy the two conditions of **Maximal Distance** w.r.t $\langle\{H,T\}, \mathcal{P}\{H,T\}, C'\rangle$. Now, let $A = \{H\}$ and $B = \{T\}$. According to the definition of $P_{im(5)}$, $\overline{P}_{im(5)}(\{H\}) = \frac{1}{4}$ and $\underline{P}_{im(5)}(\{T\}) = \frac{3}{4}$. So, $|\overline{P}_{im(5)}(\{H\}) - \underline{P}_{im(5)}(\{T\})| = \frac{1}{2}$. And according to the definition of $P_{im(1)}$, $\overline{P}_{im(1)}(\{H\}) = \frac{1}{2} = \underline{P}_{im(1)}(\{T\})$. So, $|\overline{P}_{im(1)}(\{H\}) - \underline{P}_{im(1)}(\{T\})| = 0$. Now, the L.U.B (in $\mathbb{F}$) of $\mathcal{S}'_{\{H\}} = \{x \in \mathbb{F} : 0 < x < \frac{1}{4}\}$ is $\frac{1}{4}$ while the G.L.B (in $\mathbb{F}$) of $\mathcal{S}'_{\{T\}} = \{x \in \mathbb{F} : \frac{3}{4} < x < 1\}$ is $\frac{3}{4}$. This means that the widest 'distance' among all the 'distances' between any two x and x' in $\mathcal{X}$ is $\frac{1}{2}$, which is equal to $|\overline{P}_{im(5)}(\{H\}) - \underline{P}_{im(5)}(\{T\})|$. Therefore, $P_{im(5)}$ satisfies **Maximal Distance**. But $P_{im(1)}$ does not satisfy **Maximal Distance**, as $|\overline{P}_{im(1)}(\{H\}) - \underline{P}_{im(1)}(\{T\})| = 0 < \frac{1}{2}$, i.e. $|\overline{P}_{im(1)}(\{H\}) - \underline{P}_{im(1)}(\{T\})|$ is not the widest 'distance' there is. The satisfaction of **Maximal Distance** is what distinguishes $P_{im(5)}$ from $P_{im(1)}$. And for this exact reason, $P_{im(5)}$ represents $\langle\{H,T\}, \mathcal{P}\{H,T\}, C'\rangle$ better than $P_{im(1)}$. My inclination is that all imprecise probability functions P_{im} *should* satisfy **Maximal Distance**.

Theorem 22. *Let $\langle\Omega, \mathcal{F}, C\rangle$ be any relevant token probabilistic scenario and $\mathbb{F}$ be an ordered field. If P_{im} satisfies* ***Maximal Distance*** *(and correctly represents $\langle\Omega, \mathcal{F}, C\rangle$), then for all $A \in \mathcal{F}$, $\overline{P}_{im}(A)$ is the L.U.B in $\mathbb{F}$ and $\underline{P}_{im}(A)$ is the G.L.B in $\mathbb{F}$ for $\mathcal{S}_A$.*[71]

Corollary 3. *Let $\langle\Omega, \mathcal{F}, C\rangle$ be any relevant token probabilistic scenario. If P_{im} satisfies* ***Maximal Distance*** *(and correctly represents $\langle\Omega, \mathcal{F}, C\rangle$), then P_{im} satisfies* ***Imprecise Reflection****.*

That is, **Maximal Distance** implies **Imprecise Reflection**. Recall that by **Theorem 21**, $P_{im(u)}$ does not satisfy **Imprecise Reflection**. So, by **Corollary 3**, $P_{im(u)}$ does not satisfy **Maximal Distance** too. Importantly, note that for a relevant token probabilistic scenario $\langle\Omega, \mathcal{F}, C\rangle$, if P_{im} correctly represents $\langle\Omega, \mathcal{F}, C\rangle$ and satisfies **Maximal Distance**, then for all $A \in \mathcal{F}$, the L.U.B and the G.L.B of $\mathcal{S}_A$ must exist in the given ordered field $\mathbb{F}$ (**Theorem 22**).

Finally, I can now introduce the property of tightness.

Tightness. *Let $\langle\Omega, \mathcal{F}, C\rangle$ be any relevant token probabilistic scenario and $\mathbb{F}$ be an ordered field. For all $A \in \mathcal{F}$, $\overline{P}_{im}(A)$ is the L.U.B in $\mathbb{F}$ and $\underline{P}_{im}(A)$ is the G.L.B in $\mathbb{F}$ for $\mathcal{S}_A$.*

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[69] Note that $|a|$ here denotes the absolute value of a.

[70] Note that all three sets $\mathcal{S}_A$, $\mathcal{S}_B$, $\mathcal{X}$ are subsets of $\mathbb{F}$.

[71] Recall from §4.1 that $\mathcal{S}_A = \{x \in \mathbb{F} : \exists P \in \mathcal{S}(P(A) = x)\}$ where $\mathcal{S}$ is the set of all precise probability functions P set in $\mathbb{F}$ that correctly represent $\langle\Omega, \mathcal{F}, C\rangle$.

Hereafter, say that an imprecise probability function P_{im} is *set in* $\mathbb{F}$, for some ordered field $\mathbb{F}$, just in case (i) for every A in the domain of P_{im} such that $P_{im}(A)$ is a precise value, $P_{im}(A) \in \mathbb{F}$ and (ii) for every A in the domain of P_{im} such that $P_{im}(A)$ is an interval, $P_{im}(A) \subseteq \mathbb{F}$. Call an imprecise probability function P_{im} set in $\mathbb{F}$ *tight w.r.t* $\langle \Omega, \mathcal{F}, C \rangle$ just in case P_{im} satisfies **Tightness**. Importantly, in order for an imprecise probability function P_{im} set in $\mathbb{F}$ to be tight w.r.t the given relevant token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$, for all $A \in \mathcal{F}$, the L.U.B and the G.L.B for $\mathcal{S}_A$ must exist in $\mathbb{F}$. But this does not imply that $\mathbb{F}$ must be complete, i.e. that $\mathbb{F}$ must satisfy the **Least Upper Bound Property.** To see why, consider the token probabilistic scenario $\langle \{H, T\}, \mathcal{P}\{H, T\}, C \rangle$ from before describing tossing a coin that's biased towards tails. In this scenario, $\mathcal{S}_{\{H\}} = \{x \in \mathbb{F} : 0 < x < \frac{1}{2}\}$ and $\mathcal{S}_{\{T\}} = \{x \in \mathbb{F} : \frac{1}{2} < x < 1\}$ for some ordered field $\mathbb{F}$. Note that $\mathbb{F}$ here may very well contain positive infinitesimals, i.e. $\mathcal{S}_{\{H\}}$ contains x's such that $0 < x < \frac{1}{n}$ for all $n \in \mathbb{N}$. This means that $\mathbb{F}$ is not complete. Yet, for every $A \in \mathcal{P}\{H, T\}$, the L.U.B and the G.L.B for $\mathcal{S}_A$ exist in $\mathbb{F}$, i.e. the L.U.B and the G.L.B for $\mathcal{S}_\emptyset = \{0\}$ is 0, the L.U.B and the G.L.B for $\mathcal{S}_{\{H,T\}} = \{1\}$ is 1, the L.U.B for $\mathcal{S}_{\{H\}}$ is $\frac{1}{2}$ while the G.L.B for the same set is 0, and the L.U.B for $\mathcal{S}_{\{T\}}$ is 1 while the G.L.B for the same set is $\frac{1}{2}$.

Corollary 4. *Let $\langle \Omega, \mathcal{F}, C \rangle$ be any relevant token probabilistic scenario. If P_{im} set in $\mathbb{F}$ satisfies **Maximal Distance** (and correctly represents $\langle \Omega, \mathcal{F}, C \rangle$), then P_{im} is tight w.r.t $\langle \Omega, \mathcal{F}, C \rangle$.*

As I've argued above, I think that all imprecise probability functions P_{im} *should* satisfy **Maximal Distance**. By **Corollary 4**, for a given relevant token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$, all imprecise probability functions P_{im} correctly representing $\langle \Omega, \mathcal{F}, C \rangle$ should therefore be tight w.r.t $\langle \Omega, \mathcal{F}, C \rangle$.

Theorem 23. *Let $\langle \Omega, \mathcal{F}, C \rangle$ be any relevant token probabilistic scenario. There is at most one P_{im} set in $\mathbb{F}$ that is tight w.r.t $\langle \Omega, \mathcal{F}, C \rangle$.*

Call an interval-based model $\mathcal{M}_{im} = \langle \Omega, \mathcal{F}, C, \Sigma \rangle$ of a relevant token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ where $\Sigma = \{(A, P_{im}(A)) : A \in \mathcal{F}\}$ for some imprecise probability function P_{im} *tight* just in case P_{im} is tight w.r.t $\langle \Omega, \mathcal{F}, C \rangle$. I will now focus on the token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$, as defined in §3.2. Recall from **Theorem 16** that $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$ is relevant. Since I think that all imprecise probability functions P_{im} *should* satisfy **Maximal Distance**, by **Corollary 4**, all imprecise probability functions P_{im} correctly representing $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$ should therefore be tight w.r.t $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$. Consequently, all interval-based models $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$ should be tight.

Theorem 24. *There are no tight interval-based models $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$.*

Recall the model $\mathcal{M}_{im(u)} = \left\langle \mathbb{N}, \mathcal{F}^\#, C^\#, \Sigma_{im(u)} \right\rangle$ where $\Sigma_{im(u)} = \{(A, P_{im(u)}(A)) : A \in \mathcal{F}^\#\}$, as defined in §4.2. According to **Theorem 19**, $P_{im(u)}$ correctly represents $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$

under **Comparing Imprecise Probabilities** and **Imprecise Representation**. So, $\mathcal{M}_{im(u)}$ is a correct interval-based model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. But, as proven in **Theorem 24**, $\mathcal{M}_{im(u)}$ is still not a tight interval-based model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Since all interval-based models $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ should be tight, as I've argued above, there is reason to be suspicious of $\mathcal{M}_{im(u)}$.

In fact, even if I'm *wrong* in that all imprecise probability functions P_{im} correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ should be tight w.r.t $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, there is still some reason to be suspicious of $\mathcal{M}_{im(u)}$.

Theorem 25. *There are no correct interval-based models $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ that satisfy **Imprecise Reflection**, i.e. letting $\mathcal{M}_{im} = \langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}, \Sigma \rangle$ where $\Sigma = \{(A, P_{im}(A)) : A \in \mathcal{F}^{\#}\}$ for some imprecise probability function P_{im} correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, P_{im} does not satisfy **Imprecise Reflection**.*

Even for such a *minimal* constraint like **Imprecise Reflection**, there are still no correct interval-based models $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ that satisfy **Imprecise Reflection**. This means that $\mathcal{M}_{im(u)}$ does not satisfy **Imprecise Reflection**. So, even if I'm wrong in that all imprecise probability functions P_{im} correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ should be tight w.r.t $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, there is still some reason to be suspicious of $\mathcal{M}_{im(u)}$.

How do all these discussions relate to **Regularity** and **Qualitative Regularity**? Since I think that all imprecise probability functions P_{im} *should* at least satisfy **Imprecise Reflection**, for a given relevant token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$, all correct interval-based models $\mathcal{M}_{im}$ of $\langle \Omega, \mathcal{F}, C \rangle$ should therefore at least satisfy **Imprecise Reflection**. Now, consider the particular token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Because $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is a relevant token probabilistic scenario (**Theorem 16**), all correct interval-based models $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ should at least satisfy **Imprecise Reflection**. But, according to **Theorem 25**, $\mathcal{M}_{im(u)}$, which is a correct interval-based model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, does *not* satisfy **Imprecise Reflection**. So, in virtue of there being no correct interval-based models that satisfy **Imprecise Reflection** among all the existing correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, at least one constraint in $C^{\#}$ must be *rejected*. Now, to be sure, there *are* relevant token probabilistic scenarios $\langle \Omega, \mathcal{F}, C \rangle$ that admit correct interval-based models that don't satisfy **Imprecise Reflection**. After all, I didn't include **Imprecise Reflection** in the definition of an imprecise probability function in §4.1, for the sake of not prematurely ruling out Benci et al. [2018]'s and Pruss [2021]'s proposals of allowing probabilities to be imprecise. But if I were to include **Imprecise Reflection** in the definition of an imprecise probability function, then, by **Theorem 25**, there would be no correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ that satisfy **Imprecise Reflection**. Thus, Benci et al. [2018]'s and Pruss [2021]'s proposals of allowing probabilities to be imprecise would not even get off the ground for $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$.

But excluding **Imprecise Reflection** from the definition of an imprecise probability function implies that there are relevant token probabilistic scenarios $\langle \Omega, \mathcal{F}, C\rangle$ that admit correct interval-based models that don't satisfy **Imprecise Reflection**. For example, recall the token probabilistic scenario $\langle \{H,T\}, \mathcal{P}\{H,T\}, C\rangle$ from before describing tossing a coin that's biased towards tails. In this token probabilistic scenario, $\mathcal{S}_{\{H\}} = \{x \in \mathbb{F} : 0 < x < \frac{1}{2}\}$ and $\mathcal{S}_{\{T\}} = \{x \in \mathbb{F} : \frac{1}{2} < x < 1\}$ for some ordered field $\mathbb{F}$. Recall also that the function $P_{im(1)}$ with $\mathcal{P}\{H,T\}$ as its domain is such that $P_{im(1)}(\emptyset) = 0$, $P_{im(1)}(\{H,T\}) = 1$, $P_{im(1)}(\{H\}) = (0, \frac{1}{2})$, and $P_{im(1)}(\{T\}) = (\frac{1}{2}, 1)$. And the function $P_{im(2)}$ with $\mathcal{P}\{H,T\}$ as its domain is such that $P_{im(2)}(\emptyset) = 0$, $P_{im(2)}(\{H,T\}) = 1$, $P_{im(2)}(\{H\}) = (0,1)$, and $P_{im(2)}(\{T\}) = (0,1)$. As explained above, while $P_{im(1)}$ satisfies **Imprecise Reflection**, $P_{im(2)}$ does not. Consequently, the interval-based model of $\langle \{H,T\}, \mathcal{P}\{H,T\}, C\rangle$ derived from $P_{im(1)}$ satisfies **Imprecise Reflection**, but the interval-based model derived from $P_{im(2)}$ does not. So, $\langle \{H,T\}, \mathcal{P}\{H,T\}, C\rangle$ is a relevant token probabilistic scenario that admits correct interval-based models, some of which satisfy **Imprecise Reflection**, but some of which don't. Since I think that all correct interval-based models of $\langle \{H,T\}, \mathcal{P}\{H,T\}, C\rangle$ should at least satisfy **Imprecise Reflection**, there is reason to be suspicious of the interval-based model of $\langle \{H,T\}, \mathcal{P}\{H,T\}, C\rangle$ derived from $P_{im(2)}$. Namely, it doesn't satisfy **Imprecise Reflection**. But this may not be that big of an issue for $\langle \{H,T\}, \mathcal{P}\{H,T\}, C\rangle$. After all, there is still a correct interval-based model of $\langle \{H,T\}, \mathcal{P}\{H,T\}, C\rangle$ that satisfies **Imprecise Reflection**, e.g. the model derived from $P_{im(1)}$. If we want to apply probability theory to—say, calculate the expected utility of performing an action associated with $\langle \{H,T\}, \mathcal{P}\{H,T\}, C\rangle$, we can always *choose* a correct interval-based model that satisfies **Imprecise Reflection** over one that doesn't, in order to perform that application. In fact, the satisfaction of **Imprecise Reflection** by the former model gives us good reason to choose it over the latter model. Put simply, even though $\langle \{H,T\}, \mathcal{P}\{H,T\}, C\rangle$ admits correct interval-based models that do not satisfy **Imprecise Reflection**, there is no harm done, as we can always choose an alternative model that *does* satisfy **Imprecise Reflection**. And there *are* such models for $\langle \{H,T\}, \mathcal{P}\{H,T\}, C\rangle$.

The situation for $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}\rangle$ is radically *different*. Unlike $\langle \{H,T\}, \mathcal{P}\{H,T\}, C\rangle$, there are *no* correct interval-based models $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}\rangle$ that satisfy **Imprecise Reflection**, by **Theorem 25**. So, we can never choose a correct interval-based model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}\rangle$ that satisfies **Imprecise Reflection** for some application of probability theory, as such a model doesn't even exist. Since all correct interval-based models $\mathcal{M}_{im}$ of a given relevant token probabilistic scenario $\langle \Omega, \mathcal{F}, C\rangle$ should at least satisfy **Imprecise Reflection**, we should choose a correct interval-based model that satisfies **Imprecise Reflection** for some application of probability theory. But in the case of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}\rangle$, we have no choice but to choose a correct interval-based model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}\rangle$ that *violates* **Imprecise Reflection**, as none of the existing correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}\rangle$ satisfy **Imprecise Reflection**. So, there is reason to be suspicious of some of the constraints in $C^{\#}$. In particular, these constraints *should* not be included in the C's of any $\langle \Omega, \mathcal{F}, C\rangle$ where $\Omega = \mathbb{N}$ and **Fairness** is one of the constraints in C. For if these constraints were included, then there may be no correct interval-based models that satisfy **Imprecise Reflection** at all, thus forcing us again to choose a correct interval-based model that violates **Imprecise Reflection**. Therefore, these constraints should be ruled out as legitimate constraints on a fair lottery over $\mathbb{N}$.

Before determining which constraints in $C^{\#}$ to reject, here's another point of clarification. Note that there are relevant token probabilistic scenarios $\langle \Omega, \mathcal{F}, C\rangle$ that do not admit any correct interval-based models that satisfy **Imprecise Reflection**, because there are simply *no* correct interval-based models for these scenarios, which in turn is because $\langle \Omega, \mathcal{F}, C\rangle$ admits only one correct *precise* model. Consider fair *finite* lotteries for example. For a token probabilistic scenario $\langle \Omega, \mathcal{P}\Omega, C\rangle$ where Ω is finite, $\mathcal{P}\Omega$ is the powerset of Ω, and **Fairness** is one of the constraints in C, there is only one correct precise model. Consequently, there are no correct interval-based models of $\langle \Omega, \mathcal{P}\Omega, C\rangle$. The situation for $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}\rangle$ is again radically different. While it is true that both $\langle \Omega, \mathcal{P}\Omega, C\rangle$ and $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}\rangle$ do not admit any correct interval-based models that satisfy **Imprecise Reflection**, $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}\rangle$, on the one hand, admits correct interval-based models that *violate* **Imprecise Reflection**, but $\langle \Omega, \mathcal{P}\Omega, C\rangle$, on the other hand, does not even admit any correct interval-based models. In this section, I'm only interested in relevant token probabilistic scenarios, for which there are correct interval-based models, but every such model violates **Imprecise Reflection**.

Which constraints in $C^{\#}$ should we reject? Recall that $C^{\#}$ is a collection of 8 constraints: **Non-Negativity**, **Non-Triviality**, **Transitivity**, **Additivity**, **Fairness**, **Comparing Finite and Infinite Sets**, **Qualitative Regularity**, and **Partition**. Since the first 6 appear in every C of $\langle \Omega, \mathcal{F}, C\rangle$ where $\Omega = \mathbb{N}$ and **Fairness** is one of the constraints in C, I regard them as indispensable. Moreover, these 6 constraints intuitively hold in a fair lottery over $\mathbb{N}$ as well. So, we're down to two options: **Qualitative Regularity** and **Partition**. As explained in §3.2.1, the first condition of **Partition** is indispensable. After all, in a fair lottery over $\mathbb{N}$, for any two sets A and B such that the asymptotic density of A is less than that of B, $A \preceq B$. This means that we can only reject either **Qualitative Regularity** or the second condition of **Partition**. But, as I've explained in §3.2.1, the second condition of **Partition** is added for the sole purpose of ensuring that **Qualitative Regularity** is satisfied. This means that between **Qualitative Regularity** and the second condition of **Partition**, we should reject **Qualitative Regularity** and rule it out as a legitimate constraint on a fair lottery over $\mathbb{N}$. That is, we should exclude **Qualitative Regularity** from every C of $\langle \Omega, \mathcal{F}, C\rangle$ where $\Omega = \mathbb{N}$ and **Fairness** is one of the constraints in C.

Theorem 26. *Let $\langle \Omega, \mathcal{F}, C\rangle$ be any relevant token probabilistic scenario such that Ω is infinite and **Fairness** is one of the constraints in C. Suppose:*

1. *$\emptyset \neq \mathcal{S}_{\{\omega\}} = \{x \in \mathbb{F} : \forall n \in \mathbb{N}(0 < x < \frac{1}{n})\}$ for all $\omega \in \Omega$, where $\mathbb{F}$ is a non-Archimedean ordered field.*

2. *For all $n \in \mathbb{N}$, there exists an $A \in \mathcal{F}$ such that:*

   (a) $\frac{1}{n+1} \in \mathcal{S}_A$.

   (b) For all $a \in \mathcal{S}_A$, $a \geq \frac{1}{n+1}$.

*Then, there are no correct interval-based models $\mathcal{M}_{im}$ of $\langle \Omega, \mathcal{F}, C\rangle$ that satisfy **Imprecise Reflection**.*

What does **Theorem 26** show? It shows that for *any* fair infinite lottery over Ω fulfilling the two properties below, there are no correct interval-based models $\mathcal{M}_{im}$ of that lottery that satisfy **Imprecise Reflection**.

1. **Qualitative Regularity** is included as a constraint in the specification of that lottery and there are no *monadic* constraints[72] that restrict the probabilities assigned to singleton sets.

2. There exists an event $A \in \mathcal{F}$ that satisfies the conditions spelled out in **Theorem 26** for every $n \in \mathbb{N}$.

By property 1, $\emptyset \neq \mathcal{S}_{\{\omega\}} = \{x \in \mathbb{F} : \forall n \in \mathbb{N}(0 < x < \frac{1}{n})\}$ for all $\omega \in \Omega$, fulfilling the first condition of **Theorem 26**. Here's why. In a fair *infinite* lottery, every singleton set must be assigned equal probability and any probability assigned to a singleton set must be less than $\frac{1}{n}$ for all $n \in \mathbb{N}$. So, $\mathcal{S}_{\{\omega\}} \neq \emptyset$ and there does not exist an $x \in \mathcal{S}_{\{\omega\}}$ such that $x \geq \frac{1}{n}$ for some $n \in \mathbb{N}$. In other words, $\mathcal{S}_{\{\omega\}} \subseteq \{x \in \mathbb{F} : \forall n \in \mathbb{N}(0 < x < \frac{1}{n})\} \cup \{0\}$ for some non-Archimedean ordered field $\mathbb{F}$. Since **Qualitative Regularity** is included as a constraint in the specification of that lottery, $\mathcal{S}_{\{\omega\}} \subseteq \{x \in \mathbb{F} : \forall n \in \mathbb{N}(0 < x < \frac{1}{n})\}$. But because there are no monadic constraints that restrict the probabilities assigned to singleton sets, $\{x \in \mathbb{F} : \forall n \in \mathbb{N}(0 < x < \frac{1}{n})\} \subseteq \mathcal{S}_{\{\omega\}}$. Therefore, $\mathcal{S}_{\{\omega\}} = \{x \in \mathbb{F} : \forall n \in \mathbb{N}(0 < x < \frac{1}{n})\}$ and is nonempty. If, in addition, the given fair infinite lottery fulfills property 2, then that lottery satisfies both conditions of **Theorem 26**. By **Theorem 26**, there are no correct interval-based models $\mathcal{M}_{im}$ of that lottery that satisfy **Imprecise Reflection**.[73]

It is easy to verify that $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}\rangle$ fulfills the two properties above. Since $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}\rangle$ is a fair lottery over $\mathbb{N}$, **Theorem 26** applies to it. Therefore, there are no correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}\rangle$ that satisfy **Imprecise Reflection**. This suggests that the

[72]See §3.4 for the definition of a monadic constraint.

[73]Note that any fair infinite lotteries that fulfill the two properties above admit correct interval-based models, in virtue of $\emptyset \neq \mathcal{S}_{\{\omega\}} = \{x \in \mathbb{F} : \forall n \in \mathbb{N}(0 < x < \frac{1}{n})\}$ for all $\omega \in \Omega$, where $\mathbb{F}$ is a non-Archimedean ordered field. But of course, none of these correct interval-based models satisfy **Imprecise Reflection**, as proven in **Theorem 26**. So, these lotteries are not like fair finite lotteries, for which there are no correct interval-based models that satisfy **Imprecise Reflection**, but that's because fair finite lotteries don't even admit any correct interval-based models in the first place.

two properties stated above are the main contributing factors to why there are no such models. Now, let's take a step back and examine the situation. Recall that in §3.3, I've shown that $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ admits multiple correct precise models, choosing one over another is problematic w.r.t an end result R. But, as argued by Benci et al. [2018] and Pruss [2021], there is no reason to assign only precise probabilities to the events in a fair infinite lottery; imprecise probabilities can be assigned to some events too. So, the chosen correct numerical model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ need not be a precise model. Instead, it can be an imprecise model too. However, as I've shown above in §4.2, there are in fact uncountably many correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. I will argue in §5 that choosing any one of these models over another is problematic w.r.t an end result R. So, it seems that allowing probabilities to be imprecise did not solve the problem from before when we were just considering precise models. Furthermore, there is an additional problem in choosing a correct interval-based model for $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. According to **Theorem 25**, there are no correct interval-based models $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ that satisfy **Imprecise Reflection**, which is a constraint that all imprecise probabilities should intuitively at least satisfy. Given that every correct interval-based model $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ does not satisfy **Imprecise Reflection**, we have no choice but to choose a correct interval-based model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ that *violates* **Imprecise Reflection**. Are there any ways out of this predicament?

I can think of four ways, the combination of which is *far* from exhaustive. First, a monadic constraint can be added to $C^{\#}$ to form $C^{'}$ in order to restrict the precise probabilities assigned to a singleton set. That is, $\mathcal{S}^{'}_{\{i\}} \neq \{x \in \mathbb{F} : \forall n \in \mathbb{N}(0 < x < \frac{1}{n})\}$, i.e. it is *false* that for every positive infinitesimal ε, there is a precise probability function P correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{'} \rangle$ such that $P(\{i\}) = \varepsilon$. Adding such a monadic constraint to $C^{\#}$ falsifies the first property above. And since the first property is one of the main contributing factors to why there are no correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ that satisfy **Imprecise Reflection**, adding a monadic constraint to $C^{\#}$ to falsify this property may well do the trick in ensuring the existence of a correct interval-based model that satisfies **Imprecise Reflection**. Indeed, there is such a monadic constraint. Here's an example:

Upper Bound of the Probabilities to be Assigned to Singleton Sets. *For all $i \in \mathbb{N}$, $0 < P(\{i\}) \leq \varepsilon$ for some positive infinitesimal ε.*

Let $C^{'}$ contain the constraints in $C^{\#}$ and **Upper Bound of the Probabilities to be Assigned to Singleton Sets**. For the token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{'} \rangle$, $\mathcal{S}^{'}_{\{i\}} = \{x \in \mathbb{F} : 0 < x \leq \varepsilon\}$ for a non-Archimedean field $\mathbb{F}$. Note that the L.U.B and the G.L.B for $\mathcal{S}^{'}_{\{i\}}$ exist in $\mathbb{F}$, namely ε and 0 respectively. Define a function $P_{im(6)}$ with $\mathcal{F}^{\#}$ as its domain such that:

1. $P_{im(6)}(\emptyset) = 0$.

2. $P_{im(6)}(\mathbb{N}) = 1$.

3. For all sets $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) = 0$, $P_{im(6)}(A) = d(A)$.

4. For all sets $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) > 0$, $P_{im(6)}(A) = \big(d(A), d(A) + (\varepsilon \cdot N(A))\big]$.

5. For all sets $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) < 0$, $P_{im(6)}(A) = \big[d(A) + (\varepsilon \cdot N(A)), d(A)\big)$.

For case 4, since $N(A) > 0$, $d(A) < 1$. Note that since $\varepsilon \cdot N(A)$ for this case is a positive infinitesimal, $d(A) + (\varepsilon \cdot N(A)) < 1$. For case 5, note that because $N(A) < 0$, $\varepsilon \cdot N(A) < 0$. So, $d(A) + (\varepsilon \cdot N(A)) < d(A)$. Further note that $\varepsilon \cdot N(A)$ in this case is a *negative* infinitesimal, i.e. $-\frac{1}{n} < \varepsilon \cdot N(A) < 0$ for all $n \in \mathbb{N}$. Also, in case 5, since $N(A) < 0$, $d(A) > 0$. So, $d(A) + (\varepsilon \cdot N(A)) > 0$ for this case. It is easy to verify that $P_{im(6)}$ is an imprecise probability function.

According to the definition of $P_{im(6)}$, $P_{im(6)}(\{i\}) = (0, \varepsilon]$, $P_{im(6)}(\{1\} \cup \{2, 4, 6, \ldots\}) = (\frac{1}{2}, \frac{1}{2} + \varepsilon]$, and $P_{im(6)}(\{10, 15, 20, \ldots\}) = [\frac{1}{5} - \varepsilon, \frac{1}{5})$. It is easy to verify that $P_{im(6)}$ satisfies **Imprecise Reflection**. In fact, $P_{im(6)}$ is also tight w.r.t $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$. So, the interval-based model $\mathcal{M}_{im(6)}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ derived from $P_{im(6)}$ is tight and satisfies **Imprecise Reflection**. Since there exists a correct interval-based model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ that satisfies **Imprecise Reflection**, we are no longer forced to choose a correct interval-based model that *violates* **Imprecise Reflection**. Thus, the predicament above has been avoided. However, a problem with this approach is that, like **The Probability of a Singleton Set**, there is no intuitive justification that can be provided for adding **Upper Bound of the Probabilities to be Assigned to Singleton Sets** to $C^{\#}$. Why is the upper bound of the probabilities to be assigned to singleton sets ε, and not 2ε, ε^2, $\varepsilon^2 + 2\varepsilon$, or for that matter, any other positive infinitesimal? After all, nothing about a fair lottery over $\mathbb{N}$ uniquely determines ε as the upper bound and nothing about the lottery rules out every other positive infinitesimal as an upper bound. So, it's difficult to justify why **Upper Bound of the Probabilities to be Assigned to Singleton Sets** should be included in the specification of a fair lottery over $\mathbb{N}$. Unless the reader can provide an intuitive justification for doing so, I will not be convinced of utilizing this approach to ensure the existence of at least one correct interval-based model that satisfies **Imprecise Reflection**.

Here's the second way out of the predicament: ensure that there exists an $n \in \mathbb{N}$ such that there does *not* exist an $A \in \mathcal{F}^{\#}$ such that (i) $\frac{1}{n+1} \in \mathcal{S}_A$ and (ii) for all $a \in \mathcal{S}_A$, $a \geq \frac{1}{n+1}$. Doing so will falsify the second property above. And since the second property is one of the main contributing factors to why there are no correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ that satisfy **Imprecise Reflection**, ensuring that there exists an $n \in \mathbb{N}$ such that there does *not* exist an $A \in \mathcal{F}^{\#}$ fulfilling (i) and (ii) may well do the trick in ensuring the existence of a correct interval-based model that satisfies **Imprecise Reflection**. However, I'm skeptical of the success of this approach. First, recall that for every $n \in \mathbb{N}$, there exists an $A \in \mathcal{F}^{\#}$ such that $d(A) = \frac{1}{n}$ and $N(A) = 0$. For such sets A, $P(A) = \frac{1}{n}$ for all precise probability functions P correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$.[74] This means that for every $n \in \mathbb{N}$, there exists an $A \in \mathcal{F}^{\#}$ such that $\mathcal{S}_A = \{\frac{1}{n}\}$. So, for every $n \in \mathbb{N}$, there exists an $A \in \mathcal{F}^{\#}$ fulfilling (i) and (ii). In order to ensure that this is *false*, there are two approaches. First, we can tweak the constraints in $C^{\#}$ to form a new set $C^{'}$ such that $\mathcal{S}^{'}_A \neq \{\frac{1}{n}\}$ for all sets $A \in \mathcal{F}^{\#}$ such that $\mathcal{S}_A = \{\frac{1}{n}\}$ for a particular $n \in \mathbb{N}$. But I don't even have a vague idea what such a $C^{'}$ looks like. Furthermore, I'm doubtful that the new token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{'} \rangle$ is fully specified. In particular, I'm doubtful that the constraints in $C^{'}$ fully determine the order of events in $\mathcal{F}^{\#}$. Consequently, $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{'} \rangle$ is not even relevant. Here's the second approach. We can replace $\mathcal{F}^{\#}$ with another algebra $\mathcal{F}^{'}$ such that $A \notin \mathcal{F}^{'}$ for all sets $A \in \mathcal{F}^{\#}$ such that $\mathcal{S}_A = \{\frac{1}{n}\}$ for a particular $n \in \mathbb{N}$. This means that the set $\{n, 2n, 3n, \ldots\} \notin \mathcal{F}^{'}$, which in turn implies that $\{n, 2n, 3n, \ldots\}$ cannot be compared to $\{m, 2m, 3m, \ldots\}$. But this is counterintuitive. After all, in a fair lottery over $\mathbb{N}$, if $n > m$, we want to say that $\{n, 2n, 3n, \ldots\} \prec \{m, 2m, 3m, \ldots\}$. And in order to say that, both $\{n, 2n, 3n, \ldots\}$ and $\{m, 2m, 3m, \ldots\}$ must be in the algebra $\preceq$ is defined on.

Here's the third way out of the predicament. Recall **Irregular Ordering**, as defined in §3.3. Let C^{*} contain only **Irregular Ordering**. I've shown in §3.3 that the token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$ admits only one correct precise model (**Theorem 11**). Hence, there is no need to choose a correct model (precise or interval-based) over another to represent it. If a choice has to be made, we must choose the only existing correct numerical model. In a sense, the situation with $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$ is exactly like the situation with $\langle \Omega, \mathcal{P}\Omega, C \rangle$ where Ω is finite, $\mathcal{P}\Omega$ is the powerset of Ω, and **Fairness** is one of the constraints in C, i.e. $\langle \Omega, \mathcal{P}\Omega, C \rangle$ describes a fair finite lottery. Both $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$ and $\langle \Omega, \mathcal{P}\Omega, C \rangle$ admit only one correct precise model each. And just like choosing the only correct precise model of $\langle \Omega, \mathcal{P}\Omega, C \rangle$ to model it isn't problematic, choosing the only correct precise model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$ to model it isn't problematic either. So, the predicament spelled out above doesn't even arise for $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$. Therefore, in the interest of avoiding the predicament spelled out above, we should reject the constraints in $C^{\#}$ that are not satisfied by C^{*}. And the difference between $C^{\#}$ and C^{*}, as demonstrated by **Theorems 12** and **13**, is *essentially* that while **Qualitative Regularity** is a constraint in $C^{\#}$, C^{*} implies a violation of **Qualitative Regularity**, i.e. according to **Irregular Ordering**, for every finite set $A \in \mathcal{F}^{\#}$, $\emptyset \sim^{'} A$. Consequently, **Qualitative Regularity** should be rejected as a legitimate constraint on a fair lottery over $\mathbb{N}$. That is, we should exclude **Qualitative Regularity** from every C of $\langle \Omega, \mathcal{F}, C \rangle$ where $\Omega = \mathbb{N}$ and **Fairness** is one of the constraints in C. For if **Qualitative Regularity** were included, then we would run

[74]See **Lemma 9** for the proof.

the risk of getting trapped again in the predicament spelled out above.

Indeed, this third way of avoiding the predicament spelled out above is my favourite among all the three ways just listed. After all, there are plenty of independent reasons for rejecting **Regularity** and **Qualitative Regularity** as constraints on (numerical and qualitative) probabilities , e.g. see Hájek [MS], Williamson [2007], Hájek [2012], Easwaran [2014a], Pruss [2014], Parker [2019], Pruss [2021], etc.[75] So, while there seems to be no intuitive justification for adding **Upper Bound of the Probabilities to be Assigned to Singleton Sets** to $C^{\#}$, there seem to be plenty of good reasons for *removing* **Qualitative Regularity** from $C^{\#}$. And avoiding the predicament spelled out above is just another good reason for doing the latter.

However, note that rejecting **Qualitative Regularity** as a constraint is only half the job done in avoiding the predicament spelled out above. Here's why. Consider the token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$ again. By **Theorem 11**, for all precise probability functions P that correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$, $P(A) = d(A)$. There are *no* precise probability functions P that correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$ such that $P(A) \neq d(A)$. This implies that there are no precise probability functions P that correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$ such that $P(\{i\}) = \varepsilon$ for some positive infinitesimal ε, as $d(\{i\}) = 0$. So, in order to avoid the predicament spelled out above, not only must we allow the probability function P such that $P(\{i\}) = 0$ for all $i \in \mathbb{N}$ to correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$, we must also *rule out* other probability functions $P^{'}$ such that $P^{'}(\{i\}) > 0$ for all $i \in \mathbb{N}$ as correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$. This is why rejecting **Qualitative Regularity** as a constraint is only half the job done in avoiding the predicament spelled out above. For if we were to *merely* reject **Qualitative Regularity** as a constraint, we would only be allowing the probability function P such that $P(\{i\}) = 0$ for all $i \in \mathbb{N}$ to correctly represent a fair lottery over $\mathbb{N}$. But that's not enough. We need to also rule out other probability functions $P^{'}$ such that $P^{'}(\{i\}) > 0$ for all $i \in \mathbb{N}$ as correctly representing a fair lottery over $\mathbb{N}$, in order to avoid the predicament spelled out above. So, **Qualitative Regularity** cannot just be rejected; it must be *replaced*.

What constraint can replace **Qualitative Regularity** to ensure that other probability functions $P^{'}$ such that $P^{'}(\{i\}) > 0$ for all $i \in \mathbb{N}$ are ruled out as correctly representing a fair lottery over $\mathbb{N}$? First, note that **Fairness** will rule out all the probability functions $P^{'}$ such that it is *false* that $P^{'}(\{i\}) = P^{'}(\{j\})$ for all $i, j \in \mathbb{N}$ as correctly representing a fair lottery over $\mathbb{N}$. After passing through the filter of **Fairness**, all that remains of the candidate probability functions are functions $P^{'}$ such that $P^{'}(\{i\}) = P^{'}(\{j\})$ for all $i, j \in \mathbb{N}$. Now, note that for all probability functions $P^{'}$ that assign every singleton set equal positive probability, $P^{'}$ must be *non-Archimedean*, as $P^{'}(\{i\})$ must be a positive infinitesimal.

[75]Note that most of the citations above contain arguments against **Regularity** but not **Qualitative Regularity**. Though some of these arguments cannot be straightforwardly translated into arguments against **Qualitative Regularity**, there are some that can, e.g. some of the symmetry arguments found in Parker [2019]. Also, Williamson [2007] has argued against **Qualitative Regularity** *directly*.

So, to rule these probability functions out as correctly representing a fair lottery over $\mathbb{N}$, we just need to include a constraint to the effect that probability functions correctly representing a fair lottery over $\mathbb{N}$ must be *real-valued*, i.e. **Archimedean Probability**. This constraint states that for all events A such that $P(A) > 0$, there exists an $n \in \mathbb{N}$ such that $P(A) > \frac{1}{n}$. By **Archimedean Probability**, every probability function P' that assigns every singleton set equal positive probability is ruled out as correctly representing a fair lottery over $\mathbb{N}$, in accordance with what is required to avoid the predicament spelled out above. After all, these functions assign $\{i\}$ a positive infinitesimal, which is strictly smaller than $\frac{1}{n}$ for all $n \in \mathbb{N}$, in violation of **Archimedean Probability**. Importantly, note that the probability function P such that $P(\{i\}) = 0$ for all $i \in \mathbb{N}$ *isn't* ruled out by **Archimedean Probability** as correctly representing a fair lottery over $\mathbb{N}$, as $P(\{i\}) \not> 0$.

Is there a qualitative analogue of **Archimedean Probability**? At least for a token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C \rangle$ such that (i) $\mathcal{F} \supseteq \mathcal{F}^{\#}$, (ii) **Non-Negativity**, **Non-Triviality**, **Additivity**, **Transitivity**, **Comparing Finite and Infinite Sets**, and **Fairness** are constraints in C, and (iii) the constraints in C imply the principle below, there is.

Partition*. *Let $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ be a partition of $\mathbb{N}$ such that:*

$$G_1 = \{1, 1+n, 1+2n, \ldots\}.$$

$$G_2 = \{2, 2+n, 2+2n, \ldots\}.$$

$$\ldots$$

$$G_n = \{n, 2n, 3n, \ldots\}.$$

For all partitions $\mathcal{G}$ of $\mathbb{N}$ that fulfill the form above, $G_i \sim G_j$.

Here's a qualitative analogue of **Archimedean Probability** for all token probabilistic scenarios $\langle \mathbb{N}, \mathcal{F}, C \rangle$ that satisfy (i)-(iii):

Archimedean Probability (Qualitative). *For all events $A \in \mathcal{F}$ such that $\emptyset \prec A$, there exists an $n \in \mathbb{N}$ such that $\{n, 2n, 3n, \ldots\} \prec A$.*

Theorem 27. ***Archimedean Probability (Qualitative)*** *is satisfied in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^* \rangle$ where C^* contains only* ***Irregular Ordering****, i.e.* ***Irregular Ordering*** *implies* ***Archimedean Probability (Qualitative)****.*

Theorem 28. *Under* ***Precise Representation****, for all precise probability functions P that correctly represent a token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C \rangle$ that satisfies (i)-(iii) above and* ***Archimedean Probability (Qualitative)****, $P(\{i\}) = 0$.*

Theorem 28 shows that **Archimedean Probability (Qualitative)** rules out all precise probability functions P' such that $P'(\{i\}) > 0$ for all $i \in \mathbb{N}$ as correctly representing a token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C\rangle$ satisfying (i)-(iii) above and **Archimedean Probability (Qualitative)**, just as intended.

In the comments I've received on previous drafts of this thesis, Alan Hájek has suggested a fourth way of avoiding the predicament spelled out above. Notice first that the entire discussion in this section is predicated on the assumption that the chosen correct numerical model must either be a precise model or an interval-based (imprecise) model. But as Hájek points out, these two kinds of models are not the only kinds of models. There can be a model that is neither precise nor interval-based. Hájek gives the following example. Consider again the token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}\rangle$. Instead of assigning $\{i\}$ as a probability in this scenario an interval, e.g. $P_{im(u)}(\{i\}) = (0, u)$, or a precise value, e.g. $P(\{i\}) = \varepsilon$ for some positive infinitesimal ε, why not just assign the *set* $\mathcal{S}_{\{i\}} = \{x \in \mathbb{F} : \forall n \in \mathbb{N}(0 < x < \frac{1}{n})\}$, where $\mathbb{F}$ is some non-Archimedean field, to $\{i\}$ as a probability? That is, the probability assigned to $\{i\}$ is simply the set of all positive infinitesimals. More generally, define a model $\mathcal{M}_{\mathcal{S}} = \langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}, \Sigma_{\mathcal{S}}\rangle$ such that $\Sigma_{\mathcal{S}} = \{(A, \mathcal{S}_A) : A \in \mathcal{F}^{\#}\}$. Call $\mathcal{M}_{\mathcal{S}}$ a *set-based* model, which is different from precise models and interval-based models. Given this third kind of model, it doesn't seem reasonable to restrict ourselves to choose only between precise models and interval-based models. Why not also allow the chosen correct model to be a set-based model? After all, while there are multiple correct precise models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}\rangle$, as proven in §3.3, and multiple correct interval-based models of the same scenario, as proven in §4.2, there is only *one* correct set-based model of the same scenario, i.e. $\mathcal{M}_{\mathcal{S}}$. Since there is only one correct set-based model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}\rangle$, this model 'stands out' from the rest of the models and this may give us good enough reason to choose it. Therefore, there seems to be some merit in allowing the chosen correct model to be a set-based model.

Furthermore, allowing the chosen correct model to be a set-based model re-ignites the hope that Benci et al. [2018] and Pruss [2021] had when they allowed probabilities to be imprecise. That is, by allowing probabilities to be imprecise, they hoped that there may be a unique correct interval-based model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}\rangle$. Of course, as I've shown in §4.2, this is false; there are in fact multiple correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}\rangle$. However, if we allow the chosen correct model to be a set-based model, this hope is re-ignited, as the set-based model $\mathcal{M}_{\mathcal{S}}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#}\rangle$ is unique in its own kind; there seems to be no other set-based models of the same scenario other than $\mathcal{M}_{\mathcal{S}}$. But, while I'm happy to accord this proposal *prima facie* plausibility, I'm still hesitant to give this proposal a stamp of approval, or at least until all the precise technical details have been spelled out to my satisfaction. That is, the lack of technical details is the only thing that's barring me from fully embracing this proposal. I would want to know whether the usual applications of probability theory, e.g. calculation of expected utilities, calculation of the expected value, probabilistic logic, predictive modelling, risk management, reliability theory, etc., still work if a set-based model is adopted. As this proposal currently stands, it isn't obvious that the usual applications of probability theory still work. And it would be a huge disadvantage of set-based models if we could not utilise probability theory for its usual purposes. Furthermore, I will argue in §7 that set-based models face a prima facie difficulty, of which I see no straightforward solution. However, to be clear, my argument against **Qualitative Regularity** is far from a knock-down argument against the principle, but I do hope that the discussions in this section prompt more research into alternative proposals like the one suggested by Hájek here.

5 Choosing a Correct Imprecise Model

In §4.2, I showed that there are in fact uncountably many correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Then, in §4.3, I proved that none of these correct interval-based models satisfy **Imprecise Reflection**. Consequently, none of them are tight. In this section, I will put aside the issue of correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ violating **Imprecise Reflection**, and instead focus on showing that our choosing a particular correct interval-based model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ over another is problematic w.r.t an end result R. So, even if I'm *wrong* about **Imprecise Reflection** being an intuitive minimal constraint on all imprecise probability functions, there is still a problem with *choosing* a particular correct interval-based model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ over uncountably many others in order to perform an application of probability theory. But in order to show this, I will first need to state when two correct interval-based models have the same representational power in representing the given relevant token probabilistic scenario. This will be done in §5.1. Then, in the same subsection, I will prove that choosing $\mathcal{M}_{im(u)}$ over $\mathcal{M}_{im(u')}$, as defined in §4.2, to model $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ satisfies the conditions spelled out in **Problematic Choice** w.r.t an end result R. Therefore, by **Problematic Choice**, doing so is problematic w.r.t an end result R.

In §5.2, I will switch gears. The entire discussion, up until §5.2, has been centred around numerical models (precise or imprecise). But in §5.2, I will instead talk about *qualitative* models. Here, **Infinitesimal Underdetermination** and **Same Representational Power** will be generalised using the framework of qualitative probabilities. Then, in the framework of qualitative probabilities I'm working with, I will show that two different correct qualitative models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ can be defined that satisfy all the conditions of the generalised version of **Infinitesimal Underdetermination**. Again, choosing one correct qualitative model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ over the other satisfies the conditions spelled out in **Problematic Choice** w.r.t an end result R. So, by **Problematic Choice**, doing so is problematic w.r.t an end result R. Note that this subsection is relatively more speculative than other sections and I fully acknowledge that there may be issues with the arguments I put forth here.

Finally, in §5.3, I will show that while there are multiple correct qualitative models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, there is only one correct qualitative model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$, where C^{*} contains only **Irregular Ordering**. Then, I will prove in the same subsection that **Precise Representation** 'links' the only existing correct qualitative model and the only existing correct precise model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$.

5.1 Different Correct Imprecise Models Having the Same Representational Power

Here are the sufficient conditions for when two imprecise probability functions $P_{im(1)}$ and $P_{im(2)}$ have the same representational power in representing a relevant token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$, without explication first.

Same Representational Power (Imprecise). *With respect to a relevant token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$, imprecise probability functions $P_{im(1)}$ and $P_{im(2)}$, both of which are set in a shared ordered field $\mathbb{F}$, have the same representational power in representing $\langle \Omega, \mathcal{F}, C \rangle$ if all of the following hold:*

1. *Both $P_{im(1)}$ and $P_{im(2)}$ are defined on $\mathcal{F}$ and both satisfy all the constraints in C.*
2. *For all events A and B in $\mathcal{F}$, $P_{im(1)}(A) \leq P_{im(1)}(B)$ iff $P_{im(2)}(A) \leq P_{im(2)}(B)$.*
3. *For all events $A \in \mathcal{F}$, $|\overline{P}_{im(1)}(A) - \overline{P}_{im(2)}(A)| < \frac{1}{n}$ and $|\underline{P}_{im(1)}(A) - \underline{P}_{im(2)}(A)| < \frac{1}{n}$ for all $n \in \mathbb{N}$.*[76]
4. *Both $P_{im(1)}$ and $P_{im(2)}$ are not tight w.r.t $\langle \Omega, \mathcal{F}, C \rangle$.*

Explication is in order. I'll start with the easiest condition: condition 1. This condition states that $P_{im(1)}$ and $P_{im(2)}$ assign the same events probabilities, in virtue of both of them being defined on $\mathcal{F}$. However, note that it isn't necessary for $P_{im(1)}$ and $P_{im(2)}$ to assign the same events the *same* probabilities, i.e. it is possible that $P_{im(1)} \neq P_{im(2)}$. And in virtue of the second part of condition 1, both $P_{im(1)}$ and $P_{im(2)}$ satisfy all the constraints in C. So, both functions correctly represent $\langle \Omega, \mathcal{F}, C \rangle$.

Now, consider condition 2. This condition states that $P_{im(1)}$ and $P_{im(2)}$ order the same events the same way. Next, consider condition 3. This condition states that $P_{im(1)}$ and $P_{im(2)}$ are *infinitely close* to each other, in that for all events $A \in \mathcal{F}$, $|\overline{P}_{im(1)}(A) - \overline{P}_{im(2)}(A)| < \frac{1}{n}$ and $|\underline{P}_{im(1)}(A) - \underline{P}_{im(2)}(A)| < \frac{1}{n}$ for all $n \in \mathbb{N}$. The reason why I've included conditions 1 to 3 in **Same Representational Power (Imprecise)** is simple: each condition is analogous to a corresponding condition in **Same Representational Power**. Since I consider conditions 1 to 3 of **Same Representational Power (Imprecise)** to be analogues of those conditions in **Same Representational Power**, conditions 1 to 3 are included in **Same Representational Power (Imprecise)** for the very same reasons the corresponding conditions are included in **Same Representational Power**. Therefore, I will not expend any more discussion justifying why I've included conditions 1 to 3 in **Same Representational Power (Imprecise)**. The reader can refer to §1.2 for the reasons.

[76]Note that $|a|$ here denotes the absolute value of a.

As for condition 4, more explication is required. It's best to use an example to explain why I've included condition 4 in **Same Representational Power (Imprecise)**. Recall the token probabilistic scenario $\langle\{H,T\},\mathcal{P}\{H,T\},C'\rangle$ where C' contains the probability axioms, **Regularity**, the statement 'The coin is biased towards tails', and some other statement to the effect that $\mathcal{S}'_{\{H\}} = \{x \in \mathbb{F} : 0 < x < \frac{1}{4}\}$ and $\mathcal{S}'_{\{T\}} = \{x \in \mathbb{F} : \frac{3}{4} < x < 1\}$ for some ordered field $\mathbb{F}$. Recall also the function $P_{im(5)}$ with $\mathcal{P}\{H,T\}$ as its domain such that $P_{im(5)}(\emptyset) = 0$, $P_{im(5)}(\{H,T\}) = 1$, $P_{im(5)}(\{H\}) = (0, \frac{1}{4})$, and $P_{im(5)}(\{T\}) = (\frac{3}{4}, 1)$. Now, define another function $P_{im(7)}$ with $\mathcal{P}\{H,T\}$ as its domain such that $P_{im(7)}(\emptyset) = 0$, $P_{im(7)}(\{H,T\}) = 1$, $P_{im(7)}(\{H\}) = (0, \frac{1}{4} + \varepsilon)$, and $P_{im(7)}(\{T\}) = (\frac{3}{4}, 1)$ for some positive infinitesimal ε. It is easy to verify that $P_{im(7)}$ is an imprecise probability function and that both $P_{im(5)}$ and $P_{im(7)}$ satisfy conditions 1 to 3 in **Same Representational Power (Imprecise)**. But notice that while $P_{im(5)}$ is tight w.r.t $\langle\{H,T\},\mathcal{P}\{H,T\},C'\rangle$, $P_{im(7)}$ isn't. This is because the L.U.B of $\mathcal{S}'_{\{H\}}$ is $\frac{1}{4}$. So, although $\frac{1}{4} + \varepsilon$ is an upper bound for $\mathcal{S}'_{\{H\}}$, it isn't the L.U.B. This explains why $P_{im(7)}$ isn't tight w.r.t $\langle\{H,T\},\mathcal{P}\{H,T\},C'\rangle$. Now, I hope the reader shares my intuition that $P_{im(5)}$ represents $\langle\{H,T\},\mathcal{P}\{H,T\},C'\rangle$ *better* than $P_{im(7)}$, i.e. $P_{im(5)}$ has *more* representational power than $P_{im(7)}$ in representing $\langle\{H,T\},\mathcal{P}\{H,T\},C'\rangle$, even though both functions satisfy conditions 1 to 3 in **Same Representational Power (Imprecise)**. So, a further condition must be added to **Same Representational Power (Imprecise)**, in order to spell out when two imprecise probability functions have the same representational power in representing a relevant token probabilistic scenario $\langle\Omega,\mathcal{F},C\rangle$. Condition 4 is the result.

In general, I'm inclined to think that for any relevant token probabilistic scenario $\langle\Omega,\mathcal{F},C\rangle$, an imprecise probability function P_{im} set in $\mathbb{F}$ that is tight w.r.t $\langle\Omega,\mathcal{F},C\rangle$ has *more* representational power in representing $\langle\Omega,\mathcal{F},C\rangle$ than any other imprecise probability functions set in $\mathbb{F}$ that correctly represent $\langle\Omega,\mathcal{F},C\rangle$. Therefore, in order to spell out when two imprecise probability functions set in $\mathbb{F}$ have the *same* representational power in representing $\langle\Omega,\mathcal{F},C\rangle$, condition 4 has to be included. Furthermore, recall that for any relevant token probabilistic scenario $\langle\Omega,\mathcal{F},C\rangle$, there is at most one imprecise probability function set in $\mathbb{F}$ that is tight w.r.t $\langle\Omega,\mathcal{F},C\rangle$ (**Theorem 23**). So, there cannot exist for a relevant token probabilistic scenario $\langle\Omega,\mathcal{F},C\rangle$ *two* imprecise probability functions set in $\mathbb{F}$ that are tight w.r.t $\langle\Omega,\mathcal{F},C\rangle$.

Let $\langle\Omega,\mathcal{F},C\rangle$ be any relevant token probabilistic scenario. Define a correct interval-based model $\mathcal{M}_{im(1)} = \langle\Omega,\mathcal{F},C,\Sigma_1\rangle$ of $\langle\Omega,\mathcal{F},C\rangle$ such that $\Sigma_1 = \{(A, P_{im(1)}(A)) : A \in \mathcal{F}\}$ where $P_{im(1)}$ correctly represents $\langle\Omega,\mathcal{F},C\rangle$. Similarly, define a correct interval-based model $\mathcal{M}_{im(2)} = \langle\Omega,\mathcal{F},C,\Sigma_2\rangle$ of $\langle\Omega,\mathcal{F},C\rangle$ such that $\Sigma_2 = \{(A, P_{im(2)}(A)) : A \in \mathcal{F}\}$ where $P_{im(2)}$ correctly represents $\langle\Omega,\mathcal{F},C\rangle$. Say that $\mathcal{M}_{im(1)}$ and $\mathcal{M}_{im(2)}$ have the same representational power in representing $\langle\Omega,\mathcal{F},C\rangle$ just in case both $P_{im(1)}$ and $P_{im(2)}$ have the same representational power in representing $\langle\Omega,\mathcal{F},C\rangle$.

Strictly speaking, a generalisation of **Infinitesimal Underdetermination** for imprecise probabilities isn't required for the purposes of this subsection. However, for the sole sake of completeness, I will just include it here.

Infinitesimal Underdetermination (Imprecise). *With respect to a relevant token probabilistic scenario* $\langle \Omega, \mathcal{F}, C \rangle$ *where* Ω *is infinite and* ***Fairness*** *is one of the constraints, though not necessarily the only constraint, in* C*, imprecise infinitesimal underdetermination arises if and only if there exist at least two imprecise probability functions* $P_{im(1)}$ *and* $P_{im(2)}$*, both of which are set in a shared ordered field* $\mathbb{F}$*, such that all of the following hold:*

1. *Both* $P_{im(1)}$ *and* $P_{im(2)}$ *are defined on* $\mathcal{F}$ *and both satisfy all the constraints in* C*.*

2. *For all events* A *and* B *in* $\mathcal{F}$*,* $P_{im(1)}(A) \leq P_{im(1)}(B)$ *iff* $P_{im(2)}(A) \leq P_{im(2)}(B)$*.*

3. $P_{im(1)} \neq P_{im(2)}$*, i.e. there exists a* $C \in \mathcal{F}$ *such that* $\overline{P}_{im(1)}(C) \neq \overline{P}_{im(2)}(C)$ *or* $\underline{P}_{im(1)}(C) \neq \underline{P}_{im(2)}(C)$*.*

4. *For all events* $A \in \mathcal{F}$*,* $|\overline{P}_{im(1)}(A) - \overline{P}_{im(2)}(A)| < \frac{1}{n}$ *and* $|\underline{P}_{im(1)}(A) - \underline{P}_{im(2)}(A)| < \frac{1}{n}$ *for all* $n \in \mathbb{N}$*.*

5. *Both* $P_{im(1)}$ *and* $P_{im(2)}$ *are not tight w.r.t* $\langle \Omega, \mathcal{F}, C \rangle$*.*

I hope the reader understands from my explanation above why I've included conditions 1, 2, 4, and 5 in **Infinitesimal Underdetermination (Imprecise)**. As for condition 3, in order for there to be underdetermination, there must be at least two *different* imprecise probability functions. For if there were only one imprecise probability function defined on $\mathcal{F}$ that satisfied all the constraints in C, then there would be *no* underdetermination, as the constraints in C would fully determine the imprecise probability function.

Let's consider now the following two correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$:

1. $\mathcal{M}_{im(u)}$, as defined in §4.2, for a particular $u \in \mathbb{F}$ such that u is a positive real number less than 1, where $\mathbb{F}$ is a non-Archimedean ordered field.

2. $\mathcal{M}_{im(u+\varepsilon)}$, where u here is the particular u used to define $\mathcal{M}_{im(u)}$ above and ε is a positive infinitesimal in $\mathbb{F}$.

To be sure, I'll list the probability assignments in each model again. According to $\mathcal{M}_{im(u)}$,

1. $P_{im(u)}(\emptyset) = 0$.

2. $P_{im(u)}(\mathbb{N}) = 1$.

3. For all sets $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) = 0$, $P_{im(u)}(A) = d(A)$.

4. For all sets $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) > 0$, if $d(A) + u \leq 1$, then $P_{im(u)}(A) = (d(A), d(A) + u)$. If $d(A) + u > 1$, then $P_{im(u)}(A) = (d(A), 1)$.

5. For all sets $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) < 0$, if $d(A) - u \geq 0$, then $P_{im(u)}(A) = (d(A) - u, d(A))$. If $d(A) - u < 0$, then $P_{im(u)}(A) = (0, d(A))$.

And according to $\mathcal{M}_{im(u+\varepsilon)}$,

1. $P_{im(u+\varepsilon)}(\emptyset) = 0$.
2. $P_{im(u+\varepsilon)}(\mathbb{N}) = 1$.
3. For all sets $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) = 0$, $P_{im(u+\varepsilon)}(A) = d(A)$.
4. For all sets $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) > 0$, if $d(A) + u + \varepsilon \leq 1$, then $P_{im(u+\varepsilon)}(A) = (d(A), d(A)+u+\varepsilon)$. If $d(A)+u+\varepsilon > 1$, then $P_{im(u+\varepsilon)}(A) = (d(A), 1)$.
5. For all sets $A \in \mathcal{F}^{\#} - \{\emptyset, \mathbb{N}\}$ such that $N(A) < 0$, if $d(A) - u - \varepsilon \geq 0$, then $P_{im(u+\varepsilon)}(A) = (d(A)-u-\varepsilon, d(A))$. If $d(A)-u-\varepsilon < 0$, then $P_{im(u+\varepsilon)}(A) = (0, d(A))$.

It is easy to verify that both $P_{im(u)}$ and $P_{im(u+\varepsilon)}$ are set in $\mathbb{F}$. By **Theorem 19**, both $P_{im(u)}$ and $P_{im(u+\varepsilon)}$ correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, under **Comparing Imprecise Probabilities** and **Imprecise Representation**. So, both functions are defined on $\mathcal{F}^{\#}$ and both satisfy all the conditions in $C^{\#}$. This means that condition 1 of **Same Representational Power (Imprecise)** is fulfilled.

Furthermore, for all events A and B in $\mathcal{F}^{\#}$, $P_{im(u)}(A) \leq P_{im(u)}(B)$ iff $P_{im(u+\varepsilon)}(A) \leq P_{im(u+\varepsilon)}(B)$. To see why, consider any two events A and B in $\mathcal{F}^{\#}$ such that $A \preceq B$, according to the constraints in $C^{\#}$. Since $P_{im(u)}$ satisfies all the constraints in $C^{\#}$, by **Imprecise Representation**, $P_{im(u)}(A) \leq P_{im(u)}(B)$. The same goes for $P_{im(u+\varepsilon)}$, i.e. $P_{im(u+\varepsilon)}(A) \leq P_{im(u+\varepsilon)}(B)$. Note that by **Corollary 1**, there do not exist events A and B in $\mathcal{F}^{\#}$ such that $A \npreceq B$ and $A \nsucceq B$, according to the constraints in $C^{\#}$. Therefore, condition 2 of **Same Representational Power (Imprecise)** is fulfilled.

It is easy to verify that condition 3 of **Same Representational Power (Imprecise)** is fulfilled too. After all, ε is a positive infinitesimal. And by **Theorem 24**, there are no tight interval-based models $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. So, the functions $P_{im(u)}$ and $P_{im(u+\varepsilon)}$, as defined above, are not tight w.r.t $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. This means that condition 4 of **Same Representational Power (Imprecise)** is fulfilled. All conditions of **Same Representational Power (Imprecise)** are fulfilled. Therefore, by **Same Representational Power (Imprecise)**, $P_{im(u)}$ and $P_{im(u+\varepsilon)}$ have the same representational power in representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Consequently, both models $\mathcal{M}_{im(u)}$ and $\mathcal{M}_{im(u+\varepsilon)}$ have the same representational power in representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$.

Consider the following bets:

Bet A: If the number chosen in a fair lottery over $\mathbb{N}$ is 1, then you'll be awarded $\frac{2}{2u+\varepsilon}$ units of utility. Otherwise, no units of utility will be awarded or lost.

Bet B: If the result of a fair coin toss is heads, then you'll be awarded 2 units of utility. Otherwise, no units of utility will be awarded or lost.

Bet B's expected utility is $\frac{1}{2} \cdot 2 = 1$ unit. Depending on exactly which correct interval-based model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is chosen, the L.U.B of bet A's expected utility changes. If the chosen correct interval-based model is $\mathcal{M}_{im(u)}$, then the probability assigned to $\{1\}$ is $(0, u)$. So, the L.U.B of bet A's expected utility is $u \cdot \frac{2}{2u+\varepsilon} = \frac{2u}{2u+\varepsilon} = 1 - \frac{\varepsilon}{2u+\varepsilon}$ unit. In particular, bet A's *imprecise* expected utility is $(0, 1 - \frac{\varepsilon}{2u+\varepsilon})$. Since $\frac{\varepsilon}{2u+\varepsilon} > 0$, $1 - \frac{\varepsilon}{2u+\varepsilon} < 1$. This means that bet A's expected utility is *less than* 1 unit. As prescribed by expected utility reasoning, we should choose bet B over bet A. After all, bet B's expected utility is 1 unit while bet A's expected utility is less than 1 unit, i.e. bet B's expected utility is higher than bet A's. However, if the chosen correct interval-based model is $\mathcal{M}_{im(u+\varepsilon)}$ instead, then the probability assigned to $\{1\}$ is $(0, u+\varepsilon)$. So, the L.U.B of bet A's expected utility is $(u + \varepsilon) \cdot \frac{2}{2u+\varepsilon} = \frac{2u+2\varepsilon}{2u+\varepsilon} = 1 + \frac{\varepsilon}{2u+\varepsilon}$ unit. In particular, bet A's *imprecise* expected utility is $(0, 1 + \frac{\varepsilon}{2u+\varepsilon})$. Since $\frac{\varepsilon}{2u+\varepsilon} > 0$, $1 + \frac{\varepsilon}{2u+\varepsilon} > 1$. This means that bet A's expected utility is *not less than* 1 unit, but neither is it more than or equal to 1 unit. After all, bet A's expected utility is an imprecise interval. Here, expected utility reasoning goes silent; it doesn't give a verdict on which bet we should choose. Therefore, depending on exactly which correct interval-based model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is chosen, the prescribed course of action by expected utility reasoning changes.

Let R be the bet that we should take between bets A and B, as detailed above. Following the explanation above, if $\mathcal{M}_{im(u)}$ is chosen, then $R = B$. However, if $\mathcal{M}_{im(u+\varepsilon)}$ is chosen instead, then $R \neq B$. So, R is sensitive to our choice between $\mathcal{M}_{im(u)}$ and $\mathcal{M}_{im(u+\varepsilon)}$. Is our choosing $\mathcal{M}_{im(u)}$ over $\mathcal{M}_{im(u+\varepsilon)}$ to model $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ problematic w.r.t R? Let's check whether all the conditions of **Problematic Choice** are fulfilled. First, $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is fully specified (**Corollary 2**) with an infinite set as the first item in the triple, and **Fairness** is one of the constraints in $C^{\#}$. Second, $\mathcal{M}_{im(u)}$ and $\mathcal{M}_{im(u+\varepsilon)}$ have the same representational power in modelling $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Third, R is sensitive to our choice between $\mathcal{M}_{im(u)}$ and $\mathcal{M}_{im(u+\varepsilon)}$. So, all the conditions of **Problematic Choice** are fulfilled. By **Problematic Choice**, our choosing $\mathcal{M}_{im(u)}$ over $\mathcal{M}_{im(u+\varepsilon)}$ to model $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is therefore problematic w.r.t R.

Before switching gears to *qualitative* models in the next subsection, let's take stock of the situation. Recall that in §3.3, I've shown that $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ admits multiple correct precise models, choosing between which is problematic w.r.t an end result R. But, as argued by Benci et al. [2018] and Pruss [2021], there is no reason to assign only precise probabilities to the events in a fair infinite lottery; imprecise probabilities can be assigned to some events too. So, the chosen correct numerical model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ need not be a precise model. Instead, it can be an imprecise model too. However, as I've shown above in §4.2, there are in fact uncountably many correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, choosing between which is problematic w.r.t an end result R, as shown in this subsection. So, it seems that allowing probabilities to be imprecise did not solve the problem from before when we were just considering precise models. Furthermore, there is an additional problem in choosing a correct interval-based model for $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. According to **Theorem 25**, there are no correct interval-based models $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ that satisfy **Imprecise Reflection**, which is a constraint that all imprecise probabilities should intuitively at least satisfy. Given that every correct interval-based model $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ does not satisfy **Imprecise Reflection**, we have no choice but to choose a correct interval-based model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ that *violates* **Imprecise Reflection**. And don't forget, doing so is still problematic w.r.t an end result R. The way I see it, when only precise models are considered, we are stuck with *one* problem. But when we consider imprecise models, we are instead stuck with *two* problems. So, perhaps we should have just kept probabilities precise for the sake of modelling a fair lottery over $\mathbb{N}$.

5.2 Qualitative Models

Before beginning this subsection, note that the content of this subsection is highly speculative. In fact, I'll go even further to say that this subsection is meant more as an *exploration* on how to extend my project from numerical models to qualitative models. And as with all ventures into new and unexplored territories, there will inevitably be some challenges along the way. Therefore, I fully acknowledge that there may be issues with the arguments I'm about to present here. But why then did I include this subsection? Well, it's because I used to think that qualitative probabilities solve all the underdetermination issues faced by regular non-Archimedean probabilities. After all, qualitative probabilities aren't even *numerical*. Now, I'm skeptical about whether qualitative probabilities really solve all the underdetermination issues faced by regular non-Archimedean probabilities. This subsection details the reasons why I changed my mind.

Here's the guiding thought for this subsection. Consider two separate lotteries held over $\mathbb{N}$. So, the tickets in each lottery are the same, i.e. ticket 1, ticket 2, ticket 3, etc. In the first lottery, a ticket will be randomly chosen and that ticket will be the winning ticket. The winning ticket in the second lottery will be chosen the following way. Ticket i wins in the second lottery just in case ticket $2i - 1$ or $2i$ wins in the first lottery. That is, ticket 1 wins in the second lottery just in case ticket 1 or 2 wins in the first lottery, ticket 2 wins in the second lottery just in case ticket 3 or 4 wins in the first lottery, etc. Therefore, the winning ticket in the first lottery completely determines the winning ticket in the second lottery, but not vice versa. Intuitively, ticket i winning in the second lottery is *twice* as probable as ticket i winning in the first lottery.[77] Or simply, ticket i winning in the second lottery is *more* probable than ticket i winning in the first lottery. After all, since the first lottery is fair, ticket $2i-1$ winning in the first lottery is as probable as ticket $2i$ winning in the first lottery, both of which are in turn individually as probable as ticket i winning in the first lottery. Because **Qualitative Regularity** is assumed, all relevant

[77]See Stefánsson [2018] for proposals on how to make sense of statements like 'A is twice as probable as B' in terms of qualitative probabilities.

events here are more probable than $\emptyset$. So, ticket $2i-1$ or $2i$ winning in the first lottery is twice as probable as ticket i winning in the first lottery. But since ticket i winning in the second lottery is as probable as ticket $2i-1$ or $2i$ winning in the first lottery, ticket i winning in the second lottery is twice as probable as ticket i winning in the first lottery, i.e. ticket i winning in the second lottery is more probable than ticket i winning in the first lottery.[78]

Let $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ describe the first lottery and $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{'} \rangle$ describe the second lottery, where $C^{'}$ contains only the statement 'Ticket i wins just in case ticket $2i-1$ or $2i$ wins in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$'. For every $A \in \mathcal{F}^{\#}$, define $A_{even} = \{i \in \mathbb{N} : \exists j \in A(i = 2j)\}$ and $A_{odd} = \{i \in \mathbb{N} : \exists j \in A(i = 2j-1)\}$. According to $C^{'}$, $A \in \mathcal{F}^{\#}$ contains the winning ticket in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{'} \rangle$ just in case the winning ticket in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is contained within $A_{even} \cup A_{odd}$.

Theorem 29. *For all $A \in \mathcal{F}^{\#}$, $A_{even} \in \mathcal{F}^{\#}$ and $A_{odd} \in \mathcal{F}^{\#}$.*

Since $\mathcal{F}^{\#}$ is an algebra (**Theorem 5**), **Theorem 29** implies that $A_{even} \cup A_{odd} \in \mathcal{F}^{\#}$.

Theorem 30. *For all $A \in \mathcal{F}^{\#}$, $d(A_{even} \cup A_{odd}) = d(A)$ and $N(A_{even} \cup A_{odd}) = 2 \cdot N(A)$.*

Now, let $\preceq_1$ denote the order of events in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, as implied by $C^{\#}$, and $\preceq_2$ denote the order of events in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{'} \rangle$, as implied by $C^{'}$.

Theorem 31. *For any two events $A \in \mathcal{F}^{\#}$ and $B \in \mathcal{F}^{\#}$, $A \preceq_1 B$ iff $A \preceq_2 B$.*

Theorem 31 shows that both lotteries order the same events the same way.

But is $\preceq_1$ *identical* to $\preceq_2$? My intuition says no. After all, though $\preceq_1$ and $\preceq_2$ are defined on $\mathcal{F}^{\#}$ and $A \preceq_1 B$ iff $A \preceq_2 B$ (**Theorem 31**), $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ and $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{'} \rangle$ are still *different* lotteries over $\mathbb{N}$, i.e. $C^{\#} \neq C^{'}$. Moreover, as explained above, $\{i\}$ is more probable according to $\preceq_2$ than according to $\preceq_1$, i.e. ticket i winning in the second lottery is more probable than ticket i winning in the first lottery. If $\preceq_1$ were really identical to $\preceq_2$, then there would not exist an event A such that A is more probable according to $\preceq_2$ than according to $\preceq_1$. But there *is* such an event A, e.g. $A = \{i\}$. So, $\preceq_1$ is *not* identical to $\preceq_2$. Having established that $\preceq_1$ is not identical to $\preceq_2$, here's the crucial step in proving that there exist multiple qualitative probabilities correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Notice that since $\preceq_1$ denotes the order of events in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ and that $A \preceq_1 B$ iff $A \preceq_2 B$ (**Theorem 31**), $\preceq_2$ correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ as well; both qualitative probabilities $\preceq_1$ and $\preceq_2$ are defined on $\mathcal{F}^{\#}$ and both satisfy the constraints in $C^{\#}$. From these two qualitative probabilities, two different correct qualitative models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ can be defined.

[78]This scenario is due to Pruss [2021].

5.2.1 Introducing the Concept of a Cross-Comparative Qualitative Probability

But before defining what a qualitative model is, there are two preliminary issues that need to be addressed first. Here's the first. Recall that the constraints in $C^{\#}$ are in terms of a qualitative probability. So, intuitively, there seems to be only one qualitative probability that correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$—namely, *the* qualitative probability that the constraints in $C^{\#}$ are phrased in terms of. Therefore, I'm mistaken in thinking that there are multiple qualitative probabilities that correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, or so it is argued. Here's my reply. There are two kinds of qualitative probabilities: *type* qualitative probabilities and *token* qualitative probabilities, very much like how there are type probabilistic scenarios and token probabilistic scenarios. Just like a type probabilistic scenario, a type qualitative probability is a set of token qualitative probabilities that meet a common specification, e.g. satisfaction of all the constraints in $C^{\#}$. The qualitative probability that the constraints in $C^{\#}$ are phrased in terms of is a *type* qualitative probability, not a token qualitative probability. And there may be multiple *token* qualitative probabilities that correctly represent that particular type qualitative probability, in terms of which the constraints in $C^{\#}$ are phrased. More details to come soon.

Here's the second issue. Let P and P' be two precise probability functions set in the same ordered field $\mathbb{F}$ that correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. While it is easy to compare $P(A)$ and $P'(A)$, it isn't obvious what the analogue of such a comparison is in terms of token qualitative probabilities. To be sure, comparing events A and B within the same token qualitative probability is easy. But that's not what I'm talking about. Instead, I'm talking about comparing events across *different* token qualitative probabilities. That is, how do we make sense of statements like '$\{i\}$ is more probable according to $\preceq_2$ than according to $\preceq_1$'? In order to do so, a new technical framework has to be introduced.

Let $\langle \Omega_1, \mathcal{F}_1, C_1 \rangle$ and $\langle \Omega_2, \mathcal{F}_2, C_2 \rangle$ be any two fully specified token probabilistic scenarios and let $\preceq_1$ be a token qualitative probability that correctly represents $\langle \Omega_1, \mathcal{F}_1, C_1 \rangle$ and $\preceq_2$ be a token qualitative probability that correctly represents $\langle \Omega_2, \mathcal{F}_2, C_2 \rangle$.

Definition 2. *Say that an event $A \in \mathcal{F}$ is $\frac{m}{n} \in \mathbb{Q} \cap [0,1]$ times as probable as Ω just in case*

1. $\frac{m}{n} > 0$ *iff*
   (a) *There exists a partition $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ of Ω such that $G_i \sim G_j$.*
   (b) *$A \sim \bigcup_{i=1}^{m} G_i$ where $\bigcup_{i=1}^{m} G_i \neq \emptyset$.*
2. $\frac{m}{n} = 0$ *iff $A \sim \emptyset$.*[79]

[79] This definition is inspired by Stefánsson [2018].

Definition 2 is *not* meant to be a general definition of when A is $\frac{m}{n}$ times as probable as Ω. In fact, as we will see soon, it is instead a definition specially tailored for the particular token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Therefore, I will use $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ to detail the rationale behind **Definition 2**. Consider a partition $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ of $\mathbb{N}$ that fulfills the conditions spelled out in **Definition 2**, e.g. $\mathcal{G} = \{G_1, \ldots, G_n\}$ such that:

$$G_1 = \{1, 1+n, 1+2n, \ldots\}.$$

$$G_2 = \{2, 2+n, 2+2n, \ldots\}.$$

$$\ldots$$

$$G_n = \{n, 2n, 3n, \ldots\}.$$

Intuitively, $\mathbb{N}$ can be thought of as being made up of n equiprobable 'blocks', with each $G_i \in \mathcal{G}$ as a 'block'. Since all these 'blocks' are equiprobable and make up $\mathbb{N}$, one 'block' is $\frac{1}{n}$ times as probable as $\mathbb{N}$. Therefore, any combination of m 'blocks' is $\frac{m}{n}$ times as probable as $\mathbb{N}$. If A is equiprobable to such a combination, then, intuitively, A is also $\frac{m}{n}$ times as probable as $\mathbb{N}$. This is the rationale behind the first condition of **Definition 2**. I believe that the second condition of **Definition 2** is straightforward. So, I will leave it to the reader to figure out why I've included the second condition in **Definition 2**.

Define $\mathcal{F}^* = \{(A,1) : A \in \mathcal{F}_1\} \cup \{(A,2) : A \in \mathcal{F}_2\}$. Let $\preceq^*$ be an ordering of the members of $\mathcal{F}^*$. Call $\preceq^*$ a *cross-comparative qualitative probability* (relating $\langle \Omega_1, \mathcal{F}_1, C_1 \rangle$ and $\langle \Omega_2, \mathcal{F}_2, C_2 \rangle$) just in case $\preceq^*$ satisfies all of the following[80]:

Compatibility. $(A,i) \preceq^* (B,i)$ *iff A is at most as probable as B, according to the constraints in C_i.*[81]

Cross-Non-Negativity. $(\emptyset, i) \preceq^* (A, j)$.[82]

Cross-Transitivity. *If* $(A,i) \preceq^* (B,j)$ *and* $(B,j) \preceq^* (C,k)$*, then* $(A,i) \preceq^* (C,k)$.[83]

Cross-Additivity. *Suppose $A \cap B = C \cap D = \emptyset$ and the following:*

1. $(A,i) \preceq^* (C,j)$.

2. $(B,i) \preceq^* (D,j)$.

Then, $(A \cup B, i) \preceq^* (C \cup D, j)$.[84]

[80]The numbers '1' and '2' can be replaced with any convenient symbols chosen in order to differentiate the two fully specified token probabilistic scenarios.

[81]Instead of 'A is at most as probable as B', why didn't I just say that $A \preceq_i B$? It's because the qualitative probability that the constraints in C_i are phrased in terms of is a *type* qualitative probability, while $\preceq_i$ is a *token* qualitative probability.

[82]It is possible that $i = j$.

[83]It is possible that $i = j$, $i = k$, $j = k$, or $i = j = k$.

[84]It is possible that $i = j$.

Probability Ratios. *Suppose $A \in \mathcal{F}_i$ is $\frac{m}{n}$ times as probable as Ω_i and $B \in \mathcal{F}_j$ is $\frac{p}{q}$ times as probable as Ω_j. Then, $(A, i) \preceq^* (B, j)$ iff $\frac{m}{n} \leq \frac{p}{q}$.*[85]

Explication is in order. First of all, what does '$(A, i) \preceq^* (B, j)$' denote? It's best to use an example to answer this question. Consider the token probabilistic scenarios $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ and $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$, as detailed above. Following the explanation given above, $\{i\}$ is more probable according to $\preceq_2$ than according to $\preceq_1$. But how can we make sense of this statement? Well, we *can* with cross-comparative qualitative probabilities. Let $\mathcal{F}^* = \{(A, 1) : A \in \mathcal{F}^{\#}\} \cup \{(A, 2) : A \in \mathcal{F}^{\#}\}$. Intuitively, the relation '$(\{i\}, 1) \prec^* (\{i\}, 2)$' represents the statement that $\{i\}$ is less probable according to $\preceq_1$ than according to $\preceq_2$. Here's another example. Consider the events $\{i\}$ and $\{2i - 1, 2i\}$ in $\mathcal{F}^{\#}$. According to C', ticket i wins in $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ just in case ticket $2i - 1$ or $2i$ wins in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. So, intuitively, $\{2i - 1, 2i\}$, according to $\preceq_1$, is as probable as $\{i\}$, according to $\preceq_2$. Using cross-comparative qualitative probabilities, this statement can thus be translated into the relation '$(\{2i - 1, 2i\}, 1) \sim^* (\{i\}, 2)$'. Put simply, for any two events A and B in $\mathcal{F}^{\#}$, the relation '$(A, i) \preceq^* (B, j)$' represents the statement that A, according to $\preceq_i$, is at most as probable as B, according to $\preceq_j$. In other words, it encodes information about how the ordering of events by $\preceq_1$ relates to the ordering of events by $\preceq_2$.

I'll explain the constraints now. Hopefully, the reader finds **Cross-Non-Negativity**, **Cross-Transitivity**, and **Cross-Additivity** intuitive and self-explanatory. So, I will skip explaining these constraints and just provide the rationales behind **Compatibility** and **Probability Ratios**. Consider **Compatibility**. It expresses the idea that a cross-comparative qualitative probability $\preceq^*$ (relating $\langle \Omega_1, \mathcal{F}_1, C_1 \rangle$ and $\langle \Omega_2, \mathcal{F}_2, C_2 \rangle$) 'inherits' the ordering of events, as implied by the constraints in C_1 and C_2, in each respective token probabilistic scenario. This constraint is included to ensure that $\preceq^*$ indeed encodes information about how the ordering of events in $\langle \Omega_1, \mathcal{F}_1, C_1 \rangle$ relates to the ordering of events in $\langle \Omega_2, \mathcal{F}_2, C_2 \rangle$. As for **Probability Ratios**, suppose that $A \in \mathcal{F}_i$ is $\frac{m}{n}$ times as probable as Ω_i and $B \in \mathcal{F}_j$ is $\frac{p}{q}$ times as probable as Ω_j. In terms of numerical probabilities, this means that $P_i(A) = \frac{m}{n}$ and $P_j(B) = \frac{p}{q}$. Intuitively, A, according to P_i, is at most as probable as B, according to P_j, just in case $P_i(A) = \frac{m}{n} \leq \frac{p}{q} = P_j(B)$. **Probability Ratios** is just the qualitative analogue of this observation.

5.2.2 Defining a Cross-Comparative Qualitative Probability

Having detailed the rationale behind each constraint of a cross-comparative qualitative probability, I can now resume working towards the main goal of this subsection: showing that there exist two different correct qualitative models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Let $\mathcal{F}^* = \{(A, 1) : A \in \mathcal{F}^{\#}\} \cup \{(A, 2) : A \in \mathcal{F}^{\#}\}$. Define a function $\psi : \mathcal{F}^* \to \mathbb{R} \times \mathbb{R}$ such that:

[85]It is possible that $i = j$.

$$\psi((A,i)) = \begin{cases} (d(A), N(A)) & \text{if } i = 1. \\ (d(A), 2 \cdot N(A)) & \text{if } i = 2. \end{cases} \tag{2}$$

For the sake of simplifying notation, instead of writing '$\psi((A,i))$', the inner brackets will be dropped hereafter and I will simply write '$\psi(A,i)$'. Let ψ be lexicographically ordered. That is, let $\psi(A,i) = (x_1, x_2)$ and $\psi(B,j) = (y_1, y_2)$. $\psi(A,i) \leq \psi(B,j)$ iff (i) $x_1 < y_1$ or (ii) $x_1 = y_1$ and $x_2 \leq y_2$. $\psi(A,i) < \psi(B,j)$ iff $\psi(A,i) \leq \psi(B,j)$ and $\psi(A,i) \ngeq \psi(B,j)$. $\psi(A,i) = \psi(B,j)$ iff $\psi(A,i) \leq \psi(B,j)$ and $\psi(A,i) \geq \psi(B,j)$.

Theorem 32. *If $\psi(A,i) \leq \psi(B,j)$ and $\psi(B,j) \leq \psi(C,k)$, then $\psi(A,i) \leq \psi(C,k)$.*

Theorem 33. *Let $A \cap B = C \cap D = \emptyset$. Suppose:*

1. *$\psi(A,i) \leq \psi(C,j)$.*

2. *$\psi(B,i) \leq \psi(D,j)$.*

Then, $\psi(A \cup B, i) \leq \psi(C \cup D, j)$.

Theorem 34. *$\psi(\emptyset, i) = (0,0)$ and $\psi(\mathbb{N}, i) = (1,0)$.*

Theorem 35. *For all $A \in \mathcal{F}^{\#}$, $\psi(\emptyset, i) \leq \psi(A,j)$.*

With ψ, a cross-comparative qualitative probability $\preceq^*$ that encodes information about how the ordering of events in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ relates to the ordering of events in $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ can be defined.

Definition 3. *For any two (A,i) and (B,j) in $\mathcal{F}^*$, $(A,i) \preceq^* (B,j)$ iff $\psi(A,i) \leq \psi(B,j)$.*

Here's the rationale behind **Definition 3**. Note that from ψ, two precise probability functions can be defined, with one correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ and the other correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$. Let P_1 denote the former and P_2 denote the latter. For an $A \in \mathcal{F}^{\#}$, let $\psi(A,1) = (x_1, x_2)$ and $P_1(A) = x_1 + (\varepsilon \cdot x_2)$ for some $0 < \varepsilon < \frac{1}{n}$ for all $n \in \mathbb{N}$. To see why P_1 correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, recall that for all $A \in \mathcal{F}^{\#}$, $\psi(A,1) = (d(A), N(A))$. So, $P_1(A) = d(A) + \big(\varepsilon \cdot N(A)\big)$. And by **Theorem 9**, P_1 correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ (under **Precise Representation**).

Now, let $\psi(A,2) = (y_1, y_2)$ and $P_2(A) = y_1 + (\varepsilon \cdot y_2)$. To see why P_2 correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$, recall that for all $A \in \mathcal{F}^{\#}$, $\psi(A,2) = (d(A), 2 \cdot N(A))$. So, $P_2(A) = d(A) + \big(\varepsilon \cdot 2 \cdot N(A)\big) = d(A) + \big(2\varepsilon \cdot N(A)\big)$. Let A and B be any two events in $\mathcal{F}^{\#}$ such that $A \preceq_2 B$, according to C'. In order for P_2 to correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ under **Precise Representation**, it must be proven that $P_2(A) \leq P_2(B)$. Here's how. Since $A \preceq_2 B$ according to C', $(A_{even} \cup A_{odd}) \preceq_1 (B_{even} \cup B_{odd})$. After all, the winning ticket in $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ is contained in A (or B) just in case the winning ticket in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is contained in $A_{even} \cup A_{odd}$ (or $B_{even} \cup B_{odd}$). Because P_1 correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$

under **Precise Representation**, $(A_{even} \cup A_{odd}) \preceq_1 (B_{even} \cup B_{odd})$ implies that $P_1(A_{even} \cup A_{odd}) \leq P_1(B_{even} \cup B_{odd})$. This is equivalent to $P_1(A_{even}) + P_1(A_{odd}) \leq P_1(B_{even}) + P_1(B_{odd})$. Note that $P_1(A_{even}) = P_1(A_{odd}) = \frac{d(A)}{2} + \big(\varepsilon \cdot N(A)\big)$, as $\psi(A_{even}, 1) = \psi(A_{odd}, 1) = \big(\frac{d(A)}{2}, N(A)\big)$.[86] Similarly, $P_1(B_{even}) = P_1(B_{odd}) = \frac{d(B)}{2} + \big(\varepsilon \cdot N(B)\big)$. This means that $P_1(A_{even}) + P_1(A_{odd}) = d(A) + \big(2\varepsilon \cdot N(A)\big) = P_2(A)$ and $P_1(B_{even}) + P_1(B_{odd}) = d(B) + \big(2\varepsilon \cdot N(B)\big) = P_2(B)$. From $P_1(A_{even}) + P_1(A_{odd}) \leq P_1(B_{even}) + P_1(B_{odd})$, $P_2(A) \leq P_2(B)$, which was to be demonstrated. Therefore, P_2 correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ (under **Precise Representation**).

Importantly, notice that $P_1(\{i\}) = \varepsilon$ while $P_2(\{i\}) = 2\varepsilon$, which means that $P_1(\{i\}) < P_2(\{i\})$. This is in line with what I said above about ticket i being *twice* as probable in winning in $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ than in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. In fact, generally speaking, A is at most as probable as B in containing the winning ticket in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ just in case $P_1(A) \leq P_1(B)$; A is at most as probable as B in containing the winning ticket in $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ just in case $P_2(A) \leq P_2(B)$; A is at most as probable in containing the winning ticket in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ as B in $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ just in case $P_1(A) \leq P_2(B)$; and A is at most as probable in containing the winning ticket in $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ as B in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ just in case $P_2(A) \leq P_1(B)$. Combining all these facts gets us **Definition 3**.

Theorem 36. *$\preceq^*$, as defined in **Definition 3**, is a cross-comparative qualitative probability relating $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ and $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$.*

5.2.3 Two Different Correct Qualitative Models

Recall that $\preceq_1$ denotes the order of events in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, as implied by $C^{\#}$, and $\preceq_2$ denotes the order of events in $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$, as implied by C', where both $\preceq_1$ and $\preceq_2$ are token qualitative probabilities. Towards the end of §5.2, I argued based on intuition that $\preceq_1$ is *not* identical to $\preceq_2$. But perhaps intuition isn't enough. No worries, with $\preceq^*$ as defined in **Definition 3**, I can now provide more precise definitions of $\preceq_1$ and $\preceq_2$, in a way that exhibits their difference. Let A and B be events in $\mathcal{F}^{\#}$.

Definition 4. $\preceq_1 := \{(A, B) : (A, 1) \preceq^* (B, 1)\} \cup \{(A, \sim^*) : (A, 1) \sim^* (A, 2)\} \cup \{(A, \prec^*) : (A, 1) \prec^* (A, 2)\} \cup \{(A, \succ^*) : (A, 1) \succ^* (A, 2)\}$. *Say that $A \preceq_1 B$ iff $(A, B) \in \preceq_1$.*

Definition 5. $\preceq_2 := \{(A, B) : (A, 2) \preceq^* (B, 2)\} \cup \{(A, \sim^*) : (A, 2) \sim^* (A, 1)\} \cup \{(A, \prec^*) : (A, 2) \prec^* (A, 1)\} \cup \{(A, \succ^*) : (A, 2) \succ^* (A, 1)\}$. *Say that $A \preceq_2 B$ iff $(A, B) \in \preceq_2$.*[87,88]

Readers who are familiar with the literature on qualitative probabilities may find the definitions above strange. That's understandable; as I will explain more in §7, the

[86] Refer to the proof of **Theorem 30** to see why $\psi(A_{even}, 1) = \psi(A_{odd}, 1) = \big(\frac{d(A)}{2}, N(A)\big)$.

[87] There do not exist events A and B in $\mathcal{F}^{\#}$ such that $(A, 1) \npreceq^* (A, 2)$ and $(A, 1) \nsucceq^* (A, 2)$. This is because for any two events A and B in $\mathcal{F}^{\#}$, either $\psi(A, i) \leq \psi(B, j)$ or $\psi(A, i) \geq \psi(B, j)$. After all, ψ is lexicographically ordered.

[88] An example of a *type* qualitative probability $\preceq$ is the set containing $\preceq_1$ and $\preceq_2$, as defined respectively in **Definitions 4** and **5**. Say that $A \preceq B$ iff for all token qualitative probabilities $\preceq_i$ in the set, $A \preceq_i B$.

two definitions above in fact go against the usual understanding of what a qualitative probability is in the literature. But I disagree with this usual understanding, which explains the 'strange' definitions above. More details to come soon.

On the one hand, the last two sets $\{(A, \prec^*) : (A, i) \prec^* (A, j)\}$ and $\{(A, \succ^*) : (A, i) \succ^* (A, j)\}$ where $i \neq j$ in **Definitions 4** and **5** tell us where $\preceq_1$ differs from $\preceq_2$. Put simply, these two sets tell us exactly which events are less or more probable according to $\preceq_1$ than according to $\preceq_2$. If $(A, \prec^*) \in \preceq_1$, then A, according to $\preceq_1$, is less probable than A, according to $\preceq_2$. Importantly, note that $(A, \prec^*) \in \preceq_1$ implies that $(A, \succ^*) \in \preceq_2$, as $(A, 1) \prec^* (A, 2)$ is equivalent to $(A, 2) \succ^* (A, 1)$. So, it is strictly impossible for $(A, \prec^*) \in \preceq_1$ and $(A, \prec^*) \in \preceq_2$ as it is a contradiction to say that $(A, 1) \prec^* (A, 2)$ and $(A, 1) \succ^* (A, 2)$. The same goes for $(A, \succ^*) \in \preceq_1$ and $(A, \succ^*) \in \preceq_2$. Now, is $\preceq_1$ distinct from $\preceq_2$? Well, yes, because $\{(A, \prec^*) : (A, i) \prec^* (A, j)\} \cup \{(A, \succ^*) : (A, i) \succ^* (A, j)\} \neq \emptyset$ for both $\preceq_1$ and $\preceq_2$. To see why, consider the set $\{i\}$. It is easy to verify that $\psi(\{i\}, 1) = (0, 1)$ and $\psi(\{i\}, 2) = (0, 2)$. So, $\psi(\{i\}, 1) < \psi(\{i\}, 2)$. By **Definition 3**, $(\{i\}, 1) \prec^* (\{i\}, 2)$. This means that $(\{i\}, \prec^*) \in \preceq_1$ but $(\{i\}, \prec^*) \notin \preceq_2$; instead, $(\{i\}, \succ^*) \in \preceq_2$. Therefore, $\preceq_1$ is distinct from $\preceq_2$.

On the other hand, the first two sets $\{(A, B) : (A, i) \preceq^* (B, i)\}$ and $\{(A, \sim^*) : (A, i) \sim^* (A, j)\}$ where $i \neq j$ in **Definitions 4** and **5** tell us the similarities between $\preceq_1$ and $\preceq_2$. In fact, $\{(A, B) : (A, 1) \preceq^* (B, 1)\} = \{(A, B) : (A, 2) \preceq^* (B, 2)\}$. After all, for any two events A and B in $\mathcal{F}^\#$, $\psi(A, 1) = (d(A), N(A)) \leq (d(B), N(B)) = \psi(B, 1)$ iff $\psi(A, 2) = \big(d(A), 2 \cdot N(A)\big) \leq \big(d(B), 2 \cdot N(B)\big) = \psi(B, 2)$. Apply **Definition 3** to get $(A, 1) \preceq^* (B, 1)$ iff $(A, 2) \preceq^* (B, 2)$. This explains why $\{(A, B) : (A, 1) \preceq^* (B, 1)\} = \{(A, B) : (A, 2) \preceq^* (B, 2)\}$. Therefore, **Definitions 4** and **5** are in line with **Theorem 31**, which states that for any two events A and B in $\mathcal{F}^\#$, $A \preceq_1 B$ iff $A \preceq_2 B$. Put simply, $\{(A, B) : (A, 1) \preceq^* (B, 1)\} = \{(A, B) : (A, 2) \preceq^* (B, 2)\}$ implies that $\preceq_1$ and $\preceq_2$ order the *same* events the *same* way. What does $\{(A, \sim^*) : (A, i) \sim^* (A, j)\}$ tell us about $\preceq_1$ and $\preceq_2$? It tells us exactly which events are as probable according to $\preceq_1$ as according to $\preceq_2$. If $(A, \sim^*) \in \preceq_1$, then A, according to $\preceq_1$, is as probable as A, according to $\preceq_2$. Importantly, note that $(A, \sim^*) \in \preceq_1$ implies that $(A, \sim^*) \in \preceq_2$, as $(A, 1) \sim^* (A, 2)$ is equivalent to $(A, 2) \sim^* (A, 1)$. Also, note that $\{(A, \sim^*) : (A, i) \sim^* (A, j)\} \neq \emptyset$. To see why, consider the set $\mathbb{N}$. It is easy to verify that $\psi(\mathbb{N}, i) = (1, 0)$. So, $\psi(\mathbb{N}, 1) = \psi(\mathbb{N}, 2)$. Apply **Definition 3** to get $(\mathbb{N}, 1) \sim^* (\mathbb{N}, 2)$. Therefore, $(\mathbb{N}, \sim^*) \in \preceq_1$ and $(\mathbb{N}, \sim^*) \in \preceq_2$. So, $\{(A, \sim^*) : (A, i) \sim^* (A, j)\} \neq \emptyset$.

With **Definitions 4** and **5**, it's easy to see that more than one token qualitative probability correctly represents $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$, i.e. both $\preceq_1$ and $\preceq_2$ correctly represent the same token probabilistic scenario. In virtue of $\{(A, B) : (A, 1) \preceq^* (B, 1)\} = \{(A, B) : (A, 2) \preceq^* (B, 2)\}$, both $\preceq_1$ and $\preceq_2$ are defined on $\mathcal{F}^\#$. And further in virtue of $\preceq_1$ denoting the order of events in $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$, as implied by $C^\#$, both $\preceq_1$ and $\preceq_2$ satisfy all the constraints in $C^\#$. After all, $\{(A, B) : (A, 1) \preceq^* (B, 1)\} = \{(A, B) : (A, 2) \preceq^* (B, 2)\}$

implies that $\preceq_1$ and $\preceq_2$ order the *same* events the *same* way. Consequently, both $\preceq_1$ and $\preceq_2$ correctly represent $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$. From these two token qualitative probabilities, two different correct qualitative models of $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$ can be defined. Let $\mathcal{M}_{\preceq_1}$ denote the first and $\mathcal{M}_{\preceq_2}$ denote the second. Define $\mathcal{M}_{\preceq_1} = \langle \mathbb{N}, \mathcal{F}^\#, C^\#, \preceq_1 \rangle$ and $\mathcal{M}_{\preceq_2} = \langle \mathbb{N}, \mathcal{F}^\#, C^\#, \preceq_2 \rangle$, where $\preceq_1$ and $\preceq_2$ are respectively defined in **Definitions 4** and **5**. As explained above, $\preceq_1$ is distinct from $\preceq_2$. This means that $\mathcal{M}_{\preceq_1} \neq \mathcal{M}_{\preceq_2}$.

5.2.4 Qualitative Versions of Same Representational Power and Infinitesimal Underdetermination

In this subsection, I will generalise **Same Representational Power** and **Infinitesimal Underdetermination** in a way that's applicable to qualitative probabilities. But before doing so, here's a useful theorem:

Theorem 37. *Let $\mathbb{F}$ be an ordered field. For any two $x, y \in \mathbb{F}$, $|x - y| < \frac{1}{n}$ for all $n \in \mathbb{N}$ iff there do not exist $\frac{m}{n}, \frac{p}{q} \in \mathbb{Q}$ where $\frac{m}{n} < \frac{p}{q}$ such that either (i) $x \leq \frac{m}{n} < \frac{p}{q} \leq y$ or (ii) $y \leq \frac{m}{n} < \frac{p}{q} \leq x$.*[89]

Why is **Theorem 37** useful? It's best to use an example to explain. Consider the token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$. Recall that in **Same Representational Power** and **Infinitesimal Underdetermination**, there is a condition that states that for all $A \in \mathcal{F}^\#$, $|P_1(A) - P_2(A)| < \frac{1}{n}$ for all $n \in \mathbb{N}$, where P_1 and P_2 correctly represent $\langle \mathbb{N}, \mathcal{F}^\#, C^\# \rangle$. So, in order to formulate qualitative versions of **Same Representational Power** and **Infinitesimal Underdetermination**, a qualitative analogue of the condition just stated has to be given. But how can a qualitative analogue of such a condition be given? After all, qualitative probabilities aren't even *numerical*. Well, **Theorem 37** is here to help.

According to **Theorem 37**, the condition just stated is equivalent to the following condition: for all $A \in \mathcal{F}^\#$ such that $P_1(A) < P_2(A)$ or $P_1(A) > P_2(A)$, there do not exist events B and C in $\mathcal{F}^\#$ such that all of the following hold:

1. $P_1(B) = P_2(B) = \frac{m}{n}$.
2. $P_1(C) = P_2(C) = \frac{p}{q}$.
3. $\frac{m}{n} < \frac{p}{q}$.
4. $P_1(A) \leq P_i(B) < P_j(C) \leq P_2(A)$ or $P_2(A) \leq P_i(B) < P_j(C) \leq P_1(A)$.

Now, the condition stated this way *can* be given a qualitative analogue. Let $\mathcal{G} = \{G_1, \ldots, G_n\}$ and $\mathcal{G}' = \{G'_1, \ldots, G'_q\}$ be partitions of $\mathbb{N}$ such that:

$$G_1 = \{1, 1+n, 1+2n, \ldots\}.$$

[89]Note that $|a|$ here denotes the absolute value of a.

$$G_2 = \{2, 2+n, 2+2n, \dots\}.$$

$$\dots$$

$$G_n = \{n, 2n, 3n, \dots\}.$$

and

$$G_1^{'} = \{1, 1+q, 1+2q, \dots\}.$$

$$G_2^{'} = \{2, 2+q, 2+2q, \dots\}.$$

$$\dots$$

$$G_q^{'} = \{q, 2q, 3q, \dots\}.$$

It is possible that $n = q$. In that case, $\mathcal{G} = \mathcal{G}^{'}$. According to **Partition**, $\left(\bigcup_{i=1}^{m} G_i\right)$ is at most as probable as $\left(\bigcup_{i=1}^{p} G_i^{'}\right)$ iff $\frac{m}{n} \leq \frac{p}{q}$. And it is easy to verify that $\psi\Big(\left(\bigcup_{i=1}^{m} G_i\right), 1\Big) = \psi\Big(\left(\bigcup_{i=1}^{m} G_i\right), 2\Big) = (\frac{m}{n}, 0)$ and $\psi\Big(\left(\bigcup_{i=1}^{p} G_i^{'}\right), 1\Big) = \psi\Big(\left(\bigcup_{i=1}^{p} G_i^{'}\right), 2\Big) = (\frac{p}{q}, 0)$. By **Definition 3**, $\Big(\left(\bigcup_{i=1}^{m} G_i\right), 1\Big) \sim^* \Big(\left(\bigcup_{i=1}^{m} G_i\right), 2\Big)$ and $\Big(\left(\bigcup_{i=1}^{p} G_i^{'}\right), 1\Big) \sim^* \Big(\left(\bigcup_{i=1}^{p} G_i^{'}\right), 2\Big)$. Letting $B = \left(\bigcup_{i=1}^{m} G_i\right)$ and $C = \left(\bigcup_{i=1}^{p} G_i^{'}\right)$, the last two relations can be simplified to $(B, 1) \sim^* (B, 2)$ and $(C, 1) \sim^* (C, 2)$, which are qualitative analogues of the first two conditions stated above.

What about the last two conditions? Since $\psi(B, i) = (\frac{m}{n}, 0)$ and $\psi(C, j) = (\frac{p}{q}, 0)$, stating that $\frac{m}{n} < \frac{p}{q}$ is equivalent to stating that $(B, i) \prec^* (C, j)$, by **Definition 3**. As for the last condition, it can be qualitatively translated into $(A, 1) \preceq^* (B, i) \prec^* (C, j) \preceq^* (A, 2)$ or $(A, 2) \preceq^* (B, i) \prec^* (C, j) \preceq^* (A, 1)$. Combining everything and simplifying *considerably*, here's the qualitative analogue of the entire condition for the particular token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. For all $A \in \mathcal{F}^{\#}$ such that $(A, 1) \prec^* (A, 2)$ or $(A, 1) \succ^* (A, 2)$, there do not exist partitions $\mathcal{G}$ and $\mathcal{G}^{'}$ of $\mathbb{N}$ such that $(A, 1) \preceq^* (B, i) \prec^* (C, j) \preceq^* (A, 2)$ or $(A, 2) \preceq^* (B, i) \prec^* (C, j) \preceq^* (A, 1)$, where $B = \left(\bigcup_{i=1}^{m} G_i\right)$ and $C = \left(\bigcup_{i=1}^{p} G_i^{'}\right)$. In virtue of $B = \left(\bigcup_{i=1}^{m} G_i\right)$ and $C = \left(\bigcup_{i=1}^{p} G_i^{'}\right)$, by **Definition 2**, B is $\frac{m}{n}$ times as probable as $\mathbb{N}$ and C is $\frac{p}{q}$ times as probable as $\mathbb{N}$. This encodes the information contained within the first two conditions stated above. And in virtue of stating that $(B, i) \prec^* (C, j)$, $\frac{m}{n} < \frac{p}{q}$, which is the third condition. Finally, the information contained within the last condition is encoded by stating that $(A, 1) \preceq^* (B, i) \prec^* (C, j) \preceq^* (A, 2)$ or $(A, 2) \preceq^* (B, i) \prec^* (C, j) \preceq^* (A, 1)$.

Intuitively, this qualitative analogue states that if $A \in \mathcal{F}^{\#}$, according to $\preceq_1$, is less probable than A, according to $\preceq_2$, then A, according to $\preceq_1$, is only *infinitesimally* less probable than A, according to $\preceq_2$. After all, in terms of numerical probabilities, B being $\frac{m}{n}$ times as probable as $\mathbb{N}$ implies that B is assigned a numerical probability of $\frac{m}{n}$. The same goes for C. C being $\frac{p}{q}$ times as probable as $\mathbb{N}$ implies that C is assigned a numerical probability of $\frac{p}{q}$. Since there do not exist B's and C's such that $(A, 1) \preceq^*$

$(B, i) \prec^* (C, j) \preceq^* (A, 2)$, A, according to $\preceq_1$, is only infinitesimally less probable than A, according to $\preceq_2$.

With everything in place, I can now state the qualitative generalization of **Same Representational Power**. However, for the sake of simplicity, the qualitative version of **Same Representational Power** I'm about to present only applies to the particular token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$.

Same Representational Power (Qualitative). *With respect to the token probabilistic scenario* $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, *token qualitative probabilities* $\preceq_1$ *and* $\preceq_2$ *have the same representational power in representing* $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ *if all of the following hold:*

1. $\preceq_1$ *and* $\preceq_2$ *are both defined on* $\mathcal{F}^{\#}$ *and both satisfy all the constraints in* $C^{\#}$.
2. *For all events* A *and* B *in* $\mathcal{F}^{\#}$, $A \preceq_1 B$ *iff* $A \preceq_2 B$.
3. *There exists a cross-comparative qualitative probability* $\preceq^*$ *such that for all events* A *and* B *in* $\mathcal{F}^{\#}$, $(A, i) \preceq^* (B, i)$ *iff* $A \preceq_i B$, *where* $i = 1$ *or* $i = 2$.
4. *For all* $A \in \mathcal{F}^{\#}$ *such that* $(A, 1) \prec^* (A, 2)$ *or* $(A, 1) \succ^* (A, 2)$, *there do not exist partitions* $\mathcal{G}$ *and* $\mathcal{G}'$, *as defined above, of* $\mathbb{N}$ *such that* $(A, 1) \preceq^* (B, i) \prec^* (C, j) \preceq^* (A, 2)$ *or* $(A, 2) \preceq^* (B, i) \prec^* (C, j) \preceq^* (A, 1)$, *where* $B = \left(\bigcup_{i=1}^{m} G_i\right)$ *and* $C = \left(\bigcup_{i=1}^{p} G'_i\right)$.

Explication is in order. The reason why I've included conditions 1, 2, and 4 in **Same Representational Power (Qualitative)** is simple: each condition is analogous to a corresponding condition in **Same Representational Power**. Since I consider conditions 1, 2, and 4 of **Same Representational Power (Qualitative)** to be analogues of those conditions in **Same Representational Power**, conditions 1, 2, and 4 are included in **Same Representational Power (Qualitative)** for the very same reasons the corresponding conditions are included in **Same Representational Power**. Therefore, I will not expend any more discussion justifying why I've included conditions 1, 2, and 4 in **Same Representational Power (Qualitative)**. The reader can refer to §1.2 for the reasons.

As for condition 3, more explication is needed. Intuitively, condition 3 expresses the idea that $\preceq^*$ 'inherits' the ordering of events from both $\preceq_1$ and $\preceq_2$. Put in another way, condition 3 ensures that $\preceq^*$ indeed encodes information about how the ordering of events by $\preceq_1$ relates to the ordering of events by $\preceq_2$. To be sure, the only reason why I've included condition 3 in **Same Representational Power (Qualitative)** is so that I can state condition 4.

Strictly speaking, a qualitative generalisation of **Infinitesimal Underdetermination** isn't required for the purposes of this subsection. However, for the sole sake of completeness, I will just include it here.

Infinitesimal Underdetermination (Qualitative). *With respect to the token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, qualitative infinitesimal underdetermination arises if and only if there exist at least two token qualitative probabilities $\preceq_1$ and $\preceq_2$ such that all of the following hold:*

1. *$\preceq_1$ and $\preceq_2$ are both defined on $\mathcal{F}^{\#}$ and both satisfy all the constraints in $C^{\#}$.*

2. *For all events A and B in $\mathcal{F}^{\#}$, $A \preceq_1 B$ iff $A \preceq_2 B$.*

3. *There exists a cross-comparative qualitative probability $\preceq^*$ such that for all events A and B in $\mathcal{F}^{\#}$, $(A, i) \preceq^* (B, i)$ iff $A \preceq_i B$, where $i = 1$ or $i = 2$.*

4. *$\preceq_1 \neq \preceq_2$, i.e. there exists an event $C \in \mathcal{F}^{\#}$ such that $(C, 1) \prec^* (C, 2)$ or $(C, 1) \succ^* (C, 2)$.*

5. *For all $A \in \mathcal{F}^{\#}$ such that $(A, 1) \prec^* (A, 2)$ or $(A, 1) \succ^* (A, 2)$, there do not exist partitions $\mathcal{G}$ and $\mathcal{G}'$, as defined above, of $\mathbb{N}$ such that $(A, 1) \preceq^* (B, i) \prec^* (C, j) \preceq^* (A, 2)$ or $(A, 2) \preceq^* (B, i) \prec^* (C, j) \preceq^* (A, 1)$, where $B = \left(\bigcup_{i=1}^{m} G_i\right)$ and $C = \left(\bigcup_{i=1}^{p} G_i'\right)$.*

I hope the reader understands from my explanation above why I've included conditions 1, 2, 3, and 5 in **Infinitesimal Underdetermination (Qualitative)**. As for condition 4, in order for there to be underdetermination, there must be at least two *different* token qualitative probabilities. For if there were only one token qualitative probability defined on $\mathcal{F}^{\#}$ that satisfied all the constraints in $C^{\#}$, then there would be *no* underdetermination, as the constraints in $C^{\#}$ would fully determine the token qualitative probability.

Consider the two token qualitative probabilities $\preceq_1$ and $\preceq_2$ I've defined respectively in **Definitions 4** and **5**. It is easy to verify that $\preceq_1$ and $\preceq_2$ satisfy all the conditions of **Same Representational Power (Qualitative)**. As shown towards the end of §5.2.3, both $\preceq_1$ and $\preceq_2$ correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, i.e. $\preceq_1$ and $\preceq_2$ are both defined on $\mathcal{F}^{\#}$ and both satisfy all the constraints in $C^{\#}$. So, the first condition is satisfied. By **Theorem 31**, the second condition is satisfied. In virtue of **Definitions 3**, **4**, and **5**, the third condition is satisfied.

According to **Definition 3**, if $(A, 1) \prec^* (A, 2)$ or $(A, 1) \succ^* (A, 2)$, then either (i) $\psi(A, 1) = (d(A), N(A)) < (d(A), 2 \cdot N(A)) = \psi(A, 2)$ or (ii) $\psi(A, 1) = (d(A), N(A)) > (d(A), 2 \cdot N(A)) = \psi(A, 2)$. For reductio, assume that there exist partitions $\mathcal{G}$ and $\mathcal{G}'$, as defined above, of $\mathbb{N}$ such that $(A, 1) \preceq^* (B, i) \prec^* (C, j) \preceq^* (A, 2)$ or $(A, 2) \preceq^* (B, i) \prec^* (C, j) \preceq^* (A, 1)$, where $B = \left(\bigcup_{i=1}^{m} G_i\right)$ and $C = \left(\bigcup_{i=1}^{p} G_i'\right)$. As proven above, $\psi(B, i) = (\frac{m}{n}, 0)$ and $\psi(C, j) = (\frac{p}{q}, 0)$. Apply **Definition 3** to the former relation to get $\psi(A, 1) \leq \psi(B, i) < \psi(C, j) \leq \psi(A, 2)$, which is equivalent to $(d(A), N(A)) \leq (\frac{m}{n}, 0) < (\frac{p}{q}, 0) \leq (d(A), 2 \cdot N(A))$. Similarly, **Definition 3** can be applied to the latter relation to get $\psi(A, 2) \leq \psi(B, i) < \psi(C, j) \leq \psi(A, 1)$, which is equivalent to $(d(A), 2 \cdot N(A)) \leq (\frac{m}{n}, 0) <$

$(\frac{p}{q}, 0) \leq (d(A), N(A))$. If (i) holds, then, by the lexicographical ordering of ψ, $N(A) < 2 \cdot N(A)$, which in turn implies that $N(A) > 0$. Given that $N(A) > 0$, $(d(A), N(A)) \leq (\frac{m}{n}, 0) < (\frac{p}{q}, 0) \leq (d(A), 2 \cdot N(A))$ implies that $d(A) < \frac{m}{n} < \frac{p}{q} \leq d(A)$. Similarly, given that $N(A) > 0$, $(d(A), 2 \cdot N(A)) \leq (\frac{m}{n}, 0) < (\frac{p}{q}, 0) \leq (d(A), N(A))$ implies that $d(A) < \frac{m}{n} < \frac{p}{q} \leq d(A)$. Either way, $d(A) < d(A)$. Contradiction. So, (i) cannot hold. Assume now that (ii) holds. Then, by the lexicographical order of ψ, $N(A) > 2 \cdot N(A)$, which in turn implies that $N(A) < 0$. Given than $N(A) < 0$, $(d(A), N(A)) \leq (\frac{m}{n}, 0) < (\frac{p}{q}, 0) \leq (d(A), 2 \cdot N(A))$ implies that $d(A) \leq \frac{m}{n} < \frac{p}{q} < d(A)$. Similarly, given that $N(A) < 0$, $(d(A), 2 \cdot N(A)) \leq (\frac{m}{n}, 0) < (\frac{p}{q}, 0) \leq (d(A), N(A))$ implies that $d(A) \leq \frac{m}{n} < \frac{p}{q} < d(A)$. Either way, $d(A) < d(A)$. Contradiction. So, (ii) cannot hold. Both options (i) and (ii) led to a contradiction. Therefore, there do not exist partitions $\mathcal{G}$ and $\mathcal{G}'$, as defined above, of $\mathbb{N}$ such that $(A, 1) \preceq^* (B, i) \prec^* (C, j) \preceq^* (A, 2)$ or $(A, 2) \preceq^* (B, i) \prec^* (C, j) \preceq^* (A, 1)$, where $B = \left(\bigcup_{i=1}^{m} G_i\right)$ and $C = \left(\bigcup_{i=1}^{p} G_i'\right)$. This means that the fourth condition of **Same Representational Power (Qualitative)** is satisfied. All conditions of **Same Representational Power (Qualitative)** are satisfied. Therefore, $\preceq_1$ and $\preceq_2$ have the same representational power in representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Consequently, the two correct qualitative models $\mathcal{M}_{\preceq_1}$ and $\mathcal{M}_{\preceq_2}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, as defined in §5.2.3, have the same representational power in modelling $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$.

Let R be the bet that we should take between bets A and B, such that if $\mathcal{M}_{\preceq_1}$ is chosen to represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, then $R = A$; however, if $\mathcal{M}_{\preceq_2}$ is chosen instead, then $R \neq A$. Of course, it isn't clear how decision-theoretic procedures, e.g. expected utility reasoning, will proceed in the framework of qualitative probabilities. But because of space limitations, no further details can be devoted to such a discussion. Consequently, concrete examples of bets A and B cannot be provided; I only maintain the possibility of such examples. That said, Fine [1973], Easwaran [2014b], and Pedersen [2014] have developed decision theories in which qualitative probabilities play a central role. Perhaps in those alternative decision theories, an example can be constructed. The reader may reasonably wonder whether there are truly examples of bets A and B. After all, I've not provided any concrete examples of bets A and B. To this, I have no answer, but that's only because there isn't enough space here to devote an entire discussion to formulating decision theories in the framework of qualitative probabilities, and not because there are no such examples. Indeed, as we've seen in §3.3 and in §5.1, there are examples of bets A and B, between which the bet we should take is sensitive respectively to the chosen correct precise model or the chosen correct imprecise model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. I conjecture that choosing a correct qualitative model to represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ will be no different, i.e. there exist bets A and B, between which the bet we should take is sensitive to the chosen correct qualitative model.

Is our choosing $\mathcal{M}_{\preceq_1}$ over $\mathcal{M}_{\preceq_2}$ to represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ problematic w.r.t R? To answer this question, we need to first provide a qualitative version of **Problematic Choice**.

Problematic Choice (Qualitative). *For a token probabilistic scenario* $\langle \Omega, \mathcal{F}, C \rangle$ *where* Ω *is infinite and* ***Fairness*** *is one of the constraints, though not necessarily the only constraint, in* C*, choosing* $\mathcal{M}_1$ *over* $\mathcal{M}_2$*, both of which are correct qualitative models of* $\langle \Omega, \mathcal{F}, C \rangle$*, is problematic w.r.t an end result* R *if all of the following hold:*

1. $\langle \Omega, \mathcal{F}, C \rangle$ *is fully specified.*

2. $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *have the same representational power in modelling* $\langle \Omega, \mathcal{F}, C \rangle$*.*

3. R *changes if* $\mathcal{M}_2$ *were chosen instead of* $\mathcal{M}_1$*.*

Let's check whether all the conditions of **Problematic Choice (Qualitative)** are fulfilled. First, $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is fully specified (**Corollary 2**) with an infinite set as the first item in the triple, and **Fairness** is one of the constraints in $C^{\#}$. Second, $\mathcal{M}_{\preceq_1}$ and $\mathcal{M}_{\preceq_2}$ have the same representational power in modelling $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Third, R is sensitive to our choice between $\mathcal{M}_{\preceq_1}$ and $\mathcal{M}_{\preceq_2}$. So, all the conditions of **Problematic Choice (Qualitative)** are fulfilled. By **Problematic Choice (Qualitative)**, our choosing $\mathcal{M}_{\preceq_1}$ over $\mathcal{M}_{\preceq_2}$ to model $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is therefore problematic w.r.t R.

Let's take stock of the situation. Recall that in §3.3, I've shown that $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ admits multiple correct precise models, choosing between which is problematic w.r.t an end result R. But, as argued by Benci et al. [2018] and Pruss [2021], there is no reason to assign only precise probabilities to the events in a fair infinite lottery; imprecise probabilities can be assigned to some events too. So, the chosen correct numerical model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ need not be a precise model. Instead, it can be an imprecise model too. However, as I've shown above in §4.2, there are in fact uncountably many correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, choosing between which is problematic w.r.t an end result R, as shown in §5.1. So, it seems that allowing probabilities to be imprecise did not solve the problem from before when we were just considering precise models. Here in this subsection, we allowed the chosen correct model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ to be a qualitative model. But, as proven above, there are in fact multiple correct qualitative models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, choosing between which is still problematic w.r.t an end result R. In light of the persistence of this problem, I suggest (and even recommend) that we seriously entertain the idea of rejecting at least one constraint in $C^{\#}$. And as I've argued in §1.5, §3.3, and §4.3, the most suspicious constraint in $C^{\#}$ is **Qualitative Regularity**. After all, it is this very constraint that gives rise to a multiplicity of correct precise models, a multiplicity of correct imprecise models, and a multiplicity of correct qualitative models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ in the first place. Furthermore, there are already plenty of independent reasons for rejecting **Regularity** and **Qualitative Regularity** as constraints on probabilities in the literature, e.g. Hájek [MS], Williamson [2007], Hájek [2012], Easwaran [2014a], Pruss [2014], Parker [2019], Pruss [2021], etc. Avoiding the problem spelled out above is just another reason for doing so.

To drive the point home, in the next subsection, I will prove that $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$, where C^{*} contains only **Irregular Ordering**, admits only one correct qualitative model. Hence, there is no need to choose a correct qualitative model over another to represent it. If a choice has to be made, we must choose the only existing correct qualitative model. So, choosing the only correct qualitative model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$ to model it isn't problematic. The problem spelled out above for $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ doesn't even arise for $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$. Therefore, in the interest of avoiding the problem spelled out above, we should reject the constraints in $C^{\#}$ that are not satisfied by C^{*}. And the difference between $C^{\#}$ and C^{*}, as demonstrated by **Theorems 12** and **13**, is *essentially* that while **Qualitative Regularity** is a constraint in $C^{\#}$, C^{*} implies a violation of **Qualitative Regularity**, i.e. according to **Irregular Ordering**, for every finite set $A \in \mathcal{F}^{\#}$, A is equiprobable to $\emptyset$. Consequently, **Qualitative Regularity** should be rejected as a legitimate constraint on a fair lottery over $\mathbb{N}$.

5.3 Reducing the Number of Correct Qualitative Models

Consider the token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$, where C^{*} contains only **Irregular Ordering**.

Theorem 38. *In the token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$, for all $A \in \mathcal{F}^{\#}$, A is $\frac{m}{n}$ times as probable as $\mathbb{N}$ for some $\frac{m}{n} \in \mathbb{Q}$.*

Now, let $\preceq_3$ and $\preceq_4$ be token qualitative probabilities that correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$, i.e. both $\preceq_3$ and $\preceq_4$ are defined on $\mathcal{F}^{\#}$ and both satisfy **Irregular Ordering**. Define $\mathcal{F}^{**} = \{(A, 3) : A \in \mathcal{F}^{\#}\} \cup \{(A, 4) : A \in \mathcal{F}^{\#}\}$.

Theorem 39. *Suppose $\preceq^{**}$ is a cross-comparative qualitative probability defined on $\mathcal{F}^{**}$ such that for all events A and B in $\mathcal{F}^{\#}$, $(A, i) \preceq^{**} (B, i)$ iff $d(A) \leq d(B)$, where $i = 3$ or $i = 4$. Then, for all $A \in \mathcal{F}^{\#}$, $(A, 3) \sim^{**} (A, 4)$.*

In similar spirit as **Definitions 4** and **5**, here are the definitions of token qualitative probabilities $\preceq_3$ and $\preceq_4$:

Definition 6. $\preceq_3 := \{(A, B) : (A, 3) \preceq^{**} (B, 3)\} \cup \{(A, \sim^{**}) : (A, 3) \sim^{**} (A, 4)\} \cup \{(A, \prec^{**}) : (A, 3) \prec^{**} (A, 4)\} \cup \{(A, \succ^{**}) : (A, 3) \succ^{**} (A, 4)\}$. *Say that $A \preceq_3 B$ iff $(A, B) \in \preceq_3$.*

Definition 7. $\preceq_4 := \{(A, B) : (A, 4) \preceq^{**} (B, 4)\} \cup \{(A, \sim^{**}) : (A, 4) \sim^{**} (A, 3)\} \cup \{(A, \prec^{**}) : (A, 4) \prec^{**} (A, 3)\} \cup \{(A, \succ^{**}) : (A, 4) \succ^{**} (A, 3)\}$. *Say that $A \preceq_4 B$ iff $(A, B) \in \preceq_4$.*

However, notice that in virtue of **Theorem 39**, $\{(A, \prec^{**}) : (A, 3) \prec^{**} (A, 4)\} \cup \{(A, \succ^{**}) : (A, 3) \succ^{**} (A, 4)\} = \{(A, \prec^{**}) : (A, 4) \prec^{**} (A, 3)\} \cup \{(A, \succ^{**}) : (A, 4) \succ^{**} (A, 3)\} = \emptyset$ this time. Furthermore, $\{(A, B) : (A, 3) \preceq^{**} (B, 3)\} = \{(A, B) : (A, 4) \preceq^{**} (B, 4)\}$, in virtue of $(A, i) \preceq^{**} (B, i)$ iff $d(A) \leq d(B)$ for all events A and B in $\mathcal{F}^{\#}$. And in virtue of

Theorem 39, $\{(A, \sim^{**}) : (A, 3) \sim^{**} (A, 4)\} = \{(A, \sim^{**}) : (A, 4) \sim^{**} (A, 3)\} = \{(A, \sim^{**}) : A \in \mathcal{F}^{\#}\}$. Combining everything, $\preceq_3 = \{(A, B) : d(A) \leq d(B)\} \cup \{(A, \sim^{**}) : A \in \mathcal{F}^{\#}\} = \preceq_4$, i.e. $\preceq_3$ is *identical* to $\preceq_4$. This means that the token qualitative probability that represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$ is *unique*. Consequently, there is only one correct qualitative model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$. Contrast this situation with $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. While there is only one correct qualitative model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$, there are multiple correct qualitative models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, as proven in §5.2.3.

Hence, there is no need to choose a correct qualitative model over another to model $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$. If a choice has to be made, we must choose the only existing correct qualitative model. So, choosing the only correct qualitative model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$ to model it isn't problematic, unlike choosing a correct qualitative model to model $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. In fact, the only existing correct qualitative model and the only existing correct precise model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$ *aren't* disparate; there is a principle that 'links' the two models together. First, recall from **Theorem 11** that under **Precise Representation**, for all probability functions P that correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$, $P(A) = d(A)$. So, the only existing correct numerical model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$ is $\mathcal{M} = \langle \mathbb{N}, \mathcal{F}^{\#}, C^{*}, \Sigma \rangle$, where $\Sigma = \{(A, d(A)) : A \in \mathcal{F}^{\#}\}$. Let P be the probability assignment in $\mathcal{M}$, i.e. for all $A \in \mathcal{F}^{\#}$, $P(A) = x$ iff $(A, x) \in \Sigma$. Based on the definition of Σ, $P(A) = d(A)$. Now, let $\mathcal{M}'$ denote the only existing correct qualitative model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$, i.e. $\mathcal{M}' = \langle \mathbb{N}, \mathcal{F}^{\#}, C^{*}, \preceq_3 \rangle$ where $\preceq_3$ is defined in **Definition 6**.

Theorem 40. *For all events A and B in $\mathcal{F}^{\#}$, $A \preceq_3 B$ iff $P(A) \leq P(B)$.*

The principle stated in **Theorem 40** is the 'linking' principle between $\mathcal{M}$ and $\mathcal{M}'$. This shows that $\mathcal{M}$ and $\mathcal{M}'$ are not disparate models.

What does the entire discussion from §1 to §5 suggest about $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$? We've considered correct precise models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, of which there are uncountably many, as proven in §3.3. And in the same subsection, I proved that choosing a particular correct precise model over another to represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is problematic w.r.t an end result R. Then, in §4, we considered correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, of which there are uncountably many, as proven in §4.2. And in §5.1, I proved that choosing a particular correct interval-based model over another to represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is still problematic w.r.t an end result R. So, it seems that allowing probabilities to be imprecise did not solve the problem from before when we were just considering precise models. Furthermore, there is an additional problem in choosing a correct interval-based model for $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. According to **Theorem 25**, there are no correct interval-based models $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ that satisfy **Imprecise Reflection**, which is a constraint that all imprecise probabilities should intuitively at least satisfy. Given that every correct interval-based model $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ does not satisfy **Imprecise Reflection**, we have no choice but to choose a correct interval-based model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ that *vio-*

lates **Imprecise Reflection**. Lastly, in §5.2, we considered correct qualitative models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, of which there are multiple, as proven in §5.2.3. And in §5.2.4, I proved that choosing a particular correct qualitative model over another to represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is still problematic w.r.t an end result R.

Given such a persistent problem, how should we get rid of it? One way is to explore *even* more kinds of models, other than just precise models, interval-based models, and qualitative models. For example, recall from the discussion towards the end of §4.3 a *set-based* model, in which $\{i\}$ is simply assigned as a probability the entire set of all positive infinitesimals in a given non-Archimedean ordered field $\mathbb{F}$. Following the discussion in §4.3, given an $\mathbb{F}$, there is only one set-based model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. So, the problem spelled out above for precise models, interval-based models, and qualitative models won't even arise for set-based models. But while I'm happy to accord this proposal *prima facie* plausibility, I'm still hesitant to give this proposal a stamp of approval, or at least until all the precise technical details have been spelled out to my satisfaction. That is, the lack of technical details is the only thing that's barring me from fully embracing this proposal. I would want to know whether the usual applications of probability theory, e.g. calculation of expected utilities, calculation of the expected value, probabilistic logic, predictive modelling, risk management, reliability theory, etc., still work if a set-based model is adopted. As this proposal currently stands, it isn't obvious that the usual applications of probability theory still work. And it would be a huge disadvantage of set-based models if we could not utilise probability theory for its usual purposes. Furthermore, I will argue in §7 that set-based models face a prima facie difficulty, of which I see no straightforward solution.

Here's another way out of the problem spelled out above: reject **Qualitative Regularity** as a legitimate constraint on a fair lottery over $\mathbb{N}$. That is, we should exclude **Qualitative Regularity** from every C of $\langle \Omega, \mathcal{F}, C \rangle$ where $\Omega = \mathbb{N}$ and **Fairness** is one of the constraints in C. For if **Qualitative Regularity** were included, then we would run the risk of there being a multiplicity of correct models all over again, choosing between which may be problematic w.r.t an end result R. However, as explained in §4.3, rejecting **Qualitative Regularity** as a constraint is only half the job done in avoiding the problem spelled out above. So, **Qualitative Regularity** cannot just be rejected; it must be *replaced*. But what constraint should replace **Qualitative Regularity**? Well, in accordance with the discussion in §4.3, at least for a token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C \rangle$ such that (i) $\mathcal{F} \supseteq \mathcal{F}^{\#}$, (ii) **Non-Negativity**, **Non-Triviality**, **Additivity**, **Transitivity**, **Comparing Finite and Infinite Sets**, and **Fairness** are constraints in C, and (iii) the constraints in C imply **Partition***, **Archimedean Probability (Qualitative)** should replace **Qualitative Regularity**. Indeed, as **Theorem 27** shows, **Archimedean Probability (Qualitative)** is satisfied in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$. And as shown in this subsection, $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$ admits only one correct precise model and only one correct qualitative model, between which there is a 'linking' principle (**Theorem 40**). Therefore, the problem spelled out above for $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is avoided by rejecting **Qualitative Regularity** as a legitimate constraint on a fair lottery over $\mathbb{N}$ and further replacing it with **Archimedean Probability (Qualitative)**.

6 Multiple Correct Real-Valued Models

In this section, let $\langle \Omega, \mathcal{F}, C \rangle$ be any relevant token probabilistic scenario such that (i) there are multiple correct precise models of $\langle \Omega, \mathcal{F}, C \rangle$ and (ii) for any numerical model $\mathcal{M} = \langle \Omega, \mathcal{F}, C, \Sigma \rangle$, there exists a proof (or, at least an effective procedure) to determine whether $\mathcal{M}$ is a correct or incorrect model of $\langle \Omega, \mathcal{F}, C \rangle$.[90] The purpose of (ii) is to rule out from C any constraints that are semantically vague or indeterminate. For example, consider the statement 'The coin is *slightly* biased towards tails'. There are some probability assignments that clearly satisfy or clearly violate this constraint. Here's a probability assignment that clearly satisfies this constraint: $\{H\}$ is assigned 0.49 probability and $\{T\}$ is assigned 0.51 probability. Here's another probability assignment that clearly violates this constraint: $\{H\}$ is assigned 0.1 probability and $\{T\}$ is assigned 0.9 probability. But there are some other probability assignments that *do not* clearly satisfy and *do not* clearly violate this constraint, e.g. $\{H\}$ is assigned 0.4 probability and $\{T\}$ is assigned 0.6 probability. Consequently, a numerical model derived from the first probability assignment is definitely a correct model of $\langle \{H, T\}, \mathcal{P}\{H, T\}, C \rangle$ where $\mathcal{P}\{H, T\}$ is the powerset of $\{H, T\}$ and C contains only the statement 'The coin is *slightly* biased towards tails'. A numerical model derived from the second probability assignment is definitely an *incorrect* model of $\langle \{H, T\}, \mathcal{P}\{H, T\}, C \rangle$. But, in virtue of the third probability assignment not clearly satisfying or violating the constraint, it isn't clear that a numerical model derived from the third probability assignment is a correct model of $\langle \{H, T\}, \mathcal{P}\{H, T\}, C \rangle$. Similarly, it also isn't clear that this model is an incorrect model of $\langle \{H, T\}, \mathcal{P}\{H, T\}, C \rangle$. In other words, there does not exist a proof (or an effective procedure) to determine whether the third model is a correct or incorrect model of $\langle \{H, T\}, \mathcal{P}\{H, T\}, C \rangle$. In this section, I'm only interested in relevant token probabilistic scenarios that satisfy (i) and (ii).

Up to this point, probabilities are allowed to be non-Archimedean. After all, in order for a precise probability function to correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ under **Precise Representation**, that probability function must be non-Archimedean, as $C^{\#}$ contains **Fairness** and **Qualitative Regularity**. And in §3.3, I've shown that there are uncountably many correct precise models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, choosing between which is problematic w.r.t an end result R. Now, the reader may be wondering whether this problem is unique to non-Archimedean probabilities. That is, if probabilities were stipulated to be *real-valued*, would the problems concerning our choosing a particular correct numerical model over another to represent a relevant token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ (that satisfies (i) and (ii)) then dissipate?

To be sure, no real-valued probabilities correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, under **Precise Representation**. But there seem to be plenty of other relevant token probabilistic scenarios $\langle \Omega, \mathcal{F}, C \rangle$ that admit uncountably many correct precise models expressed exclu-

[90]See §4.1 for the definition of a relevant token probabilistic scenario.

sively in terms of real-valued probabilities. For example, let $\mathcal{P}\mathbb{N}$ be the powerset of $\mathbb{N}$ and C contain the probability axioms (**Non-Negativity (Numerical)**, **Normalization**, and **Finite Additivity**), **Regularity**, and the statement '$P(\{i\}) = 2 \cdot P(\{i+1\})$'. Put simply, the token probabilistic scenario $\langle \mathbb{N}, \mathcal{P}\mathbb{N}, C \rangle$ describes a lottery over $\mathbb{N}$, where ticket 1 is twice as probable as ticket 2, and ticket 2 is twice as probable as ticket 3, etc. Define a probability function P_1 with $\mathcal{P}\mathbb{N}$ as its domain such that $P_1(\{i\}) = \frac{1}{2^i}$, i.e. $P_1(\{1\}) = \frac{1}{2}$, $P_1(\{2\}) = \frac{1}{4}$, etc. It is easy to verify that P_1 correctly represents $\langle \mathbb{N}, \mathcal{P}\mathbb{N}, C \rangle$. Now, define another probability function P_2 with $\mathcal{P}\mathbb{N}$ as its domain such that $P_2(\{i\}) = \frac{1}{2^{i+1}}$, i.e. $P_2(\{1\}) = \frac{1}{4}$, $P_2(\{2\}) = \frac{1}{8}$, etc. It is similarly easy to verify that P_2 correctly represents $\langle \mathbb{N}, \mathcal{P}\mathbb{N}, C \rangle$. Recall from footnote 1 that probability functions are merely finitely additive.[91] And from these two probability functions, two different correct precise models of $\langle \mathbb{N}, \mathcal{P}\mathbb{N}, C \rangle$ expressed exclusively in terms of real-valued probabilities can be defined, i.e. $\mathcal{M}_1 = \langle \mathbb{N}, \mathcal{P}\mathbb{N}, C, \Sigma_1 \rangle$, where $\Sigma_1 = \{(A, P_1(A)) : A \in \mathcal{P}\mathbb{N}\}$, and $\mathcal{M}_2 = \langle \mathbb{N}, \mathcal{P}\mathbb{N}, C, \Sigma_2 \rangle$, where $\Sigma_2 = \{(A, P_2(A)) : A \in \mathcal{P}\mathbb{N}\}$. In fact, since there are uncountably many real-valued (and finitely additive) probability functions that correctly represent $\langle \mathbb{N}, \mathcal{P}\mathbb{N}, C \rangle$, there are uncountably many correct precise models of $\langle \mathbb{N}, \mathcal{P}\mathbb{N}, C \rangle$ expressed exclusively in terms of real-valued probabilities too.

Isn't choosing a particular correct precise model over another to represent $\langle \mathbb{N}, \mathcal{P}\mathbb{N}, C \rangle$ still problematic w.r.t an end result R? Can't the same argument be run for this case? If so, then the problems concerning our choosing a particular correct numerical model over another to represent a relevant token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ will *not* dissipate, even after stipulating that probabilities are real-valued. Here's my reply. Recall that in order to argue that our choosing a particular correct numerical model over another to represent $\langle \Omega, \mathcal{F}, C \rangle$ is problematic w.r.t an end result R, we must first show that the models have the same representational power in representing $\langle \Omega, \mathcal{F}, C \rangle$. And in the case of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, I used **Same Representational Power** to show this. However, note that the same principle cannot be used to show that $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same representational power in representing $\langle \mathbb{N}, \mathcal{P}\mathbb{N}, C \rangle$. This is because it is *false* that for

[91] The reader may be wondering why I didn't include **Countable Additivity** as a constraint on probability functions. After all, if **Countable Additivity** were included as a constraint on probability functions, then P_2 would not correctly represent $\langle \mathbb{N}, \mathcal{P}\mathbb{N}, C \rangle$, as $\frac{1}{4} + \frac{1}{8} + \ldots < 1$. Here's my reply. As I've explained in §3.1, since the main object of interest in this thesis is non-Archimedean probabilities, **Countable Additivity** cannot be assumed as a constraint on numerical probabilities, as non-Archimedean ordered fields are not even complete (**Theorem 2**). Having said that, note that Easwaran [2013] has put forth arguments on why **Countable Additivity** should be a constraint on probability functions. Indeed, if **Countable Additivity** should be a constraint on probability functions, then **Regularity** cannot be a constraint on probability functions. To see why, consider any fair infinite lottery (countable or uncountable) over Ω. Since this lottery is fair, $P(\{\omega\}) = P(\{\omega'\})$ for all $\omega, \omega' \in \Omega$. By **Countable Additivity**, $P(\bigcup_{i=1}^{\infty} \{\omega_i\}) = P(\bigcup_{i=2}^{\infty} \{\omega_i\})$. This means that $P(\{\omega_1\}) = 0$, violating **Regularity**. So, if **Countable Additivity** should be a constraint on probability functions, then **Regularity** cannot be a constraint on probability functions. Put simply, if Easwaran [2013] is right about **Countable Additivity**, then **Regularity** goes out the window. However, Stewart & Nielsen [2021] have argued that Easwaran [2013]'s arguments have little persuasive force for a proponent of merely finitely additive probabilities.

all $A \in \mathcal{P}\mathbb{N}$, $|P_1(A) - P_2(A)| < \frac{1}{n}$ for all $n \in \mathbb{N}$, e.g. $P_1(\{1\}) - P_2(\{1\}) = \frac{1}{4}$. In fact, **Same Representational Power** cannot be used to show that two *real-valued* probability functions have the same representational power in representing a given relevant token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$.[92]

So, in order to show that $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same representational power in representing $\langle \mathbb{N}, \mathcal{P}\mathbb{N}, C \rangle$, a different principle, other than **Same Representational Power**, has to be utilised. And it is important and interesting to ask whether this other principle is even defensible in the first place. But for the sake of argument, let's just grant that $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same representational power in representing $\langle \mathbb{N}, \mathcal{P}\mathbb{N}, C \rangle$, and also that choosing $\mathcal{M}_1$ over $\mathcal{M}_2$ to represent $\langle \mathbb{N}, \mathcal{P}\mathbb{N}, C \rangle$ is problematic w.r.t an end result R. Are there any ways out of this problem?

My answer is *yes*, as we can always allow probabilities to be imprecise. In a way, once we stipulate that probabilities are real-valued, Benci et al. [2018]'s and Pruss [2021]'s proposals of allowing probabilities to be imprecise work like a charm in getting us out of the problems concerning our choosing a particular correct numerical model over another to represent a relevant token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ (that satisfies (i) and (ii)). To see why, stipulate that probabilities are real-valued and let $\langle \Omega, \mathcal{F}, C \rangle$ be any relevant token probabilistic scenario that satisfies (i) and (ii), as detailed above in the beginning of this section. Define $\mathcal{I}$ to be an index set of all the correct precise models (expressed exclusively in terms of real-valued probabilities) of $\langle \Omega, \mathcal{F}, C \rangle$. That is, $\mathcal{I}$ is the set of all i's such that there exists a correct precise model $\mathcal{M}_i = \langle \Omega, \mathcal{F}, C, \Sigma_i \rangle$ (expressed exclusively in terms of real-valued probabilities) of $\langle \Omega, \mathcal{F}, C \rangle$. Next, let P_i be the probability assignment in $\mathcal{M}_i$,

[92] So, **Same Representational Power** fails to deliver the verdict that P_1 and P_2 have the same representational power in representing $\langle \mathbb{N}, \mathcal{P}\mathbb{N}, C \rangle$. Now, the reader may be wondering whether this failure casts doubt on the truth of **Same Representational Power**. After all, it seems intuitive that P_1 and P_2 have the same representational power in representing $\langle \mathbb{N}, \mathcal{P}\mathbb{N}, C \rangle$; both are defined on $\mathcal{P}\mathbb{N}$ and both satisfy all the constraints in C. Since **Same Representational Power** cannot deliver this intuitive verdict, **Same Representational Power** is therefore *false*, or so it is thought. Here's my reply. Let's say that w.r.t a token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$, if two probability functions P and P' are defined on $\mathcal{F}$ and both satisfy the constraints in C, then P and P' have the same representational power in representing $\langle \Omega, \mathcal{F}, C \rangle$. As I've explained in footnote 13, if the reader thinks that this is true, then, by entailment, **Same Representational Power** is *true*.

More importantly, recall that **Same Representational Power** details a set of conditions, which when fulfilled together, is *sufficient*, not necessary, for two probability functions to have the same representational power in representing the given token probabilistic scenario. For all I've said so far, there may very well be another set of conditions, which when fulfilled together, is sufficient for P_1 and P_2 to have the same representational power in representing $\langle \mathbb{N}, \mathcal{P}\mathbb{N}, C \rangle$. But this doesn't mean that **Same Representational Power** is therefore false. Instead, all this shows is that **Same Representational Power** doesn't apply to P_1 and P_2. In general, the statement 'if A, then C' being true doesn't imply that the statement 'if B, then C' is false. To see why, consider these examples: 'if I leave a lit candle near the curtains on a windy day, then a fire will start' and 'if there is an electrical fault, then a fire will start'. One being true doesn't imply that the other is false. In fact, both are true, as leaving a lit candle near the curtains on a windy day and the presence of an electrical fault are both sufficient conditions for a fire to start. Just because B is a sufficient condition for C's occurrence doesn't mean that A is no longer a sufficient condition for C's occurrence. The same goes for **Same Representational Power**. The existence of another set of conditions, which when fulfilled together, is sufficient for P_1 and P_2 to have the same representational power in representing $\langle \mathbb{N}, \mathcal{P}\mathbb{N}, C \rangle$ doesn't imply that **Same Representational Power** is false.

i.e. for all $A \in \mathcal{F}$, $P_i(A) = x$ iff $(A, x) \in \Sigma_i$. In accordance with the definition provided in §4.1, $\mathcal{S} = \{P_i : i \in \mathcal{I}\}$, i.e. $\mathcal{S}$ is the set of all precise probability functions P_i that correctly represent $\langle \Omega, \mathcal{F}, C \rangle$. For each $A \in \mathcal{F}$, define $\mathcal{S}_A = \{x \in \mathbb{R} : \exists P_i \in \mathcal{S}(P_i(A) = x)\}$, i.e. $\mathcal{S}_A$ contains all the various outputs of every $P_i \in \mathcal{S}$ at A.

Since $\mathbb{R}$ satisfies the **Least Upper Bound Property**, $\mathcal{S}_A \subseteq \mathbb{R}$, and $\mathcal{S}_A$ is bounded above by 1 and bounded below by 0, the L.U.B and the G.L.B for $\mathcal{S}_A$ exist in $\mathbb{R}$. Now, define an imprecise probability function P_{im} with $\mathcal{F}$ as its domain such that $\overline{P}_{im}(A)$ is the L.U.B for $\mathcal{S}_A$ and $\underline{P}_{im}(A)$ is the G.L.B for $\mathcal{S}_A$. So, $P_{im}(A) \in \mathbb{R}$ iff $\overline{P}_{im}(A) = \underline{P}_{im}(A)$; $P_{im}(A)$ is an interval iff $\overline{P}_{im}(A) \neq \underline{P}_{im}(A)$. By the definition of P_{im}, P_{im} is tight w.r.t $\langle \Omega, \mathcal{F}, C \rangle$. In fact, according to **Theorem 23**, P_{im} is the only existing imprecise probability function correctly representing $\langle \Omega, \mathcal{F}, C \rangle$ that is tight w.r.t $\langle \Omega, \mathcal{F}, C \rangle$. Define an interval-based model $\mathcal{M}_{im} = \langle \Omega, \mathcal{F}, C, \Sigma_{im} \rangle$, where $\Sigma_{im} = \{(A, P_{im}(A)) : A \in \mathcal{F}\}$. It is easy to verify that $\mathcal{M}_{im}$ is a correct interval-based model of $\langle \Omega, \mathcal{F}, C \rangle$ and in virtue of P_{im} being the only existing imprecise probability function correctly representing $\langle \Omega, \mathcal{F}, C \rangle$ that is tight w.r.t $\langle \Omega, \mathcal{F}, C \rangle$, $\mathcal{M}_{im}$ is the only existing correct interval-based model of $\langle \Omega, \mathcal{F}, C \rangle$ that is tight.

So, although there are many correct precise models of $\langle \Omega, \mathcal{F}, C \rangle$ expressed exclusively in terms of real-valued probabilities, there is only one correct interval-based model that is tight, i.e. $\mathcal{M}_{im}$. Recall from my discussion in §4.3 that w.r.t a relevant token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$ that satisfies (i) and (ii), all imprecise probability functions correctly representing this scenario should intuitively be tight w.r.t $\langle \Omega, \mathcal{F}, C \rangle$. Therefore, in this sense, $\mathcal{M}_{im}$ not only 'stands out' from the many correct precise models of $\langle \Omega, \mathcal{F}, C \rangle$, it also 'stands out' from the many other correct interval-based models of $\langle \Omega, \mathcal{F}, C \rangle$. This fact, by itself, gives us good enough reason to choose $\mathcal{M}_{im}$ to represent $\langle \Omega, \mathcal{F}, C \rangle$.

7 The Nature of a Qualitative Probability

In this section, disregard for a moment the distinction between type qualitative probabilities and token qualitative probabilities. The reason for doing so will become clear in a little while. As the title of this section suggests, here's the important question to be answered: what is the nature of a qualitative probability? Well, according to the usual understanding of qualitative probabilities in the literature, a qualitative probability is just the *ordering* of the events in its domain. So, according to this line of thought, for any two qualitative probabilities $\preceq$ and $\preceq'$, $\preceq = \preceq'$ iff (i) $\preceq$ and $\preceq'$ share the same domain $\mathcal{F}$, and (ii) for all events A and B in $\mathcal{F}$, $A \preceq B$ iff $A \preceq' B$. Put simply, $\preceq = \preceq'$ iff $\preceq$ and $\preceq'$ order the *same* events the *same* way, according to this line of thought.

Now, recall the two qualitative probabilities $\preceq_1$ and $\preceq_2$, as defined in **Definitions 4** and **5** respectively. As shown in §5.2.3, both $\preceq_1$ and $\preceq_2$ correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, i.e. $\preceq_1$ and $\preceq_2$ order the *same* events the *same* way. Therefore, according to the line of thought spelled out above, $\preceq_1 = \preceq_2$, *contra* my discussion in §5.2.3. Consequently, the two qualitative models $\mathcal{M}_{\preceq_1}$ and $\mathcal{M}_{\preceq_2}$, as defined towards the end of §5.2.3, of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ are *identical*, i.e. $\mathcal{M}_{\preceq_1} = \mathcal{M}_{\preceq_2}$. In fact, since the constraints in $C^{\#}$ fully determine the order of events in $\mathcal{F}^{\#}$, only one qualitative model correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, *if* a qualitative probability is indeed just the ordering of the events in its domain. *Contra* my discussion in §5.2.4, there is therefore no need to choose a particular correct qualitative model over another to represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$; there is only one such model. And choosing the only existing correct qualitative model to represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ isn't problematic at all.[93] Importantly, if a qualitative probability is indeed just the ordering of the events in its domain, then the distinction between type qualitative probabilities and token qualitative probabilities collapses. Recall that a type qualitative probability is a set of token qualitative probabilities meeting a common specification, e.g. every token qualitative probability in the set must order the same events the same way. But if a qualitative probability is nothing more than the ordering of events, then there is no difference between the type qualitative probability and the token qualitative probabilities in the set.

Is this usual understanding of qualitative probabilities accurate? I don't think so. To see why, recall the two lotteries over $\mathbb{N}$, as detailed in §5.2. To refresh the reader's memory, here they are again. Consider two separate lotteries held over $\mathbb{N}$. So, the tickets in each lottery are the same, i.e. ticket 1, ticket 2, ticket 3, etc. In the first lottery, a ticket will be randomly chosen and that ticket will be the winning ticket. The winning ticket in the second lottery will be chosen the following way. Ticket i wins in the second lottery just in case ticket $2i - 1$ or $2i$ wins in the first lottery. That is, ticket 1 wins in the second lottery just in case ticket 1 or 2 wins in the first lottery, ticket 2 wins in

[93]Thanks to Nicholas DiBella for pressing me on this point. It is interesting to note though that I've also received comments to the contrary, i.e. $\preceq_1$ and $\preceq_2$ are so obviously different that most of the technical details in §5.2 aren't required to bring out the difference.

the second lottery just in case ticket 3 or 4 wins in the first lottery, etc. Therefore, the winning ticket in the first lottery completely determines the winning ticket in the second lottery, but not vice versa. Intuitively, ticket i winning in the second lottery is *twice* as probable as ticket i winning in the first lottery. Or simply, ticket i winning in the second lottery is *more* probable than ticket i winning in the first lottery. After all, since the first lottery is fair, ticket $2i-1$ winning in the first lottery is as probable as ticket $2i$ winning in the first lottery, both of which are in turn individually as probable as ticket i winning in the first lottery. Because **Qualitative Regularity** is assumed, all relevant events here are more probable than $\emptyset$. So, ticket $2i-1$ or $2i$ winning in the first lottery is twice as probable as ticket i winning in the first lottery. But since ticket i winning in the second lottery is as probable as ticket $2i-1$ or $2i$ winning in the first lottery, ticket i winning in the second lottery is twice as probable as ticket i winning in the first lottery, i.e. ticket i winning in the second lottery is more probable than ticket i winning in the first lottery. Let $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ describe the first lottery and $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ describe the second lottery, where C' contains only the statement 'Ticket i wins just in case ticket $2i-1$ or $2i$ wins in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$'. According to **Theorem 31**, both lotteries order the same events the same way.

Intuitively, these two lotteries over $\mathbb{N}$ are *not* one and the same lottery. After all, the winning ticket in the first lottery completely determines the winning ticket in the second lottery, but not vice versa. If these two lotteries were one and the same lottery, then the winning ticket in the second lottery would also completely determine the winning ticket in the first lottery; but it doesn't. Moreover, ticket i winning in the second lottery is twice as probable as ticket i winning in the first lottery. If these two lotteries were one and the same lottery, then ticket i winning in the second lottery would be as probable as ticket i winning in the first lottery; but it isn't. So, these two lotteries over $\mathbb{N}$ are *distinct* lotteries.

Importantly, notice that numerical probability functions can distinguish between these two lotteries easily. Recall from §5.2.2 the precise probability functions P_1 and P_2, i.e. for all $A \in \mathcal{F}^{\#}$, $P_1(A) = d(A) + \big(\varepsilon \cdot N(A)\big)$ and $P_2(A) = d(A) + \big(2\varepsilon \cdot N(A)\big)$. And as shown in the same subsection, P_1 correctly represents the first lottery over $\mathbb{N}$ and P_2 correctly represents the second lottery. It is easy to verify that $P_1 \neq P_2$, e.g. $P_1(\{i\}) = \varepsilon$ and $P_2(\{i\}) = 2\varepsilon$, in line with ticket i winning in the second lottery being twice as probable as ticket i winning in the first lottery. This shows that precise probability functions have no problem in distinguishing between the two lotteries. In fact, the same goes for *imprecise* probability functions. For a given non-Archimedean ordered field $\mathbb{F}$, there exist two imprecise probability functions $P_{im(u)}$ and $P_{im(2u)}$ with $\mathcal{F}^{\#}$ as their domains such that $P_{im(u)}(\{i\}) = (0, u)$ and $P_{im(2u)}(\{i\}) = (0, 2u)$, where $2u \in \mathbb{F}$ is a positive real number less than 1.[94] According to **Theorem 19**, both $P_{im(u)}$ and $P_{im(2u)}$ correctly represent

[94]See §4.2 for the full definitions of $P_{im(u)}$ and $P_{im(2u)}$.

$\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. And since $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ and $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ order the same events the same way, $P_{im(2u)}$ correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$. This shows that even imprecise probability functions have no problem in distinguishing between the two lotteries.

I would expect nothing less of qualitative probabilities. However, if a qualitative probability is just the *ordering* of the events in its domain, then, in virtue of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ and $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ ordering the same events the same way, any qualitative probability $\preceq$ correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is identical to any qualitative probability $\preceq'$ correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$, i.e. $\preceq = \preceq'$. Therefore, given the usual understanding of a qualitative probability as the ordering of the events in its domain, qualitative probabilities cannot distinguish between the two lotteries. Yet, these two lotteries are still distinct. In fact, *set-based* probability assignments, as defined towards the end of §4.3, cannot distinguish between the two lotteries too. To see why, recall first that according to set-based probabilities, $\{i\}$ in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is simply assigned as a probability the entire set $\mathcal{S}_{\{i\}}$ of all positive infinitesimals in a given non-Archimedean ordered field $\mathbb{F}$, i.e. $\mathcal{S}_{\{i\}} = \{x \in \mathbb{F} : \forall n \in \mathbb{N}(0 < x < \frac{1}{n})\}$. And as explained above, ticket i winning in $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ is twice as probable as ticket i winning in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. So, according to set-based probabilities, $\{i\}$ in $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ is simply assigned as a probability the set $\mathcal{S}'_{\{i\}} = \{x \in \mathbb{F} : \exists y \in \mathcal{S}_{\{i\}}(x = 2y)\}$. But $\mathcal{S}'_{\{i\}} = \mathcal{S}_{\{i\}}$.[95] This means that $\{i\}$ is assigned the same set-based probability $\mathcal{S}_{\{i\}}$ in both $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ and $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$. As a matter of fact, for any $A \in \mathcal{F}^{\#}$, A is assigned the same set-based probability $\mathcal{S}_A$ in both $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ and $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$, implying that set-based probabilities cannot distinguish between the two lotteries. *This* is the prima facie difficulty I've been alluding to in §4.3 and §5.3 whenever set-based probabilities are considered, and I see no straightforward way out of this difficulty.

However, why bother distinguishing between $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ and $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ in the first place? Well, it's because we should prefer to be playing for a prize in $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ than in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. After all, ticket i winning in $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ is twice as probable as ticket i winning in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. In other words, our chances of winning the prize is *doubled* if we choose to play in $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$, instead of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. And it seems only rational to prefer to play in a lottery where our chances of winning are higher. In order to reflect this preference, probabilities must be able to distinguish between the two lotteries over $\mathbb{N}$. Having said that, it is important to note that the above is only true if it is stipulated that we own a finite number of tickets, no matter which lottery we choose to play in. If we can own—say, all the even-numbered tickets, then it makes no difference which

[95] Let x be an arbitrary member in $\mathcal{S}'_{\{i\}}$. This means that there exists a $y \in \mathcal{S}_{\{i\}}$ such that $x = 2y$. Since $0 < y < \frac{1}{n}$ for all $n \in \mathbb{N}$, $0 < 2y < \frac{1}{n}$ for all $n \in \mathbb{N}$ too, which in turn implies that $2y \in \mathcal{S}_{\{i\}}$. But because $x = 2y$, $x \in \mathcal{S}_{\{i\}}$. So, $\mathcal{S}'_{\{i\}} \subseteq \mathcal{S}_{\{i\}}$. Now, I'll prove that $\mathcal{S}_{\{i\}} \subseteq \mathcal{S}'_{\{i\}}$. Let y be an arbitrary member in $\mathcal{S}_{\{i\}}$. This means that $0 < y < \frac{1}{n}$ for all $n \in \mathbb{N}$, which in turn implies that $0 < \frac{y}{2} < \frac{1}{n}$ for all $n \in \mathbb{N}$. So, $\frac{y}{2} \in \mathcal{S}_{\{i\}}$. Because $\frac{y}{2} \in \mathcal{S}_{\{i\}}$ and $\mathcal{S}'_{\{i\}} = \{x \in \mathbb{F} : \exists y \in \mathcal{S}_{\{i\}}(x = 2y)\}$, $y \in \mathcal{S}'_{\{i\}}$. Therefore, $\mathcal{S}_{\{i\}} \subseteq \mathcal{S}'_{\{i\}}$. Combining with $\mathcal{S}'_{\{i\}} \subseteq \mathcal{S}_{\{i\}}$, $\mathcal{S}'_{\{i\}} = \mathcal{S}_{\{i\}}$.

lottery we choose to play in. To see why, recall that in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, $\{2,4,6,\ldots\}$ is as probable as $\{1,3,5,\ldots\}$ in containing the winning ticket. Intuitively, this means that $\{2,4,6,\ldots\}$ is assigned $\frac{1}{2}$ probability in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Now, in $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$, the winning ticket is contained in $\{2,4,6,\ldots\}$ iff the winning ticket in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is contained in $\{3,7,11,\ldots\} \cup \{4,8,12,\ldots\}$. But in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, $\{1,5,9,\ldots\}$, $\{2,6,10,\ldots\}$, $\{3,7,11,\ldots\}$, and $\{4,8,12,\ldots\}$ are all equiprobable in containing the winning ticket. This means that $\{3,7,11,\ldots\} \cup \{4,8,12,\ldots\}$ is assigned $\frac{1}{2}$ probability in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, which in turn implies that $\{2,4,6,\ldots\}$ is assigned $\frac{1}{2}$ probability in $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ too. This shows that switching between the lotteries will not increase (or decrease) our chances of winning the prize if we own all the even-numbered tickets. So, if we own all the even-numbered tickets, we should be indifferent about which lottery to play in.

The discussion above suggests that in order for qualitative probabilities to distinguish between the two lotteries, a qualitative probability cannot just be the ordering of the events in its domain, contrary to the usual understanding of qualitative probabilities in the literature. Instead, a qualitative probability must be the combination of the ordering and something else over and above the ordering. Here's an analogy to better illustrate the point.[96] Consider the properties 'being a cordate' and 'being a renate'. A cordate is a creature with a heart while a renate is a creature with a kidney. In the actual world, any creature with a heart has a kidney too, and vice versa. So, the two properties have the same extension. As Quine [1951] has pointed out, if one thinks that the two properties are different, then those properties cannot be defined extensionally. For if they were defined extensionally, then both properties would be identical, as they have the same extension. But they aren't identical, so they cannot be defined extensionally. Instead, both properties must be defined in terms of their extension and something else over and above their extensions.[97] The same goes for qualitative probabilities. Since $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ and $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ order the same events the same way, in order for qualitative probabilities to distinguish between these two lotteries, a qualitative probability must be more than just the ordering of events.

But what is this that's over and above the ordering of events? **Definitions 4** and **5** clearly tell us what it is. It is the union of the last three sets $\{(A, \sim^*) : (A,i) \sim^* (A,j)\} \cup \{(A, \prec^*) : (A,i) \prec^* (A,j)\} \cup \{(A, \succ^*) : (A,i) \succ^* (A,j)\}$, where $i \neq j$. The

[96]Thanks to Nicholas DiBella for providing this analogy.

[97]Lewis [1986] thought that the properties 'being a cordate' and 'being a renate' are different. So, instead of defining them extensionally, he defined them *intensionally*, i.e. property A is the set of all things (actual *and* possible) that satisfy A. That is, the property 'being a cordate' is the set of all creatures in the actual world and in other possible worlds that have a heart. Since it is not necessary that a cordate is a renate, there are possible worlds in which there exist creatures with a heart but without a kidney, and vice versa. So, the set of all things (actual and possible) that have a heart isn't identical to the set of all things (actual and possible) that have a kidney, i.e. the property 'being a cordate' is different from the property 'being a renate'. In particular, according to Lewis [1986], properties A and B are identical iff both pick out the same extension throughout *all* possible worlds, i.e. two properties are identical iff they are *co-intensional*.

presence of the last three sets in the definitions of $\preceq_1$ and $\preceq_2$ is what sets $\preceq_1$ apart from $\preceq_2$. Now, as with all heterodox views, there will inevitably be objections; I will address two of them. *Objection* 1. The first objection goes something like the following:

> 'Since a qualitative probability is nothing more than just the ordering of events, the two lotteries over $\mathbb{N}$ are in fact one and the same lottery, based solely on the fact that they order the same events the same way.'

Here's my reply. As hard as I try with all my might, I cannot bring myself to accept that the two lotteries are one and the same lottery. It is hard to shake off the intuition that there is something different between the two lotteries. After all, there is some sort of asymmetry between the two lotteries: the winning ticket in the first lottery completely determines the winning ticket in the second lottery, but not vice versa. This should suggest that the two lotteries are not one and the same lottery. If the reader doesn't agree with me or share my intuition, I'm afraid there's nothing more I can say but to give the incredulous stare of disbelief.

Objection 2. The second objection goes something like the following:

> 'I'm assuming that a *single* qualitative probability can only represent *one* token probabilistic scenario. But why think that? Why not allow a single qualitative probability to represent *multiple* token probabilistic scenarios?'

Here's my reply. Now, this is an interesting objection. One of my goals in this thesis is to show that there may be multiple qualitative probabilities correctly representing a given token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$. For example, both $\preceq_1$ and $\preceq_2$ correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. That is, for *one* token probabilistic scenario, there may be *multiple* qualitative probabilities correctly representing it. Objection 2 is essentially the flip side of this goal. That is, for *one* qualitative probability, there may be *multiple* token probabilistic scenarios correctly represented by it. This seems to give rise to an interesting variant of underdetermination, which may very well be problematic. Regardless, the idea that a single qualitative probability may represent multiple token probabilistic scenarios is still interesting and worthy of consideration.

Although I'm inclined to think that a qualitative probability is more than just the ordering of events, *contra* the usual understanding of a qualitative probability, I'm perfectly fine if I'm completely *mistaken* about this. After all, as I've said in the beginning of §5.2 where I showed that there are multiple qualitative probabilities correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, the content of that subsection is highly speculative. In fact, that subsection is meant more as an *exploration* on how to extend my project from numerical models to qualitative models. But perhaps there is no such extension. And if there is no such extension, I stand corrected.

8 Conclusion

This thesis is centred around the particular token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, as defined in §3.2. First, we considered correct precise models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, of which there are uncountably many, as proven in §3.3. And in the same subsection, I proved that choosing a particular correct precise model over another to represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is problematic w.r.t an end result R. Then, in §4, we considered correct interval-based models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, of which there are uncountably many, as proven in §4.2. And in §5.1, I proved that choosing a particular correct interval-based model over another to represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is still problematic w.r.t an end result R. So, *contra* Benci et al. [2018] and Pruss [2021], it seems that allowing probabilities to be imprecise did not solve the problem from before when we were just considering precise models. Furthermore, there is an additional problem in choosing a correct interval-based model for $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. According to **Theorem 25**, there are no correct interval-based models $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ that satisfy **Imprecise Reflection**, which is a constraint that all imprecise probabilities should intuitively at least satisfy. Given that every correct interval-based model $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ does not satisfy **Imprecise Reflection**, we have no choice but to choose a correct interval-based model of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ that *violates* **Imprecise Reflection**.

Lastly, in §5.2, we considered correct qualitative models of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, of which there are multiple, as proven in §5.2.3. And in §5.2.4, I proved that choosing a particular correct qualitative model over another to represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is still problematic w.r.t an end result R. Given such a persistent problem, how should we get rid of it? One way is to explore *even* more kinds of models, other than just precise models, interval-based models, and qualitative models. In §4.3, we explored the possibility of modelling $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ using a *set-based* model, in which $\{i\}$ is simply assigned as a probability the entire set of all positive infinitesimals in a given non-Archimedean ordered field $\mathbb{F}$. But while I'm happy to accord this proposal *prima facie* plausibility, I'm still hesitant to give this proposal a stamp of approval, as it isn't obvious whether the usual applications of probability theory still work if a set-based model is adopted. Furthermore, as I've argued in §7, set-based probabilities cannot distinguish between the two lotteries over $\mathbb{N}$, as detailed in §5.2. And there seems to be no straightforward way out of this difficulty for set-based probabilities. So, we should welcome a better solution, for the sake of avoiding the problems concerning our choosing a particular correct model over another to represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. Fortunately, there is a better solution.

We can (and should) reject **Qualitative Regularity** as a legitimate constraint on a fair lottery over $\mathbb{N}$. That is, we should exclude **Qualitative Regularity** from every C of $\langle \Omega, \mathcal{F}, C \rangle$ where $\Omega = \mathbb{N}$ and **Fairness** is one of the constraints in C. For if **Qualitative Regularity** were included, then we would run the risk of there being a multiplicity of correct models all over again, choosing between which may be problematic w.r.t an end result R. However, as explained in §4.3, rejecting **Qualitative Regularity** as a constraint is only half the job done in avoiding the problem spelled out above. So, **Qualitative Regularity** cannot just be rejected; it must be *replaced*.

But what constraint should replace **Qualitative Regularity**? Well, in accordance with the discussion in §4.3, at least for a token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C \rangle$ such that (i) $\mathcal{F} \supseteq \mathcal{F}^{\#}$, (ii) **Non-Negativity**, **Non-Triviality**, **Additivity**, **Transitivity**, **Comparing Finite and Infinite Sets**, and **Fairness** are constraints in C, and (iii) the constraints in C imply **Partition***, **Archimedean Probability (Qualitative)** should replace **Qualitative Regularity**. Indeed, as **Theorem 27** shows, **Archimedean Probability (Qualitative)** is satisfied in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$, where C^{*} contains only **Irregular Ordering**. And as shown in §5.3, $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$ admits only one correct precise model and only one correct qualitative model, between which there is a 'linking' principle (**Theorem 40**). Therefore, the problems concerning our choosing a particular correct model over another to represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ don't even arise for $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$.

For the sake of getting out of these problems, we should reject **Qualitative Regularity** as a legitimate constraint on a fair lottery over $\mathbb{N}$ and further replace it with **Archimedean Probability (Qualitative)**. In fact, as shown in §6, for any relevant token probabilistic scenarios $\langle \Omega, \mathcal{F}, C \rangle$ such that (i) there are multiple correct precise models of $\langle \Omega, \mathcal{F}, C \rangle$ and (ii) for any numerical model $\mathcal{M} = \langle \Omega, \mathcal{F}, C, \Sigma \rangle$, there exists a proof (or, at least an effective procedure) to determine whether $\mathcal{M}$ is a correct or incorrect model of $\langle \Omega, \mathcal{F}, C \rangle$, once we stipulate that probabilities are *real-valued* and allow for probabilities to be imprecise, there exists a correct interval-based model $\mathcal{M}_{im}$ of $\langle \Omega, \mathcal{F}, C \rangle$ that is tight. So, not only does $\mathcal{M}_{im}$ 'stand out' from the many correct precise models of $\langle \Omega, \mathcal{F}, C \rangle$, it also 'stands out' from the many other correct interval-based models of $\langle \Omega, \mathcal{F}, C \rangle$. This fact, by itself, gives us good enough reason to choose $\mathcal{M}_{im}$ to represent $\langle \Omega, \mathcal{F}, C \rangle$.

9 Appendix

Proof of ***Theorem 1****.* For reductio, assume that there are two real-valued probability functions P_1 and P_2 where $P_1 \neq P_2$ that satisfy all four conditions of **Infinitesimal Underdetermination**. By condition 3 of **Infinitesimal Underdetermination**, there exists an event C in $\mathcal{F}$ such that $P_1(C) < P_2(C)$ or $P_1(C) > P_2(C)$. Because P_1 and P_2 are both real-valued, $|P_1(C) - P_2(C)| > \frac{1}{n}$ for some $n \in \mathbb{N}$, violating condition 4 of **Infinitesimal Underdetermination**. Contradiction. Therefore, there are no real-valued probability functions P_1 and P_2 where $P_1 \neq P_2$ that satisfy all four conditions of **Infinitesimal Underdetermination**. □

Proof of ***Theorem 2****.* Assume that $(F, +, \cdot, \leq)$ is a complete ordered field. For reductio, assume that $(F, +, \cdot, \leq)$ is not Archimedean. This means that there are $x, y \in F$ where $0 < x$ and $0 < y$ but there does not exist an $n \in \mathbb{N}$ such that $x < n \cdot y$. Because $(F, +, \cdot, \leq)$ is an ordered field, the previous sentence in turn implies that for all $n \in \mathbb{N}$, $n \cdot y \leq x$. Now, define $S = \{n \cdot y : n \in \mathbb{N}\}$. Since $n \cdot y \leq x$ for all $n \in \mathbb{N}$, x is an upper bound for S. But because $(F, +, \cdot, \leq)$ is complete, S has a least upper bound in F. Let $b \in F$ be the least upper bound for S.

Now, because b is the least upper bound for S, for all $n \in \mathbb{N}$,

$$(n + 1) \cdot y \leq b.$$

By **Distributivity**,

$$n \cdot y + y \leq b.$$

For any $x \in F$, let $-x$ be the additive inverse of x. Define $x - y = x + (-y)$ for all $x, y \in F$. Since $-y$ is the additive inverse of y, by **Inverses**, $y + (-y) = y - y = 0$. Apply **Addition Preserves Order** to the last inequality to get

$$n \cdot y + y + (-y) \leq b + (-y).$$

This is equivalent to

$$n \cdot y + y - y \leq b - y.$$

Because $y - y = 0$,

$$n \cdot y \leq b - y.$$

Note that this inequality is true for all $n \in \mathbb{N}$. So, $b - y$ is an upper bound for S too. By stipulation, $0 < y$. Apply **Addition Preserves Order** to get $-y = 0 + (-y) < y + (-y) = y - y = 0$. Therefore, $-y < 0$. Apply **Addition Preserves Order** again to get $b - y = -y + b < 0 + b = b$. However, this is a contradiction, as b is supposed to be the least upper bound for S. Yet, there is an upper bound for S which is smaller than b, namely $b - y$. Therefore, $(F, +, \cdot, \leq)$ must be Archimedean. $\square$

Lemma 1. *Suppose $A \cap B = C \cap D = \emptyset$ and the following:*

1. *$A \preceq C$.*
2. *$B \preceq D$.*

Then, $A \cup B \preceq C \cup D$.

Proof. See Krantz et al. [1971, pp. 211-212]. $\square$

Lemma 2. *If $A \subseteq B$, then $A \preceq B$. For a qualitative probability $\preceq$ that satisfies **Qualitative Regularity**, if $A \subsetneq B$, then $A \prec B$.*

Proof. Assume that $A \subseteq B$. By **Non-Negativity**, $\emptyset \preceq B - A$. Apply **Additivity** to $\emptyset \preceq B - A$ to get $A = \emptyset \cup A \preceq (B - A) \cup A = B$. The first part of the lemma has been proven.

Now, I will prove the second part. Assume **Qualitative Regularity** and $A \subsetneq B$. Since $A \subsetneq B$, $B - A \neq \emptyset$. By **Qualitative Regularity**, $\emptyset \prec B - A$. Apply **Additivity** to $\emptyset \prec B - A$ to get $A = \emptyset \cup A \prec (B - A) \cup A = B$. $\square$

*Proof of **Theorem 3**.* Let A and B be any two finite subsets of $\mathbb{N}$ such that $\#(A) \leq \#(B)$. Assume the following constraints: **Non-Negativity**, **Non-Triviality**, **Transitivity**, **Additivity**, **Fairness**, and **Comparing Finite and Infinite Sets**. Note that $\#(A) = \#(A - B) + \#(A \cap B)$ and $\#(B) = \#(B - A) + \#(A \cap B)$. So, $\#(A) \leq \#(B)$ implies that $\#(A - B) \leq \#(B - A)$. Let $C \subseteq B - A$ be such that $\#(C) = \#(A - B)$. By **Fairness**, for every member $\omega \in A - B$ and every member $\omega' \in C$, $\{\omega\} \sim \{\omega'\}$. Apply **Lemma 1** repeatedly to get $A - B \preceq C$. Since $C \subseteq B - A$, $C \preceq B - A$, by **Lemma 2**. Apply **Transitivity** to $A - B \preceq C$ and $C \preceq B - A$ to get $A - B \preceq B - A$. Finally, apply **Additivity** to $A - B \preceq B - A$ to get $A = (A - B) \cup (A \cap B) \preceq (B - A) \cup (A \cap B) = B$. $\square$

Lemma 3. *Let A be a group of maximal arithmetic progressions sharing a common difference of m. For an $x \in \mathbb{N}$, if $x \notin A$, then $x + nm \notin A$ for all $n \in \mathbb{N}$.*

Proof. Assume that $x \notin A$. Next, assume for reductio that there is some $n \in \mathbb{N}$ such that $x + nm \in A$. Let S denote the maximal arithmetic progression in A containing $x + nm$. Note that because $x > 0$, $m \leq nm < x + nm$ for all $n \in \mathbb{N}$. So, $m < x + nm$ for all $n \in \mathbb{N}$. Let a_1 denote the first term of S. Because S is maximal, $a_1 \leq m < x + nm$ for all $n \in \mathbb{N}$. So, $a_1 < x + m$. This means that $x + m \in A$ since $x + nm \in A$. In fact, $x + m$ will appear in S either as the second, or third, or fourth, etc. term. This means that the preceding term is x. So, $x \in A$. Contradiction. Therefore, $x + nm \notin A$ for all $n \in \mathbb{N}$. $\square$

*Proof of **Theorem 4**.* Let $A \in \mathcal{H}$ such that A fulfills both forms $\{A_1, A_2, A_3\}$ and $\{A_1^{'}, A_2^{'}, A_3^{'}\}$, which are both ordered triples. That is, $A = (A_1 - A_2) \cup A_3 = (A_1^{'} - A_2^{'}) \cup A_3^{'}$. For reductio, assume that $A_1 \neq A_1^{'}$. This means that their extensions are different. Say that there exists an $x \in A_1$ but $x \notin A_1^{'}$. Let $m(A_1)$ and $m(A_1^{'})$ denote the shared common differences among the maximal arithmetic progressions in A_1 and $A_1^{'}$ respectively. Since $x \in A_1$, x is a term in one of the maximal arithmetic progressions in A_1. Let m^* be the lowest common multiple (LCM) between $m(A_1)$ and $m(A_1^{'})$. Since x is a term in one of the maximal arithmetic progressions in A_1, $x + nm^*$ is also a term in the same progression for all $n \in \mathbb{N}$. That is, $x + nm^* \in A_1$ for all $n \in \mathbb{N}$. As A_2 is finite, an infinite subset of $\{x, x + m^*, x + 2m^*, \ldots\}$ is a subset of A. Call this subset B. So, $B \subseteq \{x, x + m^*, x + 2m^*, \ldots\}$ and $B \subseteq A$.

Now, since $B \subseteq A$, $B \subseteq (A_1^{'} - A_2^{'}) \cup A_3^{'}$. But because B is an infinite set, $A_1^{'} \neq \emptyset$. For if it were that $A_1^{'} = \emptyset$, then $(A_1^{'} - A_2^{'}) \cup A_3^{'} = A_3^{'}$, which is finite. And infinite sets cannot be subsets of finite sets. So, $A_1^{'} \neq \emptyset$. Recall that $x \notin A_1^{'}$. By **Lemma 3**, $x + (n \times m(A_1^{'})) \notin A_1^{'}$ for all $n \in \mathbb{N}$. This includes $x + nm^*$ for all $n \in \mathbb{N}$, i.e. for all $n \in \mathbb{N}$, $x + nm^* \notin A_1^{'}$. So, $\{x, x+m^*, x+2m^*, \ldots\} \cap (A_1^{'} - A_2^{'}) = \emptyset$. But since $B \subseteq (A_1^{'} - A_2^{'}) \cup A_3^{'}$, $B \subseteq A_3^{'}$. However, this is false, as B is infinite but $A_3^{'}$ is finite. Contradiction. Therefore, there isn't an x such that $x \in A_1$ but $x \notin A_1^{'}$. The proof proceeds similarly if it is instead assumed that there exists an $x \in A_1^{'}$ but $x \notin A_1$. Therefore, $A_1 = A_1^{'}$. It is trivial to prove that $A_2 = A_2^{'}$ and $A_3 = A_3^{'}$ after it has been proven that $A_1 = A_1^{'}$. So, it will be left to the reader. □

Lemma 4. *If $A \in \mathcal{H}$, then $(A - \{a_1, \ldots, a_i\}) \cup \{b_1, \ldots, b_j\} \in \mathcal{H}$ where $\{a_1, \ldots, a_i\} \subseteq A$ and $A \cap \{b_1, \ldots, b_j\} = \emptyset$.*

Proof. Assume that $A \in \mathcal{H}$ and let $A^{'} = (A - \{a_1, \ldots, a_i\}) \cup \{b_1, \ldots, b_j\}$. This means that A fulfills the form $\{A_1, A_2, A_3\}$, which is an ordered triple. This in turn means that $A = (A_1 - A_2) \cup A_3$. Let $B = \{a_1, \ldots, a_i\} \cap A_1$, $C = \{a_1, \ldots, a_i\} - A_1$, $D = \{b_1, \ldots, b_j\} \cap A_1$, and $E = \{b_1, \ldots, b_j\} - A_1$. Then, we have $A^{'} = [A_1 - ((A_2 \cup B) - D)] \cup [(A_3 - C) \cup E]$. Note that $(A_2 \cup B) - D$ is finite and $(A_2 \cup B) - D \subseteq A_1$, as well as $(A_3 - C) \cup E$ is finite and $[(A_3 - C) \cup E] \cap A_1 = \emptyset$. So, $A^{'}$ fulfills the form $\{A_1, (A_2 \cup B) - D, (A_3 - C) \cup E\}$, which is an ordered triple. Therefore, $A^{'} \in \mathcal{H}$. □

Lemma 5. *Let $\emptyset \neq A \subsetneq \mathbb{N}$. If A is a group of maximal arithmetic progressions, then $\neg A$ is a group of maximal arithmetic progressions.*

Proof. Assume that $\emptyset \neq A \subsetneq \mathbb{N}$ is a group of maximal arithmetic progressions $S_1, \ldots, S_k$ sharing the same common difference m where $k \leq m$. Since $A \neq \mathbb{N}$, $k < m$. So, $\mathbb{N} = S_1 \cup \ldots \cup S_k \cup S_{k+1} \cup \ldots \cup S_m$, where $S_{k+1}, \ldots, S_m$ are maximal arithmetic progressions sharing the same common difference m. Every maximal arithmetic progression $S_1, \ldots, S_m$ shares the same common difference m. Since $A = S_1 \cup \ldots \cup S_k$, $\neg A = S_{k+1} \cup \ldots \cup S_m$.

Note that $m - k \leq m$. It is easy to verify that $\neg A$ is a group of maximal arithmetic progressions. $\square$

Proof of ***Theorem 5****.* It is trivial to prove that $\mathbb{N} \in \mathcal{H}$, i.e. let $A_1 = \mathbb{N}$ and $A_2 = A_3 = \emptyset$. Also, it is trivial to prove that $\emptyset \in \mathcal{H}$, i.e. let $A_1 = A_2 = A_3 = \emptyset$.

Now, I will prove that if $A \in \mathcal{H}$, then $\neg A \in \mathcal{H}$. Assume that $A \in \mathcal{H}$. If A is either $\mathbb{N}$ or $\emptyset$, then $\neg A \in \mathcal{H}$. The remaining cases are when $\emptyset \neq A \subsetneq \mathbb{N}$. Let $A = (A_1 - A_2) \cup A_3$. Either $A_1 = \emptyset$ or not. Assume the former first: $A_1 = \emptyset$. Then, $\neg A = \mathbb{N} - A_3$. This fulfills the form, so $\neg A \in \mathcal{H}$. Now, assume the latter: $A_1 \neq \emptyset$. If $A_1 = \mathbb{N}$, then $A_3 = \emptyset$ since $A_1 \cap A_3 = \emptyset$. So, $\neg A = A_2$. This fulfills the form, so $\neg A \in \mathcal{H}$. Consider now when $A_1 \neq \mathbb{N}$. By **Lemma 5**, since A_1 is a group of maximal arithmetic progressions, $\neg A_1$ is also a group of maximal arithmetic progressions. So, $\neg A = (\neg A_1 - A_3) \cup A_2$. This fulfills the form, and therefore $\neg A \in \mathcal{H}$.

Now, I will prove that if $A \in \mathcal{H}$ and $B \in \mathcal{H}$, then $A \cup B \in \mathcal{H}$. Let $A = (A_1 - A_2) \cup A_3$ and $B = (B_1 - B_2) \cup B_3$. The important case to consider is when $A_2 = A_3 = B_2 = B_3 = \emptyset$. Assume first that $A_1 \neq \emptyset$ and $B_1 \neq \emptyset$. Let $k(A)$ and $k(B)$ denote the number of maximal arithmetic progressions in A_1 and B_1 respectively and let $m(A)$ and $m(B)$ denote the common differences shared by the arithmetic progressions in A_1 and B_1 respectively. Let m' denote the lowest common multiple (LCM) of $m(A)$ and $m(B)$. Define $C = \{i \in A_1 \cup B_1 : i \leq m'\}$ and $\frac{k(A \cup B)}{m(A \cup B)} = \frac{\#(C)}{m'}$ where $\#(C)$ is the cardinality of C. Since $A_2 = A_3 = B_2 = B_3 = \emptyset$, $A \cup B$ is the union of maximal arithmetic progressions $S_1, \ldots, S_{k(A \cup B)}$ sharing the same common difference $m(A \cup B)$, where the first term of each progression is a member of C. Let $D_1 = S_1 \cup \ldots \cup S_{k(A \cup B)}$. $A \cup B = (D_1 - D_2) \cup D_3$ where $D_2 = D_3 = \emptyset$. It is easy to verify that $(D_1 - D_2) \cup D_3$ fulfills the form. So, $A \cup B \in \mathcal{H}$.

Now, consider when either $A_1 = \emptyset$ or $B_1 = \emptyset$. Say that $A_1 = \emptyset$. Since $A_2 = A_3 = B_2 = B_3 = \emptyset$, $A = \emptyset$. Then, $A \cup B \in \mathcal{H}$ because $A \cup B = B$ and it is assumed that $B \in \mathcal{H}$. The proof proceeds the same way if $B_1 = \emptyset$ instead.

The rest of the cases where it isn't true that $A_2 = A_3 = B_2 = B_3 = \emptyset$ are handled by **Lemma 4**. First, note that $A_1 \in \mathcal{H}$ and $B_1 \in \mathcal{H}$. As proven above, this means that $A_1 \cup B_1 \in \mathcal{H}$. If at least one of A_2, A_3, B_2, or B_3 is nonempty, then $A \cup B = [(A_1 \cup B_1) - \{a_1, \ldots, a_i\}] \cup \{b_1, \ldots, b_j\}$ for some $\{a_1, \ldots, a_i\}$ and $\{b_1, \ldots, b_j\}$ where $\{a_1, \ldots, a_i\} \subseteq A_1 \cup B_1$ and $(A_1 \cup B_1) \cap \{b_1, \ldots, b_j\} = \emptyset$. So, since $A_1 \cup B_1 \in \mathcal{H}$, $A \cup B \in \mathcal{H}$ too, by **Lemma 4**. $\square$

Lemma 6. *Let* $A = (A_1 - A_2) \cup A_3 \in \mathcal{F}^{\#}$ *and* $B = (B_1 - B_2) \cup B_3 \in \mathcal{F}^{\#}$*. If* $A \cap B = \emptyset$*, then* $A_1 \cap B_1 = A_3 \cap B_3 = \emptyset$*. Moreover, if* $A \cap B = \emptyset$*, then* $A_3 \cap B_1 = A_3 \cap B_2$ *and* $B_3 \cap A_1 = B_3 \cap A_2$*.*

Proof. Assume that $A \cap B = \emptyset$. Next, assume for reductio that $A_1 \cap B_1 \neq \emptyset$. This means that there exists an $x \in A_1 \cap B_1$. Let $m(A)$ and $m(B)$ denote the common differences shared by the arithmetic progressions in A_1 and B_1 respectively and let $m^{'}$ be the lowest common multiple (LCM) of $m(A)$ and $m(B)$. Since $x \in A_1 \cap B_1$, $x + nm^{'} \in A_1 \cap B_1$ for all $n \in \mathbb{N}$. This means that there are infinitely many members in the intersection $A_1 \cap B_1$. All these members have to be members of $A_2 \cup B_2$ for $A \cap B = \emptyset$. But since $A_2 \cup B_2$ is finite, there is a contradiction. Therefore, $A_1 \cap B_1 = \emptyset$.

Now, assume for reductio that $A_3 \cap B_3 \neq \emptyset$. This means that there exists an $x \in A_3 \cap B_3$. So, $x \in A_3$ and $x \in B_3$. This means that $x \in A$ and $x \in B$, so $x \in A \cap B$. But $A \cap B = \emptyset$. Contradiction. Therefore, $A_3 \cap B_3 = \emptyset$.

I will now prove the second part of the lemma. Assume that $A \cap B = \emptyset$. In order to prove that $A_3 \cap B_1 = A_3 \cap B_2$, I will prove that $A_3 \cap B_1 \subseteq A_3 \cap B_2$ and $A_3 \cap B_2 \subseteq A_3 \cap B_1$. Proving the latter is trivial as $B_2 \subseteq B_1$. So, I will just prove the former. Let $x \in A_3 \cap B_1$. So, $x \in A_3$ and $x \in B_1$. It must be that $x \in B_2$. For if it were that $x \in B_1 - B_2$, then $x \in B$. Since $x \in A_3$, $x \in A$ too. This would mean that $x \in A \cap B$. But $A \cap B = \emptyset$. Contradiction. So, it must be that $x \in B_2$. Because $x \in A_3$ and $x \in B_2$, $x \in A_3 \cap B_2$. Therefore, $A_3 \cap B_1 \subseteq A_3 \cap B_2$. Since $A_3 \cap B_2 \subseteq A_3 \cap B_1$ too, $A_3 \cap B_1 = A_3 \cap B_2$. It can be proven in similar ways that $B_3 \cap A_1 = B_3 \cap A_2$. ☐

Lemma 7. *Let* $A = (A_1 - A_2) \cup A_3 \in \mathcal{F}^{\#}$, $B = (B_1 - B_2) \cup B_3 \in \mathcal{F}^{\#}$, $C_1 = A_1 \cup B_1$, $C_2 = (A_2 - B_3) \cup (B_2 - A_3)$, *and* $C_3 = (A_3 - B_1) \cup (B_3 - A_1)$. *If* $A \cap B = \emptyset$, *then the ordered triple* $\{C_1, C_2, C_3\}$ *is the form* $A \cup B$ *fulfills.*

Proof. I will first prove that C_1, C_2, and C_3 fulfill the conditions spelled out in the definition of $\mathcal{H}$. Assume that $A \cap B = \emptyset$. This means that $A_1 \cap B_1 = \emptyset$ by **Lemma 6**. Assume first that $A_1 \neq \emptyset$ and $B_1 \neq \emptyset$. Let $m(A)$ and $m(B)$ denote the common differences shared among the maximal arithmetic progressions in A_1 and B_1 respectively. Let $m^{'}$ denote the lowest common multiple (LCM) of $m(A)$ and $m(B)$. Also, let $k(A)$ and $k(B)$ denote the number of arithmetic progressions in A_1 and B_1 respectively. Define $D = \{i \in A_1 \cup B_1 : i \leq m^{'}\}$ and $\frac{k(A \cup B)}{m(A \cup B)} = \frac{\#(D)}{m^{'}}$. $C_1 = A_1 \cup B_1$ is the union of maximal arithmetic progressions $S_1, \ldots, S_{k(A \cup B)}$ sharing the same common difference $m(A \cup B)$, where the first term of each progression is a member of D. Therefore, C_1 is a group of maximal arithmetic progressions, which in turn implies that C_1 fulfills the conditions spelled out in the definition of $\mathcal{H}$.

Now, consider when either $A_1 = \emptyset$ or $B_1 = \emptyset$. Say that $A_1 = \emptyset$. Then, $C_1 = B_1$ as $A_1 \cup B_1 = B_1$. If, instead, $B_1 = \emptyset$, then $C_1 = A_1$. Either way, C_1 fulfills the conditions spelled out in the definition of $\mathcal{H}$, as both A_1 and B_1 fulfill those conditions.

I will prove that $C_2 \subseteq C_1$. Let $x \in C_2 = (A_2 - B_3) \cup (B_2 - A_3)$. Either $x \in A_2 - B_3$ or $x \in B_2 - A_3$. In either case, $x \in A_1 \cup B_1 = C_1$. So, $C_2 \subseteq C_1$. Note that C_2 is finite too. Therefore, C_2 fulfills the conditions spelled out in the definition of $\mathcal{H}$.

Also, $C_1 \cap C_3 = \emptyset$. For reductio, assume that $x \in C_1 \cap C_3$. Then, $x \in C_1 = A_1 \cup B_1$ and $x \in C_3 = (A_3 - B_1) \cup (B_3 - A_1)$. Either $x \in A_1$ or $x \in B_1$. If the former is true, then $x \notin A_3 - B_1$ because $A_1 \cap A_3 = \emptyset$. Also, $x \notin B_3 - A_1$. This means that $x \notin C_3$. Contradiction. If the latter is true, then $x \notin A_3 - B_1$. Also, $x \notin B_3 - A_1$ because $B_1 \cap B_3 = \emptyset$. This means that $x \notin C_3$. Contradiction. Either way, there is a contradiction. Therefore, $C_1 \cap C_3 = \emptyset$. Note also that C_3 is finite. So, C_3 fulfills the conditions spelled out in the definition of $\mathcal{H}$.

The proof above shows that C_1, C_2, and C_3 fulfill the conditions spelled out in the definition of $\mathcal{H}$. All that remains to be proven is that $A \cup B = (C_1 - C_2) \cup C_3$. Since the proof of this claim is trivial but tedious, it will be left to the reader. □

*Proof of **Theorem 6**.* Let $A = (A_1 - A_2) \cup A_3$ and $B = (B_1 - B_2) \cup B_3$. Assume that $A \cap B = \emptyset$. $N(A) = \#(A_3) - \#(A_2)$ and $N(B) = \#(B_3) - \#(B_2)$. By **Lemma 7**, $A \cup B = (C_1 - C_2) \cup C_3$, as defined there. $N(A \cup B) = \#(C_3) - \#(C_2)$. Recall that, by **Lemma 6**, if $A \cap B = \emptyset$, then $A_1 \cap B_1 = A_3 \cap B_3 = \emptyset$. This means that $A_2 \cap B_2 = \emptyset$ too. So, the following are true:

$$\#(C_2) = \#(A_2 - B_3) + \#(B_2 - A_3). \tag{3}$$

$$\#(C_3) = \#(A_3 - B_1) + \#(B_3 - A_1). \tag{4}$$

These equations are equivalent to the following equations respectively.

$$\#(C_2) = (\#(A_2) - \#(A_2 \cap B_3)) + (\#(B_2) - \#(B_2 \cap A_3)). \tag{5}$$

$$\#(C_3) = (\#(A_3) - \#(A_3 \cap B_1)) + (\#(B_3) - \#(B_3 \cap A_1)). \tag{6}$$

As $A_3 \cap B_1 = A_3 \cap B_2$ and $B_3 \cap A_1 = B_3 \cap A_2$ by **Lemma 6**, the latter equation is equivalent to

$$\#(C_3) = (\#(A_3) - \#(A_3 \cap B_2)) + (\#(B_3) - \#(B_3 \cap A_2)). \tag{7}$$

This means that

$$\#(C_3) - \#(C_2) = \#(A_3) + \#(B_3) - \#(A_2) - \#(B_2). \tag{8}$$

$\#(A_3) + \#(B_3) - \#(A_2) - \#(B_2) = N(A) + N(B)$. So, $\#(C_3) - \#(C_2) = N(A) + N(B)$. Since $N(A \cup B) = \#(C_3) - \#(C_2)$, $N(A \cup B) = N(A) + N(B)$. □

Lemma 8. *Assume the constraints in $C^{\#}$. For any two finite sets $A \in \mathcal{F}^{\#}$ and $B \in \mathcal{F}^{\#}$, if $A \sim B$, then $\#(A) = \#(B)$.*

Proof. Assume that $A \sim B$. For reductio, assume that $\#(A) < \#(B)$. This implies that $\#(A - B) < \#(B - A)$. Let $C \subseteq B - A$ such that $\#(C) = \#(A - B)$. By **Fairness**, for every

$\omega \in C$ and $\omega' \in A-B$, $\{\omega\} \sim \{\omega'\}$. Apply **Lemma 1** repeatedly to get $C \sim A-B$. Since $\#(A-B) < \#(B-A)$, $(B-A)-C \neq \emptyset$. By **Qualitative Regularity**, $\emptyset \prec (B-A)-C$. Apply **Lemma 1** to $A-B \sim C$ and $\emptyset \prec (B-A)-C$ to get $A-B \prec B-A$. Apply **Additivity** to get $A = (A-B) \cup (A \cap B) \prec (B-A) \cup (A \cap B) = B$. Contradiction, as $A \sim B$. So, $\#(A) \not< \#(B)$. It can be similarly proven that $\#(A) \not> \#(B)$. Therefore, $\#(A) = \#(B)$. □

Proof of ***Theorem 7****.* Assume the constraints in $C^{\#}$. ($\Rightarrow$) Now, assume that $A \preceq B$. There are 4 cases to consider.

Case 1: both A and B are finite. Then, $d(A) = d(B) = 0$. For reductio, assume that $N(A) > N(B)$. Note that for all finite sets $A \in \mathcal{F}^{\#}$, $N(A) = \#(A)$. So, $N(A) > N(B)$ implies that $\#(A) > \#(B)$. By **Theorem 3**, $A \succeq B$. Since $A \preceq B$ and $A \succeq B$, by **Definition 1**, $A \sim B$. By **Lemma 8**, $\#(A) = \#(B)$. Contradiction, as $\#(A) > \#(B)$. Therefore, $N(A) \leq N(B)$. We have $d(A) = d(B)$ and $N(A) \leq N(B)$, which is condition 2 of **Regular Ordering**.

Case 2: A is finite and B is infinite. Then, $d(A) = 0$. Since $B = (B_1 - B_2) \cup B_3$ is infinite, $B_1 \neq \emptyset$. By the definition of d, $d(B) = \frac{k}{m}$ as $B_1 \neq \emptyset$ where k is the number of maximal arithmetic progressions in B_1 and m is the shared common difference among all such arithmetic progressions. This implies that $d(B) > 0$. So, $0 = d(A) < d(B)$, which is condition 1 of **Regular Ordering**.

Case 3: A is infinite and B is finite. By **Comparing Finite and Infinite Sets**, $A \succeq B$. Since $A \preceq B$ and $A \succeq B$, by **Definition 1**, $A \sim B$. Let $C \subseteq A$ such that $\#(C) = \#(B)$. By **Theorem 3**, $C \preceq B$ and $B \preceq C$. By **Definition 1**, $B \sim C$. Apply **Transitivity** to $A \sim B$ and $B \sim C$ to get $A \sim C$. But since C is finite and A is infinite, $C \subsetneq A$, i.e. $A - C \neq \emptyset$. By **Qualitative Regularity**, $\emptyset \prec A - C$. Apply **Additivity** to get $C = \emptyset \cup C \prec (A - C) \cup C = A$. Contradiction, as $C \sim A$. So, this case can be ignored.

Case 4: both $A = (A_1 - A_2) \cup A_3$ and $B = (B_1 - B_2) \cup B_3$ are infinite. Since both A and B are infinite, $A_1 \neq \emptyset$ and $B_1 \neq \emptyset$. This implies that $d(A) > 0$ and $d(B) > 0$, by the definition of d. **Partition** will yield both conditions of **Regular Ordering**. This completes the proof of the left-to-right direction of **Regular Ordering**.

($\Leftarrow$) First, assume that $d(A) < d(B)$. By condition 1 of **Partition**, $A \preceq B$. Now, assume that $d(A) = d(B)$ and $N(A) \leq N(B)$. By condition 2 of **Partition**, $A \preceq B$. This completes the proof of the right-to-left direction of **Regular Ordering**. □

Proof of ***Theorem 8****.* The proof is straightforward. Just notice that for any two sets $A \in \mathcal{F}^{\#}$ and $B \in \mathcal{F}^{\#}$, either $d(A) \leq d(B)$ or $d(A) \geq d(B)$. Also, either $N(A) \leq N(B)$ or $N(A) \geq N(B)$. □

Proof of ***Corollary 1****.* This corollary is a direct consequence of **Theorem 7** and **Theorem 8**. □

*Proof of **Corollary 2**.* Since each item in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is specified and $C^{\#}$ fully determines the order of events in $\mathcal{F}^{\#}$ (**Corollary 1**), $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is fully specified. $\square$

*Proof of **Theorem 9**.* Assume **Precise Representation**. Showing that P_1 satisfies **Non-Negativity**, **Non-Triviality**, **Transitivity**, and **Additivity** is trivial. Therefore, the proofs of these claims will be left to the reader.

By the definition of P_1, $P_1(\{i\}) = P_1(\{j\}) = \varepsilon$. After all, $N(\{i\}) = N(\{j\}) = 1$ and $d(\{i\}) = d(\{j\}) = 0$. Apply **Precise Representation** to get $\{i\} \sim \{j\}$. Therefore, P_1 satisfies **Fairness**.

Let $A \in \mathcal{F}^{\#}$ be a finite set and $B = (B_1 - B_2) \cup B_3 \in \mathcal{F}^{\#}$ be an infinite set. Since A is a finite set, $d(A) = 0$. Since B is an infinite set, $B_1 \neq \emptyset$. This means that $d(B) > 0$. So, $0 = d(A) < d(B)$. Since $d(A) < d(B)$, $P_1(A) < P_1(B)$, by the definition of P_1. Apply **Precise Representation** to get $A \preceq B$. Therefore, P_1 satisfies **Comparing Finite and Infinite Sets**.

By the definition of P_1, $P_1(\emptyset) = 0$. After all, $d(\emptyset) = 0$ and $N(\emptyset) = 0$. Consider a nonempty finite set $A = (A_1 - A_2) \cup A_3 \in \mathcal{F}^{\#}$ first. Since A is finite, $A_1 = \emptyset$. This means that $d(A) = 0$. So, $N(A) = \#(A_3) = \#(A)$. Since A is nonempty, $N(A) > 0$. So, for all nonempty finite sets $A \in \mathcal{F}^{\#}$, $P_1(A) > 0$, by the definition of P_1. Now, consider an infinite set $B = (B_1 - B_2) \cup B_3 \in \mathcal{F}^{\#}$. Since B is infinite, $B_1 \neq \emptyset$. This means that $d(B) > 0$. So, for all infinite sets $B \in \mathcal{F}^{\#}$, $P_1(B) > 0$, by the definition of P_1. This means that for all nonempty sets $A \in \mathcal{F}^{\#}$, $P_1(A) > 0 = P(\emptyset)$. Apply **Precise Representation** to get $\emptyset \prec A$. Therefore, P_1 satisfies **Qualitative Regularity**.

Consider the sets $(A - A^{'}) \cup A^{*}$ and $(B - B^{'}) \cup B^{*}$, as defined in **Partition**. Note that $P_1\big((A - A^{'}) \cup A^{*}\big) = P_1(A) - P_1(A^{'}) + P(A^{*})$. Similarly, $P_1\big((B - B^{'}) \cup B^{*}\big) = P_1(B) - P_1(B^{'}) + P(B^{*})$. Since A is the union of p maximal arithmetic progressions, all of which share the same common difference of n, $d(A) = \frac{p}{n}$ and $N(A) = 0$. So, $P_1(A) = \frac{p}{n}$. The same reasoning applies to B, i.e. $P_1(B) = \frac{q}{m}$. Because $A^{'}$, A^{*}, $B^{'}$, and B^{*} are finite, each set has 0 asymptotic density. Furthermore, $N(A^{'}) = \#(A^{'})$, $N(A^{*}) = \#(A^{*})$, $N(B^{'}) = \#(B^{'})$, and $N(B^{*}) = \#(B^{*})$. This means that $P_1(A^{'}) = \varepsilon \cdot \#(A^{'})$, $P_1(A^{*}) = \varepsilon \cdot \#(A^{*})$, $P_1(B^{'}) = \varepsilon \cdot \#(B^{'})$, and $P_1(B^{*}) = \varepsilon \cdot \#(B^{*})$. So, $P_1\big((A - A^{'}) \cup A^{*}\big) = \frac{p}{n} - \big(\varepsilon \cdot \#(A^{'})\big) + \big(\varepsilon \cdot \#(A^{*})\big)$ and $P_1\big((B - B^{'}) \cup B^{*}\big) = \frac{q}{m} - \big(\varepsilon \cdot \#(B^{'})\big) + \big(\varepsilon \cdot \#(B^{*})\big)$.

$P_1\big((A - A^{'}) \cup A^{*}\big) \leq P_1\big((B - B^{'}) \cup B^{*}\big)$ iff either of the following holds:

1. $\frac{p}{n} < \frac{q}{m}$.
2. $\frac{p}{n} = \frac{q}{m}$ and $\big(\varepsilon \cdot \#(A^{*})\big) - \big(\varepsilon \cdot \#(A^{'})\big) \leq \big(\varepsilon \cdot \#(B^{*})\big) - \big(\varepsilon \cdot \#(B^{'})\big)$.

The second inequality in the second condition above can be simplified by dividing ε throughout to get $\#(A^{*}) - \#(A^{'}) \leq \#(B^{*}) - \#(B^{'})$. Apply **Precise Representation** to see that P_1 satisfies **Partition**. $\square$

Proof of ***Theorem 10***. Assume **Precise Representation**. Let's start with the easiest condition of **Infinitesimal Underdetermination**: condition 3. Consider the set $\{i\}$. Note that $d(\{i\}) = 0$ and $N(\{i\}) = 1$. By the definition of P_1, $P_1(\{i\}) = \varepsilon$. But by the definition of P', $P'(\{i\}) = r \cdot \varepsilon$ where $r \in \mathbb{R}^+ - \{1\}$. So, $P_1(\{i\}) \neq P'(\{i\})$. Therefore, P_1 and P' satisfy condition 3 of **Infinitesimal Underdetermination**.

Consider condition 2 of **Infinitesimal Underdetermination** now. Recall the definitions of P_1 and P', i.e. $P_1(A) = d(A) + \big(\varepsilon \cdot N(A)\big)$ and $P'(A) = d(A) + \big(r \cdot \varepsilon \cdot N(A)\big)$. For $P_1(A) \leq P_1(B)$, either (i) $d(A) < d(B)$ or (ii) $d(A) = d(B)$ and $\big(\varepsilon \cdot N(A)\big) \leq \big(\varepsilon \cdot N(B)\big)$. The same goes for P', i.e. for $P'(A) \leq P'(B)$, either (i) $d(A) < d(B)$ or (ii) $d(A) = d(B)$ and $\big(r \cdot \varepsilon \cdot N(A)\big) \leq \big(r \cdot \varepsilon \cdot N(B)\big)$. Assume that $P_1(A) \leq P_1(B)$. If $d(A) < d(B)$, then $P'(A) \leq P'(B)$ too. If $d(A) = d(B)$ and $\big(\varepsilon \cdot N(A)\big) \leq \big(\varepsilon \cdot N(B)\big)$, then, by multiplying r on both sides of the latter, $\big(r \cdot \varepsilon \cdot N(A)\big) \leq \big(r \cdot \varepsilon \cdot N(B)\big)$. This means that $P'(A) \leq P'(B)$. Either way, if $P_1(A) \leq P_1(B)$, then $P'(A) \leq P'(B)$. It can be similarly proven that if $P'(A) \leq P'(B)$, then $P_1(A) \leq P_1(B)$. So, $P_1(A) \leq P_1(B)$ iff $P'(A) \leq P'(B)$. Therefore, P_1 and P' satisfy condition 2 of **Infinitesimal Underdetermination**.

Consider condition 1 of **Infinitesimal Underdetermination** now. In virtue of P_1 and P' satisfying condition 2 of **Infinitesimal Underdetermination** and P_1 satisfying all the constraints in $C^{\#}$ under **Precise Representation** (**Theorem 9**), P' satisfies all the constraints in $C^{\#}$ under **Precise Representation** too. By the definitions of P_1 and P', both functions are defined on $\mathcal{F}^{\#}$. Therefore, P_1 and P' satisfy condition 1 of **Infinitesimal Underdetermination**.

Last, consider condition 4 of **Infinitesimal Underdetermination**. First, note that

$$P_1(A) - P'(A) = \big(\varepsilon \cdot N(A)\big) - \big(r \cdot \varepsilon \cdot N(A)\big) = \big(\varepsilon \cdot N(A)\big) \cdot (1 - r). \tag{9}$$

So, $|P_1(A) - P'(A)| = |\big(\varepsilon \cdot N(A)\big) \cdot (1 - r)|$. Since $|\big(\varepsilon \cdot N(A)\big)| < \frac{1}{n}$ for all $n \in \mathbb{N}$, $|\big(\varepsilon \cdot N(A)\big) \cdot (1 - r)| < \frac{1}{n}$ for all $n \in \mathbb{N}$. This means that $|P_1(A) - P'(A)| < \frac{1}{n}$ for all $n \in \mathbb{N}$. Therefore, P_1 and P' satisfy condition 4 of **Infinitesimal Underdetermination**. □

Proof of ***Theorem 11***. Assume **Precise Representation**. Let P be a probability function correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{*} \rangle$. Now, recall the four kinds of subsets of $\mathbb{N}$ that are in $\mathcal{F}^{\#}$, as listed at the top of §3.2.2. I will consider each in turn.

The first kind: all finite subsets of $\mathbb{N}$. Let A and B be any two finite subsets of $\mathbb{N}$. Note that $d(A) = d(B) = 0$. So, by **Irregular Ordering** (applied twice), $A \sim' B$. Apply **Precise Representation** to get $P(A) = P(B)$. Since A and B can be any two finite subsets of $\mathbb{N}$, let A be any finite subset and $B = \emptyset$. So, $P(A) = P(\emptyset) = 0 = d(A)$.

The second kind: all cofinite subsets of $\mathbb{N}$. Let A be any cofinite subset of $\mathbb{N}$. Note that $d(A) = 1$. Since $\neg A$ is a finite subset, $P(\neg A) = 0$, as proven above. So, $P(A) = 1 = d(A)$.

The third kind: every set G that is a nonempty union of any number of G_i's, as defined below. Let $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ be a partition of $\mathbb{N}$ such that:

$$G_1 = \{1, 1+n, 1+2n, \ldots\}.$$

$$G_2 = \{2, 2+n, 2+2n, \ldots\}.$$

$$\ldots$$

$$G_n = \{n, 2n, 3n, \ldots\}.$$

Note that $d(G_i) = \frac{1}{n}$. So, by **Irregular Ordering**, $G_i \sim' G_j$. Apply **Precise Representation** to get $P(G_i) = P(G_j)$. Since $\sum_{i=1}^{n} P(G_i) = 1$, $P(G_i) = \frac{1}{n}$. Let $\mathcal{I} = \{i : G_i \subseteq G\}$. Because $P(G_i) = \frac{1}{n}$ for $1 \leq i \leq n$, $P(G) = \frac{\#(\mathcal{I})}{n}$. Note that $d(G) = \frac{\#(\mathcal{I})}{n}$. Therefore, $P(G) = d(G)$.

The fourth kind: every set H such that $H = (G - G') \cup G^*$, as defined in §3.2.2. Note that $P(H) = P(G) - P(G') + P(G^*)$. Since G' and G^* are finite sets, $P(G') = P(G^*) = 0$, as proven above. So, $P(H) = P(G) = \frac{\#(\mathcal{I})}{n} = d(G)$. Because G' and G^* are finite sets, $d(H) = d(G)$. Therefore, $P(H) = d(G) = d(H)$. □

*Proof of **Theorem 12**.* The proof is straightforward and therefore left to the reader. □

*Proof of **Theorem 13**.* I will first prove that $\preceq'$ does not satisfy **Qualitative Regularity**. Consider the set $\{i\}$. Note that $d(\{i\}) = 0 = d(\emptyset)$. Therefore, by **Irregular Ordering**, $\emptyset \sim' \{i\}$, violating **Qualitative Regularity**.

Now, I will prove that $\preceq'$ does not satisfy **Partition**. According to **Partition**, $\{4, 6, 8, \ldots\}$ is less probable than $\{2, 4, 6, \ldots\}$. But according to **Irregular Ordering**, $\{4, 6, 8, \ldots\} \sim' \{2, 4, 6, \ldots\}$ since $d(\{4, 6, 8, \ldots\}) = d(\{2, 4, 6, \ldots\}) = \frac{1}{2}$. □

*Proof of **Theorem 14**.* Assume **Comparing Imprecise Probabilities** and **Imprecise Representation**. By the definition of $P_{im(4)}$, $P_{im(4)}$ is defined on $\mathcal{F}^{\#}$. All that remains to be proven is that $P_{im(4)}$ satisfies all the constraints in $C^{\#}$. Take any two sets $A \in \mathcal{F}^{\#}$ and $B \in \mathcal{F}^{\#}$ such that according to the constraints in $C^{\#}$, $A \preceq B$. Now, in order to prove that $P_{im(4)}$ satisfies all the constraints in $C^{\#}$ under **Imprecise Representation**, it needs to be shown that $P_{im(4)}(A) \leq P_{im(4)}(B)$. And to show that $P_{im(4)}(A) \leq P_{im(4)}(B)$, it needs to be shown that for every probability function $P \in \mathcal{S}$, $P(A) \leq P(B)$, by **Comparing Imprecise Probabilities**.

Now, let P be an arbitrary probability function in $\mathcal{S}$. Since $\mathcal{S}$ is the set of all precise probability functions that correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, P satisfies all the constraints in $C^{\#}$. Recall that $A \preceq B$, according to the constraints in $C^{\#}$. By **Precise Representation**, $P(A) \leq P(B)$. The proof proceeds similarly for any two sets $A \in \mathcal{F}^{\#}$ and $B \in \mathcal{F}^{\#}$ such that according to the constraints in $C^{\#}$, $A \succeq B$. Note that according to **Theorem 8**, there do not exist sets $A \in \mathcal{F}^{\#}$ and $B \in \mathcal{F}^{\#}$ such that $A \not\preceq B$ and $A \not\succeq B$. □

*Proof of **Theorem 15**.* The proof proceeds in a similar way as that of **Theorem 9**. Just replace 'ε' with 'x'. $\square$

Lemma 9. *For any precise probability function P that correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ under **Precise Representation**, $P(A) = d(A) + \big(x \cdot N(A)\big)$ for some x such that $0 < x < \frac{1}{n}$ for all $n \in \mathbb{N}$.*

Proof. Let P be a precise probability function that correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ under **Precise Representation**. Consider any singleton set $\{i\}$. Since **Qualitative Regularity** is one of the constraints in $C^{\#}$, $\emptyset \prec \{i\}$. By **Precise Representation**, $0 = P(\emptyset) < P(\{i\})$. Because $\mathbb{N}$ is infinite, $P(\{i\}) < \frac{1}{n}$ for all $n \in \mathbb{N}$. So, $0 < P(\{i\}) < \frac{1}{n}$ for all $n \in \mathbb{N}$.

Now, consider any set $A = (A_1 - A_2) \cup A_3 \in \mathcal{F}^{\#}$ fulfilling all the conditions spelled out in §3.2.2. Note that $P(A) = P(A_1) - P(A_2) + P(A_3)$. But since A_2 and A_3 are finite sets, $P(A_2) = \#(A_2) \cdot P(\{i\})$ and $P(A_3) = \#(A_3) \cdot P(\{i\})$.

$$P(A) = P(A_1) - P(A_2) + P(A_3) = P(A_1) - \big(\#(A_2) \cdot P(\{i\})\big) + \big(\#(A_3) \cdot P(\{i\})\big). \tag{10}$$

This equation is equivalent to

$$P(A) = P(A_1) + P(\{i\}) \cdot \big(\#(A_3) - \#(A_2)\big). \tag{11}$$

Recall that $N(A) = \#(A_3) - \#(A_2)$. So, the equation above is equivalent to

$$P(A) = P(A_1) + \big(P(\{i\}) \cdot N(A)\big). \tag{12}$$

Consider now the following partition $\mathcal{G} = \{G_1, \ldots, G_n\}$ of $\mathbb{N}$:

$$G_1 = \{1, 1+n, 1+2n, \ldots\}.$$

$$G_2 = \{2, 2+n, 2+2n, \ldots\}.$$

$$\cdots$$

$$G_n = \{n, 2n, 3n, \ldots\}.$$

Since A_1 is a group of maximal arithmetic progressions, A_1 must be the union of a number of G_i's for some $\mathcal{G}$, i.e. $A_1 = \bigcup_{k=1}^{p} G_{i_k}$ for some $\mathcal{G}$. Note that $d(G_i) = \frac{1}{n}$ and thus $d\big(\bigcup_{k=1}^{p} G_{i_k}\big) = \frac{p}{n} = d(A_1)$. Further note that $N(G_i) = 0$.

Recall that the constraints in $C^{\#}$ imply **Regular Ordering** (**Theorem 7**). So, by **Regular Ordering**, $G_i \sim G_j$. After all, $d(G_i) = d(G_j) = \frac{1}{n}$ and $N(G_i) = N(G_j) = 0$. Apply **Precise Representation** to $G_i \sim G_j$ to get $P(G_i) = P(G_j)$. Since $\sum_{i=1}^{n} P(G_i) = 1$, $P(G_i) = \frac{1}{n}$. This means that $P\big(\bigcup_{k=1}^{p} G_{i_k}\big) = \frac{p}{n} = P(A_1)$. Since $d(A_1) = \frac{p}{n}$, $P(A_1) =$

$d(A_1)$. Furthermore, because $A = (A_1 - A_2) \cup A_3$ where $A_2 \cup A_3$ is finite, $d(A) = d(A_1)$. So, $P(A_1) = d(A_1) = d(A)$.

Therefore, equation 12 above is equivalent to

$$P(A) = d(A) + \big(P(\{i\}) \cdot N(A)\big). \tag{13}$$

Let $P(\{i\}) = x$ for some x such that $0 < x < \frac{1}{n}$ for all $n \in \mathbb{N}$. We have $P(A) = d(A) + \big(x \cdot N(A)\big)$. $\square$

*Proof of **Theorem 16**.* By **Corollary 2**, $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is fully specified. Now, let P_x and $P_{x'}$ be two precise probability functions correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. By **Lemma 9**, $P_x(A) = d(A) + \big(x \cdot N(A)\big)$ for some x such that $0 < x < \frac{1}{n}$ for all $n \in \mathbb{N}$. Similarly, $P_{x'}(A) = d(A) + \big(x' \cdot N(A)\big)$ for some x' such that $0 < x' < \frac{1}{n}$ for all $n \in \mathbb{N}$. Stipulate that $P_x(A) < P_{x'}(A)$. Let a^* be a value such that $P_x(A) \leq a^* \leq P_{x'}(A)$.

$$d(A) + \big(x \cdot N(A)\big) \leq a^* \leq d(A) + \big(x' \cdot N(A)\big). \tag{14}$$

Based on the inequality above, $a^* = d(A) + \big(x^* \cdot N(A)\big)$ for some x^* such that $0 < x^* < \frac{1}{n}$ for all $n \in \mathbb{N}$. By **Lemma 9**, there exists a precise probability function P_{x^*} such that $P_{x^*}(A) = d(A) + \big(x^* \cdot N(A)\big)$. $\square$

*Proof of **Theorem 17**.* ($\Rightarrow$) Assume that $0 \leq u \leq 1$ is an upper bound for $\mathcal{S}_{\{i\}}$. Next, assume for reductio that $u \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. This means that $u \neq \frac{1}{n}$ for all $n \in \mathbb{N}$. So, $u < \frac{1}{n}$ for all $n \in \mathbb{N}$. Since $\mathcal{S}_{\{i\}}$ is the set of all positive infinitesimals in $\mathbb{F}$ and u is an upper bound for $\mathcal{S}_{\{i\}}$, $u > 0$. Therefore, $0 < u < \frac{1}{n}$ for all $n \in \mathbb{N}$. Now, take an arbitrary $n \in \mathbb{N}$. This means that $2n \in \mathbb{N}$ too. Because $0 < u < \frac{1}{n}$ for all $n \in \mathbb{N}$, $0 < u < \frac{1}{2n}$. Multiply 2 throughout to get $0 < 2u < \frac{1}{n}$. Since n is arbitrary, $0 < 2u < \frac{1}{n}$ for all $n \in \mathbb{N}$. This implies that $2u \in \mathcal{S}_{\{i\}}$. Because $u > 0$, $2u > u$. There exists a member in $\mathcal{S}_{\{i\}}$ that is greater than u. But it was assumed that u is an upper bound for $\mathcal{S}_{\{i\}}$. Contradiction. Therefore, $u > \frac{1}{n}$ for some $n \in \mathbb{N}$.

($\Leftarrow$) Proving this direction is straightforward. $\square$

*Proof of **Theorem 18**.* Let $P \in \mathcal{S}$ be an arbitrary precise probability function correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ under **Precise Representation**. By **Lemma 9**, $P(A) = d(A) + \big(x \cdot N(A)\big)$ for some x such that $0 < x < \frac{1}{n}$ for all $n \in \mathbb{N}$. By the definition of P, for all sets $A \in \mathcal{F}^{\#}$ such that $N(A) = 0$, $P(A) = d(A)$. So, for all such A's, $\mathcal{S}_A = \{d(A)\}$. By the definition of $P_{im(u)}$, for all such A's, $P_{im(u)}(A) = d(A)$, in accordance with the definition of an imprecise probability function.

Now, consider sets $A \in \mathcal{F}^{\#}$ such that $N(A) > 0$. By the definition of P, for all such A's, $P(A) = d(A) + \big(x \cdot N(A)\big) > d(A)$. Fix an arbitrary $n \in \mathbb{N}$. This means that $n \cdot N(A) \in \mathbb{N}$ too. Since $0 < x < \frac{1}{n}$ for all $n \in \mathbb{N}$, $x < \frac{1}{n \cdot N(A)}$. Multiply $N(A)$ throughout to get $x \cdot N(A) < \frac{1}{n}$. Add $d(A)$ on both sides to get $P(A) = d(A) + \big(x \cdot N(A)\big) < d(A) + \frac{1}{n}$.

Since n is arbitrary, $P(A) < d(A) + \frac{1}{n}$ for all $n \in \mathbb{N}$. So, for all such A's, $\mathcal{S}_A = \{x \in \mathbb{F} : \forall n \in \mathbb{N}(d(A) < x < d(A) + \frac{1}{n})\}$. A lower bound for $\mathcal{S}_A$ is $d(A)$ and an upper bound for the same set is $d(A) + u$. After all, u is an upper bound for $\mathcal{S}_{\{i\}}$, i.e. $u > \frac{1}{n}$ for some $n \in \mathbb{N}$ (**Theorem 17**). Add $d(A)$ on both sides to get $d(A) + u > d(A) + \frac{1}{n}$. This means that $d(A) + u$ is an upper bound for $\mathcal{S}_A$ for all such A's. By the definition of $P_{im(u)}$, for all such A's, $P_{im(u)}(A) = (d(A), d(A) + u)$, if $d(A) + u \leq 1$. If $d(A) + u > 1$, then $P_{im(u)}(A) = (d(A), 1)$. This is in accordance with the definition of an imprecise probability function.

Now, consider sets $A \in \mathcal{F}^{\#}$ such that $N(A) < 0$. By the definition of P, for all such A's, $P(A) = d(A) + \big(x \cdot N(A)\big) < d(A)$. Fix an arbitrary $n \in \mathbb{N}$. This means that $-n \cdot N(A) \in \mathbb{N}$ too. Since $0 < x < \frac{1}{n}$ for all $n \in \mathbb{N}$, $x < \frac{1}{-n \cdot N(A)}$. Multiply $N(A)$ throughout to get $x \cdot N(A) > -\frac{1}{n}$. Add $d(A)$ on both sides to get $P(A) = d(A) + \big(x \cdot N(A)\big) > d(A) - \frac{1}{n}$. Since n is arbitrary, $P(A) > d(A) - \frac{1}{n}$ for all $n \in \mathbb{N}$. So, for all such A's, $\mathcal{S}_A = \{x \in \mathbb{F} : \forall n \in \mathbb{N}(d(A) - \frac{1}{n} < x < d(A))\}$. An upper bound for $\mathcal{S}_A$ is $d(A)$ and a lower bound for the same set is $d(A) - u$. After all, u is an upper bound for $\mathcal{S}_{\{i\}}$, i.e. $u > \frac{1}{n}$ for some $n \in \mathbb{N}$ (**Theorem 17**). So, $-u < -\frac{1}{n}$. Add $d(A)$ on both sides to get $d(A) - u < d(A) - \frac{1}{n}$. This means that $d(A) - u$ is a lower bound for $\mathcal{S}_A$ for all such A's. By the definition of $P_{im(u)}$, for all such A's, $P_{im(u)}(A) = (d(A) - u, d(A))$, if $d(A) - u \geq 0$. If $d(A) - u < 0$, then $P_{im(u)}(A) = (0, d(A))$. This is in accordance with the definition of an imprecise probability function. $P_{im(u)}$ satisfies all the conditions in the definition of an imprecise probability function and is therefore an imprecise probability function. □

*Proof of **Theorem 19**.* Assume **Comparing Imprecise Probabilities** and **Imprecise Representation**. By the definition of $P_{im(u)}$, $P_{im(u)}$ is defined on $\mathcal{F}^{\#}$. All that remains to be proven is that $P_{im(u)}$ satisfies all the constraints in $C^{\#}$. Take any two sets $A \in \mathcal{F}^{\#}$ and $B \in \mathcal{F}^{\#}$ such that according to the constraints in $C^{\#}$, $A \preceq B$. Now, in order to prove that $P_{im(u)}$ satisfies all the constraints in $C^{\#}$ under **Imprecise Representation**, it needs to be shown that $P_{im(u)}(A) \leq P_{im(u)}(B)$. And to show that $P_{im(u)}(A) \leq P_{im(u)}(B)$, it needs to be shown that for every probability function $P_x \in \mathcal{S}$, $P_x(A) \leq P_x(B)$, by **Comparing Imprecise Probabilities**.

Now, let P_x be an arbitrary precise probability function in $\mathcal{S}$. By **Theorem 15**, P_x satisfies all the constraints in $C^{\#}$ under **Precise Representation**. Recall that $A \preceq B$, according to the constraints in $C^{\#}$. By **Precise Representation**, $P_x(A) \leq P_x(B)$. The proof proceeds similarly for any two sets $A \in \mathcal{F}^{\#}$ and $B \in \mathcal{F}^{\#}$ such that according to the constraints in $C^{\#}$, $A \succeq B$. Note that according to **Theorem 8**, there do not exist sets $A \in \mathcal{F}^{\#}$ and $B \in \mathcal{F}^{\#}$ such that $A \not\preceq B$ and $A \not\succeq B$. □

*Proof of **Theorem 20**.* Let $\langle \Omega, \mathcal{F}, C \rangle$ be any relevant token probabilistic scenario. For some $A \in \mathcal{F}$ and $B \in \mathcal{F}$, assume that A and B fulfill the two conditions stated in **Imprecise Reflection**. Next, assume for reductio that there exists an $x \in \mathcal{S}_A$ such that

$x \geq y$ for some $y \in \mathcal{S}_B$. Since $\mathcal{S}_A \cap \mathcal{S}_B = \emptyset$, $x \neq y$. So, it must be that $x > y$. Let P be a probability function that assigns A probability x, i.e. $P(A) = x$. In virtue of the first condition of **Imprecise Reflection**, $P(B) \neq y$. So, a probability function P' that assigns B probability y must be different from P, i.e. $P \neq P'$. By the first condition of **Imprecise Reflection**, $x' = P'(A) \leq P'(B) = y$. So, $x' \leq y < x$, which implies that $x' < x$. Because $\langle \Omega, \mathcal{F}, C \rangle$ is relevant, for any x^* such that $x' \leq x^* \leq x$, there exists a probability function P^* correctly representing $\langle \Omega, \mathcal{F}, C \rangle$ such that $P^*(A) = x^*$. Let $x^* = y$, i.e. $P^*(A) = y$. This implies that $y \in \mathcal{S}_A$, which in turn implies that $y \in \mathcal{S}_A \cap \mathcal{S}_B$. Contradiction. Therefore, for every $x \in \mathcal{S}_A$, $x < y$ for any $y \in \mathcal{S}_B$. □

*Proof of **Theorem 21**.* Consider the set $\{i\}$. Note that $d(\{i\}) = 0$ and $\mathcal{S}_{\{i\}} = \{x \in \mathbb{F} : \forall n \in \mathbb{N}(0 < x < \frac{1}{n})\}$ for some non-Archimedean field $\mathbb{F}$. Now, consider a set $A \in \mathcal{F}^{\#}$ such that $d(A) = \frac{1}{n}$ and $N(A) = 0$. Note that $\mathcal{S}_A = \{\frac{1}{n}\}$. So, $\mathcal{S}_{\{i\}} \cap \mathcal{S}_A = \emptyset$. Further note that for all $P_x \in \mathcal{S}$, $P_x(\{i\}) \leq P_x(A)$.[98] All the conditions of **Imprecise Reflection** are thus satisfied, but its consequent isn't. By the definition of $P_{im(u)}$, $P_{im(u)}(\{i\}) = (0, u)$ for some $u \in \mathbb{F}$ such that $0 \leq u \leq 1$ and $u > \frac{1}{n}$ for some $n \in \mathbb{N}$. This means that $\overline{P}_{im(u)}(\{i\}) = u$. And by the definition of $P_{im(u)}$, $P_{im(u)}(A) = \frac{1}{n}$. This means that $\underline{P}_{im(u)}(A) = \frac{1}{n}$. Therefore, $\overline{P}_{im(u)}(\{i\}) = u > \frac{1}{n} = \underline{P}_{im(u)}(A)$, in violation of **Imprecise Reflection**. □

*Proof of **Theorem 22**.* Let $\langle \Omega, \mathcal{F}, C \rangle$ be any relevant token probabilistic scenario and $\mathbb{F}$ be an ordered field. Assume that P_{im} satisfies **Maximal Distance**. Consider the sets $\emptyset$ and $A \in \mathcal{F}$. Note that $\mathcal{S}_\emptyset = \{0\}$. So, the L.U.B in $\mathbb{F}$ of $\mathcal{S}_\emptyset$ is 0. Further note that for all $P \in \mathcal{S}$, $P(\emptyset) \leq P(A)$. Now, define $\mathcal{X}$ to be the set of all $x \in \mathbb{F}$ such that x is an upper bound for $\mathcal{S}_\emptyset$ and a lower bound for $\mathcal{S}_A$. It is obvious that $0 \in \mathcal{X}$. There are two cases to consider: (i) $\mathcal{S}_\emptyset \cap \mathcal{S}_A = \emptyset$ and (ii) $\mathcal{S}_\emptyset \cap \mathcal{S}_A \neq \emptyset$. Assume (i) first. Since P_{im} satisfies **Maximal Distance**, $|\overline{P}_{im}(\emptyset) - \underline{P}_{im}(A)| \geq |x - x'|$ for any two x and x' in $\mathcal{X}$. But because $0 \in \mathcal{X}$, we can let $x' = 0$. So, $|\overline{P}_{im}(\emptyset) - \underline{P}_{im}(A)| \geq |x|$ for any $x \in \mathcal{X}$. By the definition of P_{im}, $P_{im}(\emptyset) = 0$, which implies that $\overline{P}_{im}(\emptyset) = 0$. Therefore, $|-\underline{P}_{im}(A)| \geq |x|$ for any $x \in \mathcal{X}$. Note that $\underline{P}_{im}(A) \geq 0$, so $|-\underline{P}_{im}(A)| = \underline{P}_{im}(A)$. Further note that for all $x \in \mathcal{X}$, $x \geq 0$, which implies that $|x| = x$. Therefore, $|-\underline{P}_{im}(A)| \geq |x|$ for any $x \in \mathcal{X}$ can be simplified to $\underline{P}_{im}(A) \geq x$ for any $x \in \mathcal{X}$. And since x is a lower bound for $\mathcal{S}_A$, this implies that $\underline{P}_{im}(A)$ is greater than or equal to any lower bound for $\mathcal{S}_A$. Finally, by the definition of P_{im}, $\underline{P}_{im}(A)$ must be a lower bound for $\mathcal{S}_A$ too. Combining all the pieces of information, $\underline{P}_{im}(A)$ is the G.L.B for $\mathcal{S}_A$.

Now, assume (ii): $\mathcal{S}_\emptyset \cap \mathcal{S}_A \neq \emptyset$. This means that $0 \in \mathcal{S}_A$, which in turn implies that the G.L.B of $\mathcal{S}_A$ is 0. After all, for all $a \in \mathcal{S}_A$, $a \geq 0$. By the definition of P_{im}, $\underline{P}_{im}(A)$ must be a lower bound for $\mathcal{S}_A$ and be non-negative. So, the only value $\underline{P}_{im}(A)$ can take is 0, which is the G.L.B of $\mathcal{S}_A$. Either way, $\underline{P}_{im}(A)$ is the G.L.B for $\mathcal{S}_A$.

[98] See the top of §4.2 for the definition of P_x.

Consider the sets Ω and A from before. Note that $\mathcal{S}_\Omega = \{1\}$. So, the G.L.B in $\mathbb{F}$ of $\mathcal{S}_\Omega$ is 1. Further note that for all $P \in \mathcal{S}$, $P(A) \leq P(\Omega)$. Now, define $\mathcal{X}$ to be the set of all $x \in \mathbb{F}$ such that x is an upper bound for $\mathcal{S}_A$ and a lower bound for $\mathcal{S}_\Omega$. It is obvious that $1 \in \mathcal{X}$. There are two cases to consider: (i) $\mathcal{S}_A \cap \mathcal{S}_\Omega = \emptyset$ and (ii) $\mathcal{S}_A \cap \mathcal{S}_\Omega \neq \emptyset$. Assume (i) first. Since P_{im} satisfies **Maximal Distance**, $|\overline{P}_{im}(A) - \underline{P}_{im}(\Omega)| \geq |x - x^{'}|$ for any two x and $x^{'}$ in $\mathcal{X}$. But because $1 \in \mathcal{X}$, we can let $x = 1$. So, $|\overline{P}_{im}(A) - \underline{P}_{im}(\Omega)| \geq |1 - x^{'}|$ for any $x^{'}$ in $\mathcal{X}$. By the definition of P_{im}, $P_{im}(\Omega) = 1$, which implies that $\underline{P}_{im}(\Omega) = 1$. Therefore, $|\overline{P}_{im}(A) - 1| \geq |1 - x^{'}|$ for any $x^{'}$ in $\mathcal{X}$. Note that $\overline{P}_{im}(A) \leq 1$, so $|\overline{P}_{im}(A) - 1| = 1 - \overline{P}_{im}(A)$. Further note that for all $x \in \mathcal{X}$, $x \leq 1$, which implies that $|1 - x^{'}| = 1 - x^{'}$. Therefore, $|\overline{P}_{im}(A) - 1| \geq |1 - x^{'}|$ for any $x^{'}$ in $\mathcal{X}$ can be simplified to $1 - \overline{P}_{im}(A) \geq 1 - x^{'}$ for any $x^{'}$ in $\mathcal{X}$, which can be further simplified to $\overline{P}_{im}(A) \leq x^{'}$ for any $x^{'}$ in $\mathcal{X}$. And since $x^{'}$ is an upper bound for $\mathcal{S}_A$, this implies that $\overline{P}_{im}(A)$ is less than or equal to any upper bound for $\mathcal{S}_A$. Finally, by the definition of P_{im}, $\overline{P}_{im}(A)$ must be an upper bound for $\mathcal{S}_A$ too. Combining all the pieces of information, $\overline{P}_{im}(A)$ is the L.U.B for $\mathcal{S}_A$.

Now, assume (ii): $\mathcal{S}_A \cap \mathcal{S}_\Omega \neq \emptyset$. This means that $1 \in \mathcal{S}_A$, which in turn implies that the L.U.B of $\mathcal{S}_A$ is 1. After all, for all $a \in \mathcal{S}_A$, $a \leq 1$. By the definition of P_{im}, $\overline{P}_{im}(A)$ must be an upper bound for $\mathcal{S}_A$ and be less than or equal to 1. So, the only value $\overline{P}_{im}(A)$ can take is 1, which is the L.U.B of $\mathcal{S}_A$. Either way, $\overline{P}_{im}(A)$ is the L.U.B for $\mathcal{S}_A$. $\square$

Lemma 10. *Let $\mathbb{F}$ be an ordered field. Consider any two sets $A \subseteq \mathbb{F}$ and $B \subseteq \mathbb{F}$ such that (i) for every $x \in A$, $x < y$ for any $y \in B$, and (ii) the G.L.B and the L.U.B for both sets exist in $\mathbb{F}$. Let a be the L.U.B for A and b be the G.L.B for B. Then, $a \leq b$.*

Proof. For reductio, assume that $a > b$. This means that a is not a lower bound for B.[99] So, there exists a $y \in B$ such that $a > y$. In virtue of $y \in B$ and (i), y is an upper bound for A. But this means that a is not the L.U.B for A, as $a > y$. Contradiction. Therefore, $a \leq b$. $\square$

Proof of ***Corollary 3****.* Let $\langle \Omega, \mathcal{F}, C \rangle$ be any relevant token probabilistic scenario. Assume that P_{im} satisfies **Maximal Distance**. By **Theorem 22**, this means that for any $A \in \mathcal{F}$, $\overline{P}_{im}(A)$ is the L.U.B for $\mathcal{S}_A$ and $\underline{P}_{im}(A)$ is the G.L.B for $\mathcal{S}_A$. So, the L.U.B and the G.L.B for $\mathcal{S}_A$ exist in the given ordered field $\mathbb{F}$ for all $A \in \mathcal{F}$. Now, consider any two sets A and B in $\mathcal{F}$ that satisfies the two conditions of **Imprecise Reflection**. By **Theorem 20**, for every $x \in \mathcal{S}_A$, $x < y$ for any $y \in \mathcal{S}_B$. So, the conditions of **Lemma 10** are satisfied. By **Lemma 10**, $\overline{P}_{im}(A) \leq \underline{P}_{im}(B)$, in accordance with **Imprecise Reflection**. $\square$

Proof of ***Corollary 4****.* This is a direct consequence of **Theorem 22** and **Tightness**. $\square$

[99]For if it were, then b wouldn't be the G.L.B for B, as $a > b$. This results in a contradiction.

*Proof of **Theorem 23**.* Let $\langle \Omega, \mathcal{F}, C \rangle$ be any relevant token probabilistic scenario. In virtue of the fact that for each $\mathcal{S}_A$, the L.U.B and the G.L.B for $\mathcal{S}_A$ are unique in $\mathbb{F}$, there is at most one P_{im} set in $\mathbb{F}$ that is tight w.r.t $\langle \Omega, \mathcal{F}, C \rangle$. $\square$

*Proof of **Theorem 24**.* Recall that in order for an imprecise probability function P_{im} set in $\mathbb{F}$ to be tight w.r.t the given relevant token probabilistic scenario $\langle \Omega, \mathcal{F}, C \rangle$, for all $A \in \mathcal{F}$, the L.U.B and the G.L.B for $\mathcal{S}_A$ must exist in $\mathbb{F}$. As proven in §4.2, $\mathcal{S}_{\{i\}} = \{x \in \mathbb{F} : \forall n \in \mathbb{N}(0 < x < \frac{1}{n})\}$, i.e. $\mathcal{S}_{\{i\}}$ is the set of all positive infinitesimals in $\mathbb{F}$. By **Theorem 17**, for every $u \in \mathbb{F}$ such that $0 \leq u \leq 1$, u is an upper bound for $\mathcal{S}_{\{i\}}$ iff $u > \frac{1}{n}$ for some $n \in \mathbb{N}$. This implies that the L.U.B for $\mathcal{S}_{\{i\}}$ does not exist in $\mathbb{F}$, as for every upper bound $0 \leq u \leq 1$ for $\mathcal{S}_{\{i\}}$ such that $u > \frac{1}{n}$ for some $n \in \mathbb{N}$, there is always another $u' < u$ that is also an upper bound for $\mathcal{S}_{\{i\}}$, e.g. $u' = \frac{1}{n+1}$. Since there is a set $A \in \mathcal{F}^{\#}$ such that the L.U.B for $\mathcal{S}_A$ doesn't exist in $\mathbb{F}$, there are no imprecise probability functions P_{im} set in $\mathbb{F}$ that are tight w.r.t $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. And because $\mathbb{F}$ is an arbitrary ordered field, for all $\mathbb{F}$, there are no imprecise probability functions P_{im} set in $\mathbb{F}$ that are tight w.r.t $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. This means that there are no imprecise probability functions P_{im} that are tight w.r.t $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. And because there are no imprecise probability functions P_{im} that are tight w.r.t $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, there are consequently no tight interval-based models $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$. $\square$

*Proof of **Theorem 25**.* Consider the set $\{i\}$. As proven in §4.2, $\mathcal{S}_{\{i\}} = \{x \in \mathbb{F} : \forall n \in \mathbb{N}(0 < x < \frac{1}{n})\}$ for some ordered field $\mathbb{F}$. By the definition of an imprecise probability function, $\overline{P}_{im}(\{i\})$ must be an upper bound for $\mathcal{S}_{\{i\}}$. But by **Theorem 17**, for every $u \in \mathbb{F}$ such that $0 \leq u \leq 1$, u is an upper bound for $\mathcal{S}_{\{i\}}$ iff $u > \frac{1}{n}$ for some $n \in \mathbb{N}$. So, an upper bound $0 \leq u \leq 1$ for $\mathcal{S}_{\{i\}}$ must be such that $u > \frac{1}{n}$ for some $n \in \mathbb{N}$. Let $A \in \mathcal{F}^{\#}$ such that $d(A) = \frac{1}{n}$ and $N(A) = 0$. Recall that by **Lemma 9**, for any precise probability function P that correctly represents $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ under **Precise Representation**, $P(A) = d(A) + \big(x \cdot N(A)\big)$ for some x such that $0 < x < \frac{1}{n}$ for all $n \in \mathbb{N}$. So, for all precise probability functions P that correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ under **Precise Representation**, $P(A) = \frac{1}{n}$. This means that $\mathcal{S}_A = \{\frac{1}{n}\}$. By the definition of an imprecise probability function, $P_{im}(A) = \frac{1}{n}$, which means that $\underline{P}_{im}(A) = \frac{1}{n}$. Note that for all precise probability functions P that correctly represent $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ under **Precise Representation**, $P(\{i\}) \leq P(A)$. Further note that $\mathcal{S}_{\{i\}} \cap \mathcal{S}_A = \emptyset$. So, both conditions of **Imprecise Reflection** are fulfilled. By **Imprecise Reflection**, $\overline{P}_{im}(\{i\}) \leq \underline{P}_{im}(A)$. But this is false, as $\overline{P}_{im}(\{i\}) > \frac{1}{n} = \underline{P}_{im}(A)$. Therefore, P_{im} does not satisfy **Imprecise Reflection**. Since P_{im} is arbitrary, for all P_{im} correctly representing $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, P_{im} does not satisfy **Imprecise Reflection**. Consequently, there are no correct interval-based models $\mathcal{M}_{im}$ of $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ that satisfy **Imprecise Reflection**. $\square$

*Proof of **Theorem 26**.* Let $\langle \Omega, \mathcal{F}, C \rangle$ be any relevant token probabilistic scenario such that Ω is infinite and **Fairness** is one of the constraints in C. Suppose:

1. $\emptyset \neq \mathcal{S}_{\{\omega\}} = \{x \in \mathbb{F} : \forall n \in \mathbb{N}(0 < x < \frac{1}{n})\}$ for all $\omega \in \Omega$, where $\mathbb{F}$ is a non-Archimedean ordered field.

2. For all $n \in \mathbb{N}$, there exists an $A \in \mathcal{F}$ such that:

   (a) $\frac{1}{n+1} \in \mathcal{S}_A$.

   (b) For all $a \in \mathcal{S}_A$, $a \geq \frac{1}{n+1}$.

The proof of **Theorem 17** still works for $\mathcal{S}_{\{\omega\}}$, i.e. for every $u \in \mathbb{F}$ such that $0 \leq u \leq 1$, u is an upper bound for $\mathcal{S}_{\{\omega\}}$ iff $u > \frac{1}{n}$ for some $n \in \mathbb{N}$. That is, an upper bound $0 \leq u \leq 1$ for $\mathcal{S}_{\{\omega\}}$ must be greater than $\frac{1}{n}$ for some $n \in \mathbb{N}$. By the definition of an imprecise probability function P_{im}, $P_{im}(\{\omega\}) = (0, u)$ for some $0 \leq u \leq 1$ such that $u > \frac{1}{n}$ for some $n \in \mathbb{N}$. This means that $\overline{P}_{im}(\{\omega\}) = u$.

Now, consider an $A \in \mathcal{F}$ that satisfies the conditions above with respect to that particular n for which $u > \frac{1}{n}$. In virtue of the conditions above, $\frac{1}{n}$ is not a lower bound for $\mathcal{S}_A$, as $\frac{1}{n} > \frac{1}{n+1}$. By the definition of an imprecise probability function, $\underline{P}_{im}(A) < \frac{1}{n}$. Now, note that in virtue of the conditions above, for all precise probability functions P that correctly represent $\langle \Omega, \mathcal{F}, C \rangle$, $P(\{\omega\}) \leq P(A)$. Further note that $\mathcal{S}_{\{\omega\}} \cap \mathcal{S}_A = \emptyset$. Both conditions of **Imprecise Reflection** are satisfied. Therefore, by **Imprecise Reflection**, $\overline{P}_{im}(\{\omega\}) \leq \underline{P}_{im}(A)$. But this is false, as $\overline{P}_{im}(\{\omega\}) = u > \frac{1}{n} > \underline{P}_{im}(A)$. So, P_{im} does not satisfy **Imprecise Reflection**. But since P_{im} is arbitrary, there are no imprecise probability functions correctly representing $\langle \Omega, \mathcal{F}, C \rangle$ that satisfy **Imprecise Reflection**. Consequently, there are no correct interval-based models $\mathcal{M}_{im}$ of $\langle \Omega, \mathcal{F}, C \rangle$ that satisfy **Imprecise Reflection**. □

*Proof of **Theorem 27**.* Assume **Irregular Ordering**. Now, consider an $A \in \mathcal{F}^{\#}$ such that $\emptyset \prec' A$. By **Irregular Ordering**, $0 = d(\emptyset) < d(A)$. So, $d(A)$ is a positive real number. This means that there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < d(A)$. Note that $d(\{n, 2n, 3n, \ldots\}) = \frac{1}{n}$. Therefore, $d(\{n, 2n, 3n, \ldots\}) < d(A)$. By **Irregular Ordering**, $\{n, 2n, 3n, \ldots\} \prec' A$, in accordance with **Archimedean Probability (Qualitative)**. □

*Proof of **Theorem 28**.* Assume **Precise Representation**. Let P be an arbitrary precise probability function that correctly represents a token probabilistic scenario $\langle \mathbb{N}, \mathcal{F}, C \rangle$ that satisfies (i)-(iii) and **Archimedean Probability (Qualitative)**. For reductio, assume that $\emptyset \prec \{i\}$. By **Archimedean Probability (Qualitative)**, this means that $\{n, 2n, 3n, \ldots\} \prec \{i\}$. Since $\{n\} \subseteq \{n, 2n, 3n, \ldots\}$, by **Lemma 2**, $\{n\} \preceq \{n, 2n, 3n, \ldots\}$. Apply **Transitivity** to $\{n\} \preceq \{n, 2n, 3n, \ldots\}$ and $\{n, 2n, 3n, \ldots\} \prec \{i\}$ to get $\{n\} \prec \{i\}$. But by **Fairness**, $\{n\} \sim \{i\}$. Contradiction. Therefore, $\emptyset \not\prec \{i\}$. By **Non-Negativity**, $\emptyset \preceq \{i\}$. This means that $\emptyset \sim \{i\}$. Since both $\emptyset$ and $\{i\}$ are in $\mathcal{F}^{\#}$, by **Precise Representation**, $0 = P(\emptyset) = P(\{i\})$. □

*Proof of **Theorem 29**.* Let A be an arbitrary set in $\mathcal{F}^{\#}$. According to the definition of $\mathcal{F}^{\#}$, $A = (A_1 - A_2) \cup A_3$. Either $A_1 = \emptyset$ or $A_1 \neq \emptyset$. Assume the former first. Then, $A = A_3$. It is easy to verify that $\{i \in \mathbb{N} : \exists j \in A_3(i = 2j)\} \in \mathcal{F}^{\#}$ and $\{i \in \mathbb{N} : \exists j \in A_3(i = 2j - 1)\} \in \mathcal{F}^{\#}$. So, $A_{even} \in \mathcal{F}^{\#}$ and $A_{odd} \in \mathcal{F}^{\#}$.

Now, assume the latter: $A_1 \neq \emptyset$. Let S be a maximal arithmetic progression in A_1 with common difference m. Notice that $S_{even} = \{i \in \mathbb{N} : \exists j \in S(i = 2j)\}$ is also a maximal arithmetic progression but with common difference $2m$. Let $A_{1(even)}$ be the union of all S_{even}'s for each maximal arithmetic progression S in A_1. Further define $A_{2(even)} = \{i \in \mathbb{N} : \exists j \in A_2(i = 2j)\}$ and $A_{3(even)} = \{i \in \mathbb{N} : \exists j \in A_3(i = 2j)\}$. It is easy to verify that $A_{2(even)} \subseteq A_{1(even)}$, $A_{3(even)} \cap A_{1(even)} = \emptyset$, and $A_{2(even)} \cup A_{3(even)}$ is finite. By the definition of $\mathcal{F}^{\#}$, $\big(A_{1(even)} - A_{2(even)}\big) \cup A_{3(even)} \in \mathcal{F}^{\#}$. Note that $A_{even} = \big(A_{1(even)} - A_{2(even)}\big) \cup A_{3(even)}$, so $A_{even} \in \mathcal{F}^{\#}$. The proof proceeds similarly for A_{odd}. Therefore, $A_{even} \in \mathcal{F}^{\#}$ and $A_{odd} \in \mathcal{F}^{\#}$. $\square$

*Proof of **Theorem 30**.* Let A be an arbitrary set in $\mathcal{F}^{\#}$. According to the definition of $\mathcal{F}^{\#}$, $A = (A_1 - A_2) \cup A_3$. And according to **Theorem 29**, $A_{even} \cup A_{odd} \in \mathcal{F}^{\#}$. Define the following sets:

1. $A_{1(even)} = \{i \in \mathbb{N} : \exists j \in A_1(i = 2j)\}$.
2. $A_{2(even)} = \{i \in \mathbb{N} : \exists j \in A_2(i = 2j)\}$.
3. $A_{3(even)} = \{i \in \mathbb{N} : \exists j \in A_3(i = 2j)\}$.
4. $A_{1(odd)} = \{i \in \mathbb{N} : \exists j \in A_1(i = 2j - 1)\}$.
5. $A_{2(odd)} = \{i \in \mathbb{N} : \exists j \in A_2(i = 2j - 1)\}$.
6. $A_{3(odd)} = \{i \in \mathbb{N} : \exists j \in A_3(i = 2j - 1)\}$.

It is easy to verify that $A_{even} \cup A_{odd} = \big((A_{1(even)} \cup A_{1(odd)}) - (A_{2(even)} \cup A_{2(odd)})\big) \cup (A_{3(even)} \cup A_{3(odd)})$.

Now, either $A_1 = \emptyset$ or $A_1 \neq \emptyset$. Assume the former first. This means that $A_{1(even)} \cup A_{1(odd)} = \emptyset$. So, $A_{even} \cup A_{odd} = A_{3(even)} \cup A_{3(odd)}$, which in turn implies that $A_{even} \cup A_{odd}$ is a finite set. Therefore, $d(A_{even} \cup A_{odd}) = 0$. Because $A_1 = \emptyset$, $A = A_3$. This means that A is a finite set. So, $d(A) = 0$. Consequently, $d(A_{even} \cup A_{odd}) = d(A)$.

Note that $N(A_{even} \cup A_{odd}) = \#(A_{3(even)} \cup A_{3(odd)}) - \#(A_{2(even)} \cup A_{2(odd)})$. But because $A_1 = \emptyset$, $A_{2(even)} \cup A_{2(odd)} = \emptyset$. Therefore, $N(A_{even} \cup A_{odd}) = \#(A_{3(even)} \cup A_{3(odd)}) = \#(A_{3(even)}) + \#(A_{3(odd)})$. Since A_3 is a finite set, $\#(A_3) = \#(A_{3(even)}) = \#(A_{3(odd)})$. This implies that $N(A_{even} \cup A_{odd}) = 2 \cdot \#(A_3)$. Furthermore, since $A = A_3$, $\#(A) = \#(A_3)$. Substituting, $N(A_{even} \cup A_{odd}) = 2 \cdot \#(A)$. But note that since $A_1 = \emptyset$, $A_2 = \emptyset$. Therefore, $N(A) = \#(A_3) - \#(A_2) = \#(A_3)$. Further note that $A = A_3$, so $\#(A) = \#(A_3)$. All these imply that $N(A) = \#(A)$. Because $N(A_{even} \cup A_{odd}) = 2 \cdot \#(A)$, $N(A_{even} \cup A_{odd}) = 2 \cdot N(A)$, by substitution.

Now, assume the latter: $A_1 \neq \emptyset$. First, note that $d(A_{1(even)}) = d(A_{1(odd)}) = \frac{d(A)}{2}$. Since $A_{1(even)} \cap A_{1(odd)} = \emptyset$, $d(A_{1(even)} \cup A_{1(odd)}) = d(A_{1(even)}) + d(A_{1(odd)}) = d(A)$. Further note that $d(A_{even} \cup A_{odd}) = d(A_{1(even)} \cup A_{1(odd)}) = d(A)$.

Recall that $N(A_{even} \cup A_{odd}) = \#(A_{3(even)} \cup A_{3(odd)}) - \#(A_{2(even)} \cup A_{2(odd)})$. This is equivalent to

$$N(A_{even} \cup A_{odd}) = \#(A_{3(even)}) + \#(A_{3(odd)}) - \#(A_{2(even)}) - \#(A_{2(odd)}). \tag{15}$$

Note that since A_2 and A_3 are finite sets, $\#(A_{3(even)}) = \#(A_{3(odd)}) = \#(A_3)$ and $\#(A_{2(even)}) = \#(A_{2(odd)}) = \#(A_2)$. So, equation 15 is equivalent to

$$N(A_{even} \cup A_{odd}) = \big(2 \cdot \#(A_3)\big) - \big(2 \cdot \#(A_2)\big) = 2 \cdot \big(\#(A_3) - \#(A_2)\big). \tag{16}$$

Because $N(A) = \#(A_3) - \#(A_2)$, $N(A_{even} \cup A_{odd}) = 2 \cdot N(A)$. $\square$

*Proof of **Theorem 31**.* $(\Rightarrow)$ Assume that $A \preceq_1 B$. By **Regular Ordering**, either (i) $d(A) < d(B)$ or (ii) $d(A) = d(B)$ and $N(A) \leq N(B)$. Assume (i) first. According to C', in order for $A \preceq_2 B$, it needs to be proven that $(A_{even} \cup A_{odd}) \preceq_1 (B_{even} \cup B_{odd})$. By **Theorem 30**, $d(A_{even} \cup A_{odd}) = d(A)$ and $d(B_{even} \cup B_{odd}) = d(B)$. Since $d(A) < d(B)$, $d(A_{even} \cup A_{odd}) < d(B_{even} \cup B_{odd})$. Apply **Regular Ordering** to get $(A_{even} \cup A_{odd}) \preceq_1 (B_{even} \cup B_{odd})$. Therefore, $A \preceq_2 B$.

Assume (ii) now: $d(A) = d(B)$ and $N(A) \leq N(B)$. This means that $d(A_{even} \cup A_{odd}) = d(B_{even} \cup B_{odd})$. By **Theorem 30**, $N(A_{even} \cup A_{odd}) = 2 \cdot N(A)$ and $N(B_{even} \cup B_{odd}) = 2 \cdot N(B)$. Since $N(A) \leq N(B)$, $2 \cdot N(A) \leq 2 \cdot N(B)$. Substituting, $N(A_{even} \cup A_{odd}) \leq N(B_{even} \cup B_{odd})$. Apply **Regular Ordering** to get $(A_{even} \cup A_{odd}) \preceq_1 (B_{even} \cup B_{odd})$. Therefore, $A \preceq_2 B$.

$(\Leftarrow)$ Assume that $A \preceq_2 B$. According to C', $(A_{even} \cup A_{odd}) \preceq_1 (B_{even} \cup B_{odd})$. By **Theorem 30**, $d(A_{even} \cup A_{odd}) = d(A)$ and $d(B_{even} \cup B_{odd}) = d(B)$. And by the same theorem, $N(A_{even} \cup A_{odd}) = 2 \cdot N(A)$ and $N(B_{even} \cup B_{odd}) = 2 \cdot N(B)$. Since $(A_{even} \cup A_{odd}) \preceq_1 (B_{even} \cup B_{odd})$, by **Regular Ordering**, either (i) $d(A_{even} \cup A_{odd}) < d(B_{even} \cup B_{odd})$ or (ii) $d(A_{even} \cup A_{odd}) = d(B_{even} \cup B_{odd})$ and $N(A_{even} \cup A_{odd}) \leq N(B_{even} \cup B_{odd})$. (i) is equivalent to $d(A) < d(B)$ and (ii) is equivalent to $d(A) = d(B)$ and $N(A) \leq N(B)$. Either way, **Regular Ordering** can be applied to get $A \preceq_1 B$. $\square$

*Proof of **Theorem 32**.* Assume that $\psi(A, i) \leq \psi(B, j)$ and $\psi(B, j) \leq \psi(C, k)$. Let $\psi(A, i) = (x_1, x_2)$, $\psi(B, j) = (y_1, y_2)$, and $\psi(C, k) = (z_1, z_2)$. Either $\psi(A, i) \leq \psi(B, j)$ in virtue of (i) $x_1 < y_1$ or (ii) $x_1 = y_1$ and $x_2 \leq y_2$.

Assume the former first: $x_1 < y_1$. Either $\psi(B, j) \leq \psi(C, k)$ in virtue of (iii) $y_1 < z_1$ or (iv) $y_1 = z_1$ and $y_2 \leq z_2$. Either way, $y_1 \leq z_1$. So, $x_1 < y_1 \leq z_1$. This implies that

$x_1 < z_1$. Because ψ is lexicographically ordered, $\psi(A,i) \leq \psi(C,k)$.

Now, assume the latter: $x_1 = y_1$ and $x_2 \leq y_2$. Either $\psi(B,j) \leq \psi(C,k)$ in virtue of (iii) $y_1 < z_1$ or (iv) $y_1 = z_1$ and $y_2 \leq z_2$. If the former is true, then $x_1 = y_1 < z_1$. This implies that $x_1 < z_1$. Because ψ is lexicographically ordered, $\psi(A,i) \leq \psi(C,k)$. If the latter is true, then $x_1 = y_1 = z_1$ and $x_2 \leq y_2 \leq z_2$. These imply that $x_1 = z_1$ and $x_2 \leq z_2$. Because ψ is lexicographically ordered, $\psi(A,i) \leq \psi(C,k)$. $\square$

Proof of ***Theorem 33***. Let $A \cap B = C \cap D = \emptyset$. Suppose:

1. $\psi(A,i) \leq \psi(C,j)$.

2. $\psi(B,i) \leq \psi(D,j)$.

If $i = j = 1$ or $i = j = 2$, then the theorem is a straightforward consequence of **Theorem 6** and the fact that asymptotic densities are finitely additive. So, the proofs for these cases will be left to the reader.

Assume now that $i \neq j$. Without loss of generality, let $i = 1$ and $j = 2$. By the definition of ψ, $\psi(A,1) = (d(A), N(A))$, $\psi(B,1) = (d(B), N(B))$, $\psi(C,2) = (d(C), 2 \cdot N(C))$, and $\psi(D,2) = (d(D), 2 \cdot N(D))$. Since ψ is lexicographically ordered, $\psi(A,1) \leq \psi(C,2)$ in virtue of (i) $d(A) < d(C)$ or (ii) $d(A) = d(C)$ and $N(A) \leq 2 \cdot N(C)$. Similarly, $\psi(B,1) \leq \psi(D,2)$ in virtue of (iii) $d(B) < d(D)$ or (iv) $d(B) = d(D)$ and $N(B) \leq 2 \cdot N(D)$. Either way, $d(B) \leq d(D)$. Assume that (i) holds first: $d(A) < d(C)$. So, $d(A \cup B) < d(C \cup D)$. Because ψ is lexicographically ordered, $\psi(A \cup B, 1) \leq \psi(C \cup D, 2)$.

Now, assume that (ii) holds: $d(A) = d(C)$ and $N(A) \leq 2 \cdot N(C)$. If (iii) holds, then $d(A \cup B) < d(C \cup D)$. Because ψ is lexicographically ordered, $\psi(A \cup B, 1) \leq \psi(C \cup D, 2)$. If (iv) holds instead, then $d(A \cup B) = d(C \cup D)$ and $N(A) + N(B) \leq 2 \cdot (N(C) + N(D))$. By **Theorem 6**, the second inequality is equivalent to $N(A \cup B) \leq 2 \cdot N(C \cup D)$. Because ψ is lexicographically ordered, $\psi(A \cup B, 1) \leq \psi(C \cup D, 2)$. $\square$

Proof of ***Theorem 34***. $\psi(\emptyset, 1) = (d(\emptyset), N(\emptyset)) = (0,0)$ while $\psi(\emptyset, 2) = (d(\emptyset), 2 \cdot N(\emptyset)) = (0,0)$. Therefore, $\psi(\emptyset, i) = (0,0)$.

$\psi(\mathbb{N}, 1) = (d(\mathbb{N}), N(\mathbb{N})) = (1,0)$ while $\psi(\mathbb{N}, 2) = (d(\mathbb{N}), 2 \cdot N(\mathbb{N})) = (1,0)$. Therefore, $\psi(\mathbb{N}, i) = (1,0)$. $\square$

Proof of ***Theorem 35***. Let $A \in \mathcal{F}^{\#}$ be an arbitrary event. Either $d(A) > 0$ or $d(A) = 0$. If the former is true, then, by **Theorem 34**, $\psi(\emptyset, i) = (0,0) \leq (d(A), N(A)) = \psi(A,1)$ since ψ is lexicographically ordered. Also, $\psi(\emptyset, i) = (0,0) \leq (d(A), 2 \cdot N(A)) = \psi(A,2)$ since ψ is lexicographically ordered. Therefore, $\psi(\emptyset, i) \leq \psi(A,j)$.

Now, assume the latter: $d(A) = 0$. This means that $N(A) \geq 0$. This is because according to the definition of $\mathcal{F}^{\#}$, $A = (A_1 - A_2) \cup A_3$. Since $d(A) = 0$, $A_1 = \emptyset$, which implies that $A_2 = \emptyset$. $N(A) = \#(A_3) - \#(A_2) = \#(A_3)$ since $\#(A_2) = 0$. $\psi(\emptyset, i) = (0,0) \leq (0, N(A)) = \psi(A,1)$ since ψ is lexicographically ordered. Also, $\psi(\emptyset, i) = (0,0) \leq (0, 2 \cdot N(A)) = \psi(A,2)$ since ψ is lexicographically ordered. Therefore, $\psi(\emptyset, i) \leq \psi(A,j)$. $\square$

*Proof of **Theorem 36**.* First, I will prove that $\preceq^*$ satisfies **Compatibility**. ($\Rightarrow$) Assume that $(A,1) \preceq^* (B,1)$. By **Definition 3**, $\psi(A,1) = (d(A), N(A)) \leq (d(B), N(B)) = \psi(B,1)$. Since ψ is lexicographically ordered, either (i) $d(A) < d(B)$ or (ii) $d(A) = d(B)$ and $N(A) \leq N(B)$. Recall that the constraints in $C^{\#}$ imply **Regular Ordering** (**Theorem 7**). So, either way, by **Regular Ordering**, A is at most as probable as B, according to $C^{\#}$.

Now, assume that $(A,2) \preceq^* (B,2)$. By **Definition 3**, $\psi(A,2) = (d(A), 2 \cdot N(A)) \leq (d(B), 2 \cdot N(B)) = \psi(B,2)$. Since ψ is lexicographically ordered, either (i) $d(A) < d(B)$ or (ii) $d(A) = d(B)$ and $2 \cdot N(A) \leq 2 \cdot N(B)$. Assume that (i) is true. Recall from **Theorem 30** that $d(A_{even} \cup A_{odd}) = d(A)$ and $d(B_{even} \cup B_{odd}) = d(B)$. So, $d(A_{even} \cup A_{odd}) < d(B_{even} \cup B_{odd})$. By **Regular Ordering**, $A_{even} \cup A_{odd}$ is at most as probable as $B_{even} \cup B_{odd}$, according to $C^{\#}$. This means that A is at most as probable as B, according to C'. After all, the winning ticket in $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ is contained in A (or B) just in case the winning ticket in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is contained in $A_{even} \cup A_{odd}$ (or $B_{even} \cup B_{odd}$).

Assume that (ii) holds: $d(A) = d(B)$ and $2 \cdot N(A) \leq 2 \cdot N(B)$. Recall from **Theorem 30** that $d(A_{even} \cup A_{odd}) = d(A)$ and $d(B_{even} \cup B_{odd}) = d(B)$, as well as $N(A_{even} \cup A_{odd}) = 2 \cdot N(A)$ and $N(B_{even} \cup B_{odd}) = 2 \cdot N(B)$. So, $d(A_{even} \cup A_{odd}) = d(B_{even} \cup B_{odd})$ and $N(A_{even} \cup A_{odd}) \leq N(B_{even} \cup B_{odd})$. By **Regular Ordering**, $A_{even} \cup A_{odd}$ is at most as probable as $B_{even} \cup B_{odd}$, according to $C^{\#}$. This means that A is at most as probable as B, according to C'. After all, the winning ticket in $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ is contained in A (or B) just in case the winning ticket in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is contained in $A_{even} \cup A_{odd}$ (or $B_{even} \cup B_{odd}$).

($\Leftarrow$) Assume that A is at most as probable as B, according to $C^{\#}$. Since the constraints in $C^{\#}$ imply **Regular Ordering** (**Theorem 7**), either (i) $d(A) < d(B)$ or (ii) $d(A) = d(B)$ and $N(A) \leq N(B)$. Either way, $\psi(A,1) = (d(A), N(A)) \leq (d(B), N(B)) = \psi(B,1)$, since ψ is lexicographically ordered. Apply **Definition 3** to get $(A,1) \preceq^* (B,1)$.

Assume that A is at most as probable as B, according to C'. This means that $A_{even} \cup A_{odd}$ is at most as probable as $B_{even} \cup B_{odd}$, according to $C^{\#}$. After all, the winning ticket in $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$ is contained in A (or B) just in case the winning ticket in $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$ is contained in $A_{even} \cup A_{odd}$ (or $B_{even} \cup B_{odd}$). Since the constraints in $C^{\#}$ imply **Regular Ordering** (**Theorem 7**), either (i) $d(A_{even} \cup A_{odd}) < d(B_{even} \cup B_{odd})$ or (ii) $d(A_{even} \cup A_{odd}) = d(B_{even} \cup B_{odd})$ and $N(A_{even} \cup A_{odd}) \leq N(B_{even} \cup B_{odd})$. Recall from **Theorem 30** that $d(A_{even} \cup A_{odd}) = d(A)$ and $d(B_{even} \cup B_{odd}) = d(B)$, as well as $N(A_{even} \cup A_{odd}) = 2 \cdot N(A)$ and $N(B_{even} \cup B_{odd}) = 2 \cdot N(B)$. This means that either (i) $d(A) < d(B)$ or (ii) $d(A) = d(B)$ and $2 \cdot N(A) \leq 2 \cdot N(B)$. Either way, $\psi(A,2) = (d(A), 2 \cdot N(A)) \leq (d(B), 2 \cdot N(B)) = \psi(B,2)$, since ψ is lexicographically ordered. Apply **Definition 3** to get $(A,2) \preceq^* (B,2)$. Thus, $\preceq^*$ satisfies **Compatibility**.

It is easy to verify that $\preceq^*$ satisfies **Cross-Non-Negativity**. This is a direct consequence of **Theorem 35** and **Definition 3**.

It is easy to verify that $\preceq^*$ satisfies **Cross-Transitivity**. This is a direct consequence of **Theorem 32** and **Definition 3**.

It is easy to verify that $\preceq^*$ satisfies **Cross-Additivity**. This is a direct consequence of **Theorem 33** and **Definition 3**.

Finally, I will prove that $\preceq^*$ satisfies **Probability Ratios**. Suppose $A \in \mathcal{F}^{\#}$ is $\frac{m}{n}$ times as probable as $\mathbb{N}$ and $B \in \mathcal{F}^{\#}$ is $\frac{p}{q}$ times as probable as $\mathbb{N}$. I will first prove that $\psi(A, i) = (\frac{m}{n}, 0)$ and $\psi(B, j) = (\frac{p}{q}, 0)$. In $\langle \mathbb{N}, \mathcal{F}^{\#}, C^{\#} \rangle$, by **Definition 2**, either (i) $\frac{m}{n} > 0$, and there exists a partition $\mathcal{G} = \{G_1, \ldots, G_n\}$ of $\mathbb{N}$ such that G_i is equiprobable to G_j and A is equiprobable to $\bigcup_{i=1}^{m} G_i$, or (ii) $\frac{m}{n} = 0$ and A is equiprobable to $\emptyset$. Similarly, by **Definition 2**, either (iii) $\frac{p}{q} > 0$, and there exists a partition $\mathcal{G}' = \{G'_1, \ldots, G'_q\}$ of $\mathbb{N}$ such that G'_i is equiprobable to G'_j and B is equiprobable to $\bigcup_{i=1}^{p} G_i$, or (iv) $\frac{p}{q} = 0$ and B is equiprobable to $\emptyset$. Assume (i) holds. Since A is equiprobable to $\bigcup_{i=1}^{m} G_i$, by **Regular Ordering**, $d(A) = d(\bigcup_{i=1}^{m} G_i) = \frac{m}{n}$ and $N(A) = N(\bigcup_{i=1}^{m} G_i) = 0$. So, $\psi(A, 1) = (d(A), N(A)) = (\frac{m}{n}, 0)$. Now, assume that (ii) holds. Because **Qualitative Regularity** is a constraint in $C^{\#}$, A being equiprobable to $\emptyset$ implies that $A = \emptyset$. By **Theorem 34**, $\psi(A, 1) = \psi(\emptyset, 1) = (0, 0) = (\frac{m}{n}, 0)$. Either way, $\psi(A, 1) = (\frac{m}{n}, 0)$. The proof proceeds similarly for B, i.e. $\psi(B, 1) = (\frac{p}{q}, 0)$.

In $\langle \mathbb{N}, \mathcal{F}^{\#}, C' \rangle$, by **Definition 2**, either (i) $\frac{m}{n} > 0$, and there exists a partition $\mathcal{G} = \{G_1, \ldots, G_n\}$ of $\mathbb{N}$ such that G_i is equiprobable to G_j and A is equiprobable to $\bigcup_{i=1}^{m} G_i$, or (ii) $\frac{m}{n} = 0$ and A is equiprobable to $\emptyset$. Similarly, by **Definition 2**, either (iii) $\frac{p}{q} > 0$, and there exists a partition $\mathcal{G}' = \{G'_1, \ldots, G'_q\}$ of $\mathbb{N}$ such that G'_i is equiprobable to G'_j and B is equiprobable to $\bigcup_{i=1}^{p} G_i$, or (iv) $\frac{p}{q} = 0$ and B is equiprobable to $\emptyset$. Assume (i) holds. Since A is equiprobable to $\bigcup_{i=1}^{m} G_i$, according to C', this means that $A_{even} \cup A_{odd}$ is equiprobable to $(\bigcup_{i=1}^{m} G_i)_{even} \cup (\bigcup_{i=1}^{m} G_i)_{odd}$, according to $C^{\#}$. By **Regular Ordering**, this in turn implies that $d(A_{even} \cup A_{odd}) = d\big((\bigcup_{i=1}^{m} G_i)_{even} \cup (\bigcup_{i=1}^{m} G_i)_{odd}\big)$ and $N(A_{even} \cup A_{odd}) = N\big((\bigcup_{i=1}^{m} G_i)_{even} \cup (\bigcup_{i=1}^{m} G_i)_{odd}\big)$. Recall from **Theorem 30** that $d(A_{even} \cup A_{odd}) = d(A)$ and $d\big((\bigcup_{i=1}^{m} G_i)_{even} \cup (\bigcup_{i=1}^{m} G_i)_{odd}\big) = d(\bigcup_{i=1}^{m} G_i)$, as well as $N(A_{even} \cup A_{odd}) = 2 \cdot N(A)$ and $N\big((\bigcup_{i=1}^{m} G_i)_{even} \cup (\bigcup_{i=1}^{m} G_i)_{odd}\big) = 2 \cdot N(\bigcup_{i=1}^{m} G_i)$. Therefore, $d(A) = d(\bigcup_{i=1}^{m} G_i) = \frac{m}{n}$ and $2 \cdot N(A) = 2 \cdot N(\bigcup_{i=1}^{m} G_i) = 0$. This means that $\psi(A, 2) = (d(A), 2 \cdot N(A)) = (\frac{m}{n}, 0)$. Now, assume that (ii) holds. Since A is equiprobable to $\emptyset$, according to C', this means that $A_{even} \cup A_{odd}$ is equiprobable to $\emptyset$, according to $C^{\#}$. Because **Qualitative Regularity** is a constraint in $C^{\#}$, $A_{even} \cup A_{odd}$ being equiprobable to $\emptyset$ implies that $A_{even} \cup A_{odd} = \emptyset$, which in turn implies that $A = \emptyset$. By **Theorem 34**, $\psi(A, 2) = \psi(\emptyset, 2) = (0, 0) = (\frac{m}{n}, 0)$. Either way, $\psi(A, 2) = (\frac{m}{n}, 0)$. The proof proceeds similarly for B, i.e. $\psi(B, 2) = (\frac{p}{q}, 0)$. Because $\psi(A, 1) = \psi(A, 2) = (\frac{m}{n}, 0)$, $\psi(A, i) = (\frac{m}{n}, 0)$. The same goes for B, i.e. $\psi(B, j) = (\frac{p}{q}, 0)$.

($\Rightarrow$) Assume that $(A, i) \preceq^* (B, j)$. By **Definition 3**, this means that $\psi(A, i) = (\frac{m}{n}, 0) \leq (\frac{p}{q}, 0) = \psi(B, j)$. Because ψ is lexicographically ordered, $\frac{m}{n} \leq \frac{p}{q}$.

($\Leftarrow$) Assume that $\frac{m}{n} \leq \frac{p}{q}$. Because ψ is lexicographically ordered, $\psi(A, i) = (\frac{m}{n}, 0) \leq (\frac{p}{q}, 0) = \psi(B, j)$. Apply **Definition 3** to get $(A, i) \preceq^* (B, j)$. Thus, $\preceq^*$ satisfies **Probability Ratios**. $\square$

*Proof of **Theorem 37**.* Let $\mathbb{F}$ be an ordered field and $x, y \in \mathbb{F}$. ($\Rightarrow$) Assume that $|x-y| < \frac{1}{n}$ for all $n \in \mathbb{N}$. For reductio, assume that there exist $\frac{m}{n}, \frac{p}{q} \in \mathbb{Q}$ where $\frac{m}{n} < \frac{p}{q}$ such that either (i) $x \leq \frac{m}{n} < \frac{p}{q} \leq y$ or (ii) $y \leq \frac{m}{n} < \frac{p}{q} \leq x$. This means that $|\frac{m}{n} - \frac{p}{q}| > \frac{1}{n}$ for some $n \in \mathbb{N}$. Since $|x - y| \geq |\frac{m}{n} - \frac{p}{q}|$, $|x - y| > \frac{1}{n}$. Contradiction. Therefore, there do not exist $\frac{m}{n}, \frac{p}{q} \in \mathbb{Q}$ where $\frac{m}{n} < \frac{p}{q}$ such that either (i) $x \leq \frac{m}{n} < \frac{p}{q} \leq y$ or (ii) $y \leq \frac{m}{n} < \frac{p}{q} \leq x$.

($\Leftarrow$) Assume that there do not exist $\frac{m}{n}, \frac{p}{q} \in \mathbb{Q}$ where $\frac{m}{n} < \frac{p}{q}$ such that either (i) $x \leq \frac{m}{n} < \frac{p}{q} \leq y$ or (ii) $y \leq \frac{m}{n} < \frac{p}{q} \leq x$. This means that $|x-y| < |\frac{m}{n} - \frac{p}{q}|$ for all $\frac{m}{n}, \frac{p}{q} \in \mathbb{Q}$ where $\frac{m}{n} < \frac{p}{q}$. Let $\frac{m}{n} = 0$ and $\frac{p}{q} = \frac{1}{n}$. So, $|\frac{m}{n} - \frac{p}{q}| = \frac{1}{n}$. Consequently, $|x - y| < \frac{1}{n}$ for all $n \in \mathbb{N}$. □

*Proof of **Theorem 38**.* Let $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ be a partition of $\mathbb{N}$ such that:

$$G_1 = \{1, 1+n, 1+2n, \ldots\}.$$

$$G_2 = \{2, 2+n, 2+2n, \ldots\}.$$

$$\cdots$$

$$G_n = \{n, 2n, 3n, \ldots\}.$$

It is easy to verify that $d(G_i) = d(G_j) = \frac{1}{n}$. So, by **Irregular Ordering**, G_i is equiprobable to G_j. Also, $d\left(\bigcup_{i=1}^{m} G_i\right) = \frac{m}{n}$. Now, consider an arbitrary infinite set $A \in \mathcal{F}^{\#}$. $d(A) = \frac{m}{n} > 0$ for some $\frac{m}{n} \in \mathbb{Q}$. So, for each infinite set $A \in \mathcal{F}^{\#}$, there exists a partition $\mathcal{G}$, as defined above, such that $d(A) = d\left(\bigcup_{i=1}^{m} G_i\right)$. Apply **Irregular Ordering** to get that A is equiprobable to $\bigcup_{i=1}^{m} G_i$. By **Definition 2**, this means that A is $\frac{m}{n}$ times as probable as $\mathbb{N}$.

Now, consider an arbitrary finite set $A \in \mathcal{F}^{\#}$. Note that $d(A) = 0 = d(\emptyset)$. By **Irregular Ordering**, A is equiprobable to $\emptyset$. By **Definition 2**, this means that A is $\frac{m}{n} = 0$ times as probable as $\mathbb{N}$. So, for all $A \in \mathcal{F}^{\#}$, A is $\frac{m}{n}$ times as probable as $\mathbb{N}$ for some $\frac{m}{n} \in \mathbb{Q}$. □

*Proof of **Theorem 39**.* This theorem is a direct consequence of **Probability Ratios** and **Theorem 38**. □

*Proof of **Theorem 40**.* ($\Rightarrow$) Assume that $A \preceq_3 B$. By **Definition 6**, $(A, B) \in \preceq_3$. This means that $(A, 3) \preceq^{**} (B, 3)$, which in turn implies that $d(A) \leq d(B)$. Recall that for all $A \in \mathcal{F}^{\#}$, $P(A) = d(A)$. So, $d(A) \leq d(B)$ is equivalent to $P(A) \leq P(B)$.

($\Leftarrow$) Assume that $P(A) \leq P(B)$. Recall that for all $A \in \mathcal{F}^{\#}$, $P(A) = d(A)$. So, $P(A) \leq P(B)$ is equivalent to $d(A) \leq d(B)$. This means that $(A, 3) \preceq^{**} (B, 3)$. By **Definition 6**, $(A, B) \in \preceq_3$, i.e. $A \preceq_3 B$. □

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