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A Perturbation Method for Elliptic and Parabolic Equations

Guji Tian * and Xu-Jia Wang†

Abstract

In this paper we recall an elementary perturbation method which was used to prove the $C^{2,\alpha}$ estimates and partial $C^{2,\alpha}$ estimates for linear and nonlinear elliptic and parabolic equations. This method also gives sharp estimates for the modulus of continuity of second derivatives when the inhomogeneous term is Dini continuous or Lipschitz continuous. In this paper we also present some improvements to our earlier works on the topic.

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1 Introduction

A perturbation method was introduced by Campanato [6, 7] to prove the $C^{2,\alpha}$ estimates for linear elliptic and parabolic equations. The method was used by Safanov [25, 26] and Caffarelli [3] to prove the $C^{2,\alpha}$ estimate for fully nonlinear elliptic equations, and was extended to fully nonlinear parabolic equations by Lihe Wang [32, 33].

The main ingredient of the perturbation method [6, 7, 26, 3] is to estimate the distance from the solution to a quadratic polynomial and prove a convergence rate of the polynomials. The perturbation argument is greatly simplified in [34]. It is shown that the $C^{2,\alpha}$ estimate can be easily obtained from the interior regularity of solutions with constant coefficients, by simply measuring the difference of the second derivatives of two consecutive approximating solutions. This perturbation technique has a number of nice properties which we would like to point out.

*Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071, China. Guji Tian was supported by NSFC Grants 11171339 and 11071246. E-mail: tian-guji@wipm.ac.cn

†Centre for Mathematics and Its Applications, Australian National University, Canberra, ACT 0200, Australia. Xu-Jia Wang was supported by the Australian Research Council. E-mail: Xu-Jia.Wang@anu.edu.au

- It uses only the maximum principle and the interior regularity for equations with constant coefficients. Hence it is not only elementary but also applies to both linear and fully nonlinear elliptic and parabolic equations, including the Monge-Ampere equation [34, 13].
- It gives a unified proof for the cases when the inhomogeneous term f is Dini continuous, Hölder continuous, or Lipschitz continuous.
- It gives sharp estimates for the modulus of continuity of second derivatives in the above mentioned cases.
- The method also yields a partial regularity of solutions for certain equations. That is if the coefficients and inhomogeneous term are Hölder continuous in a given direction ξ, then the second derivative $u_{\xi\xi}$ is Hölder continuous.

The perturbation method can also be used to establish different kind of estimates. The purpose of this article is to recall the perturbation method, and as well to introduce the partial regularity, to the readers. In this paper we also make some improvements to our earlier works [34, 13]. Namely, for linear elliptic equations and the Monge-Ampere equation, we are able to replace the uniform norm ω_f by the VMO norm $\Upsilon_{f,1}$, where $\Upsilon_{f,p}$ ($p \in [1, \infty]$) is defined in (1.5).

We say that a solution is *partially regular* if it is smooth in some directions but fails in others. For example, the function $u(x) = [\max(0, x_n)]^2$ is smooth in $x_1, \cdots, x_{n-1}$ but fail in x_n. Note that this partial regularity is different from the one in geometric measure theory. In geometric measure theory, a solution is partially regular if it is smooth in a subset but not in the whole domain. Partial regularity in geometric measure theory has been a subject of extensive studies for decades. Partial regularity in the above sense is also of interest and is useful in applications such as image processing [1]. Therefore it deserves a more detailed investigation.

Consider the Possion equation

$$\Delta u = f(x). \tag{1.1}$$

If f is Hölder continuous in a direction ξ, then one can prove that $u_{x\xi}$ is Hölder continuous in all variables. Hence the Possion equation satisfies our the partial regularity. Note that the partial regularity also implies the classical Schauder estimate when f is Hölder continuous in all directions. A question is whether this partial regularity holds for general linear and nonlinear elliptic and parabolic equations. By our perturbation method, we can give an affirmative answer for linear elliptic and parabolic equations when the coefficients are continuous.

There are numerous works on second derivative estimates for elliptic and parabolic equations. We will mention some related researches as we proceed. Here we would like to mention different proofs for the $C^{2,\alpha}$ estimate for the Possion equation (1.1). Traditionally the $C^{2,\alpha}$ estimate is built upon the Newton potential theory. Different proofs were found by Campanato [6, 7], who introduced a perturbation argument; Peetre[23], who used the convolution of functions; Trudinger [30], who used the mollification of functions; and Simon [27], who used a blow-up argument. Another different proof by the maximum principle was introduced by Brandt [2]. For the partial regularity we refer the reader to [11, 28, 9].

Next we introduce some notations and terminology needed below. Let ξ be a unit vector in R^n and ϕ be a function defined in Ω. Denote

$$\begin{aligned}\omega_{\phi,\xi}(r) &= \sup\{|\phi(x)-\phi(x+t\xi)|: \ x, x+t\xi\in\Omega, |t|<r\},\\ \omega_\phi(r) &= \sup\{\omega_{\phi,\xi}(r): \ \xi\in R^n, \ |\xi|=1\}.\end{aligned} \tag{1.2}$$

We say ϕ is Dini continuous in the direction ξ if

$$\int_0^1 \frac{\omega_{\phi,\xi}(r)}{r}dr<\infty. \tag{1.3}$$

We say ϕ is Hölder continuous in direction ξ with Hölder exponent α if $\omega_{\phi,\xi}\in C^\alpha$, and denote as $\phi\in C^\alpha_\xi(\Omega)$ (or $C^{0,1}_\xi(\Omega)$ if $\alpha=1$), with the norm

$$\|\phi\|_{C^\alpha_\xi(\Omega)}=\sup_{x\in\Omega}|\phi(x)|+\sup_{t>0}\frac{\omega_{\phi,\xi}(t)}{t^\alpha}. \tag{1.4}$$

We also denote

$$\begin{aligned}\Upsilon_{f,p}(x_0,r) &= \left(|B_r(x_0)|^{-1}\int_{B_r(x_0)}|f(x)-\bar f_{x_0,r}|^p dx\right)^{\frac{1}{p}},\\ \Upsilon_{f,p}(r) &= \sup\nolimits_{x\in\Omega}\Upsilon_{f,p}(x,r),\end{aligned} \tag{1.5}$$

where $1\le p\le\infty$ is a constant, $B_r(x_0)\subset\Omega$ is a ball,

$$\bar f_{x_0,r}=|B_r(x_0)|^{-1}\int_{B_r(x_0)}f(x)dx$$

is the average of f in $B_r(x_0)$. By the definition we have

$$\Upsilon_{f,p}(r)\le\Upsilon_{f,\infty}(r)=\omega_f(r), \quad \forall\, p\ge1. \tag{1.6}$$

The quantity $\Upsilon_{f,\infty}(r)$ represents the L^∞-oscillation of f in a ball B_r and $\Upsilon_{f,1}(r)$ represents the L^1-oscillation of f in B_r. By extending f to R^n, we may assume that the function $\Upsilon_{f,p}(x_0,r)$ is defined for all $r>0$. When $p=1$, we will drop the subscript p in the notation $\Upsilon_{f,p}$.

For pointwise $C^{2,\alpha}$ estimates of solutions to linear elliptic equations and the Monge-Ampere equation, we show that the Dini and Hölder continuity of f can be replaced by

$$\int_0^1\frac{\Upsilon_f(x,r)}{r}dr<\infty, \qquad \text{and} \qquad \Upsilon_f(x,r)\le Cr^\alpha. \tag{1.7}$$

The paper is organized as follows. In Section 2 we adapt the perturbation argument from [34] to prove an interior second derivative estimate for linear elliptic equations, including the Possion equation (1.1), assuming that f satisfies (1.7). In Section 3, we apply the perturbation method to obtain the $C^{2+\alpha}$ estimate for fully nonlinear elliptic equations, including the Monge-Ampere equation [13]. In Section 4, we discuss the partial $C^{2+\alpha}$ estimates for elliptic equations obtained in [28]. The last Section 5 is devoted to the Hölder continuity of $\partial_x^2 u$ for fully nonlinear parabolic equations when $f=f(x,t)$ is Hölder continuous in x but only measurable and bounded in t [29]. This paper is dedicated to the memory of Professor Chern with full respect and admiration.

2 Estimates for linear equations

We first prove a pointwise estimate for the modulus of continuity for the second derivatives of solutions to the Possion equation (1.1). In Theorem 2.1 below we will replace the oscillation ω_f in earlier works by the quantity Υ_f. In our proof we may assume the solution is smooth. For weak solutions, we may define $D^2u(0) = \lim_{k\to\infty} D^2u_k(0)$, as long as the integral on the right hand side of (2.1) is convergent, where u_k is the solution of (2.2).

Theorem 2.1. *Let u be a solution to* (1.1). *Then $\forall\, y, z \in \Omega_\delta$, we have the estimate*

$$|\partial_x^2 u(y) - \partial_x^2 u(z)| \le Cd\Bigg(\sup_\Omega |u| + \|f\|_{L^1(\Omega)} + \frac{1}{d}\int_0^d \frac{\Upsilon_f(y,r)+\Upsilon_f(z,r)}{r}dr + \int_d^1 \frac{\Upsilon_f(y,r)}{r^2}dr\Bigg), \quad (2.1)$$

where ∂_x denotes partial derivative in any direction, $d = |y-z|$, $\delta > 0$ is any given constant, $\Omega_\delta = \{x \in \Omega : \ dist(x, \partial\Omega) > \delta\}$, $C > 0$ is a constant depending only on n, Ω and δ.

Proof. Let us assume that $y = 0$ the origin. Denote $B_k = B_{\rho^k}(0)$ (here we take $\rho = \frac{1}{2}$). For $k = 0, 1, 2, \cdots$, let u_k be the solution of

$$\begin{aligned} \Delta u_k &= \bar f_{0,\rho^k} \quad \text{in } B_k, \\ u_k &= u \quad \text{on } \partial B_k, \end{aligned} \quad (2.2)$$

where $\bar f_{0,\rho^k}$ is the average of f in $B_k(0)$. Then

$$\begin{aligned} \Delta(u_k - u) &= \bar f_{0,\rho^k} - f(x) \quad \text{in } B_k, \\ u_k - u &= 0 \quad \text{on } \partial B_k. \end{aligned}$$

Using the fundamental solution and a rescaling argument, we have

$$\frac{1}{|B_k|}\int_{B_k} |u_k - u| dx \le C\rho^{2k}\Upsilon_f(0,\rho^k). \quad (2.3)$$

We get the estimate

$$\frac{1}{|B_{k+1}|}\int_{B_{k+1}} |u_k - u_{k+1}| dx \le C\rho^{2k}\Upsilon_f(0,\rho^k).$$

Notice that

$$\Delta(u_k - u_{k+1}) = \bar f_{0,\rho^k} - \bar f_{0,\rho^{k+1}} \quad \text{in } B_{k+1}.$$

By the mean value formula for the harmonic function

$$w := u_k - u_{k+1} - \frac{1}{2n}(\bar f_{0,\rho^k} - \bar f_{0,\rho^{k+1}})|x|^2,$$

we have

$$\sup_{B_{k+2}(0)}|u_k - u_{k+1}| \le C\left(\rho^{2k}|\bar{f}_{0,\rho^k} - \bar{f}_{0,\rho^{k+1}}| + |B_k|^{-1}\|u_k - u_{k+1}\|_{L^1(B_{k+1}(0))}\right).$$

To estimate $|\bar{f}_{0,\rho^k} - \bar{f}_{0,\rho^{k+1}}|$, by subtracting a constant we assume that $\bar{f}_{0,\rho^k} = 0$. Then

$$\begin{aligned}|\bar{f}_{0,\rho^k} - \bar{f}_{0,\rho^{k+1}}| &= |B_{k+1}(0)|^{-1}\int_{B_{k+1}(0)} |\bar{f}_{0,\rho^k} - \bar{f}_{0,\rho^{k+1}}|dx \\ &= |B_{k+1}(0)|^{-1}\int_{B_{k+1}(0)} \left(|f(x)| + |f(x) - \bar{f}_{0,\rho^{k+1}}|\right)dx \\ &\le |B_{k+1}(0)|^{-1}\int_{B_k(0)} |f(x)|dx + \Upsilon_f(0,\rho^{k+1}) \\ &\le C\Upsilon_f(0,\rho^k).\end{aligned} \tag{2.4}$$

Hence we obtain

$$\|u_k - u_{k+1}\|_{L^\infty(B_{k+2})} \le C\rho^{2k}\Upsilon_f(0,\rho^k). \tag{2.5}$$

Since the function w is harmonic, we have

$$|\partial_x^j w(0)| \le C\rho^{-kj}\|w\|_{L^\infty(B_k)}.$$

It follows that

$$\begin{aligned}|\partial_x^j(u_k - u_{k+1})(0)| &\le C\rho^{-kj}\|u_k - u_{k+1}\|_{L^\infty(B_{k+1})} + C\rho^{-k(j-2)}\Upsilon_f(0,\rho^k) \\ &\le C\rho^{-k(j-2)}\Upsilon_f(0,\rho^k)\end{aligned} \tag{2.6}$$

for any integer $j \ge 0$, where C depends on j. In particular,

$$|\partial_x^2(u_k - u_{k+1})(0)| \le C\Upsilon_f(0,\rho^k). \tag{2.7}$$

If u is smooth, so are f and u_k. Let $Q = u(0) + Du(0)x + \frac{1}{2}\sum_{i,j=1}^n u_{x_ix_j}(0)x_ix_j$. Then

$$|u_k - Q| \le |u_k - u| + |u - Q| = o(r^2) \quad \text{in } B_r.$$

Hence

$$\lim_{k\to\infty}\partial_x^2 u_k(0) = \partial_x^2 u(0). \tag{2.8}$$

If u is not smooth, we may use (2.8) to define $\partial_x^2 u(0)$.

Let z be a point near 0. We have

$$|\partial_x^2 u(z) - \partial_x^2 u(0)| \le I_1 + I_2 + I_3, \tag{2.9}$$

where

$$\begin{aligned}I_1 &= |\partial_x^2 u_m(0) - \partial_x^2 u(0)|, \\ I_2 &= |\partial_x^2 u_m(z) - \partial_x^2 u_m(0)|, \\ I_3 &= |\partial_x^2 u(z) - \partial_x^2 u_m(z)|.\end{aligned}$$

We choose $m \geq 1$ such that

$$\rho^{m+4} \leq |z| < \rho^{m+3}. \tag{2.10}$$

First we estimate I_1. By (2.7) we have

$$\begin{aligned} I_1 &\leq \sum_{k\geq m} |\partial_x^2 u_k(0) - \partial_x^2 u_{k+1}(0)| \\ &\leq C\sum_{k\geq m} \Upsilon_f(0,\rho^k) \\ &\leq C\int_0^d \frac{\Upsilon_f(0,r)}{r}dr, \end{aligned} \tag{2.11}$$

where the constant C may change from line to line.

Next we estimate I_2. Denote $h_k = u_k - u_{k-1}$. When $k \leq m$, z is an interior point of B_k. By (2.6)

$$\begin{aligned} |\partial_x^2 h_k(z) - \partial_x^2 h_k(0)| &\leq |z||\partial_x^3 h_k(\hat{z})| \\ &\leq C|z|\rho^{-3k}\|u_k - u_{k-1}\|_{L^\infty(B_{k+1})} \\ &\leq C|z|\rho^{-k}\Upsilon_f(0,\rho^k), \end{aligned} \tag{2.12}$$

where $\hat{z}$ is some point between 0 and z. Since $\Delta u_0 = \bar{f}_{0,1}$ is a constant, we have

$$\begin{aligned} |\partial_x^2 u_0(z) - \partial_x^2 u_0(0)| &\leq |z||\partial_x^3 u_0(\hat{z})| \\ &\leq C|z|(\sup_{\partial B_1(0)}|u_0| + |\bar{f}_{0,1}|) \\ &\leq C|z|(\|u\|_{L^\infty(B_1(0))} + \|f\|_{L^1(B_1(0))}), \end{aligned} \tag{2.13}$$

where $\hat{z}$ is some point between 0 and z. Combining (2.12) with (2.13), we obtain

$$\begin{aligned} I_2 &= |\partial_x^2 u_m(z) - \partial_x^2 u_m(0)| \\ &\leq |\partial_x^2 u_{m-1}(z) - \partial_x^2 u_{m-1}(0)| + |\partial_x^2 h_m(z) - \partial_x^2 h_m(0)| \\ &\leq |\partial_x^2 u_0(z) - \partial_x^2 u_0(0)| + \sum_{k=1}^m |\partial_x^2 h_k(z) - \partial_x^2 h_k(0)| \\ &\leq |\partial_x^2 u_0(z) - \partial_x^2 u_0(0)| + C|z|\sum_{k=1}^m \rho^{-k}\Upsilon_f(0,\rho^k) \\ &\leq C|z|\left(\|u\|_{L^\infty(B_1(0))} + \|f\|_{L^1(B_1(0))} + \int_{|z|}^1 \frac{\Upsilon_f(0,r)}{r^2}\right). \end{aligned} \tag{2.14}$$

Finally we estimate I_3. Let $\hat{u}_m$ be the solution of

$$\begin{aligned} \Delta\hat{u}_m &= \bar{f}_{z,\rho^m} \quad \text{in } B_{\rho^k}(z), \\ \hat{u}_m &= u \qquad \text{on } \partial B_{\rho^k}(z). \end{aligned}$$

We have

$$I_3 \leq |\partial_x^2 u_m(z) - \partial_x^2\hat{u}_m(z)| + |\partial_x^2 u(z) - \partial_x^2\hat{u}_m(z)|.$$

Similarly to (2.11), we have

$$|\partial_x^2 u(z) - \partial_x^2\hat{u}_m(z)| \leq C\int_0^d \frac{\Upsilon_f(z,r)}{r}dr. \tag{2.15}$$

To estimate $|\partial_x^2 u_m(z) - \partial_x^2 \hat{u}_m(z)|$, we denote $w = u_m - \hat{u}_m$. Then

$$\Delta w = \bar{f}_{0,\rho^m} - \bar{f}_{z,\rho^m} \quad \text{in} \quad B_{\rho^{m+1}}(0).$$

By (2.3),

$$\begin{aligned}\|w\|_{L^1(B_{m+1}(z))} &\leq \|u_m - u\|_{L^1(B_{m+1}(z))} + \|\hat{u}_m - u\|_{L^1(B_{m+1}(z))} \\ &\leq \|u_m - u\|_{L^1(B_m(0))} + \|\hat{u}_m - u\|_{L^1(B_m(z))} \\ &\leq C\rho^{(n+2)m}\left(\Upsilon_f(0,\rho^m) + \Upsilon_f(z,\rho^m)\right).\end{aligned} \tag{2.16}$$

Hence similarly to (2.5) we have

$$\|w\|_{L^\infty(B_{m+2}(z))} \leq C\rho^{2m}\left(\Upsilon_f(0,\rho^m) + \Upsilon_f(z,\rho^m)\right). \tag{2.17}$$

It follows that

$$|\partial_x^2 w(z)| \leq C\left(\Upsilon_f(0,\rho^m) + \Upsilon_f(z,\rho^m)\right). \tag{2.18}$$

Therefore we obtain

$$I_3 \leq C\int_0^d \frac{\Upsilon_f(z,r)}{r}dr + C\Upsilon_f(0,\rho^m). \tag{2.19}$$

Combining the estimates (2.11), (2.14) and (2.19), we obtain (2.1). This completes the proof. □

Note that

$$\Upsilon_f(x_0, r) \leq \omega_f(r) \quad \forall x_0 \in \Omega_\delta.$$

Hence we obtain

Corollary 2.1. ([34]) *Let $u \in C^2(\Omega)$ be a solution of (1.1). Then $\forall\, x, z \in \Omega_\delta$, we have the estimate*

$$|\partial_x^2 u(x) - \partial_x^2 u(z)| \leq Cd\left(\sup_\Omega |u| + \|f\|_{L^1(\Omega)} + \frac{1}{d}\int_0^d \frac{\omega_f(r)}{r} + \int_d^1 \frac{\omega_f(r)}{r^2}\right), \tag{2.20}$$

where $C > 0$ is a constant depending only on n, Ω and δ.

Estimate (2.20) yields a continuity estimate for the second derivatives $\partial_x^2 u$ in terms of the Dini continuity of f. That is if f is Dini continuous,

$$|\partial_x^2 u(x) - \partial_x^2 u(z)| = o(1) \quad \text{as} \quad d \to 0. \tag{2.21}$$

If $f \in C^\alpha (0 < \alpha < 1)$, from (2.20) we have

$$\|\partial_x^2 u\|_{C^\alpha(\Omega_\delta)} \leq C\left(\sup_\Omega |u| + \frac{\|f\|_{C^\alpha(\Omega)}}{\alpha(1-\alpha)}\right). \tag{2.22}$$

If $f \in C^{0,1}$, from (2.20) it follows that

$$|\partial_x^2 u(x) - \partial_x^2 u(z)| \leq Cd\left(\sup_\Omega |u| + \|f\|_{C^{0,1}}|\log d|\right). \tag{2.23}$$

The perturbation method also applies to the boundary $C^{2,\alpha}$ estimate. That is

Corollary 2.2. ([34]) *Let $u \in C^2(B_1 \cap \{x_n \geq 0\})$ be a solution of* (1.1) *with $\Omega = B_1 \cap \{x_n > 0\}$. If $u = 0$ on $\{x_n = 0\}$ and f is Dini continuous. Then $\forall x, y \in B_{1/2} \cap \{x_n \geq 0\}$, the estimate* (2.20) *holds.*

Corollary 2.2 is a direct consequence of Corollary 2.1, provided we replace B_{ρ^k} by $B_{\rho^k} \cap \{x_n \geq 0\}$ and notice that if w is a harmonic function in $B_1 \cap \{x_n \geq 0\}$ and $w = 0$ on $\{x_n = 0\}$, then w is harmonic in B_1 after odd extension in x_n−axis.

From the estimate of I_1 for $m = 0$, we also have

Corollary 2.3. *Let $u \in C^2(\Omega)$ be a solution of* (1.1). *Then $\forall\ x \in \Omega_\delta$, we have the estimate*

$$|\partial_x^2 u(x)| \leq C \left(\sup_\Omega |u| + \|f\|_{L^1(\Omega)} + \int_0^1 \frac{\Upsilon_f(0,r)}{r} \right) \tag{2.24}$$

where $C > 0$ is a constant depending only on n, Ω and δ.

The above perturbation argument also applies to the first derivatives. In particular if $f \in L^\infty(\Omega)$, it yields a log-Lipschitz estimate for gradient of solutions to the Possion equation [34].

Theorem 2.2. *Let $u \in C^1(\Omega)$ be a solution of* (1.1). *Then $\forall x, y \in \Omega_\delta$,*

$$\begin{aligned} |Du(y) - Du(z)| \leq Cd \Bigg(& \sup_\Omega |u| + \|f\|_{L^1(\Omega)} \\ & + \frac{1}{d} \int_0^d (\Upsilon_f(y,r) + \Upsilon_f(z,r)) dr + \int_d^1 \frac{\Upsilon_f(y,r)}{r} dr \Bigg), \end{aligned} \tag{2.25}$$

where C and d are the the same as those in Theorem 2.1.

Next we consider the linear elliptic equations with variable coefficients,

$$\sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j} = f(x) \quad \text{in} \quad \Omega. \tag{2.26}$$

By the standard freezing coefficient argument, one can extend Corollary 2.1 to equation (2.26). Here we apply the perturbation and the freezing coefficients techniques simultaneously to get the estimate (2.1) for equation (2.26). In particular in the estimate (2.27) below, we can replace the usual oscillation ω_a by the quantity Υ_a.

Theorem 2.3. *Let $u \in C^2(\Omega)$ be a solution of* (2.26). *Then $\forall\ y, z \in \Omega_\delta$,*

$$\begin{aligned} |\partial_x^2 u(y) - \partial_x^2 u(z)| \leq C \sup_\Omega |\partial_x^2 u| & \left(\int_0^d \frac{\Upsilon_a(y,r) + \Upsilon_a(z,r)}{r} dr + d \int_d^1 \frac{\Upsilon_a(y,r)}{r^2} dr \right) \\ & + Cd \left(\sup_\Omega |u| + \|f\|_{L^1(\Omega)} + \frac{1}{d} \int_0^d \frac{\Upsilon_f(y,r) + \Upsilon_f(z,r)}{r} dr \right. \\ & \left. + \int_d^1 \frac{\Upsilon_f(y,r)}{r^2} dr, \right. \end{aligned} \tag{2.27}$$

where $\Upsilon_a(y,r) = \max_{i,j} \Upsilon_{a_{ij}}(y,r)$, $C > 0$ is a constant depending only on n, Ω and δ.

Proof. The proof is similar to that of Theorem 2.1. Let us assume that $\Omega = B_1(0)$, $y = 0$ and z near 0. Denote $B_k = B_{\rho^k}(0)$ ($\rho = \frac{1}{2}$). For $k = 0, 1, 2, \cdots$, let u_k be the solution of

$$\begin{aligned} \sum_{i,j=1}^{n} \bar{a}_{ij}^k u_{x_i x_j} &= \bar{f}_{0,\rho^k} \quad \text{in } B_k, \\ u_k &= u \qquad \text{on } \partial B_k, \end{aligned} \tag{2.28}$$

where $\bar{a}_{ij}^k = |B_k|^{-1} \int_{B_k} a_{ij}$. We point out that if $\bar{a}_{ij}^k$ in (2.28) is replaced by $a_{ij}(0)$, the following proof can be simplified but that the quantity Υ_a in (2.27) needs to be replaced by ω_a.

From (2.26) and (2.28), we have

$$\begin{aligned} \bar{a}_{ij}^k (u - u_k)_{x_i x_j} &= f(x) - \bar{f}_{0,\rho^k} + (a_{ij}(x) - \bar{a}_{ij}^k) u_{x_i x_j} \quad \text{in } B_k, \\ u_k - u &= 0 \qquad \text{on } \partial B_k. \end{aligned} \tag{2.29}$$

Similar to (2.3) we have

$$\begin{aligned} \int_{B_k} |u_k - u|(x)dx &\le C\rho^{2k} \int_{B_k} \left(|f(x) - \bar{f}_{0,\rho^k}| + |(a_{ij}(x) - \bar{a}_{ij}^k) u_{x_i x_j}| \right) dx \\ &\le C\rho^{(2+n)k} \left(\Upsilon_f(0, \rho^k) + \|D^2 u\|_{L^\infty(B_1)} \Upsilon_a(0, \rho^k) \right). \end{aligned} \tag{2.30}$$

Hence

$$\int_{B_{k+1}} |u_k - u_{k+1}|(x)dx \le C\rho^{(2+n)k} \left(\Upsilon_f(0, \rho^k) + \|D^2 u\|_{L^\infty(B_1)} \Upsilon_a(0, \rho^k) \right). \tag{2.31}$$

The function $w := u_k - u_{k+1}$ satisfies the equation

$$a_{ij}^{k+1} w_{x_i x_j} = (\bar{f}_{0,\rho^k} - \bar{f}_{0,\rho^{k+1}}) + (a_{ij}^{k+1} - a_{ij}^k)(u_k)_{x_i x_j}. \tag{2.32}$$

By (2.4) we have

$$\begin{aligned} &|\bar{f}_{0,\rho^k} - \bar{f}_{0,\rho^{k+1}}| \le C\Upsilon_f(0, \rho^k), \\ &|(a_{ij}^{k+1} - a_{ij}^k)(u_k)_{x_i x_j}| \le C\Upsilon_a(0, \rho^k) \|D^2 u_k\|_{L^\infty(B_{k+1})} \quad \text{in } B_{k+1}. \end{aligned}$$

Let w_1 be a harmonic function satisfying $w_1 = w$ on ∂B_{k+1}, and w_2 be the function satisfying equation (2.32) in B_{k+1} and vanishing on ∂B_{k+1}. By the decomposition $w = w_1 + w_2$, we obtain

$$\|u_k - u_{k+1}\|_{L^\infty(B_{k+1})} \le C\rho^{2k} \left(\Upsilon_f(0, \rho^k) + \|D^2 u_k\|_{L^\infty(B_{k+1})} \Upsilon_a(0, \rho^k) \right). \tag{2.33}$$

Hence similarly to (2.6) we have

$$|\partial_x^j (u_k - u_{k+1})(0)| \le C\rho^{-k(j-2)} \left(\Upsilon_f(0, \rho^k) + \|D^2 u_k\|_{L^\infty(B_{k+1})} \Upsilon_a(0, \rho^k) \right) \tag{2.34}$$

for any integer $j \ge 0$. By equation (2.28) and noticing that $\bar{f}_{0,\rho^k}$ is a constant, we have

$$\|D^2 u_k\|_{L^\infty(B_{k+1})} \le C(|\bar{f}_{0,\rho^k}| + \|D^2 u\|_{B_1}) \le C\|D^2 u\|_{B_1}.$$

Hence we obtain

$$|\partial_x^j(u_k - u_{k+1})(0)| \leq C\rho^{-k(j-2)} \left(\Upsilon_f(0,\rho^k) + \|D^2u\|_{L^\infty(B_1)}\Upsilon_a(0,\rho^k)\right). \quad (2.35)$$

Let z be a point near 0. As (2.9) we have

$$\begin{aligned}|\partial_x^2 u(z) - \partial_x^2 u(0)| &\leq I_1 + I_2 + I_3 \\ &:= |\partial_x^2 u_m(0) - \partial_x^2 u(0)| + |\partial_x^2 u_m(z) - \partial_x^2 u_m(0)| \\ &\quad + |\partial_x^2 u(z) - \partial_x^2 u_m(z)|. \end{aligned} \quad (2.36)$$

Choose $m \geq 1$ such that $\rho^{m+4} \leq |z| < \rho^{m+3}$.

Similarly to (2.11), we have

$$I_1 \leq C \int_0^d \frac{\Upsilon_f(0,r) + \|D^2u\|_{L^\infty(B_1)}\Upsilon_a(0,r)}{r} dr. \quad (2.37)$$

I_2 and I_3 can be estimated similarly and we have

$$I_2 \leq C|z| \left(\|u\|_{L^\infty(B_1(0))} + \|f\|_{L^\infty(B_1(0))} + \int_{|z|}^1 \frac{\Upsilon_f(0,r) + \|D^2u\|_{L^\infty(B_1)}\Upsilon_a(0,r)}{r^2} \right).$$

$$I_3 \leq C \int_0^d \frac{\Upsilon_f(z,r) + \|D^2u\|_{L^\infty(B_1)}\Upsilon_a(z,r)}{r} dr + C\left(\Upsilon_f(0,r) + \|D^2u\|_{L^\infty(B_1)}\Upsilon_a(0,r)\right).$$

Combining the above estimates, we obtain (2.27). This completes the proof. □

From (2.27) we also infer that if a_{ij}, f are Hölder continuous with exponent $\alpha \in (0,1)$, then

$$\|\partial_x^2 u\|_{C^\alpha(\Omega_\delta)} \leq C \left(\sup_\Omega |u| + \frac{\|f\|_{C^\alpha(\Omega)}}{\alpha(1-\alpha)} \right). \quad (2.38)$$

If $a_{ij}, f \in C^{0,1}$, we have the log-Lipschitz continuity of D^2u,

$$|\partial_x^2 u(x) - \partial_x^2 u(z)| \leq Cd \left(\sup_\Omega |u| + \|f\|_{C^{0,1}} |\log d| \right). \quad (2.39)$$

Remark. The reader may have noticed that the term $\|f\|_{L^1(\Omega)}$ in the estimates (2.1), (2.24), and (2.27) can be dropped, as it can be controlled by other terms on the right hand side of these estimates. For example, if f is a constant, then the second and third derivatives of u can be controlled by $\|u\|_{L^\infty}$, which in turn means f can be controlled by $\|u\|_{L^\infty}$.

3 Estimates for fully nonlinear equations

In this section we use the perturbation argument to establish the corresponding estimates for the Monge-Ampere equation

$$\det D^2 u = f \quad \text{in} \quad B_1(0), \quad (3.1)$$

where f satisfies

$$C_1 \leq f \leq C_2 \quad (3.2)$$

for some positive constants $C_2 \geq C_1 > 0$.

Theorem 3.1. *Let $u \in C^2$ be a solution of (3.1). Then $\forall\, x, y \in B_{1/2}(0)$,*

$$|\partial_x^2 u(y) - \partial_x^2 u(z)| \le Cd\Bigg(\sup_{\Omega}|u| + \|f\|_{L^1(\Omega)} + \frac{1}{d}\int_0^d \frac{\Upsilon_f(y,r)+\Upsilon_f(z,r)}{r}dr + \int_d^1 \frac{\Upsilon_f(y,r)}{r^2}dr\Bigg), \quad (3.3)$$

where $d = |y-z|$, and $C > 0$ depends only on n, m and C_1, C_2.

Note that estimate (3.3) is in the same form as (2.1), which improves the corresponding estimate in [13] by replacing ω_f by Υ_f. For the Monge-Ampere equation, we will define the quantity Υ_f in a different way, namely

$$\Upsilon_f(x_0, r) = |S_h^0(x_0)|^{-1}\int_{S_h^0(x_0)} |f(x) - \bar{f}_{x_0,h}|dx, \quad (3.4)$$

where $h = r^2$ and $\bar{f}_{x_0,r}$ is the average of f in the sub-level set $S_h^0(x_0)$, defined in (3.9) below. The quantity $\Upsilon_f(x_0, r)$ measures the L^1 difference of f and its averages in the sub-level sets of the solution u, rather than in balls. This new definition is more convenient for the perturbation argument below. When $\Upsilon_f(r)/r$ is integrable, the second derivatives of u is bounded and the definition of Υ_f in (3.4) is equivalent to that in Section 1. From Theorem 3.1 it follows that

(i) If f is Dini continuous, then we obtain an estimate for the modulus of continuity of D^2u.

(ii) If $f \in C^\alpha(B_1)$ and $\alpha \in (0,1)$, then

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \le C\left(\sup_{B_1}|u| + \frac{\|f\|_{C^\alpha(B_1)}}{\alpha(1-\alpha)}\right). \quad (3.5)$$

(iii) If $f \in C^{0,1}(B_1)$, then

$$|\partial_x^2 u(x) - \partial_x^2 u(y)| \le C\Big(\sup_{B_1}|u| + \|f\|_{C^{0,1}(B_1)}|\log d|\Big). \quad (3.6)$$

The interior $C^{2,\alpha}$ estimate for the Monge-Ampere equation was pointed out in [4]. Estimate of the form (3.5) and its detailed proof were given in [13, 34].

In Theorem 3.1 we denote by m the modulus of convexity of u, which is defined by

$$m(t) := \inf\{u(x) - \ell_z(x) : |x - z| > t\}, \quad (3.7)$$

where $t > 0$, ℓ_z is the tangent plane of u at z. Obviously m is a nonnegative function. When u is strictly convex, it is a positive function. It was proved in [3] that if u is a convex solution of (3.1), vanishes on $\partial\Omega$, then u is strictly convex. Note that in the estimate (3.3), the constant C depends also on $\sup_{B_1}(u - \ell_0)$, which is in turn determined by m, C_1 and C_2. The proof of Theorem 3.1 is similar to that of Theorem 2.1, except that we need to normalize the level sets of the strictly convex solution, and use level sets instead of balls.

Lemma 3.1. *Let Ω be a bounded convex domain in R^n. Then there is a unique minimum ellipsoid containing Ω, which attains the minimum volume among all ellipsoids containing Ω.*

Minimum ellipsoid is very useful in studying the Monge-Ampere equation, as the equation is invariant under the change $x \to Tx$ and $u \to (\det T)^{\frac{2}{n}} u$, for any non-degenerate matrix T. We say a convex set Ω is normalized if its minimum ellipsoid is a ball. When Ω is normalized, one has $B_{r/n} \subset \Omega \subset B_r$ for concentrated balls $B_{r/n}$ and B_r. Therefore for any bounded convex domain Ω, there is a unique unimodular linear transformation T (namely $\det T = 1$) such that $T(\Omega)$ is normalized. Choose an appropriate coordinate system such that the minimum ellipsoid of Ω is given by $E = \left\{\sum_{i=1}^n \frac{x_i^2}{a_i^2} < 1\right\}$ with $a_1 \geq a_2 \geq \cdots \geq a_n$. Then we say Ω has a good shape if

$$\lambda_n c^* \leq \lambda_1 \tag{3.8}$$

for some constant c^* under control. If Ω has a good shape, then there exist two concentrated balls B_r and B_R with $R \leq nc^* r$ such that $B_r \subset \Omega \subset B_R$.

Let u be a convex function defined in a bounded convex domain Ω. For any $y \in \Omega$, denote by

$$S^0_{h,u}(y) = \{x \in \Omega : u(x) < l_y(x) + h\} \tag{3.9}$$

the sub-level set of u and denote by $S_{h,u}(y) = \partial S^0_{h,u}(y)$ its boundary, where l_y is the tangent plane of u at y. When no confuses arises we will drop the subscript u, and when y is the minimum point of u, we will simply write the level set as S^0_h.

In the proof of Theorem 2.1, we have used strongly the interior regularity of harmonic functions. For the Monge-Ampere equation, correspondingly we have the following regularity.

Lemma 3.2. *Let Ω be a bounded convex domain in R^n. Let u be a convex solution of $\det D^2u = 1$ in Ω, vanishing on $\partial\Omega$. If $B_r(0) \subset \Omega \subset B_R(0)$, then for any $\Omega' \subset \Omega$, there is a constant $C > 0$, depending only on n, r, R, and dist(Ω', Ω), such that*

$$\|u\|_{C^4(\Omega')} \leq C. \tag{3.10}$$

By Lemma 3.2, we then have

Lemma 3.3. *Let $u_i, i = 1, 2$, be two strictly convex solutions of $\det D^2u = 1$ in $B_1(0)$. Suppose $\|u_i\|_{C^4} \leq C_0$. Then if $u_1 - u_2 \leq \delta$ in $B_1(0)$ for some constant $\delta > 0$, we have, for $1 \leq k \leq 3$,*

$$|D^k(u_1 - u_2)| \leq C\delta \quad in \quad B_{1/2}(0). \tag{3.11}$$

To see estimate (3.11), it suffices to notice that $u_2 - u_1$ satisfies the linearized equation of the Monge-Ampere equation, and by Lemma 3.2, the linearized equation is uniformly elliptic and has smooth coefficients.

Lemma 3.2 also implies that if u is a convex solution of $\det D^2u = 1$ in Ω which vanishes on $\partial\Omega$, and if $D^2u(0)$ is the unit matrix (or uniformly bounded), then the domain Ω must have a good shape. We refer the reader to [13] for more details.

To prove Theorem 3.1, we need to replace the oscillation ω_f in [13] by the quantity Υ_f, for which we need the following lemma.

Lemma 3.4. *Let Ω be a bounded convex domain in R^n with good shape. Let u, v be solutions to*

$$\begin{aligned} \det D^2 u = 1 \quad & in \;\; \Omega, \\ \det D^2 v = f \quad & in \;\; \Omega, \\ u = v = 0 \quad & on \;\; \partial\Omega, \end{aligned} \tag{3.12}$$

where f satisfies (3.2). *Then we have the estimate*

$$\sup_{\Omega} |u - v| \le C|f - 1|_{L^1(\Omega)}, \tag{3.13}$$

where C depends on n, C_1, C_2 and Ω.

To prove (3.13), noticing that $u-v$ satisfies a linearized equation and $f \in L^\infty$, hence by Aleksandrov's maximum principle, we have

$$\sup_{\Omega} |u - v| \le C|f - 1|_{L^n(\Omega)} \le C'|f - 1|_{L^1(\Omega)}.$$

Proof of Theorem 3.1. The proof is based on the same perturbation technique as in Section 2. Here we point out the differences in the argument.

The first difference is that we must choose an appropriate sub-level set to start. By subtracting a linear function we suppose

$$u(0) = 0, Du(0) = 0.$$

Let $h > 0$ small such that S_h^0 is compactly supported in $B_1(0)$. By Lemma 3.1 there is a unique unimodular linear transform T_h such that $T_h(S_h^0)$ is normalized. Hence by making the change $x \to \frac{T_h x}{\sqrt{h}}$ and $u \to \frac{u}{h}$, we may suppose $h = 1$, S_1^0 is normalized, and

$$\int_0^1 \frac{\Upsilon_f(x_0, r)}{r} \le \epsilon \tag{3.14}$$

for any $x_0 \in B_{1/2}(0)$, where $\epsilon > 0$ can be as small as we want, provided h is sufficiently small.

The second difference is that we let u_k, where $k = 1, 2, \cdots$, be solutions of the Monge-Ampere equation in the sub-level set $S_{4^{-k}}^0$ rather than in balls. Namely u_k is the solution of

$$\begin{cases} \det D^2 u_k = \bar{f}_k & \text{in } \; S^0_{4^{-k},u}, \\ u_k = u (= 4^{-k}) & \text{on } \; \partial S^0_{4^{-k},u}, \end{cases} \tag{3.15}$$

where $\bar{f}_k$ is the average of f in $S^0_{4^{-k},u}$.

When ϵ is sufficiently small, we can show by induction that the sub-level sets $S^0_{4^{-k},u}$ has a good shape for all k. Hence by Lemma 3.2, u_k are uniformly smooth. By the uniform regularity of u_k and with the help of Lemmas 3.3 and 3.4, we can proceed in the same way as in Section 2 to obtain the estimate (3.3). We omit the details here. □

We point out that the perturbation argument has also been applied to Monge-Ampere type equation arising in optimal transportation [21].

Similar estimate holds for the fully nonlinear elliptic equation

$$F(D^2u, x) = f(x) \quad \text{in } B_1(0). \tag{3.16}$$

The perturbation argument requires the interior regularity of solutions to the equation

$$F(D^2u, x_0) = f(x_0) \quad \text{in } B_1(0), \tag{3.17}$$

for any fixed points $x_0 \in B_1(0)$. Interior a priori estimates for (3.17) was established by Evans and Krylov [10, 15], provided F is concave in the variables D^2u. For convenience we state the interior regularity as an assumption, so that it covers more general equations for which the interior regularity is available.

Assumption (a). There are constants $0 < \bar{\alpha} \leq 1$ and $\overline{C} > 0$ such that for any symmetric matrix M with $F(M, 0) = 0$ and any $w_0 \in C(\partial B_r)$, there exists a solution $w \in C^2(B_r) \cap C(\bar{B}_r)$ to

$$\begin{aligned} F(D^2w(x) + M, 0) &= 0 \quad \text{in } B_r(0), \\ w &= w_0 \quad \text{on } \partial B_r(0) \end{aligned}$$

which satisfies the estimate

$$\|u\|_{C^{2,\overline{\alpha}}(B_{r/2})} \leq \overline{C} r^{-2-\overline{\alpha}} \|u\|_{L^\infty(B_1)}, \tag{3.18}$$

where $\overline{C}$ is independent of M and r. Denote

$$\begin{aligned} \beta(y;\ x) &= \sup_M \left| \frac{F(M, y) - F(M, x)}{\|M\|} \right| \\ \Gamma(y, r) &= \left(|B_r(y)|^{-1} \int_{B_r(y)} \beta(y; x) dx \right)^{1/n}. \end{aligned}$$

Theorem 3.2. *Let $u \in C^2$ be a solution of* (3.16). *Then $\forall\ z, y \in B_{1/2}(0)$,*

$$\begin{aligned} |\partial_x^2 u(y) - \partial_x^2 u(z)| \leq C \Bigg(& d^{\bar{\alpha}} \sup_{B_1} |u| + d^{\bar{\alpha}} \sup_{B_1} |f| \\ & + \int_0^d \frac{\Upsilon_{f,n}(y, r) + \Upsilon_{f,n}(z, r)}{r} dr + d^{\bar{\alpha}} \int_d^1 \frac{\Upsilon_{f,n}(y, r)}{r^{1+\bar{\alpha}}} dr \Bigg) \\ & + C \sup_{B_1} |\partial_x^2 u| \left(\int_0^d \frac{\Gamma(y, r) + \Gamma(z, r)}{r} + d^{\bar{\alpha}} \int_d^1 \frac{\Gamma(y, r)}{r^{1+\bar{\alpha}}} \right), \end{aligned} \tag{3.19}$$

where $d = |z - y|$, and C depends on $n, \bar{\alpha}, \bar{C}$, and the ellipticity constants of F.

It follows that of $F(\cdot, x)$ and $f(x)$ are Hölder continuous, with exponent $\alpha \in (0,1]$, then we have

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \le C\Big(\sup_{B_1}|u| + \|f\|_{C^\alpha(B_1)}\Big), \qquad \text{if } 0<\alpha<\bar\alpha,$$

$$|\partial_x^2 u(x) - \partial_x^2 u(y)| \le C d^{\bar\alpha}\Big(\sup_{B_1}|u| + \|f\|_{C^\alpha(B_1)}|\log d|\Big), \qquad \text{if } \alpha=\bar\alpha,$$

$$\|u\|_{C^{2,\bar\alpha}(B_{1/2})} \le C\Big(\sup_{B_1}|u| + \|f\|_{C^\alpha(B_1)}\Big), \qquad \text{if } 0<\bar\alpha<\alpha\le 1.$$

Proof. The proof is similar to that of Theorem 2.3. Denote $B_k = B_{\rho^k}(0)$ $(\rho = \frac{1}{2})$. For $k = 0, 1, 2, \cdots$, let u_k be the solution of

$$\begin{aligned} F(D^2u_k, 0) &= f(0) \quad \text{in } B_k, \\ u_k &= u \qquad\ \ \text{on } \partial B_k. \end{aligned} \tag{3.20}$$

Then

$$F(D^2u, 0) - F(D^2u_k, 0) = f(x) - f(0) + (F(D^2u, 0) - F(D^2u, x)).$$

The left hand side can be written as a linearized equation of F for $u - u_k$. Hence by Aleksandrov's maximum principle,

$$\|u_k - u\|_{L^\infty(B_k)} \le C\rho^{2k}\left(\Upsilon_f(0,\rho^k) + \|D^2u\|_{L^\infty(B_1)}\Gamma(0,\rho^k)\right).$$

By Assumption (a), $u_k - u_{k+1}$ satisfies a linearized equation of F with coefficients in $C^{\bar\alpha}$. Hence by the Schauder estimate for linear elliptic equations,

$$\begin{aligned} \|D^2(u_k - u_{k+1})\|_{B_{k+2}} &\le C\rho^{-2k}\|u_k - u_{k+1}\|_{L^\infty(B_k)} \\ &\le C\left(\Upsilon_f(0,\rho^k) + \|D^2u\|_{L^\infty(B_1)}\Gamma(0,\rho^k)\right). \end{aligned} \tag{3.21}$$

Now one can proceed in the same way as the proof of Theorem 2.1. The only difference is that (2.6) should be replaced by

$$|\partial_x^2 h_k(z) - \partial_x^2 h_{k+1}(z)| \le C|z|^{\bar\alpha}\rho^{-k\bar\alpha}\left(\Upsilon_f(0,\rho^k) + \|D^2u\|_{L^\infty(B_1)}\Gamma(0,\rho^k)\right),$$

where $h_k = u_k - u_{k-1}$. □

4 Partial regularity for elliptic equations

An interesting question for elliptic and parabolic equations is whether the solution is smooth in a direction if the data (the coefficients and inhomogeneous term of the equation) are smooth in the direction. More precisely, consider the Possion equation

$$\Delta u = f(x) \quad \text{in } \Omega, \tag{4.1}$$

if f is Hölder continuous in a given direction, is the second derivative $u_{\xi\xi}$ Hölder continuous in ξ? This problem can be asked for more general linear and nonlinear equations.

In this section we consider this problem. To illustrate our results we first consider the Possion equation (4.1). More general equations will be discussed at the end of the section.

Theorem 4.1. *Let u be a smooth solution of* (4.1). *Then $\forall\ x, z \in \Omega_\delta$, we have the estimate*

$$|\partial_\xi \partial_x u(y) - \partial_\xi \partial_x u(z)| \le Cd \left(\sup_\Omega |u| + \frac{1}{d}\int_0^d \frac{\omega_{f,\xi}(r)}{r} + \int_d^1 \frac{\omega_{f,\xi}(r)}{r^2} + \int_\Omega |f| \right), \tag{4.2}$$

where the notations ∂_x, d, $\delta > 0$, and Ω_δ are the same as in Theorem 2.1, *and $C > 0$ is a constant depending only on n, Ω and δ.*

Estimate (4.2) yields a continuity estimate of the second derivatives $\partial_\xi \partial_x u$ in terms of the Dini continuity of f. When f is Hölder continuous in a direction ξ with exponent $\alpha \in (0, 1)$, then we have

$$\|\partial_\xi \partial_x u\|_{C^\alpha(\Omega_\delta)} \le C \left(\sup_\Omega |u| + \frac{\|f\|_{C^\alpha_\xi(\Omega)}}{\alpha(1-\alpha)} \right). \tag{4.3}$$

If f is Lipschitz continuous in ξ, then we have

$$|\partial_\xi \partial_x u(x) - \partial_\xi \partial_x u(z)| \le Cd \left(\sup_\Omega |u| + \|f\|_{C^{0,1}_\xi} |\log d| \right). \tag{4.4}$$

Moreover, by approximation, estimate (4.2) holds for weak solutions.

Proof. The proof is similar to that of Theorem 2.1. As before we assume that $\Omega = B_1(0)$ and $y = 0$. Without loss of generality we assume that $\xi = e_n := (0, \cdots, 0, 1)$. Denote $B_k = B_{\rho^k}(0)$ $(\rho = \frac{1}{2})$. For $k = 0, 1, 2, \cdots$, let u_k be the solution of

$$\begin{aligned} \Delta u_k &= f(x', 0) \quad \text{in } B_k, \\ u_k &= u \qquad\quad\ \ \text{on } \partial B_k, \end{aligned} \tag{4.5}$$

where $x' = (x_1, \cdots, x_{n-1})$. As we assume u is smooth, so f and u_k are also smooth.

Let z be a point near 0. We have

$$|\partial_{x_n} \partial_x u(z) - \partial_{x_n} \partial_x u(0)| \le I_1 + I_2 + I_3, \tag{4.6}$$

where

$$\begin{aligned} I_1 &= |\partial_{x_n} \partial_x u_m(0) - \partial_{x_n} \partial_x u(0)|, \\ I_2 &= |\partial_{x_n} \partial_x u_m(z) - \partial_{x_n} \partial_x u_m(0)|, \\ I_3 &= |\partial_{x_n} \partial_x u(z) - \partial_{x_n} \partial_x u_m(z)|. \end{aligned}$$

We choose $m \ge 1$ such that $\rho^{m+4} \le |z| < \rho^{m+3}$.

First we estimate I_1. By equation (4.5),

$$\begin{aligned} \Delta(u_k - u) &= f(x',0) - f(x',x_n) \quad \text{in } B_k, \\ u_k - u &= 0 \quad \text{on } \partial B_k, \end{aligned}$$

where $|f(x',0) - f(x',x_n)| \le \omega_{f,\xi}(\rho^k)$ in B_k. Hence by the maximum principle,

$$\|u_k - u\|_{L^\infty(B_k)} \le C\rho^{2k}\omega_{f,\xi}(\rho^k). \tag{4.7}$$

We get the estimate

$$\|u_k - u_{k+1}\|_{L^\infty(B_{k+1})} \le C\rho^{2k}\omega_{f,\xi}(\rho^k). \tag{4.8}$$

Since $u_k - u_{k+1}$ is harmonic in B_{k+1}, we have the estimate

$$|\partial_x^j(u_k - u_{k+1})(0)| \le C_{n,j}\rho^{-kj}\|u_k - u_{k+1}\|_{L^\infty(B_{k+1})}, \tag{4.9}$$

for any integer $j \ge 0$, where $C_{n,j}$ depends only on n and j. Recalling that $u \in C^2$, we have

$$\begin{aligned} |\partial_x^2 u_m(0) - \partial_x^2 u(0)| &\le \sum_{k \ge m} |\partial_x^2 u_k(0) - \partial_x^2 u_{k+1}(0)| \\ &\le C \sum_{k \ge m} \omega_{f,\xi}(\rho^k) \\ &\le C \int_0^d \frac{\omega_{f,\xi}(r)}{r} dr. \end{aligned} \tag{4.10}$$

This estimate holds for all second derivatives, including the mixed derivatives $\partial_{x_n}\partial_x u$.

Next we estimate I_2. Denote $h_k = u_k - u_{k-1}$. When $k \le m$, z is an interior point of B_k. Hence by (4.8) and (4.9),

$$\begin{aligned} |\partial_x^2 h_k(z) - \partial_x^2 h_k(0)| &\le |z||\partial_x^3 h_k(\hat z)| \\ &\le C|z|\rho^{-3k}\|u_k - u_{k+1}\|_{L^\infty(B_{k+1})} \\ &\le C|z|\rho^{-k}\omega_{f,\xi}(\rho^k), \end{aligned} \tag{4.11}$$

where $\hat z$ is some point between 0 and z. Hence

$$\begin{aligned} |\partial_x^2 u_m(z) - \partial_x^2 u_m(0)| &\le |\partial_x^2 u_{m-1}(z) - \partial_x^2 u_{m-1}(0)| + |\partial_x^2 h_m(z) - \partial_x^2 h_m(0)| \\ &\le |\partial_x^2 u_0(z) - \partial_x^2 u_0(0)| + \sum_{k=1}^m |\partial_x^2 h_k(z) - \partial_x^2 h_k(0)| \\ &\le |\partial_x^2 u_0(z) - \partial_x^2 u_0(0)| + C|z|\sum_{k=1}^m \rho^{-k}\omega_{f,\xi}(\rho^k) \\ &\le |\partial_x^2 u_0(z) - \partial_x^2 u_0(0)| + C|z|\int_{|z|}^1 \frac{\omega_{f,\xi}(r)}{r^2}. \end{aligned} \tag{4.12}$$

Again the above estimate holds for all second derivatives. But to estimate the second derivatives of u_0, we restrict to the mixed derivative $\partial_{x_n}\partial_x u_0$. By equation (4.5) we have,

$$u_0(x) = \int_{B_1(0)} \frac{f(y',0)}{|x-y|^{n-2}} dy + g(x)$$

where g is a bounded harmonic function. As mentioned before, we may assume that $f \in C^0(\Omega)$, $u \in C^2(\Omega)$. Differentiating the above integral in x_n and using Fubini Theorem, we get

$$\sup_{x\in B_{1/2}}|\partial_{x_n}u_0(x)| \le C\sup_{x\in B_1}|u(x)| + C\|f\|_{L^1(B_1)}. \tag{4.13}$$

As $\partial_{x_n}u_0$ is harmonic, as in (4.9), we have

$$\sup_{x\in B_{1/4}}|\partial_x^m\partial_{x_n}u_0(x)| \le C_{n,m}\sup_{B_{1/2}}|\partial_{x_n}u_0|. \tag{4.14}$$

Hence

$$\begin{aligned}|\partial_{x_n}\partial_x u_0(z) - \partial_{x_n}\partial_x u_0(0)| &\le |z||\partial_{x_n}\partial_x^2 u_0(\hat z)| \\ &\le C|z|\sup_{B_{1/2}}|\partial_{x_n}u_0| \\ &\le C|z|(\sup_{x\in B_1}|u(x)| + C\|f\|_{L^1(B_1)})\end{aligned}$$

for some point $\hat z$ on the segment $\overline{oz}$. Hence we obtain

$$I_2 \le C|z|\left(\sup_{x\in B_1}|u(x)| + \|f\|_{L^1(B_1)} + \int_{|z|}^1 \frac{\omega_{f,\xi}(r)}{r^2}\right). \tag{4.15}$$

The above proof is essentially parallel to that of Theorem 2.1, and one can similarly estimate I_3. Namely

$$I_3 \le C\int_0^d \frac{\omega_{f,\xi}(r)}{r}dr + C\omega_{f,\xi}(\rho^k) \le C\int_0^d \frac{\omega_{f,\xi}(r)}{r}dr. \tag{4.16}$$

Combining the above estimates we obtain (4.2). □

Note that in estimate (4.3), f is assumed to be Hölder continuous in the direction ξ only, but $\partial_x\partial_\xi u$ is Hölder continuous in all directions. Such kind of regularity is not only of interest itself but has applications in other areas such as image processing [1]. For related research we refer the reader to [9, 11].

For the linear elliptic equations with variable coefficients,

$$L[u] := \sum_{i,j=1}^n a_{ij}(x)u_{x_ix_j} = f \quad \text{in } \Omega, \tag{4.17}$$

where the matrix (a_{ij}) is symmetric, positive definite, and satisfies

$$\lambda|\zeta|^2 \le \sum a_{ij}\zeta_i\zeta_j \le \Lambda|\zeta|^2, \quad \forall\ \zeta \in \mathbb{R}^n. \tag{4.18}$$

We have a similar partial regularity result [28].

Theorem 4.2. *Let u be a smooth solution to (4.17). Suppose $a_{ij} \in C(\Omega)$ and are Lipschitz continuous in the ξ direction. Suppose $f \in L^p(\Omega)$ for some $p > n$. Then $\forall\ x, z \in \Omega_\delta$, we have the estimate*

$$|\partial_\xi\partial_x u(y) - \partial_\xi\partial_x u(z)| \le Cd^{\bar\alpha}\left(\sup_\Omega|u| + \int_d^1 \frac{\omega_{f,\xi}(r)}{r^{1+\bar\alpha}} + \|f\|_{L^p(\Omega)}\right) + C\int_0^d \frac{\omega_{f,\xi}(r)}{r}, \tag{4.19}$$

where $d = |y-z|$, $C > 0$ depends only on $n, \delta, p, \Omega, \lambda, \Lambda$ and the modulus of continuity of a_{ij}.

The Lipschitz condition of a_{ij} in the ξ-direction can be relaxed. It suffices to assume that $\partial_\xi a_{ij} \in L^q(\Omega)$ for some $q > 1$ such that $(\partial_\xi a_{ij})u_{x_i x_j} \in L^{p'}$ for some $p' > n$. We can also drop the continuity of a_{ij} in any one direction [18]. In particular in dimension two, we can drop the continuity condition of a_{ij} and extend the above estimate to fully nonlinear elliptic equations [28]. Recently with Dongsheng Li we observed that the Lipschitz condition $a_{ij} \in C^{0,1}_\xi$ can be relaxed to $a_{ij} \in C^\alpha_\xi$ for $\alpha \in (0,1)$, provided $f \in L^p(\Omega)$ for some $p > n/\alpha$.

5 Estimates for parabolic equations

The perturbation argument also applies to parabolic equations and yields similar estimates. In this paper we will not repeat these results for parabolic equations. But we would like to present an interesting regularity result which is special for parabolic equations.

We consider the regularity of solutions to the fully nonlinear, second order, parabolic equation

$$u_t - F(D^2u) = f(x,t) \quad \text{in} \quad Q := B_1(0) \times (0,1]. \tag{5.1}$$

We assume

(a) The operator F is uniformly elliptic, i.e. $a_{ij} := \frac{\partial}{\partial r_{ij}} F(r)$ satisfies (4.18).
(b) F is either concave or convex in r.
(c) The function f is Hölder continuous in x, (with Hölder exponent $\overline{\alpha} \in (0,1]$), and bounded and measurable in t.

Theorem 5.1. *Let $u \in C^{4,2}_{x,t}(Q)$ be a solution to* (1.1). *Assume conditions* (a)–(c). *Then there exists $\alpha \in (0,\overline{\alpha})$ depending on λ and Λ, such that*

$$\|D^2_x u\|_{C^{\alpha,\alpha/2}_{x,t}(Q_r)} \le C\Big(\|u\|_{L^\infty(Q)} + \sup_{t\in(0,1]} \|f(\cdot,t)\|_{C^{\overline{\alpha}}(B_1)}\Big), \tag{5.2}$$

where $r \in (0,1)$, $Q_r = B_r(0) \times (1-r^2, 1]$, and C is a constant depending on $n, r, \alpha, \lambda, \Lambda, M$.

Theorem 5.1 was proved in [29], where we considered more general operator $F = F(D^2u, Du, u, x, t)$. For simplicity we consider a simpler model here. When F depends on the gradient Du, we need to assume certain growth condition on Du [8], and the linear relation in (5.2) may be violated. The assumption $u \in C^{4,2}_{x,t}(Q)$ in Theorem 5.1 implies that f is continuous in t. By this assumption we can avoid the introduction of viscosity solutions and also we can conveniently differentiate equation (5.1) in our proof. But our estimate (5.2) does not depend on the modulus of continuity of f in t. Therefore by approximation, estimate (5.2) also holds for viscosity solutions as long as the uniqueness of solutions, subject to appropriate boundary condition, can be proved. We also note that the exponent α in (5.2) is related to the Hölder continuity of linear parabolic equations of general form, so it depends also on λ and Λ. It may be strictly smaller than $\overline{\alpha}$.

Theorem 5.1 is of interest itself but our work [29] was motivated by applications in curvature and gradient flows with volume constraints. In these applications one has a fully nonlinear parabolic equation with the right hand side

$$f = f(x, u, \int G(D_x^2 u, D_x u)dx),$$

where f, G are smooth functions. Usually one can establish the $C^{2,1}_{x,t}$ estimate for u, which implies that f is smooth in x but only measurable in t. Hence one cannot use Krylov's regularity theory directly.

For the linear uniformly parabolic equation,

$$u_t - \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x,t)u_{x_i} + c(x,t)u = f(x,t), \tag{5.3}$$

estimate (5.2) was obtained by Knerr [14], who improved early result of Brandt [2]. This estimate was later reproved and discussed in [20, 17, 22]. For linear parabolic equation, the coefficients a_{ij}, b_i and c are allowed to be discontinuous in t. More precisely, assuming

(i) $\lambda I \le \{a_{ij}(x,t)\} \le \Lambda I \ \forall\ (x,t) \in Q$,
(ii) $a_{ij}, b_i, c, f \in L^\infty(Q)$,
(iii) $a_{ij}(\cdot,t), b_i(\cdot,t), c(\cdot,t), f(\cdot,t) \in C^\alpha(B_1),\ \forall\ t \in (0,1]$,

they proved the following estimate.

Theorem 5.2. ([14]) *Let $u \in C^{4,2}_{x,t}(Q)$ be a solution to* (5.3). *Assume the conditions* (i)–(iii) *above. Then we have the estimate*

$$\|D_x^2 u\|_{C^{\alpha,\alpha/2}_{x,t}(Q_r)} \le C\Big(\|u\|_{L^\infty(Q)} + \sup_{t\in(0,1]} \|f(\cdot,t)\|_{C^\alpha(B_1)}\Big), \tag{5.4}$$

where C depends on $n, r, \alpha, \lambda, \Lambda$, $\sup_{t\in(0,1]} \|a_{ij}(\cdot,t), b_i(\cdot,t), c(\cdot,t)\|_{C^\alpha(B_1)}$.

Our proof of Theorems 5.1 and 5.2 uses the same perturbation argument as above, which enables us to reduce the estimates (5.2) and (5.4) to the case when a_{ij}, b_i, c and f are constants (for any fixed t).

For the fully nonlinear parabolic equation (5.1), if f is independent of x, one notices that $v(x,t) := u(x,t) - \int f(s)ds$ satisfies the equation $v_t = F(D^2 v)$ and hence Krylov's regularity theory implies that higher regularity of v and u in x. For the linear parabolic equation (5.3), one may differentiate the equation in x directly, and use Krylov and Safonov's Hölder continuity for linear parabolic equations [19] to get higher regularity in x. We will not repeat the details here.

As was discussed before, the perturbation argument applies not only to the case when f is Hölder continuous, but also to the case when f is Dini continuous in x. That is if $u \in C^{4,2}_{x,t}(Q)$ is a solution of (5.1), where F satisfies conditions (a)–(b), f is Dini continuous in x, and bounded and measurable in t, then for any $r \in (0,1)$, $D_x^2 u$ is continuous in Q_r and the modulus of continuity of $D_x^2 u$ depends only on $n, r, \alpha, \lambda, \Lambda, M, \|u\|_{L^\infty(Q)}, \|f\|_{L^\infty(Q)}$, and the Dini continuity of f in x.

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