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Leaderless Synchronization of Linear Multi-Agent Systems under Directed Switching Topologies: An Invariance Approach

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Abstract— **In this paper, we study the synchronization problem for a class of identical linear, high order systems. The interconnections between the systems are assumed to be time-varying, directed, and the underlying graph is not necessarily connected at any time instant. We show that in this setup, even destabilization may occur through diffusive couplings. In order to overcome this problem, we present a distributed control law, which is able to guarantee positive invariance of energy level-sets, and we investigate its synchronizing properties.**

I. INTRODUCTION

Consensus and synchronization have become an important field of interest in systems and control theory in recent years. Typically, a distributed control scheme is applied to a *multi-agent system*(MAS), i.e. a network of individual systems called *agents*. That means, for each agent in the network a separate *local controller* is implemented that can only receive information from the agent and its neighbors.

In the past few years, much attention has switched from the consensus of integrator agents, see e.g. [1], [2], and double-integrator agents, see e.g. [3], to synchronization of high-order linear multi-agent systems, in which each agent is modeled by generic linear system dynamics $\dot{x} = Ax + Bu$ [4]. This is mainly due to their ability to explain more complicated scenarios.

When the interconnection between the agents do not change over time, having a directed spanning tree has been shown to be the weakest condition that should be imposed on the interconnection topology to achieve synchronization [4], [5]. For the case with time-varying interconnection topology, a point of interest is whether the condition on the interconnection topology can be relaxed to the more general case where the interconnection topology does not necessarily have a directed spanning tree at any time instant, but only jointly over some time-interval. In the consensus case with integrator systems, this question has be answered by [6], [7].

For linear high-order systems two essentially different approaches can be distinguished: First, dynamic feedback controllers may be applied in order to deal with partial state outputs as seen in [8] and [9], and they are proven to be capable to deal with different time-varying topologies under certain assumptions. In [10], the authors consider a framework, where it is known how the mean of the time-varying topology will evolve over time and the topology switches fast enough. Under these assumptions, the proposed controller is shown to be able to synchronize a large class of systems. Secondly, static feedback controller design can be investigated for time-varying topologies. Some approaches have been recently made by [11], [12]. There, the authors consider undirected topologies and undirected topologies with an additional external leader. [10] can also be applied in the static feedback controller case and presents a controller for general directed graphs if again, it is known how the mean of the time-varying topology will evolve over time and the topology switches fast enough.

In this paper, we address the problem of synchronizing a class of linear MAS under directed switching topologies, where we allow arbitrary switching. As it will be shown in the paper, there are structural difficulties when directed couplings occur arbitrarily that might even lead to destabilization of stable MAS. In order to prevent this, we introduce a coupling function that is able to "turn off" couplings if required. Then, we analyze the synchronizing properties of the resulting controller.

The rest of the paper is organized as follows: Section II introduces the preliminaries including notation, graph theory, and states the synchronization problem at hand. At the end of Section II, we present an example that motivates the main results, which are shown in Section III. There, we propose a coupling function that is applied to the controller, such that no destabilization will occur. Moreover, we use a Lyapunov-type approach to show invariance of sets and analyze, under which conditions the controller will synchronize the MAS. In Section IV, we pick up the motivating example from Section II and simulate it using the proposed controller.

II. PRELIMINARIES

Throughout the paper the following notation is used: I_n denotes the $n \times n$ identity matrix. Let A be a quadratic matrix. $\mathrm{Ker}(A)$ denotes the kernel of A. $A > 0$ and $A \geq 0$ denote A to be positive definite, and positive semi-definite, respectively. For a positive definite matrix P, $\|x\|_P$ denotes a norm of the vector x, defined as $\|x\|_P = \sqrt{x^T P x}$. $A \otimes B$ denotes the Kronecker-Product of two matrices A and B and $\langle v_1, v_2 \rangle$ denotes the scalar product between to vectors v_1 and v_2. $\mathbb{R}^+$ and $\overline{\mathbb{R}}^+$ are the sets of positive and non-negative real numbers, respectively.

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A. Communication graphs

In this section we briefly review some definitions on communication graphs. We use a directed, unweighted, switching graph $\mathscr{G}_{\sigma(t)} = (\mathscr{V}, \mathscr{E}_{\sigma(t)}, \mathscr{A}_{\sigma(t)})$ to describe the interconnections between the individual agents. The switching signal $\sigma : \overline{\mathbb{R}}^+ \to \mathscr{P} = \{1,2,...,p\}$, is piecewise constant, where p is a positive integer. The switching time instants are denoted by $\{\tau_r | r = 1,2,...\}$ and $\sigma(t)$ is right continuous on each interval $[\tau_r, \tau_{t+1}), r = 1,2,....$ $\mathscr{V} = \{v_1,...,v_N\}$ is the set of vertices, where $v_k \in \mathscr{V}$ represents the k-th agent. $\mathscr{E}_{\sigma(t)} \subset V \times V$ is the set of edges, which models information flow, i.e. the k-th agent receives information from the j-th agent at time t if and only if $(v_j, v_k) \in \mathscr{E}_{\sigma(t)}$. $\mathscr{A}_{\sigma(t)}$ is the adjacency matrix, which encodes the edges, where $a_{kj,\sigma(t)} = 1$ if $(v_j, v_k) \in \mathscr{E}_{\sigma(t)}$ and $a_{kj,\sigma(t)} = 0$ otherwise. $\mathscr{A}_{\sigma(t)}$ is piecewise constant and bounded. Note that since the number of vertices N is finite, the set of different graphs, which can occur, is also finite. Thus, $p < \infty$. A *path* from vertex v_{i_1} to vertex v_{i_l} at time t is a sequence of vertices $\{v_{i_1},...,v_{i_l}\}$ such that $\{v_{i_j}, v_{i_{(j+1)}}\} \in \mathscr{E}_{\sigma(t)}, j = 1,...,l-1$.

Given some time interval $I = [t_1, t_2]$, we denote the *union graph* as $\overline{\mathscr{G}}(I) = (\mathscr{V}, \overline{\mathscr{E}}(I), \overline{A}_{\overline{\mathscr{G}}}(I))$, where $\overline{\mathscr{E}}(I) = \{\bigcup_{t \in I} \mathscr{E}_{\sigma(t)}\}$ and $\overline{A}_{\overline{\mathscr{G}}}(I)$ is the respective adjacency matrix.

Definition 1: $\mathscr{G}_{\sigma(t)}$ is *strongly connected* at time t, if for every vertex v_k and every vertex v_j, $k, j = 1,...,N, j \neq k$, there exists a path from v_k to v_j at time t. $\mathscr{G}_{\sigma(t)}$ is *jointly strongly connected* if there exist a finite time horizon T such that for all t the union graph $\overline{\mathscr{G}}([t, t+T])$ is strongly connected.

B. Problem Statement

Consider a network of N LTI agents connected over a switching interconnection topology that is described by a switching graph $\mathscr{G}_{\sigma(t)} = (\mathscr{V}, \mathscr{E}_{\sigma(t)}, \mathscr{A}_{\sigma(t)})$. The agents are modeled as

$$\dot{x}_k = Ax_k + Bu_k \tag{1}$$

with state variable $x_k \in \mathbb{R}^n$, input variable $u_k \in \mathbb{R}^p$ for $k = 1,...,N$ and matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times p}$. Without loss of generality, we assume that the initial time t_0 is 0 in the remainder of the paper, and denote the initial condition $x(0) = x_0$.

Definition 2: State synchronization takes place when $(x_j - x_k) \to 0$ for all $k, j = 1,...,N$ and $t \to \infty$.

For each agent we introduce a local controller given by

$$u_k = K \sum_{j=1}^{N} a_{kj,\sigma(t)} q(x_j, x_k), \tag{2}$$

where $K \in \mathbb{R}^{p \times n}$ is a controller gain matrix and $q(\cdot)$ is a coupling function $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with $q(x_k, x_j) = 0$ for $x_k = x_j$.

These definitions yield the following description of the problem to be considered in this paper:

Problem 1 (Synchronization): Let N individual agents be modeled by (1) for $k = 1,...,N$. Let the interconnection topology be defined by a graph $\mathscr{G}_{\sigma(t)} = (\mathscr{V}, \mathscr{E}_{\sigma(t)}, \mathscr{A}_{\sigma(t)})$. Find a controller gain K and a coupling function $q(\cdot)$, such that $(x_j - x_k) \to 0$ for all $k, j = 1,...,N$ and all initial conditions $x_k(0) \in \mathbb{R}^n$.

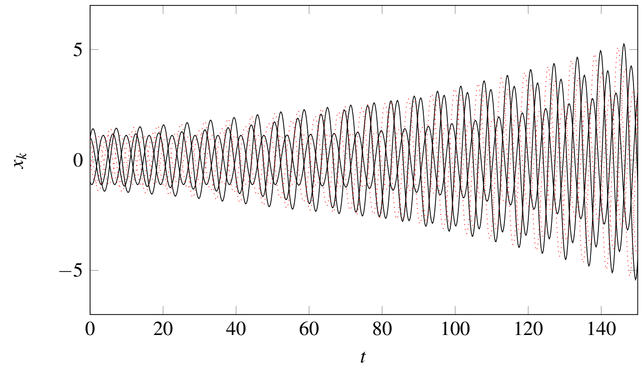


Fig. 1. Destabilization of the agents states is displayed. The black lines represent the first state, the dotted red lines represent the second state.

The following assumptions are needed for the results presented in this paper.

Assumption 1: The open loop systems (1) with $u_k = 0$ for $k = 1,...,N$ are Lyapunov stable in the sense that there exists a positive definite matrix $P > 0$ and a positive-semidefinite matrix $Q \geq 0$, which satisfies the Lyapunov equation $A^T P + PA = -Q$. Furthermore, (A, B) is stabilizable and B has full column rank.

Remark 1: Lyapunov stability of $\dot{x} = Ax$ holds, if all eigenvalues of A lie in the closed left half complex plane $\mathbb{C}^-$, and for all eigenvalues lying on the imaginary axis, their algebraic multiplicity is equal to their geometric multiplicity.

Assumption 2: The switching signal $\sigma(t)$ has non-vanishing dwell time, i.e. there exists a $\tau_0 > 0$, such that the switching instants $\{\tau_r | r = 1,2,...\}$ satisfy

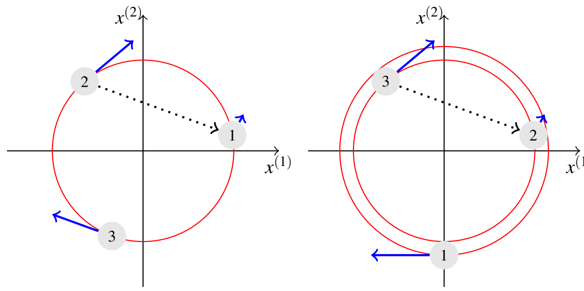
$$\inf_{r \geq 1} (\tau_{r+1} - \tau_r) \geq \tau_0.$$

C. A motivating example

We consider a group of $N = 3$ harmonic oscillators

$$\dot{x}_k = \begin{bmatrix} \dot{x}_k^{(1)} \\ \dot{x}_k^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_k^{(1)} \\ \dot{x}_k^{(2)} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k. \tag{3}$$

Assumption 1 is satisfied as (A, B) is stabilizable and $A^T P + PA = 0$ with $P = I_2$. We choose $q(x_j, x_k) = x_j - x_k$. In the special case of undirected graphs, the controller $u_k = B^T P \sum_{j=1}^{N} a_{kj,\sigma(t)} (x_j - x_k)$ synchronizes the agents [12]. The simulation result for the same controller under some directed graphs is shown in Figure 1. As it can be seen, the controller is even able to destabilize the system, if the communication topology switches unfortunately. In order to depict the problem at hand, consider the situations displayed in Figure 2. Loosely speaking, at t_1, agent 1 follows agent 2, but the control input u_1 can only act in $x^{(2)}$-direction. As a result, agent 1 increases its distance to the origin. At a later time instant t_2, agent 2 follows agent 3 and also increases its distance to the origin. Subsequently, without knowing,



(a) $t_1 : a_{14} = 1$ (b) $t_2 : a_{12} = 1$

Fig. 2. Directed diffusive coupling may have destabilizing effects. The blue arrows denote the current velocity vector of the agents.

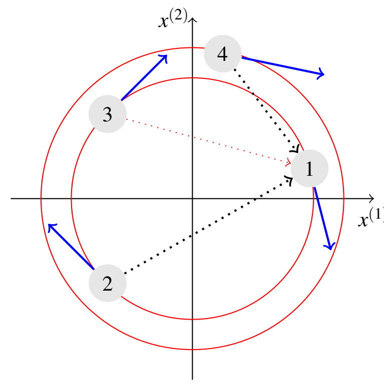


Fig. 3. The connection of agent 1 to agent 3 (red dashed line) is cancelled as $\widetilde{q}(x_j,x_k) = 0$.

how the topology of the multi-agent system evolves with t, diffusive coupling may destabilize a multi-agent system in the case of a directed switching graph. This applies to any controller of the form $u_k = K\sum_{j=1}^{N} a_{kj,\sigma(t)}(x_j - x_k)$ for the harmonic oscillators (3).

III. MAIN RESULTS

In this section we use a Lyapunov type approach in order to show that by a suitable choice of controller gain K and coupling function $q(\cdot)$, the evolution of the maximum energy function can be bounded, which overcomes the destabilizing behavior presented in the previous section. In the second theorem of this section, the synchronizing properties of the proposed controller are investigated.

We now introduce the positive definite, locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$ with

$$V(x) = \max_{1\le k\le N} V_k(x_k) = \max_{1\le k\le N} x_k^T P x_k, \tag{4}$$

where P solves the Lyapunov-Equation $A^T P + PA \le 0$. Note that $V(x)$ is continuous, but not differentiable in x. The controller (2) is proposed as

$$u_k = \varepsilon B^T P \sum_{j=1}^{N} a_{kj,\sigma(t)} q(x_j,x_k) \tag{5}$$

i.e. $K = \varepsilon B^T P$, where $\varepsilon > 0$ is an arbitrary gain. The coupling function is proposed as

$$\begin{aligned} q(x_j,x_k) &= (x_j - x_k)\cdot \widetilde{q}(x_j,x_k) \\ \widetilde{q}(x_j,x_k) &= \max(\|x_j\|_P - \|x_k\|_P, 0)^2 \\ &\quad + \max(\langle -B^T P(x_j - x_k), B^T P x_k\rangle, 0)^2. \end{aligned} \tag{6}$$

Note that $\widetilde{q}(x_j,x_k)$ is a scalar, non-negative factor. The coupling gain $\widetilde{q}(x_j,x_k)$ can be interpreted in the way that it cancels "harmful" connections, as depicted in Figure 3.

Let $x = \begin{bmatrix} x_1^T & \cdots & x_N^T \end{bmatrix}^T \in \mathbb{R}^{n\times N}$ be the stacked state vector, then the proposed controller u_k is continuously differentiable in x, and piecewise continuously differentiable in t. Therefore, we know that the systems (1) with the local controllers (5) will have an unique and absolutely continuous solution $x(t,x_0)$ in the sense of *Carathéodory*, i.e. $x(t,x_0)$ satisfies the differential equation (1), (5) almost everywhere. In the remainder of the paper, whenever we speak of a solution $x(t,x_0)$, we mean the solution in the sense of *Carathéodory*.

Define the weighted adjacency matrix

$$\widetilde{\mathscr{A}}_{\sigma(t)}(x) = \begin{bmatrix} 0 & \widetilde{a}_{12,\sigma(t)}(\cdot) & \cdots & \widetilde{a}_{1N,\sigma(t)}(\cdot) \\ \widetilde{a}_{21,\sigma(t)}(\cdot) & 0 & \cdots & \widetilde{a}_{2N,\sigma(t)}(\cdot) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{a}_{N1,\sigma(t)}(\cdot) & \widetilde{a}_{N2,\sigma(t)}(\cdot) & \cdots & 0 \end{bmatrix},$$

where the coefficients are $\widetilde{a}_{kj,\sigma(t)}(x_j,x_k) = \varepsilon a_{kj,\sigma(t)}\widetilde{q}(x_j,x_k)$. Let $\widetilde{\mathscr{D}}_{\sigma(t)}(x)$ be the weighted in-degree matrix, i.e. the diagonal matrix with the diagonal elements

$$\widetilde{d}_k(x) = \sum_{j=1}^{N} \widetilde{a}_{kj}(x_j,x_k),$$

and the weighted Laplacian matrix

$$\widetilde{\mathscr{L}}_{\sigma(t)}(x) = \widetilde{\mathscr{D}}_{\sigma(t)}(x) - \widetilde{\mathscr{A}}_{\sigma(t)}(x). \tag{7}$$

With (1), (5), (6) and (7), the closed loop system can now be written as the nonlinear switched system

$$\dot{x} = (I_N \otimes A - \widetilde{\mathscr{L}}_{\sigma(t)}(x) \otimes (BB^T P))x. \tag{8}$$

In order to study the convergence properties of the solutions $x(t,x_0)$, we investigate how $V(x)$ evolves along the solution of (8). According to the definition of $V(x)$ given by (4), we can see that $V(x(t))$ is not continuously differentiable at every time instant t. However, it can be shown that $x(t)$ is locally Lipschitz in t, and therefore $V(x(t))$ is also locally Lipschitz in t. The convergence properties of $V(x(t))$ can now be studied with Dini derivatives.

The upper Dini derivative of $V(x(t))$ is defined as

$$D^+V(x(t)) = \limsup_{s\to 0^+} \frac{V(t+s) - V(t)}{s},$$

and it can be calculated with the following Lemma, which is also used in [13] and refers to [14].

Lemma 1: Let $V(x(t))$ and $V_k(x_k(t))$be defined as in (4) and let the index set $I(t)$ be defined as

$$I(t) = \{k : V(x(t)) = V_k(x(t))\} \tag{9}$$

Then

$$D^+V(x(t)) = \max_{k\in I(t)} \dot{V}_k(x_k(t)),$$

with the Lie-derivative being defined as $\dot{V}_k(x_k(t)) = \frac{\partial V_k}{\partial x_k}\dot{x}_k = \frac{\partial V_k}{\partial x_k}(Ax_k + Bu_k)$.

With (8) and Lemma 1, we can deduce the main results on the monotony of $V(x(t))$ and invariance of its level-sets.

Theorem 1: Given the LTI systems (1), the local controllers (5), and the function V defined in (4), then for any $t \in \mathbb{R}$ it holds that

(a) $D^+V(x(t)) \leq 0$ and
(b) $D^+V(x(t)) = 0$ only if there exists a $k \in I(t)$, such that $u_k(t) = 0$.

Proof:

(a) First, we consider the positive definite C^1 functions $V_k(x_k)$. Along trajectories $x_k(t)$, the Lie-derivatives of $V_k(x_k)$ can be obtained as

$$\begin{aligned}\frac{\partial V_k}{\partial x_k}\dot{x}_k =& \underbrace{x_k^T(A^TP + PA)x_k}_{\leq 0} + x_k^TPBu_k + u_k^TB^TPx_k \\ \leq& 2x_k^TPBu_k\end{aligned} \tag{10}$$

Let $\widetilde{\mathscr{L}}^{(k)}_{\sigma(t)}(x)$ be the k-th row of the matrix $\widetilde{\mathscr{L}}_{\sigma(t)}(x)$, which is defined in (7), then

$$u_k = \widetilde{\mathscr{L}}^{(k)}_{\sigma(t)}(x) \otimes (B^TP)x.$$

When $k \in I(t)$, with $I(t)$ as defined in (9), then we know that

$$\|x_j\|_P - \|x_k\|_P \leq 0 \text{ for all } j \neq k.$$

Thus, the coupling function (6) is simply

$$\widetilde{q}(x_j, x_k) = \max(\langle -B^TP(x_j - x_k), B^TPx_k\rangle, 0)^2. \tag{11}$$

For Equation (10), it holds that

$$\begin{aligned}\frac{\partial V_k}{\partial x_k}\dot{x}_k \leq& 2x_k^TPB\widetilde{\mathscr{L}}^{(k)}_{\sigma(t)}(x) \otimes (B^TP)x \\ =& \sum_{j=1}^{N} \widetilde{a}_{kj,\sigma(t)}(x_j, x_k)x_k^TPBB^TP(x_j - x_k) \\ =& \sum_{j=1}^{N} \varepsilon a_{kj,\sigma(t)} \underbrace{\widetilde{q}(x_j, x_k)x_k^TPBB^TP(x_j - x_k)}_{\leq 0}\end{aligned} \tag{12}$$

with Equation (11). Therefore, $\frac{\partial V_k}{\partial x_k}\dot{x}_k \leq 0$ for all $k \in I(t)$, and with Lemma 1 we can conclude

$$D^+V(x(t)) \leq 0, t \in \mathbb{R}.$$

(b) Suppose that $D^+V(x(t)) = 0$. From Lemma 1 and Equation (10) it follows that there exists a $k \in I(t)$ such that

$$\dot{V}_k = \underbrace{x_k^T(A^TP + PA)x_k}_{\leq 0} + \underbrace{2x_k^TPBu_k}_{\leq 0} = 0,$$

and therefore $2x_k^TPBu_k = 0$.

Now suppose $u_k \neq 0$. Then there exists at least one coefficient $\widetilde{a}_{kj,\sigma(t)}(x_j, x_k) \neq 0$ and therefore $\widetilde{q}(x_j(t), x_k(t)) > 0$. With Equation (6) this means that

$$\langle -B^TP(x_j - x_k), B^TPx_k\rangle > 0,$$

which contradicts

$$2x_k^TPBu_k = 2x_k^TPB\left(\widetilde{\mathscr{L}}^{(k)}_{\sigma(t)}(x) \otimes (B^TP)\right)x(t) = 0.$$

■

From the proof of Theorem 1, we can also derive a specific result on u_k in case when all $V_k(x_k)$ have the same value.

Corollary 1: Given the LTI systems (1), the local controllers (5), and the function V_k defined in (4). Suppose $\|x_k\|_P = \|x_j\|_P$ for all $j, k = 1, ..., N$, then it holds for every $k = 1, ..., N$ that $\dot{V}_k(x_k) = 0$ only if $u_k = 0$.

In fact, since $V(x(t))$ is continuous in t, it is non-increasing if $D^+V(x(t)) \leq 0$ for any t. Furthermore, $V(x(t))$ is lower bounded by 0, therefore, we can immediately conclude that $V(x(t)) \to c$ for $t \to \infty$ and a constant $c \in \overline{\mathbb{R}}^+$. Moreover, following Lemma on the convergence of $D^+V(x(t))$ holds:

Lemma 2: Given the switched system (8) and the positive definite, locally Lipschitz function (4), where $D^+V(x(t)) \leq 0$, then $D^+V(x(t)) \to 0$ for $t \to \infty$.

Proof: This Lemma can be proven analogously to Lemma 1 in [12] by exploiting the fact that every solution $x(t, x_0)$ is continuous and bounded, where the latter follows from $D^+V(x(t)) \leq 0$ for all $t \in \mathbb{R}$, and the fact that every $V_k(x_k(t))$ as defined in (4) is differentiable twice if t is not a switching time instant τ_r. ■

Even though the system we consider is a switched system, due to the existence of function $V(x)$, which is positive definite and non-increasing along all solutions $x(t, x_0)$, an invariance principle can still be found. Some results on LaSalle's Invariance Principle for switched nonlinear systems have been presented in [15], [16], [17]. The linear case has been specially treated in [18] and [19].

However, these results are not directly applicable to this problem because the function $V(x)$ is not in C^1. Instead, we directly seek an invariance principle for the problem at hand, based on Lemmas 1, 2, Theorem 1 and Corollary 1.

Before we present the theorem analyzing the synchronization behavior, we introduce one more adjacency matrix. Let $\widehat{\mathscr{A}}(t)$ be defined as

$$\widehat{\mathscr{A}}(t) = \begin{bmatrix} 0 & \widehat{a}_{12}(t) & \cdots & \widehat{a}_{1N}(t) \\ \widehat{a}_{21}(t) & 0 & \cdots & \widehat{a}_{2N}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{a}_{N1}(t) & \widehat{a}_{N2}(t) & \cdots & 0 \end{bmatrix}, \tag{13}$$

where $\widehat{a}_{kj}(t) = 1$ for $t = [\tau_r, \tau_{r+1})$ if one of the following two conditions is satisfied:

a) There exists a τ^* such that $\tau_r \leq \tau^* < (\tau^* + \tau_0) < \tau_{r+1}$ and $\widetilde{a}_{kj,\sigma(t)}(x_j(t), x_k(t)) > \alpha(\|x_k(t)\|_P - \|x_j(t)\|_P)^2$ for all $t \in [\tau^*, \tau^* + \tau_0)$.

τ_0 and $\alpha \in \mathbb{R}^+$ are fixed but arbitrarily small parameters, where τ_0 is a lower bound for the dwell-time of the switching signal $\sigma(t)$ as defined in Assumption 2.

b) The solutions of the agents are synchronous, i.e. $x_j(t) = x_k(t)$ for $t = [\tau_r, \tau_{r+1})$.

If none of these two conditions are satisfied, then $\widehat{a}_{kj}(t) = 0$ for $t = [\tau_r, \tau_{r+1})$. We denote $\widehat{\mathscr{E}}(t)$ as the time-varying set of edges corresponding to $\widehat{\mathscr{A}}(t)$. This means $\widehat{\mathscr{A}}(t)$ is the time-dependent unweighted adjacency matrix encoding only the active interconnections of $\mathscr{A}_{\sigma(t)}$ over time t and the synchronous agents. Like $\mathscr{A}_{\sigma(t)}$, $\widehat{\mathscr{A}}(t)$ is piecewise constant and has the same switching time instants $\tau_r, r = 1, 2, \ldots$. This will be relevant for the following statement on synchronization.

Theorem 2: Let N individual agents be modeled by (1) and let the interconnection topology be defined by an jointly strongly connected graph $\mathscr{G}_{\sigma(t)} = (\mathscr{V}, \mathscr{E}_{\sigma(t)}, \mathscr{A}_{\sigma(t)})$. Suppose Assumptions 1 and 2 are satisfied. Then the controller (5) solves Problem 1 for every $\varepsilon > 0$, if the graph $\widehat{\mathscr{G}}(t) = (\mathscr{V}, \widehat{\mathscr{E}}(t), \widehat{\mathscr{A}}(t))$ is jointly strongly connected.

Remark 2: Theorem 2 presents an *implicit* condition for synchronization because the connectivity of $\widehat{\mathscr{G}}(t)$ depends on the solution of the closed loop system (8). Therefore, Theorem 2 does not represent a sufficient condition for synchronization that can easily be checked when designing a controller. Instead, it represents an analysis result that explains, how synchronization and interconnection between the agents correspond to each other.

Proof: We assume that the graph $\widehat{\mathscr{G}}(t) = (\mathscr{V}, \widehat{\mathscr{E}}(t), \widehat{\mathscr{A}}(t))$ is jointly strongly connected. The proof is then done in three steps:

(P1) Show that $V_k(x_k(t)) - V_j(x_j(t)) \to 0$ for all j,k and $t \to \infty$.

(P2) Show that on the set $Z_c = \{x | V_j(x_j) = V_k(x_k) = c \text{ for all } j,k\}$, $c \in \mathbb{R}, c > 0$, the largest invariant subset with respect to the closed loop system (8) is the synchronized set $S_c = \{x | x_j = x_k \text{ and } V(x_k) = c \text{ for all } j,k\}$.

(P3) Use an invariance argument to show synchronization.

In the following we address these three steps one-by-one:

(P1)

From Lemma 2, we know that $V(x(t)) \to c$ for $t \to \infty$ and $D^+V(x(t)) \to 0$ for $t \to \infty$. Let $c > 0$, as the case where $c = 0$ is trivial.

Now assume that a set Z is invariant for the closed loop system (8) for some jointly strongly connected topology and $V(x) = c$ for all $x \in Z$. Moreover, assume that at some time instant $t_0 \in \overline{\mathbb{R}}^+$ there exists a j such that $V_j(x_j) < c$. Note that, for the proposed controller (5), $V_j(x_j(t)) < c$ for all $t > t_0$. Furthermore, with (12), it holds for all k with $V_k(x_k) = c$ that

$$\frac{\partial V_k}{\partial x_k} \dot{x}_k \leq \sum_{j=1}^{N} \widetilde{a}_{kj,\sigma(t)}(x_j, x_k) x_k^T PBB^T P(x_j - x_k),$$

which can be estimated with (13) and yields

$$\frac{\partial V_k}{\partial x_k} \dot{x}_k \leq -\sum_{j=1}^{N} \widehat{a}_{kj}(t) \widehat{\alpha} (\|x_k\|_P - \|x_j\|_P)^3$$

for some time interval $t \in [\tau^*, \tau^* + \tau_0)$ and $\widehat{\alpha} > 0$. Obviously, the right-hand side is negative, if $\widehat{a}_{kj}(t) = 1$ for an j with $V_k(x_k) > V_j(x_j)$. Exploiting the properties discussed above and assuming that the topology is jointly strongly connected, we can derive that this contradicts the invariance of Z.

(P2)

Assume that $V_k(x_k) \equiv c > 0$ for all $k = 1, \ldots, N$, i.e. $x \in Z_c$, where x is governed by the closed loop system (8). Obviously, for $x \in Z_c$ it holds that $\|x_k\|_P = \|x_j\|_P$ for all $j,k = 1, \ldots, N$. From Corollary 1, we know that if $\dot{V}_k(x_k) = 0$, then $u_k = 0$ and $x_k \in \ker(Q)$ for all $k = 1, \ldots, N$, where $A^T P + PA = -Q$ from Assumption 1. Suppose the solution $x(t, x_0) \in Z_c$ for all $t \in \overline{\mathbb{R}}^+$, i.e. $\dot{x} = (I_N \otimes A)x$.

In order for $\widehat{\mathscr{G}}(t) = (\mathscr{V}, \widehat{\mathscr{E}}(t), \widehat{\mathscr{A}}(t))$ to be jointly strongly connected, either all agents are synchronized with $x_k(t) = x_j(t)$ for all $t \in \overline{\mathbb{R}}^+$, or there exists one edge in $\widehat{\mathscr{E}}(t)$, i.e. $\widetilde{a}_{jk,\sigma(t)}(x_j, x_k) \neq 0$ for some time interval $t \in [\tau^*, \tau^* + \tau_0)$. Therefore, $\widetilde{q}(x_j(t), x_k(t)) > 0$ and with (12) we obtain

$$\dot{V}_k = \sum_{j=1}^{N} \varepsilon a_{kj,\sigma(t)} \underbrace{\widetilde{q}(x_j, x_k) x_k^T PBB^T P(x_j - x_k)}_{<0}.$$

Thus, under the assumption that $\widehat{\mathscr{G}}(t) = (\mathscr{V}, \widehat{\mathscr{E}}(t), \widehat{\mathscr{A}}(t))$ is jointly strongly connected, the synchronized set $S_c = \{x | x_j = x_k \text{ and } V(x_k) = c \text{ for all } j,k\}$ is the largest invariant subset with respect to the closed loop system (8), which is contained in $Z_c = \{x | V_j(x_j) = V_k(x_k) = c \text{ for all } j,k\}$, $c > 0$.

(P3)

Let $x(t, x_0)$ be a solution of (8) for a given initial condition x_0. Let $V(x)$ be defined as in (4). From Theorem 1 we know that $D^+V(x(t))$ is non-increasing. Thus, the compact set $\Omega = \{x | V(x) < V(x_0)\}$ is positively invariant. Let $\omega(x_0)$ be the positive limit set of the solution $x(t, x_0)$. Note that $\omega(x_0)$ exists because Ω is compact. For each $\xi \in \omega(x_0)$, there exists an increasing sequence t_i with $t_i \to \infty$ and $x(t_i) \to \xi$ for $i \to \infty$.

From Theorem 1 and Lemma 2 we know that $\lim_{i \to \infty} V(x(t_i)) = c$ and $\lim_{i \to \infty} D^+V(x(t_i)) = 0$. By continuity of $V(x)$, we conclude that $V(\xi) = c$. Moreover, from (P1), we know that $\xi \in Z_c$ for each $\xi \in \omega(x_0)$. Therefore, $\omega(x_0) \subset Z_c$. Additionally, we know that $\omega(x_0)$ is an invariant set. Thus, $\omega(x_0)$ is included in the largest invariant subset of Z_c, which is the synchronized set S_c according to (P2). ■

As mentioned in Remark 2, Theorem 2 presents an analysis of how connectivity of the graph under the nonlinear coupling (6) correspond to synchronization of the agents. Here, the difficulty arises that the nonlinear coupling $\widetilde{q}(x_j, x_k)$ may "cancel" edges that consist in the Graph $\mathscr{G}_{\sigma(t)}$.

Therefore, there is no easy way to directly relate connectivity of $\mathscr{G}_{\sigma(t)}$ to $\widehat{\mathscr{G}}(t)$ yet. However, simulations show, that uniform strong connectivity of $\widehat{\mathscr{G}}(t)$ is usually given, when $\mathscr{G}_{\sigma(t)}$ is uniformly strongly connected.

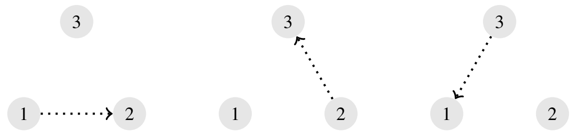


Fig. 4. Periodic sequence of connections.

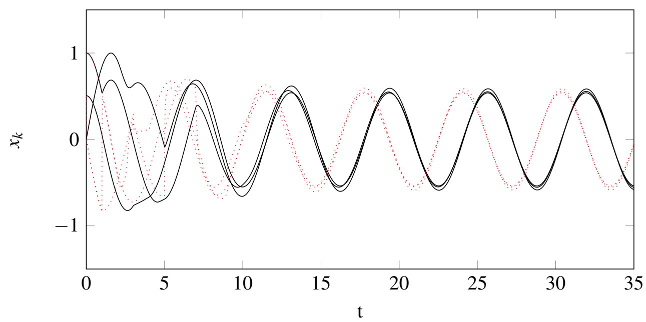


Fig. 5. Synchronization of the agents' states is displayed. The black lines represent the first state, the dotted red lines represent the second state.

IV. SIMULATION EXAMPLE

Same as in Section II-C, we consider a group of $N = 3$ harmonic oscillators

$$\dot{x}_k = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k.$$

We choose u_k as the controller (5), and the interconnection topology as a directed cycle, where only one edge is active at a time, as depicted in Figure 4.

The controller (5) with the coupling function (6) is implemented and leads to synchronization in both states of the agents, as shown in Figure 5. Note that the convergence speed is not exponential due to the nonlinear coupling (6) and the states converge very slowly once they are close to each other. As expected, the function $V(x(t))$ monotonically converges to a positive value, as seen in Figure 6. Note that at times, $V_k(x_k)$ drastically decreases and increases again, e.g. at $t = 5$. This happens, when agents are connected to other agents on the opposite side of the cycle and therefore are “drawn” towards the origin and then “drawn away” from the origin again.

V. CONCLUSION

In this paper, we studied the synchronization problem for a class of identical linear, high order systems. We showed that classic diffusive coupling is not suitable to handle the considered synchronization problem under time-varying, directed topologies, and arbitrary switching. A distributed nonlinear control law was developed, which guarantees boundedness of the solutions, and also has synchronizing properties. Future work will include extending the setup to leader-follower topologies and heterogenous networks.

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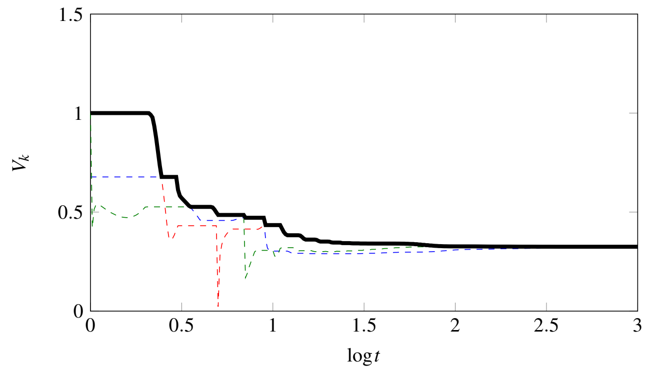

Fig. 6. The bold black line represents $V(x)$ and is monotonically decreasing. The colored dashed lines represent $V_k(x_k), k = 1,2,3$.

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