Structural characterization of linear quantum systems with application to back-action evading measurement

Guofeng Zhang, Ian R. Petersen, *Fellow, IEEE*, and Jinghao Li

***Abstract*—The purpose of this paper is to study the structure of quantum linear systems in terms of their Kalman canonical form, which was proposed in a recent paper ([17]). The spectral structure of quantum linear systems is explored, which indicates that a quantum linear system is both controllable and observable provided that it is Hurwitz stable. A new parameterization method for quantum linear systems is proposed. This parameterization is designed for the Kalman canonical form directly. Consequently, the parameters involved are in a blockwise form in correspondence with the blockwise structure of the Kalman canonical form. This parameter structure can be used to simplify various quantum control design problems. For example, necessary and sufficient conditions for the realization of quantum back-action evading (BAE) measurements are given in terms of these new parameters. Due to their blockwise nature, a small number of parameters are required for realizing quantum BAE measurements.**

Index Terms— Quantum linear systems; Kalman canonical form; back-action evading (BAE) measurements.

I. Introduction

The last two decades have witnessed a fast growth in the theoretical investigation and experimental demonstration of quantum control as it is an essential ingredient of quantum information technologies, including quantum communication, quantum computation, quantum cryptography, quantum ultra-precision metrology, and nano-electronics. Quantum linear systems play an important role in quantum control theory. In the field of quantum optics, linear systems are widely used as they are easy to manipulate and, more importantly, they are often good approximations to more general dynamics [12], [10]. Besides their wide applications in quantum optical systems, quantum linear models have found important and successful applications for many other quantum dynamical systems such as opto-mechanical systems [5], [8], circuit quantum electro-dynamical (circuit QED) systems [7], and atomic ensembles [9], [8].

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G. Zhang is with the Department of Applied Mathematics, Hong Kong Polytechnic University, Hong Kong. (e-mail: Guofeng.Zhang@polyu.edu.hk).

I.R. Petersen is with the Research School of Engineering, Australian National University, Canberra ACT 2601, Australia. (e-mail: i.r.petersen@gmail.com).

J. Li is with the Institute of Systems Science, Northeastern University, Shenyang 110819, China. (e-mail: lijinghao@stumail.neu.edu.cn).

In [17], a Kalman canonical form is proposed for quantum linear systems. More specifically, given a quantum linear system, an orthogonal and blockwise symplectic transformation matrix is constructed which transforms the original system into four subsystems: the controllable and observable (co) subsystem, the controllable and unobservable ($c\bar{o}$) subsystem, the uncontrollable and observable ($\bar{c}o$) subsystem, and the uncontrollable and unobservable ($\bar{c}\bar{o}$) subsystem. In [17], the quantities $\boldsymbol{x}_{co}, \boldsymbol{x}_{\bar{c}\bar{o}}, \boldsymbol{q}_h, \boldsymbol{p}_h$ are used to denote the quadrature operators of the $co, \bar{c}\bar{o}, c\bar{o}, \bar{c}o$ subsystems respectively.

On the basis of the Kalman canonical form proposed in [17], in this paper we aim to explore deeper the structure of quantum linear systems. An open quantum system can be described by a set of quantum stochastic differential equations (QSDEs). Due to the nature of quantum-mechanical systems, there are constraints on the coefficients of these QSDEs, which are called physical realizability conditions of quantum systems [6]. In this paper we present physical realizability conditions for quantum linear systems in their Kalman canonical form; see Lemma 3.1. Interestingly, these conditions allow us to expose the nice structure of the spectrum of a quantum linear system; see Propositions 3.2 and 3.3, and Example 3.1. Moreover, it is shown in Theorem 3.1 that if a quantum linear system is Hurwitz stable,, then it is both controllable and observable. A new parameterization method for quantum linear systems is proposed in the paper. We express the system Hamiltonian and the coupling operator explicitly in terms of the partitioned system variables $\boldsymbol{x}_{co}, \boldsymbol{x}_{\bar{c}\bar{o}}, \boldsymbol{q}_h, \boldsymbol{p}_h$. Specifically, let $\boldsymbol{x} = [\boldsymbol{q}_h^\top \ \boldsymbol{p}_h^\top \ \boldsymbol{x}_{co}^\top \ \boldsymbol{x}_{\bar{c}\bar{o}}^\top]^\top$. Then the system Hamiltonian is $\boldsymbol{H} = \boldsymbol{x}^\top H \boldsymbol{x}/2$ where H is a real symmetric matrix, and the coupling operator is $\boldsymbol{L} = \Gamma \boldsymbol{x}$ with Γ being a complex matrix. Due to the special structure of the system matrices in the Kalman canonical form, if $\boldsymbol{H}$ and $\boldsymbol{L}$ generate the Kalman canonical form, then the matrices H and Γ should be of specific form. This form is given in Lemma 3.2. Moreover, we also establish the converse: If the matrices H and Γ are of the given specific form, then the resulting QSDEs are formally in the Kalman canonical form; see Lemma 3.3. Finally, as the Kalman canonical form is obtained based on the notions of controllability and observability, we derive further conditions on H and Γ such that the resulting system is indeed the quantum Kalman canonical form; see Theorem 3.2.

A measurement process often involves measurement noise from the surrounding environment. In quantum mechanics, environmental noise can be represented by two conjugate quadrature operators. A fundamental fact in quantum mechanics is that these two noise quadrature operators do not commute. This gives rise to the so-called standard quantum limit (SQL). However, if a measurement process suffers from a noise quadrature (shot noise), but not from the conjugate quadrature noise (measurement back-action noise), then it is called a BAE measurement, [1], [11, Fig. 2(a)], [13], [15]. As a result, a BAE measurement may be able to beat the SQL, thus enabling extremely high precision measurement. In fact, the idea of BAE measurement originates from the study of gravitational wave detection [5]. In the language of linear systems theory, a BAE measurement is realized if the transfer function from the measurement back-action noise to the measured output is zero. On the basis of the proposed new parameterization method for quantum linear systems, in this paper, necessary and sufficient conditions for the realization of BAE measurements by means of quantum linear systems are given in Theorem 4.1.

The rest of the paper is organized as follows. The notation used in this paper is summarized in Ssection I-A. Preliminaries are given in Section II, which include quantum linear systems and their Kalman canonical form. The structural properties of the Kalman canonical form are studied in Section III. An application to the realization of quantum BAE measurements is studied in Section IV. An Example is given in Section V. Concluding remarks are given in Section VI.

A. Notation

1) x^* denotes the complex conjugate of a complex number x or the adjoint of an operator x.
2) For a matrix $X=[x_{ij}]$ with number or operator entries, denote $X^\#=[x_{ij}^*]$, $X^\top=[x_{ji}]$, and $X^\dagger=(X^\#)^\top$. Moreover, let $\breve{X}=\left[\begin{smallmatrix} X \\ X^\# \end{smallmatrix}\right]$. $\mathrm{Re}(X)$ and $\mathrm{Im}(X)$ denote the real part and imaginary part of a matrix X, respectively.
3) The commutator of two operators X and Y is defined as $[X,Y]\triangleq XY-YX$. If X and Y are two vectors of self-adjoint operators, then their commutator is defined as the matrix of operators $[X,Y^\top]\triangleq XY^\top-(YX^\top)^\top$.
4) I_k is the identity matrix and 0_k the zero matrix in $\mathbb{C}^{k\times k}$. δ_{ij} denotes the Kronecker delta. Let $J_k=\mathrm{diag}(I_k,-I_k)$. For a matrix $X\in\mathbb{C}^{2k\times 2r}$, define its $\flat$-adjoint by $X^\flat\triangleq J_rX^\dagger J_k$.
5) Given two matrices $U,\ V\in\mathbb{C}^{k\times r}$, define $\Delta(U,V)\triangleq\left[\begin{smallmatrix} U & V \\ U^\# & V^\# \end{smallmatrix}\right]$. A matrix with this structure will be called *doubled-up* [3].
6) A matrix $T\in\mathbb{C}^{2k\times 2k}$ is called *Bogoliubov* if it is doubled-up and satisfies $TT^\flat=T^\flat T=I_{2k}$.
7) Let $\mathbb{J}_k=\left[\begin{smallmatrix} 0_k & I_k \\ -I_k & 0_k \end{smallmatrix}\right]$. A matrix $S\in\mathbb{C}^{2k\times 2k}$ is called *symplectic*, if it satisfies $S\mathbb{J}_kS^\dagger=S^\dagger\mathbb{J}_kS=\mathbb{J}_k$.

II. Quantum linear systems and their Kalman canonical form

In this section, quantum linear systems are briefly introduced, and their Kalman canonical form, recently derived in [17], is also presented for completeness.

A. Quantum linear systems

An open quantum linear system G can be used to model a collection of n quantum harmonic oscillators interacting with m input boson fields. The j-th oscillator, $j=1,\dots,n$, may be represented by its annihilation operator $\boldsymbol{a}_j$ and creation operator $\boldsymbol{a}_j^*$ (the adjoint operator of $\boldsymbol{a}_j$). These are operators on an infinite-dimensional Hilbert space and satisfy the *canonical commutation relations (CCRs)* $[\boldsymbol{a}_j(t),\boldsymbol{a}_k(t)]=[\boldsymbol{a}_j^*(t),\boldsymbol{a}_k^*(t)]=0$, and $[\boldsymbol{a}_j(t),\boldsymbol{a}_k^*(t)]=\delta_{jk}$, $\forall j,k=1,\dots n,\forall t\in\mathbb{R}^+$. Let $\boldsymbol{a}=[\boldsymbol{a}_1\ \cdots\ \boldsymbol{a}_n]^\top$. The system Hamiltonian is given by $\boldsymbol{H}=(1/2)\breve{\boldsymbol{a}}^\dagger\Omega\breve{\boldsymbol{a}}$, where $\breve{\boldsymbol{a}}=[\boldsymbol{a}^\top\ \ (\boldsymbol{a}^\#)^\top]^\top$, and $\Omega=\Delta(\Omega_-,\Omega_+)\in\mathbb{C}^{2n\times 2n}$ is a Hermitian matrix with $\Omega_-,\Omega_+\in\mathbb{C}^{n\times n}$. The coupling of the system to the input fields is described by the operator $\boldsymbol{L}=[C_-\ C_+]\breve{\boldsymbol{a}}$, with $C_-,C_+\in\mathbb{C}^{m\times n}$. The k-th input boson field, $k=1,\dots,m$, is represented in terms of its annihilation operator $\boldsymbol{b}_k(t)$ and creation operator $\boldsymbol{b}_k^*(t)$ (the adjoint operator of $\boldsymbol{b}_k(t)$). These are operators on a symmetric Fock space (a special kind of infinite-dimensional Hilbert space). The operators $\boldsymbol{b}_k(t)$ and $\boldsymbol{b}_k^*(t)$ satisfy the *singular commutation relations* $[\boldsymbol{b}_j(t),\boldsymbol{b}_k(r)]=[\boldsymbol{b}_j^*(t),\boldsymbol{b}_k^*(r)]=0$, and $[\boldsymbol{b}_j(t),\ \boldsymbol{b}_k^*(r)]=\delta_{jk}\delta(t-r)$, $\forall j,k=1,\dots,m$, $\forall t,r\in\mathbb{R}$. Let $\boldsymbol{b}(t)=[\boldsymbol{b}_1(t)\ \cdots\ \boldsymbol{b}_m(t)]^\top$ and $\breve{\boldsymbol{b}}(t)=[\boldsymbol{b}(t)^\top\ (\boldsymbol{b}(t)^\#)^\top]^\top$.

The dynamics of an open quantum linear system is described by the following QSDEs, ([3, Eq. (26)])

$$\dot{\breve{\boldsymbol{a}}}(t)=\mathcal{A}\breve{\boldsymbol{a}}(t)+\mathcal{B}\breve{\boldsymbol{b}}(t),\ \breve{\boldsymbol{b}}_{\rm out}(t)=\mathcal{C}\breve{\boldsymbol{a}}(t)+\breve{\boldsymbol{b}}(t),\quad t\geq 0,\ (1)$$

where the system matrices are parametrized by the physical parameters of the Hamiltonian $\boldsymbol{H}$ an coupling $\boldsymbol{L}$, which are $\mathcal{C}=\Delta(C_-,\ C_+)$, $\mathcal{B}=-\mathcal{C}^\flat$, $\mathcal{A}=-\imath J_n\Omega-\frac{1}{2}\mathcal{C}^\flat\mathcal{C}$. These system matrices satisfy

$$\mathcal{A}+\mathcal{A}^\flat+\mathcal{B}\mathcal{B}^\flat=0,\quad \mathcal{B}=-\mathcal{C}^\flat. \tag{2}$$

On the other hand, if matrices $\mathcal{A},\mathcal{B},\mathcal{C}$ satisfy Eq. (2), then the system Hamiltonian $\boldsymbol{H}=(1/2)\breve{\boldsymbol{a}}^\dagger\Omega\breve{\boldsymbol{a}}$ is completely determined as the matrix Ω can be computed via $\Omega=\frac{\imath}{2}(J_n\mathcal{A}-\mathcal{A}^\dagger J_n)$. Moreover, the coupling $\boldsymbol{L}=[C_-\ C_+]\breve{\boldsymbol{a}}$ is also determined once the matrix $\mathcal{C}$ is given. In this case, the mathematical model (1) is said to be *physically realizable* as it could in principle be physically realized [6].

B. The quantum Kalman canonical form

The Kalman decomposition of quantum linear systems, recently developed in [17], is presented in this subsection.

A unitary and blockwise Bogoliubov coordinate transformation matrix T is defined in [17, Eq. (47)], which is

$$T\triangleq\left[\begin{array}{cc|cc|cc} Z_3 & 0 & Z_1 & 0 & Z_2 & 0 \\ 0 & Z_3^\# & 0 & Z_1^\# & 0 & Z_2^\# \end{array}\right],$$

where $Z_1\in\mathbb{C}^{n\times n_1}$, $Z_2\in\mathbb{C}^{n\times n_2}$, and $Z_3\in\mathbb{C}^{n\times n_3}$ ($n_1,n_2,n_3\geq 0$ and $n_1+n_2+n_3=n$). T is called blockwise Bogoliubov as it satisfies

$$T^\dagger J_nT=\begin{bmatrix} J_{n_3} & 0 & 0 \\ 0 & J_{n_1} & 0 \\ 0 & 0 & J_{n_2} \end{bmatrix}. \tag{3}$$

Define the unitary matrix $\Pi\in\mathbb{C}^{2n_3\times 2n_3}$ by

$$\Pi\triangleq\begin{bmatrix} I_{n_a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{n_b} \\ 0 & 0 & I_{n_a} & 0 \\ 0 & I_{n_b} & 0 & 0 \end{bmatrix},$$

where $0 \leq n_a, n_b \leq n_3$, and $n_a + n_b = n_3$. Let $\tilde{V}_{n_3} = \Pi V_{n_3}$, where $V_k \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & I_k \\ -\imath I_k & \imath I_k \end{bmatrix}$, $k \in \mathbb{N}$ is a unitary matrix. Define two more unitary matrices

$$\tilde{V}_n \triangleq \begin{bmatrix} \tilde{V}_{n_3} & & \mathbf{0} \\ & V_{n_1} & \\ \mathbf{0} & & V_{n_2} \end{bmatrix}$$

and $\hat{T} \triangleq T\tilde{V}_n^\dagger$. The following result is proved in [17], which puts the quantum linear system (1) in the Kalman canonical form.

Proposition 2.1: [17, Theorem 4.4] The coordinate transformations

$$\begin{bmatrix} \boldsymbol{q}_h \\ \boldsymbol{p}_h \\ \hline \boldsymbol{x}_{co} \\ \hline \boldsymbol{x}_{\bar{c}\bar{o}} \end{bmatrix} \equiv \boldsymbol{x} \triangleq \hat{T}^\dagger \breve{\boldsymbol{a}},$$
$$\begin{bmatrix} \boldsymbol{q}_{\rm in} \\ \boldsymbol{p}_{\rm in} \end{bmatrix} \equiv \boldsymbol{u} \triangleq V_m \breve{\boldsymbol{b}}, \quad \begin{bmatrix} \boldsymbol{q}_{\rm out} \\ \boldsymbol{p}_{\rm out} \end{bmatrix} \equiv \boldsymbol{y} \triangleq V_m \breve{\boldsymbol{b}}_{\rm out} \tag{4}$$

convert the system (1) into the real quadrature form

$$\dot{\boldsymbol{x}}(t) = \bar{A}\boldsymbol{x}(t) + \bar{B}\boldsymbol{u}(t), \quad \boldsymbol{y}(t) = \bar{C}\boldsymbol{x}(t) + \boldsymbol{u}(t), \tag{5}$$

where the real matrices $\bar{A}, \bar{B}, \bar{C}$ are of the form

$$\bar{A} \triangleq \left[\begin{array}{cc|c|c} A_h^{11} & A_h^{12} & A_{12} & A_{13} \\ 0 & A_h^{22} & 0 & 0 \\ \hline 0 & A_{21} & A_{co} & 0 \\ \hline 0 & A_{31} & 0 & A_{\bar{c}\bar{o}} \end{array}\right],$$
$$\bar{B} \triangleq \begin{bmatrix} B_h \\ 0 \\ \hline B_{co} \\ \hline 0 \end{bmatrix}, \quad \bar{C} \triangleq \left[\begin{array}{cc|c|c} 0 & C_h & C_{co} & 0 \end{array}\right]. \tag{6}$$

The corresponding system block diagram is shown in [17, Fig. 2].

III. Structure of the quantum Kalman canonical form

In this section, we investigate the structure of the quantum Kalman canonical form. A useful lemma is given in Subsection III-A, the spectral structure of the quantum Kalman canonical form is presented in Subsection III-B, a new parametrization method for quantum Kalman canonical form is proposed in Subsection III-C, and finally a refined form of the quantum Kalman canonical form is presented in Subsection III-D.

A. A useful lemma

In this subsection, we present a lemma, Lemma 3.1, which is useful for the future development of this section.

The following result is an immediate consequence of the coordinate transformations (4) and the identity $\hat{T}^\dagger J_n \hat{T} = \imath \bar{\mathbb{J}}_n$, where

$$\bar{\mathbb{J}}_n \triangleq \begin{bmatrix} \mathbb{J}_{n_3} & 0 & 0 \\ 0 & \mathbb{J}_{n_1} & 0 \\ 0 & 0 & \mathbb{J}_{n_2} \end{bmatrix}.$$

Proposition 3.1: For the Kalman canonical form (5), the following conditions hold.

$$\bar{A}\bar{\mathbb{J}}_n + \bar{\mathbb{J}}_n\bar{A}^\top + \bar{B}\mathbb{J}_m\bar{B}^\top = 0, \quad \bar{B} = \bar{\mathbb{J}}_n\bar{C}^\top\mathbb{J}_m. \tag{7}$$

Substituting system matrices $\bar{A}, \bar{B}$ and $\bar{C}$ in Eq. (6) into Eq. (7) we can obtain the following result which presents the real-quadrature counterpart of physical realizability conditions (2).

Lemma 3.1: For the Kalman canonical form (5), the following conditions hold. $A_h^{22^\top} = -A_h^{11}$, $-A_h^{12} + A_h^{12^\top} + B_h\mathbb{J}_m B_h^\top = 0$, $-A_{12} + A_{21}^\top\mathbb{J}_{n_1} + B_h\mathbb{J}_m B_{co}^\top\mathbb{J}_{n_1} = 0$, $A_{31}^\top\mathbb{J}_{n_2} = A_{13}$, $\mathbb{J}_{n_1}A_{co} + A_{co}^\top\mathbb{J}_{n_1} - \mathbb{J}_{n_1}B_{co}\mathbb{J}_m B_{co}^\top\mathbb{J}_{n_1} = 0$, $\mathbb{J}_{n_2}A_{\bar{c}\bar{o}} + A_{\bar{c}\bar{o}}^\top\mathbb{J}_{n_2} = 0$, $B_h = C_h^\top\mathbb{J}_m$, $B_{co} = \mathbb{J}_{n_1}C_{co}^\top\mathbb{J}_m$.

B. Spectral structure of the quantum Kalman canonical form

The specific relations among the components of the system matrices $\bar{A}, \bar{B}$ and $\bar{C}$, established in Lemma 3.1, can be used to explore the spectral structure of quantum linear systems, which is the focus of this subsection. We denote the set of eigenvalues of a matrix A by $\sigma(A)$.

Let us first look at the $\bar{c}\bar{o}$ subsystem by ignoring the other modes in the Kalman canonical form (5), which is

$$\dot{\boldsymbol{x}}_{\bar{c}\bar{o}}(t) = A_{\bar{c}\bar{o}}\boldsymbol{x}_{\bar{c}\bar{o}}(t). \tag{8}$$

The following result shows that the poles of this subsystem are symmetric about both the real and imaginary axes.

Proposition 3.2: If $\lambda \in \sigma(A_{\bar{c}\bar{o}})$, then $-\lambda, \lambda^*, -\lambda^* \in \sigma(A_{\bar{c}\bar{o}})$.

In the Kalman canonical form (5), if we ignore the $\bar{c}\bar{o}$ and *co* subsystems, we obtain the following subsystem

$$\begin{bmatrix} \dot{\boldsymbol{q}}_h(t) \\ \dot{\boldsymbol{p}}_h(t) \end{bmatrix} = \begin{bmatrix} A_h^{11} & A_h^{12} \\ 0 & A_h^{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_h(t) \\ \boldsymbol{p}_h(t) \end{bmatrix} + \begin{bmatrix} B_h \\ 0 \end{bmatrix} \boldsymbol{u}(t),$$
$$\boldsymbol{y}(t) = \begin{bmatrix} 0 & C_h \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_h(t) \\ \boldsymbol{p}_h(t) \end{bmatrix} + \boldsymbol{u}(t). \tag{9}$$

In this paper, the system (9) is called the "h" subsystem.

Proposition 3.3: For the "h" subsystem (9), we have:

1) The set of the poles is given by $\sigma\left(A_h^{11}\right) \cup \sigma\left(-A_h^{11}\right)$;
2) If λ is a pole of this subsystem, then so are $-\lambda, \lambda^*, -\lambda^*$.

If $C_+ = 0$ and $\Omega_+ = 0$, the resulting quantum linear system (1) is said to be *passive* ([16], [15]). In the passive case, the existence of purely imaginary poles is equivalent to the existence of the $\bar{c}\bar{o}$ subsystem, as has been proved in [17, Theorem 3.2]. In the general (not necessarily passive) case, according to Propositions 3.2-3.3, the poles of the "h" subsystem and $\bar{c}\bar{o}$ subsystem are symmetric about the real and imaginary axes. However, this spectral property does not guarantee the existence of an "h" subsystem or a $\bar{c}\bar{o}$ subsystem. This is shown by the following counter-example.

Example 3.1: Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix}.$$

It is easy to see that the two poles of the system (A, B, C) are -1 and 1, which are symmetric about the real and imaginary axes. Moreover, this system is physically realizable as it satisfies Eq. (7). However, this system is both controllable and observable. Thus it is neither an "h" subsystem nor a $\bar{c}\bar{o}$ subsystem.

Finally, we look at the co subsystem by ignoring the other modes in the Kalman canonical form (5), which is

$$\dot{\boldsymbol{x}}_{co}(t) = A_{co}\boldsymbol{x}_{co}(t) + B_{co}\boldsymbol{u}(t), \ \ \boldsymbol{y}(t) = C_{co}\boldsymbol{x}_{co}(t) + \boldsymbol{u}(t). \tag{10}$$

In general, the poles of a co subsystem are not symmetric about the real and imaginary axes. For example, let $n = m = 1$, $\Omega_- = C_+ = 0$, and $C_- = \Omega_+ = 1$. It is easy to see that the resulting quantum linear system is both controllable and observable; in other words, it is a co system. However, the poles of this system are $1/2$ and $-3/2$, which are not symmetric about the real and imaginary axes.

The spectral structure of the Kalman canonical form established above implies the following interesting result.

Theorem 3.1: If a quantum linear system is Hurwitz stable, then it is both controllable and observable.

Theorem 3.1 tells us that Hurwitz stability implies controllability and observability for quantum linear systems; in general the converse is not true, as shown by Example 3.1 above. However, for the passive case, Theorem 3.1 can be strengthened to the following result, which has already been proved in [4, Lemma 2].

Corollary 3.1: For a passive quantum linear system, the properties of Hurwitz stability, controllability and observability are all equivalent.

We end this section with a final remark.

Remark 3.1: A $2d \times 2d$ real matrix N is said to be a Hamiltonian matrix if the matrix $\mathbb{J}_d N$ is symmetric. If a Hamiltonian matrix has λ as an eigenvalue, then $-\lambda, \lambda^*, -\lambda^*$ are also its eigenvalues. Later in Lemma 3.3 and Remark 3.3 we will show that both the matrices $A_{\bar{c}\bar{o}}$ and $\begin{bmatrix} A_h^{11} & 0 \\ 0 & A_h^{22} \end{bmatrix}$ are Hamiltonian matrices, while in general the matrix A_{co} is not.

C. Parameterization for the quantum Kalman canonical form

For the quantum Kalman canonical form (5), let the system Hamiltonian be

$$\boldsymbol{H} = \frac{1}{2}\boldsymbol{x}^\top H \boldsymbol{x}, \tag{11}$$

where the real-quadrature operator $\boldsymbol{x}$ satisfies the CCRs $[\boldsymbol{x}(0),\ \boldsymbol{x}(0)^\top] = \imath \mathbb{J}_n$, and the real matrix $H \in \mathbb{R}^{2n\times 2n}$ is symmetric. Let the coupling operator be

$$\boldsymbol{L} = \Gamma \boldsymbol{x}, \tag{12}$$

where $\Gamma \in \mathbb{C}^{m\times 2n}$. In this subsection, we aim to find conditions on the matrices H and Γ such that the QSDEs generated by the system Hamiltonian $\boldsymbol{H}$ in Eq. (11) and the coupling operator $\boldsymbol{L}$ in Eq. (12) are exactly the Kalman canonical form (5).

1) The necessary condition: By means of Lemma 3.1 given in subsection III-A, we can derive the following result, which presents a necessary condition for the system Hamiltonian $\boldsymbol{H}$ in Eq. (11) and the coupling operator $\boldsymbol{L}$ in Eq. (12) to generate the quantum Kalman canonical form (5).

Lemma 3.2: If the system Hamiltonian $\boldsymbol{H}$ in Eq. (11) and the coupling operator $\boldsymbol{L}$ in Eq. (12) lead to QSDEs in the Kalman canonical form (5), then the real symmetric matrix H must be of the form

$$H = \left[\begin{array}{cc|c|c} 0 & H_h^{12} & 0 & 0 \\ H_h^{12\top} & H_h^{22} & H_{12} & H_{13} \\ \hline 0 & H_{12}^\top & H_{co} & 0 \\ \hline 0 & H_{13}^\top & 0 & H_{\bar{c}\bar{o}} \end{array}\right], \tag{13}$$

where $H_h^{12} = -A_h^{22}$, $H_h^{22} = A_h^{12} - B_h \mathbb{J}_m B_h^\top/2$, $H_{12} = A_{12} - B_h \mathbb{J}_m B_{co}^\top \mathbb{J}_{n_1}/2$, $H_{13} = A_{13}$, $H_{co} = -\mathbb{J}_{n_1} A_{co} + \mathbb{J}_{n_1} B_{co} \mathbb{J}_m B_{co}^\top \mathbb{J}_{n_1}/2$, $H_{\bar{c}\bar{o}} = -\mathbb{J}_{n_2} A_{\bar{c}\bar{o}}$, and the complex matrix Γ must satisfy

$$\begin{bmatrix} \Gamma \\ \Gamma^\# \end{bmatrix} = \left[\begin{array}{c|c|c} 0 \quad \Gamma_h & \Gamma_{co} & 0 \end{array}\right], \tag{14}$$

where

$$\Gamma_h = V_m^\dagger C_h, \quad \Gamma_{co} = V_m^\dagger C_{co}. \tag{15}$$

Remark 3.2: The matrices Γ_{co} and Γ_h in Eq. (15) are of the form

$$\Gamma_{co} = \begin{bmatrix} \Gamma_{co,q} & \Gamma_{co,p} \\ \Gamma_{co,q}^\# & \Gamma_{co,p}^\# \end{bmatrix} \in \mathbb{C}^{2m\times 2n_1}, \tag{16a}$$

and

$$\Gamma_h = \begin{bmatrix} \Gamma_{h,p} \\ \Gamma_{h,p}^\# \end{bmatrix} \in \mathbb{C}^{2m\times n_3}, \tag{16b}$$

respectively.

2) The sufficient condition: We have shown in Lemma 3.2 that if the system Hamiltonian $\boldsymbol{H}$ in Eq. (11) and the coupling operator $\boldsymbol{L}$ in Eq. (12) generate QSDEs in the Kalman canonical form (5), then the real symmetric matrix H has the form (13) and the matrix Γ satisfies (14). In this subsection, we establish the converse result.

Lemma 3.3: If the real symmetric matrix H for the system Hamiltonian (11) is of the form (13) and the complex matrix Γ for the coupling operator (12) satisfies Eq. (14), then the QSDEs generated are of the form (5), with the matrices $\bar{A}$, $\bar{B}$ and $\bar{C}$ in Eq. (6) given by

$$\begin{aligned} &A_h^{11} = H_h^{12\top}, \quad A_h^{12} = H_h^{22} - \imath\Gamma_h^\dagger J_m \Gamma_h/2, \\ &A_h^{22} = -H_h^{12}, \quad A_{12} = H_{12} - \imath\Gamma_h^\dagger J_m \Gamma_{co}/2, \\ &A_{13} = H_{13}, \quad A_{co} = \mathbb{J}_{n_1} H_{co} - \imath\mathbb{J}_{n_1}\Gamma_{co}^\dagger J_m \Gamma_{co}/2, \\ &A_{\bar{c}\bar{o}} = \mathbb{J}_{n_2} H_{\bar{c}\bar{o}}, \quad A_{21} = \mathbb{J}_{n_1} H_{12}^\top - \imath\mathbb{J}_{n_1}\Gamma_{co}^\dagger J_m \Gamma_h/2, \\ &A_{31} = \mathbb{J}_{n_2} H_{13}^\top, \quad B_h = \Gamma_h^\dagger V_m^\dagger \mathbb{J}_m, \quad B_{co} = \mathbb{J}_{n_1}\Gamma_{co}^\dagger V_m^\dagger \mathbb{J}_m, \end{aligned} \tag{17}$$

and $C_h = V_m \Gamma_h$, $\ C_{co} = V_m \Gamma_{co}$.

Remark 3.3: By Eq. (17), $\mathbb{J}_{n_2} A_{\bar{c}\bar{o}} = -H_{\bar{c}\bar{o}}$ is symmetric. Thus, $A_{\bar{c}\bar{o}}$ is a Hamiltonian matrix. Similarly, by Eq. (17), the matrix $\begin{bmatrix} A_h^{11} & 0 \\ 0 & A_h^{22} \end{bmatrix}$ is also a Hamiltonian matrix. On the other hand, if the matrix A_{co} in Eq. (17) is a Hamiltonian matrix, then $\Gamma_{co}^{\dagger} J_m \Gamma_{co} = \Gamma_{co}^{\top} J_m \Gamma_{co}^{\#}$ must hold. However, by Eq. (16a) it can be readily shown that $\Gamma_{co}^{\dagger} J_m \Gamma_{co} + \Gamma_{co}^{\top} J_m \Gamma_{co}^{\#} = 0$. Therefore, in general the matrix A_{co} is not a Hamiltonian matrix. This remark, together with Remark 3.1, describes the spectral structure of the co, $\bar{c}\bar{o}$, and "h" subsystems.

Given the real symmetric matrix H in Eq. (13) and complex matrix Γ satisfying Eq. (14), Lemma 3.3 provides a way for constructing the system matrices $\bar{A}$, $\bar{B}$ and $\bar{C}$ in the Kalman canonical form. However, to guarantee that the QSDEs are indeed the quantum Kalman canonical form (5), certain controllability and observability conditions have to be satisfied. In what follows, we investigate this problem.

Lemma 3.4: For the system (5), (A_h^{11}, B_h) is controllable if and only if (A_h^{22}, C_h) is observable if and only if (H_h^{12}, Γ_h) is observable.

Lemma 3.5: For the system (5), (A_{co}, B_{co}) is controllable if and only if (A_{co}, C_{co}) is observable if and only if $(\mathbb{J}_{n_1} H_{co}, \Gamma_{co})$ is observable.

Lemma 3.6: For the system (5), the following statements are equivalent.

(i) $\left(\begin{bmatrix} \mathbb{J}_{n_1} H_{co} & 0 \\ H_{12} & H_h^{12\top} \end{bmatrix}, \begin{bmatrix} \mathbb{J}_{n_1} \Gamma_{co}^{\dagger} \\ \Gamma_h^{\dagger} \end{bmatrix} \right)$ is controllable;

(ii) $\left(\begin{bmatrix} \mathbb{J}_{n_1} H_{co} & \mathbb{J}_{n_1} H_{12}^{\top} \\ 0 & -H_h^{12} \end{bmatrix}, \begin{bmatrix} \Gamma_{co} & \Gamma_h \end{bmatrix} \right)$ is observable.

Combining Lemma 3.2-3.6, we obtain the main result of this section.

Theorem 3.2: Suppose that the real matrix H in Eq. (13) and complex matrix Γ in Eq. (14) satisfy the following conditions:

(i) $H_h^{22} = H_h^{22\top}$, $H_{\rm co} = H_{\rm co}^{\top}$, and $H_{\bar{\rm c}\bar{\rm o}} = H_{\bar{\rm c}\bar{\rm o}}^{\top}$;

(ii) $\left(\begin{bmatrix} \mathbb{J}_{n_1} H_{co} & \mathbb{J}_{n_1} H_{12}^{\top} \\ 0 & -H_h^{12} \end{bmatrix}, \begin{bmatrix} \Gamma_{co} & \Gamma_h \end{bmatrix} \right)$ is observable.

Then the resulting QSDEs are in the Kalman canonical form (5). In other words, $\boldsymbol{x}_{co}$ is both controllable and observable, $\boldsymbol{x}_{\bar{c}\bar{o}}$ is neither controllable nor observable, $\boldsymbol{q}_h$ is controllable and unobservable, and $\boldsymbol{p}_h$ is uncontrollable and observable. Conversely, if the system Hamiltonian $\boldsymbol{H}$ in Eq. (11) and the coupling operator $\boldsymbol{L}$ in Eq. (12) generate the QSDEs in the Kalman canonical form (5) , then the conditions (13), (14) and (i)-(ii) above must be satisfied.

D. A refinement of the quantum Kalman canonical form

It follows from the form of the matrix H in (13) that the "h" subsystem (9), in general, interacts with the "co" subsystem (10) and "$\bar{c}\bar{o}$" subsystem (8) via the sub-matrices H_{12} and H_{13}, respectively. As far as the Kalman canonical form is concerned, the sub-matrices H_{12} and H_{13} are free parameters; see also the interconnections among subsystems. In this subsection, we explore the structures of these and other matrices to refine the Kalman canonical form (5); in particular, we reveal its noiseless and invariant subsystems. To this end, we first introduce the following concept for quantum linear systems.

Definition 3.1: If a quantum linear system G can be written in a concatenation form[1] $G = G_1 \boxplus G_2$, then we say that G_1 and G_2 are *invariant* subsystems of G. Moreover, an invariant subsystem is called a *noiseless* subsystem if it is completely isolated from the environment.

By means of the (H, Γ) representation in Eqs. (13) and (14), we are in a position to construct the noiseless subsystem and the invariant subsystems arising in the quantum Kalman canonical form (5). To begin with, let us consider the noiseless subsystem.

Lemma 3.7: The "$\bar{c}\bar{o}$" subsystem (8) of the Kalman canonical form (5) has a noiseless subsystem $G_{\bar{c}\bar{o}}$ if there exists an orthogonal and blockwise symplectic matrix $\mathcal{P}_{\bar{c}\bar{o}}$ such that

$$\mathcal{P}_{\bar{c}\bar{o}} \boldsymbol{x}_{\bar{c}\bar{o}} = \begin{bmatrix} \boldsymbol{x}_{\bar{c}\bar{o}1} \\ \boldsymbol{x}_{\bar{c}\bar{o}2} \end{bmatrix}, \ \mathcal{P}_{\bar{c}\bar{o}} H_{\bar{c}\bar{o}} \mathcal{P}_{\bar{c}\bar{o}}^{\top} = \begin{bmatrix} H_{\bar{c}\bar{o}1} & 0 \\ 0 & H_{\bar{c}\bar{o}2} \end{bmatrix}, \tag{18a}$$

$$H_{13} \mathcal{P}_{\bar{c}\bar{o}}^{\top} = \begin{bmatrix} H_{131} & 0 \end{bmatrix}, \tag{18b}$$

where the system variables $\boldsymbol{x}_{\bar{c}\bar{o}1}$ and $\boldsymbol{x}_{\bar{c}\bar{o}2}$ satisfy the following CCRs

(A1) $\left[\begin{bmatrix} \boldsymbol{x}_{\bar{c}\bar{o}1} \\ \boldsymbol{x}_{\bar{c}\bar{o}2} \end{bmatrix}, \begin{bmatrix} \boldsymbol{x}_{\bar{c}\bar{o}1} \\ \boldsymbol{x}_{\bar{c}\bar{o}2} \end{bmatrix}^{\top} \right] = \imath \begin{bmatrix} \mathbb{J}_{n_2 - n_4} & 0 \\ 0 & \mathbb{J}_{n_4} \end{bmatrix}$

with $n_4 > 0$. In this case, the noiseless subsystem $G_{\bar{c}\bar{o}}$ is given by

$$\dot{\boldsymbol{x}}_{\bar{c}\bar{o}2}(t) = \mathbb{J}_{n_4} H_{\bar{c}\bar{o}2} \boldsymbol{x}_{\bar{c}\bar{o}2}(t). \tag{19}$$

Remark 3.4: In order to construct a noiseless subsystem which is itself a quantum-mechanical system, the entries of $\boldsymbol{x}_{\bar{c}\bar{o}}$ need to be combined in an appropriate way, as has been done by the first equation in (18a). Moreover, condition (A1) in Lemma 3.7 gives the CCRs for the physical quantities $\boldsymbol{x}_{\bar{c}\bar{o}1}$ and $\boldsymbol{x}_{\bar{c}\bar{o}2}$. Finally, it can be readily seen from Eqs. (18a)-(18b) that the noiseless subsystem is indeed the one in Eq. (19).

Compared with noiseless subsystems, general invariant subsystems are more complicated as the interaction between the quantum subsystem and the fields also needs to be considered. To make this clearer, we will study these invariant subsystems contained in the Kalman canonical form (5).

Lemma 3.8: The "co" subsystem (10) of the Kalman canonical form (5) has an invariant subsystem G_{co} if there exists an orthogonal and blockwise symplectic matrix $\mathcal{P}_{co}$ such that

$$\begin{aligned} &\mathcal{P}_{co} \boldsymbol{x}_{co} = \begin{bmatrix} \boldsymbol{x}_{co1} \\ \boldsymbol{x}_{co2} \end{bmatrix}, \ \Gamma_{co} \mathcal{P}_{co}^{\top} = \begin{bmatrix} \Gamma_{co1} & \Gamma_{co2} \end{bmatrix}, \\ &\mathcal{P}_{co} H_{co} \mathcal{P}_{co}^{\top} = \begin{bmatrix} H_{co1} & 0 \\ 0 & H_{co2} \end{bmatrix}, H_{12} \mathcal{P}_{co}^{\top} = \begin{bmatrix} H_{121} & 0 \end{bmatrix}, \end{aligned} \tag{20}$$

where the system variables $\boldsymbol{x}_{co1}$ and $\boldsymbol{x}_{co2}$ satisfy the condition

(B1) $\left[\begin{bmatrix} \boldsymbol{x}_{co1} \\ \boldsymbol{x}_{co2} \end{bmatrix}, \begin{bmatrix} \boldsymbol{x}_{co1} \\ \boldsymbol{x}_{co2} \end{bmatrix}^{\top} \right] = \imath \begin{bmatrix} \mathbb{J}_{n_1 - n_5} & 0 \\ 0 & \mathbb{J}_{n_5} \end{bmatrix}$ with $n_5 > 0$;

and the constant matrices Γ_{co1} and Γ_{co2} satisfy the condition

[1]Given two open quantum systems $G_1 \triangleq (\boldsymbol{S}_1, \boldsymbol{L}_1, \boldsymbol{H}_1)$ and $G_2 \triangleq (\boldsymbol{S}_2, \boldsymbol{L}_2, \boldsymbol{H}_2)$, their concatenation product is defined to be $G_1 \boxplus G_2 \triangleq \left(\begin{bmatrix} \boldsymbol{S}_1 & 0 \\ 0 & \boldsymbol{S}_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{L}_1 \\ \boldsymbol{L}_2 \end{bmatrix}, \boldsymbol{H}_1 + \boldsymbol{H}_2 \right)$. See [2] for more details.

(B2) each row of $\left[\begin{array}{cc|c}\Gamma_h & \Gamma_{co1} & \Gamma_{co2}\end{array}\right]$ is in the form of either $\left[\begin{array}{cc|c}\Gamma_h^i & \Gamma_{co1}^i & 0\end{array}\right]$ or $\left[\begin{array}{cc|c}0 & 0 & \Gamma_{co2}^i\end{array}\right]$, where Γ_h^i, Γ_{co1}^i and Γ_{co2}^i, $i=1,\cdots,2m$, are the ith rows of the matrices Γ_h, Γ_{co1} and Γ_{co2}.

Denote the set of indices of nonzero rows of Γ_{co2} by $\mathbb{I}_{co}=\{i_1,\cdots,i_{2m_1}\}$, where $m_1\le m$, and $1\le i_1<\cdots<i_{2m_1}\le 2m$. Define

$$\hat{\Gamma}_{co2}\triangleq\begin{bmatrix}\Gamma_{co2}^{i_1}\\ \vdots\\ \Gamma_{co2}^{i_{2m_1}}\end{bmatrix},\ \boldsymbol{y}_{co}(t)\triangleq\begin{bmatrix}\boldsymbol{y}_{i_1}(t)\\ \vdots\\ \boldsymbol{y}_{i_{2m_1}}(t)\end{bmatrix},$$
$$\boldsymbol{u}_{co}(t)\triangleq\left[\begin{array}{ccc}\boldsymbol{u}_{i_1}(t) & \cdots & \boldsymbol{u}_{i_{2m_1}}(t)\end{array}\right],$$

where $\boldsymbol{u}_i(t)$ and $\boldsymbol{y}_i(t)$ are the ith column and ith row of $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$, respectively. In this case, the invariant controllable and observable subsystem G_{co} is given by

$$\begin{aligned}\dot{\boldsymbol{x}}_{co2}(t)&=(\mathbb{J}_{n_5}H_{co2}-\frac{\imath}{2}\mathbb{J}_{n_5}\hat{\Gamma}_{co2}^{\dagger}J_{m_1}\hat{\Gamma}_{co2})\boldsymbol{x}_{co2}(t)\\&\quad+\mathbb{J}_{n_5}\hat{\Gamma}_{co2}^{\dagger}V_{m_1}^{\dagger}\mathbb{J}_{m_1}\boldsymbol{u}_{co}(t),\\ \boldsymbol{y}_{co}(t)&=V_{m_1}\hat{\Gamma}_{co2}\boldsymbol{x}_{co2}(t)+\boldsymbol{u}_{co}(t).\end{aligned}\tag{21}$$

In a similar way, we can derive the following result for the “h” subsystem.

Lemma 3.9: The “h” subsystem (9) of the Kalman canonical form (5) has an invariant subsystem G_h if there exists an orthogonal and blockwise symplectic matrix $\mathcal{P}_h$ such that

$$\mathcal{P}_h\begin{bmatrix}\boldsymbol{q}_h\\ \boldsymbol{p}_h\end{bmatrix}=\left[\begin{array}{c}\boldsymbol{q}_{h1}\\ \boldsymbol{p}_{h1}\\ \hline \boldsymbol{q}_{h2}\\ \boldsymbol{p}_{h2}\end{array}\right],\ \Gamma_h\mathcal{P}_h^{\top}=\left[\begin{array}{cc}\Gamma_{h1} & \Gamma_{h2}\end{array}\right],$$

$$\mathcal{P}_h\begin{bmatrix}0 & H_h^{12}\\ H_h^{12^T} & H_h^{22}\end{bmatrix}\mathcal{P}_h^{\top}=\begin{bmatrix}0 & H_{h1}^{12} & 0 & 0\\ H_{h1}^{12^\top} & H_{h1}^{22} & 0 & 0\\ 0 & 0 & 0 & H_{h2}^{12}\\ 0 & 0 & H_{h2}^{12^\top} & H_{h2}^{22}\end{bmatrix},\tag{22a}$$

$$\mathcal{P}_h\begin{bmatrix}0\\ H_{12}\end{bmatrix}=\begin{bmatrix}0\\ H_{12}^1\\ 0\\ 0\end{bmatrix},\ \mathcal{P}_h\begin{bmatrix}0\\ H_{13}\end{bmatrix}=\begin{bmatrix}0\\ H_{13}^1\\ 0\\ 0\end{bmatrix},\tag{22b}$$

where the system variables $\boldsymbol{q}_{h1}$, $\boldsymbol{p}_{h1}$, $\boldsymbol{q}_{h2}$, and $\boldsymbol{p}_{h2}$ satisfy the conditions

(C1) $$\left[\begin{bmatrix}\boldsymbol{q}_{h1}\\ \boldsymbol{p}_{h1}\\ \boldsymbol{q}_{h2}\\ \boldsymbol{p}_{h2}\end{bmatrix},\begin{bmatrix}\boldsymbol{q}_{h1}\\ \boldsymbol{p}_{h1}\\ \boldsymbol{q}_{h2}\\ \boldsymbol{p}_{h2}\end{bmatrix}^{\top}\right]=\imath\begin{bmatrix}\mathbb{J}_{n_3-n_6} & 0\\ 0 & \mathbb{J}_{n_6}\end{bmatrix}$$
with $n_6>0$;

and constant matrices Γ_{h1} and Γ_{h2} satisfy the condition

(C2) each row of $\left[\begin{array}{cc|c}\Gamma_{h1} & \Gamma_{co} & \Gamma_{h2}\end{array}\right]$ is in the form $\left[\begin{array}{cc|c}\Gamma_{h1}^i & \Gamma_{co}^i & 0\end{array}\right]$ or $\left[\begin{array}{cc|c}0 & 0 & \Gamma_{h2}^i\end{array}\right]$, where Γ_{h1}^i, Γ_{h2}^i and Γ_{co}^i, $i=1,\cdots,2m$, are the ith rows of the matrices Γ_{h1}, Γ_{h2} and Γ_{co}.

Denote the set of indices of nonzero rows of Γ_{h2} by $\mathbb{I}_h=\{i_1,\cdots,i_{2m_2}\}$, where $m_2\le m$ and $1\le i_1<\cdots<i_{2m_2}\le 2m$. Define

$$\hat{\Gamma}_{h2}\triangleq\begin{bmatrix}\Gamma_{h2}^{i_1}\\ \vdots\\ \Gamma_{h2}^{i_{2m_2}}\end{bmatrix},\ \boldsymbol{y}_h(t)\triangleq\begin{bmatrix}\boldsymbol{y}_{i_1}(t)\\ \vdots\\ \boldsymbol{y}_{i_{2m_2}}(t)\end{bmatrix},$$
$$\boldsymbol{u}_h(t)\triangleq\left[\begin{array}{ccc}\boldsymbol{u}_{i_1}(t) & \cdots & \boldsymbol{u}_{i_{2m_2}}(t)\end{array}\right],$$

where $\boldsymbol{u}_i(t)$ and $\boldsymbol{y}_i(t)$ are the ith column and ith row of $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$, respectively. Then the invariant subsystem G_h is given by

$$\begin{aligned}\begin{bmatrix}\dot{\boldsymbol{q}}_{h2}(t)\\ \dot{\boldsymbol{p}}_{h2}(t)\end{bmatrix}&=\begin{bmatrix}H_{h2}^{12^\top} & H_{h2}^{22}-\frac{\imath}{2}\hat{\Gamma}_{h2}^{\dagger}J_{m_2}\hat{\Gamma}_{h2}\\ 0 & -H_{h2}^{12}\end{bmatrix}\begin{bmatrix}\boldsymbol{q}_{h2}(t)\\ \boldsymbol{p}_{h2}(t)\end{bmatrix}\\&\quad+\begin{bmatrix}\hat{\Gamma}_{h2}^{\dagger}V_{m_2}^{\dagger}\mathbb{J}_{m_2}\\ 0\end{bmatrix}\boldsymbol{u}_h(t),\\ \boldsymbol{y}_h(t)&=V_{m_2}\hat{\Gamma}_{h2}\boldsymbol{p}_{h2}(t)+\boldsymbol{u}_h(t).\end{aligned}\tag{23}$$

By removing subsystems $G_{\bar{c}\bar{o}},G_{co},G_h$ from the Kalman canonical form (5), the remaining subsystem, denoted by G_m, is clearly an invariant subsystem. Next, we introduce this invariant subsystem. Let $m_3=m-m_1-m_2$. Denote the set of indices of nonzero rows of $\left[\begin{array}{cc}\Gamma_{h1} & \Gamma_{co1}\end{array}\right]$ by $\mathbb{I}_m=\{i_1,\cdots,i_{2m_3}\}$, $1\le i_1<\cdots<i_{2m_3}\le 2m$. Define

$$\left[\hat{\Gamma}_{h1}\ \hat{\Gamma}_{co1}\right]\triangleq\begin{bmatrix}\Gamma_{h1}^{i_1} & \Gamma_{co1}^{i_1}\\ \vdots & \vdots\\ \Gamma_{h1}^{i_{2m_3}} & \Gamma_{co1}^{i_{2m_3}}\end{bmatrix},\ \boldsymbol{x}_m(t)\triangleq\left[\begin{array}{c}\boldsymbol{q}_{h1}(t)\\ \boldsymbol{p}_{h1}(t)\\ \hline \boldsymbol{x}_{co1}(t)\\ \hline \boldsymbol{x}_{\bar{c}\bar{o}1}(t)\end{array}\right]$$

$$\boldsymbol{y}_m(t)\triangleq\begin{bmatrix}\boldsymbol{y}_{i_1}(t)\\ \vdots\\ \boldsymbol{y}_{i_{2m_3}}(t)\end{bmatrix},\ \boldsymbol{u}_m(t)\triangleq\left[\boldsymbol{u}_{i_1}(t)\ \cdots\ \boldsymbol{u}_{i_{2m_3}}(t)\right],$$

where Γ_{h1}^i, Γ_{co1}^i, $i=1,\cdots,2m$, are the ith row of matrices Γ_{h1}, Γ_{co1}, and $\boldsymbol{u}_i(t)$, $\boldsymbol{y}_i(t)$ are the ith column and ith row of $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$, respectively. The invariant subsystem G_m is of the form

$$\dot{\boldsymbol{x}}_m(t)=\vec{A}\boldsymbol{x}_m(t)+\vec{B}\boldsymbol{u}_m(t),\ \boldsymbol{y}_m(t)=\vec{C}\boldsymbol{x}_m(t)+\boldsymbol{u}_m(t),\tag{24}$$

where

$$\vec{A}\triangleq\left[\begin{array}{cc|c|c}A_{h1}^{11} & A_{h1}^{12} & A_{m12} & A_{m13}\\ 0 & A_{h1}^{22} & 0 & 0\\ \hline 0 & A_{m21} & A_{co1} & 0\\ \hline 0 & A_{m31} & 0 & A_{\bar{c}\bar{o}1}\end{array}\right],\ \vec{B}\triangleq\left[\begin{array}{c}B_{h1}\\ 0\\ \hline B_{co1}\\ \hline 0\end{array}\right],$$
$$\vec{C}\triangleq\left[\begin{array}{cc|c|c}0 & C_{h1} & C_{co1} & 0\end{array}\right],$$

with $A_{h1}^{11}=H_{h1}^{12^\top}$, $A_{h1}^{12}=H_{h1}^{22}-\frac{\imath}{2}\hat{\Gamma}_{h1}^{\dagger}J_{m_3}\hat{\Gamma}_{h1}$, $A_{h1}^{22}=-H_{h1}^{12}$, $A_{m12}=H_{m12}-\frac{\imath}{2}\hat{\Gamma}_{h1}^{\dagger}J_{m_3}\hat{\Gamma}_{co1}$, $A_{m13}=H_{m13}$, $A_{co1}=\mathbb{J}_{n_1-n_5}H_{co1}-\frac{\imath}{2}\mathbb{J}_{n_1-n_5}\hat{\Gamma}_{co1}^{\dagger}J_{m_3}\hat{\Gamma}_{co1}$, $A_{\bar{c}\bar{o}1}=\mathbb{J}_{n_2-n_4}H_{\bar{c}\bar{o}1}$, $A_{m21}=\mathbb{J}_{n_1-n_5}H_{m12}^{\top}-\frac{\imath}{2}\mathbb{J}_{n_1-n_5}\hat{\Gamma}_{co1}^{\dagger}J_{m_3}\hat{\Gamma}_{h1}$, $A_{m31}=\mathbb{J}_{n_2-n_4}H_{m13}^{\top}$, $B_{h1}=\hat{\Gamma}_{h1}^{\dagger}V_{m_3}\mathbb{J}_{m_3}$, $B_{co1}=\mathbb{J}_{n_1-n_5}\hat{\Gamma}_{co1}^{\dagger}V_{m_3}\mathbb{J}_{m_3}$, $C_{h1}=V_{m_3}\hat{\Gamma}_{h1}$, and $C_{co1}=V_{m_3}\hat{\Gamma}_{co1}$,

Based on Lemmas 3.7-3.9 and the subsystem G_m given in (24), we are now in a position to propose the following result.

Theorem 3.3: The quantum Kalman canonical form (5) can be put in the concatenation form

$$G = G_{\bar{c}\bar{o}} \boxplus G_{co} \boxplus G_h \boxplus G_m, \tag{25}$$

where $G_{\bar{c}\bar{o}}$ is the noiseless subsystem given in Lemma 3.7, G_{co} is the invariant controllable and observable subsystem given in Lemma 3.8, G_h is the invariant subsystem given in Lemma 3.9, and G_m is given in (24), provided that

(i) there exist orthogonal and blockwise symplectic matrices $\mathcal{P}_{\bar{c}\bar{o}}$, $\mathcal{P}_{co}$ and $\mathcal{P}_h$ satisfying (18a), (20), (22a), and

$$\mathcal{P}_h \begin{bmatrix} 0 \\ H_{12} \end{bmatrix} \mathcal{P}_{co}^\top = \begin{bmatrix} 0 & 0 \\ H_{m12} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathcal{P}_h \begin{bmatrix} 0 \\ H_{13} \end{bmatrix} \mathcal{P}_{\bar{c}\bar{o}}^\top = \begin{bmatrix} 0 & 0 \\ H_{m13} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix};$$

(ii) (A1), (B1)-(B2), and (C1)-(C2) hold;

(iii) each row of $\left[\begin{array}{cc|c|c} \Gamma_{h1} & \Gamma_{co1} & \Gamma_{h2} & \Gamma_{co2} \end{array}\right]$ is in one of the following forms $\left[\begin{array}{cc|c|c} \Gamma_{h1}^i & \Gamma_{co1}^i & 0 & 0 \end{array}\right]$, or $\left[\begin{array}{cc|c|c} 0 & 0 & \Gamma_{h2}^i & 0 \end{array}\right]$, or $\left[\begin{array}{cc|c|c} 0 & 0 & 0 & \Gamma_{co2}^i \end{array}\right]$, where Γ_{h1}^i, Γ_{h2}^i, Γ_{co1}^i and Γ_{co2}^i, $i = 1, \cdots, 2m$, are the ith row of matrices Γ_{h1}, Γ_{h2}, Γ_{co1} and Γ_{co2}.

Remark 3.5: We have the following observations on Theorem 3.3.

(i) The noiseless subsystem $G_{\bar{c}\bar{o}}$ is a subsystem of the $\bar{c}\bar{o}$ subsystem, as can be seen from Eq. (19); similarly, the invariant subsystem G_{co} is a subsystem of the *co* subsystem, as can be seen from Eq. (21); and the invariant subsystem G_h is a subsystem of the "h" subsystem, as can be seen from Eq. (23);

(ii) The invariant subsystem G_m is a mixture of "h", *co*, and $\bar{c}\bar{o}$ subsystems, as can be seen from Eq. (24).

(iii) The system (25) in Theorem 3.3 involves partitioning system inputs and outputs, while the original Kalman canonical form (5) does not.

(iv) It is worthwhile to notice that the proposed system decomposition is very general. In some cases, some of the subsystems may not exist; this can be easily seen from the conditions in Lemmas 3.7-3.9 and Theorem 3.3. On the other hand, there might be more than one invariant *co*, $\bar{c}\bar{o}$, or "h" subsystems. Indeed, the system in Example below can be decomposed into two invariant *co* subsystems, each of which is a harmonic oscillator driven by a single input field.

It is interesting to see that the subsystem G_m is in the Kalman canonical form (5), while $G_{\bar{c}\bar{o}}$, G_h, and G_{co} are in the form of (8), (9), and (10) respectively. Therefore, the quantum Kalman canonical form (5) is decomposed into four subsystems which are decoupled from each other, and one of which itself is a smaller Kalman canonical form. This means that the Kalman canonical form (5), in general, may not reveal the noiseless subsystem and the invariant subsystems of a given quantum linear system. In Theorem 3.3, a refined decomposition of the matrices H and Γ shows that a quantum linear system can be expressed in the form (25) by appropriate coordinate transformations. Finally, the following should be noted. In the Heisenberg picture of quantum mechanics, a quantum linear system G may be put into a concatenation form, where four possible subsystems are decoupled form each other. However, the initial state of the whole system G may still be a state superposed among all these subsystems.

IV. Applications to quantum BAE measurements

In this section, we consider the realization of BAE measurements. We present necessary and sufficient conditions for the quantum Kalman canonical form (5) to realize BAE measurements. These necessary and sufficient conditions are given explicitly in terms of the physical parameters H and Γ.

As BAE measurements are an input-output property, they are determined completely by the *co* subsystem. For the quantum linear system (5), the transfer function from $\boldsymbol{u}$ to $\boldsymbol{y}$ is $\Xi_{\boldsymbol{u}\to\boldsymbol{y}}(s) = C_{co}(sI - A_{co})^{-1}B_{co} + I$. Partition the matrices B_{co} and C_{co} as $B_{co} = [B_{co,q} \;\; B_{co,p}]$, $C_{co} = \begin{bmatrix} C_{co,q} \\ C_{co,p} \end{bmatrix}$, respectively. Then the transfer function from $\boldsymbol{p}_{\rm in}$ to $\boldsymbol{q}_{\rm out}$ is $\Xi_{\boldsymbol{p}_{\rm in}\to\boldsymbol{q}_{\rm out}}(s) = C_{co,q}(sI - A_{co})^{-1}B_{co,p}$. Similarly, the transfer function from $\boldsymbol{q}_{\rm in}$ to $\boldsymbol{p}_{\rm out}$ is $\Xi_{\boldsymbol{q}_{\rm in}\to\boldsymbol{p}_{\rm out}}(s) = C_{co,p}(sI - A_{co})^{-1}B_{co,q}$.

The following is the main result of this section, which gives necessary and sufficient conditions for the realization of BAE measurements by the quantum linear system (5).

Theorem 4.1:

(i) The quantum Kalman canonical form (5) realizes the BAE measurements of $\boldsymbol{q}_{\rm out}$ with respect to $\boldsymbol{p}_{\rm in}$; i.e., $\Xi_{\boldsymbol{p}_{\rm in}\to\boldsymbol{q}_{\rm out}}(s) \equiv 0$ if and only if

$$\begin{aligned} &\left[\, \mathrm{Re}\,(\Gamma_{co,q}) \quad \mathrm{Re}\,(\Gamma_{co,p}) \,\right] (sI - \mathbb{J}_{n_1} H_{co})^{-1} \\ &\times \begin{bmatrix} \mathrm{Re}(\Gamma_{co,p}^\top) \\ -\mathrm{Re}(\Gamma_{co,q}^\top) \end{bmatrix} \equiv 0; \end{aligned} \tag{26}$$

(ii) The quantum Kalman canonical form (5) realizes the BAE measurements of $\boldsymbol{p}_{\rm out}$ with respect to $\boldsymbol{q}_{\rm in}$; i.e., $\Xi_{\boldsymbol{q}_{\rm in}\to\boldsymbol{p}_{\rm out}}(s) \equiv 0$ if and only if

$$\begin{aligned} &\left[\, \mathrm{Im}\,(\Gamma_{co,q}) \quad \mathrm{Im}\,(\Gamma_{co,p}) \,\right] (sI - \mathbb{J}_{n_1} H_{co})^{-1} \\ &\times \begin{bmatrix} \mathrm{Im}(\Gamma_{co,p}^\top) \\ -\mathrm{Im}(\Gamma_{co,q}^\top) \end{bmatrix} \equiv 0. \end{aligned} \tag{27}$$

The following corollary presents a special case of Theorem 4.1.

Corollary 4.1: Let a quantum linear system be parametrized by the Hamiltonian $\boldsymbol{H} = \frac{1}{2}\boldsymbol{x}^\top H \boldsymbol{x}$ and the coupling operator $\boldsymbol{L} = \Gamma\boldsymbol{x}$, where $\boldsymbol{x}$ satisfies the CCRs $[\boldsymbol{x},\ \boldsymbol{x}^\top] = \imath\mathbb{J}_n$, $H = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$ and $\Gamma = [\Gamma_q \;\; \Gamma_p]$ with $\Gamma_q, \Gamma_p \in \mathbb{C}^{m\times n}$. We have:

(i) The system is controllable and observable and $\Xi_{\boldsymbol{p}_{\rm in}\to\boldsymbol{q}_{\rm out}}(s) \equiv 0$ if and only if $\mathrm{Re}\,(\Gamma_q) \perp \mathrm{Re}\,(\Gamma_p)$, and

$$\mathrm{rank}\left(\begin{bmatrix} \check{\Gamma} \\ \check{\Gamma} J_n \end{bmatrix}\right) = 2n, \tag{28}$$

where $\breve{\Gamma} \triangleq \begin{bmatrix} \Gamma \\ \Gamma^{\#} \end{bmatrix}$;

(ii) The system is controllable and observable and $\Xi_{\boldsymbol{q}_{\rm in}\to\boldsymbol{p}_{\rm out}}(s) \equiv 0$ if and only if $\mathrm{Im}\,(\Gamma_q) \perp \mathrm{Im}\,(\Gamma_p)$ and (28) hold.

V. Examples

In this section, we use an example to illustrate the theoretical results derived in this paper.

This example considers the Michelson's interferometer which is one of the simplest devices for gravitational wave detection, see [15, Fig. 3(c)]. The interferometer contains two identical mechanical oscillators with position quadratures $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, and momentum quadratures $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, respectively. The resonant frequency and mass of the mechanical oscillators are denoted by ω_m and m, respectively. The input coherent light field (the probe field $\hat{W}_1$ in [15, Fig. 3(c)]) and the input vacuum light field ($\hat{W}_2$ in [15, Fig. 3(c)]) are described by their respective position quadratures $\boldsymbol{q}_{\rm in,1}$, $\boldsymbol{q}_{\rm in,2}$, and momentum quadratures $\boldsymbol{p}_{\rm in,1}$, $\boldsymbol{p}_{\rm in,2}$. Let λ be the coupling strength between the probe field and the mechanical oscillators. It is assumed that the mechanical oscillators are subjected to forces F and $-F$. Then the dynamics of the system, given in [15, Eq. (19)], is described by the following QSDEs.

$$\begin{aligned}
\begin{bmatrix} \dot{\boldsymbol{q}}_1 \\ \dot{\boldsymbol{q}}_2 \\ \dot{\boldsymbol{p}}_1 \\ \dot{\boldsymbol{p}}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 1/m & 0 \\ 0 & 0 & 0 & 1/m \\ -m\omega_m^2 & 0 & 0 & 0 \\ 0 & -m\omega_m^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_2 \\ \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \end{bmatrix} \\
&\quad + \sqrt{\lambda} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{\rm in,1} \\ \boldsymbol{q}_{\rm in,2} \\ \boldsymbol{p}_{\rm in,1} \\ \boldsymbol{p}_{\rm in,2} \end{bmatrix}, \\
\begin{bmatrix} \boldsymbol{q}_{\rm out,1} \\ \boldsymbol{q}_{\rm out,2} \\ \boldsymbol{p}_{\rm out,1} \\ \boldsymbol{p}_{\rm out,2} \end{bmatrix} &= \sqrt{\lambda} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_2 \\ \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{q}_{\rm in,1} \\ \boldsymbol{q}_{\rm in,2} \\ \boldsymbol{p}_{\rm in,1} \\ \boldsymbol{p}_{\rm in,2} \end{bmatrix}.
\end{aligned} \tag{29}$$

Note that the system (29) is both controllable and observable. It follows from Lemma 3.2 that

$$\begin{aligned}
H &= \begin{bmatrix} m\omega_m^2 & 0 & 0 & 0 \\ 0 & m\omega_m^2 & 0 & 0 \\ 0 & 0 & 1/m & 0 \\ 0 & 0 & 0 & 1/m \end{bmatrix}, \\
\Gamma &= \sqrt{\frac{\lambda}{2}} \begin{bmatrix} \imath & \imath & 0 & 0 \\ \imath & -\imath & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{30}$$

It is easy to check that H and Γ in Eq. (30) satisfy the condition (26), but not the condition (27). Indeed, denote $\boldsymbol{q}_{\rm in} = \begin{bmatrix} \boldsymbol{q}_{\rm in,1} \\ \boldsymbol{q}_{\rm in,2} \end{bmatrix}$, $\boldsymbol{p}_{\rm in} = \begin{bmatrix} \boldsymbol{p}_{\rm in,1} \\ \boldsymbol{p}_{\rm in,2} \end{bmatrix}$, $\boldsymbol{q}_{\rm out} = \begin{bmatrix} \boldsymbol{q}_{\rm out,1} \\ \boldsymbol{q}_{\rm out,2} \end{bmatrix}$, and $\boldsymbol{p}_{\rm out} = \begin{bmatrix} \boldsymbol{p}_{\rm out,1} \\ \boldsymbol{p}_{\rm out,2} \end{bmatrix}$. It turns out that $\Xi_{\boldsymbol{p}_{\rm in}\to\boldsymbol{q}_{\rm out}}(s) \equiv 0$, $\Xi_{\boldsymbol{q}_{\rm in}\to\boldsymbol{p}_{\rm out}}(s) = \frac{2}{m(s^2+\omega_m^2)} I_2 \neq 0$. Therefore the QSDEs (29) only realizes the BAE measurements of $\boldsymbol{q}_{\rm out}$ with respect to $\boldsymbol{p}_{\rm in}$. Finally, by Lemma 3.8, it can be easily verified that the orthogonal and blockwise symplectic matrix

$$\mathcal{P} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

transforms system (29) to two controllable and observable subsystems which are decoupled from each other, i.e., $G_{co} \boxplus G_{co}$.

VI. Conclusion

In this paper, we have investigated the structure of quantum linear systems by means of their Kalman canonical form. In particular, a new parametrization method has been proposed which generates the quantum Kalman canonical form directly. Necessary and sufficient conditions for realizing quantum BAE measurements have also been proposed in terms of these physical parameters. The *system analysis* results presented in this paper may be useful for quantum control engineering, e.g., of opto-mechanical systems.

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