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Essays on the Microfoundations of Asset Liquidity and Resale Premia

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Declaration

I, **Simon Mishricky**, declare that this thesis, titled *Essays on the Microfoundations of Asset Liquidity and Resale Premia*, is my own work. It has not been submitted for any other award, and all sources are acknowledged.

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Abstract

Chapter 1 of the thesis reviews and synthesises the literature on the role of money and assets in models with trading frictions. The main focus of the chapter is on how liquidity and resale premia shape asset prices and monetary transmission. In frictionless Walrasian settings, neither money nor financial assets are essential for exchange, and asset values reflect only discounted dividends. Once search and decentralised trade are introduced, however, assets acquire additional value as media of exchange or as instruments of resale, leading to systematic deviations from standard asset pricing predictions. By comparing these two strands of theory, the chapter highlights the mechanisms through which frictions in asset exchange alter equilibrium valuations, monetary policy transmission, and the structure of risk premia in modern financial economies.

Chapter 2 of the thesis examines asset pricing with collateral and resale premia. Asset prices respond ambiguously to monetary policy: easing sometimes lowers dividend yields, as in traditional and collateral/liquidity-based monetary-style asset models, and sometimes raises them, as in monetary resale premium models. This chapter reconciles these views by developing a monetary over-the-counter (OTC) model where assets serve both as collateral for goods and asset purchases and as resaleable securities. The framework generates both collateral and resale premia, explaining why dividend yields can move in opposite directions under monetary easing/tightening. Further, we show that policy remains effective in the cashless limit, and that the cashless limit equilibrium differs from the nonmonetary equilibrium, highlighting that money's role cannot be abstracted from asset pricing.

Chapter 3 of the thesis considers asset price dispersion, monetary policy, and macroprudential regulation. Following the 2007–2009 Global Financial Crisis, sustained monetary expansion and tighter financial regulation have left financial markets thinner, less resilient, and more prone to instability. This chapter develops a monetary model of decentralised financial exchange to account for these outcomes. The framework links search frictions and costly posting to the joint effects of monetary and regulatory policy on asset prices, quoting behaviour, and market stability. Increases in inflation or posting costs reduce quoting intensity, widen the distribution of executable prices, and raise the probability of trading breakdowns. The model replicates key post-crisis patterns such as wider spreads, higher execution costs, and an increased likelihood of flash-crash events, showing that the interaction between monetary and regulatory policy can unintentionally increase financial market fragility.

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Chapter 1

Assets and Money: A Guided Tour of Liquidity and Resale Premia

ABSTRACT

This chapter reviews and synthesises the literature on the role of money and assets in models with trading frictions, focusing on how liquidity and resale premia shape asset prices and monetary transmission. In frictionless Walrasian settings, neither money nor financial assets are essential for exchange, and asset values reflect only discounted dividends. Once search and decentralised trade are introduced, however, assets acquire additional value as media of exchange or as instruments of resale, leading to systematic deviations from standard asset pricing predictions. By comparing these two strands of theory, this chapter highlights the mechanisms through which frictions in asset exchange alter equilibrium valuations, monetary policy transmission, and the structure of risk premia in modern financial economies.

1.1 Introduction

The study of money and asset prices has long stood at the intersection of macroeconomics and finance. In the canonical Walrasian framework, exchange occurs frictionlessly along budget constraints, implying that neither money nor financial assets are essential for trade. In such environments, liquidity plays no intrinsic role: asset prices reflect only discounted dividend flows, and monetary policy affects real allocations solely through nominal rigidities, if any are assumed. Once search frictions and decentralised trade are introduced, however, this neutrality breaks down. In these frictional environments, money and assets acquire value as media of exchange or through their resaleability. Their pricing therefore reflects not only their dividend payoffs but also the liquidity services and/or resale opportunities they provide.

Two distinct but complementary strands of the modern literature formalise these liquidity effects. The first emphasises *liquidity premia*, arising when assets directly facilitate exchange in decentralised markets. Models in this tradition, such as Geromichalos et al. (2007); Lagos and Rocheteau (2009); Lagos (2010); Mattesini and Nosal (2016); Geromichalos et al. (2023), show that financial assets—particularly government bonds—earn a premium because of their superior ability to relax trading constraints. Consequently, assets that serve as better media of exchange exhibit higher valuations and lower expected returns. The second strand focuses on *resale premia*, where the value of an asset stems from its potential to be resold in the future to investors with higher valuations, as in Lagos and Zhang (2019a,b, 2020); Geromichalos and Jung (2019); Geromichalos et al. (2021). These models, often featuring over-the-counter (OTC) market intermediation, demonstrate how heterogeneous investor valuations and limited participation create additional sources of asset value beyond dividends.

Although both frameworks attribute deviations from the standard asset pricing model to frictions in trading, they generate distinct implications for the response of asset prices to monetary and financial conditions. Assets with a *liquidity premium* tend to appreciate when monetary liquidity becomes scarce, since their ability to facilitate exchange becomes more valuable. In contrast, assets with a *resale premium*, like equities traded in OTC markets, depreciate when liquidity tightens, because higher financing costs or inflation reduce the expected gains from future resale. Hence, the direction of the price response to policy or liquidity shocks depends on whether the asset's value derives from its role as a medium of exchange or from its resale option.

The objective of this review is to synthesise these two approaches—liquidity and resale premia—within a common theoretical structure. We first examine models in which assets and money serve as liquidity instruments and derive the implications for bond and equity pricing. We then turn to environments in which assets trade in intermediated OTC markets, generating resale premia through heterogeneous valuations and trading frictions. Together, these literatures reveal how frictions in asset exchange fundamentally shape valuations, the transmission of monetary policy, and the nature of equilibrium price differentials across financial instruments.

The remainder of this chapter proceeds as follows. Section 1.2 reviews models in which assets yield liquidity premia, emphasising how frictions in exchange alter asset prices and monetary policy transmission. Section 1.3 turns to resale premia, where heterogeneous valuations and over-the-counter trading generate value from the option to resell assets. Section 1.4 compares the two mechanisms and highlights their distinct empirical implications. Section 1.5 concludes.

1.2 Liquidity Premia

Having established the motivation for studying money and assets in frictional settings, we now turn to the first class of models that formalise how liquidity enters asset prices. These *liquidity premium* models show that assets can acquire value not only from their dividend payoffs but also from their ability to facilitate exchange when agents face trading constraints. By embedding a standard asset pricing structure within a Lagos and Wright (2005)–style monetary framework, these models demonstrate how liquidity services alter equilibrium pricing, returns, and the transmission of monetary policy.

We begin by examining a stripped-down version of Lagos (2010). We first consider an economy with a single equity asset that serves as the medium of exchange. We then extend the framework to allow two assets, equity and money, both usable as media of exchange. Finally, we replace money with government bonds to study how the risk-free rate interacts with equity returns.

1.2.1 Equities with Liquidity Premia

Consider an economy in which a single equity share represents ownership of an active production unit.[1] Time is discrete, and in each period agents participate sequentially in two markets: a decentralised market in the first subperiod and a competitive, centralised market in the second. There is a unit measure of investors, and an equal measure of production units. Each active unit produces a non-storable *dividend good* in the first subperiod of period t. The dividend follows a Markov process, and it is the only source of aggregate uncertainty. Investors derive utility from consuming *special goods* traded in the decentralised market, *general goods* purchased in the centralised market, and experience disutility from producing the special good. Preferences are

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[u(q_t) - e(\tilde{q}_t) + c_t - h_t \right], \tag{1.1}$$

where $\beta \in (0,1)$ is the discount factor. Here, $q_t \in \mathbb{R}_+$ and $\tilde{q}_t \in \mathbb{R}_+$ denote the quantities of special goods consumed and produced in decentralised trade, $c_t \in \mathbb{R}_+$ denotes consumption of general goods, and $h_t \in \mathbb{R}_+$ represents hours worked in the second subperiod. Expectations are taken with respect to information at time 0. $u : \mathbb{R}_+ \to \mathbb{R}_+$ is the investor's utility function for the consumption of the special good in the first subperiod, $e : \mathbb{R}_+ \to \mathbb{R}_+$ is the investor's cost of producing the special good in the first subperiod. We assume $u(0) = e(0) = 0$, $u' > 0 > u''$, $e' > 0$, and $e'' \geq 0$. Every active unit yields an exogenous dividend good $y_t \in \mathbb{R}_+$. Dividends evolve over time according to a Markov process, so that the distribution of y_{t+1} depends only on the current period dividend y_t.

Let $W_t(a_t^s)$ denote the value of entering the centralised market with $a_t^s \in \mathbb{R}_+$ shares when dividends are y_t, and $V_t(a_t^s)$ the corresponding value of entering the decentralised market. At each date t, the investor chooses consumption of general goods $c_t \in \mathbb{R}_+$, labor supply $h_t \in \mathbb{R}_+$,

[1] Often referred to in the literature as "Lucas trees". However, following Lagos and Zhang (2019a,b, 2020) we refer to them as *active production units*.

and an end-of-period equity position $a^s_{t+1} \in \mathbb{R}_+$. The centralised market problem is:

$$W_t(a^s_t) = \max_{(c_t, h_t, a^s_{t+1}) \in \mathbb{R}^3_+} \left[c_t - h_t + \beta \mathbb{E}_t V_{t+1}(a^s_{t+1})\right] \tag{1.2}$$
$$\text{s.t. } c_t + \phi^s_t a^s_{t+1} \leq (\phi^s_t + y_t) a^s_t + h_t,$$

where $\mathbb{E}_t$ is the conditional expectations operator over the next-period realisation of the dividend, and ϕ^s_t is the (*ex-dividend*) share price. Using the budget constraint, we can substitute for h_t in the objective function, so that the centralised market problem becomes:

$$W_t(a^s_t) = \lambda_t a^s_t + \max_{a^s_{t+1} \in \mathbb{R}_+} \Big[-\phi^s_t a^s_{t+1} + \beta \mathbb{E}_t V_{t+1}(a^s_{t+1}) \Big], \tag{1.3}$$

where $\lambda_t \equiv \phi^s_t + y_t$. During the decentralised trading period, with probability $\alpha \in (0,1)$, a buyer with equity holdings a^s_t meets a seller with holdings $\tilde{a}^s_t$. Their terms of trade are given by $[q_t(a^s_t, \tilde{a}^s_t), p_t(a^s_t, \tilde{a}^s_t)]$, where $q_t(\cdot) \in \mathbb{R}_+$ is the quantity of special goods exchanged and $p_t(\cdot) \in \mathbb{R}_+$ is the corresponding asset transfer. The value of entering the decentralised market with holdings a^s_t when dividends are y_t is then characterised by,

$$V_t(a^s_t) = \alpha \int \{u\left[q_t(a^s_t, \tilde{a}^s_t)\right] + W_t\left[a^s_t - p_t(a^s_t, \tilde{a}^s_t)\right]\} \, dG(\tilde{a}^s_t)$$
$$+ \alpha \int \{-e\left[q_t(\tilde{a}^s_t, a^s_t)\right] + W_t\left[a^s_t + p_t(\tilde{a}^s_t, a^s_t)\right]\} \, dG(\tilde{a}^s_t) + (1 - 2\alpha) W_t(a^s_t), \tag{1.4}$$

where G denotes the distribution of share holdings. Consider a meeting in the decentralised market between a buyer holding a^s_t and a seller holding $\tilde{a}^s_t$. Following Lagos (2010), the terms of trade (q_t, p_t) are determined by Nash bargaining where the buyer has all the bargaining power:

$$\max_{q_t \in \mathbb{R}_+, p_t \leq a^s_t} \left[u(q_t) + W_t(a^s_t - p_t) - W_t(a^s_t)\right] \tag{1.5}$$
$$\text{s.t. } -e(q_t) + W_t(\tilde{a}^s_t + p_t) \geq W_t(\tilde{a}^s_t).$$

The constraint $p_t \leq a^s_t$ indicates that the buyer cannot spend more assets than he owns. Notice that $W_t(a_t^s + p_t) - W_t(a_t^s) = \lambda_t p_t$, so the bargaining problem reduces to:

$$\max_{q_t \in \mathbb{R}_+, p_t \leq a_t^s} [u(q_t) - \lambda_t p_t] \quad \text{s.t.} \quad -e(q_t) + \lambda_t p_t \geq 0. \tag{1.6}$$

In the first subperiod, the buyer's allocation depends only on the product $\lambda_t a_t^s$. If $\lambda_t a_t^s \geq e(q^*)$, the buyer exchanges $p_t = e(q^*)/\lambda_t \leq a_t^s$ shares for the first-best quantity q^*. Otherwise, the buyer transfers all of his shares in exchange for a quantity q_t satisfying $e(q_t) = \lambda_t a_t^s$. Thus, as shown in Lagos (2010), the solution to the bargaining problem is given by

$$q(\lambda_t a_t^s) = \begin{cases} q^* & \text{if } \lambda_t a_t^s \geq e(q^*) \\ e^{-1}(\lambda_t a_t^s) & \text{if } \lambda_t a_t^s < e(q^*). \end{cases}$$

Using the bargaining solution and the linearity of $W_t(a_t^s)$, the value of search can be expressed as,

$$V_t(a_t^s) = \mathcal{S}(\lambda_t a_t^s) + W_t(a_t^s), \tag{1.7}$$

where $\mathcal{S}(\lambda_t a_t^s) \equiv \alpha\big\{u[q(\lambda_t a_t^s)] - e[q(\lambda_t a_t^s)]\big\}$. The function $\mathcal{S}$ is twice differentiable, with $\mathcal{S}' \geq 0$ and $\mathcal{S}'' \leq 0$, both strict for $\lambda_t a_t^s < e(q^*)$. As in Lagos (2010), we can now solve the Euler equation for the asset. The first-order condition for the choice of a_{t+1}^s is

$$\phi_t^s = \beta \mathbb{E}_t \left\{ \left[1 + \alpha \left(\frac{u'\left[q(\lambda_{t+1} a_{t+1}^s)\right]}{e'\left[q(\lambda_{t+1} a_{t+1}^s)\right]} - 1 \right) \right] \lambda_{t+1} \right\}. \tag{1.8}$$

The market-clearing conditions imply $a_{t+1}^s = 1$, and letting $\mathcal{L}(\Lambda_{t+1}) \equiv [1 + \mathcal{S}'(\Lambda_{t+1})]$, where $\Lambda_{t+1} \equiv \lambda_{t+1}$. The liquidity premium can thus be written as

$$\mathcal{L}(\Lambda) = 1 + \alpha \left\{ \frac{u'\left[q\left(\Lambda\right)\right]}{e'\left[q\left(\Lambda\right)\right]} - 1 \right\}$$

Collecting the preceding results, equilibrium is defined by a price sequence $\{\phi_t^s\}_{t=0}^{\infty}$ that satisfies the optimality condition:

$$\phi_t^s = \beta \mathbb{E}_t \left[\mathcal{L}(\Lambda_{t+1})(\phi_{t+1}^s + y_{t+1}) \right]. \tag{1.9}$$

While the equity-only framework clarifies how liquidity affects the pricing of a single asset, it abstracts from the coexistence of multiple liquid instruments that characterises real economies.[2] In particular, government-issued money competes with financial assets in providing transaction services. To capture this interaction, the next section extends the analysis by introducing fiat money alongside equity. This extension reveals how monetary policy—through its control of liquidity supply—jointly determines the value of money and the pricing of financial assets.

1.2.2 Equities and Money with Liquidity Premia

This section investigates how monetary policy affects asset prices. Following Lagos (2010), we introduce a second asset into the model called *money.* Money is issued by a government, which at $t = 0$ commits to a monetary policy described by a sequence of positive real-valued growth rates $\{\mu_t\}_{t=0}^{\infty}$. Given an initial money stock $A_0^m > 0$, the policy generates a money supply process $\{A_t^m\}_{t=0}^{\infty}$ according to $A_{t+1}^m = \mu_t A_t^m$. The government implements the policy via lump-sum transfers or taxes in the second subperiod of each period, so that along a sample path, $A_{t+1}^m = A_t^m + T_t$, where T_t is the transfer (or tax, if negative).

Both money and equity are perfectly recognisable, cannot be forged, and can be traded in both centralised and decentralised markets. At $t = 0$, each investor is endowed with a_0^s equity shares and a_0^m units of fiat money.

Let $\mathbf{a}_t \equiv (a_t^m, a_t^s)$ denote an investor's portfolio, consisting of $a_t^m \in \mathbb{R}_+$ units of money and $a_t^s \in \mathbb{R}_+$ equity shares. Let $W_t(\mathbf{a}_t)$ and $V_t(\mathbf{a}_t)$ be the maximum attainable expected discounted utility of an investor entering the centralised and decentralised markets, respectively, at time t

[2] In Lagos (2010) the labour market is modelled explicitly. We abstract from these labour market details. Lagos (2010) present (1.9) in a form that nests equation (6) of Lucas (1978) when there is no liquidity premium (i.e., $\mathcal{L}(\Lambda_t) = 1$ for all t) or when investors have no liquidity needs (i.e., $\alpha = 0$). Under our specification, (1.9) can equivalently be written as $U'(y_t)\phi_t^s = \beta \mathbb{E}_t \left[\mathcal{L}(\Lambda_{t+1}) U'(y_{t+1}) (\phi_{t+1}^s + y_{t+1}) \right]$ in order to correspond to Lagos (2010), where $U(c_t) = c_t$ in our case and $c_t = y_t$ in equilibrium.

with portfolio $\mathbf{a}_t$. The centralised market problem is

$$W_t(\mathbf{a}_t) = \max_{(c_t,h_t,\mathbf{a}_{t+1})\in\mathbb{R}^4_+} \Big[c_t - h_t + \beta\mathbb{E}_t V_{t+1}(\mathbf{a}_{t+1})\Big] \tag{1.10}$$

$$\text{s.t. } c_t + \phi^s_t a^s_{t+1} + \phi^m_t a^m_{t+1} \leq (\phi^s_t + y_t)a^s_t + \phi^m_t(a^m_t + T_t) + h_t.$$

That is, upon entering the centralised market in period t with portfolio $\mathbf{a}_t$, the investor chooses consumption of the general good c_t, labor supply h_t, and the next-period portfolio $\mathbf{a}_{t+1}$ to maximise expected discounted utility. Again, $\mathbb{E}_t$ is the conditional expectations operator over the next-period realisation of the dividend, ϕ^s_t is the ex-dividend share price, and ϕ^m_t is the real price of money in terms of the general consumption good.

Consider a meeting in the decentralised market at period t between a buyer with portfolio $\mathbf{a}_t$ and a seller with portfolio $\tilde{\mathbf{a}}_t$. Let $[q_t(\mathbf{a},\tilde{\mathbf{a}}), \mathbf{p}_t(\mathbf{a},\tilde{\mathbf{a}})]$ denote the terms at which the buyer trades with the seller, where $q_t(\mathbf{a},\tilde{\mathbf{a}}) \in \mathbb{R}_+$ is the quantity of the special good exchanged and $\mathbf{p}_t(\mathbf{a},\tilde{\mathbf{a}}) \equiv [p^m_t(\mathbf{a},\tilde{\mathbf{a}}), p^s_t(\mathbf{a},\tilde{\mathbf{a}})] \in \mathbb{R}^2_+$ is the portfolio transfer from buyer to seller. The buyer makes a take-it-or-leave-it offer, choosing $(q_t, \mathbf{p}_t)$ to solve

$$\max_{q_t,\mathbf{p}_t\leq\mathbf{a}_t} \Big[u(q_t) + W_t(\mathbf{a}_t - \mathbf{p}_t) - W_t(\mathbf{a}_t)\Big] \tag{1.11}$$

$$\text{s.t. } -e(q_t) + W_t(\tilde{\mathbf{a}}_t + \mathbf{p}_t) \geq W_t(\tilde{\mathbf{a}}_t),$$

ensuring that the offer is acceptable to the seller and that the buyer cannot spend more assets than he owns.

Define $\boldsymbol{\lambda}_t \equiv (\lambda^m_t, \lambda^s_t)$ where $\lambda^m_t = \phi^m_t$ and $\lambda^s_t = (\phi^s_t + y_t)$. Intuitively, λ^s_t is the real *cum-dividend* value of equity in terms of general goods, and λ^m_t is the real value of money in terms of general goods. And thus the solution to the bargaining problem is

$$q(\boldsymbol{\lambda}_t\mathbf{a}_t) = \begin{cases} q^* & \text{if } \boldsymbol{\lambda}_t\mathbf{a}_t \geq e(q^*) \\ e^{-1}(\boldsymbol{\lambda}_t\mathbf{a}_t) & \text{if } \boldsymbol{\lambda}_t\mathbf{a}_t < e(q^*). \end{cases}$$

With this bargaining solution, the value entering the decentralised market for an investor is similar to (1.4) with portfolio $\mathbf{a}_t$, and is simplified to yield,

$$V_t(\mathbf{a}_t) = \mathcal{S}(\boldsymbol{\lambda}_t \mathbf{a}_t) + W_t(\mathbf{a}_t), \tag{1.12}$$

where $\mathcal{S}$ again represents the expected gain from trade. Substituting the budget constraint into $W_t(\mathbf{a}_t)$ yields

$$W_t(\mathbf{a}_t) = \boldsymbol{\lambda}_t \mathbf{a}_t + \tau_t + \max_{\mathbf{a}_{t+1} \geq 0} \Big[- \boldsymbol{\phi}_t \mathbf{a}_{t+1} + \beta \mathbb{E}_t V_{t+1}(\mathbf{a}_{t+1}) \Big],$$

where $\tau_t \equiv \lambda_t^m T_t$ and $\boldsymbol{\phi}_t \equiv (\phi_t^m, \phi_t^s)$. This provides a compact expression for the maximum expected discounted utility of an investor entering the centralised market with portfolio $\mathbf{a}_t$.

As shown in Lagos (2010), the market clearing conditions imply that money and equity holdings satisfy $\{a_{t+1}^m, a_{t+1}^s\}_{t=0}^{\infty} = \{A_{t+1}^m, 1\}_{t=0}^{\infty}$, and where $\Lambda_{t+1} \equiv \lambda_{t+1}^s + \lambda_{t+1}^m A_{t+1}^m$ represents the real value of the portfolio each investor brings into the decentralised market at period $t+1$.

Given a money supply process $\{A_t^m\}_{t=0}^{\infty}$, and again defining $\mathcal{L}(\Lambda_{t+1}) \equiv 1 + \mathcal{S}'(\Lambda_{t+1})$, a monetary equilibrium can be summarised by a sequence of prices $\{\phi_t^m, \phi_t^s\}_{t=0}^{\infty}$ that satisfy the following necessary and sufficient conditions for individual optimisation:

$$\phi_t^m = \beta \mathbb{E}_t \Big[\mathcal{L}(\Lambda_{t+1}) \phi_{t+1}^m \Big] \tag{1.13}$$

$$\phi_t^s = \beta \mathbb{E}_t \Big[\mathcal{L}(\Lambda_{t+1}) (\phi_{t+1}^s + y_{t+1}) \Big]. \tag{1.14}$$

These are the Euler equations for money and equity, respectively.[3] Because both assets can serve as a medium of exchange in bilateral trades, the usual stochastic discount factor is augmented by the liquidity factor $\mathcal{L}(\Lambda_{t+1})$, which captures the marginal liquidity value of the asset—i.e., the extent to which it facilitates transactions.

It is convenient to introduce a notion of the nominal interest rate. As in Lagos (2010), suppose the economy included an additional asset: an illiquid nominal bond, i.e., a one-period, risk-free government bond that pays one unit of money in the centralised market and cannot be used in decentralised trade. Let ϕ_t^n denote the price of this bond. In equilibrium, its price must

[3]The transversality conditions require that $\lim_{t\to\infty} \mathbb{E}_0[\beta^t \phi_t^m A_{t+1}^m] = 0$ and $\lim_{t\to\infty} \mathbb{E}_0[\beta^t \phi_t^s] = 0$.

satisfy

$$\phi_t^n = \beta \mathbb{E}_t\big[\phi_{t+1}^m\big]. \tag{1.15}$$

Since ϕ_t^n/ϕ_t^m represents the money price of the bond, the net nominal interest rate in a monetary equilibrium is defined as

$$i_t = \frac{\phi_t^m}{\phi_t^n} - 1, \tag{1.16}$$

or equivalently,

$$i_t = \frac{\mathbb{E}_t\big[\mathcal{L}(\Lambda_{t+1})\lambda_{t+1}^m\big]}{\mathbb{E}_t\big[\lambda_{t+1}^m\big]} - 1. \tag{1.17}$$

Liquidity considerations therefore generate a negative relationship between the nominal interest rate (and, by extension, the inflation rate) and equity returns. Intuitively, a higher nominal interest rate implies that buyers are, on average, short of liquidity, making equity more valuable as it helps relax trading constraints. This additional liquidity value reduces the expected real return on equity when interest rates are high.

The joint analysis of money and equity demonstrates how monetary policy influences asset prices through liquidity conditions. However, in most economies government bonds, rather than money, are the dominant liquid instrument used to absorb and redistribute liquidity across investors. Extending the framework to include bonds therefore allows us to distinguish between assets that provide liquidity services and those that do not, and to explore how differences in liquidity across assets shape their relative prices and returns.

1.2.3 Equities and Bonds with Liquidity Premia

In this section, we follow Lagos (2010) and extend the framework to include a government-issued, one-period, risk-free real bond, hereafter referred to simply as a *bond*. Money is no longer considered, so the bond serves as the only government liability. Let A_t^b denote the stock of bonds outstanding in period t, which are redeemed before the centralised trading session of that period. At the end of period t, the government issues A^b_{t+1} in the centralised market. The government's budget constraint is therefore,

$$A^b_t = \phi^b_t A^b_{t+1} + \tau_t, \tag{1.18}$$

where $\phi^b_t \in \mathbb{R}_+$ is the bond price and τ_t represents net transfers (positive for transfers, negative for taxes). Now let $\mathbf{a}_t \in \mathbb{R}^2_+$ denote the portfolio of an investor in period t, consisting of two assets: $\mathbf{a}_t \equiv (a^s_t, a^b_t)$, where $a^s_t \in \mathbb{R}_+$ and $a^b_t \in \mathbb{R}_+$ represent the holdings of shares and bonds, respectively. Let $\boldsymbol{\phi}_t \equiv (\phi^s_t, \phi^b_t)$ denote the corresponding real asset prices, with ϕ^s_t again as the ex-dividend real price of a share and ϕ^b_t the real price of a bond. The value function of an investor entering the centralised market with portfolio $\mathbf{a}_t$ and dividend income y_t is given by

$$W_t(\mathbf{a}_t) = \max_{(c_t, h_t, \mathbf{a}_{t+1}) \in \mathbb{R}^4_+} \Big[c_t - h_t + \beta \mathbb{E}_t V_{t+1}(\mathbf{a}_{t+1}) \Big] \tag{1.19}$$
$$\text{s.t. } c_t + \boldsymbol{\phi}_t \mathbf{a}_{t+1} \leq (\phi^s_t + y_t) a^s_t + a^b_t + h_t - \tau_t.$$

The investor chooses consumption of the general good c_t, labor supply h_t, and the next-period portfolio $\mathbf{a}_{t+1}$ to maximise expected discounted utility. Again, $\mathbb{E}_t$ is the conditional expectations operator over the next-period realisation of the dividend, and $\boldsymbol{\phi}_t \mathbf{a}_t$ denotes the dot product of $\boldsymbol{\phi}_t$ and $\mathbf{a}_t$.

Consider a meeting in the decentralised market at period t between a buyer with portfolio $\mathbf{a}_t$ and a seller with portfolio $\tilde{\mathbf{a}}_t$. Let $[q_t(\mathbf{a}, \tilde{\mathbf{a}}), \mathbf{p}_t(\mathbf{a}, \tilde{\mathbf{a}})]$ denote the terms at which the buyer trades with the seller, where $q_t(\mathbf{a}, \tilde{\mathbf{a}}) \in \mathbb{R}_+$ is the quantity of the special good exchanged and $\mathbf{p}_t(\mathbf{a}, \tilde{\mathbf{a}}) \equiv [p^b_t(\mathbf{a}, \tilde{\mathbf{a}}), p^s_t(\mathbf{a}, \tilde{\mathbf{a}})] \in \mathbb{R}^2_+$ is the portfolio transfer from buyer to seller. The buyer makes a take-it-or-leave-it offer, choosing $(q_t, \mathbf{p}_t)$ to solve

$$\max_{q_t, \mathbf{p}_t \leq \mathbf{a}_t} \Big[u(q_t) + W_t(\mathbf{a}_t - \mathbf{p}_t) - W_t(\mathbf{a}_t) \Big] \tag{1.20}$$
$$\text{s.t. } -e(q_t) + W_t(\tilde{\mathbf{a}}_t + \mathbf{p}_t) \geq W_t(\tilde{\mathbf{a}}_t).$$

The resulting allocation is

$$q(\boldsymbol{\lambda}_t \mathbf{a}_t^s) = \begin{cases} q^*, & \text{if } \boldsymbol{\lambda}_t \mathbf{a}_t^s \geq e(q^*) \\ e^{-1}(\boldsymbol{\lambda}_t \mathbf{a}_t^s), & \text{if } \boldsymbol{\lambda}_t \mathbf{a}_t^s < e(q^*). \end{cases}$$

Using this bargaining solution and the linearity of $W_t(\mathbf{a}_t)$, the value function of the first subperiod can be expressed as

$$V_t(\mathbf{a}_t) = \mathcal{S}(\boldsymbol{\lambda}_t \mathbf{a}_t) + W_t(\mathbf{a}_t). \tag{1.21}$$

The market clearing conditions imply that equity and bond holdings satisfy $\{a_{t+1}^s, a_{t+1}^b\}_{t=0}^{\infty} = \{1, A_{t+1}^b\}_{t=0}^{\infty}$, and where $\Lambda_{t+1} \equiv \lambda_{t+1}^s + \lambda_{t+1}^b A_{t+1}^b$ represents the real value of the portfolio each investor brings into the decentralised market at period $t+1$. An equilibrium can be summarised by a sequence of prices $\{\phi_t^b, \phi_t^s\}_{t=0}^{\infty}$ that satisfy the following necessary and sufficient conditions for individual optimisation:

$$\phi_t^b = \beta \mathbb{E}_t \Big[\mathcal{L}(\Lambda_{t+1}) \Big] \tag{1.22}$$

$$\phi_t^s = \beta \mathbb{E}_t \Big[\mathcal{L}(\Lambda_{t+1})(\phi_{t+1}^s + y_{t+1}) \Big]. \tag{1.23}$$

If we set $\alpha = 0$ (an economy with no liquidity needs), then conditions (1.22) and (1.23) collapse to a special case of the standard equations for asset prices derived in Lucas (1978) and Mehra and Prescott (1985). In this special case, the equilibrium price of a share in an economy with bonds is identical to that in an economy without bonds. The bond supply is irrelevant when liquidity plays no role.[4]

By contrast, when investors are occasionally liquidity constrained, variations in the outstanding stock of bonds affect the equilibrium prices of both shares and bonds. Since both assets can be freely exchanged in the decentralised market, they yield the same liquidity return in every state.

As detailed in Lagos (2010), when investors are occasionally liquidity constrained, asset

[4] Recall the earlier footnote on Lagos (2010) and Lucas (1978), which explains the notation mapping between our simplified set up here and Lucas (1978); Mehra and Prescott (1985).

pricing replaces intrinsic payoffs with *full returns* that bundle payoffs with a state-contingent liquidity premium. The bond Euler condition then prices the risk-free asset off the marginal rate of substitution scaled by the expected liquidity term. When liquidity is valuable on average, this scaling lowers the implied risk-free rate, helping rationalise the risk-free rate puzzle. If bonds are acceptable in a wider set of meetings than equity, they earn a larger liquidity premium; to clear markets, equity must compensate for its relative illiquidity with a higher average excess return, contributing to the equity premium puzzle. Empirically, this perspective suggests working with liquidity-adjusted returns rather than bare payoffs and testing whether the implied premia reconcile the observed negative risk-free Euler residual and positive equity premium residual.

1.3 Resale Premia

The liquidity premium framework augments the canonical asset pricing model with a liquidity term that rises with the asset's capacity to relax trading constraints. Yet, it attributes all deviations from fundamental values to liquidity services provided during trade. In many financial markets, however, assets derive additional value from their *resaleability*. Investors purchase them not to use directly in transactions but because they can be resold later to other investors with higher valuations. To capture this mechanism, we turn to models featuring *resale premia*, where trade occurs in decentralised OTC markets and asset value arises not only from dividends but also from the option to resell under heterogeneous valuations.[5]

In this section we examine key elements of Lagos and Zhang (2020). We derive pricing conditions that decompose equity value into dividend and resale premia, and then study how changes in money growth or the nominal interest rate affect real balances and equity returns.

[5] A further conceptual refinement of the resale channel concerns the distinction between *direct* and *indirect* asset liquidity. An asset is said to provide direct liquidity when it is itself used as a medium of exchange in decentralised trade, whereas indirect liquidity arises when the asset's value stems from the possibility of reselling it for some other object—money, a different financial instrument, or the numeraire good—that is itself liquid. This distinction, formalised in Geromichalos and Herrenbrueck (2016) and Herrenbrueck and Geromichalos (2017), has important implications for how resale value should be interpreted: the magnitude and even the direction of the resale premium may depend on which object lies on the other side of the secondary-market transaction. In the models reviewed below, equity is resold for money in OTC markets, so the resale premium operates through an indirect liquidity channel. Geromichalos et al. (2021) endogenise the boundary between direct and indirect liquidity by allowing the acceptability of assets in exchange to be determined in equilibrium.

1.3.1 Equities and Money with Resale Premia

We consider a financial asset, such as equity or a real bond, that delivers a dividend stream in consumption goods and is demanded by investors with time-varying heterogeneous valuations. To capture the gains from trade arising from these valuation differences, investors participate in a bilateral market with random search, intermediated by specialised dealers who have access to a competitive interdealer market.

Again, time is represented by a sequence of periods indexed by $t = 0, 1, \ldots$ Each period is divided into two subperiods, where different activities take place. Alongside investors, there is also a continuum of agents called *dealers*, each identified with a point in the set $\mathcal{D} = [0, 1]$.

There is, again, a continuum of active production units, this time with measure $A^s \in \mathbb{R}_{++}$. As above, every active unit yields an exogenous dividend $y_t \in \mathbb{R}_+$ at the end of the first subperiod of period t.

Relative to our setup of Lagos (2010) in Section 2, the version of Lagos and Zhang (2020) developed here adds the following features. At the start of each period, every active unit faces an independent idiosyncratic shock that renders it permanently unproductive with probability $1 - \delta \in [0, 1)$. Conditional on remaining active, the dividend follows $y_t = \gamma_t y_{t-1}$, where γ_t is nonnegative with cumulative distribution function Γ so that $\mathbb{P}\{\gamma_t \leq \gamma\} = \Gamma(\gamma)$, and mean $\bar{\gamma} \in \big(0, (\beta\delta)^{-1}\big)$.[6] The period-$t$ dividend y_t is publicly revealed at the beginning of period t. At that time, each failed unit is replaced by a new unit that pays y_t in its first period and then evolves according to the same process, with the initial dividend $y_0 \in \mathbb{R}_{++}$ given at $t = 0$. In the second subperiod of every period, all agents have access to a linear technology that converts effort into a perishable consumption good.

At the beginning of every period $t \geq 1$, each investor receives $(1 - \delta)A^s$ equity shares corresponding to the new production units. When a production unit fails, its equity share disappears. Like the previous section, there is a second financial instrument, *money*, that is intrinsically useless.

[6] The bound $\bar{\gamma} < (\beta\delta)^{-1}$ ensures the present value of a surviving unit's dividend stream is finite and the dynamic program is well posed. If instead $\beta\delta\,\bar{\gamma} \geq 1$, the marginal value of postponing payoffs does not decay, and agents strictly prefer shifting resources to the future; economically, discounting is overpowered by growth, ruling out a stationary monetary equilibrium and violating the usual transversality condition. This is a standard restriction in monetary search/OTC models (see Lagos and Zhang (2019a, 2020) which impose the same type of bound).

The stock of money at time t is denoted A_t^m. The initial stock of money, $A_0^m \in \mathbb{R}_{++}$, is given and $A_{t+1}^m = \mu A_t^m$, with $\mu \in \mathbb{R}_{++}$. A monetary authority injects or withdraws money via lump-sum transfers or taxes to investors in the second subperiod of every period. At the beginning of period $t = 0$, each investor is endowed with a portfolio of equity shares and money. All financial instruments are perfectly recognisable, cannot be forged, and can be traded in every subperiod.

In the second subperiod of every period, all agents can trade the consumption good produced in that subperiod, equity shares, and money in a spot Walrasian market. In the first subperiod of every period, trading is organised as follows. Investors trade equity and money with dealers through a random bilateral OTC market, while dealers are also able to rebalance their portfolios in a competitive Walrasian dealer market. Let $\alpha \in [0, 1]$ denote the probability that an investor meets a dealer in the OTC market (and symmetrically, a dealer meets an investor with probability α). When a meeting occurs, the investor and dealer negotiate both the quantity of assets to be intermediated through the dealer market and the associated intermediation fee. The terms of trade are determined through Nash bargaining, where the investor holds bargaining weight $\theta \in [0, 1]$.

The timing is that the round of OTC trades takes place in the first subperiod and ends before production units yield dividends. Hence, equity is traded cum-dividend in the OTC market (and dealer market) of the first subperiod and ex-dividend in the Walrasian market of the second subperiod. Asset purchases in the OTC market cannot be financed by borrowing in this case. This assumption and the structure of preferences described below create the need for a medium of exchange in the OTC market. An individual dealer's preferences are represented by

$$\mathbb{E}_0^d \sum_{t=0}^{\infty} \beta^t (c_{dt} - h_{dt}), \tag{1.24}$$

where c_{dt} is his consumption of the general good in the second subperiod of period t, and h_{dt} is the utility costs from exerting h_{dt} units of effort to produce this good. The expectation operator $\mathbb{E}_0^d$ is with respect to the probability measure induced by the dividend process and the random trading process in the OTC market. Dealers get no utility from the dividend good. An individual investor's preferences are represented by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\varepsilon_{it} y_{it} + c_{it} - h_{it}\right), \tag{1.25}$$

where y_{it} is the quantity of the dividend good that investor i consumes at the end of the first subperiod of period t, c_{it} is his consumption of the general good in the second subperiod of period t, and h_{it} is the utility costs from exerting h_{it} units of effort to produce this good. The variable ε_{it} denotes the realisation of a valuation shock that is distributed independently over time and across agents, with a differentiable cumulative distribution function G on the support $[\varepsilon_L, \varepsilon_H] \subset [0, \infty]$, and $\bar{\varepsilon} \equiv \int \varepsilon dG(\varepsilon)$. Investor i learns his realisation ε_{it} at the beginning of period t, before the OTC trading round. The expectation operator $\mathbb{E}_0$ is with respect to the probability measure induced by the dividend process, the investor's valuation shock, and the random trading process in the OTC market.[7]

Consider the determination of the terms of trade in a bilateral meeting in the OTC round of period t between a dealer with portfolio $\mathbf{a}_{dt} \equiv (a^m_{dt}, a^s_{dt})$ and an investor with portfolio $\mathbf{a}_{it} \equiv (a^m_{it}, a^s_{it})$ and valuation ε. Let $\bar{\mathbf{a}}_t \equiv (\bar{a}^m_t, \bar{a}^s_t)$ denote the investor's post-trade portfolio consisting of $\bar{a}^m_t \in \mathbb{R}_+$ units of money and $\bar{a}^s_t \in \mathbb{R}_+$ units of equity shares, and let $k_t \in \mathbb{R}_+$ denote the fee the dealer charges for his intermediation services. The fee is expressed in terms of the second subperiod consumption good and paid by the investor in the second subperiod.

Let $\hat{W}^D_t(\mathbf{a}_{dt}, k_t)$ denote the maximum expected discounted payoff of a dealer with portfolio $\mathbf{a}_{dt}$ and earned fee k_t when he reallocates his portfolio in the dealer market of period t. Let $W^I_t(\mathbf{a}_{it}, k_t)$ denote the maximum expected discounted payoff at the beginning of the second subperiod of period t (after the production units have borne dividends) of an investor who is holding portfolio $\mathbf{a}_{it}$ and has to pay a fee k_t.

Following Lagos and Zhang (2020), for each t, define a pair of functions $\bar{a}^k_t : \mathbb{R}^2_+ \times [\varepsilon_L, \varepsilon_H] \to \mathbb{R}_+$ for $k = \{m, s\}$ and a function $k_t : \mathbb{R}^2_+ \times [\varepsilon_L, \varepsilon_H] \to \mathbb{R}_+$, and let $\bar{\mathbf{a}}_t(\mathbf{a}_{it}, \varepsilon) = (\bar{a}^m_t(\mathbf{a}_{it}, \varepsilon), \bar{a}^s_t(\mathbf{a}_{it}, \varepsilon))$ for each $(\mathbf{a}_{it}, \varepsilon) \in \mathbb{R}^2_+ \times [\varepsilon_L, \varepsilon_H]$. We use $[\bar{\mathbf{a}}_t(\mathbf{a}_{it}, \varepsilon), k_t(\mathbf{a}_{it}, \varepsilon)]$ to represent the bargaining outcome for a bilateral meeting at time t between an investor with portfolio $\mathbf{a}_{it}$ and valuation ε, and a

[7] The valuation shock stands in for the various idiosyncratic reasons why individual investors may wish to hold different quantities of a certain asset at different points in time, such as differences in their liquidity needs, financing or financial-distress costs, or hedging needs.

dealer with portfolio $\mathbf{a}_{dt}$. That is $[\bar{\mathbf{a}}_t(\mathbf{a}_{it}, \varepsilon), k_t(\mathbf{a}_{it}, \varepsilon)]$ solves

$$\max_{(\bar{\mathbf{a}}_t, k_t) \in \mathbb{R}^3_+} \Big\{ \left[\varepsilon y_t \bar{a}^s_t + W^I_t(\bar{\mathbf{a}}_t, k_t) - \varepsilon y_t a^s_{it} - W^I_t(\mathbf{a}_{it}, 0)\right]^{\theta} \times \left[\hat{W}^D_t(\mathbf{a}_{dt}, k_t) - \hat{W}^D_t(\mathbf{a}_{dt}, 0)\right]^{1-\theta} \Big\} \tag{1.26}$$

$$\text{s.t. } \bar{a}^m_t + p_t \bar{a}^s_t \leq a^m_{it} + p_t a^s_{it}$$
$$\hat{W}^D_t(\mathbf{a}_{dt}, 0) \leq \hat{W}^D_t(\mathbf{a}_{dt}, k_t)$$
$$\varepsilon y_t a^s_{it} + W^I_t(\mathbf{a}_{it}, 0) \leq \varepsilon y_t \bar{a}^s_t + W^I_t(\bar{\mathbf{a}}_t, k_t),$$

where p_t is the dollar price of an equity share in the dealer market of period t. Let $W^D_t(\mathbf{a}_t, k_t)$ denote the maximum expected discounted payoff of a dealer who has earned fee k_t in the OTC round of period t and, at the beginning of the second subperiod of period t, is holding portfolio $\mathbf{a}_t$. Then the dealer's value of trading in the dealer market is

$$\hat{W}^D_t(\mathbf{a}_t, k_t) = \max_{\hat{\mathbf{a}}_t \in \mathbb{R}^2_+} W^D_t(\hat{\mathbf{a}}_t, k_t) \tag{1.27}$$
$$\text{s.t. } \hat{a}^m_t + p_t \hat{a}^s_t \leq a^m_t + p_t a^s_t.$$

Let $V^D_t(\mathbf{a}_t)$ denote the maximum expected discounted payoff of a dealer who enters the OTC round of period t with portfolio $\mathbf{a}_t = (a^m_t, a^s_t)$. Again, let $\boldsymbol{\phi}_t \equiv (\phi^m_t, \phi^s_t)$, where ϕ^m_t is the real price of money and ϕ^s_t is the real ex-dividend price of equity in the second subperiod of period t. Then,

$$W^D_t(\mathbf{a}_t, k_t) = \max_{(c_t, h_t, \tilde{\mathbf{a}}_{t+1}) \in \mathbb{R}^4_+} \left[c_t - h_t + \beta \mathbb{E}_t V^D_{t+1}(\tilde{a}^m_{t+1}, \delta \tilde{a}^s_{t+1})\right] \tag{1.28}$$
$$\text{s.t. } c_t + \boldsymbol{\phi}_t \tilde{\mathbf{a}}_{t+1} \leq h_t + k_t + \boldsymbol{\phi}_t \mathbf{a}_t,$$

where $\tilde{\mathbf{a}}_{t+1} \equiv (\tilde{a}^m_{t+1}, \tilde{a}^s_{t+1})$, $\mathbb{E}_t$ is the conditional expectations operator over the next-period realisation of the dividend, and $\boldsymbol{\phi}_t \mathbf{a}_t$ denotes the dot product of $\boldsymbol{\phi}_t$ and $\mathbf{a}_t$. Similarly, let $V^I_t(\mathbf{a}_t, \varepsilon)$ denote the maximum expected discounted payoff of an investor with valuation ε and portfolio $\mathbf{a}_t$ at the beginning of the OTC round of period t. Then,

$$W_t^I(\mathbf{a}_t, k_t) = \max_{(c_t, h_t, \tilde{\mathbf{a}}_{t+1}) \in \mathbb{R}_+^4} \left\{ c_t - h_t + \beta \mathbb{E}_t \int V_{t+1}^I[\tilde{a}_{t+1}^m, \delta \tilde{a}_{t+1}^s + (1-\delta)A^s, \varepsilon'] dG(\varepsilon') \right\} \quad (1.29)$$

$$\text{s.t. } c_t + \boldsymbol{\phi}_t \tilde{\mathbf{a}}_{t+1} \leq h_t - k_t + \boldsymbol{\phi}_t \mathbf{a}_t + T_t,$$

where $T_t \in \mathbb{R}$ is the real value of the time t lump-sum monetary transfer.

The value function of an investor who enters the OTC round of period t with portfolio $\mathbf{a}_t$ and valuation ε is

$$V_t^I(\mathbf{a}_t, \varepsilon) = \alpha \left\{ \varepsilon y_t \bar{a}_t^s(\mathbf{a}_t, \varepsilon) + W_t^I \left[\bar{\mathbf{a}}_t(\mathbf{a}_t, \varepsilon), k_t(\mathbf{a}_t, \varepsilon) \right] \right\}$$
$$+ (1-\alpha) \left[\varepsilon y_t a_t^s + W_t^I(\mathbf{a}_t, 0) \right]. \quad (1.30)$$

The value function of a dealer who enters the OTC round of period t with portfolio $\mathbf{a}_t$ is:

$$V_t^D(\mathbf{a}_t) = \alpha \int \hat{W}_t^D \left[\mathbf{a}_t, k_t(\mathbf{a}_{it}, \varepsilon) \right] dH_{It}(\mathbf{a}_{it}, \varepsilon) + (1-\alpha) \hat{W}_t^D(\mathbf{a}_t, 0), \quad (1.31)$$

where H_{It} is the cumulative distribution function over the portfolios and valuations of the investors the dealer may contact in the OTC market of period t.

Let $j \in \{D, I\}$ index agent type, where D denotes dealers and I denotes investors. For each $j \in \{D, I\}$, let A_{jt}^m and A_{jt}^s represent the total holdings of money and equity shares, respectively, by agents of type j at the beginning of the OTC round in period t (after the replacement of production units). Formally, $A_{jt}^m = \int a_t^m \, dF_{jt}(\mathbf{a}_t)$ and $A_{jt}^s = \int a_t^s \, dF_{jt}(\mathbf{a}_t)$, where F_{jt} denotes the distribution of portfolios $\mathbf{a}_t = (a_t^m, a_t^s)$ across agents of type j at the start of the OTC round.

For $j \in \{D, I\}$, let $\tilde{A}_{jt+1}^m$ and $\tilde{A}_{jt+1}^s$ denote the total quantities of money and equity shares held by all agents of type j at the end of period t. Formally, $\tilde{A}_{Dt+1}^k = \int_{\mathcal{D}} \tilde{a}_{dt+1}^k \, dl$ and $\tilde{A}_{It+1}^k = \int_{\mathcal{I}} \tilde{a}_{it+1}^k \, di$ for $k \in \{m, s\}$, $l \in \mathcal{D}$ and $i \in \mathcal{I}$. These end-of-period holdings evolve according to $A_{Dt+1}^m = \tilde{A}_{Dt+1}^m$, $A_{Dt+1}^s = \delta \, \tilde{A}_{Dt+1}^s$, $A_{It+1}^m = \tilde{A}_{It+1}^m$, and $A_{It+1}^s = \delta \, \tilde{A}_{It+1}^s + (1-\delta)A^s$, where $\delta \in (0, 1]$ is the survival probability of production units and A^s is the measure of newly issued shares.

Let $\bar{A}_{Dt}^m$ and $\bar{A}_{Dt}^s$ denote the total quantities of money and shares held by dealers immediately after the OTC trading round in period t, and let $\bar{A}^m_{It}$ and $\bar{A}^s_{It}$ denote the corresponding post-OTC holdings of those investors who trade in the first subperiod. For $k \in \{m, s\}$, $\bar{A}^k_{Dt} = \int \hat{a}^k_t(\mathbf{a}_t)\, dF_{Dt}(\mathbf{a}_t)$ and $\bar{A}^k_{It} = \alpha \int \bar{a}^k_t(\mathbf{a}_t, \varepsilon)\, dH_{It}(\mathbf{a}_t, \varepsilon)$, where $\hat{a}^k_t(\mathbf{a}_t)$ denotes the dealer's post-OTC portfolio choice, $\bar{a}^k_t(\mathbf{a}_t, \varepsilon)$ denotes the investor's post-OTC portfolio conditional on valuation ε, F_{Dt} is the distribution over dealer portfolios at the start of the OTC round, H_{It} is the joint distribution of investor portfolios and valuations at the start of the OTC round, and $\alpha \in [0, 1]$ is the probability that an investor is able to trade in the OTC market.

As shown in Lagos and Zhang (2020), the following result characterises the equilibrium post-trade portfolios of dealers and investors in the OTC market, taking beginning-of-period portfolios as given. Define $\varepsilon^*_t \equiv (p_t\phi^m_t - \phi^s_t)/y_t$ and

$$\chi(\varepsilon^*_t, \varepsilon) \begin{cases} = 1 & \text{if } \varepsilon^*_t < \varepsilon \\ \in [0,1] & \text{if } \varepsilon^*_t = \varepsilon \\ = 0 & \text{if } \varepsilon < \varepsilon^*_t. \end{cases}$$

Consider a bilateral meeting in the OTC round of period t between a dealer and an investor with portfolio $\mathbf{a}_t$ and valuation ε. The investor's post-trade portfolio, $[\bar{a}^m_t(\mathbf{a}_t, \varepsilon), \bar{a}^s_t(\mathbf{a}_t, \varepsilon)]$ is given by

$$\bar{a}^m_t(\mathbf{a}_t, \varepsilon) = [1 - \chi(\varepsilon^*_t, \varepsilon)]\,(a^m_t + p_t a^s_t) \tag{1.32}$$

$$\bar{a}^s_t(\mathbf{a}_t, \varepsilon) = \chi(\varepsilon^*_t, \varepsilon)(1/p_t)(a^m_t + p_t a^s_t), \tag{1.33}$$

and the intermediation fee charged by the dealer is

$$k_t = (1-\theta)(\varepsilon - \varepsilon^*_t)\left[\chi(\varepsilon^*_t, \varepsilon)\frac{1}{p_t}a^m_t - [1 - \chi(\varepsilon^*_t, \varepsilon)]\,a^s_t\right] y_t. \tag{1.34}$$

A dealer who enters the OTC market with portfolio $\mathbf{a}_{dt}$ exits the OTC market with portfolio $[\hat{a}^m_t(\mathbf{a}_{dt}), \hat{a}^s_t(\mathbf{a}_{dt})] = [\bar{a}^m_t(\mathbf{a}_{dt}, 0), \bar{a}^s_t(\mathbf{a}_{dt}, 0)]$. The bargaining outcome hinges on the investor's valuation ε relative to a cutoff ε^*_t, which identifies the *marginal investor* who is indifferent between holding equity and money. If $\varepsilon > \varepsilon^*_t$, the investor values equity more than the marginal investor and therefore spends all available cash to acquire additional shares. Conversely, if

$\varepsilon < \varepsilon_t^*$, the investor values liquidity more than equity and sells all of his shares in exchange for money. The dealer receives a share $(1-\theta)$ of the investor's surplus from this reallocation as an intermediation fee. The dealer's post-trade portfolio is the same as that of an investor with $\varepsilon = 0$.

We restrict attention to *recursive equilibria*, i.e., equilibria in which aggregate equity holdings are constant over time, so that $A_{Dt}^s = A_D^s$ and $A_{It}^s = A_I^s$ for all t. Asset prices are assumed to be linear functions with respect to the dividend, implying $\phi_t^s = \phi^s y_t$, $p_t \phi_t^m \equiv \bar{\phi}_t^s = \bar{\phi}^s y_t$, $\phi_t^m A_{It}^m = Z y_t$, and $\phi_t^m A_{Dt}^m = Z_D y_t$, where Z and Z_D are positive constants.

In such an equilibrium, the marginal investor's valuation cutoff is time-invariant and satisfies $\varepsilon_t^* = \bar{\phi}^s - \phi^s \equiv \varepsilon^*$. Moreover, price ratios evolve with fundamentals: $\phi_{t+1}^s/\phi_t^s = \bar{\phi}_{t+1}^s/\bar{\phi}_t^s = \gamma_{t+1}$, $\phi_t^m/\phi_{t+1}^m = \mu/\gamma_{t+1}$, and $p_{t+1}/p_t = \mu$. As discussed in Lagos and Zhang (2020), it is convenient to define

$$\bar{\mu} \equiv \bar{\beta}\left[1 + \frac{\alpha\theta(1-\bar{\beta}\delta)(\bar{\varepsilon}-\varepsilon_L)}{\bar{\beta}\delta\bar{\varepsilon} + (1-\bar{\beta}\delta)\varepsilon_L}\right] \tag{1.35}$$

$$\hat{\mu} \equiv \bar{\beta}\left[1 + \frac{(1-\alpha\theta)(1-\bar{\beta}\delta)(\hat{\varepsilon}-\bar{\varepsilon})}{\hat{\varepsilon}}\right] \tag{1.36}$$

where $\hat{\varepsilon} \in [\bar{\varepsilon}, \varepsilon_H]$ is the unique solution to

$$\bar{\varepsilon} - \hat{\varepsilon} + \alpha\theta \int_{\varepsilon_L}^{\hat{\varepsilon}} (\hat{\varepsilon} - \varepsilon) dG(\varepsilon) = 0. \tag{1.37}$$

Note that $\hat{\mu} < \bar{\mu}$. A nonmonetary equilibrium, i.e., $\phi_t^m = 0$, exists for any parametrisation, and there is no recursive monetary equilibrium if $\mu \geq \bar{\mu}$. In the nonmonetary equilibrium, $A_I^s = A^s - A_D^s = A^s$ (i.e., only investors hold equity), there is no trade in the OTC market, and the equity price in the second subperiod is

$$\phi^s = \frac{\bar{\beta}\delta}{1-\bar{\beta}\delta}\bar{\varepsilon}.$$

As derived in Lagos and Zhang (2020), if $\mu \in (\bar{\beta}, \hat{\mu}]$ there is one recursive monetary equilibrium; asset holdings of dealers and investors at the beginning of the OTC round of period t are $A_{Dt}^m = A_t^m - A_{It}^m = 0$ and $A_D^s = A^s - A_I^s \leq \delta A^s$ (which reflects that dealers hold equity across periods rather than acting as brokers), and asset price is

$$\phi^s = \frac{\bar{\beta}\delta}{1-\bar{\beta}\delta}\varepsilon^*.$$

Otherwise, if $\mu \in (\hat{\mu}, \bar{\mu})$, then there is one recursive monetary equilibrium; asset holdings of dealers and investors at the beginning of the OTC round of period t are $A^m_{Dt} = A^m_t - A^m_{It} = 0$ and $A^s_D = A^s - A^s_I = 0$ and asset price is[8]

$$\phi^s = \frac{\bar{\beta}\delta}{1-\bar{\beta}\delta}\left[\bar{\varepsilon} + \alpha\theta \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon) dG(\varepsilon)\right],$$

and for any $\mu \in (\hat{\mu}, \bar{\mu})$, $\varepsilon^* \in (\varepsilon_L, \varepsilon_H)$ is the unique solution to

$$\frac{(1-\bar{\beta}\delta)\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon - \varepsilon^*)dG(\varepsilon)}{\varepsilon^* + \bar{\beta}\delta\left[\bar{\varepsilon} - \varepsilon^* + \alpha\theta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^* - \varepsilon)dG(\varepsilon)\right]\mathbb{I}_{\{\hat{\mu}<\mu\}}} - \frac{\mu - \bar{\beta}}{\bar{\beta}\alpha\theta} = 0.$$

In the nonmonetary equilibrium, dealers are inactive and all equity shares are held solely by investors. Since money has no value in this case, investors and dealers cannot exploit the gains from trade that arise from heterogeneity in investor valuations during the first subperiod. The real asset price is therefore

$$\phi^s = \frac{\bar{\beta}\delta}{1-\bar{\beta}\delta}\,\bar{\varepsilon},$$

which corresponds to the expected discounted value of the dividend stream, as the equity share is not traded. Although shares may be traded in the Walrasian market in the second subperiod, the gains from trade at that stage are zero.

A recursive monetary equilibrium exists only if the inflation rate is not excessively high, i.e., if $\mu < \bar{\mu}$. Unlike the nonmonetary case, the OTC market is active in the monetary equilibrium. It is straightforward to show that the marginal valuation ε^* is strictly decreasing in the inflation rate,

$$\frac{\partial \varepsilon^*}{\partial \mu} < 0.$$

Intuitively, as inflation increases, the real value of money falls, lowering the marginal investor's

[8] In Lagos and Zhang (2020) the object Z is derived of the form $Z = \frac{\alpha\, G(\varepsilon^*) A^s_I + A^s_D}{\alpha\,[1-G(\varepsilon^*)]}\,(\varepsilon^* + \phi^s)$.

valuation. The investor who was previously indifferent between holding cash and equity under a lower inflation rate now strictly prefers equity.

Let $q^B_{t,k}$ denote the nominal price in the second subperiod of period t of an N-period risk-free pure discount nominal bond maturing in period $t+k$, for $k = 0, 1, 2, \ldots, N$, where k represents the time to maturity. Suppose the bond cannot serve as a means of payment in the first subperiod. Even though it cannot be traded for equity in the OTC market, it can be redeemed for money at the end of the period at no cost. In a recursive monetary equilibrium we have $q^B_{t,k} = (\bar{\beta}/\mu)^k$, implying a nominal yield to maturity $i = \mu/\bar{\beta} - 1$. Because the gross inflation rate satisfies $\phi^m_t/\phi^m_{t+1} = \mu(y_t/y_{t+1}) \equiv 1 + \pi_{t+1}$, it follows that $1 + i = \mu/\bar{\beta} \equiv (1 + r)(1 + \pi)$, where $1 + \pi \equiv \left[\mathbb{E}_t\left(\frac{1}{1+\pi_{t+1}}\right)\right]^{-1} = \mu/\bar{\gamma}$.

We now examine how monetary policy influences asset prices within the model. The real price of equity in a monetary equilibrium is partly shaped by the option available to low valuation investors to resell their holdings to higher valuation investors. When the growth rate of the money supply (and thus the inflation rate) rises, equilibrium real money balances decline, and the marginal investor valuation ε^* falls. Intuitively, under higher inflation, the investor who was previously indifferent between holding cash or equity now strictly prefers holding equity. Because the marginal investor in the OTC market has a lower valuation, the resale premium declines, which lowers the real price of equity, both the ex-dividend price ϕ^s and the cum-dividend price $\bar{\phi}^s$. At the same time, a higher growth rate of the money supply reduces the real value of money ϕ^m_t.

Following Lagos and Zhang (2020), in the recursive monetary equilibrium, this implies that ex-dividend and cum-dividend equity prices, as well as the real value of money, are all decreasing functions of inflation:

$$\frac{\partial \phi^s}{\partial \mu} < 0, \quad \frac{\partial \bar{\phi}^s}{\partial \mu} < 0, \quad \frac{\partial \phi^m_t}{\partial \mu} < 0.$$

Likewise, all three quantities decline with the real interest rate:

$$\frac{\partial \phi^s}{\partial r} < 0, \quad \frac{\partial \bar{\phi}^s}{\partial r} < 0, \quad \frac{\partial \phi^m_t}{\partial r} < 0.$$

Next, we provide the economic intuition behind the results. In this setting, as derived by Lagos and Zhang (2020), the Euler equations for money and equity are

$$\phi_t^m = \frac{1}{1+r}\,\mathbb{E}_t\left[\phi_{t+1}^m + \alpha\theta\int_{\varepsilon^*}^{\varepsilon_H}\frac{(\varepsilon-\varepsilon^*)y_{t+1}}{p_{t+1}}\,dG(\varepsilon)\right] \tag{1.38}$$

$$\phi_t^s = \frac{1}{1+r}\,\delta\,\mathbb{E}_t\left[\bar{\varepsilon}y_{t+1} + \phi_{t+1}^s + \alpha\theta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)y_{t+1}\,dG(\varepsilon)\right]. \tag{1.39}$$

In the first equation, (1.38), the left-hand side, ϕ_t^m, represents the cost (in terms of second-subperiod consumption) of acquiring one additional unit of money in the competitive market during the second subperiod of period t. The right-hand side gives the discounted expected value of holding that unit until period $t+1$. This marginal value has two components: first, the expected value of holding the dollar through to the end of period $t+1$, ϕ_{t+1}^m, and second, the expected value of the option to exchange the dollar for equity in the OTC market at the start of period $t+1$.

This exchange opportunity is realised only if the investor gains access to the dealer market (with probability α), receives a share θ of the trade surplus with the dealer, and holds a valuation above the marginal level ε^*. The resulting expected gain from exercising the option, i.e., the resale premium, is therefore

$$\alpha\theta\int_{\varepsilon^*}^{\varepsilon_H}\frac{(\varepsilon-\varepsilon^*)y_{t+1}}{p_{t+1}}\,dG(\varepsilon).$$

Next, consider the second equation, (1.39). On the left-hand side, ϕ_t^s represents the cost of acquiring one additional equity share in the competitive market during the second subperiod of period t. The right-hand side gives the discounted, net-of-depreciation expected value of holding that share until period $t+1$.

As explained in Lagos and Zhang (2020), this marginal value comprises two components. First, the expected cum-dividend value from holding the equity through the end of period $t+1$, given by $\bar{\varepsilon}y_{t+1} + \phi_{t+1}^s$. Second, the expected value of the option to resell the equity in the OTC market at the beginning of period $t+1$. This resale opportunity is realised if the investor gains access to the dealer market (with probability α), receives a fraction θ of the trading surplus, and encounters a counterparty with valuation below the marginal level ε^*. The expected value of this resale premium is therefore

$$\alpha\theta \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon) y_{t+1} \, dG(\varepsilon).$$

This additional term, which raises the equilibrium real value of an equity share above the discounted expected dividend, is analogous to the resale premium value described in Harrison and Kreps (1978). The key novelty in the Lagos and Zhang (2020) framework is that the value of this resale option is endogenous to monetary policy. It depends on how policy affects asset reallocation and determines the marginal investor valuation, ε^*, in the OTC market.

Given $p_t \phi_t^m \equiv (\varepsilon^* + \phi^s) y_t$, $\phi_t^m / \phi_{t+1}^m = \mu / \gamma_{t+1}$, $\mathbb{E}_t y_{t+1} = \mathbb{E}_t \gamma_{t+1} y_t = \bar{\gamma} y_t$, the money Euler equation (1.38) can be expressed as

$$1 = \frac{1}{1+i} \left[1 + \alpha\theta \int_{\varepsilon^*}^{\varepsilon_H} \frac{\varepsilon - \varepsilon^*}{\varepsilon^* + \phi^s} \, dG(\varepsilon) \right].$$

The left-hand side represents the real cost of acquiring one additional unit of real money balances, while the right-hand side gives the discounted expected value of holding that unit into the OTC round of the following period.

As Lagos and Zhang (2020) highlight, an increase in the nominal interest rate, i, functions as a tax on real money balances. It reduces the incentive to hold money, thereby lowering the valuation of the marginal investor, ε^*. Consequently, the real equity price declines. Importantly, this result holds whether the increase in i is driven by higher expected inflation, π, or by a higher real interest rate, r. The underlying intuition is that, although a change in i affects both money and equity valuations in general equilibrium, it disproportionately alters the incentive to hold money, which in turn lowers the marginal investor's valuation of equity.[9]

The equity Euler equation (1.39) does not respond directly to π, but only indirectly through its general equilibrium effect on real money balances. The direct effect of expected inflation on the real value of money dominates the indirect effect on equity prices. As a result, the marginal investor valuation ε^* decreases. The investor who was previously marginal under the

[9]To see this more clearly, consider first the case where the increase in i is associated with higher expected inflation, π, or equivalently, a higher money growth rate, μ. The money pricing equation above shows that the primary (partial equilibrium) effect of higher π is to reduce the real value of a dollar, which occurs through a decline in ε^*. In other words, the equilibrium cum-dividend value of an equity share—purchased using money—falls.

lower inflation rate shifts his portfolio away from money toward equity in order to mitigate the inflation tax.

To illustrate this from another perspective, consider a setting where monetary policy follows the Friedman rule, i.e., $i = 0$. In this case, the opportunity cost of holding money is zero, so investors are willing to hold enough real balances to meet any random liquidity demand in the OTC round. Formally, this implies $\varepsilon^* = \varepsilon_H$ when $i = 0$. Intuitively, the highest valuation investors (with $\varepsilon = \varepsilon_H$) can absorb the entire supply of equity in the dealer market, and the equilibrium equity price reflects only their valuations.

If the inflation rate exceeds the Friedman rule target, i.e., $i > 0$, holding money becomes costly, and even the highest valuation investors are budget constrained in the OTC round. That is, they no longer hold sufficient real balances to remain unconstrained across all possible next-period valuations, so $\varepsilon^* < \varepsilon_H$ when $i > 0$. Consequently, investors with lower valuations can now purchase some equity, and the marginal investor who determines the equilibrium equity price has a valuation $\varepsilon^* < \varepsilon_H$. In short, higher inflation reduces aggregate real balances, limits the ability of high valuation investors to fully express their preferences, and shifts pricing and holdings toward investors with lower valuations.

Similarly, consider the case where an increase in i stems from a rise in the real interest rate, r. Here, ε^* also declines, reflecting that an increase in the real rate diminishes the value of money more than the value of equity. Since equity provides a real dividend while money does not, maintaining a positive value of money in equilibrium requires that money serve as a relatively better store of value than equity (formally, $\mu < \bar{\mu}$). Therefore, although higher r reduces valuations of both equity and money, the effect on the incentive to hold equity is relatively smaller.

The analysis in Lagos and Zhang (2020) shows that resale premia hinge on liquidity conditions, which are in turn shaped by monetary policy.[10]

[10] Nevertheless, the model abstracts from the use of collateral and credit, which are essential for understanding the interaction between monetary policy and asset markets in practice. In particular, when investors can issue bonds collateralised by their asset holdings, the distinction between liquidity and resale channels becomes blurred. The following section incorporates collateralised bonds into the resale premium framework, bridging the analysis of credit, liquidity, and asset pricing.

1.4 Contrasting Mechanisms

The preceding sections have examined two distinct mechanisms through which frictions in trade alter asset valuation: *liquidity premia*, arising from the use of assets as media of exchange, and *resale premia*, generated by the option to transfer assets in future secondary markets. Although both frameworks depart from the frictionless Walrasian benchmark by introducing search and decentralised trading, they yield fundamentally different predictions for how asset prices respond to monetary conditions.

In liquidity premium models, such as the model in Lagos (2010) discussed in Section 2, assets are valued not only for their dividend flows but also for their capacity to relax trading constraints in decentralised markets. The equilibrium Euler equation features a liquidity factor, $\mathcal{L}(\Lambda_{t+1})$, that augments the standard stochastic discount factor, reflecting the marginal utility value of liquidity services. When monetary policy tightens and real balances fall, the liquidity premium rises. Assets that function as exchange media therefore appreciate in real terms, as investors bid up their value to facilitate trade. Their expected returns fall accordingly.

By contrast, resale premium models, such as the model in Lagos and Zhang (2020) described in Section 3, explain asset valuation through heterogeneity in investor valuations and the intermediation of trade via OTC markets. Here, assets derive additional value from the option to resell them to higher valuation investors in the future. The resale premium thus depends on the matching probability, the trade surplus received by the investor in bilateral exchange (both exogenous in the Lagos and Zhang (2020) setup), and the marginal investor's valuation of the asset. When liquidity tightens—because inflation rises/policy contracts—the marginal investor's valuation falls, diminishing the expected resale option. Consequently, assets characterised by resale premia, such as equities and illiquid bonds, tend to depreciate under restrictive monetary conditions.

Viewed jointly, these frameworks reveal that the asset price response to monetary shocks depends on the source of its value. Assets that serve primarily as exchange media gain value when liquidity becomes scarce, while those whose value stems from speculative resale opportunities lose value. This contrast provides a theoretical basis for the empirical observation that interest rate increases or liquidity contractions are often accompanied by falling equity prices but rising prices of liquid government securities or that asset prices often display ambiguous responses to monetary policy.

Moreover, both mechanisms underscore the central insight that trading frictions—not preference heterogeneity or risk aversion alone—govern the cross-sectional structure of returns. This perspective unifies the liquidity and resale literatures within a broader frictional asset pricing paradigm, where monetary policy, market structure, and asset heterogeneity jointly determine valuations.

The synthesis also highlights several open questions that motivate avenues for subsequent research. First, when both mechanisms operate simultaneously—as in modern financial systems where assets provide liquidity services yet also trade in secondary markets—their interaction may generate non-linear and state-dependent asset price responses to monetary policy. Second, monetary policy may influence not only the level of liquidity premia but also the relative strength of liquidity versus resale channels, thereby shaping the transmission of policy to asset prices and risk spreads.

1.5 Conclusion

This review has surveyed search-based asset pricing models with decentralised trade, clarifying how liquidity premia arise from the ability of assets to relax trading constraints and how resale premia reflect the expected gains from reallocating claims across heterogeneous reservation values. We have seen that models incorporating search frictions and decentralised trading fundamentally reshape asset valuations, assigning them a liquidity function and/or resale premium that extends beyond their intrinsic dividend payoffs.

The literature demonstrates that liquidity and resale considerations shed new light on long-standing asset pricing puzzles. By accounting for liquidity needs and resale opportunities, these models generate return patterns more consistent with empirical observations, highlighting the centrality of trading frictions for understanding asset prices.

Overall, these insights underscore the limitations of the standard Walrasian benchmark, which abstracts from trading frictions. Liquidity and resale considerations are key determinants of equilibrium outcomes and help explain persistent anomalies in financial markets.

Chapter 2

Asset Prices with Collateral and Resale Premia

ABSTRACT

This chapter examines asset pricing with collateral and resale premia. Asset prices respond ambiguously to monetary policy: easing sometimes lowers dividend yields, as in traditional and collateral/liquidity-based monetary-style asset models, and sometimes raises them, as in monetary resale premium models. This chapter reconciles these views by developing a monetary over-the-counter (OTC) model where assets serve both as collateral for goods and asset purchases and as resaleable securities. The framework generates both collateral and resale premia, explaining why dividend yields can move in opposite directions under monetary easing/tightening. Further, we show that policy remains effective in the cashless limit, and that the cashless-limit equilibrium differs from the nonmonetary equilibrium, highlighting that money's role cannot be abstracted from asset pricing.

2.1 Introduction

Asset prices in modern markets display ambiguous responses to monetary policy. In some episodes, monetary easing coincides with lower dividend yields, while in others easing is associated with higher dividend yields. Empirical studies document that equity valuations often rise when monetary policy and bond yields fall, consistent with the intuition behind the "Fed model" (see, Bekaert and Engstrom (2010); Asness (2003)). Conversely, other evidence shows episodes and sectors in which policy changes produce the opposite response in equity valuations and dividend yields (e.g., Rigobon and Sack (2004); Bernanke and Kuttner (2005); Gürkaynak et al. (2005)).[1] Such mixed patterns stand in contrast to the uniform predictions of conventional theories and point to the need for a richer framework linking monetary policy to asset prices.

Traditional asset pricing models, and indeed most monetary-style asset pricing models, predict that loose monetary policy reduces dividend yields, and vice versa under tightening. In pure asset-pricing frameworks, the mechanism is discounting: lower interest rates raise discount factors and therefore increase asset prices, driving down dividend yields (see, Gordon and Shapiro (1956); Gordon (1959)). Likewise, in monetary models where assets facilitate exchange or collateralised trade (e.g., Lagos and Rocheteau (2009); Geromichalos et al. (2007); Lester et al. (2012); Nosal and Rocheteau (2013)), an increase in the rate of money growth increases the liquidity premium, raising asset prices and lowering dividend yields, while a decrease does the reverse.[2] By contrast, monetary models with resale premia imply the opposite pattern: an increase in money growth reduces asset prices and therefore raises dividend yields, while a decrease does the reverse (e.g., Lagos and Zhang (2015, 2019a,b, 2020)). In these settings, assets carry a resale option value that declines when liquidity conditions improve, lowering its price relative to dividends.

Yet empirically, both patterns are observed: dividend yields sometimes fall with monetary easing and sometimes rise, and likewise under monetary tightening. This presents a puzzle

[1] For example, during the 2022 tightening cycle, Energy rose strongly while Communication Services and Consumer Discretionary fell (see, S&P Dow Jones Indices (2023)).

[2] We state this comparative static in terms of the money growth rate to be precise about the policy instrument in the models cited here. In these frameworks, the nominal interest rate is identified with the yield on an illiquid bond linked to money growth via a Fisher equation. The comparative statics of an interest rate change may differ depending on whether the rate corresponds to a liquid or illiquid instrument; see Geromichalos and Herrenbrueck (2022) and Herrenbrueck and Wang (2023)

that no single strand of the existing literature can resolve. The contribution of this chapter is to reconcile these responses by building a unified asset pricing model that incorporates both collateral and resale premia. By embedding both functions of assets into a monetary over-the-counter (OTC) framework, we show how the relative strength of collateral and resale motives determines whether monetary easing lowers or raises dividend yields.

A recurrent critique holds that monetary models have limited usefulness because agents economise on cash, leaving little scope for monetary frictions to influence asset pricing (see, Woodford (1998, 2003)). We address this critique by developing a monetary asset-pricing framework that integrates both collateral and margin lending. This design serves four purposes. First, it clarifies the role of bonds in asset pricing. Bonds act not only as discounting instruments but also as funding and settlement objects that support collateralised purchases and margin borrowing, so their pricing is integral to monetary policy transmission. Second, it delivers the full set of premia that matter for policy analysis in asset markets, generating within one environment resale, collateral, margin, and liquidity premia, and thereby allowing mechanisms usually studied in separate literatures to be examined jointly. Third, it delivers a cashless limit equilibrium which is distinct from the nonmonetary equilibrium, showing that monetary considerations remain operative even when money holdings are economised. Fourth, the monetary environment provides the natural home for resale and collateral mechanisms. Resale is liquidation into money, and goods are acquired using credit or money that ultimately settles in money, so a monetary model supplies the right primitives for studying how these premia shape asset prices and how policy shifts them.

In this chapter, therefore, we develop a monetary OTC asset-pricing model in which assets perform three functions: (*i*) they provide a stream of dividends, (*ii*) they serve as collateral for goods purchases and margin borrowing, and (*iii*) they trade as resaleable securities in financial markets. By embedding all three functions in one environment, the framework reconciles the mixed dividend-yield responses to monetary easing/tightening, organises the complete set of asset-pricing premia within a single structure, and maintains the results in the cashless limit.

This chapter relates to four research areas: search-theoretic models of money, search-theoretic models of financial trade in OTC markets, collateral theories of asset pricing and resale theories of asset pricing. From a methodological standpoint, this chapter helps bridge the search-theoretic monetary literature that has largely focused on macroeconomic issues, and the search-theoretic financial OTC literature that focuses on microeconomic considerations. The basic structure of the model builds on Lagos and Wright (2005). Specifically, we follow Lagos and Zhang (2019b) which adopts OTC market structures for the transaction of assets of Duffie et al. (2005). Lagos and Zhang (2022) use a similar structure where agents and banks participate in Nash bargains over loans for the purpose of consuming goods. In short, banks are set up in way in Lagos and Zhang (2022) that mimic the OTC structure for brokers in Lagos and Zhang (2019b). In this chapter, money serves as a medium of exchange for financial assets and consumption, whereas in Lagos and Zhang (2022) framework, it is only used as a medium of exchange for consumption goods and Lagos and Zhang (2019b) it is solely used financial asset purchases. We use this OTC structure from Duffie et al. (2005) because it is amenable to general equilibrium analysis, and delivers a natural transmission mechanism through which monetary policy influences asset prices and consumption and the standard measures of financial liquidity that are the main focus of the microeconomic strand of the literature. Geromichalos et al. (2007), Jacquet and Tan (2012), Lagos and Rocheteau (2009), Lester et al. (2012) and Nosal and Rocheteau (2013), introduce a real asset that can (at least to some extent) be used along with money as a medium of exchange for consumption goods in variants of Lagos and Wright (2005). These papers identify the liquidity value of the asset with its usefulness in exchange, and find that when the asset is valuable as a medium of exchange, this manifests itself as a "liquidity premium" that makes the real price higher than the expected present discounted value of its financial dividend.

In common with five recent papers, Geromichalos and Herrenbrueck (2016), Lagos and Zhang (2019b,a, 2020, 2022), this chapter shares the general interest in bringing model of OTC trade in financial markets into the realm of modern monetary general equilibrium theory. Lagos and Zhang (2020) offers an in-depth analysis of a model that nests Duffie et al. (2005) and the prototypical monetary search model. Lagos and Zhang (2019b) develops a model where investors have access to margin accounts and can trade with money and bargain in an OTC market for assets and bonds. The frictional nature of OTC trade gives rise to a monetary policy channel even in a cashless-limit economy. The asset prices in their paper incorporate a resale premium and collateral premium since investors can use their assets to obtain margin loans. Lagos and Zhang (2022) operates similarly, though in their case, consumers bargain with a bank over loans in order to purchase consumption goods. Consumers in their model purchase goods via unsecured loans since there are no assets to use as collateral. They also show that monetary policy influences consumption goods even in the cashless-limit economy. In the formulation studied here, investors have access to both equity and goods markets and may have additional access to bond markets facilitated by bond brokers. Asset prices therefore have an additional collateral premium for the purchase of consumption goods through the use of a security-based line of credit (SBLOC), representing a departure from both Lagos and Zhang (2019b) and Lagos and Zhang (2022).

The rest of this chapter is organised as follows. Section 2.2 presents the model and market structure. Section 2.3 characterises equilibrium. Section 2.4 studies the theoretical results in the cashless limit, and Section 2.5 concludes. Appendix A contains some further modelling assumptions. Appendix B contains extensions. Appendix C contains all proofs.

2.2 Model

Time is represented as a sequence of periods indexed by $t = 0, 1, \ldots$. Each period is split into two subperiods devoted to distinct activities. There is a continuum of infinitely-lived *investors*, each corresponding to a point in the set $\mathcal{I} = [0, N_I]$, with $N_I \in \mathbb{R}_+$. Similarly, there is a continuum of infinitely-lived *producers*, each corresponding to a point in the set $\mathcal{P} = [0, N_P]$, with $N_P \in \mathbb{R}_+$, and a continuum of infinitely-lived *brokers*, each corresponding to a point in the set $\mathcal{B} = [0, N_B]$, with $N_B \in \mathbb{R}_+$. All agents discount across periods with factor, $\beta \in (0, 1)$.

There is a continuum of assets of total measure $A^s \in \mathbb{R}_{++}$ active each period. Each active asset delivers at the end of the first subperiod a perishable exogenous *dividend* $y_t \in \mathbb{R}_+$. Since dividends are identical across active assets, the aggregate dividend is $y_t A^s$. At the start of each period, every active asset independently faces an idiosyncratic shock that makes it permanently unproductive with probability $1 - \eta \in [0, 1)$. Conditional on remaining productive, its dividend at time t evolves according to $y_t = \gamma_t y_{t-1}$, where γ_t is a nonnegative random variable with a cumulative distribution function Γ and a mean $\bar{\gamma} \in (0, (\beta\eta)^{-1})$.[3] At the start of period t, the dividend realisation y_t is publicly observed. Any asset that has failed by then is replaced one-for-one with a new asset that pays y_t in the first subperiod and thereafter follows the same stochastic law as incumbent assets. In the second subperiod, each agent has access to a linear technology that transforms effort into a perishable, homogeneous consumption good (the *general good*).

Each active equity share confers the right to collect dividends to the investor who owns it. At the beginning of period $t \geq 1$, each investor is endowed with $(1-\eta)A^s$ equity shares. There is also a second financial instrument, *money*, which is intrinsically useless. The money supply at time t is denoted A_t^m. The initial money supply, $A_0^m \in \mathbb{R}_{++}$, is given, and evolves according to $A_{t+1}^m = \mu A_t^m$, where $\mu \in \mathbb{R}_{++}$. A monetary authority injects or withdraws money via lump-sum transfers or taxes applied to investors and producers in the second subperiod of each period. At the start of period $t = 0$, each investor is endowed with a portfolio of equity and money, while each producer is endowed with money only.

To ensure money remains essential in the model, we follow the standard monetary theory device that agents are anonymous and cannot commit. With no enforcement at all, borrowing would be infeasible and first-subperiod purchases could only be financed with money. To permit credit for equity and goods, we follow Lagos and Zhang (2019b) and introduce limited enforcement. During the first subperiod of period t, a subset of investors and producers issue bonds, each promising one unit of the general good for delivery in the second subperiod of t. These bonds are collateralised so that, upon default at the start of the second subperiod, the creditor seizes a fraction of the debtor's equity—$\kappa \in [0,1]$ for equity purchases and $\lambda \in [0,1]$ for goods purchases.[4]

The market structure operates as follows. In the second subperiod, all agents trade the consumption good produced in that period, as well as equity shares and money in a spot Walrasian

[3]The bound $\bar{\gamma} < (\beta\eta)^{-1}$ ensures the present value of a surviving unit's dividend stream is finite and the dynamic program is well posed. If instead $\beta\eta\bar{\gamma} \geq 1$, the marginal value of postponing payoffs does not decay, and agents strictly prefer shifting resources to the future; economically, discounting is overpowered by growth, ruling out a stationary monetary equilibrium and violating the usual transversality condition. This is a standard restriction in monetary search/OTC models (see Lagos and Zhang (2019a, 2020) which impose the same type of bound).

[4]This credit arrangement is similar to the one described in Lagos and Zhang (2019b) and Lagos and Zhang (2022). In contrast with Lagos and Zhang (2022) investors in our model cannot obtain unsecured loans, so loans in this model must be secured with asset holdings. This ties the goods market to the asset price.

market. In the first subperiod, trade in equity, special goods, money, and collateralised bonds is organised as follows. Three distinct Walrasian markets function concurrently: a *bond market*, an *equity market*, and a *special goods market*. All bond brokers participate in the bond market, where they trade collateralised bonds and money. A proportion $\delta \in [0, 1]$ of investors have access to the equity market, where equity shares and money are traded. Additionally, some of these investors gain indirect access to the bond market through bilateral trades with brokers they meet randomly. Specifically, with probability $1 - \alpha \in [0, 1]$, an investor meets a broker and can trade equity, money, and bonds. With probability α, the investor does not meet a broker and can only trade equity and money in the equity market. This meeting process is independent across investors and time. Similarly, a measure $1 - \delta$ of investors have access to the special goods market, where they trade special goods and money. A fraction $1 - \alpha$ of these investors also gain indirect access to the bond market through random encounters with brokers, just as in the equity market.

When a broker and an investor meet, they bargain over the quantity of bonds and money the broker will trade on behalf of the investor in the bond market during the first subperiod, along with an intermediation fee for the broker's services. The terms of this trade are determined by Nash bargaining, with the investor having a bargaining power $\theta \in [0, 1]$.[5] The timing is such that trading in the first subperiod concludes before production units pay dividends, meaning equity is traded *cum-dividend* in the first subperiod and *ex-dividend* in the second subperiod.[6]

The preferences of an individual agent of type $j \in \{I, P, B\}$ (where "I" denotes "investor", "P" denotes "producer" and "B" denotes "broker") are represented by

$$\mathbb{E}_0^j \sum_{t=0}^{\infty} \beta^t \left\{ [u(q_t) + \varepsilon_t y_t] \mathbb{I}_{\{j=I\}} - \varrho q_t \mathbb{I}_{\{j=P\}} + c_t - h_t \right\}$$

where c_t is consumption of the homogeneous good that is produced, traded, and consumed in the second subperiod of period t, and h_t is the utility cost from exerting h_t units of effort to

[5]The assumption that investors have direct access to competitive equity and special goods markets is for analytical simplicity. It amounts to regarding the equity and goods markets as a conventional Walrasian markets where investors bear zero transaction costs, which is a good approximation to organised exchanges such as the New York Stock Exchange (in the case of the equity market), and in the case of the goods market is a standard assumption for consumption markets. There are many examples of works in which frictions play an important role in equity or goods markets however, in this chapter we seek to investigate frictions in credit markets.

[6]As in standard search models of financial over-the-counter markets (e.g., Lagos and Zhang (2020) and Lagos and Zhang (2019b)), an investor must own the equity share to consume the dividend.

produce this good. $u : \mathbb{R}_+ \to \mathbb{R}$ is the investor's utility function for the consumption good of the first subperiod, $\varrho \in \mathbb{R}_{++}$ is the producer's marginal (disutility) cost of producing the consumption good in the first subperiod. q_t is the investor's consumption (if $j = I$) or production (if $j = P$) of the first subperiod consumption good. We assume $u'' < u(0) = 0 < u'$ and that $\underline{\varrho} < \varrho$. The variable y_t is the quantity of the dividend good that an investor consumes at the end of the first subperiod of period t, and ε_t denotes the realisation of a valuation shock that is distributed independently over time and across investors, with a differentiable cumulative distribution function G on the support $[\varepsilon_L, \varepsilon_H] \subset [0, \infty]$, and $\bar{\varepsilon} \equiv \int \varepsilon dG(\varepsilon)$. Each investor learns the realisation ε_t at the beginning of period t. $\mathbb{I}_{\{.\}}$ is an indicator function that equals 1 if the condition in the subscript is satisfied, and 0 otherwise.[7] The expectation operator $\mathbb{E}_0^I$ is with respect to the probability measure induced by the dividend process, the investor's valuation shock, and the investor's random trading process in the first subperiod.

2.2.1 Bargaining and Portfolio Problems

Investors may find themselves in one of four trading situation in the OTC round of trade. With probability $\alpha\delta$, an investor is only able to trade equity and money competitively in the equity market. With probability $(1-\alpha)\delta$, an investor simultaneously trades money and equity in the equity market, while also bargaining with a broker over the quantities of bonds and money the broker will trade on the investor's behalf in the bond market, along with the broker's intermediation fee. With $\alpha(1-\delta)$, an investor is only able to purchase goods for money in the competitive goods market. And with probability $(1-\alpha)(1-\delta)$, an investor simultaneously trades money and goods with a producer, and bargains with a broker over the quantities of bonds and money that the broker trades in the bond market on behalf of the investor, as well as over the broker's intermediation fee. In these cases, the outcome of the negotiation with the broker is determined by Nash bargaining with investor bargaining power θ. The broker's fees are expressed in terms of the general good and paid by the investor in the second subperiod. To simplify the exposition, we assume brokers are mere matchmakers and cannot hold assets (equity,

[7]Notice that this reflects the assumption that brokers get no utility from the dividend good or the first subperiod consumption good. This implies that brokers have no direct motive for holding equity or purchase consumption goods. This is easy to relax however, but it is adopted here because it is the standard benchmark in the search-based literature on over-the-counter markets (see Duffie et al. (2005), Lagos and Zhang (2019b) and Lagos and Zhang (2022)).

money, or bonds) for their own account. An individual investor's portfolio at the beginning of period t is represented by a vector $\mathbf{a}_t \equiv (a_t^m, a_t^s) \in \mathbb{R}_+^2$, i.e., it consists of $a_t^m \in \mathbb{R}_+$ units of money and $a_t^s \in \mathbb{R}_+$ equity shares.[8]

Let $W_t^I(\mathbf{a}_t, a_t^b, k_t)$ denote the maximum expected discounted payoff at the beginning of the second subperiod of period t of an investor who is holding portfolio $(\mathbf{a}_t, a_t^b)$, with $a_t^b \in \mathbb{R}$ units of bonds and has to pay a fee $k_t \in \mathbb{R}_+$. Consider an investor who enters period t with (pre-trade) portfolio $\mathbf{a}_t = (a_t^m, a_t^s)$ and valuation ε. With probability $\alpha\delta$, the investor is only able to trade in the equity market and their post-trade portfolio is $\hat{\mathbf{a}}_t(\mathbf{a}_t, \varepsilon) \equiv (\hat{a}_t^m(\mathbf{a}_t, \varepsilon), \hat{a}_t^s(\mathbf{a}_t, \varepsilon), \hat{a}_t^b(\mathbf{a}_t, \varepsilon))$, with $\hat{a}_t^b(\mathbf{a}_t, \varepsilon) = 0$ and

$$(\hat{a}_t^m(\mathbf{a}_t, \varepsilon), \hat{a}_t^s(\mathbf{a}_t, \varepsilon)) = \underset{(\hat{a}_t^m, \hat{a}_t^s) \in \mathbb{R}_+^2}{\arg\max} \ \varepsilon y_t \hat{a}_t^s + W_t^I(\hat{a}_t^m, \hat{a}_t^s, \mathbf{0}) \tag{2.1}$$

$$\text{s.t. } \hat{a}_t^m + p_t^s \hat{a}_t^s \leq a_t^m + p_t^s a_t^s,$$

where $\mathbf{0}$ represents $(0, 0)$. In this case the investor's bond holding is fixed at its beginning-of-period value of 0 since he has no access to the bond market.

With probability $(1-\alpha)\delta$, the investor can simultaneously trade in the equity market and bargain over a bond-market trade with a broker. In this case, the bargaining outcome consists of an intermediation fee to the broker, $k_t(\mathbf{a}_t, \varepsilon)$, and a post-trade portfolio, $\bar{\mathbf{a}}_t(\mathbf{a}_t, \varepsilon) \equiv (\bar{a}_t^m(\mathbf{a}, \varepsilon), \bar{a}_t^s(\mathbf{a}_t, \varepsilon), \bar{a}_t^b(\mathbf{a}_t, \varepsilon))$, that solves

$$\max_{(\bar{\mathbf{a}}_t, k_t) \in \mathbb{R}_+^2 \times \mathbb{R} \times \mathbb{R}_+} [\varepsilon y_t \bar{a}_t^s + W_t^I(\bar{\mathbf{a}}_t, k_t) - \varepsilon y_t \hat{a}_t^s(\mathbf{a}_t, \varepsilon) - W_t^I(\hat{a}_t^m(\mathbf{a}_t, \varepsilon), \hat{a}_t^s(\mathbf{a}_t, \varepsilon), \mathbf{0})]^\theta k_t^{1-\theta} \tag{2.2}$$

$$\text{s.t. } \bar{a}_t^m + p_t^s \bar{a}_t^s + p_t^{sb} \bar{a}_t^b \leq a_t^m + p_t^s a_t^s$$
$$\varepsilon y_t \hat{a}_t^s(\mathbf{a}_t, \varepsilon) + W_t^I(\hat{a}_t^m(\mathbf{a}_t, \varepsilon), \hat{a}_t^s(\mathbf{a}_t, \varepsilon), \mathbf{0}) \leq \varepsilon y_t \bar{a}_t^s + W_t^I(\bar{\mathbf{a}}_t, k_t)$$
$$W_t^I(\bar{a}_t^m, (1-\kappa)\bar{a}_t^s, 0, k_t) \leq W_t^I(\bar{\mathbf{a}}_t, k_t),$$

where $\bar{\mathbf{a}}_t \equiv (\bar{a}_t^m, \bar{a}_t^s, \bar{a}_t^b)$. If the investor and the broker were unable to reach an agreement, the

[8] Since bonds issued in the OTC round of period $t-1$ are settled in the second subperiod of $t-1$, there are no bonds outstanding at the beginning of period t.

investor can still trade equity and cash in the equity market at the terms specified by (2.1). Therefore (2.1) is the investor's outside option in his bargaining problem with the broker. The investor's gain from trade corresponding to an outcome $(\bar{\mathbf{a}}_t, k_t)$ consists of the payoff $\varepsilon y_t \bar{a}_t^s + W_t^I(\bar{\mathbf{a}}_t, k_t)$, minus the investor's outside option, $\varepsilon y_t \hat{a}_t^s(\mathbf{a}_t, \varepsilon) + W_t^I(\hat{a}_t^m(\mathbf{a}_t, \varepsilon), \hat{a}_t^s(\mathbf{a}_t, \varepsilon), \mathbf{0})$, i.e., the payoff achieved in (2.1). The first constraint on (2.2) is the budget constraint the investor faces in the OTC round when he is able to trade in the equity and the bond market simultaneously. The second constraint ensures the trade is incentive compatible for the investor, and $k_t \in \mathbb{R}_+$ ensures the trade is incentive compatible for the broker. The third constraint ensures the investor will repay any debt incurred in the previous OTC round in the following subperiod, rather than default and forfeit a fraction κ of his post-trade equity holding.

With probability $\alpha(1-\delta)$, the investor is only able to purchase goods with their own money and their post-trade allocation is

$$(\hat{q}_t(a_t^m), \hat{a}_t^m(a_t^m)) = \underset{(\hat{q}_t, \hat{a}_t^m) \in \mathbb{R}_+^2}{\arg\max}\ u(\hat{q}_t) + W_t^I(\hat{a}_t^m, \mathbf{0}) \tag{2.3}$$

$$\text{s.t. } \hat{a}_t^m + p_t^q \hat{q}_t \leq a_t^m,$$

with $\hat{a}_t^b(\mathbf{a}_t) = 0$ and where $\mathbf{0} = (0, 0, 0)$. The investor in this case still receives the divided on their assets, yet it does not impact on the investor's decision for consumption goods. Again, the investor's bond holding is fixed at its beginning-of-period value of 0 since he has no access to the bond market.

With probability $(1-\alpha)(1-\delta)$, the investor can simultaneously purchase goods in the goods market and bargain over a bond-market trade with a broker. In this case, the bargaining outcome consists of an intermediation fee to the broker, $k_t(\mathbf{a}_t)$, and a post-trade allocation $(\bar{q}_t(\mathbf{a}_t), \bar{a}_t^m(\mathbf{a}_t), \bar{a}_t^b(\mathbf{a}_t))$, that solves

$$\max_{(\bar{q}_t, \bar{a}_t^m, \bar{a}_t^b, k_t) \in \mathbb{R}_+^2 \times \mathbb{R} \times \mathbb{R}_+} \{u(\bar{q}_t) + W_t^I(\bar{a}_t^m, 0, \bar{a}_t^b, k_t) - u[\hat{q}(a_t^m)] - W_t^I(\hat{a}_t^m(a_t^m), \mathbf{0})\}^\theta k_t^{1-\theta} \tag{2.4}$$

$$\text{s.t. } \bar{a}_t^m + p_t^q \bar{q}_t + p_t^{qb} \bar{a}_t^b \leq a_t^m$$

$$u[\hat{q}(a_t^m)] + W_t^I(\hat{a}(a_t^m), \mathbf{0}) \leq u(\bar{q}_t) + W_t^I(\bar{a}_t^m, 0, \bar{a}_t^b, k_t)$$

$$W_t^I(\bar{a}_t^m, (1-\lambda)a_t^s, 0, k_t) \leq W_t^I(\bar{a}_t^m, a_t^s, 0, k_t),$$

Again, notice that if the investor and the broker were unable to reach an agreement, the investor can still purchase goods in the goods market at the terms specified by (2.3). Therefore (2.3) is the investor's outside option in his bargaining problem with the broker. The investor's gain from trade corresponding to an outcome $(\bar{q}_t, \bar{a}_t^m, k_t)$ consists of the payoff $u(\bar{q}_t) + W_t^I(\bar{a}_t^m, 0, \bar{a}_t^b, k_t)$, minus the investor's outside option, $u[\hat{q}_t(a_t^m)] + W_t^I(\hat{a}_t^m(a_t^m), \mathbf{0})$, i.e., the payoff the investor achieves in (2.3). The first constraint on (2.4) is the budget constraint the investor faces in the OTC round when he is able to purchase goods and trade in the bond market simultaneously. The second constraint ensures the trade is incentive compatible for the investor, and $k_t \in \mathbb{R}_+$ ensures the trade is incentive compatible for the broker. The third constraint ensures the investor will prefer to repay in the following subperiod any debt incurred in the previous OTC round, rather than default and forfeit a fraction λ of his post-trade equity holding.

Let $V_t^I(\mathbf{a}_t, \varepsilon)$ denote the maximum expected discounted payoff of an investor with valuation ε and portfolio $\mathbf{a}_t = (a_t^m, a_t^s)$ at the beginning of the OTC round of period t. Define $\boldsymbol{\phi}_t \equiv (\phi_t^m, \phi_t^s)$, where ϕ_t^m denotes the real price of money, and ϕ_t^s the real *ex dividend* price of equity in the second subperiod of period t (both expressed in terms of the general good). In the second subperiod, the investor chooses consumption of the homogeneous good c_t, the utility cost of production h_t, and the end-of-subperiod portfolio $\tilde{\mathbf{a}}_{t+1} \equiv (\tilde{a}_{t+1}^m, \tilde{a}_{t+1}^s)$, recognising that at the end of the second subperiod a fraction $1-\eta$ of equity is rendered unproductive and replaced, i.e., $\mathbf{a}_{t+1} = (\tilde{a}_{t+1}^m, \eta\tilde{a}_{t+1}^s + (1-\eta)A^s)$. Therefore, we have,

$$W_t^I(\mathbf{a}_t, a_t^b, k_t) = \max_{(c_t, h_t, \tilde{\mathbf{a}}_{t+1}) \in \mathbb{R}_+^4} \left[c_t - h_t + \beta \mathbb{E}_t \int V_{t+1}^I(\mathbf{a}_{t+1}, \varepsilon) dG(\varepsilon) \right] \tag{2.5}$$

$$\text{s.t. } c_t + \boldsymbol{\phi}_t \tilde{\mathbf{a}}_{t+1} \leq h_t + \boldsymbol{\phi}_t \mathbf{a}_t + a_t^b - k_t + T_t,$$

where $\mathbb{E}_t$ is the conditional expectation over the next-period realisation of the dividend, $\boldsymbol{\phi}_t \mathbf{a}_t$

denotes the dot product of $\boldsymbol{\phi}$ and $\mathbf{a}_t$ and $T_t \in \mathbb{R}$ is the real value of the time-t lump-sum monetary transfer.

The value function of an investor who enters the OTC round of period t with portfolio $\mathbf{a}_t$ and valuation ε is

$$\begin{aligned} V_t^I(\mathbf{a}_t, \varepsilon) &= \alpha\delta[\varepsilon y_t \hat{a}_t^s(\mathbf{a}_t, \varepsilon) + W_t^I(\hat{\mathbf{a}}_t(\mathbf{a}_t, \varepsilon), 0)] \\ &+ (1-\alpha)\delta[\varepsilon y_t \bar{a}_t^s(\mathbf{a}_t, \varepsilon) + W_t^I(\bar{\mathbf{a}}_t(\mathbf{a}_t, \varepsilon), k_t(\mathbf{a}_t, \varepsilon))] \\ &+ \alpha(1-\delta)\{u[\hat{q}_t(a_t^m)] + \varepsilon y_t a_t^s + W_t^I(\hat{a}_t^m(a_t^m), a_t^s, \mathbf{0})\} \\ &+ (1-\alpha)(1-\delta)\{u[\bar{q}_t(\mathbf{a}_t)] + \varepsilon y_t a_t^s + W_t^I(\bar{a}_t^m(\mathbf{a}_t), a_t^s, \bar{a}_t^b(\mathbf{a}_t), k_t(\mathbf{a}_t))\}. \end{aligned} \tag{2.6}$$

With probability $1-\delta$ producers enter the OTC round and produce goods. Once producers have produced the goods, they find themselves in one of two trading situations in the OTC round. With probability α, a producer is only able to trade goods and money competitively in the goods market. With probability $1-\alpha$, a producer simultaneously trades money and goods in the goods market, and bargains with a broker over the quantities of bonds and money that the broker trades in the bond market on behalf of the investor, as well as over the broker's intermediation fee. Again, in this case, the outcome of the negotiation with the broker is determined by Nash bargaining with producer bargaining power θ. The broker's fee is expressed in terms of the general good and paid by the producer in the second subperiod. An individual producer holds $a_t^m \in \mathbb{R}_+$ units of money and $q_t \in \mathbb{R}_+$ units of production of the good.

Let $W_t^P(a_t^m, a_t^g, a_t^b, k_t)$ denote the maximum expected discounted payoff at the beginning of the second subperiod of period t of a producer who is holding (a_t^m, a_t^g, a_t^b) and has to pay a fee k_t. Consider a producer who enters period t with (pre-trade) money and production (a_t^m, q_t). With probability α, the producer is only able to trade in the goods market and his post-trade allocation is $(\hat{q}_t(a_t^m, q_t), \hat{a}_t^m(a_t^m, q_t), \hat{a}_t^b(a_t^m, q_t))$ with $\hat{a}_t^b(a_t^m, q_t) = 0$. Note that $\hat{a}_t^g(a_t^m, q_t) \equiv (q_t - \hat{q}_t(a_t^m, q_t))\underline{\varrho}$ is the remaining unsold inventory of the producer and

$$(\hat{q}_t(a_t^m, q_t), \hat{a}_t^m(a_t^m, q_t)) = \underset{(\hat{q}_t, \hat{a}_t^m) \in \mathbb{R}_+^2}{\arg\max}\; W_t^P(\hat{a}_t^m, \hat{a}_t^g, \mathbf{0}) \tag{2.7}$$

$$\text{s.t. } \hat{a}_t^m \leq a_t^m + p_t^q \hat{q}_t$$
$$\hat{q}_t \leq q_t.$$

In this case, the producer's bond holding is fixed at its beginning-of-period value of 0 since he has no access to the bond market.

With probability $1-\alpha$, the producer can simultaneously trade in the goods market and bargain over a bond-market trade with a broker. In this case, the bargaining outcome consists of an intermediation fee to the broker, $k_t(a_t^m, q_t)$, and a post-trade allocation, $(\bar{a}_t^m(a_t^m, q_t), \bar{q}_t(a_t^m))$ with bond holdings $\bar{a}_t^b(a_t^m, q_t)$, that solves

$$\max_{(\bar{a}_t^m, \bar{q}_t, \bar{a}_t^b) \in \mathbb{R}_+^2 \times \mathbb{R}} [W_t^P(\bar{a}_t^m, \bar{a}_t^g, \bar{a}_t^b, k_t) - W_t^P(\hat{a}_t^m(a_t^m, q_t), \hat{a}_t^g(a_t^m, q_t), \mathbf{0})] \tag{2.8}$$

$$\text{s.t. } \bar{a}_t^m + p_t^{qb} \bar{a}_t^b \leq a_t^m + p_t^q \bar{q}_t$$
$$\bar{q}_t \leq q_t$$
$$W_t^P(\hat{a}_t^m(a_t^m, q_t), \hat{a}_t^g(a_t^m, q_t), \mathbf{0}) \leq W_t^P(\bar{a}_t^m, \bar{a}_t^g, \bar{a}_t^b, k_t).$$

Notice that if the producer and the broker were unable to reach an agreement, the producer can still trade goods and money in the goods market at the terms specified by (2.7). Hence, the outcome (2.7) is the producer's outside option in his bargaining problem with the broker. The producer's gain from trade consists of the continuation payoff $W_t^P(\bar{a}_t^m, \bar{a}_t^g, \bar{a}_t^b, k_t)$, minus the producer's outside option, $W_t^P(\hat{a}_t^m(a_t^m, q_t), \hat{a}_t^g(a_t^m, q_t), \mathbf{0})$, namely the payoff the producer achieves in (2.7). The first constraint on (2.8) is the budget constraint the producer faces in the OTC round when he is able to trade in the goods and bond market simultaneously. The second constraint ensures only goods that are produced can be sold. The third constraint ensures that trade is incentive compatible for the producer, and the restriction $k_t \in \mathbb{R}_+$ ensures the trade is incentive compatible for the broker.

Let $V_t^P(a_t^m)$ denote the maximum expected discounted payoff of a producer with money a_t^m

at the beginning of the OTC round of period t. Then,

$$W_t^P(a_t^m, a_t^g, a_t^b, k_t) = \max_{(c_t, h_t, \tilde{a}_{t+1}^m) \in \mathbb{R}_+^3} [c_t - h_t + \beta V_{t+1}^P(\tilde{a}_{t+1}^m)] \tag{2.9}$$

$$\text{s.t. } c_t + \phi_t^m \tilde{a}_{t+1}^m \leq h_t + \phi_t^m a_t^m + a_t^b + a_t^g - k_t + T_t,$$

where $T_t \in \mathbb{R}$ is again the real value of the time-t lump-sum monetary transfer.

The value function of a producer who enters the OTC round of period t with money holdings a_t^m is,

$$\begin{aligned} V_t^P(a_t^m) = (1-\delta) \max_{q_t \in \mathbb{R}_+} [&-\varrho q_t + \alpha W_t^P(\hat{a}_t^m(a_t^m, q_t), \hat{a}_t^g(a_t^m, q_t), \mathbf{0}) \\ &+ (1-\alpha) W_t^P(\bar{a}_t^m(a_t^m, q_t), \bar{a}_t^g(a_t^m, q_t), \bar{a}_t^b(a_t^m, q_t), k_t(a_t^m, q_t))] + \delta W_t^P(\mathbf{0}). \end{aligned} \tag{2.10}$$

Similarly, let $W_t^B(k_t)$ denote the maximum expected discounted payoff at the beginning of the second subperiod of period t of a broker who has earned fee k_t. Then,

$$W_t^B(k_t) = k_t + \beta \mathbb{E}_t V_{t+1}^B, \tag{2.11}$$

where V_{t+1}^B denotes the maximum expected discounted payoff of a broker at the beginning of the OTC round of period $t+1$.

The value function of a broker who enters the OTC round of period t is,

$$\begin{aligned} V_t^B = (1-\alpha)\delta \int W_t^B \left[k_t^{is}(\tilde{\mathbf{a}}_t, \varepsilon)\right] dH_{It}(\tilde{\mathbf{a}}_t, \varepsilon) + (1-\alpha)(1-\delta) \int W_t^B[k_t^{iq}(\tilde{\mathbf{a}}_t)] dF_{It}(\tilde{\mathbf{a}}_t) \\ + (1-\alpha)(1-\delta) \int W_t^B[k_t^{pq}(\tilde{a}_t^m, q_t(\tilde{a}_t^m))] dF_{Pt}(\tilde{a}_t^m) + (1-2\alpha) W_t^B(0), \end{aligned} \tag{2.12}$$

where H_{It} is the joint cumulative distribution function over the portfolios and valuations of the investors that contact brokers in the OTC round of trade of period t. F_{It} is the cumulative distribution function over portfolios $\mathbf{a}_t = (a_t^m, a_t^s)$ held by investors at the beginning of the OTC round of period t and F_{Pt} is the cumulative distribution function over money holdings a_t^m held by producers at the beginning of the OTC round of period t.

2.3 Equilibrium

Let A^m_{It} and A^s_{It} denote the quantities of money and equity shares, respectively, held by investors at the beginning of the OTC round of period t (after depreciated production units have been replaced). Also let A^m_{Pt} denote the quantity of money held by producers at the beginning of the OTC round of period t. That is $A^m_{It} = N_I \int a^m_t dF_{It}(\mathbf{a}_t)$, $A^s_{It} = N_I \int a^s_t dF_{It}(\mathbf{a}_t)$ and $A^m_{Pt} = N_P \int a^m_t dF_{Pt}(a^m_t)$.

Let $\tilde{A}^m_{It+1}$ and $\tilde{A}^s_{It+1}$ denote the total quantities of money and shares held by all investors at the end of period t, i.e., $\tilde{A}^k_{It+1} = \int_{\mathcal{I}} \tilde{a}^k_{it+1} di$ for $k \in \{m, s\}$, where $\tilde{a}^k_{it+1}$ denotes the quantity of asset k held at the end of period t by the individual investor identified with the point $i \in \mathcal{I}$. Let $\tilde{A}_{Pt+1}$ denote the total quantity of money held by all producers at the end of period t, i.e., $\tilde{A}^m_{Pt+1} = \int_{\mathcal{P}} \tilde{a}^m_{pt+1} dp$, where $\tilde{a}^m_{pt+1}$ denotes the quantity of money held at the end of period t by the individual producer identified with the point $p \in \mathcal{P}$. Thus $A^m_{It+1} = \tilde{A}^m_{It+1}$, $A^s_{It+1} = \eta \tilde{A}^s_{It+1} + (1-\eta) A^s$ and $A^m_{Pt+1} = \tilde{A}^m_{Pt+1}$.

In the equity market of the OTC round of period t, for asset $k \in \{b, m, s\}$, let $\bar{A}^k_{It} = (1-\alpha)\delta N_I \int \bar{a}^k_t(\mathbf{a}_t, \varepsilon) dH^s_{It}(\mathbf{a}_t, \varepsilon)$, and $\hat{A}^k_{It} = \alpha\delta N_I \int \hat{a}^k_t(\mathbf{a}_t, \varepsilon) dH^s_{It}(\mathbf{a}_t, \varepsilon)$, with $\hat{a}^b_t(\mathbf{a}_t, \varepsilon) = 0$.

In the goods market of the OTC round of period t, for investors and for asset $k \in \{b, m\}$, let $\bar{A}^k_{It} = (1-\alpha)(1-\delta) N_I \int \bar{a}^k_t(\mathbf{a}_t) dF_{It}(\mathbf{a}_t)$, $\bar{Q}_{It} = (1-\alpha)(1-\delta) N_I \int \bar{q}_t(\mathbf{a}_t) dF_{It}(\mathbf{a}_t)$, $\hat{A}^m_{It} = \alpha(1-\delta) N_I \int \hat{a}^m_t(a^m_t) dF_{It}(\mathbf{a}_t)$ and $\hat{Q}_{It} = \alpha(1-\delta) N_I \int \hat{q}_t(a^m_t) dF_{It}(\mathbf{a}_t)$. In the goods market of the OTC round of perioid t, for producers and for asset $k \in \{b, m\}$, let $\bar{A}^k_{Pt} = (1-\alpha)(1-\delta) N_P \int \bar{a}^k_t(a^m_t, q_t(a^m_t)) dF_{Pt}(a^m_t)$, $\bar{Q}_{Pt} = (1-\alpha)(1-\delta) N_P \int \bar{q}_t(a^m_t, q_t(a^m_t)) dF_{Pt}(a^m_t)$, $\hat{A}^m_{Pt} = \alpha(1-\delta) N_P \int \hat{a}^m_t(a^m_t) dF_{Pt}(a^m_t)$ and $\hat{Q}_{Pt} = \alpha(1-\delta) N_P \int \hat{q}_t(a^m_t) dF_{Pt}(a^m_t)$.

We are now ready to define equilibrium.

Definition 1. *An* equilibrium *is a sequence of prices, $\{p^s_t, p^q_t, p^{sb}_t, p^{qb}_t \phi^m_t, \phi^s_t\}^\infty_{t=0}$, portfolio allocations and fees for investors in the equity market of period t, $\{\hat{\mathbf{a}}_t(.), \bar{\mathbf{a}}_t(.), k_t(.)\}^\infty_{t=0}$, allocations and fees for investors in the goods market period t, $\{\hat{a}^m_t(.), \hat{q}_t(.)\}^\infty_{t=0}$ and $\{\bar{a}^m_t(.), \bar{a}^b_t(.), \bar{q}_t(.), k_t(.)\}^\infty_{t=0}$, end-of-day portfolios for investors, $\{\tilde{\mathbf{a}}_{t+1}\}^\infty_{t=0}$, allocations and fees for producers in the goods market of the OTC round of period t, $\{\hat{a}^m_t(.), \hat{q}_t(.)\}^\infty_{t=0}$ and $\{\bar{a}^m_t(.), \bar{a}^b_t(.), \bar{q}_t(.), k_t(.)\}^\infty_{t=0}$, end-of-day money holdings for producers $\{\tilde{a}^m_{t+1}\}^\infty_{t=0}$ such that for all t:*

(i) The portfolios and fees for investors in the equity market of the OTC round of period t solve (2.1) and (2.2).

(ii) The allocations and fees for investors in the goods market of the OTC round of period t solve (2.3) and (2.4).

(iii) Taking prices and the bargaining protocol as given, the end-of-period portfolios for investors solve (2.5).

(iv) The allocations and fees for producers in the goods market of the OTC round of period t solve (2.7) and (2.8).

(v) Taking prices and the bargaining protocol as given, the end-of-period portfolios for producers solve (2.9).

(vi) Prices are such that all Walrasian markets clear, i.e., $\tilde{A}^s_{It+1} = A^s$, $\tilde{A}^m_{It+1} = A^m_{t+1}$, $\hat{A}^s_{It} + \bar{A}^s_{It} = \delta A^s$, $\hat{Q}_{It} + \bar{Q}_{It} = \hat{Q}_{Pt} + \bar{Q}_{Pt}$, *and* $(\hat{A}^m_{It} + \bar{A}^m_{It} + \hat{A}^m_{Pt} + \bar{A}^m_{Pt} - A^m_t)\mathbb{I}_{\{\phi^m_t > 0\}} = 0$.

An equilibrium is *monetary* if $\phi^m_t > 0$ for all t. The first step toward characterising equilibrium is to find the bargaining outcomes. For any $(y, z) \in \mathbb{R}^2$, it is convenient to define the "indicator correspondence" $\chi : \mathbb{R}^2 \to [0, 1]$ by

$$\chi(y,z) \begin{cases} = 1 & \text{if } y < z \\ \in [0,1] & \text{if } y = z \\ = 0 & \text{if } z < y \end{cases} \tag{2.13}$$

The following lemma characterises post-trade allocations in the OTC market for an economy with no money.

Lemma 1. *Consider the economy with no money and*

(I) Consider an investor who enters the OTC round of period t and with probability δ, the investor enters the equity market with equity holding a^s_t and valuation ε, and let

$$\varepsilon^n_t \equiv \frac{\bar{\phi}^s_t - \phi^s_t}{y_t}, \tag{2.14}$$

where $\bar{\phi}_t^s$ *denotes the price of an equity share in terms of bonds in the equity market. Then:*

(i) If the investor does not contact a broker, the post-trade equity holding is $\hat{a}_t^s(a_t^s) = a_t^s$*.*

(ii) If the investor contacts a broker, the bargaining problem has a solution only if

$$\kappa < \frac{\bar{\phi}_t^s}{\phi_t^s}, \tag{2.15}$$

the post-trade portfolio is

$$\bar{a}_t^s(a_t^s, \varepsilon) = \chi(\varepsilon_t^n, \varepsilon) \frac{\bar{\phi}_t^s}{\bar{\phi}_t^s - \kappa \phi_t^s} a_t^s \tag{2.16}$$

$$\bar{a}_t^b(a_t^s, \varepsilon) = \bar{\phi}_t^s \left[1 - \chi(\varepsilon_t^n, \varepsilon) \frac{\bar{\phi}_t^s}{\bar{\phi}_t^s - \kappa \phi_t^s} \right] a_t^s, \tag{2.17}$$

and the intermediation fee for the broker is

$$k_t(a_t^s, \varepsilon) = (1 - \theta)(\varepsilon - \varepsilon_t^n) y_t \left[\chi(\varepsilon_t^n, \varepsilon) \frac{\bar{\phi}_t^s}{\bar{\phi}_t^s - \kappa \phi_t^s} \right] a_t^s. \tag{2.18}$$

(II) Consider an investor who enters the OTC round of period t *and with probability* $1 - \delta$*, the investor enters the goods market with equity holding* a_t^s*. Then:*

(i) If the investor does not contact a broker, the post-trade goods allocation is $\hat{q}_t(a_t^s) = 0$

(ii) If the investor contacts a broker, the bargaining problem post-trade allocation is

$$\bar{q}_t(a_t^s) = \min \left\{ \mathrm{D}(\varphi_t^q), \frac{\lambda \phi_t^s a_t^s}{\varphi_t^q} \right\} \tag{2.19}$$

$$\bar{a}_t^b(a_t^s) = -\varphi_t^q \bar{q}_t(a_t^s), \tag{2.20}$$

where for any $\varphi \in \mathbb{R}_+$, $\mathrm{D}(\varphi) \equiv u'^{-1}(\varphi)$*, and the intermediation fee for the broker is*

$$k_t(a_t^s) = (1 - \theta)\{u[\bar{q}_t(a_t^s)] - \varphi_t^q \bar{q}_t(a_t^s)\}, \tag{2.21}$$

where φ_t^q *denotes the price of goods in terms of bonds in the goods market.*

(III) Consider a producer who enters the OTC round of period t *and with probability* $1 - \delta$*, the*

producer enters the goods market after producing q_t. Then:

(i) If the producer does not contact a broker, the post-trade goods allocation is $\hat{q}_t = 0$.

(ii) If the producer does contact a broker, the bargaining problem post-trade allocation is

$$\bar{q}_t(q_t) = \chi(\underline{\varrho}, \varphi_t^q)q_t \tag{2.22}$$

$$\bar{a}_t^b(q_t) = \varphi_t^q \chi(\underline{\varrho}, \varphi_t^q)q_t, \tag{2.23}$$

and the intermediation fee for the broker is

$$k_t(q_t) = (1-\theta)(\varphi_t^q - \underline{\varrho})\chi(\underline{\varrho}, \varphi_t^q)q_t. \tag{2.24}$$

Section (I) of Lemma 1 corresponds to Lemma 1 in Lagos and Zhang (2019b) and covers the allocation of an investor who enters the equity market in the OTC round of period t. Part (i) of section (I) of Lemma 1 states that in a nonmonetary economy, investors who enter the equity market but fail to contact a broker are unable to trade in the OTC round.[9] Part (ii) of section (I) states that an investor who enters the equity market and contacts a broker can both buy and sell equity, as well as take long or short positions in bonds. From (2.16) and (2.17), if $\varepsilon < \varepsilon_t^n$, the investor sells all his equity for bonds. Conversely, if $\varepsilon_t^n < \varepsilon$, the investor shorts the bond in order to take a long position in equity. Condition (2.18) indicates the broker earns a fee on these transactions as long as $\varepsilon \neq \varepsilon_t^n$.

Section (II) of Lemma 1 produces results reminiscent of Lagos and Zhang (2022) and covers the allocation of an investor who enters the goods market in the OTC round of period t. Part (i) of section (II) of Lemma 1 states that in a nonmonetary economy, investors who enter the goods market who do not contact a broker are unable to purchase goods in the OTC round.[10] Part (ii) of section (II) of Lemma 1 states that an investor who enters the goods market who contacts a broker can purchase goods by taking a short position in the bond.[11] From (2.19)

[9] In a nonmonetary economy, investors entering the equity market without access to credit cannot buy equity, as they lack a means of payment, nor can they sell equity, since they have no means of receiving payment from buyers.

[10] In a nonmonetary economy, investors who enter the goods market with no access to credit have no medium of exchange to facilitate a purchase.

[11] Notice that in equilibrium, investors who enter the goods market and who make contact with a broker do not take long positions in the bond.

the investor is constrained by the collateral constraint and can only purchase up to a certain amount of the good. Condition (2.20) states that the size of the short position of the bond is equal to the value of the purchased good. Condition (2.21) indicates the broker earns a fee on these transactions.

Section (III) of Lemma 1 again produces results reminiscent of Lagos and Zhang (2022) covers the allocation of a producer who enters the goods market of the OTC round of period t. Part (i) of section (III) of Lemma 1 states that in a nonmonetary economy, producers who do not contact a broker are unable to trade in the OTC round.[12] Part (ii) of section (III) states that a producer who contacts a broker can only sell goods by taking a long position in the bond.[13] From (2.22) if $\underline{\varrho} < \varphi_t^q$, the producer sells all of their goods inventory in the goods market. Conversely, if $\varphi_t^q < \underline{\varrho}$, the producer retains all of their inventory and does not sell any goods in the goods market. Condition (2.23) states that the size of the long position of the bond is equal to the value of the sold goods. Lastly, condition (2.24) indicates the broker earns a fee on these transactions as long as $\underline{\varrho} \leq \varphi_t^q$.

Similar to Lagos and Zhang (2019b), notice that in the nonmonetary benchmark, trade can occur in both the equity and goods markets of the OTC round, but only for investors and producers with access to credit. Although the equity, goods, and bond markets are segmented (only equity trades in the equity market, only goods in the goods market, and only bonds in the bond market), agents who can access either the equity or goods market and the bond market can trade in both simultaneously by using bonds as the settlement asset. Hence, such investors can effectively exchange securities at a relative price of $\bar{\phi}_t^s$ bonds per equity share or purchase goods at φ_t^q bonds per unit. Likewise, producers with access to both the goods and bond markets can sell goods at the price φ_t^q bonds per unit.

As in Lagos and Zhang (2019b), since each bond in the equity market entitles its holder to 1 unit of the second subperiod consumption good, $\bar{\phi}_t^s$ can be interpreted as the real cum-dividend OTC price of an equity share, expressed in units of the second subperiod good. In the nonmonetary economy, bonds traded in the equity market carry an implicit real interest rate

[12] In a nonmonetary economy, producers with no access to credit cannot sell goods because they have no way to collect payment from buyers.

[13] Notice that in equilibrium, producers who enter the goods market and make contact with a broker do not take short positions in the bond.

(in units of equity shares), denoted i_t^{sn}. In the OTC round, a credit-access investor can use 1 unit of equity to acquire $\bar{\phi}_t^s$ bonds. These bonds deliver $\bar{\phi}_t^s$ general goods in the next subperiod, when the relative price of general goods in terms of equity shares is $1/\phi_t^s$. Thus, $i_t^{sn} \equiv \bar{\phi}_t^s/\phi_t^s - 1$, or using (2.14),

$$i_t^{sn} = \frac{\varepsilon_t^n y_t}{\phi_t^s}. \tag{2.25}$$

Notice that with (2.25), (2.15) becomes $\kappa < 1 + i_t^{sn}$, so (2.15) can only be violated if the net real interest rate is negative, which will not be the case in equilibrium.[14] The following lemma characterises post-trade allocations in the OTC market for an economy with money.

Lemma 2. *Consider the economy with money and*

(I) Consider an investor who enters the OTC round of period t and with probability δ the investor enters the equity market with equity $\mathbf{a}_t$ and valuation ε, and let

$$\varepsilon_t^* \equiv (p_t^s \phi_t^m - \phi_t^s)\frac{1}{y_t} \tag{2.26}$$

$$\varepsilon_t^{**} \equiv \varepsilon_t^* + (1 - p_t^{sb}\phi_t^m)\left[\mathbb{I}_{\{p_t^{sb}\phi_t^m<1\}}\frac{p_t^s}{p_t^{sb}} + \mathbb{I}_{\{1<\kappa\phi_t^s\}}\right]\frac{1}{y_t}. \tag{2.27}$$

Then:

(i) If the investor does not contact a broker, the post-trade portfolio is

$$\hat{a}_t^m(\mathbf{a}_t, \varepsilon) = [1 - \chi(\varepsilon_t^*, \varepsilon)](a_t^m + p_t^s a_t^s) \tag{2.28}$$

$$\hat{a}_t^s(\mathbf{a}_t, \varepsilon) = \chi(\varepsilon_t^*, \varepsilon)\frac{1}{p_t^s}(a_t^m + p_t^s a_t^s). \tag{2.29}$$

[14] As in Lagos and Zhang (2019b), to justify why (2.15) is necessary for bargaining outcomes to be well defined, consider the budget and collateral constraints of an investor who contacts a broker, i.e., $\bar{\phi}_t^s \bar{a}_t^s + \bar{a}_t^b = \bar{\phi}_t^s a_t^s$, and $-\kappa\phi_t^s \bar{a}_t^s \leq \bar{a}_t^b$. These two conditions imply the borrowing constraint $-\kappa\phi_t^s a_t^s \leq (1 - \kappa\phi_t^s/\bar{\phi}_t^s)\bar{a}_t^b$. This constraint would be slack for all $\bar{a}_t^b < 0$ if (2.15) is violated, meaning that an investor with $\varepsilon > \varepsilon_t^n$ would be able (and willing) to take an infinitely long position in the stock. If (2.15) is violated, then an investor who starts with no assets can sell b collateralised bonds to purchase $b/\bar{\phi}_t^s$ equity shares and this purchase would leave the investor's borrowing constraint nonbinding, since $b < \kappa\phi_t^s b/\bar{\phi}_t^s$.

(ii) If the investor contacts a broker, the bargaining problem has a solution if

$$\kappa < \frac{p_t^s}{p_t^{sb}\phi_t^s}, \tag{2.30}$$

and in that case the post-trade portfolio is

$$\bar{a}_t^m(\mathbf{a}_t,\varepsilon) = \left\{\mathbb{I}_{\{1<p_t^{sb}\phi_t^m\}}[1-\chi(\varepsilon_t^{**},\varepsilon)] + \mathbb{I}_{\{p_t^{sb}\phi_t^m=1\}}\mathbb{I}_{\{\varepsilon<\varepsilon_t^{**}\}}[1-\chi(p_t^{sb}\phi_t^m,1)]\right\}(a_t^m + p_t^s a_t^s)$$
$$+ \mathbb{I}_{\{p_t^{sb}\phi_t^m=1\}}\mathbb{I}_{\{\varepsilon=\varepsilon_t^{**}\}}\tilde{a}_t^m \tag{2.31}$$
$$\bar{a}_t^s(\mathbf{a}_t,\varepsilon) = \left\{\mathbb{I}_{\{p_t^{sb}\phi_t^m=1\}}\mathbb{I}_{\{\varepsilon_t^{**}<\varepsilon\}} + [1-\mathbb{I}_{\{p_t^{sb}\phi_t^m=1\}}]\chi(\varepsilon_t^{**},\varepsilon)\right\}\frac{a_t^m + p_t^s a_t^s}{p_t^s - \kappa p_t^{sb}\phi_t^s}$$
$$+ \mathbb{I}_{\{p_t^{sb}\phi_t^m=1\}}\mathbb{I}_{\{\varepsilon=\varepsilon_t^{**}\}}\tilde{a}_t^s \tag{2.32}$$
$$\bar{a}_t^b(\mathbf{a}_t,\varepsilon) = -\frac{1}{p_t^{sb}}\{[\bar{a}_t^m(\mathbf{a}_t,\varepsilon) - a_t^m] + p_t^s[\bar{a}_t^s(\mathbf{a}_t,\varepsilon) - a_t^s]\}, \tag{2.33}$$

where

$$(\tilde{a}_t^m, \tilde{a}_t^s) \in \left\{\mathbb{R}_+^2 : \tilde{a}_t^m + (p_t^s - \kappa p_t^{sb}\phi_t^s)\tilde{a}_t^s = a_t^m + p_t^s\right\},$$

and the intermediation fee is

$$k_t^{is}(\mathbf{a}_t,\varepsilon) = (1-\theta)\left\{(\varepsilon y_t + \phi_t^s)[\bar{a}_t^s(\mathbf{a}_t,\varepsilon) - \hat{a}_t^s(\mathbf{a}_t,\varepsilon)] + \phi_t^m[\bar{a}_t^m(\mathbf{a}_t,\varepsilon) - \hat{a}_t^m(\mathbf{a}_t,\varepsilon)] + \bar{a}_t^b(\mathbf{a}_t,\varepsilon)\right\}. \tag{2.34}$$

(II) Consider an investor who enters the OTC round of period t and with probability $1-\delta$ the investor enters the goods market with portfolio $\mathbf{a}_t$. Then:

(i) If the investor does not contact a broker, the post-trade allocation is

$$\hat{q}_t(a_t^m) = \min\left\{\mathrm{D}(\varphi_t^m), \frac{a_t^m}{p_t^q}\right\} \tag{2.35}$$
$$\hat{a}_t^m(a_t^m) = a_t^m - p_t^q\hat{q}_t(a_t^m). \tag{2.36}$$

(ii) If the investor contacts a broker, the bargaining problem post-trade allocation is

$$\bar{a}_t^m(\mathbf{a}_t) = \mathbb{I}_{\{1 \leq p_t^{qb}\phi_t^m\}}\tilde{a}_t^m \tag{2.37}$$

$$\bar{q}_t(\mathbf{a}_t) = \mathbb{I}_{\{1 < p_t^{qb}\phi_t^m\}}\mathrm{D}(\varphi_t^m) + \mathbb{I}_{\{p_t^{qb}\phi_t^m \leq 1\}} \min\left\{\mathrm{D}(\varphi_t^q), \frac{a_t^m + \lambda p_t^{qb}\phi_t^s a_t^s}{p_t^q}\right\} \tag{2.38}$$

$$\bar{a}_t^b(\mathbf{a}_t) = -\frac{1}{p_t^{qb}}\mathbb{I}_{\{p_t^{qb}\phi_t^m < 1\}}\{[\bar{a}_t^m(\mathbf{a}_t) - a_t^m] + p_t^q\bar{q}_t(\mathbf{a}_t)\} + \mathbb{I}_{\{1 = p_t^{qb}\phi_t^m\}}\tilde{a}_t^b, \tag{2.39}$$

where

$$(\tilde{a}_t^m, \tilde{a}_t^b) \in \left\{\mathbb{R}_+^2 : \tilde{a}_t^m + p_t^{qb}\tilde{a}_t^b \leq a_t^m + \lambda\phi_t^s a_t^s - p_t^q\mathrm{D}(\varphi_t^q)\right\},$$

and the intermediation fee is

$$k_t^{iq}(\mathbf{a}_t) = (1-\theta)\left\{u[\bar{q}_t(\mathbf{a}_t)] - u[\hat{q}_t(a_t^m)] + \phi_t^m[\bar{a}_t^m(\mathbf{a}_t) - \hat{a}_t^m(a_t^m)] + \bar{a}_t^b(\mathbf{a}_t)\right\}. \tag{2.40}$$

(III) Consider a producer who enters the OTC round of period t and with probability $1-\delta$ the producer enters the goods market after producing q_t and money holding a_t^m. Then:

(i) If the producer does not contact a broker, the post-trade allocation is

$$\hat{q}_t(a_t^m, q_t) = \chi(\underline{\varrho}, \varphi_t^m)q_t \tag{2.41}$$

$$\hat{a}_t^m(a_t^m, q_t) = a_t^m + p_t^q\chi(\underline{\varrho}, \varphi_t^m)q_t. \tag{2.42}$$

(ii) If the producer contacts a broker, the post-trade allocation is

$$\bar{a}_t^m(a_t^m, q_t)\begin{cases} \to \infty & \textit{if } 1 < p_t^{qb}\phi_t^m \\ \in [0,\infty) & \textit{if } p_t^{qb}\phi_t^m = 1 \\ = 0 & \textit{if } p_t^{qb}\phi_t^m < 1 \end{cases} \tag{2.43}$$

$$\bar{q}_t(a_t^m, q_t) = \chi(\underline{\varrho}, \varphi_t^q)q_t \tag{2.44}$$

$$\bar{a}_t^b(a_t^m, q_t) = \frac{1}{p_t^b}[a_t^m + p_t^q\bar{q}_t(a_t^m, q_t) - \bar{a}_t^m(a_t^m, q_t)], \tag{2.45}$$

and the intermediation fee is

$$k_t^{pq}(a_t^m, q_t) = (1-\theta)\left\{ i_t^{qm}\phi_t^m a_t^m + [(\varphi_t^q - \underline{\varrho})\chi(\underline{\varrho}, \varphi_t^q) - (\varphi_t^m - \underline{\varrho})\chi(\underline{\varrho}, \varphi_t^m)]q_t \right\}. \quad (2.46)$$

Section (I) of Lemma 2 corresponds to Lemma 2 in Lagos and Zhang (2019b) and covers the allocation of an investor who enters the equity market in the OTC round of period t. Part (i) of section (I) states that an individual investor with valuation ε who does not contact a broker in the OTC trading round uses all of their money balances to purchase equity when $\varepsilon_t^* < \varepsilon$, and sells all of their equity for money when $\varepsilon < \varepsilon_t^*$. Part (ii) describes the bargaining outcome and post-trade portfolio of an investor who contacts a broker and can therefore trade both bonds and equity simultaneously. For ease of interpretation, suppose $p_t^{sb}\phi_t^m < 1$, which holds in equilibrium. In this case, if $\varepsilon_t^{**} < \varepsilon$, the investor short-sells the bond up to the collateral constraint and uses the proceeds, together with all pre-trade money balances, to purchase equity. Conversely, if $\varepsilon < \varepsilon_t^{**}$, the investor sells all pre-trade equity holdings and uses the proceeds, along with all pre-trade money balances, to purchase bonds. The broker charges a fee whenever the investor earns a positive gain from trade.

Section (II) of Lemma 2 produces similar results to Lagos and Zhang (2022) adapted to our setting and covers the allocation of an investor who enters the goods market in the OTC round of period t. Part (i) of section (II) states that an investor who enters the goods market who does not contact a broker in the OTC round of trade chooses to consume subject to the money constraint. Part (ii) of section (II) of Lemma 2 states that an investor who enters the goods market and contacts a broker can simultaneously trade goods and bonds. To offer an interpretation of the post-trade allocation suppose again that $p_t^{sb}\phi_t^m < 1$. In this case, if the investor is not liquidity constrained then they consume the optimal quantity. If, on the other hand the investor is liquidity constrained then the consumer will take on a short position of the bond as allowed by the collateral constraint, and use the proceeds from the short sale along with all his pre-trade money balance to buy goods. Condition (2.40) states that the broker earns a fee when the investor has positive gain from trade.

Section (III) of Lemma 2 again produces similar results to Lagos and Zhang (2022) and covers the allocation of a producer who enters the goods market in the OTC round of period t. Part (i) of section (III) states that a producer who enters the goods market who does not contact a broker in the OTC round of trade can sell goods for money. From (2.41) if $\underline{\varrho} < \varphi_t^m$, the producer sells all of their goods inventory in exchange for money. Conversely, if $\varphi_t^m < \underline{\varrho}$, the producer retains all of their inventory and does not sell any goods in the goods market. Part (ii) of section (III) of Lemma 2 states that a producer who enters the goods market and contacts a broker can simultaneously trade goods and bonds. To offer an interpretation of the post-trade allocation suppose again that $p_t^{qb}\phi_t^m < 1$. In this case, the producer sells all of his inventory of goods and uses the proceeds and money holdings to take a long position in bond. Condition (2.46) states that the broker earns a fee when the producer has positive gain from trade.

Along the same lines as Lagos and Zhang (2019b), notice that with 1 unit of money an investor can buy $\frac{1}{p_t^{kb}}$ bonds, which yields $\frac{1}{p_t^{kb}}$ general goods in the following subperiod. This is equivalent to $\frac{1}{p_t^{kb}\phi_t^m}$ dollars, for $k \in \{s, q\}$ where s represents the price of the bond in the equity market, and q represents the price of the bond in the goods market. The competitive nominal rates on collateralised loans in the equity and goods market, respectively, are

$$i_t^{sm} \equiv \frac{1}{p_t^{sb}\phi_t^m} - 1, \quad \text{and} \quad i_t^{qm} \equiv \frac{1}{p_t^{qb}\phi_t^m} - 1. \tag{2.47}$$

As in Lagos and Zhang (2019b), since the loan is repaid within the same period, the corresponding interest rate can be interpreted as a real rate, with both the loan and its repayment measured in units of the general good.[15] Another relevant notion of interest is the real interest rate on bonds measured in terms of equity shares—the same concept that applies in the economy without money. If an investor spends 1 unit of money to purchase bonds, this corresponds to trading $\frac{1}{p_t^s}$ worth of equity shares for $\frac{1}{p_t^{sb}}$ bonds that deliver $\frac{1}{p_t^{sb}}$ general goods in the second subperiod. This is equivalent to $\frac{1}{p_t^{sb}\phi_t^s}$ equity shares. The real rate on a bond (in terms of equity

[15] A bond investment of $\frac{1}{\phi_t^m}$ dollars is equivalent to a bond investment of 1 unit of the general good. The $\frac{1}{\phi_t^m}$ dollars allow to buy $\frac{1}{p_t^{kb}\phi_t^m}$ bonds, which in total yield $\frac{1}{p_t^{kb}\phi_t^m}$ general goods. So the gross real interest in terms of general goods is also $\frac{1}{p_t^{kb}\phi_t^m}$.

shares) is

$$i_t^{ss} \equiv \frac{p_t^s}{p_t^{sb}\phi_t^s} - 1. \tag{2.48}$$

With (2.47), (2.30) becomes $\kappa < 1 + i_t^{ss}$. (2.30) can only be violated if the net real interest is negative, which will not be the case in equilibrium.[16]

We focus our analysis on equilibria with an active bond market, which implies $p_t^{kb}\phi_t^m \leq 1$, i.e., the net nominal interest rate on bonds, i_t^{km}, is nonnegative.[17]

In what follows, we consider recursive equilibria, i.e., equilibria in which real equity prices (and real money balances, in a monetary equilibrium) are time-invariant linear functions of the dividend.

Definition 2. *A* recursive nonmonetary equilibrium (RNE) *is a nonmonetary equilibrium in which real equity prices (general goods per equity share) are time-invariant linear functions of the aggregate dividend, i.e.,* $\phi_t^s = \phi^s y_t$, *and* $\bar{\phi}_t^s = \bar{\phi}^s y_t$ *for some* ϕ^s, $\bar{\phi}^s \in \mathbb{R}_+$.

In a RNE, $\varepsilon_t^n = (\bar{\phi}_t^s - \phi_t^s)\frac{1}{y_t} = \bar{\phi}^s - \phi^s \equiv \varepsilon^n$, and the real interest rate on the bond, i.e., $i_t^{sn} \equiv \bar{\phi}_t^s/\phi_t^s - 1$ as defined in (2.25) is

$$i^{sn} = \frac{\varepsilon^n}{\phi^s}, \tag{2.49}$$

and the price of the first subperiod consumption good in terms of the price of the bond, i.e., $\varphi^q = p_t^q/p_t^{qb}$ is given by

$$\varphi^q \leq \hat{\varphi}^q \equiv \varrho + \frac{1-(1-\alpha)\theta}{(1-\alpha)\theta}(\varrho - \underline{\varrho}). \tag{2.50}$$

[16] As in Lagos and Zhang (2019b), to see why condition (2.30) is necessary for the bargaining outcome of an investor with access to credit to be well defined, take the budget and the collateral constraints of an investor who contacts a broker, $\bar{a}_t^m + p_t^s\bar{a}_t^s + p_t^{sb}\bar{a}_t^b = a_t^m + p_t^s a_t^s$ and $-\kappa\phi_t^s\bar{a}_t^s \leq \bar{a}_t^b$. These two conditions imply $-\kappa\phi_t^s[a_t^s + (a_t^m - \bar{a}_t^m)/p_t^s] \leq (1 - \kappa p_t^{sb}\phi_t^s/p_t^s)\bar{a}_t^b$. This constraint is nonbinding for all $\bar{a}_t^b < 0$ if (2.30) were violated, meaning that an investor with $\varepsilon > \varepsilon_t^{**}$ would be able to take an infinitely long position in the stock. If (2.30) is violated, then an investor with access to credit with no asset, can sell b collateralised bonds for money and use the proceeds to purchase bp_t^{sb}/p_t^s equity shares. This purchase would leave the investor's borrowing constraint slack, as $b < \kappa\phi_t^s bp_t^{sb}/p_t^s$.

[17] Lemma 14 in Appendix A establishes that $p_t^{kb}\phi_t^m \leq 1$ is necessary for bonds to trade in equilibrium. We focus on equilibria with an active bond market because if the bond is not traded, money is the only means of payment.

Notice that the equilibrium price satisfies $\underline{\varrho} < \varrho < \varphi^q$ and that $\varphi^q = \varrho$ only if $(1-\alpha)\theta = 1$.[18] Therefore, the price of the consumption good in the first subperiod is higher than the marginal cost of production. Consequently, whether due to the good's price or to the collateral constraint, consumption in the first subperiod is inefficiently low in the nonmonetary economy whenever either not all producers have access to credit markets, i.e., $\alpha < 1$ or brokers possess market power over the producers they trade with, i.e., $\theta < 1$. This inefficiency arises because the producer must produce the first subperiod consumption good before simultaneously selling it and bargaining with the broker. Building on the constant-returns production technology in Lagos and Zhang (2022), a producer must break even in equilibrium when producing the first-subperiod consumption good. In particular, when the producer chooses how much to produce at the start of the period, they expect the relative price of the first-subperiod consumption good to be some $\varphi \geq \varrho \geq \underline{\varrho}$, implying an expected profit equal to $\Pi^n(\varphi) \equiv R^n(\varphi) - \varrho = (1-\alpha)\theta(\varphi - \varrho) + [1 - (1-\alpha)\theta](\underline{\varrho} - \varrho)$ per unit of first subperiod consumption good. To interpret this expected profit, note that if, after producing, the producer meets a broker and has all the bargaining power, the per-unit profit is $\varphi - \varrho$. By contrast, if the producer either cannot sell in the first subperiod (for instance, because no broker is contacted, which occurs with probability α) or meets a broker who holds all the bargaining power (with probability $(1-\alpha)(1-\theta)$), then the per-unit profit falls to $\underline{\varrho} - \varrho < 0$. Hence, as long as $(1-\alpha)\theta < 1$, the zero-profit equilibrium condition $\Pi^n(\varphi) = 0$ requires $0 < \varphi - \varrho$, which means an inefficient level of consumption and production of the first subperiod consumption good.

Definition 3. *A* recursive monetary equilibrium (RME) *is a monetary equilibrium in which: (i) real equity prices (general goods per equity share) are time-invariant linear functions of the aggregate dividend, i.e.,* $\phi_t^s = \phi^s y_t$, $p_t^s \phi_t^m \equiv \bar{\phi}_{mt}^s = \bar{\phi}_m^s y_t$, *and* $p_t^s / p_t^{sb} \equiv \bar{\phi}_{bt}^s = \bar{\phi}_b^s y_t$ *for some* $\phi^s, \bar{\phi}_m^s, \bar{\phi}_b^s \in \mathbb{R}_+$*; and (ii) real money balances are a constant proportion of output, i.e.,* $\phi_t^m A_t^m = Z A^s y_t$ *for some* $Z \in \mathbb{R}_{++}$

In a RME, $\varepsilon_t^* = (p_t^s \phi_t^m - \phi_t^s)\frac{1}{y_t} = \bar{\phi}_m^s - \phi^s \equiv \varepsilon^*$, $\varepsilon_t^{**} = (p_t^s / p_t^{sb} - \phi_t^s)\frac{1}{y_t} = \bar{\phi}_b^s - \phi^s \equiv \varepsilon^{**}$,

[18] The case $\underline{\varrho} = \varrho$—which we exclude in the baseline—implies producers earn zero surplus from trading the first-subperiod consumption good. Hence a producer–broker meeting yields no gains from trade, effectively nullifying the broker's market power.

$p_t^s = \frac{(\varepsilon^* + \phi^s)A_t^m}{ZA^s}$, $\phi_t^m = \frac{ZA^s y_t}{A_t^m}$, and

$$p_t^{sb} = \frac{(\varepsilon^* + \phi^s)A_t^m}{(\varepsilon^{**} + \phi^s)ZA^s y_t}. \tag{2.51}$$

Thus, $\phi_{t+1}/\phi_t^s = \bar{\phi}_{mt+1}^s/\bar{\phi}_{mt}^s = \bar{\phi}_{bt+1}^s = \bar{\phi}_{bt}^s = \gamma_{t+1}$, $p_{t+1}^s/p_t^s = \mu$, and $\phi_t^m/\phi_{t+1}^m = \mu/\gamma_{t+1}$. In a RME, the i_t^{sm} defined in (2.47) becomes

$$i^{sm} = \frac{\varepsilon^{**} - \varepsilon^*}{\varepsilon^* + \phi^s}, \tag{2.52}$$

and the price of goods in terms of the real value of money

$$\varphi^m = \frac{\varrho}{1 + (1-\alpha)\theta i^{qm}}, \tag{2.53}$$

and the price of goods in terms of the price of bonds

$$\varphi^q = \frac{1 + i^{qm}}{1 + (1-\alpha)\theta i^{qm}}\varrho, \tag{2.54}$$

where i^{qm} is determined in equilibrium. In a monetary equilibrium, $\underline{\varrho} < \varphi^m \leq \varrho \leq \varphi^q$. Therefore, as long as $(1-\alpha)\theta < 1$ and $0 < i^{qm}$, or if the investor is collateral constrained then the first subperiod consumption good is inefficiently low in the monetary equilibrium. This inefficiency arises because the producer must commit to producing the first-subperiod consumption good before selling it and bargaining with the broker. Again following Lagos and Zhang (2022), under constant-returns production technology, any equilibrium with positive production of the first-subperiod consumption good requires producers to expect zero profits. When the producer chooses output at the start of the period, they anticipate a relative price of $\varphi^m > \underline{\varrho}$ if the good is sold for cash, and a price $\varphi^q \leq \varrho \leq \varphi^m$ if it is sold in terms of bonds. This implies an expected profit equal to

$$\Pi^m(\varphi^m, \varphi^q) \equiv R^m(\varphi^m, \varphi^q) - \varrho = [1 - (1-\alpha)\theta](\varphi^m - \varrho) + (1-\alpha)\theta(\varphi^q - \varrho),$$

per unit of the first subperiod consumption good produced, the beginning-of-period expected profit is determined by the mode of sale. If, after producing, the producer reaches a broker and holds all the bargaining power, the per-unit payoff is $\varphi^q - \varrho$. If instead the producer must sell for cash in the first subperiod (e.g., fails to reach a broker with probability α) or reaches a broker who holds all the bargaining power (with probability $(1-\alpha)(1-\theta)$), the per-unit payoff is $\varphi^m - \varrho$. Whenever $0 < i^{qm}$, it follows that $\varphi^m - \varrho < 0$. Therefore, the zero-profit condition $\Pi^m(\varphi^m, \varphi^q) = 0$ requires $\varphi^q - \varrho > 0$, which implies an inefficiently low level of consumption and production of the first-subperiod good.

Let $p^B_{t,k}$ denote the nominal price in the second subperiod of period t of an N-period risk-free nominal bond that matures in period $t+k$, for $k = 0, 1, 2, \ldots, N$ (so k is the number of periods until the bond matures). Imagine the bond is illiquid in the sense that it cannot be traded in the OTC market.[19] Then, in a stationary monetary equilibrium, $p^B_{t,k} = (\bar{\beta}/\mu)^k$, and

$$i^p = \frac{\mu - \bar{\beta}}{\bar{\beta}} \tag{2.55}$$

is the time-t nominal yield to maturity of the bond with k periods until maturity. Throughout the analysis we let $\bar{\beta} \equiv \beta\bar{\gamma}$ and maintain the assumption $\mu > \bar{\beta}$. Since there is a mapping between μ and the interest rate i^p, we can regard i^p as the nominal policy rate chosen by the monetary authority.

In what follows, we consider a formulation of the model where the length of the time period is arbitrarily short, which allows us to solve an analytical form of the model. This can interpreted as an approximation to a continuous-time version of our discrete-time economy. To this end, we first assume the utility for the first subperiod consumption good is $u_t(q_t) = \bar{\omega}_t q_t^\sigma$,[20] with $\sigma \in (0,1)$ and $\bar{\omega}_t \equiv (\phi^s_t)^{1-\sigma}$,[21] is a scaling factor.[22] Second, we generalise the discrete-time

[19] An important caveat concerns the interpretation of this illiquid-bond rate as "the" nominal interest rate. As Geromichalos and Herrenbrueck (2022) and Herrenbrueck and Wang (2023) emphasise, the yield on a liquid bond—one whose Euler equation incorporates a liquidity or resale component—may exhibit comparative statics that differ from, or even oppose, those of the illiquid-bond rate defined here. In particular, the direction of the asset price response to an "interest rate hike" depends on whether the rate in question corresponds to a liquid or illiquid instrument. Our identification of the policy rate with the illiquid-bond yield follows the convention established in Lagos and Zhang (2019a, 2020), which provides the foundational framework for the analysis in this chapter.

[20] Given the utility specification, the marginal utility can be written as $u_t'(q_t) \equiv \tilde{u}'(\tilde{q}_t)$ where $\tilde{q}_t \equiv q_t/\phi^s_t$. Define $\tilde{\mathrm{D}}(\cdot) \equiv \tilde{u}'^{-1}(\cdot)$, and henceforth use this representation for convenience.

[21] We specify $u_t(q_t) = \bar{\omega}_t q_t^\sigma$ with $\sigma \in (0,1)$ and $\bar{\omega}_t \equiv (\phi^s_t)^{1-\sigma}$. The exponent σ ensures diminishing marginal utility, while the scaling factor $\bar{\omega}_t$ introduces dependence on the asset price ϕ^s_t. Since $1-\sigma > 0$, a higher ϕ^s_t increases $\bar{\omega}_t$, capturing a wealth effect whereby higher asset prices raise utility for a given q_t.

[22] This assumption is made for analytical convenience. If the asset were instead used directly as a medium of

model by allowing the period length to be a constant, and then take the limit as this constant becomes arbitrarily small. Let Δ denote the length of the model period, and define the discount rate, r, the expected dividend growth, g, the depreciation rate, d, and the money growth rate, π, as $\beta \equiv (1+r\Delta)^{-1}$, $\bar{\gamma} \equiv 1+g\Delta$, $\eta \equiv 1-d\Delta$, $\mu \equiv 1+\pi\Delta$. Over a time period of length Δ, the dividend is $y_t\Delta$, and utility from consumption of the dividend good is $\varepsilon y_t \Delta$.[23]

We focus on recursive equilibria where, as $\Delta \to 0$, real asset prices are time-invariant linear functions of the *dividend rate*, y_t. Specifically, let $\Phi^s(\Delta)$ and $\Phi^m_t(\Delta)A^m_t$ denote the real equity price and the real aggregate money balance, respectively, in the discrete-time economy such that $\Phi^s_t(\Delta) = \Phi^s(\Delta)y_t\Delta$ and $\Phi^m_t(\Delta)A^m_t = Z(\Delta)A^s y_t\Delta$, where $\Phi^s(\Delta)$ and $Z(\Delta)$ are time-invariant functions with the property that $\lim_{\Delta\to 0}\Phi^s(\Delta)\Delta = \phi^s$ and $\lim_{\Delta\to 0} Z(\Delta)\Delta = Z$, $\phi^s, Z \in \mathbb{R}$. Hence (2.51), (2.52), and (2.53) generalise to $i^p = \frac{(r+\pi-g+r\pi\Delta)\Delta}{1+g\Delta}$, $\varphi^m = \frac{\varrho}{1+(1-\alpha)\theta\Delta\rho^{qm}}$ and $\varphi^q = \frac{1+\Delta\rho^{qm}}{1+(1-\alpha)\theta\Delta\rho^{qm}}\varrho$, respectively.[24]

Define

$$\rho^p \equiv \lim_{\Delta\to 0}\frac{i^p}{\Delta} = r+\pi-g.$$

Intuitively, ρ^p represents the policy nominal interest rate under the direct control of the monetary authority. Notice that $\rho^p = r+\bar{\pi}$ is a Fisher equation that equates the nominal interest rate to the real risk-free interest rate, r, plus an expected inflation rate, $\bar{\pi} \equiv \pi - g$.[25] We can show that $\bar{\varphi}^m = \lim_{\Delta\to 0}\frac{\varrho}{1+(1-\alpha)\theta\Delta\rho^{qm}} = \varrho$ and $\bar{\varphi}^q = \lim_{\Delta\to 0}\frac{1+\Delta\rho^{qm}}{1+(1-\alpha)\theta\Delta\rho^{qm}}\varrho = \varrho$. Notice that in the continuous-time limit in the monetary equilibrium, that the price of the good converges to the competitive market price (i.e., the marginal cost of production ϱ). This occurs because all debt obligations are paid immediately and therefore bonds become equivalent to fiat money. To see this, consider the fact that broker's bargaining power becomes essentially inconsequential since

exchange rather than as collateral, the assumption would be unnecessary. In fact, models with assets as collateral and models with assets as media of exchange deliver similar conclusions (see Lagos et al. (2017)). We adopt the collateral interpretation as it aligns more closely with common real-world arrangements.

[23] The goods-pledge option over an arbitrarily small time interval is given by $\Delta[\tilde{u}'(\min\{\tilde{\mathrm{D}}(\varphi), \bar{q}/\varphi\})/\varphi - 1]$ for given φ and $\bar{q}$. For full derivations, see Rocheteau and Wang (2023)

[24] The discrete-time formulation we laid out previously, corresponds to a special case of this formulation with $\Delta = 1$, $\Phi^s_t(1) \equiv \phi^s_t$, $\Phi^m_t(1) \equiv \phi^m_t$, $\Phi^s(1) \equiv \phi^s$, and $Z(1) \equiv Z$.

[25] The (gross) inflation rate in terms of general goods is given by $\phi^m_t/\phi^m_{t+1} = \mu\, y_t/y_{t+1} \equiv 1+\tilde{\pi}_{t+1}$, which is stochastic. A measure of average expected inflation is $[\mathbb{E}_t(1+\tilde{\pi}_{t+1})^{-1}]^{-1} = \mu/\bar{\gamma} = (1+\pi\Delta)/(1+g\Delta) \equiv 1+\bar{\pi}\Delta$. Therefore, as $\Delta \to 0$, we have $\bar{\pi} = \pi - g$ as the expected inflation rate in the nominal price of general goods. Since $p^s_{t+1}/p^s_t - 1 = \mu - 1 = \pi\Delta$, the inflation rate in the nominal price of equity is π.

investors or producers can achieve the same allocation via their outside option. Therefore in the continuous-time limit, unless the investor is collateral constrained in the goods market, there is an efficient level of consumption and production of the first subperiod consumption good.

For what follows, it is useful to let $\mathcal{Z} \equiv \rho Z$, $\iota \equiv \rho^p/\rho$, and $\varphi^n \equiv \rho\phi^s$ in a nonmonetary economy, or $\varphi \equiv \rho\phi^s$ in a monetary economy, where

$$\rho \equiv \lim_{\Delta \to 0} \frac{1}{\Delta} \frac{1-\bar{\beta}\eta}{\bar{\beta}\eta} = r + d - g.$$

The factor ρ can be interpreted as a *capitalisation rate* for equity holdings, so φ (or φ^n) and $\mathcal{Z}$ are the "flow values" of an equity share and a unit of real balances, respectively, and ι is the policy rate up to a convenient normalisation. In the remainder, we focus on the limiting economy that obtains as $\Delta \to 0$.

Proposition 1. *There exists a unique recursive nonmonetary equilibrium,* $(\varepsilon^n, \varphi^n)$. *Moreover,*

$$\varphi^n = \frac{\bar{\varepsilon} + (1-\alpha)\delta\theta \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon) dG(\varepsilon) + \frac{\kappa}{1-\kappa} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n) dG(\varepsilon)\right]}{1 - (1-\alpha)(1-\delta)\lambda\hat{\theta} \left\{ \tilde{u}' \left[\min\{\tilde{\mathrm{D}}(\hat{\varphi}^q), \lambda A^s/\hat{\varphi}^q\} \right] - \hat{\varphi}^q \right\}}, \tag{2.56}$$

where $\hat{\theta} = \frac{\theta}{\rho\hat{\varphi}^q}$ *and* $\varepsilon^n \in (\varepsilon_L, \varepsilon_H)$ *is the unique solution to*

$$G(\varepsilon^n) = \kappa. \tag{2.57}$$

In the nonmonetary equilibrium, the asset price can be broken down into four components. The first term in the numerator of (2.56) represents the expected value of the dividend flow. The second term on the numerator in (2.56) captures the expected gain from exercising the option to resell the asset in the OTC market, which we refer to as the *(value of the) resale option.* The third term on the numerator of (2.56) reflects the expected marginal value of pledging the asset as collateral to short the bond for the purpose of purchasing additional equity; we refer to this as the *(value of the) equity pledge option.* The term on the denominator is the capitalisation rate minus the liquidity premium (i.e., the net marginal gain) of the asset when it is pledge as collateral for shorting the bond in order to purchase the first subperiod consumption good. We label this component the *(value of the) goods pledge option.*

Similar to Lagos and Zhang (2019b), from (2.57), it follows that the marginal investor's valuation that clears the equity market in the OTC round ε^n increases with respect to κ. As κ increases, the collateral constraint is relaxed, enabling higher valuation investors to hold higher proportion of the asset holdings. In particular, $\varepsilon \to \varepsilon_H$ as $\kappa \to 1$. For a given ε^n, the asset price increases with κ. However, in general equilibrium, a higher κ raises ε^n, which makes it less likely that investors receive a valuation shock large enough to justify using the asset as collateral. This latter effect tends to lower the price. Nonetheless, the first effect dominates, so that φ^n remains increasing in κ even after taking into account the general equilibrium feedback (see Lemma 18 in Appendix A).

From (2.56), it is clear that if $\hat{\varphi}^q \tilde{\mathrm{D}}(\hat{\varphi}^q) \leq \lambda A^s$ that the goods pledge option becomes zero and (2.56) returns to the Lagos and Zhang (2019b) formulation. If, however, $\lambda A^s \leq \hat{\varphi}^q \tilde{\mathrm{D}}(\hat{\varphi}^q)$ then the goods pledge option is positive and therefore it can be shown that the asset price φ^n is increasing in λ, reflecting the fact that as λ increases, the collateral constraint is relaxed facilitating consumption of the first subperiod consumption good. In what follows, it is useful to define $\xi = (\kappa, \lambda, \varrho, A^s)$.[26]

Let

$$\begin{aligned}
\bar{\iota}(\xi) \equiv & \left\{ [\alpha + (1-\alpha)(1-\theta)] \int_{\varepsilon_L^+}^{\varepsilon_H} (\varepsilon - \varepsilon_L^+) dG(\varepsilon) + (1-\alpha)\theta \left[\varepsilon^n - \varepsilon_L^+ + \frac{1}{1-\kappa} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n) dG(\varepsilon) \right] \right\} \frac{\delta}{\bar{\varphi}(\xi)} \\
& + (1-\delta)\bar{\theta} \left\{ \alpha \tilde{u}' \left[\min\left\{ \tilde{\mathrm{D}}(\varrho), \epsilon A^s/\varrho \right\} \right] + (1-\alpha)\tilde{u}' \left[\min\left\{ \tilde{\mathrm{D}}(\varrho), (\epsilon+\lambda) A^s/\varrho \right\} \right] - \varrho \right\} \qquad (2.58) \\
\hat{\iota}(\xi) \equiv & \left[\alpha + (1-\alpha)\left(1 + \theta \frac{\kappa}{1-\kappa}\right) \right] \frac{\delta}{\hat{\varphi}(\xi)} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n) dG(\varepsilon) \\
& + (1-\delta)\bar{\theta} \left\{ \alpha \tilde{u}' \left[\min\left\{ \tilde{\mathrm{D}}(\varrho), \hat{z} A^s/\varrho \right\} \right] + (1-\alpha)\tilde{u}' \left[\min\left\{ \tilde{\mathrm{D}}(\varrho), (\hat{z}+\lambda) A^s/\varrho \right\} \right] - \varrho \right\}. \qquad (2.59)
\end{aligned}$$

The next proposition provides a full characterisation of the set of RME.

Proposition 2. *(i) If $\hat{\iota}(\xi) < \iota < \bar{\iota}(\xi)$ then there exists a unique recursive monetary equilibrium,*

[26] We denote the nominal interest rate bounds in terms of exogenous parameters, which change the equilibrium. By assumption, we have $\tilde{u}'(\cdot)$ is decreasing i.e., $\tilde{u}''(\cdot) < 0$, therefore as $\varepsilon^* \to \varepsilon_L$ which implies that $z \to 0$ and $u'[\min\{\tilde{\mathrm{D}}(\varrho), zA^s/\varrho\}] \to \infty$. To avoid this, suppose there exists ε_L^+ such that $z(\varepsilon_L^+) = \epsilon > 0$. This has the implication that the nominal interest cannot be such that $\varepsilon \to \varepsilon_L$. Also to simplify the exposition, we denote $\bar{\varphi}(\xi) = \varphi|_{\varepsilon^* = \varepsilon_L^+}$. Similarly, we denote $\hat{\varphi}(\xi) = \varphi|_{\varepsilon^* = \varepsilon^n}$ and $\hat{z} = z|_{\varepsilon^* = \varepsilon^n}$, for the appropriate nominal interest rate ranges. The upper bound for the nominal interest can be specified in such as way without loss of generality. We also assume that the utility function and ϵ guarentee that $\varphi|_{z=\epsilon} \geq 0$.

$(\varepsilon^*, \varepsilon^{**}, \varphi, \mathcal{Z})$. The asset prices are

$$\varphi = \frac{\bar{\varepsilon} + (1-\alpha)\delta\theta\left[\int_{\varepsilon_L}^{\varepsilon^{**}}(\varepsilon^{**}-\varepsilon)dG(\varepsilon) + \frac{\kappa}{1-\kappa}\int_{\varepsilon^{**}}^{\varepsilon_H}(\varepsilon-\varepsilon^{**})dG(\varepsilon)\right] + [\alpha+(1-\alpha)(1-\theta)]\delta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)dG(\varepsilon)}{1-(1-\alpha)(1-\delta)\lambda\bar{\theta}\left\{\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}(\varrho), (z+\lambda)A^s/\varrho\right\}\right]-\varrho\right\}} \tag{2.60}$$

$$\mathcal{Z} = \frac{\alpha G(\varepsilon^*)}{[1-G(\varepsilon^*)]\alpha + 1 - \alpha}\varphi, \tag{2.61}$$

where $z = \mathcal{Z}/\varphi$ and $\bar{\theta} = \theta/(\rho\varrho)$. The marginal valuations are $\varepsilon^{**} = \varepsilon^*$ where $\varepsilon^* \in (\varepsilon_L, \varepsilon^n)$ is the unique solution to

$$\begin{aligned}\iota = {} & [\alpha+(1-\alpha)(1-\theta)]\frac{\delta}{\varphi}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)dG(\varepsilon) + (1-\alpha)\theta\frac{\delta}{\varphi}\left[\varepsilon^n - \varepsilon^* + \frac{1}{1-\kappa}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)dG(\varepsilon)\right] \\ & + (1-\delta)\bar{\theta}\left\{\alpha\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}(\varrho), zA^s/\varrho\right\}\right] + (1-\alpha)\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}(\varrho), (z+\lambda)A^s/\varrho\right\}\right] - \varrho\right\}.\end{aligned} \tag{2.62}$$

(ii) If $0 < \iota \leq \hat{\iota}(\xi)$ then there exists a unique recursive monetary equilibrium, $(\varepsilon^*, \chi, \varphi, \mathcal{Z})$. The asset prices are

$$\varphi = \frac{\bar{\varepsilon} + \delta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)dG(\varepsilon) + (1-\alpha)\delta\theta\frac{\kappa}{1-\kappa}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)dG(\varepsilon)}{1-(1-\alpha)(1-\delta)\lambda\bar{\theta}\left\{\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}(\varrho), (z+\lambda)A^s/\varrho\right\}\right]-\varrho\right\}} \tag{2.63}$$

$$\mathcal{Z} = \frac{\alpha G(\varepsilon^*) + (1-\alpha)\frac{1}{1-\kappa}[G(\varepsilon^*)-\kappa]}{[1-G(\varepsilon^*)]\left[\alpha + (1-\alpha)\frac{1}{1-\kappa}\right]}\varphi. \tag{2.64}$$

The marginal valuations are $\varepsilon^{**} = \varepsilon^*$, where $\varepsilon^* \in [\varepsilon^n, \varepsilon_H)$ (with $\varepsilon^* = \varepsilon^n$ only if $\iota = \hat{\iota}(\xi)$) is the unique solution to

$$\begin{aligned}\iota = {} & \left[\alpha + (1-\alpha)\left(1+\theta\frac{\kappa}{1-\kappa}\right)\right]\frac{\delta}{\varphi}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)dG(\varepsilon) \\ & + (1-\delta)\bar{\theta}\left\{\alpha\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}(\varrho), zA^s/\varrho\right\}\right] + (1-\alpha)\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}(\varrho), (z+\lambda)A^s/\varrho\right\}\right] - \varrho\right\},\end{aligned} \tag{2.65}$$

and

$$\chi = \frac{\kappa}{1-\kappa}\frac{1-G(\varepsilon^*)}{G(\varepsilon^*)}, \tag{2.66}$$

is the proportion of financial wealth that investors with access to credit in the equity market with valuation lower than ε^* *hold in the form of bonds (they hold the remaining* $1-\chi$ *fraction in cash).*

If $\hat{\iota}(\xi) < \iota < \bar{\iota}(\xi)$, then credit is scarce, so the nominal loan rate ρ^{sm} is positive. In this case, investors in the equity market without access to credit who have valuations higher than ε^* spend all their money to purchase equity. Investors with access to credit who have valuations higher than ε^{**} spend all their money and short the bond up to the collateral constraint to purchase equity. Investors without access to credit who have valuations lower than ε^* sell all their equity holdings for money. Correspondingly, investors with access to credit who have valuations lower than ε^{**} sell all their equity holdings and spend all the proceeds along with their pre-trade money holdings to take a long position in bonds. Notice that if $\varepsilon_L < \varepsilon^* < \varepsilon^{**} = \varepsilon^n$ the marginal equity buyer without access to credit has a lower valuation than the marginal equity buyer with access to credit. The latter has the same valuation he would have in the nonmonetary equilibrium. The fact that $\varepsilon^* < \varepsilon^{**}$ reflects that investors with access to credit have the option of investing money in interest bearing bonds, while the option is not available to investor without access to credit.

If, on the other hand, $0 < \iota \leq \hat{\iota}(\xi)$, then real balances are plentiful and the demand for credit is weak, implying that $\rho^{sm} = 0$. Investors without access to credit who have valuations larger than ε^* spend all their money to buy equity for money. Investors with access to credit who have valuations higher than ε^* use all their money and short the bond up to the collateral constraint to purchase equity. Investors with access to credit who have valuations lower than ε^* liquidate all their equity holdings and are indifferent between holding the resulting proceeds (together with any pre-trade money balances) as money or as bonds. Notice that if $\varepsilon_L < \varepsilon^n \leq \varepsilon^* = \varepsilon^{**} < \varepsilon_H$ the marginal equity buyer without access to credit has the same valuation as the marginal equity buyer with access to credit. This marginal valuation is higher than the marginal valuation an investor with access to credit would have in the nonmonetary equilibrium. The fact that $\varepsilon^* = \varepsilon^{**}$ reflects that the nominal interest rate on inside bonds is zero. Consequently, investors with and without access to credit assign the same valuation to money in the OTC round, even though only the former can lend it in the bond market.

For all ι, if $zA^s < \varrho\tilde{\mathrm{D}}(\varrho)$ then investors without access to credit use all their money to purchase consumption goods and if $(z+\lambda)A^s < \varrho\tilde{\mathrm{D}}(\varrho)$, investors with access to credit spend all their money and short the bond up to the collateral constraint to purchase first subperiod consumption goods.

The following proposition formalises how monetary exchange raises the real asset price above its nonmonetary benchmark.

Proposition 3. *The real asset price in the monetary equilibrium is higher than in the nonmonetary equilibrium, i.e.,* $\varphi^n \leq \varphi$ *for all* ι.

If $\hat{\iota}(\xi) < \iota < \bar{\iota}(\xi)$, the equity price in the monetary equilibrium is larger than the equity price in the nonmonetary equilibrium. Notice that we can write the real equity price as

$$\varphi = \frac{\left[1-(1-\alpha)(1-\delta)\lambda\hat{\theta}\left\{\tilde{u}'\left[\min\{\tilde{\mathrm{D}}(\hat{\varphi}^q),\lambda A^s/\varphi^q\}\right]-\hat{\varphi}^q\right\}\right]\varphi^n + [\alpha+(1-\alpha)(1-\theta)]\delta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)dG(\varepsilon)}{1-(1-\alpha)(1-\delta)\lambda\bar{\theta}\left\{\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}(\varrho),(z+\lambda)A^s/\varrho\right\}\right]-\varrho\right\}}. \tag{2.67}$$

Firstly, the goods pledge option decreases with respect to price and since $\varrho < \hat{\varphi}^q$ and since $z > 0$ we have

$$\frac{\tilde{u}\left[\min\left\{\tilde{\mathrm{D}}(\hat{\varphi}^q),\lambda A^s/\hat{\varphi}^q\right\}\right]}{\hat{\varphi}^q} < \frac{\tilde{u}\left[\min\left\{\tilde{\mathrm{D}}(\varrho),(z+\lambda)A^s/\varrho\right\}\right]}{\varrho}.$$

This implies that the first term in the numerator is larger than the denomentor in (2.67). Secondly, the second term in the numerator which is multiplied by α, implies that in a monetary equilibrium there are equity-for-money trades in the equity market so the investor without access to credit can resell equity if their valuation is relatively low. This is not possible in the nonmonetary equilibrium. The third reason (captured by the term multiplied by $(1-\alpha)(1-\theta)$) is the option to sell equity for money in the equity market which the outside option of an investor with access to credit when he bargains for the terms of the trade with the broker. Notice that $\varrho < \hat{\varphi}^q = \varrho + \frac{1-(1-\alpha)\theta}{(1-\alpha)\theta}(\varrho - \underline{\varrho})$ and therefore the price of goods in the nonmonetary equilibrium is higher than in monetary equilibrium, whether or not the investor has access to credit or not. This reflects the fact that investors and producers in the nonmonetary equilibrium do not have access to a monetary outside option. This implies that if investors and producers do not reach a bargain with the broker, they are not able to trade in the goods market. This maintains the bargaining power of the broker. This is not the case in the monetary equilibrium. Investors and producers in the monetary equilibrium can transact with cash if they do not reach a bargain. In fact, in the continuous-time limit, since bonds are repaid instantly, fiat money and bonds hold the same value. Therefore the outside option for investors and producers is equivalent to the allocation in the OTC market with the broker. This dissipates the bargaining power of the broker and makes access to the credit market redundant.

If, on the other hand, $0 < \iota < \hat{\iota}(\xi)$, then again the monetary equilibrium real equity price is larger than the nonmonetary equilibrium equity price. To observe this, first note that again the denominator of (2.56) is larger than the denominator of (2.63). Again, this is because the price of the good in the nonmonetary equilibrium is higher than the price of the good in the monetary equilibrium, effectively reducing the good pledge option. Further, the investor in the monetary equilibrium is guaranteed to purchase the asset, whereas in the nonmonetary equilibrium, they can only purchase the asset if they can contact a broker. This is the reason why the numerator of (2.63) is larger than the numerator in (2.56).

The following proposition characterises how the policy nominal interest rate affects the marginal investor's valuation cutoff and the conditions under which it affects the equity price.

Proposition 4. *(i) In a monetary equilibrium, if $\hat{\iota}(\xi) < \iota < \bar{\iota}(\xi)$, then ε^* (the valuation of the marginal investor with no access to credit) is decreasing in ι and, further φ is decreasing in ι if and only if*

$$\varphi < \frac{[(1-\alpha)\theta - 1]\delta\rho\varrho^2 G(\varepsilon^*)[1-\alpha G(\varepsilon^*)]^2}{(1-\alpha)(1-\delta)\lambda\theta A^s \tilde{u}''\left[\min\left\{\tilde{\mathrm{D}}(\varrho), (z+\lambda)A^s/\varrho\right\}\right]\alpha G'(\varepsilon^*)}. \tag{2.68}$$

(ii) If $0 < \iota \leq \hat{\iota}(\xi)$, then ε^* (the valuation of the marginal investor) is decreasing in ι and, further φ is decreasing in ι if and only if

$$\varphi < \frac{\left\{(1-\alpha)\theta\frac{\kappa}{1-\kappa}[1-G(\varepsilon^*)] - G(\varepsilon^*)\right\}[1-\alpha G(\varepsilon^*)]^2\delta\rho\varrho^2}{(1-\alpha)(1-\delta)\lambda\theta A^s \tilde{u}''\left[\min\left\{\tilde{\mathrm{D}}(\varrho), (z+\lambda)A^s/\varrho\right\}\right]\alpha G'(\varepsilon^*)} \tag{2.69}$$

Proposition 4 reveals the tension that exists when the asset is brought into the equity market versus into the goods market. In the equity market, equity and fiat money are substitutes. For a given valuation investors choose to either sell all their assets for cash or use all their money to purchase equity. Increasing the opportunity cost of holding money (as represented by the nominal interest rate ι) therefore leads to a decrease in the price of the asset. Conversely, in the goods market, equity and fiat money are complements. Equity assets allow the investor to slacken their budget constraint, therefore increasing the opportunity cost of holding money increases the relative price of equity since it can be used instead to facilitate the purchase of the first subperiod consumption good. This conflict leads to our conditions in Proposition 4. If the marginal benefit of holding the asset in the equity market dominates the marginal benefit of holding the asset in the goods market, then the equity price will decrease in response to monetary policy. Otherwise, if the marginal benefit of holding the asset in the goods market dominates the marginal benefit of holding the asset in the equity market, then the equity price will increase in response to monetary policy. Crucially, conditions (2.68) and (2.69) are dependent on ε^* which is a function of ι, this implies that the response of the equity price to changes in the nominal interest rate (or inflation) depends upon the current rate of inflation/nominal interest rate.

The relationship between inflation and equity prices is complex and is influenced by several factors including inflation expectations, corporate profitability, interest rates, and inflation uncertainty. Our explanation suggests candidate conditions that conform with findings in the empirical literature. The framework presented here shows that the effect of inflation or nominal interest rates on equity prices varies based on model fundamentals and the prevailing nominal interest rate. This variation arises because assets serve multiple purposes, and these functions respond differently to inflation, creating opposing forces. The dominant effect dictates the reaction of asset prices to monetary policy, with this dominance shifting based on fundamentals in the model—such as asset quantities or production costs—or the current nominal interest rate.

2.4 Cashless Limits

In this section we examine monetary equilibrium in environments where agents can economise on cash holdings, so that the economy approximates a *pure credit* or *cashless* system. Following Lagos and Zhang (2019b), we characterise the limiting equilibrium as $\alpha \to 0$. In this limit, regardless of the equilibrium value of money, no agent requires money in order to trade. Any investor can take a long position in equity and purchase goods by shorting the bond.

For the following result it is useful to let $\hat{\varsigma}(\alpha;\delta)$ and $\bar{\varsigma}(\alpha;\delta)$ denote the bounds of the nominal interest rate, but regarded as functions of α and δ. Now let $\alpha \to 0$. In order to define $\hat{\varsigma}(0;\delta)$ and $\bar{\varsigma}(0;\delta)$, we require a definition of the asset prices in these contexts i.e.,

$$\hat{\varphi}(0;\delta) \equiv \frac{\bar{\varepsilon} + \delta\theta\left[I_1 + \frac{\kappa}{1-\kappa} I_2\right]}{1-(1-\delta)\lambda\bar{\theta}\left[\frac{\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}(\varrho),\lambda A^s/\varrho\right\}\right]}{\varrho} - 1\right]}$$
$$\bar{\varphi}(0;\delta) \equiv \hat{\varphi}(0;\delta) + \frac{\delta(1-\theta)I_1}{1-(1-\delta)\lambda\frac{\theta}{\rho}\left[\frac{\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}(\varrho),\lambda A^s/\varrho\right\}\right]}{\varrho} - 1\right]},$$

where $I_1 \equiv \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon) dG(\varepsilon)$, and $I_2 \equiv \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n) dG(\varepsilon)$. Therefore, we have,

$$\hat{\varsigma}(0;\delta) \equiv \left(1+\theta\frac{\kappa}{1-\kappa}\right)\frac{\delta}{\hat{\varphi}(0;\delta)}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)dG(\varepsilon) + (1-\delta)\bar{\theta}\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}(\varrho),\lambda A^s/\varrho\right\} - \varrho\right] \tag{2.70}$$
$$\bar{\varsigma}(0;\delta) \equiv \left\{\bar{\varepsilon} - \varepsilon_L + \theta\left[\varepsilon^n - \bar{\varepsilon} + \frac{1}{1-\kappa}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)dG(\varepsilon)\right]\right\}\frac{\delta}{\bar{\varphi}(0;\delta)}$$
$$+ (1-\delta)\bar{\theta}\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}(\varrho),\lambda A^s/\varrho\right\} - \varrho\right]. \tag{2.71}$$

Proposition 5. *Consider an economy with* $\alpha \in [0,1]$, $\kappa \in (0,1]$ *and* $\lambda \in (0,1]$. *Let*

$$\tilde{\varphi}^n \equiv \lim_{\alpha\to 0}\varphi^n = \frac{\bar{\varepsilon} + \delta\theta\left[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n-\varepsilon)dG(\varepsilon) + \frac{\kappa}{1-\kappa}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)dG(\varepsilon)\right]}{1-(1-\delta)\lambda\tilde{\theta}\left\{\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}(\tilde{\varphi}^q),\lambda A^s/\tilde{\varphi}^q\right\}\right] - \tilde{\varphi}^q\right\}} \tag{2.72}$$
$$\tilde{\varphi}^q \equiv \lim_{\alpha\to 0}\hat{\varphi}^q = \varrho + \frac{1-\theta}{\theta}(\varrho - \underline{\varrho}), \tag{2.73}$$

where $\tilde{\theta} = \frac{\theta}{\rho\tilde{\varphi}^q}$.

In the limit as $\alpha \to 0$,

(i) If $\hat{\varsigma}(0;\delta) < \iota < \bar{\varsigma}(0;\delta)$, *then*

$$\frac{\mathcal{Z}}{\varphi} \to 0 \tag{2.74}$$

$$\varphi \to \tilde{\varphi} \equiv \frac{\bar{\varepsilon} + \delta\theta \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon) dG(\varepsilon) + \frac{\kappa}{1-\kappa} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n) dG(\varepsilon) \right] + (1-\theta)\delta \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon) dG(\varepsilon)}{1 - (1-\delta)\lambda\bar{\theta} \left[\tilde{u}' \left[\min \left\{ \tilde{\mathrm{D}}(\varrho), \lambda A^s/\varrho \right\} \right] - \varrho \right]}, \tag{2.75}$$

where $\varepsilon^* \in (\varepsilon_L, \varepsilon^n)$ *is the unique solution to*[27]

$$\begin{aligned} \iota = (1-\theta)\frac{\delta}{\tilde{\varphi}} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*) dG(\varepsilon) + (1-\delta)\bar{\theta}\tilde{u}' \left[\min \left\{ \tilde{\mathrm{D}}(\varrho), \lambda A^s/\varrho \right\} - \varrho \right] \\ + \theta \frac{\delta}{\tilde{\varphi}} \left[\varepsilon^n - \varepsilon^* + \frac{1}{1-\kappa} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n) dG(\varepsilon) \right]. \end{aligned} \tag{2.76}$$

(ii) If $0 < \iota \leq \hat{\varsigma}(0;\delta)$*, then*

$$\frac{\mathcal{Z}}{\varphi} \to \frac{G(\varepsilon^*) - \kappa}{1 - G(\varepsilon^*)} \tag{2.77}$$

$$\varphi \to \tilde{\varphi} \equiv \frac{\bar{\varepsilon} + \delta \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon) dG(\varepsilon) + \delta\theta \frac{\kappa}{1-\kappa} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*) dG(\varepsilon)}{1 - (1-\delta)\lambda\bar{\theta} \left\{ \tilde{u}' \left[\min \left\{ \tilde{\mathrm{D}}(\varrho), (\hat{z} + \lambda) A^s/\varrho \right\} \right] - \varrho \right\}}, \tag{2.78}$$

where $\varepsilon^* \in [\varepsilon^n, \varepsilon_H)$ *is the unique solution to*

$$\iota = \left(1 - \theta + \theta \frac{1}{1-\kappa} \right) \frac{\delta}{\tilde{\varphi}} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*) dG(\varepsilon) + (1-\delta)\bar{\theta}\tilde{u}' \left[\min \left\{ \tilde{\mathrm{D}}(\varrho), (\hat{z} + \lambda) A^s/\varrho \right\} - \varrho \right], \tag{2.79}$$

where $\hat{z} \equiv \mathcal{Z}/\varphi$*.*

Proposition 5 studies the limiting economy in which the share of investors without credit access goes to zero. Investors can buy equity and consumption goods even if they begin the period with no cash. As set out in Proposition 2, in a monetary equilibrium the investors who arrive at the equity market without credit and who have relatively low valuations are the ones who always demand money in the OTC round. By contrast, investors with credit and relatively low valuations demand money in the OTC round only when $0 < \iota \leq \hat{\varsigma}(0;\delta)$. A similar logic holds in the goods market. Investors without credit demand money in the OTC round to purchase consumption goods, while investors with credit demand money only to the extent that it relaxes the collateral constraint. Therefore, as $\alpha \to 0$, the first order implication is that the extensive margin of money demand from investors without credit access, meaning the number of such investors who wish to hold money, tends to zero.

[27] In Appendix A, we use l'Hopital's rule to show that $\lim_{\alpha \to 0} \alpha \tilde{u}' \left[\min \left\{ \tilde{\mathrm{D}}(\varrho), zA^s/\varrho \right\} \right] \to 0$

If the nominal policy rate is relatively low, that is, if $0 < \iota \leq \hat{\varsigma}(0;\delta)$ as in part (*ii*) of Proposition 5, then the aggregate money demand from investors without access to credit vanishes in the limit, while aggregate money demand from investors with credit access and have low valuations remains positive in the limit. The source of monetary value in this limiting economy can be seen from two perspectives. First, overnight money demand can be positive because the collateral constraint is expected to bind for credit access investors in the equity market, so money allows them to take a larger long position in equity than they could achieve using only the margin loan. Second, low valuation investors with credit access are willing to hold cash at the end of the OTC round from the equity market because the nominal bond rate, which is the opportunity cost of holding cash in the OTC round, is zero when $0 < \iota \leq \hat{\varsigma}(0;\delta)$. They are therefore indifferent between holding wealth in money or in bonds and hold some of each. As a result, in economies that satisfy $0 < \iota \leq \hat{\varsigma}(0;\delta)$, meaning relatively low inflation and relatively limited scope for leverage, real balances converge to a positive limit, *i.e.*, (2.77), as $\alpha \to 0$.

If the nominal rate is relatively high, that is, if $\hat{\varsigma}(0;\delta) < \iota < \bar{\varsigma}(0;\delta)$ as in part (*i*) of Proposition 5, then real balances converge to zero, see (2.74), as $\alpha \to 0$. The mechanism is straightforward: at this policy rate, aggregate money demand in the OTC round vanishes as $\alpha \to 0$ for two reasons. First, low-valuation investors without access to credit would be willing to hold money, but their mass goes to zero as $\alpha \to 0$. Second, investors with access to credit are unwilling to hold money because money is dominated in rate of return by the collateralised bond. With virtually no agents wishing to hold money in the OTC round, money has no value in the limiting economy as $\alpha \to 0$ with $\iota \in (\hat{\varsigma}(0;\delta), \bar{\varsigma}(0;\delta))$. Thus, real balances converge to zero in this case.

$$
\begin{aligned}
&\lim_{\alpha\to 0}\frac{\mathcal{Z}}{\varphi} = 0 < \lim_{\alpha\to 0}(\varphi - \varphi^n) \\
&= \frac{\varphi^n(1-\delta)\lambda\left\{\bar{\theta}\tilde{u}\left[\min\left\{\tilde{\mathrm{D}}(\varrho), \lambda A^s/\varrho\right\}\right] - \tilde{\theta}\tilde{u}\left[\min\left\{\tilde{\mathrm{D}}(\tilde{\varphi}^q), \lambda A^s/\tilde{\varphi}^q\right\}\right]\right\} + (1-\theta)\delta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)dG(\varepsilon)}{1-(1-\delta)\lambda\bar{\theta}\left[\tilde{u}\left[\min\left\{\tilde{\mathrm{D}}(\varrho), \lambda A^s/\varrho\right\}\right] - \varrho\right]},
\end{aligned}
\tag{2.80}
$$

where

$$\varphi^n = \frac{\bar{\varepsilon} + \delta\theta\left[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n - \varepsilon)dG(\varepsilon) + \frac{\kappa}{1-\kappa}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)dG(\varepsilon)\right]}{1-(1-\delta)\lambda\tilde{\theta}\left\{\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}(\tilde{\varphi}^q), \lambda A^s/\tilde{\varphi}^q\right\}\right] - \tilde{\varphi}^q\right\}} > 0.$$

Even though real balances converge to their nonmonetary levels as $\alpha \to 0$, the real equity price in the cashless limit exceeds the corresponding nonmonetary price by the difference between the goods-pledge option values in the two equilibria plus the resale-option term in (2.80). Because ε^* depends on ι from (2.76), the asset price in the cashless limit remains responsive to monetary policy, and the magnitude of this response is bounded away from zero even as real balances vanish. This contrasts with the conventional result that monetary equilibrium prices and allocations converge to their nonmonetary counterparts when real balances go to zero.

Similar to Lagos and Zhang (2019b), the cashless limit in Proposition 5 does not coincide with the nonmonetary equilibrium because of (2.80) and because two conditions hold: (a) as $\alpha \to 0$, the marginal valuation of investors with credit access remains strictly above ε_L, and (b) $\theta < 1$. Condition (a) arises because demand from high-valuation, credit-access investors supports an asset price that, in the limit, is high enough to induce some low-valuation investors without credit access to sell the asset for cash. Condition (b) captures the market power of financial intermediaries. When $\theta = 1$, brokers have no market power and extract no intermediation fees, and the equity price in the cashless limit coincides with the nonmonetary equilibrium price.

To understand this result, it is useful to review the components of the limiting equity price

$$\lim_{\alpha\to 0}\varphi = \frac{\bar{\varepsilon} + \delta\theta\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n - \varepsilon)dG(\varepsilon) + \delta\theta\frac{\kappa}{1-\kappa}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)dG(\varepsilon) + (1-\theta)\delta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^* - \varepsilon)dG(\varepsilon)}{1-(1-\delta)\lambda\bar{\theta}\left\{\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}(\varrho), \lambda A^s/\varrho\right\}\right] - \varrho\right\}}.$$

The first term in the numerator is the expected dividend flow. The second term represents the investor's share θ of the expected value of the resale option. The third term is the investor's share θ of the expected value of the equity-pledge option, that is, the marginal value of using the asset as collateral to short the bond in order to purchase equity. The fourth term arises in the cashless equilibrium but is absent in the nonmonetary benchmark. It captures the gain in the investor's bargaining position with the bond broker in the equity market, since the outside option is now trading in the equity market without access to bonds rather than autarky. The term on the denominator is the net capitalisation rate i.e., the capitalisation rate minus the goods pledge option. This represents the investors share of the marginal net gain of purchasing the first subperiod consumption good. Notice that the price of the first subperiod consumption good in the cashless limit is lower than in the corresponding nonmonetary equilibrium, i.e., $\varrho < \underline{\varrho} + \frac{1-\theta}{\theta}(\varrho - \underline{\varrho})$. This is because producers in a monetary equilibrium are able to sell their excess inventory to investors who do not contact a broker. Also, investors who enter the goods market are able to purchase these first subperiod consumption goods as an outside option improving the bargaining position of the investor relative to the broker. The restricted supply in the nonmonetary equilibrium allows the broker to exploit their bargaining power to increase the price of the first subperiod consumption good.

Now, observe that the case with $\theta = 1$ (i.e., where bond brokers have no market power to extract intermediation fees), the real equity price in the cashless limit corresponds to the asset price in the nonmonetary equilibrium. In this case the nonmonetary equilibrium price is given by

$$\lim_{\alpha \to 0} \varphi^n = \frac{\bar{\varepsilon} + \delta \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon) dG(\varepsilon) + \delta \frac{\kappa}{1-\kappa} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n) dG(\varepsilon)}{1 - (1-\delta) \frac{\lambda}{\rho \varrho} \left[\tilde{u}' \left[\min \left\{ \tilde{\mathrm{D}}(\varrho), \lambda A^s / \varrho \right\} \right] - \varrho \right]}.$$

Thus, (2.80) holds because investors who enter the equity market with valuations $\varepsilon \in (\varepsilon_L, \varepsilon^*]$ have the option to sell equity for cash in that market. This outside option strengthens their bargaining position with bond brokers and allows them to capture a larger share of the surplus from reallocating their portfolio from equity to bonds. Similarly, investors who enter the goods market can purchase producers' excess inventories for cash, which improves their bargaining position with the bond broker and lowers the first subperiod goods price. The value of these options to an individual investor remains strictly positive as $\alpha \to 0$, even though aggregate real money balances converge to zero. Because these components remain positive in the limit, the asset price stays relatively high and, in particular, exceeds the nonmonetary equilibrium price.

As also found in Lagos and Zhang (2019b), the result (2.80) depends on two fundamental features of the environment. First, bond brokers must have at least some degree of market power, i.e., $\theta < 1$. Second, investors must have at least some ability to take on leverage, i.e., $\kappa > 0$ and $\lambda > 0$.

2.4.1 Moneyless Monetary Economics

Similar to Lagos and Zhang (2019b), our findings on the medium of exchange role of money in the transmission of monetary policy stand in contrast to a large literature that adopts a moneyless framework for monetary economics. This moneyless perspective was advanced by Woodford (1998) and, following the expositions in Woodford (2003) and Galí (2015), is now viewed by many as the standard approach in monetary theory and practice. The usual rationale for a moneyless treatment is that frictions linked to the medium of exchange function are irrelevant for how monetary policy is transmitted.[28]

The cashless limits we consider, for example in Propositions 5 and 6, follow Wicksell's idea of a "pure credit economy" and align with the motivation for Woodford's cashless limit. In general, however, our findings differ from Woodford's. We show that the medium of exchange role of money is central for monetary transmission and that this channel remains operative even in the cashless limit. As $\alpha \to 0$, real balances converge to zero, yet monetary policy continues to affect asset prices, consumption, investment, and output. Moreover, these results persist even as trade volume goes to zero, provided there is some infrequency in market access, that is, $\Delta > 0$ (see Appendix B). There is one special case that reproduces Woodford-like irrelevance of the medium of exchange in the cashless limit: when financial intermediaries have no bargaining power, $\theta = 1$. Hence, to claim that monetary frictions are irrelevant in cashless limiting economies, or nearly irrelevant in economies that are close to cashless, one must also assume that investors always capture the full gains from trade when dealing with intermediaries in financial markets.[29]

As highlighted by Lagos and Zhang (2019b), the point is that $\theta = 1$ is atypical in the empirical

[28] For a comprehensive overview of these arguments, see Lagos and Zhang (2019b) and Lagos and Zhang (2022). There are also well known critiques of reduced form models of money. Kareken and Wallace (1980) offer two. First, specifying money directly in the utility function is a form of "implicit theorising", meaning that while one can tell stories to motivate the specification, the assumptions embedded in those stories are not primitives, and without an explicit underlying environment the internal consistency of the theory cannot be evaluated. Second, reduced form specifications leave too many questions unanswered and explain too little, for example, what precisely is being called "money". Although these critiques are compelling, they are largely set aside in much of applied monetary economics.

[29] Our formulation also imposes borrowing limits for individual investors in both the equity and goods markets. This provides a second departure from what may be called frictionless financial markets. However, our cashless results continue to hold even if $\kappa \to 1$ in the equity market and $\lambda \to 1$ in the goods market.

literature. There does not seem to be any evidence that $\theta = 1$ is typical in practice.[30] For $\theta < 1$, our theory, alongside Lagos and Zhang (2019b, 2022) provide counterexamples to the typical claims used to endorse the moneyless approach.[31] The results presented in Proposition 5 and Proposition 6, as well as in related work (Lagos and Zhang (2019b) and Lagos and Zhang (2022)) indicate that traditional medium-of-exchange considerations are in fact an essential aspect of the monetary transmission mechanism-even in the cashless limit or in near-cashless economies. Any attempt to assess the macroeconomic effects of monetary policy without such considerations is necessarily incomplete.

2.5 Conclusion

This chapter has developed a model in which money is used as a medium of exchange in financial and commercial transactions and aided through access to over-the-counter markets of bond brokers. In any monetary equilibrium where investors have access to both equity and goods markets, and their corresponding loan facilities the asset prices can respond either positively or negatively to changes in monetary policy depending on certain conditions. Asset prices in this monetary equilibrium can be shown to be greater than the corresponding nonmonetary equilibrium. This implies that there are consequences to abstracting away from primitive and general micro foundational structures of monetary economics and assuming reduced-form formulations. We establish this point by showing that in cashless limit economies asset prices differ from their nonmonetary counterparts and that asset prices in cashless limit economies still respond to monetary policy even though real money balances converge to zero so long as bond brokers have some market power.

[30] For a given $\theta < 1$, the cashless limiting economy depends on the policy rate ι and on credit conditions, captured by κ in the baseline model. If the policy rate and leverage are relatively large, as in part (i) of Proposition 5, then money is dominated in rate of return, real balances converge to zero, and asset prices still respond to ι in the cashless limit. If the policy rate and leverage are relatively low, as in part (ii) of Proposition 5, then as $\alpha \to 0$ real balances are not dominated in rate of return, do not converge to zero, and monetary policy remains effective in the limit. In both cases, our conclusions about the effects of ι on prices and allocations in the pure credit limiting economy differ from the limiting irrelevance result in Woodford (2003).

[31] In fact, the claims of the theory presented in this chapter hold even if the bargaining power of brokers in the equity market and goods market are different. Further observe that if θ_1 is the bargaining power of the broker in the equity market, and θ_2 is the bargaining power of the broker in the goods market, then if $\theta_2 = 1$ and $\theta_1 < 1$ then the asset price in the cashless limit asset price is still greater than the nonmonetary equilibrium equity price and that it still is responsive to monetary policy. Similarly, if $\theta_1 = 1$ and $\theta_2 < 1$ then the results still hold except at the continuous time limit (i.e., $\Delta \to 0$), in which case the cashless limits asset price differs from the nonmonetary equilibrium asset price but is no longer responsive to monetary policy.

We conclude by mentioning what may be a promising avenue for future work. Given that the model can generate a speculative premium above the present value of discounted cash flows in a way that responds nontrivially to changes in monetary policy and expected inflation, it would be interesting to explore the quantitative implications of the theory which may shed further light on the transmission of monetary policy.

Chapter 3

Asset Price Dispersion, Monetary Policy and Macroprudential Regulation

ABSTRACT

This chapter considers asset price dispersion, monetary policy, and macroprudential regulation. Following the 2007–2009 Global Financial Crisis, sustained monetary expansion and tighter financial regulation have left financial markets thinner, less resilient, and more prone to instability. This chapter develops a monetary model of decentralised financial exchange to account for these outcomes. The framework links search frictions and costly posting to the joint effects of monetary and regulatory policy on asset prices, quoting behaviour, and market stability. Increases in inflation or posting costs reduce quoting intensity, widen the distribution of executable prices, and raise the probability of trading breakdowns. The model replicates key post-crisis patterns such as wider spreads, higher execution costs, and an increased likelihood of flash-crash events, showing that the interaction between monetary and regulatory policy can unintentionally increase financial market fragility.

3.1 Introduction

One of the striking features of modern financial markets is that the policies introduced to prevent financial instability may themselves be generating new sources of fragility. In the aftermath of the 2007–2009 financial crisis, central banks—including the Federal Reserve, the European Central Bank, and the Bank of Japan—implemented historically unprecedented expansions in the money supply (see Gertler and Karadi 2011, Chadha et al. 2020, and Duc et al. 2022). At the same time, a wave of regulatory reforms, most notably Basel III and the Dodd–Frank Act, sought to curb systemic financial risk. Among these, the Volcker Rule has had a direct impact on market functioning by restricting proprietary trading and limiting the ability of intermediaries to act as principals.

A growing body of evidence indicates that these post-crisis policy shifts have altered the structure of trading and the cost of order execution. Market transactions have become more expensive, and extreme episodes of price dislocation, often described as "flash crashes", have become more frequent (see Jovanovic and Menkveld 2022, Jovanovic and Menkveld 2024, Trebbi and Xiao 2019, Golub et al. 2012, Easley et al. 2012, and Gayduk and Nadtochiy 2018). Several studies document a persistent decline in market-making activity, particularly in fixed income markets, following the implementation of these reforms. Adrian et al. (2017) survey the evidence and find that trading costs have risen across several asset classes since the financial crisis. Anderson and Stulz (2017) show that transaction costs remain elevated for large institutional orders, and Bao et al. (2018) and Dick-Nielsen and Rossi (2019) report that the cost of executing large sales increases sharply in the post-crisis period. The role of dealers as intermediaries has also diminished: Choi et al. (2024) and Bessembinder et al. (2018) find that dealers increasingly avoid acting as principal, especially when capital constraints bind.

Beyond higher trading costs, empirical evidence points to greater market fragility. Adrian et al. (2015) show that bond markets have become more prone to sudden breakdowns in intermediation capacity, while Duffie (2012) and Duffie (2018) demonstrate that post-crisis regulation raised the cost of market making and weakened the resilience of intermediaries under stress. In equity and dealer markets, changes in price posting and execution behaviour have also been observed under new regulatory constraints (Hendershott et al. 2011, Jovanovic and Menkveld

2022, Jovanovic and Menkveld 2024).

In parallel, a growing empirical literature has linked persistently loose monetary policy to broader patterns of financial instability. Grimm et al. (2023) analyse more than a century of data across advanced economies and find that extended periods of monetary accommodation are associated with a higher probability of financial crises. This effect is strongest when accommodative policy coincides with rapid credit growth. Similar evidence is provided by Boissay et al. (2023), Schularick et al. (2021), and Jiménez et al. (2023), which show that prolonged periods of low rates amplify systemic risk and lead to more volatile asset valuations. These studies collectively suggest that the joint stance of monetary and regulatory policy has profound and sometimes unintended effects on market stability.

Existing theoretical models provide only partial explanations for these developments. Standard inventory-cost models such as Grossman and Miller (1988) explain persistent execution costs through dealers' risk aversion and capital constraints but remain static and cannot reproduce abrupt breakdowns in trading activity. Search-based models of over-the-counter trade, beginning with Duffie et al. (2005) and extended by Lagos and Rocheteau (2009), incorporate frictions in matching and bargaining that generate liquidity premia, yet they abstract from the strategic pricing behaviour of intermediaries. More recent frameworks such as Jovanovic and Menkveld (2022) introduce costly participation and random entry, capturing the incentive for market participants to refrain from quoting, but they typically focus on one side of the market and exclude the role of monetary conditions. As a result, the literature does not fully explain why trading activity and price dispersion respond jointly to monetary expansion and regulatory costs, nor why flash-crash-like events can occur endogenously even in periods of apparent stability.

This chapter develops a monetary model of decentralised financial exchange that integrates these missing elements into a single framework. The model shows how the stance of monetary policy and the cost of broker participation jointly determine the structure of bid and ask price distributions, the frequency of trading, and the occurrence of price collapses. Investors trade a real asset using fiat money through brokers who post prices under costly entry and random search, following the framework of Burdett and Judd (1983). The real quantity of money influences the number of active quotes through its role as a medium of exchange, while posting costs capture regulatory or balance-sheet constraints. Their interaction determines the intensity of quote competition, the spread between buying and selling prices, and the likelihood of periods with no active trading.

Specifically, the model links the quantity of money and the structure of market microstructure to fundamental dimensions of trading behaviour. In particular, it shows that (*i*) the structure of bid and ask price distributions is jointly determined by monetary conditions and the cost of broker participation, (*ii*) standard measures of market performance, such as execution costs and liquidity, depend systematically on these factors, (*iii*) the propensity for market instability, including flash-crash-like events, emerges endogenously and responds to changes in policy, and (*iv*) speculative premia and discounts arise as equilibrium outcomes when investors face these trading conditions.

This chapter contributes to four main strands of the literature: search-theoretic models of money, search-based models of financial trade in over-the-counter (OTC) markets, resale-option theories of asset pricing, and costly bidder participation models. By combining insights from each of these areas, we provide a unified framework that captures how monetary conditions and market frictions interact to shape asset prices, price distributions, and market stability.

From a methodological standpoint, the chapter helps bridge the gap between two bodies of work that have traditionally been developed in parallel: the search-theoretic literature on monetary economics, which emphasizes macroeconomic questions such as the efficiency of exchange and the role of liquidity in general equilibrium, and the microeconomic literature on search-based financial markets, which focuses on asset trade in decentralized settings. The core structure of our model builds on the monetary framework of Lagos and Wright (2005), and in particular follows Lagos and Zhang (2020), which adapts the OTC market framework of Duffie et al. (2005) to the context of monetary exchange. Similar hybrid structures also appear in Lagos and Zhang (2019a) and Lagos and Zhang (2019b), which incorporate bilateral asset trade with monetary exchange and frictions.

Within this class of models, several papers have introduced real assets that can serve (at least partially) as media of exchange alongside money. For example, Geromichalos et al. (2007), Jacquet and Tan (2012), Lagos and Rocheteau (2009), Lester et al. (2012), and Nosal and Rocheteau (2013) explore how assets that are useful in facilitating consumption trades acquire a liquidity premium—defined as the wedge between the market price and the fundamental value implied by future dividends. When the asset is useful in exchange, its market price exceeds the expected present discounted value of its future payoffs, reflecting its role as a store of liquidity.

A related concept is the resale premium, developed in Lagos and Zhang (2019a), Lagos and Zhang (2019b), and Lagos and Zhang (2020), which analyse how the value of an asset includes not only its dividend stream but also its usefulness in resale to future investors. This speculative component arises when investors anticipate gains from trade with others which value the asset more highly in the future. Our chapter builds on this resale-option logic but applies it in a model with richer microstructure.

Specifically, we integrate the structure of Lagos and Zhang (2020) with the noisy search and costly posting model of Burdett and Judd (1983), as well as its modern applications to monetary economics as in Head et al. (2012) and Burdett et al. (2015). The framework allows us to endogenise the distribution of prices in a tractable way while preserving the macro-monetary implications of the model. We adopt a special case of the Burdett–Judd framework that is analytically convenient and well-suited to search-theoretic environments, see, Wang et al. (2020) and Mortensen (2005).

Alternative approaches to generating price dispersion arise in the literature on endogenous bidder participation in auction theory. Jovanovic and Menkveld (2022) adapt the model of Levin and Smith (1994) by assuming bidders are uncertain about how many others choose to participate. This removes the multiplicity of asymmetric pure-strategy equilibria and leaves a unique symmetric mixed-strategy equilibrium. Their model generates bid price dispersion with a closed-form distribution that matches the empirical shape of order book bids, though it lacks a monetary channel and resale considerations. The framework in Jovanovic and Menkveld (2022) is closely related to earlier common-value auction models such as Hausch and Li (1993), and the all-pay auction setting in Baye et al. (1996), which also produce strategic dispersion in posted prices.

On the ask side, a parallel literature studies price distributions arising from uncertain customer arrival or limited trading opportunities. For example, Prescott (1975) analyzes a multi-unit auction in which the number of trading opportunities is uncertain, yielding an ask price distribution that is endogenously right-skewed. Similarly, models such as Shilony (1977), Rosenthal (1980), and Varian (1980) generate price dispersion by assuming that sellers face a mix of informed and uninformed (or "captive") customers—whether due to geography, switching costs, or informational frictions.

By bringing together these strands of research—monetary frictions, noisy price posting, speculative resale motives, and endogenous participation—we construct a unified framework capable of explaining both macro-level phenomena such as the impact of inflation and bidder participation on asset markets, and micro-level features such as price dispersion, flash crashes, and bid-ask spreads.

The rest of this chapter is organised as follows. Section 3.2 describes the basic model. Equilibrium is characterised in Section 3.3. Section 3.4 presents the theoretical implications of the model for asset prices. Section 3.5 presents the theoretical implications of the model to financial liquidity. Section 3.6 provides a discussion of the results in the context of the literature. Section 3.7 concludes. All proofs are contained in Appendix D.

3.2 Model

Time is represented by a sequence of periods indexed by $t = 0, 1, \ldots$ Each time period is divided into two subperiods in which different activities take place. There is a continuum of infinitely lived agents called *investors*, each identified with a point in the set $\mathcal{I} = [0, 1]$. There is also a continuum of infinitely lived agents called *asking brokers*, each identified with a point in the set $\mathcal{B}^A = [0, 1]$, and a continuum of infinitely lived agents called *bidding brokers*, each identified with a point in the set $\mathcal{B}^B = [0, 1]$.[1] All agents discount across periods with factor, $\beta \in (0, 1)$.

There is a continuum of production units with measure $A^s \in \mathbb{R}_{++}$ that are active every period. Every active unit yields an exogenous *dividend* $y_t \in \mathbb{R}_+$. Each active unit yields the same dividend, therefore $y_t A^s$ is the aggregate dividend. At the beginning of every period,

[1] It is possible to model asking and bidding brokers as a single representative broker type that performs both functions. However, such an approach complicates the exposition and obscures the distinction between the two sides of the market. For clarity, we treat asking and bidding brokers as separate agent types. This separation is purely expositional and does not affect the equilibrium results.

each active unit is subject to an independent idiosyncratic shock that renders it permanently unproductive with probability $1 - \eta \in [0, 1)$. If a production unit remains active, its dividend in period t is $y_t = \gamma_t y_{t-1}$, where γ_t is a nonnegative random variable with cumulative distribution function Φ, i.e., $\mathbb{P}(\gamma_t \leq \gamma) = \Phi(\gamma)$, and mean $\bar{\gamma} \in (0, (\beta\eta)^{-1})$.

The time-t dividend becomes public knowledge at the beginning of period t. At that time each failed production unit is replaced by a new unit that pays y_t in the initial period and follows the same process as other active units thereafter. The dividend of the initial set of production units, $y_0 \in \mathbb{R}_{++}$, is given at $t = 0$. In the second subperiod of every period, every agent has access to a linear production technology that transforms effort into a perishable homogeneous consumption good.

For each active production unit there exists a durable and perfectly divisible *equity share* that represents ownership of the production unit and entitles the holder to its dividends. At the start of every period $t \geq 1$, each investor receives an endowment of $(1 - \eta)A^s$ equity shares associated with the new production units (when a production unit fails, its equity share is extinguished). There is a second financial instrument, *money*, that is intrinsically useless. The quantity of money at time t is denoted A_t^m. The initial quantity of money, $A_0^m \in \mathbb{R}_{++}$, is given, and $A_{t+1}^m = \mu A_t^m$, with $\mu \in \mathbb{R}_{++}$. A monetary authority injects or withdraws money via lump-sum transfers or taxes to investors in the second subperiod of each period. At the beginning of period $t = 0$, each investor is endowed with a portfolio of equity shares and money.[2] All financial instruments are perfectly recognisable, not forgeable, and can be traded among agents in every subperiod.

In the second subperiod of every period, all investors can trade the consumption good produced in that subperiod, equity shares, and money in a spot Walrasian market. In the first subperiod of every period, trading is organised as follows: each broker posts a nominal price p, taking as given the distribution of prices posted by all other brokers (for asking brokers, this is described by the cumulative distribution function (CDF) $A_t(p)$, and for bidding brokers, by the CDF $B_t(p)$). Investors know the distributions $A_t(p)$ and $B_t(p)$ but can only contact, and

[2] We assume that brokers do not hold financial assets. This assumption allows us to abstract from the broker's portfolio problem in the first subperiod, which is not essential for the questions we study in this chapter. See Lagos and Zhang (2019b) or Lagos and Zhang (2020) for a treatment of the broker's portfolio problem in this class of models.

hence can only trade with, a random sample of brokers. An investor contacts $k \in \{0, 1, 2, \dots\}$ brokers with probability $\alpha_k \in (0, 1)$. Once an investor and broker have met, the transaction takes place at the price posted by the broker. Brokers act as intermediaries, executing trades on behalf of investors in the decentralised market. After the transaction is completed, the investor and broker part ways.[3] The timing assumption is that the round of trade between investors and brokers takes place in the first subperiod of a typical period t, and ends before the equity shares yield dividends.[4] Hence, equity is traded *cum-dividend* in the first subperiod, but *ex-dividend* in the Walrasian market of the second subperiod. We assume that agents cannot make binding commitments, that there is no enforcement, and that histories of actions are private, which precludes any borrowing or lending, so all trade must be *quid pro quo*. This assumption, together with the structure of preferences described below, creates the need for a medium of exchange.[5,6]

A broker's preferences are given by

$$\mathbb{E}_0^B \sum_{t=0}^{\infty} \beta^t (c_t - h_t),$$

where c_t and h_t denote a broker's consumption of the homogeneous good that is produced, traded, and consumed in the second subperiod of period t, and the utility cost of exerting h_t units of effort to produce this good.[7] The expectation operator $\mathbb{E}_0^B$ is with respect to the probability measure induced by the random trading process in the first subperiod.[8]

[3]See Zhang (2018) for an OTC model with long-term relationships between investors and dealers.

[4]As in previous search models of OTC markets, e.g., Duffie et al. (2005) and Lagos and Rocheteau (2009), an investor must own the equity in order to consume the dividend flow of the consumption good in the first subperiod.

[5]Under these conditions, there cannot exist a futures market for the equity share or the dividend, so an agent who wishes to consume the dividend must hold the equity share at the time the dividend is paid. A similar assumption is typically made in search models of financial OTC trade, e.g., Duffie et al. (2005), Lagos and Rocheteau (2009), Lagos and Zhang (2019b), or Lagos and Zhang (2020).

[6]See Lagos and Zhang (2019b) for a similar model where investors can buy equity with *margin loans*.

[7]Brokers receive no utility from the dividends of the equity share, so they have no motive to purchase equity on their own account in the first subperiod. This assumption can be relaxed, but it is standard in the search-based OTC literature, e.g., Duffie et al. (2005), Lagos and Rocheteau (2009), or Weill (2007).

[8]For analytical tractability, we abstract from brokers holding equity on their own account. While in reality dealer positions contribute to observed trade volume, modelling such behaviour would add unnecessary complexity and is not essential for addressing the questions pursued in this chapter. Accordingly, we do not model trade volume explicitly and instead proxy trading activity by quote intensity and execution frequency, see Lagos and Zhang (2019b).

An investor's preferences are given by

$$\mathbb{E}_0^I \sum_{t=0}^{\infty} \beta^t (\varepsilon_i y_t + c_t - h_t),$$

where c_t is consumption of the homogeneous good that is produced, traded, and consumed in the second subperiod of period t, and h_t is the utility cost of exerting h_t units of effort to produce this good. The variable y_t is the quantity of the dividend good that an investor consumes at the end of the first subperiod of period t, and ε_i denotes the realisation of a preference shock that is distributed independently over time and across agents, with cumulative distribution function G, where $\varepsilon_i \in \{\varepsilon_L, \varepsilon_H\}$ with corresponding probabilities $\pi_i \in (0,1)$ and $\bar{\varepsilon} = \sum_{i \in \{L,H\}} \pi_i \varepsilon_i$. An investor learns his realisation ε_i at the beginning of period t, before trading has commenced. The expectation operator $\mathbb{E}_0^I$ is with respect to the probability measure induced by the dividend process, the investor's preference shock, and the random trading process in the first subperiod of each period t.

3.3 Equilibrium

Consider the determination of allocations in the first subperiod of period t for an investor who enters the period with portfolio $\mathbf{a}_t \equiv (a_t^m, a_t^s)$, consisting of $a_t^m \in \mathbb{R}_+$ units of money and $a_t^s \in \mathbb{R}_+$ units of equity, and preference type ε_i.

Let $W_t(\mathbf{a}_t)$ denote the maximum expected discounted payoff of an investor holding portfolio $\mathbf{a}_t$ at the beginning of the second subperiod of period t, after dividends have been realised. Suppose the investor enters period t with (pre-trade) portfolio $\mathbf{a}_t \equiv (a_t^m, a_t^s)$ and preference type ε_i. If the investor meets a broker, then they are able to trade and their post-trade portfolio is $\bar{\mathbf{a}}_t(\mathbf{a}_t, \varepsilon_i) \equiv (\bar{a}_t^m(\mathbf{a}_t, \varepsilon_i), \bar{a}_t^s(\mathbf{a}_t, \varepsilon_i))$, and,

$$(\bar{a}_t^m(\mathbf{a}_t, \varepsilon_i), \bar{a}_t^s(\mathbf{a}_t, \varepsilon_i)) = \underset{(\bar{a}_t^m, \bar{a}_t^s) \in \mathbb{R}_+^2}{\arg\max} \; \varepsilon_i y_t \bar{a}_t^s + W_t(\bar{\mathbf{a}}_t) \tag{3.1}$$
$$\text{s.t. } \bar{a}_t^m + p\bar{a}_t^s \le a_t^m + p a_t^s.$$

where p is the dollar price of equity in the first subperiod of period t.

We now analyse the investor's problem in a typical period. Let $V_t(\mathbf{a}_t, \varepsilon_i)$ denote the maximum expected discounted payoff of an investor with type ε_i and portfolio $\mathbf{a}_t = (a_t^m, a_t^s)$ at the beginning of the first subperiod of period t. Let ϕ_t^m be the real price of money, and ϕ_t^s be the *ex-dividend* price of equity in the second subperiod of period t (both expressed in terms of the second subperiod consumption good). In the second subperiod, the investor chooses consumption of the homogeneous good c_t, the utility cost of production h_t, and the end-of-subperiod portfolio $\tilde{\mathbf{a}}_{t+1} \equiv (\tilde{a}_{t+1}^m, \tilde{a}_{t+1}^s)$, recognising that at the end of the second subperiod a fraction $1-\eta$ of equity is rendered unproductive and replaced. Therefore, we have,

$$W_t(\mathbf{a}_t) = \max_{(c_t, h_t, \tilde{\mathbf{a}}_{t+1}) \in \mathbb{R}_+^4} \left[c_t - h_t + \beta \mathbb{E}_t \sum_{i \in \{L,H\}} \pi_i V_{t+1}(\mathbf{a}_{t+1}, \varepsilon_i) \right] \tag{3.2}$$

$$\text{s.t. } c_t + \boldsymbol{\phi}_t \tilde{\mathbf{a}}_{t+1} \leq h_t + \boldsymbol{\phi}_t \mathbf{a}_t + T_t$$

$$\mathbf{a}_{t+1} = \left[\tilde{a}_{t+1}^m, \eta \tilde{a}_{t+1}^s + (1-\eta) A^s \right],$$

where $\mathbb{E}_t$ is the conditional expectation over the next-period realisation of the dividend, T_t is the real value of the time-t lump-sum monetary transfer (a tax if negative), $\boldsymbol{\phi}_t \equiv (\phi_t^m, \phi_t^s)$, and $\boldsymbol{\phi}_t \mathbf{a}_t$ denotes the dot product of $\boldsymbol{\phi}_t$ and $\mathbf{a}_t$. Since ε_i is independently and identically distributed over time, $W_t(\mathbf{a}_t)$ is independent of ε_i, and the portfolio that each investor chooses to carry into period $t+1$ is independent of ε_i.

We can now write the value function of an investor who enters the first subperiod of period t with portfolio $\mathbf{a}_t$ and preference type ε_H,

$$\begin{aligned} V_t(\mathbf{a}_t, \varepsilon_H) = \alpha_0 W_t(\mathbf{a}_t) &+ \alpha_1 \int \{ \varepsilon_H y_t \bar{a}_t^s(\mathbf{a}_t, \varepsilon_H) + W_t [\bar{\mathbf{a}}_t(\mathbf{a}_t, \varepsilon_H)] \} \, dA_t(p) \\ &+ \alpha_2 \int \{ \varepsilon_H y_t \bar{a}_t^s(\mathbf{a}_t, \varepsilon_H) + W_t [\bar{\mathbf{a}}_t(\mathbf{a}_t, \varepsilon_H)] \} \, d\left\{ 1 - [1 - A_t(p)]^2 \right\} \\ &\ldots, \end{aligned} \tag{3.3}$$

where $A_t(p)$ is the cumulative distribution function of ask prices. This expression indicates that an investor with a high valuation ε_H, and thus a desire to purchase equity (as shown in Lemma 1 below), faces the following outcomes. With probability α_0, the investor does not observe any prices and consequently enters the second subperiod of period t with an unchanged portfolio $\mathbf{a}_t$ without making a purchase. With probability α_1, the investor observes one price drawn from the ask distribution $A_t(p)$ and purchases equity at that price. With probability α_2, the investor observes two prices and chooses to trade at the lower of the two, which follows the distribution $1-[1-A_t(p)]^2$. More generally, with probability α_k, the investor observes $k \geq 1$ prices and transacts at the minimum of these observed prices, which is distributed according to $1-[1-A_t(p)]^k$. Hence, the value function in equation (3.3) can be simplified and expressed as follows:

$$V_t(\mathbf{a}_t, \varepsilon_H) = \alpha_0 W_t(\mathbf{a}_t) + \int \sum_{k=1}^{\infty} \alpha_k k[1-A_t(p)]^{k-1} \left\{\varepsilon_H y_t \bar{a}_t^s(\mathbf{a}_t, \varepsilon_H) + W_t\left[\bar{\mathbf{a}}_t(\mathbf{a}_t, \varepsilon_H)\right]\right\} dA_t(p). \tag{3.4}$$

Similarly, the value function for an investor who enters the first subperiod of period t with portfolio $\mathbf{a}_t$ and a low valuation type ε_L,

$$\begin{aligned} V_t(\mathbf{a}_t, \varepsilon_L) = \alpha_0 W_t(\mathbf{a}_t) &+ \alpha_1 \int \left\{\varepsilon_L y_t \bar{a}_t^s(\mathbf{a}_t, \varepsilon_L) + W_t\left[\bar{\mathbf{a}}_t(\mathbf{a}_t, \varepsilon_L)\right]\right\} dB_t(p) \\ &+ \alpha_2 \int \left\{\varepsilon_L y_t \bar{a}_t^s(\mathbf{a}_t, \varepsilon_L) + W_t\left[\bar{\mathbf{a}}_t(\mathbf{a}_t, \varepsilon_L)\right]\right\} d\left\{[B_t(p)]^2\right\} \\ &\ldots, \end{aligned} \tag{3.5}$$

where $B_t(p)$ is the cumulative distribution function of bid prices. This expression indicates that an investor with a low valuation ε_L, and thus a desire to sell equity (as shown in Lemma 1 below), faces the following outcomes. With probability α_0, the investor does not observe any prices and consequently enters the second subperiod of period t with an unchanged portfolio $\mathbf{a}_t$ without making a sale. With probability α_1, the investor observes one price drawn from the bid distribution $B_t(p)$ and sells equity at that price. With probability α_2, the investor observes two prices and chooses to trade at the higher of the two, which follows the distribution $[B_t(p)]^2$. More generally, with probability α_k, the investor observes $k \geq 1$ prices and transacts at the maximum of these observed prices, which is distributed according to $[B_t(p)]^k$. Hence, the value function in equation (3.4) can be simplified and expressed as follows:

$$V_t(\mathbf{a}_t, \varepsilon_L) = \alpha_0 W_t(\mathbf{a}_t) + \int \sum_{k=1}^{\infty} \alpha_k k[B_t(p)]^{k-1} \left\{\varepsilon_L y_t \bar{a}_t^s(\mathbf{a}_t, \varepsilon_L) + W_t\left[\bar{\mathbf{a}}_t(\mathbf{a}_t, \varepsilon_L)\right]\right\} dB_t(p). \tag{3.6}$$

Summarising, the value functions (3.4) and (3.6) can be written as one expression which corresponds to the value function in (3.2),

$$V_t(\mathbf{a}_t, \varepsilon_i) = \alpha_0 W_t(\mathbf{a}_t) + \int \sum_{k=1}^{\infty} \alpha_k k \left\{ \varepsilon_i y_t \bar{a}_t^s(\mathbf{a}_t, \varepsilon_i) + W_t \left[\bar{\mathbf{a}}_t(\mathbf{a}_t, \varepsilon_i) \right] \right\} dF_t^k(p, \varepsilon_i), \tag{3.7}$$

where $dF_t^k(p, \varepsilon_i) := \mathbb{I}_{\{\varepsilon_i = \varepsilon_H\}}[1 - A_t(p)]^{k-1} dA_t(p) + \mathbb{I}_{\{\varepsilon_i = \varepsilon_L\}}[B_t(p)]^{k-1} dB_t(p)$ and $\mathbb{I}_{\{\cdot\}}$ is an indicator function that takes the value 1 if ε_i satisfies its condition and 0 otherwise.

Next we analyse the broker's problem in a typical period. Let $W_t^B(\varsigma_t)$ denote the maximum expected discounted payoff at the beginning of the second subperiod of period t for a broker who has earned spread ς_t. Then,

$$W_t^B(\varsigma_t) = \phi_t^m \varsigma_t + \beta \mathbb{E}_t V_{j,t+1}^B, \tag{3.8}$$

where $V_{j,t+1}^B$ denote the maximum expected discounted payoff of either an asking or bidding broker, i.e, $j \in \{a, b\}$, where a denotes an asking broker and b denotes a bidding broker, at the beginning of the first subperiod of period $t+1$. We can now write the value function of an asking broker who enters the first subperiod of period t,

$$V_{a,t}^B = \max_{p \in \mathcal{A}} \left\{ \left[\sum_{k=1}^{\infty} \alpha_k k \left[1 - A_t(p) \right]^{k-1} \right] W_t^B \left(\frac{p - \hat{p}}{\kappa_t} \right) \right\} + \alpha_0 W_t^B(0). \tag{3.9}$$

Here, κ_t denotes the cost of posting a quote, and $\hat{p}$ represents the mid-price quote around which bids and asks are symmetrically distributed.[9,10] The broker's objective is to choose the price that maximises expected profits, taking into account the probability that a contacted investor will accept the quote.[11] The term inside the brackets, $\alpha_k k[1 - A_t(p)]^{k-1}$, represents the probability that an investor who observes the broker's quote has received $k-1$ other quotes, and is summed over all k. Intuitively, this captures the probability that an investor who meets this broker has possibly observed $k-1$ competing prices higher than p, and will therefore transact at the

[9] The symbol $\hat{p}$ denotes the mid-price quote, defined as $\hat{p} = (\bar{p} + \underline{p})/2$, where $\bar{p}$ is the highest ask price and $\underline{p}$ is the lowest bid price.

[10] Because the same underlying probability structure applies to both asking and bidding brokers, as well as to investors, the equilibrium distributions of posted prices are symmetric around the mid-price $\hat{p}$. This assumption can be generalised to allow for asymmetries across agent types, but doing so does not lead to any substantive change in the results. For ease of exposition, we therefore maintain symmetric probabilities throughout.

[11] The asking broker's price support $\mathcal{A}$ is determined in equilibrium.

broker's quote.

Similarly, the value function of a bidding broker who enters the first subperiod of period t is

$$V_{b,t}^{B} = \max_{p \in \mathcal{B}} \left\{ \left[\sum_{k=1}^{\infty} \alpha_k k \left[B_t(p) \right]^{k-1} \right] W_t^B \left(\frac{\hat{p} - p}{\kappa_t} \right) \right\} + \alpha_0 W_t^B(0). \tag{3.10}$$

The bidding broker's objective (3.10) is to choose the price that maximises expected profits, taking into account the probability that a contacted investor will accept the bid to sell equity.[12] The term inside the brackets, $\alpha_k k[B_t(p)]^{k-1}$, represents the probability that an investor who observes the broker's quote has received $k-1$ other quotes, and is summed over all k. Intuitively, this captures the probability that an investor who meets this broker has possibly observed $k-1$ competing bids lower than p, and will therefore transact at the broker's quote.

Let A_{It}^m and A_{It}^s denote the quantities of money and equity shares, respectively, held by all investors at the beginning of the period t (after production units have depreciated and been replaced). That is, $A_{It}^m = \int a_t^m dH_t(\mathbf{a}_t)$ and $A_{It}^s = \int a_t^s dH_t(\mathbf{a}_t)$, where H_t is the cumulative distribution function over portfolios $\mathbf{a}_t = (a_t^m, a_t^s)$ held by investors at the beginning at the first subperiod of period t. Let $\tilde{A}_{It+1}^m$ and $\tilde{A}_{It+1}^s$ denote the total quantities of money and shares held by all investors at the end of period t, i.e., $\tilde{A}_{It+1}^h = \int_{\mathcal{I}} \tilde{a}_{it+1}^h di$ for $h \in \{s, m\}$ with $A_{It+1} = \tilde{A}_{It+1}^m$, and $A_{It+1}^s = \eta \tilde{A}_{It+1}^s + (1-\eta) A^s$. Let $\bar{A}_{It}^m$ and $\bar{A}_{It}^s$ denote the quantities of money and shares held after the first subperiod of period t by all investors who are able to trade in the first subperiod. For asset $h \in \{s, m\}$, $\bar{A}_{It}^h = \int \sum_{k=1}^{\infty} \bar{a}_t^h(\mathbf{a}_t, \varepsilon_i) dF_t^k(p, \varepsilon_i)$. We are now ready to define equilibrium.

Definition 1. *An* equilibrium *is a sequence of prices* $\{\phi_t^m, \phi_t^s\}_{t=0}^{\infty}$*, distributions* $\{A_t(p), B_t(p)\}_{t=0}^{\infty}$*, portfolio allocations in the first subperiod* $\{\bar{\mathbf{a}}_t(\cdot)\}_{t=0}^{\infty}$*, and end-of-period portfolio* $\{\tilde{\mathbf{a}}_{t+1}\}_{t=0}^{\infty}$*, such that for all t: (i) the portfolios in the first subperiod solve* (3.1)*; (ii) taking prices and first subperiod portfolios as given, the end-of-period portfolio solves* (3.2)*; and (iii) prices are such that all Walrasian markets clear, i.e.,* $\tilde{A}_{It+1}^s = A^s$ *(the end-of-period t Walrasian market for equity clears),* $\bar{A}_{It}^s = A^s$ *(the first subperiod market for equity clears), and* $(\bar{A}_{It}^m - A_t^m)\mathbb{I}_{\{\phi_t^m > 0\}} = 0$ *(the first subperiod market for money clears); and, (iv) taking distributions as given prices solve the brokers' problems* (3.9) *and* (3.10)*.*

[12]The bidding broker's price support $\mathcal{B}$ is determined in equilibrium.

The first step toward characterising equilibrium is to find the post-trade portfolio allocations of the investor. The maximisation problem in (3.1) represents the portfolio problem of an investor with price p in the first subperiod of period t. The solution is summarised as follows:

Lemma 1. *Define* $\varepsilon_t^* \equiv \frac{p\phi_t^m - \phi_t^s}{y_t}$ *and*

$$\chi(\varepsilon_t^*, \varepsilon_i) \begin{cases} = 1 & \text{if } \varepsilon_t^* < \varepsilon_i \\ \in [0,1] & \text{if } \varepsilon_t^* = \varepsilon_i \\ = 0 & \text{if } \varepsilon_t^* > \varepsilon_i. \end{cases} \tag{3.11}$$

Consider an investor who enters the first subperiod of period t with portfolio $\mathbf{a}_t$ and valuation ε_i. The investor's post-trade portfolio, $[\bar{a}_t^m(\mathbf{a}_t, \varepsilon_i), \bar{a}_t^s(\mathbf{a}_t, \varepsilon_i)]$, is given by

$$\bar{a}_t^m(\mathbf{a}_t, \varepsilon_i) = [1 - \chi(\varepsilon_t^*, \varepsilon_i)]\,(a_t^m + pa_t^s) \tag{3.12}$$

$$\bar{a}_t^s(\mathbf{a}_t, \varepsilon_i) = \chi(\varepsilon_t^*, \varepsilon_i)(1/p)(a_t^m + pa_t^s). \tag{3.13}$$

Lemma 1 provides a complete characterisation of the post-trade portfolios of investors in the first subperiod of period t.[13] The outcome depends on whether the investor's valuation ε_i is above or below a cutoff level ε_t^*. That is, investors with preference type $\varepsilon_i > \varepsilon_t^*$ use all their cash to purchase equity, whereas if $\varepsilon_i < \varepsilon_t^*$, they sell all of their equity holdings for cash.[14]

In what follows, we specialise the analysis to recursive monetary equilibria. That is, equilibria in which asset holdings are constant over time, i.e., $A_{It}^s = A_I^s$, and real asset prices are time-invariant functions of the aggregate dividend, i.e., $\phi_t^s = \phi^s y_t$, $p\phi_t^m \equiv \bar{\phi}_t^s = \bar{\phi}^s y_t = p\varphi^m y_t$, $\kappa_t = \kappa y_t$, $\phi_t^m A_{It}^m = ZA^s y_t$, where $Z \in \mathbb{R}_+$. Hence, in a recursive monetary equilibrium $\varepsilon_t^* = p\varphi^m - \phi^s \equiv \varepsilon^*$, $\phi_{t+1}^s/\phi_t^s = \gamma_{t+1}$, and $\phi_t^m/\phi_{t+1}^m = \mu/\gamma_{t+1}$. Throughout the analysis, we let $\bar{\beta} \equiv \beta\bar{\gamma}$, $\bar{\beta}_L \equiv \pi_L\bar{\beta}$, and $\bar{\beta}_H \equiv \pi_H\bar{\beta}$ and maintain the following assumption,

Assumption 1. *The number of quotes $k \in \{0, 1, 2, \dots\}$ received by an investor is Poisson*

[13]The lemma is stated in a general form that accommodates both discrete and continuous specifications of the preference shock ε_i.

[14]Given ε_t^*, we can interpret $p\phi_t^m = \varepsilon_t^* y_t + \phi_t^s$ is the *cum-dividend* real value of equity to the marginal investor in period t.

distributed with mean $\theta \in \mathbb{R}_+$. *That is, the probability* α_k *of receiving* k *quotes is given by*

$$\alpha_k = \frac{\theta^k e^{-\theta}}{k!}. \tag{3.14}$$

Following Wang et al. (2020) and Mortensen (2005), we assume that the number of quotes k observed by each investor follows a Poisson distribution with mean $\theta \in \mathbb{R}_+$.[15] This specification captures the independence of quote arrivals across investors and over time while allowing closed-form solutions for equilibrium price distributions. The parameter θ is the *quote intensity.* Given this structure, the following result characterises the equilibrium ask and bid price distributions, denoted by $A(p)$ and $B(p)$, respectively. Building on the framework of Burdett and Judd (1983), these distributions describe the optimal pricing behaviour of brokers posting quotes in a decentralised market with random search and posting costs. Closed-form expressions for the equilibrium ask and bid distributions consistent with profit maximisation by all brokers are presented below.

Proposition 1. *If* α_k *is Poisson with mean* $\theta \in \mathbb{R}_+$, *then*

(i) There exists a unique ask price distribution consistent with profit maximisation by all asking brokers given by

$$A(p) = \frac{1}{\theta} \ln \left[\frac{\varphi^m (p - \hat{p})}{\kappa} \right], \tag{3.15}$$

with support $\mathcal{A} \equiv [\hat{p} + \kappa/\varphi^m, \bar{p}] \subset \mathbb{R}_+$ *and the corresponding density is*

$$a(p) = \frac{1}{\theta (p - \hat{p})}. \tag{3.16}$$

(ii) There exists a unique bid price distribution consistent with profit maximisation by all asking brokers given by

$$B(p) = 1 - \frac{1}{\theta} \ln \left[\frac{\varphi^m (\hat{p} - p)}{\kappa} \right], \tag{3.17}$$

[15] Since the number of quotes is Poisson with mean $\theta \in \mathbb{R}_+$, its variance also equals θ. The coefficient of variation CV_θ, defined as the standard deviation divided by the mean, is therefore $CV_\theta = 1/\sqrt{\theta}$, which strictly decreases in θ. This is intuitive since one would expect greater variability when there is less quoting activity in the market.

with support $\mathcal{B} \equiv \left[\underline{p}, \hat{p} - \kappa/\varphi^m\right] \subset \mathbb{R}_+$ *and the corresponding density is*

$$b(p) = \frac{1}{\theta\left(\hat{p} - p\right)}, \tag{3.18}$$

(iii) The equilibrium quote intensity is given by

$$\theta = \ln\left(\frac{\varphi^m \Gamma}{\kappa}\right), \tag{3.19}$$

where $\Gamma \equiv \bar{p} - \hat{p} = \hat{p} - \underline{p}$ *is the book-width.*

An asking broker faces a trade-off between posting a higher price to extract a larger surplus conditional on execution and posting a lower, more competitive price to increase the probability of being selected by an investor. The same logic applies symmetrically to bidding brokers, who trade off offering a higher bid to attract sellers against a lower bid to retain surplus when a trade occurs. The equilibrium price distributions, $A(p)$ for asks and $B(p)$ for bids, represent the mixed-strategy solutions to these optimisation problems under random matching and Poisson-distributed quote arrivals. Price dispersion arises endogenously from brokers' strategic indifference across feasible prices.[16]

The supports of $A(p)$ and $B(p)$ correspond to the bounds at which brokers are indifferent between posting the highest and lowest feasible prices consistent with trade. For the ask side, the lowest possible ask $\hat{p} + \kappa/\varphi^m$ represents the price at which the broker just breaks even after paying the posting cost κ.[17] Similarly, on the bid side, the highest possible bid $\hat{p} - \kappa/\varphi^m$ represents the price at which a broker is indifferent between posting and withdrawing from the market.

Finally, the quote intensity θ links market liquidity to monetary conditions and the cost of market participation. In particular, the assumption that quote arrival probabilities are identical on both sides of the market implies a symmetric quote intensity $\theta > 0$. Under this symmetry,

[16] Jovanovic and Menkveld (2022) generate a similar dispersion of bids, but their model considers only the bid side of the market and abstracts from both monetary policy considerations and asset price determination.

[17] As shown in Proposition 1, the equilibrium quote intensity satisfies $\theta = \ln(\varphi^m \Gamma/\kappa)$. This condition can be rearranged to yield $\kappa = e^{-\theta}(\varphi^m \Gamma)$, indicating that κ equals the expected benefit when no investor meets the broker. Hence, κ can be interpreted as the effective cost of participation or posting in the decentralised market.

the lower bound of the real value of money satisfies $\varphi_L^m = \kappa/\Gamma$.[18] This symmetry in quote arrival ensures that both sides of the market exhibit analogous statistical properties, allowing the equilibrium price distributions to be characterised by the following corollary.

Corollary 1. *If α_k is Poisson with mean $\theta \in \mathbb{R}_+$, then*

(i) The ask density $a(p)$ strictly decreases in p, is convex and prices are therefore right-skewed.

(ii) The bid density $b(p)$ strictly increases in p, is convex and prices are therefore left-skewed.

The intuition behind these results follows directly from the brokers' indifference conditions across the price support. On the ask side, the strictly decreasing and convex shape of $a(p)$ implies that higher ask prices are posted less frequently, producing a right-skewed distribution. As the ask price rises, the probability that an investor will accept the quote falls disproportionately, so asking brokers must engage in progressively stronger price shading to remain indifferent across feasible prices. Conversely, on the bid side, the strictly increasing and convex shape of $b(p)$ implies that lower bid prices are posted less frequently, yielding a left-skewed distribution. As the bid price declines, the probability of a successful trade diminishes sharply, and bidding brokers must similarly shade prices more aggressively to preserve indifference across the support.

The properties of the ask and bid distributions extend naturally to their order-statistic counterparts, the best-ask and best-bid distributions, defined as the minimum and maximum of the k observed quotes, respectively.

Corollary 2. *If α_k is Poisson with mean $\theta \in \mathbb{R}_+$, then*

(i) There exists a unique best-ask distribution given by

$$\bar{A}(p) = \begin{cases} 1 - \frac{\kappa}{\varphi^m(p-\hat{p})} & \text{if } p \in \left[\hat{p} + \frac{\kappa}{\varphi^m}, \bar{p}\right) \\ 1 & \text{if } p = \bar{p}. \end{cases} \tag{3.20}$$

The corresponding best-ask density $\bar{a}(p)$ strictly decreases in p, is convex and prices are therefore right-skewed.

[18] Since $\varphi_L^m = \kappa/\Gamma$, it follows that κ must also be bounded from above. If κ were to increase beyond a certain point, φ_L^m would approach φ_H^m, the upper bound of the real value of money given below in Proposition 2, thereby violating the interiority condition required for a monetary equilibrium.

(ii) There exists a unique best-bid distribution given by

$$\bar{B}(p) = \begin{cases} 0 & \text{if } p = \underline{p} \\ \frac{\kappa}{\varphi^m(\hat{p}-p)} & \text{if } p \in \left(\underline{p}, \hat{p} - \frac{\kappa}{\varphi^m}\right]. \end{cases} \tag{3.21}$$

The corresponding best-bid density $\bar{b}(p)$ strictly increases in p, is convex and prices are therefore left-skewed.

This result formalises how the process of quote sampling and selection translates into the distributions of executable prices in equilibrium. The best-ask distribution, $\bar{A}(p)$, represents the probability that the lowest observed ask price is less than or equal to p, while the best-bid distribution, $\bar{B}(p)$, represents the probability that the highest observed bid price is less than or equal to p. Both are derived directly from the underlying quote distributions $A(p)$ and $B(p)$ under the Poisson sampling structure given by Assumption 1.

The convexity and skewness properties of $\bar{A}(p)$ and $\bar{B}(p)$ mirror those of $A(p)$ and $B(p)$, whereby the distributions are more concentrated near the most competitive quotes. Specifically, the best-ask distribution is right-skewed, indicating that most observed best asks cluster near the lower bound of the support, while the best-bid distribution is left-skewed, with most observed best bids clustering near the upper bound.

Having characterised the distributions of quotes and executable prices, we now turn to the determination of equilibrium asset prices implied by these trading frictions. The following result establishes the existence and uniqueness of a recursive monetary equilibrium in which the real (*ex-dividend*) equity price and the real value of money are jointly determined by investors' portfolio choices, brokers' optimal pricing strategies, and the underlying monetary environment.

The result below formalises this relationship by linking the real equity price ϕ^s and the real value of money φ^m to the posting cost κ, and the inflation rate μ. For the analysis that follows, it is convenient to define

$$\bar{\mu} \equiv \bar{\beta}_L \left\{ 1 + \frac{1}{1-\bar{\beta}_H\eta} \cdot \frac{\pi_H}{1-\pi_H} \left[\varepsilon_H \frac{\Gamma}{\kappa\bar{p}} + \bar{\beta}_H \eta \frac{\underline{p}}{\bar{p}} \right] \right\}. \tag{3.22}$$

The following proposition characterises the equilibrium set of the recursive monetary equilibrium.

Proposition 2. *If α_k is Poisson with mean $\theta \in \mathbb{R}_+$, and $\bar{\beta} < \mu < \bar{\mu}$, then there exists a unique recursive monetary equilibrium where real (ex-dividend) equity price is given by*

$$\phi^s = \frac{\bar{\beta}_H \eta}{1 - \bar{\beta}_H \eta} \left\{ \varepsilon_H + \frac{\pi_L}{1 - \pi_L} \left[\varphi^m \hat{p} - \frac{\kappa \theta}{1 - e^{-\theta}} \right] \right\}, \tag{3.23}$$

and the real value of money $\varphi^m \in (\varphi^m_L, \varphi^m_H)$ is the unique solution to

$$\frac{\varepsilon_H + \phi^s}{1 - e^{-\theta}} \int_{\hat{p} + \frac{\kappa}{\varphi^m}}^{\bar{p}} \frac{\kappa}{p \left[\varphi^m (p - \hat{p})\right]^2} dp = \frac{\mu - \bar{\beta}_L}{\bar{\beta}_L}. \tag{3.24}$$

This proposition characterises the equilibrium relationship between the real value of money, the equity price, and the liquidity conditions prevailing in the decentralised exchange market. Given Poisson distributed quote arrivals with mean θ, and a monetary policy satisfying $\bar{\beta} < \mu < \bar{\mu}$, there exists a unique recursive monetary equilibrium in which the real equity price ϕ^s and the real value of money φ^m jointly satisfy the two conditions above.[19]

The equilibrium expression for ϕ^s highlights two distinct components. The first term, $\frac{\bar{\beta}_H \eta}{1 - \bar{\beta}_H \eta} \varepsilon_H$, represents the fundamental value of equity, which is the discounted present value of the future dividend flow as perceived by high-valuation investors. The second term inside the braces, $\varphi^m \hat{p} - \frac{\kappa \theta}{1 - e^{-\theta}}$, captures the real expected value of the best-bid distribution and therefore the resale premium of the asset. A low-valuation investor values the asset at the expected present value of dividends evaluated at the mean valuation across investor types. When they sell, however, the real asset price reflects the net benefit they obtain from trade. This benefit arises because the selling investor receives more than their own valuation: by transferring the asset to a high-valuation investor, they realise the additional value that the buyer places on the dividend stream, along with the resale premium captured by the expected best bid. In this way, the equilibrium real asset price reflects both the speculative gain from reallocating the asset toward higher-valuation investors and the liquidity premium generated by active trading in the market.

[19] The Appendix derives the expression for Z, given by $Z = \pi_L (1 - e^{-\theta}) \left[\pi_H \int_{\hat{p} + \kappa / \varphi^m}^{\bar{p}} \frac{\kappa}{p [\varphi^m (p - \hat{p})]^2} dp \right]^{-1}$. This expression is omitted from the main text to maintain focus on the core equilibrium relationships. It can be shown that $\partial Z / \partial \mu < 0$ and $\partial Z / \partial \kappa < 0$, indicating that higher inflation or posting costs reduce the aggregate value of money balances held by investors.

The second equilibrium condition implicitly defines the real value of money φ^m as the unique level consistent with optimal portfolio choices and monetary policy. The left-hand side captures the expected liquidity return from holding money, weighted by the probability of successful trade across the support of the ask distribution, while the right-hand side, $\frac{\mu-\bar{\beta}_L}{\beta_L}$, represents the inflation-induced opportunity cost of holding money. Equilibrium is attained when these two margins are equal, yielding an interior solution $\varphi^m \in (\varphi^m_L, \varphi^m_H)$.

3.4 Asset Prices

In this section, we examine the properties of the equilibrium asset prices derived in Proposition 2. In particular, we analyse how these prices respond to changes in monetary policy and to the degree of trading frictions in the first subperiod as captured cost of posting.

Assumption 2. *The elasticity of the value of money φ^m with respect to the posting cost κ is less than one, i.e. $\epsilon_{\varphi^m,\kappa} < 1$, where $\epsilon_{\varphi^m,\kappa} \equiv \dfrac{\partial \varphi^m}{\partial \kappa}\dfrac{\kappa}{\varphi^m}$.*

This assumption restricts the elasticity of the value of money with respect to posting cost to be strictly subunitary, i.e. changes in κ affect φ^m less than proportionally. It ensures that the effect of κ on φ^m is such that an increase in κ leads to a decrease in quoting intensity θ, which is consistent with what we would expect. Although an equilibrium also exists when this condition is violated, we focus on this case since it corresponds to the most natural setting and aligns with what is typically observed empirically.[20]

3.4.1 Inflation

Variations in the inflation rate μ influence investors' portfolio decisions by changing the relative attractiveness of holding money between trading rounds. Higher inflation causes a reduction in the value of money and a decline in real money balances. This reduction in real balances, together with the lower real value of money, leads to a fall in quote intensity θ. The decline in quote intensity reflects the fact that, under the higher inflation rate, the investor whose valuation was marginal under the lower rate is no longer indifferent between carrying cash and

[20] The comparative statics with respect to κ reverse direction in most of the following propositions when this condition is violated. These cases are not considered here, as violating the condition produces economically unnatural outcomes that are not typically observed.

equity out of the first subperiod market. With fewer investors entering the decentralised market with sufficient real balances, competition among brokers weakens. As a result, the resale option available to investors becomes less valuable, and the real equity price ϕ^s falls. Naturally, the real value of money φ^m is also decreasing in the growth rate of the money supply. These arguments are formalised in Proposition 3.

Proposition 3. *If α_k is Poisson with mean $\theta \in \mathbb{R}_+$, then in the recursive monetary equilibrium:*

(i) The real value of money φ^m is decreasing with inflation μ i.e., $\partial\varphi^m/\partial\mu < 0$.

(ii) The real (ex-dividend) price of the equity share ϕ^s is decreasing with inflation μ i.e., $\partial\phi^s/\partial\mu < 0$.

Proposition 3 is particularly useful in environments where monetary policy operates primarily through adjustments in the expected inflation rate. In such settings, changes in the nominal interest rate i are associated exclusively with variations in inflation μ, rather than with changes in the real interest rate, r.[21,22]

3.4.2 Posting Cost

Through its effect on brokers' incentives, the posting cost κ governs how actively quotes are supplied and thus how much trading occurs in the decentralised market. Higher posting costs discourage brokers from entering, which reduces the effective quote intensity θ. The decline in quote intensity reflects the fact that, under a higher κ, some brokers who were marginal at the previous cost level are no longer indifferent between posting a quote and remaining inactive. With fewer active brokers submitting quotes, investors encounter fewer trading opportunities in each round. The reduced ability to trade raises the value of holding real money balances between trading rounds, as investors must maintain higher balances to smooth consumption and meet potential trading opportunities when they arise. Consequently, the real value of money φ^m rises

[21] It is also possible to investigate the effect of policy changes in the real interest rate r, noting that the Fisher relation implies $1 + i = \mu/\bar{\beta}$, which is equivalent to $1 + i = (1 + r)(1 + \pi)$ with $1 + \pi \equiv \mu/\bar{\gamma}$, where π is the gross inflation rate. See Lagos and Zhang (2020) for a detailed discussion of this relationship. This analysis lies outside the scope of the present chapter but can be inferred directly from the relationship above.

[22] The results of Proposition 3 are consistent with the resale–premium literature of Lagos and Zhang (2015, 2019a,b, 2020), where asset prices decline with inflation, making equity a poor hedge against inflation. This empirical relationship corresponds to what is commonly known as the 'Fed model'. For an extended discussion of the Fed model and its connection to resale-premium asset-pricing frameworks, see Lagos and Zhang (2019a).

with κ. This increase in the value of money dominates the decline in quote intensity, leading to a higher resale premium and therefore a higher real equity price ϕ^s.[23] In equilibrium, both the real value of money and the real price of equity increase with the posting cost, reflecting the positive relationship between intermediation frictions and the valuation of liquid assets. These mechanisms are formalised in Proposition 4.

Proposition 4. *If α_k is Poisson with mean $\theta \in \mathbb{R}_+$, then in the recursive monetary equilibrium:*

(i) The real value of money φ^m is increasing with the posting cost κ, i.e., $\partial\varphi^m/\partial\kappa > 0$.

(ii) The real (ex-dividend) price of the equity share ϕ^s is increasing with the posting cost κ i.e., $\partial\phi^s/\partial\kappa > 0$.

Proposition 4 provides a theoretical basis for interpreting regulatory interventions as effective increases in the posting cost κ. By raising the cost of intermediation, such policies reduce quote intensity and make holding real money balances more valuable, which in turn increases the real equity price ϕ^s through the resale channel.

3.5 Financial Liquidity

In this section, we examine how monetary policy and trading frictions jointly shape equilibrium outcomes in the recursive monetary economy. We analyse how variations in monetary conditions and in the degree of market frictions influence key aspects of decentralised trade and asset pricing. Specifically, we study how these factors affect quote intensity, the stochastic dominance properties of the price distributions, the extent of price dispersion, and the determination of bid–ask spreads. We also examine how the interaction between monetary and regulatory conditions contributes to systemic vulnerabilities, including flash-crash risk, and how it shapes speculative behaviour in decentralised financial markets.

[23] Assumption 2 ensures that the effect of κ on φ^m is such that θ is decreasing in κ, as stated in Proposition 5 below. Hence, the effect of κ on φ^m does not dominate the direct effect that κ has on θ. However, this dominance holds in the case of the real equity price ϕ^s, where the increase in φ^m more than offsets the decline in quote intensity and direct effect of κ.

3.5.1 Quote Intensity

Consider first how monetary policy and posting costs affect quoting behaviour in the decentralised market. According to Propositions 2 and 3, an increase in the inflation rate μ reduces investors' incentive to maintain real balances. With fewer investors carrying adequate real balances, the value of the spread earned by brokers is reduced, which in turn lowers their incentive to post quotes. The resulting decline in quoting activity leads to a reduction in quoting intensity θ. Similarly, according to Proposition 4 and given Assumption 2, a higher posting cost κ raises the marginal cost of participation for brokers and further depresses quoting intensity θ.[24] This intuition is formalised in the following proposition.

Proposition 5. *If α_k is Poisson with mean $\theta \in \mathbb{R}_+$ and $\epsilon_{\varphi^m,\kappa} < 1$, then in the recursive monetary equilibrium, the quote intensity θ: (i) decreases with inflation μ, i.e., $\partial\theta/\partial\mu < 0$; and (ii) decreases with the posting cost κ, i.e., $\partial\theta/\partial\kappa < 0$.*

The quote intensity θ thus serves as the principal transmission channel through which monetary and regulatory policy affect outcomes in the decentralised market. As we will see, variations in θ translate policy shocks into observable changes in spreads, depth, and the overall shape of the bid and ask price distributions.

3.5.2 Stochastic Dominance

The best-ask distribution $\bar{A}(p)$ gives the probability that the lowest observed ask price is less than or equal to p, and therefore captures the behaviour of marginal sellers. When the inflation rate μ rises, the real value of money φ^m declines and investors enter the decentralised market with smaller real balances. The resulting fall in quote intensity θ reduces the number of competing ask quotes that investors observe, shifting probability mass in $\bar{A}(p)$ toward lower prices. A rise in

[24] Following the 2007–2009 crisis, a structural break in liquidity supply has been linked to post-crisis regulation. On July 21, 2010, the Dodd–Frank Act was signed into law. Section 619, the Volcker Rule, generally prohibits insured depository institutions and their affiliates from proprietary trading and from certain relationships with hedge funds and private equity funds, subject to exemptions and definitions. Restricting proprietary trading removed banks as deep-pocket market makers and reduced immediacy provision. The request for public comments closed on November 5, 2010, by which time market participants likely understood the substance of the forthcoming rule, helping to explain the observed break in liquidity supply near the end of 2010. The amended rule became effective on April 1, 2014, with full compliance required by July 21, 2015. The interval between these dates coincides with an accelerated decline in 50-basis-point depth and a reduction in the average number of middlemen, consistent with the view that the Volcker Rule accounted for a large part of the depth decline.

the posting cost κ produces a similar effect. Higher κ raises the cost of participation for brokers and lowers θ, reducing the frequency of competitive quoting and shifting the distribution of executable ask prices downward. In both cases, $\bar{A}(p)$ takes smaller values for every p, indicating a downward shift of the CDF. Formally, for any p, $\bar{A}_2(p) < \bar{A}_1(p)$ when $\mu_2 > \mu_1$ or $\kappa_2 > \kappa_1$, implying that the distribution associated with lower inflation or lower posting costs first-order stochastically dominates that under higher values of either parameter. The following proposition formalises this intuition.[25]

Proposition 6. *If α_k is Poisson with mean $\theta \in \mathbb{R}_+$ and $\epsilon_{\varphi^m,\kappa} < 1$, then in the recursive monetary equilibrium the best-ask distribution $\bar{A}(p)$ satisfies: (i) if $\mu_2 > \mu_1$ then $\bar{A}_{\mu_1,\kappa}(p)$ first-order stochastically dominates $\bar{A}_{\mu_2,\kappa}(p)$; and, (ii) if $\kappa_2 > \kappa_1$ then $\bar{A}_{\mu,\kappa_1}(p)$ first-order stochastically dominates $\bar{A}_{\mu,\kappa_2}(p)$.*

The best-bid distribution $\bar{B}(p)$ gives the probability that the highest observed bid price is less than or equal to p, and therefore captures the behaviour of marginal buyers. When the inflation rate μ rises, the real value of money φ^m declines and investors enter the decentralised market with smaller real balances. The associated fall in quote intensity θ reduces the number of competing bids that investors observe, shifting probability mass in $\bar{B}(p)$ toward higher prices. A rise in the posting cost κ produces a similar effect. Higher κ raises the cost of participation for brokers and lowers θ, reducing the frequency of competitive quoting and shifting $\bar{B}(p)$ toward higher prices. Formally, for any p, $\bar{B}_2(p) > \bar{B}_1(p)$ when $\mu_2 > \mu_1$ or $\kappa_2 > \kappa_1$, implying that the distribution associated with higher inflation or higher posting costs first-order stochastically dominates that under lower values of either parameter. The following proposition formalises this intuition.[26]

Proposition 7. *If α_k is Poisson with mean $\theta \in \mathbb{R}_+$ and $\epsilon_{\varphi^m,\kappa} < 1$, then in the recursive monetary equilibrium the best-bid distribution $\bar{B}(p)$ satisfies: (i) if $\mu_2 > \mu_1$ then $\bar{B}_{\mu_2,\kappa}(p)$ first-order*

[25] Proposition 6A in the Appendix shows that the ask distribution is decreasing in both μ and κ. Higher inflation reduces real balances, and higher posting costs discourage participation, both of which make high asking prices less likely in equilibrium. Only the results for the best-ask distribution are presented in the main text, as the analysis focuses on executable prices.

[26] Proposition 7A in the Appendix shows that the bid distribution is increasing in both μ and κ. Rising inflation lowers real balances and higher posting costs reduce quoting intensity, both of which shift bids downward and make lower bid prices more probable. Only the results for the best-bid distribution are presented in the main text, as the analysis focuses on executable prices.

stochastically dominates $\bar{B}_{\mu_1,\kappa}(p)$*; and, (ii) if* $\kappa_2 > \kappa_1$ *then* $\bar{B}_{\mu,\kappa_2}(p)$ *first-order stochastically dominates* $\bar{B}_{\mu,\kappa_1}(p)$*.*

3.5.3 Price Dispersion

The mean and variance of the best-ask distribution are given respectively by

$$\mathcal{M}_a = \hat{p} + \frac{\kappa\theta}{\varphi^m(1-e^{-\theta})} \quad \text{and} \quad \mathcal{V}_a = \left(\frac{\kappa}{\varphi^m}\right)^2 \left[e^{\theta} - \left(\frac{\theta}{1-e^{-\theta}}\right)^2\right]$$

These expressions summarise how the level and dispersion of executable ask prices vary with monetary and posting conditions. When the inflation rate μ increases, the real value of money φ^m declines and investors enter the decentralised market with smaller real balances. The resulting fall in quote intensity θ reduces the number of competing ask quotes that investors observe, leading to a higher average execution price and greater variation across observed asks. Consequently, both the mean $\mathcal{M}_a$ and the variance $\mathcal{V}_a$ rise with inflation. A similar mechanism operates when the posting cost κ increases. Higher κ discourages brokers from posting, lowers θ, and widens the distance between the most and least competitive ask quotes. As a result, $\mathcal{M}_a$ rises since the average executed ask price increases, and $\mathcal{V}_a$ rises as the distribution becomes more dispersed. Formally, both $\mathcal{M}_a$ and $\mathcal{V}_a$ are increasing in μ and κ, indicating that higher inflation or posting costs shift the best-ask distribution upward and amplify its cross-sectional spread. The following proposition formalises this intuition.

Proposition 8. *If* α_k *is Poisson with mean* $\theta \in \mathbb{R}_+$ *and* $\epsilon_{\varphi^m,\kappa} < 1$*, then in the recursive monetary equilibrium, (i) the mean* $\mathcal{M}_a$ *of the best-ask distribution is increasing in both* μ *and* κ*; and, (ii) the variance* $\mathcal{V}_a$ *of the best-ask distribution is increasing in both* μ *and* κ*.*

The mean and variance of the best-bid distribution are given respectively by

$$\mathcal{M}_b = \hat{p} - \frac{\kappa\theta}{\varphi^m(1-e^{-\theta})} \quad \text{and} \quad \mathcal{V}_b = \left(\frac{\kappa}{\varphi^m}\right)^2 \left[e^{\theta} - \left(\frac{\theta}{1-e^{-\theta}}\right)^2\right]$$

When the inflation rate μ rises, the real value of money φ^m declines, and investors enter the decentralised market with smaller real balances. The resulting fall in quote intensity θ reduces the number of competing bids that investors observe, leading to a lower average execution price.

The reduction in competition among brokers also increases the spread between high and low bids, raising the variance $\mathcal{V}_b$. A rise in the posting cost κ produces the same comparative statics. Higher κ discourages brokers from quoting, lowers θ, and widens the range of feasible bid prices. As a result, the mean $\mathcal{M}_b$ declines while the variance $\mathcal{V}_b$ increases. Formally, $\mathcal{M}_b$ is decreasing and $\mathcal{V}_b$ is increasing in both μ and κ, indicating that higher inflation or posting costs shift the best-bid distribution downward and make it more dispersed. The following proposition formalises this intuition.

Proposition 9. *If α_k is Poisson with mean $\theta \in \mathbb{R}_+$ and $\epsilon_{\varphi^m,\kappa} < 1$, then in the recursive monetary equilibrium, (i) the mean $\mathcal{M}_b$ of the best-bid distribution is decreasing in both μ and κ; and, (ii) the variance $\mathcal{V}_b$ of the best-bid distribution is increasing in both μ and κ.*

Beyond the mean and variance, a more informative way to assess dispersion is through the coefficient of variation CV. Defined as the ratio of the standard deviation to the mean, $CV_j = \sqrt{\mathcal{V}_j}/\mathcal{M}_j$ for $j \in \{a, b\}$. This statistic provides a scale-free measure of variability that captures how dispersed a distribution is relative to its average level. It serves as an important indicator of uncertainty in decentralised markets, reflecting how volatile or concentrated quotes are around their mean price. A higher coefficient of variation indicates greater heterogeneity in observed quotes and lower predictability in execution prices.

In the context of the best-bid distribution $\bar{B}(p)$, a rise in the inflation rate μ reduces the real value of money φ^m and lowers investors' real balances. The resulting decline in quote intensity θ decreases competition among brokers, leading to a fall in the mean $\mathcal{M}_b$ and a rise in the variance $\mathcal{V}_b$. Because these effects move in opposite directions, the ratio $\sqrt{\mathcal{V}_b}/\mathcal{M}_b$ increases, indicating that bids become more dispersed relative to their average level. An increase in the posting cost κ produces the same comparative statics: higher costs discourage quoting, reduce θ, and widen the spread between high and low bids. Consequently, this intuition is summarised by the following corollary.

Corollary 3. *If α_k is Poisson with mean $\theta \in \mathbb{R}_+$ and $\epsilon_{\varphi^m,\kappa} < 1$, then in the recursive monetary equilibrium, the coefficient of variation CV_b of the best-bid distribution: (i) increases with inflation, i.e., $\partial CV_b/\partial\mu > 0$; and (ii) increases with the posting cost, i.e., $\partial CV_b/\partial\kappa > 0$.*

By contrast, the coefficient of variation CV_a of the best-ask distribution $\bar{A}(p)$ exhibits no general direction with respect to these parameters. Changes in μ or κ affect the mean and variance of the best-ask distribution $\bar{A}(p)$ in offsetting ways, making the overall response of $\sqrt{\mathcal{V}_a}/\mathcal{M}_a$ ambiguous in equilibrium.

Propositions 8 and 9, together with Corollary 3, show that monetary and intermediation conditions jointly determine the degree of price dispersion in decentralised markets. These results are consistent with the patterns established in Propositions 6 and 7, but provide a more detailed characterisation of how changes in inflation and posting costs translate into differences in the distribution of executable prices and the extent of dispersion across trades.

3.5.4 Spreads

Proposition 8 shows that the mean price $\mathcal{M}_a$ with respect to the best-ask distribution $\bar{A}(p)$ captures the average price at which the marginal seller is willing to transact, conditional on being the most competitive offer at any given time. Proposition 9 shows that the mean price $\mathcal{M}_b$ with respect to the best-bid distribution $\bar{B}(p)$ captures the average price at which the marginal buyer is willing to transact, conditional on being the most competitive bid. Hence, the *nominal bid–ask spread* is $\mathcal{S}^m = |\mathcal{M}_a - \mathcal{M}_b|$ which is given by

$$\mathcal{S}^m = \frac{2\kappa\theta}{\varphi^m(1 - e^{-\theta})}$$

When the inflation rate μ increases, the real value of money φ^m declines, and investors enter the decentralised market with smaller real balances. The reduction in purchasing power lowers the quoting intensity θ, leading to fewer competing quotes and transactions over a wider range of offers. As a result, the nominal bid–ask spread $\mathcal{S}^m$ increases with inflation. Similarly, when the posting cost κ rises, brokers quote less and require greater expected compensation to recover their costs. The higher cost of participation directly widens the price differential between the average best-bid and best-ask quotes, while the fall in θ further amplifies this effect by reducing market competition.

Proposition 10. *If α_k is Poisson with mean $\theta \in \mathbb{R}_+$ and $\epsilon_{\varphi^m,\kappa} < 1$, then in the recursive mone-*

tary equilibrium, the nominal bid-ask spread $\mathcal{S}^m$: (i) increases with inflation, i.e., $\partial \mathcal{S}^m / \partial \mu > 0$; and (ii) increases with the posting cost, i.e., $\partial \mathcal{S}^m / \partial \kappa > 0$.

To evaluate how these differences translate into real economic costs, it is useful to consider the *real bid–ask spread*, defined as the nominal spread scaled by the real value of money φ^m, i.e., $\mathcal{S} \equiv \varphi^m \mathcal{S}^m$. The real spread measures the effective trading cost in terms of consumption goods and therefore captures the true resource cost of market participation. In equilibrium, variations in inflation and posting costs influence the real spread only through their effects on the quoting intensity θ, linking monetary and intermediation frictions to the efficiency of asset exchange. The following result is a corollary of Proposition 10.[27]

Corollary 4. *If α_k is Poisson with mean $\theta \in \mathbb{R}_+$ and $\epsilon_{\varphi^m,\kappa} < 1$, then in the recursive monetary equilibrium, the real bid-ask spread $\mathcal{S}$: (i) decreases with inflation, i.e., $\partial \mathcal{S} / \partial \mu < 0$; and (ii) increases with the posting cost, i.e., $\partial \mathcal{S} / \partial \kappa > 0$.*

This result is novel within the monetary resale literature, as the behaviour of the real bid–ask spread in response to monetary policy and trading frictions is generally ambiguous in earlier frameworks such as Lagos and Zhang (2015). The present framework yields unambiguous comparative statics, showing that the real bid–ask spread decreases with inflation and increases with the posting cost, consistent with the empirical literature.

3.5.5 Flash Crash Risk

A *flash crash* is defined as the event in which no quotes are observed by investors, so that no transactions take place in the decentralised market. Since α_0 denotes the proportion of investors who receive no quote in equilibrium, it corresponds directly to the probability of a flash crash. Given that the number of quotes k received by each investor is Poisson distributed with mean θ, the probability that no quotes are observed is $\mathcal{F} = e^{-\theta}$. Following Jovanovic and Menkveld (2022), this formulation captures the idea that flash crashes arise endogenously as a result of strategic inaction: when the expected gains from trade fall below the cost of posting, brokers

[27] The functional form of the real bid–ask spread used here is identical to that in Jovanovic and Menkveld (2022), which yields similar comparative statics with respect to the posting cost κ. In both settings, higher intermediation costs raise the equilibrium real bid–ask spread through its direct effect and by lowering quoting intensity. Their model, however, focuses solely on the bid side of pricing and abstracts from monetary conditions and asset valuation, which are central to the present analysis.

optimally choose not to quote, leading to episodes of market illiquidity even in the absence of exogenous shocks or informational frictions.[28]

Monetary policy affects flash crash risk through its influence on quote intensity. Higher inflation increases the opportunity cost of holding money between trading rounds, reducing the real balances investors bring into the decentralised market. With fewer investors carrying liquidity, brokers face weaker trading opportunities and lower expected gains from posting quotes. This decline in activity compresses the value of the spread, further reducing the incentive to quote. As a result, both investor participation and broker activity fall, lowering the equilibrium quote intensity θ and increasing $\mathcal{F}$. Similarly, higher posting costs κ raise the marginal cost of quoting for brokers. When these costs become more significant relative to the expected spread, fewer brokers find it profitable to post quotes, and those that do participate less actively. The resulting decline in quote intensity again raises the probability that no quotes are observed.

In equilibrium, both tighter monetary conditions and higher trading frictions weaken market participation and increase flash crash risk, making the decentralised market more fragile. The risk of a flash crash therefore increases when inflation is high or the cost of posting rises, as summarised in the following proposition.[29]

[28]Define the *half-spread* $\mathcal{H}$ as the difference between the mid-price and the expected best-ask price or the expected best-bid price: $\mathcal{H} \equiv \mathcal{M}_a - \hat{p} = \hat{p} - \mathcal{M}_b = \kappa\theta/\big(\varphi^m(1-e^{-\theta})\big) = \mathcal{S}^m/2$. Given this we can write the variance of the best-ask distribution and the best-bid distribution as $\mathcal{V} \equiv \mathcal{V}_a = \mathcal{V}_b = \kappa^2/\big((\varphi^m)^2 e^{-\theta}\big) - \mathcal{H}^2 = \kappa^2/\big((\varphi^m)^2\mathcal{F}\big) - \mathcal{H}^2$. It is also important to note that we define the *mean-squared error* (MSE) in the following way, $\text{MSE}_a = \mathbb{E}_{\bar{A}}[(p-\hat{p})^2] = \mathcal{V}_a + (\mathcal{M}_a - \hat{p})^2 = \mathcal{V} + \mathcal{H}^2 = \text{MSE}_b \equiv \text{MSE}$, and therefore $\text{MSE} = \kappa^2/\big((\varphi^m)^2\mathcal{F}\big)$. With $\theta = \ln(\varphi^m\Gamma/\kappa)$, this specification links the variance $\mathcal{V}$, the half-spread $\mathcal{H}$, the book width Γ, the non-crash pricing error MSE, and the flash-crash probability $\mathcal{F}$ in a single set of identities: $\text{MSE} = \mathcal{V} + \mathcal{H}^2 = \kappa\,\Gamma/\varphi^m$ and $\mathcal{F} = \kappa/(\varphi^m\Gamma)$. The inverse proportionality then has a precise meaning: MSE and $\mathcal{F}$ are reciprocal transforms of the same book-width object Γ (up to the fixed scale factor κ/φ^m). Put differently, increases in Γ raise MSE and lower $\mathcal{F}$, and the product $\text{MSE}\cdot\mathcal{F} = \kappa^2/(\varphi^m)^2$ is conserved. This makes the frequency–magnitude trade-off exact: doubling $\mathcal{F}$ halves MSE, and as $\mathcal{F} \downarrow 0$ we must have $\text{MSE} \uparrow \infty$. It also yields a testable and calibratable restriction (near-constancy of $\text{MSE}\cdot\mathcal{F}$) and a clear diagnostic: observed $(\mathcal{V}, \mathcal{H}^2)$ must add to $\text{MSE} = \kappa\,\Gamma/\varphi^m$, while any violation of $\text{MSE}\cdot\mathcal{F} = \kappa^2/(\varphi^m)^2$ points to misspecification or measurement error. A decline in MSE can reflect *more* probability mass in crash states (higher $\mathcal{F}$), not necessarily safer day-to-day conditions: by $\text{MSE}\cdot\mathcal{F} = \kappa^2/(\varphi^m)^2$, lower MSE must be offset by higher $\mathcal{F}$ (or a change in κ/φ^m). Hence low day-to-day noise in quotes, i.e., low non-crash pricing error (MSE small), can go hand-in-hand with elevated crash probability ($\mathcal{F}$ high). Thus warning that smaller MSE may reflect a reallocation of risk into rarer crash states rather than genuinely safer conditions.

[29]Corollary 2A in the Appendix derives the Value-at-Risk (quantile) functions of the best bid and best ask distributions. It demonstrates that these functions exhibit a piecewise structure composed of continuous segments separated by discrete mass points at the price boundaries, $\bar{p}$ for asks and $\underline{p}$ for bids. These mass points represent the non-negligible probability that no quotes are posted at the extremes of the market, corresponding to episodes of illiquidity or flash crashes. The size and location of the mass points depend on the ratio $\kappa/(\varphi^m\Gamma)$, which links the cost of posting quotes, κ, the value of money, φ^m, and the spread, Γ. When φ^m increases, the continuous intervals of the quantile functions widen, indicating a higher density of posted quotes and greater market depth, as brokers find it more profitable to supply quotes. This reduces the probability mass at the boundaries and lowers flash-crash risk. In contrast, increases in κ shrink these intervals, pushing more probability mass toward the boundaries and raising the likelihood that no quotes are posted. The contraction in quoting activity translates into thinner

Proposition 11. *If α_k is Poisson with mean $\theta \in \mathbb{R}_+$ and $\epsilon_{\varphi^m,\kappa} < 1$, then in the recursive monetary equilibrium, the flash crash risk $\mathcal{F}$: (i) increases with inflation, i.e., $\partial\mathcal{F}/\partial\mu > 0$; and (ii) increases with the posting cost, i.e., $\partial\mathcal{F}/\partial\kappa > 0$.*

This is a notable result, as it implies that policies typically designed to mitigate financial instability, such as looser monetary conditions or higher regulatory posting costs, can in this framework amplify market fragility by increasing the probability of a flash crash. When higher money growth or stricter intermediation requirements reduce quoting activity, market depth erodes and liquidity becomes more fragile, raising the likelihood that trading halts occur endogenously. This mechanism highlights a potential policy trade-off between stability in normal times and vulnerability to sudden market breakdowns.

3.5.6 Speculation

It is standard to define the *fundamental value* of an asset as the expected present discounted value of its future dividend stream, $\hat{\phi}_t^s \equiv \left[\bar{\beta}\eta/(1-\bar{\beta}\eta)\right]\bar{\varepsilon}y_t \equiv \hat{\phi}^s y_t$, and to refer to any transaction value in excess of this fundamental value as a *bubble*. One could argue, however, that the relevant notion of "fundamental value" should incorporate market aggregation across heterogeneous investor valuations and reflect the influence of monetary policy and trading frictions such as the frequency of trading opportunities and the market power of intermediaries, which together determine asset prices in equilibrium. To avoid semantic ambiguity, we follow Harrison and Kreps (1978) and define the *speculative premium (or discount)* as the difference between the market price and the fundamental value, $\mathcal{P}_t \equiv \phi_t^s - \hat{\phi}_t^s$, which captures the value investors assign to the right to resell the asset in the future. Investors exhibit speculative behaviour when this resale option induces them to pay more or less for the asset than they would if obliged to hold it indefinitely.[30] According to Proposition 2, the speculative premium/discount is $\mathcal{P}_t = \mathcal{P}y_t$,

markets and greater fragility. The piecewise structure captured by equations of Corollary 2A highlights the nonlinear response of market stability to changes in policy and participation costs: small variations in μ or κ can shift the support of the quote distributions and alter the boundary mass points in ways that sharply amplify the probability of flash crashes.

[30] Our notion of *speculative premium* corresponds to the concept of a *speculative bubble* used in the modern literature, e.g., Barlevy (2007), Abreu and Brunnermeier (2003), Scheinkman and Xiong (2003), Scheinkman (2013), and Xiong (2013), following the *resale-option theory of bubbles* introduced by Harrison and Kreps (1978).

where

$$\mathcal{P} = \frac{\bar{\beta}_H \eta}{1 - \bar{\beta}_H \eta} \left\{ \varepsilon_H + \frac{\pi_L}{1 - \pi_L} \left[\varphi^m \hat{p} - \frac{\kappa \theta}{1 - e^{-\theta}} \right] \right\} - \frac{\bar{\beta} \eta}{1 - \bar{\beta} \eta} \bar{\varepsilon}.$$

Note that, unlike Lagos and Zhang (2015, 2019b), which generate only speculative premia, the present model can produce both speculative premia and speculative discounts. This represents a novel feature of the environment and one that is consistent with findings in the empirical literature. In particular, a speculative discount $\mathcal{P} < 0$ arises when $\varphi^m \mathcal{M}_b < \varepsilon_L + \hat{\phi}^s$; equality implies no speculative component $\mathcal{P} = 0$, and otherwise there is a speculative premium $\mathcal{P} > 0$. Economically, $\varphi^m \mathcal{M}_b$ denotes the real mean best bid, the real value investors expect to obtain when selling the asset at the average bid price in the decentralised market, or, in other words, the expected *cum-dividend* price of the equity share. This value declines as inflation reduces real balances and lower quote intensity limits trading opportunities. The term $\varepsilon_L + \hat{\phi}^s$ represents the fundamental benchmark, combining the valuation of the low-type investor with the expected discounted value of future dividends.

Since Proposition 3 shows that a higher inflation rate lowers the real asset price, it follows that the speculative premium decreases with inflation. Higher inflation raises the opportunity cost of holding money and reduces the real balances investors carry into the decentralised market. With less money available, quote intensity falls and the resale option embedded in the asset loses value. Investors are therefore less willing to pay above the fundamental value, and the speculative premium declines.

In contrast, an increase in the posting cost κ raises the speculative premium. While higher posting costs reduce brokers' quoting activity and increase trading frictions, they also raise the real value of money φ^m which induces investors to hold larger real balances between trading rounds. This increase in φ^m dominates the opposing effects of κ on market participation, supporting higher asset valuations and amplifying the resale option value embedded in prices, consistent with Proposition 4.

In summary, monetary loosening reduces speculative behaviour by weakening resale opportunities, whereas higher posting costs increase it, since the decline in quote intensity is more than offset by the rise in money holdings generated by the higher real value of money that accompanies an increase in κ. These relationships are formalised in the following proposition.

Proposition 12. *If α_k is Poisson with mean $\theta \in \mathbb{R}_+$, then in the recursive monetary equilibrium, the speculative premium $\mathcal{P}$: (i) decreases with inflation, i.e., $\partial\mathcal{P}/\partial\mu < 0$; and (ii) increases with the posting cost, i.e., $\partial\mathcal{P}/\partial\kappa > 0$.*

A novel implication of the model is that speculative premia/discounts increase in response to higher intermediation costs, even though such costs reduce quote intensity. Evidence consistent with this mechanism may already be present in the empirical literature (see Duffie 2012, 2018).

3.6 Discussion

This section highlights where the model's predictions line up with established stylized facts regarding the data. First, consider the shape of buy and sell quotes. Empirical work shows that buy and sell quote distributions are asymmetric and convex: bids cluster below the midprice and asks above it. This pattern is persistent across countries and asset classes, as documented in Biais et al. (1995); Goldstein and Kavajecz (2004); Hollifield et al. (2006); Næs and Skjeltorp (2006); Degryse et al. (2015). Proposition 1 generates the same shape on both sides in the model. Strategic posting with participation costs and random matching yields ask and bid distributions that are convex and skewed. Corollaries 1 and 2 correspond to the best quote and tail behaviour reported in Zovko and Farmer (2002), Weber and Rosenow (2005), and Gu et al. (2008), where best quotes concentrate near the inside and executions display long tails.

Second, broker participation has declined following post-crisis regulatory reforms. Jovanovic and Menkveld (2022) document that the number of active intermediaries in U.S. equity markets fell by more than half over an eleven-year period, with a sharp break following the introduction of the Volcker Rule in 2013–2014. Bessembinder et al. (2018) and Choi et al. (2024) show similar changes, with dealers stepping back from principal activity and quoting becoming noticeably sparser as capital and balance-sheet constraints tightened. Proposition 5 generates this reduction in broker participation in the model. Increased regulatory requirements correspond to a higher effective posting cost, which lowers broker entry and reduces quote intensity. This results in sparser quoting, consistent with the empirical evidence.

Third, declines in broker participation have been accompanied by reductions in market depth and increases in effective trading costs. Empirical studies have shown that although quoted spreads narrowed modestly in the decade following the crisis, deeper execution metrics reveal a substantial deterioration in market depth. Adrian et al. (2017) find that while bid–ask spreads in the sample of Jovanovic and Menkveld (2022) fell by around 6 percent per year, depth near the best quote fell by almost 20 percent annually, and depth within 50 basis points of the midprice declined by more than 25 percent. Quoted spreads appeared narrow, but executing large trades became more costly. Propositions 6 and 7 account for the decline in depth: when broker participation falls, quote intensity declines and fewer quotes are executed close to the midprice. Corollary 4 explains the narrowing of the quoted spread: an increase in the money growth rate can compress the bid–ask spread even as participation falls and even as regulatory costs increase. These observations show that commonly used measures of liquidity, such as the bid–ask spread, are not sufficient to assess market conditions.

Fourth, markets occasionally experience extreme dislocations, such as flash crashes. Jovanovic and Menkveld (2022) show that while the probability of observing no quotes on a given day is small, over longer horizons it implies roughly a 31 percent chance of at least one such event in their sample. Menkveld and Yueshen (2019) documents that during the May 6, 2010 Flash Crash, prices in several U.S. equities temporarily collapsed to near zero before recovering within minutes, consistent with a brief absence of liquidity rather than a change in fundamentals. Proposition 11 places this observation in the context of variation in quote intensity: when quote intensity is low, quote arrivals become sparse and the probability of periods with no posted quotes increases. The model thus explains how seemingly stable markets can become vulnerable to sudden collapses in trading activity as an endogenous consequence of participation frictions.

Fifth, speculative episodes often feature both high valuations and high trading activity. Scheinkman and Xiong (2003) and Scheinkman (2013) emphasise that speculative premia are associated with strong resale motives and elevated trade volume. Together, Proposition 5, Corollary 3, and Proposition 12 show that this relationship arises in the inflation case. Changes in quote intensity, which serves as a proxy for trade volume, generate a positive correlation between trading activity and the size of the speculative premium when inflation varies. This positive correlation matches the empirical patterns observed in historical bubble episodes.

Taken together, the evidence suggests that the central mechanism of the model, the interaction of monetary policy, participation costs, and decentralised exchange, organises several observed features of these markets. The key margins are the broker's decision to post and the intensity with which investors search. Variations in these margins, driven by monetary and regulatory conditions, are sufficient to account for the movements in depth, stability, spreads, and speculative premia documented above.

3.7 Conclusion

This chapter develops a monetary model of financial exchange to examine how monetary and regulatory policy shape price formation, trading activity, and market stability. The framework combines search frictions and costly posting in a unified environment that links the microeconomic process of trade to the macroeconomic role of money. The analysis shows that participation costs and monetary conditions jointly determine the dispersion of bid and ask prices, execution costs, and the probability of market disruption. Increases in posting costs or reductions in real balances make quoting less frequent and prices more dispersed, producing thinner participation and greater variability in execution outcomes. Even small policy adjustments, such as moderate increases in inflation or higher regulatory posting costs, can substantially raise the likelihood of temporary trading breakdowns and flash-crash events. Overall, the results show that financial stability depends not only on aggregate policy settings but also on how these policies influence the incentives governing decentralised trade. Future work could explore how credit conditions and leverage shape the transmission of monetary and regulatory policy to asset prices, particularly through mechanisms involving margin lending and short selling. Such extensions would provide further insight into how the structure of financial intermediation amplifies or dampens the effects of policy on market stability.

Appendix A

Additional Modelling Assumptions

In modern markets, broker and dealers extend credit against securities for two main uses. *Margin credit* finances purchases of additional securities, with the investor's existing or newly acquired securities posted as collateral. *Securities-based lines of credit (SBLOCs)* similarly lend against a portfolio but allow general spending on goods and services. Interest accrues while the loan is outstanding. Use of margin and SBLOCs is widespread among sophisticated investors and subject to regulatory and firm-level risk controls.

Margin mechanics. Let an asset cost A and the loan be L. Investor equity is $E = A - L$. The margin is $\mathcal{M} = E/A$, leverage is $\mathcal{L} = A/E$, and the loan-to-value ratio is $\mathcal{R} = L/A$. The asset serves as collateral and the equity E is the haircut (down payment).

SBLOC mechanics. An SBLOC mirrors this structure—$E = A - L$, $\mathcal{M} = E/A$, $\mathcal{L} = A/E$, $\mathcal{R} = L/A$—but funds can be used broadly rather than to buy more securities. If portfolio values fall, lenders may require repayment or additional collateral to restore target LTVs (akin to a margin call).

Regulatory backdrop. In the U.S., margin accounts are governed by Federal Reserve Regulation T (50% initial margin for stock purchases, unchanged since 1974), with additional maintenance rules from FINRA and exchanges (e.g., FINRA Rule 4210's 25% minimum; brokers often set higher thresholds). Falling below maintenance triggers a margin call and potential liquidation. SBLOCs are not subject to Regulation T's 50% rule; firms manage them via internal LTV limits and maintenance calls based on collateral value.

Mapping to the model. Loans obtained via collateralised bonds in the first sub-period represent margin and SBLOC credit. If an investor borrows a fraction of collateral value—$L = \kappa A$ for margin or $L = \lambda A$ for SBLOC—then the implied margin, leverage, and LTV are $\mathcal{M} = 1 - \kappa$ or $1 - \lambda$, $\mathcal{L} = (1 - \kappa)^{-1}$ or $(1 - \lambda)^{-1}$, and $\mathcal{R} = \kappa$ or λ, respectively. Loans are repaid within the period, so $\mathcal{M}$ is interpreted as an initial margin only; there are no maintenance calls in the model. Investors with access to credit can lever to purchase securities or goods; others must self-finance.

Appendix B

Extensions

B.1 Illiquid Assets in the Cashless Limit

In this section, we study the properties of the cashless equilibrium in situations where agents no longer trade equity in the first subperiod. To this end, we characterise the limiting equilibrium as $\delta \to 0$. This limit approximates an economy where investors never enter the equity market, and assets can only be used as a collateral pledge for the purpose of purchasing first subperiod consumption goods. In what follows, we consider both the discrete-time case (i.e., $\Delta > 0$) and the continuous-time limit (i.e., $\Delta \to 0$) for a cashless limit economy with illiquid assets when $\hat{\varsigma}(0;0) < \iota < \bar{\varsigma}(0;0)$.

Proposition 1 (**Section B.1**). *Consider an economy with $\alpha \to 0$, $\kappa \in (0,1]$ and $\lambda \in (0,1]$. Let*

$$\bar{\varphi}^n \equiv \lim_{\delta \to 0} \tilde{\varphi}^n = \bar{\varepsilon} \left[1 - \lambda \tilde{\theta} \left\{ \tilde{u}' \left[\min \left\{ \tilde{\mathrm{D}}(\tilde{\varphi}^q), \lambda A^s / \tilde{\varphi}^q \right\} \right] - \tilde{\varphi}^q \right\} \right]^{-1}. \tag{B.1}$$

In the limit as $\delta \to 0$:

(i) If $\Delta > 0$, then

$$\mathcal{V} \to 0 \tag{B.2}$$

$$\frac{Z(\Delta)}{\Phi^s(\Delta)} \to 0 \tag{B.3}$$

$$\bar{\varphi}^q(\Delta) = \frac{1+\Delta\rho^{qm}}{1+\theta\Delta\rho^{qm}}\varrho \tag{B.4}$$

$$\Phi^s(\Delta) = \bar{\varepsilon}\left[\frac{(r+d-g+dg\Delta)\,\Delta}{(1+g\Delta)(1-d\Delta)} + \lambda\theta\Delta\left\{\frac{\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}[\bar{\varphi}^q(\Delta)], \lambda A^s/\bar{\varphi}^q(\Delta)\right\}\right]}{\bar{\varphi}^q(\Delta)} - 1\right\}\right]^{-1}, \tag{B.5}$$

where $\rho^m \in \mathbb{R}_+$ is the unique solution to

$$\bar{\varphi}^m(\Delta) = \frac{\varrho}{1+\theta\Delta\rho^{qm}} \tag{B.6}$$

$$\frac{(r+\pi-g+r\pi\Delta)}{1+g\Delta} = \theta\left\{\frac{\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}\left[\bar{\varphi}^m(\Delta)\right], \lambda A^s/\bar{\varphi}^q(\Delta)\right\}\right]}{\bar{\varphi}^m(\Delta)} - 1\right\}. \tag{B.7}$$

(ii) If $\Delta \to 0$, then

$$\mathcal{V} \to 0 \tag{B.8}$$

$$\frac{\mathcal{Z}}{\varphi} \to 0 \tag{B.9}$$

$$\tilde{\varphi} \to \bar{\varphi} \equiv \bar{\varepsilon}\left[1 - \lambda\bar{\theta}\left\{\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}(\varrho), \lambda A^s/\varrho\right\}\right] - \varrho\right\}\right]^{-1}, \tag{B.10}$$

where the nominal interest rate in this case is constant given by $\iota = \bar{\theta}\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}(\varrho), \lambda A^s/\varrho\right\} - \varrho\right]$.

Proposition 1 (Section B.1) considers the limiting economy as the fraction of investors who have access to the equity market vanishes. This limiting economy is one where equity is not traded at all in the first subperiod, and the asset can only be used to pledge against for purchasing consumption goods by shorting bonds. As explained in the context of Proposition 5, in a cashless monetary equilibrium, investors who have access to the equity market are the only ones who are able to trade equities in the first subperiod. Therefore the first-order implication of letting δ approach zero is that the investors no longer are able to trade equities in the first subperiod and trade volume in the equities market approaches zero.

If the length between time periods becomes arbitrarily small, i.e., $\Delta \to 0$ as in part (*ii*) of Proposition 1 (Section B.1), then the trade volume in the equities market vanishes in the limit (i.e., (B.8)), but the equilibrium equity price (i.e, (B.10)) remains positive in the limit. The reason why equity can have value in this limiting economy is that holding assets still provide dividends as well as the convenience of the goods pledge option in the goods market which is the premium associated with slackening the budget constraint of the investor in the goods market. Notably, even though real balances and trade volume converge to the (corresponding) nonmonetary equilibrium, the equity price is greater in the cashless economy with illiquid assets than in the nonmonetary economy with illiquid assets. This is because the price of goods (in terms of bonds) in the nonmonetary economy remains inefficiently high, whereas in the cashless economy the price reaches the efficient level (i.e., ϱ). Since the price of goods reaches the efficient level in the continuous-time limit of the cashless economy with illiquid assets, equity prices cease to be responsive to monetary policy and only changes in the quantity of assets, marginal costs of production or the collateral constraint λ can influence the equity price, which have all been assumed to be constant.

If the time period length is positive, i.e., $\Delta > 0$ as in part (*i*) of Proposition 1 (Section B.1), then again real balances and trade volume converge to zero (i.e., (B.2), (B.3)), but the real equity price remains positive as $\delta \to 0$. Notably, even though real balances and trade volume converge to zero as both α and δ tend to zero, the real equity price is still responsive to monetary policy. Since ρ^m is a function of π, so long as $\Delta > 0$, the magnitude of the response of the equity to changes in inflation remain bounded away from zero. The reason for this stems from the fact that the price of the good in terms of bonds can be considered a gross return on the cost of production of a single unit of the first subperiod consumption good, i.e.,

$$\bar{\varphi}^q(\Delta) = \frac{1 + \Delta\rho^m}{1 + \theta\Delta\rho^m}\varrho \equiv (1 + \Delta\tilde{\rho}^m)\,\varrho.$$

Intuitively, producers provide loans to facilitate the sale of their goods, these loans are not settled till the end of the period. Producers require compensation for this delay, and the size of the compensation is in proportion to the length of delay till settlement. This explains why in the continuous-time limit (i.e., $\Delta \to 0$) the price of the good converges to the efficient and competitive rate, i.e., $\bar{\varphi}^m = \bar{\varphi}^q = \varrho$ since money and bonds are seen as equivalent.[1] This brings us to following result.

Proposition 2 (Section B.1). *(i) The real asset price in the cashless monetary economy with illiquid assets (i.e., $\delta \to 0$) is higher than in the nonmonetary economy with illiquid assets if and only if*

$$\Delta\tilde{\rho}^m \leq \frac{(1-\theta)(\varrho - \underline{\varrho})}{\theta\varrho}. \tag{B.11}$$

(ii) In a cashless monetary equilibrium with illiquid assets (i.e., $\delta \to 0$), $\Phi^s(\Delta)$ is increasing in π, and the size of the effect depends upon Δ.

We take continuous-time to be the most realistic description of how time passes. Consequently, one should be cautious when interpreting real economies through discrete-time equilibria that differ qualitatively from their continuous-time counterparts. Still, discrete time often better reflects the timing of many activities, such as market opening frequencies, asset pricing schedules, and the timing of loan issuance. In practice, some asset, credit, or goods markets operate continuously, while others are accessed less frequently. The former are naturally modelled in continuous-time, the latter in discrete-time. Propositions 1 (Section B.1) and 2 (Section B.1) show that whenever the interval between OTC rounds is strictly positive, monetary policy has effects in a cashless economy with illiquid assets. Only in the limit as $\Delta \to 0$ does monetary policy's effects on asset prices vanish; even then, the cashless monetary economy's real equity price differs from the nonmonetary benchmark because the broker's bargaining power remains strictly positive.

[1] To see this, recall that $\varphi^q = (1+i^{qm})\varphi^m$, which implies that $\varphi^q = \varphi^m$ only when $i^{qm} = 0$, i.e., the opportunity cost of holding money instead of the inside bond for goods becomes zero.

Appendix C

Chapter 2 Proofs

C.1 Bargaining and Portfolio Problems

The investor's second subperiod value function can be written as

$$W_t^I(\mathbf{a}_t, a_t^b, k_t) = \phi_t^m a_t^m + \phi_t^s a_t^s + a_t^b - k_t + \bar{W}_t^I, \tag{C.1}$$

with

$$\bar{W}_t^I \equiv T_t + \max_{(\tilde{a}_{t+1}^m, \tilde{a}_{t+1}^s) \in \mathbb{R}_+^2} \left[-\phi_t^m \tilde{a}_{t+1}^m - \phi_{t+1}^s \tilde{a}_{t+1}^s + \beta \mathbb{E}_t \int V_{t+1}^I \left[\tilde{a}_{t+1}^m, \eta \tilde{a}_{t+1}^s + (1-\eta) A^s, \varepsilon \right] dG(\varepsilon) \right]. \tag{C.2}$$

The producer's second subperiod value function can be written as

$$W_t^P(a_t^m, a_t^g, a_t^b, k_t) = \phi_t^m a_t^m + a_t^b + a_t^g - k_t + \bar{W}_t^P, \tag{C.3}$$

with

$$\bar{W}_t^P \equiv T_t + \max_{\tilde{a}_{t+1}^m \in \mathbb{R}_+} \left[-\phi_{t+1}^m \tilde{a}_{t+1}^m + \beta V_{t+1}^P(\tilde{a}_{t+1}^m) \right], \tag{C.4}$$

and $a_t^g = (q_t - \tilde{q}_t)\underline{\varrho}$.

Lemma 1. *Consider the economy with no money, and let*

$$\varepsilon_t^n \equiv \frac{\bar{\phi}_t^s - \phi_t^s}{y_t}, \tag{C.5}$$

where $\bar{\phi}_t^s$ *denotes the price of an equity share in terms of bonds. Consider an investor who enters the OTC equity round of period* t *with equity holding* a_t^s *and valuation* ε. *Then:*

(i) If the investor does not contact a broker, the post-trade equity holding is $\hat{a}_t^s(a_t^s) = a_t^s$.

(ii) If the investor contacts a broker, the bargaining problem has a solution only if

$$\kappa < \frac{\bar{\phi}_t^s}{\phi_t^s}, \tag{C.6}$$

the post-trade portfolio is

$$\bar{a}_t^s(a_t^s, \varepsilon) = \chi(\varepsilon_t^n, \varepsilon)\frac{\bar{\phi}_t^s}{\bar{\phi}_t^s - \kappa\phi_t^s}a_t^s \tag{C.7}$$

$$\bar{a}_t^b(a_t^s, \varepsilon) = \bar{\phi}_t^s\left[1 - \chi(\varepsilon_t^n, \varepsilon)\frac{\bar{\phi}_t^s}{\bar{\phi}_t^s - \kappa\phi_t^s}\right]a_t^s, \tag{C.8}$$

and the intermediation fee for the broker is

$$k_t(a_t^s, \varepsilon) = (1-\theta)(\varepsilon - \varepsilon_t^n)y_t\left[\chi(\varepsilon_t^n, \varepsilon)\frac{\bar{\phi}_t^s}{\bar{\phi}_t^s - \kappa\phi_t^s}\right]a_t^s. \tag{C.9}$$

Proof. In a nonmonetary economy, the second subperiod value function for the investor reduces to

$$W_t^I(a_t^s, a_t^b, k_t) = \phi_t^s a_t^s + a_t^b - k_t + \bar{W}_t^I.$$

(i) In a nonmonetary equilibrium, we have $\hat{a}_t^s(a_t^s, \varepsilon) = \arg\max_{0 \le \hat{a}_t^s \le a_t^s}(\varepsilon y_t + \phi_t^s)\hat{a}_t^s$.

(ii) In a nonmonetary economy, $\left[\bar{a}_t^s(a_t^s, \varepsilon), \bar{a}_t^b(a_t^s, \varepsilon), k_t(a_t^s, \varepsilon)\right]$, is the solution to

$$\max_{(\bar{a}_t^s, k_t)\in\mathbb{R}_+^2, \bar{a}_t^b\in\mathbb{R}} \left[(\varepsilon y_t + \phi_t^s)(\bar{a}_t^s - a_t^s) + \bar{a}_t^b - k_t\right]^\theta k_t^{1-\theta}$$

$$\text{s.t. } \bar{\phi}_t^s \bar{a}_t^s + \bar{a}_t^b = \bar{\phi}_t^s a_t^s$$
$$-\kappa \phi_t^s \bar{a}_t^s \leq \bar{a}_t^b.$$

Notice that the first-order conditions with respect to k_t implies

$$k_t(a_t^s, \varepsilon) = (1-\theta)\left\{(\varepsilon y_t + \phi_t^s)[\bar{a}_t^s(a_t^s, \varepsilon) - a_t^s] + \bar{a}_t^b(a_t^s, \varepsilon)\right\},$$

so the bargaining solution can be found by solving the following auxiliary problem

$$\bar{a}_t^s(a_t^s, \varepsilon) = \arg\max_{\bar{a}_t^s \in \mathbb{R}_+} (\varepsilon - \varepsilon_t^n) y_t \bar{a}_t^s \quad \text{s.t.} \quad (\bar{\phi}_t^s - \kappa \phi_t^s) \leq \bar{\phi}_t^s a_t^s.$$

The problem has no solution (for $\varepsilon > \varepsilon_t^n$) if $\bar{\phi}_t^s - \kappa \phi_t^s \leq 0$. Provided $\bar{\phi}_t^s - \kappa \phi_t^s > 0$, the solution exists for all ε and is provided in the lemma. Given $\bar{a}_t^s(a_t^s, \varepsilon)$, $\bar{a}_t^b(a_t^s, \varepsilon) = \bar{\phi}_t^s\left[a_t^s - \bar{a}_t^s(a_t^s, \varepsilon)\right]$ and $k_t(a_t^s, \varepsilon)$ is given above. ■

Lemma 2. *Consider the economy with no money and an investor who enters the OTC goods round of period t with equity holding a_t^s. Then:*

(i) If the investor does not contact a broker, the post trade goods allocation is $\hat{q}_t(a_t^s) = 0$.

(ii) If the investor contacts a broker, the bargaining problem post-trade allocation is

$$\bar{q}_t(a_t^s) = \min\left\{\mathrm{D}(\varphi_t^q), \frac{\lambda \phi_t^s a_t^s}{\varphi_t^q}\right\} \tag{C.10}$$
$$\bar{a}_t^b(a_t^s) = -\varphi_t^q \bar{q}_t(a_t^s), \tag{C.11}$$

and the intermediation fee for the broker is

$$k_t(a_t^s) = (1-\theta)\left\{u[\bar{q}_t(a_t^s)] - \varphi_t^q \bar{q}_t(a_t^s)\right\}, \tag{C.12}$$

where φ_t^q is the price of the first subperiod consumption good in terms of bonds.

Proof. In a nonmonetary economy, the second subperiod value function for the investor reduces to

$$W_t^I(a_t^s, a_t^b, k_t) = \phi_t^s a_t^s + a_t^b - k_t + \bar{W}_t^I.$$

(i) In a nonmonetary equilibrium the investor is not able to purchase the goods in the OTC goods market.

(ii) In a nonmonetary economy, $[\bar{q}_t(a_t^s), \bar{a}_t^b(a_t^s), k_t(a_t^s)]$, is the solution to

$$\max_{(\bar{q}_t, k_t)\in\mathbb{R}_+^2, \bar{a}_t^b\in\mathbb{R}} \left[u(\bar{q}_t) + \bar{a}_t^b - k_t\right]^\theta k_t^{1-\theta}$$

$$\text{s.t. } \varphi_t^q \bar{q}_t + \bar{a}_t^b = 0$$
$$-\lambda\phi_t^s \bar{a}_t^s \leq \bar{a}_t^b.$$

Notice the first-order conditions with respect to k_t implies

$$k_t(a_t^s) = (1-\theta)\left\{u[\bar{q}_t(a_t^s)] + \bar{a}_t^b(a_t^s)\right\},$$

so the bargaining solution can be found by solving the following auxiliary problem

$$\max_{\bar{q}_t\in\mathbb{R}_+} [u(\bar{q}_t) - \varphi_t^q \bar{q}_t] \quad \text{s.t. } \lambda\phi_t^s a_t^s \geq \varphi_t^q \bar{q}_t.$$

The Lagrangian corresponding to the auxiliary problem is

$$\mathcal{L} = u(\bar{q}_t) - \varphi_t^q \bar{q}_t + \xi^b[\lambda\phi_t^s a_t^s - \varphi_t^q \bar{q}_t] + \xi^q \bar{q}_t,$$

where ξ^b, ξ^q are the multipliers on the collateral constraint and the nonnegativity constraint

$\bar{q} \geq 0$, respectively. The first-order conditions and complementary slackness conditions are

$$u'(\bar{q}_t) - \varphi_t^q - \varphi_t^q \xi^b + \xi^q = 0$$
$$\xi^b[\lambda \phi_t^s a_t^s - \varphi_t^q \bar{q}_t] = 0$$
$$\xi^q \bar{q}_t = 0.$$

Notice that $\xi^q = 0$. To see this, suppose that if $\xi^q > 0$, it implies that $\bar{q}_t = 0$, however $u'(0)$ is assumed to undefined, a contradiction. The first-order conditions become

$$u'(\bar{q}_t) - \varphi_t^q - \varphi_t^q \xi^b = 0$$
$$\xi^b[\lambda \phi_t^s a_t^s - \varphi_t^q \bar{q}_t] = 0.$$

By working out the two possible binding patterns for the multiplier ξ^b and collecting the optimal allocations along with the inequality restrictions implied by each case, we obtain the lemma. ■

Lemma 3. *Consider the economy with no money and a producer who enters the OTC goods round of period t. Then:*

(i) If the producer does not contact a broker, the post trade goods allocation is $\hat{q}_t = 0$,

(ii) If the producer does contact a broker, the bargaining problem post-trade allocation is

$$\bar{q}_t(q_t) = \chi(\underline{\varrho}, \varphi_t^q) q_t \tag{C.13}$$
$$\bar{a}_t^b(q_t) = \varphi_t^q \chi(\underline{\varrho}, \varphi_t^q) q_t, \tag{C.14}$$

and the intermediation fee for the broker is

$$k_t(q_t) = (1-\theta)(\varphi_t^q - \underline{\varrho}) \chi(\underline{\varrho}, \varphi_t^q) q_t. \tag{C.15}$$

Proof. In a nonmonetary economy, the second subperiod value function for the producer reduces to

$$W_t^P(a_t^g, a_t^b, k_t) = a_t^g + a_t^b - k_t + \bar{W}_t^P.$$

(i) Since the investor has no method of payment to purchase the good in this case, the producer sells none of the good, which implies $\hat{q} = 0$.

(ii) In a nonmonetary equilibrium, $[\bar{q}_t(q_t), \bar{a}_t^b(q_t), k_t(q_t)]$, is the solution to

$$\max_{(\bar{q}_t, k_t) \in \mathbb{R}_+^2, \bar{a}_t^b \in \mathbb{R}} \left[\bar{a}_t^b - k_t - \underline{\varrho}\bar{q}_t\right]^\theta k_t^{1-\theta}$$

$$\begin{aligned} \text{s.t. } \bar{a}_t^b &\leq \varphi_t^q \bar{q}_t \\ \bar{q}_t &\leq q_t \\ k_t + \underline{\varrho}\bar{q}_t &\leq \bar{a}_t^b. \end{aligned}$$

Notice that the first-order condition with respect to k_t implies

$$k_t(q_t) = (1-\theta)[\bar{a}_t^b(q_t) - \underline{\varrho}\bar{q}_t(q_t)],$$

so the bargaining solution can be found by solving the auxiliary problem, where $\bar{a}_t^b(q_t) = \varphi_t^q \bar{q}_t(q_t)$,

$$\max_{\bar{q}_t \in \mathbb{R}_+} [\varphi_t^q - \underline{\varrho}]\bar{q}_t \quad \text{s.t. } \bar{q}_t \leq q_t.$$

The corresponding Lagrangian to the auxiliary problem is

$$\mathcal{L} = [\varphi_t^q - \underline{\varrho}]\bar{q}_t + \xi^b[q_t - \bar{q}_t] + \xi^q \bar{q}_t,$$

where ξ^b and ξ^q are the multipliers for the budget constraint and the nonnegativity constraint, respectively. The first-order condition and the complementary slackness conditions are given by

$$\begin{aligned} [\varphi_t^q - \underline{\varrho}] - \xi^b + \xi^q &= 0 \\ \xi^b[q_t - \bar{q}_t] &= 0 \\ \xi^q \bar{q} &= 0. \end{aligned}$$

Working through the three cases yields the lemma. ■

Lemma 4. *Consider the economy with money, and let*

$$\varepsilon_t^* \equiv (p_t^s \phi_t^m - \phi_t^s)\frac{1}{y_t} \tag{C.16}$$

$$\varepsilon_t^{**} \equiv \varepsilon_t^* + (1 - p_t^{sb}\phi_t^m)\left[\mathbb{I}_{\{p_t^{sb}\phi_t^m<1\}}\frac{p_t^s}{p_t^{sb}} + \mathbb{I}_{\{1<p_t^{sb}\phi_t^m\}}\kappa\phi_t^m\right]\frac{1}{y_t}, \tag{C.17}$$

Consider an investor who enters the OTC equity round of period t with portfolio $\mathbf{a}_t$ and valuation ε. Then:

(i) If the investor does not contact a broker, the post-trade portfolio is

$$\hat{a}_t^m(\mathbf{a}_t, \varepsilon) = [1 - \chi(\varepsilon_t^*, \varepsilon)]\,(a_t^m + p_t^s a_t^s) \tag{C.18}$$

$$\hat{a}_t^s(\mathbf{a}_t, \varepsilon) = \chi(\varepsilon_t^*, \varepsilon)\frac{1}{p_t^s}(a_t^m + p_t^s a_t^s). \tag{C.19}$$

(ii) If the investor contacts a broker, the bargaining problem has a solution only if

$$\kappa < \frac{p_t^s}{p_t^{sb}\phi_t^s}, \tag{C.20}$$

and in that case the post-trade portfolio is

$$\bar{a}_t^m(\mathbf{a}_t, \varepsilon) = \left\{\mathbb{I}_{\{1<p_t^{sb}\phi_t^m\}}[1 - \chi(\varepsilon_t^{**}, \varepsilon)] + \mathbb{I}_{\{p_t^{sb}\phi_t^m=1\}}\mathbb{I}_{\{\varepsilon<\varepsilon^{**}\}}[1 - \chi(p_t^{sb}\phi_t^s, 1)]\right\}(a_t^m + p_t^s a_t^s) \tag{C.21}$$

$$+ \mathbb{I}_{\{p_t^{sb}\phi_t^m=1\}}\mathbb{I}_{\{\varepsilon=\varepsilon^{**}\}}\tilde{a}_t^m \tag{C.22}$$

$$\bar{a}_t^s(\mathbf{a}_t, \varepsilon) = \left\{\mathbb{I}_{\{p_t^{sb}\phi_t^m=1\}}\mathbb{I}_{\{\varepsilon^{**}<\varepsilon\}} + [1 - \mathbb{I}_{\{p_t^{sb}\phi_t^m=1\}}]\chi(\varepsilon_t^{**}, \varepsilon)\right\}\frac{a_t^m + p_t^s a_t^s}{p_t^s - \kappa p_t^{sb}\phi_t^s} \tag{C.23}$$

$$+ \mathbb{I}_{\{p_t^{sb}\phi_t^m=1\}}\mathbb{I}_{\{\varepsilon=\varepsilon^{**}\}}\tilde{a}_t^s \tag{C.24}$$

$$\bar{a}_t^b(\mathbf{a}_t, \varepsilon) = -\frac{1}{p_t^{sb}}\left\{[\bar{a}_t^m(\mathbf{a}_t, \varepsilon) - a_t^m] + p_t^s[\bar{a}_t^s(\mathbf{a}_t, \varepsilon) - a_t^s]\right\}, \tag{C.25}$$

where

$$(\tilde{a}_t^m, \tilde{a}_t^s) \in \left\{\mathbb{R}_+^2 : \tilde{a}_t^m + (p_t^s - \kappa p_t^{sb}\phi_t^s)\tilde{a}_t^s = a_t^m + p_t^s a_t^s\right\}, \tag{C.26}$$

and the intermediation fee is

$$k_t(\mathbf{a}_t,\varepsilon) = (1-\theta)\Big\{(\varepsilon y_t + \phi_t^s)[\bar{a}_t^s(\mathbf{a}_t,\varepsilon) - \hat{a}_t(\mathbf{a}_t,\varepsilon)] + \phi_t^m[\bar{a}_t^m(\mathbf{a}_t,\varepsilon) - \hat{a}_t^m(\mathbf{a}_t,\varepsilon)] + \bar{a}_t^b(\mathbf{a}_t,\varepsilon)\Big\}. \quad \text{(C.27)}$$

Proof. (i) With W_t^I, it is easy to show the allocation in the non-brokered case. (ii) With W_t^I, the generalised Nash bargaining problem can be written as

$$\max_{(\bar{a}_t^m,\bar{a}_t^s,k_t)\in\mathbb{R}_+^3,\bar{a}_t^b\in\mathbb{R}} \Big\{(\varepsilon y_t + \phi_t^s)[\bar{a}_t^s - \hat{a}_t^s(\mathbf{a}_t,\varepsilon)] + \phi_t^m[\bar{a}_t^m - \hat{a}_t^m(\mathbf{a}_t,\varepsilon)] + \bar{a}_t^b - k_t\Big\}^\theta k_t^{1-\theta}$$

$$\text{s.t. } \bar{a}_t^m + p_t^s\bar{a}_t^s + p_t^{sb}\bar{a}_t^b = a_t^m + p_t^s a_t^s$$
$$-\kappa\phi_t^s\bar{a}_t^s \leq \bar{a}_t^b,$$

Notice that the first-order condition with respect to k_t implies the condition in the lemma, so the bargaining solution can be found by solving the follow auxiliary problem

$$\max_{(\bar{a}_t^m,\bar{a}_t^s)\in\mathbb{R}_+^2,\bar{a}_t^b\in\mathbb{R}} \Big\{(\varepsilon y_t + \phi_t^s)[\bar{a}_t^s - \hat{a}_t^s(\mathbf{a}_t,\varepsilon)] + \phi_t^m[\bar{a}_t^m - \hat{a}_t^m(\mathbf{a}_t,\varepsilon)] + \bar{a}_t^b\Big\}$$

$$\text{s.t. } \bar{a}_t^m + p_t^s\bar{a}_t^s + p_t^{sb}\bar{a}_t^b = a_t^m + p_t^s a_t^s$$
$$-\kappa\phi_t^s\bar{a}_t^s \leq \bar{a}_t^b.$$

Once the solution $\bar{a}_t^m(\mathbf{a}_t,\varepsilon)$, $\bar{a}_t^s(\mathbf{a}_t,\varepsilon)$, and $\bar{a}_t(\mathbf{a}_t,\varepsilon)$ to this problem has been found, $k_t(\mathbf{a}_t,\varepsilon)$ can be derived. Substituting the budget constraint into the auxiliary problem is equivalent to

$$\max_{(\bar{a}_t^m,\bar{a}_t^s)\in\mathbb{R}_+^2} \left[\left(\varepsilon y_t + \phi_t^s - \frac{p_t^s}{p_t^{sb}}\right)\bar{a}_t^s + \left(\phi_t^m - \frac{1}{p_t^{sb}}\right)\bar{a}_t^m\right]$$

$$\text{s.t. } 0 \leq a_t^m + p_t^s a_t^s - \bar{a}_t^m - (p_t^s - \kappa p_t^{sb}\phi_t^s)\bar{a}_t^s.$$

This problem has no solution if $p_t^s \leq \kappa p_t^{sb}\phi_t^s$. To see this, assume $p_t^s \leq \kappa p_t^{sb}\phi_t^s$. Set $\bar{a}_t^m =$

$a_t^m + p_t^s a_t^s$, notice that the new budget constraint is satisfied by any $\bar{a}_t^s \in \mathbb{R}_+$. Thus, the value of the new objective function is bounded below by

$$\left(\phi_t^m - \frac{1}{p_t^{sb}}\right)(a_t^m + p_t^s a_t^s) + \max_{\bar{a}_t^s \in \mathbb{R}_+} [\varepsilon y_t + (1-\kappa)\phi_t^s]\bar{a}_t^s,$$

which is arbitrarily large. Hence, $\kappa < p_t^s / p_t^{sb}\phi_t^s$ is necessary for the bargaining problem to have a solution. The Lagrangian corresponding to the auxiliary problem is

$$\mathcal{L} = \left(\varepsilon y_t + \phi_t^s - \frac{p_t^s}{p_t^{sb}}\right)\bar{a}_t^s + \left(\phi_t^m - \frac{1}{p_t^{sb}}\right)\bar{a}_t^m$$
$$+ \xi^b[a_t^m + p_t^s a_t^s - \bar{a}_t^m - (p_t^s - \kappa p_t^{sb}\phi_t^s)\bar{a}_t^s] + \xi^m \bar{a}_t^m + \xi^s \bar{a}_t^s,$$

where ξ^b, ξ^m, and ξ^s are the multipliers on the above constraints.

$$\varepsilon y_t + \phi_t^s - \frac{p_t^s}{p_t^{sb}} + \xi^s - (p_t^s - \kappa p_t^{sb}\phi_t^s)\xi^b = 0$$
$$\phi_t^m - \frac{1}{p_t^{sb}} + \xi^m - \xi^b = 0.$$

By working out the eight possible binding patterns for the multiplier (ξ^b, ξ^m, ξ^s) and collecting the optimal allocations along with the inequality restriction implied by each case. ■

Lemma 5. *Consider the economy with money and an investor who enters the OTC goods round of period t with portfolio $\mathbf{a}_t$. Then:*

(i) If the investor does not contact a broker, the post-trade portfolio is

$$\hat{q}_t(a_t^m) = \min\left\{\mathrm{D}(\varphi_t^m), \frac{a_t^m}{p_t^q}\right\} \tag{C.28}$$

$$\hat{a}_t^m(a_t^m) = a_t^m - p_t^q \hat{q}_t(a_t^m). \tag{C.29}$$

(ii) If the investor contacts a broker, the bargaining problem post-trade allocation is

$$\bar{q}_t(\mathbf{a}_t) = \mathbb{I}_{\left\{p_t^{qb}\phi_t^m>1\right\}}\mathrm{D}(\varphi_t^m) + \mathbb{I}_{\left\{p_t^{qb}\phi_t^m\leq 1\right\}} \min\left\{\mathrm{D}(\varphi_t^q), \frac{a_t^m + \lambda p_t^{qb}\phi_t^s a_t^s}{p_t^q}\right\} \tag{C.30}$$

$$\bar{a}_t^m(\mathbf{a}_t) = \mathbb{I}_{\left\{p_t^{qb}\phi_t^m>1\right\}}[a_t^m + \lambda p_t^{qb}\phi_t^s a_t^s - p_t^q \mathrm{D}(\varphi_t^m)] + \mathbb{I}_{\left\{p_t^{qb}\phi_t^m=1\right\}}\tilde{a}_t^m \tag{C.31}$$

$$\bar{a}_t^b(\mathbf{a}_t) = -\frac{1}{p_t^{qb}}\left\{[\bar{a}_t^m(\mathbf{a}_t) - a_t^m] + p_t^q \bar{q}(\mathbf{a}_t)\right\}, \tag{C.32}$$

where

$$(\tilde{a}_t^m, \tilde{a}_t^b) \in \left\{\mathbb{R}_+^2 : \tilde{a}_t^m + p_t^{qb}\tilde{a}_t^b \leq a_t^m + \lambda\phi_t^s a_t^s - p_t^q \mathrm{D}(\varphi_t^q)\right\}, \tag{C.33}$$

and the intermediation fee is

$$k_t(\mathbf{a}_t) = (1-\theta)\{u(\bar{q}_t) - u[\hat{q}_t(a_t^m)] + \phi_t^m[\bar{a}_t^m - \hat{a}_t^m(a_t^m)] + \bar{a}_t^b\}. \tag{C.34}$$

Proof. (i) The solution is simple to derive in the non-brokered case. (ii) With W_t^I, the generalised Nash bargaining problem can be written as

$$\max_{(\bar{q}_t,\bar{a}_t^m,k_t)\in\mathbb{R}_+^3,\bar{a}_t^b\in\mathbb{R}_+} [u(\bar{q}_t) - u[\hat{q}_t(a_t^m)] + \phi_t^m[\bar{a}_t^m - \hat{a}_t^m(a_t^m)] + \bar{a}_t^b - k_t]^\theta k_t^{1-\theta}$$

$$\text{s.t. } \bar{a}_t^m + p_t^{qb}\bar{a}_t^b + p_t^q\bar{q}_t = a_t^m$$

$$-\lambda\phi_t^s a_t^s \leq \bar{a}_t^b.$$

Notice that the first-order condition with respect to k_t implies

$$k_t(\mathbf{a}_t) = (1-\theta)\{u(\bar{q}_t) - u[\hat{q}_t(a_t^m)] + \phi_t^m[\bar{a}_t^m - \hat{a}_t^m(a_t^m)] + \bar{a}_t^b\},$$

so the bargaining solution can be found by solving the following auxiliary problem

$$\max_{(\bar{q}_t,\bar{a}_t^m)\in\mathbb{R}_+^2}\left[u(\bar{q}_t) - \varphi_t^q\bar{q}_t + \left(\phi_t^m - \frac{1}{p_t^b}\right)\bar{a}_t^m\right]$$

$$\text{s.t. } 0 \leq a_t^m - \bar{a}_t^m - p_t^q \bar{q}_t + \lambda p_t^{qb} \phi_t^s a_t^s.$$

Substituting the budget constraint into the objective problem yields the Lagrangian

$$\mathcal{L} = u(\bar{q}_t) - \varphi_t^q \bar{q}_t + \left(\phi_t^m - \frac{1}{p_t^{qb}} \right) \bar{a}_t^m$$
$$+ \xi^b [a_t^m - \bar{a}_t^m - p_t^q \bar{q}_t + \lambda p_t^{qb} \phi_t^s a_t^s] + \xi^q \bar{q}_t + \xi^m \bar{a}_t^m,$$

where ξ^b, ξ^m and ξ^q are the multipliers on the above constraints. The first-order conditions are given by

$$u'(\bar{q}_t) - \varphi_t^q - \xi^b p_t^q = 0$$
$$\phi_t^m - \frac{1}{p_t^{qb}} - \xi^b + \xi^m = 0.$$

Notice that $\xi^q = 0$ since $u'(\bar{q}_t)$ is undefined for $\bar{q}_t = 0$. Therefore, there are four possible binding patterns for the multiplier (ξ^b, ξ^m). Case 1. Assume $0 < \xi^m$, $0 < \xi^b$. Then $\bar{a}_t^m = 0$, $\bar{a}_t^b = -\lambda \phi_t^s a_t^s$, $\bar{q}_t = \frac{1}{p_t^q}[a_t^m + \lambda p_t^{qb} \phi_t^s a_t^s]$, $u'(\bar{q}_t) - \varphi_t^q = p_t^q \xi^b$ and $u'(\bar{q}_t) - \varphi_t^m = p_t^q \xi^m$. This kind of solution is only possible $u'(\bar{q}_t) > \max\{\varphi_t^q, \varphi_t^m\}$. Case 2. Assume $0 < \xi^m$, $0 = \xi^b$. Then $\bar{a}_t^m = 0$, $\bar{q}_t = \mathrm{D}(\varphi_t^q)$ and $\bar{a}_t^b = \frac{1}{p_t^{qb}} [a_t^m - p_t^q \mathrm{D}(\varphi_t^q)]$. This kind of solution is only possible if $p_t^{qb} \phi_t^m < 1$. Case 3. Assume $0 = \xi^m$, $0 < \xi^b$. Then $\bar{a}_t^m = a_t^m + \lambda p_t^{qb} \phi_t^s a_t^s - p_t^q \mathrm{D}(\varphi_t^m)$, $\bar{q}_t = \mathrm{D}(\varphi_t^m)$ and $\bar{a}_t^b = -\lambda \phi_t^s a_t^s$. This kind of solution is only possible is $p_t^{qb} \phi_t^m > 1$. Case 4. Assume $0 = \xi^m$, $0 = \xi^b$. Then $\bar{q}_t = \mathrm{D}(\varphi_t^q)$ and $(\bar{a}_t^m, \bar{a}_t^b) \in [0, \infty) \times [-\lambda \phi_t^s a_t^s, \infty]$ and $\bar{a}_t^m + p_t^{qb} \bar{a}_t^b + p_t^q \mathrm{D}(\varphi_t^q) = a_t^m$. This kind of solution is only possible if $p_t^{qb} \phi_t^m = 1$. By collecting the solutions along with the inequality restrictions implied by the four cases obtains the lemma. ■

Lemma 6. *Consider the economy with money and a producer who enters the OTC goods round of period t with money and output (a_t^m, q_t). Then:*

(i) If the producer does not contact a broker, the post-trade allocation is

$$\hat{q}_t(a_t^m, q_t) = \chi(\underline{\varrho}, \varphi_t^m) q_t \tag{C.35}$$

$$\hat{a}_t^m(a_t^m, q_t) = a_t^m + p_t^q \chi(\underline{\varrho}, \varphi_t^m) q_t. \tag{C.36}$$

(ii) If the producer contacts a broker, the post-trade allocation is

$$\bar{a}_t^m(a_t^m, q_t) \begin{cases} \to \infty & \textit{if } 1 < p_t^{qb}\phi_t^m \\ \in [0, \infty) & \textit{if } p_t^{qb}\phi_t^m = 1 \\ = 0 & \textit{if } p_t^{qb}\phi_t^m < 1 \end{cases} \tag{C.37}$$

$$\bar{q}_t(a_t^m, q_t) = \chi(\underline{\varrho}, \varphi_t^q)q_t \tag{C.38}$$

$$\bar{a}_t^b(a_t^m, q_t) = \frac{1}{p_t^{qb}}[a_t^m + p_t^q\bar{q}_t(a_t^m, q_t) - \bar{a}_t^m(a_t^m, q_t)], \tag{C.39}$$

and the intermediation fee is

$$k_t(a_t^m, q_t) = (1-\theta)\left\{i_t^{qm}\phi_t^m a_t^m + [(\varphi_t^q - \underline{\varrho})\chi(\underline{\varrho}, \varphi_t^q) - (\varphi_t^m - \underline{\rho})\chi(\underline{\varrho}, \varphi_t^m)]q_t\right\}. \tag{C.40}$$

Proof. (i) In a monetary equilibrium $[\hat{a}_t^m(a_t^m, q_t), \hat{q}_t(a_t^m, q_t)]$ is the solution to

$$\max_{(\hat{a}_t^m, \hat{q}_t)\in\mathbb{R}_+^2} [\phi_t^s\hat{a}_t^m + (q_t - \hat{q}_t)\underline{\varrho}]$$

$$\text{s.t. } \hat{a}_t^m = p_t^q\hat{q}_t + a_t$$

$$\hat{q}_t \leq q_t.$$

Substituting the budget constraint into the objective problem yields the following Lagrangian

$$\mathcal{L} = [\varphi_t^m - \underline{\varrho}]\hat{q}_t + \xi^q[q_t - \hat{q}_t] + \varsigma^q\hat{q}_t.$$

Notice that the first-order condition with respect to $\hat{q}_t$ implies

$$\varphi_t^m - \underline{\varrho} - \xi^q + \varsigma^q = 0.$$

Working through the binding patterns for the multipliers (ξ^q, ς^q) and collecting the optimal allocations yields the allocations provided in the lemma.

(ii) With $\bar{W}_t^P$, the generalised Nash problem can be written as

$$\max_{(\bar{a}_t^m,\bar{q}_t^m,k_t)\in\mathbb{R}_+^2,\bar{a}_t^b\in\mathbb{R}} \left[\phi_t^m(\bar{a}_t^m-\hat{a}_t^m)-(\bar{q}_t-\hat{q}_t)\underline{\varrho}+\bar{a}_t^b-k_t\right]^\theta k_t^{1-\theta}$$

$$\text{s.t. } \bar{a}_t^m+p_t^{qb}\bar{a}_t^b\le a_t^m+p_t^q\bar{q}_t$$

$$\bar{q}_t\le q_t.$$

Notice that the first-order condition with respect to k_t implies

$$k_t(a_t^m,q_t)=(1-\theta)\left\{\phi_t^m[\bar{a}_t^m-\hat{a}_t^m]-[\bar{q}_t-\hat{q}_t]\underline{\varrho}+\bar{a}_t^b\right\},$$

so the bargaining solution can be found by solving the follow auxiliary problem

$$\max_{(\bar{a}_t^m,\bar{q}_t)\in\mathbb{R}_+^2,\bar{a}_t^b\in\mathbb{R}} \left\{\phi_t^m[\bar{a}_t^m-\hat{a}_t^m(a_t^m,q_t)]-[\bar{q}_t-\hat{q}_t(a_t^m,q_t)]\underline{\varrho}+\bar{a}_t^b\right\}$$

$$\bar{a}_t^m+p_t^{qb}\bar{a}_t^b=a_t^m+p_t^q\bar{q}_t$$

$$\bar{q}_t\le q_t,$$

Once the solution $\bar{a}_t(a_t^m,q_t)$, $\bar{q}_t(a_t^m,q_t)$ and $\bar{a}_t^b(a_t^m,q_t)$ to this problem has been found, $k_t(a_t^m,q_t)$ yields the intermediation fee. Substituting the budget constraint into the auxiliary problem is equivalent to

$$\max_{(\bar{a}_t^m,\bar{q}_t)\in\mathbb{R}_+^2} \left[\left(\phi_t^m-\frac{1}{p_t^{qb}}\right)\bar{a}_t^m-\left(\underline{\varrho}-\varphi_t^q\right)\bar{q}_t\right]$$

$$\text{s.t. } \bar{q}_t\le q_t.$$

Solving the auxiliary problem yields

$$\bar{a}_t^m(a_t^m, q_t) \begin{cases} \to \infty & \text{if } 1 < p_t^{qb}\phi_t^m \\ \in [0, \infty) & \text{if } p_t^{qb}\phi_t^m = 0 \\ = 0 & \text{if } p_t^{qb}\phi_t^m < 1, \end{cases}$$

and $\bar{q}(a_t^m, q_t) = \chi(\underline{\varrho}, \varphi_t^q)q_t$. Substituting the budget constraint into k_t

$$k_t(a_t^m, q_t) = (1-\theta)\left\{\frac{1}{p_t^{qb}}a_t^m + \left(\phi_t^m - \frac{1}{p_t^{qb}}\right)\bar{a}_t^m - \phi_t^m\hat{a}_t^m + (\varphi_t^q - \underline{\varrho})\bar{q}_t + \underline{\varrho}\hat{q}_t\right\}.$$

Substituting $\hat{q}_t(a_t^m, q_t) = \chi(\underline{\varrho}, \varphi_t^m)q_t$, $\hat{a}_t^m(a_t^m, q_t) = a_t^m + p_t^q\chi(\underline{\varrho}, \varphi_t^m)q_t$ and $\bar{q}_t(a_t^m, q_t) = \chi(\underline{\varrho}, \varphi_t^q)q_t$ yields the condition in the lemma. ■

C.2 Value Functions

In this section we derive the value functions for brokers, producers and investors, in a monetary economy (Lemma 7) and in a nonmonetary economy (Lemma 8).

Lemma 7. *Consider an economy with money:*

(i) The value function of a broker at the beginning of the OTC round of period t is

$$V_t^B = \Xi_t^I + \Xi_t^P + \bar{W}_t^B, \tag{C.41}$$

where $\bar{W}_t^B \equiv \beta\mathbb{E}_t V_{t+1}^B$, and

$$\Xi_t^I \equiv (1-\alpha)\left[\delta\int k_t^{is}(\tilde{\mathbf{a}}_t, \varepsilon)dH_t^I(\tilde{\mathbf{a}}_t, \varepsilon) + (1-\delta)\int k_t^{iq}(\tilde{\mathbf{a}}_t)dF_t^I(\tilde{\mathbf{a}}_t)\right] \tag{C.42}$$

$$\Xi_t^P \equiv (1-\alpha)(1-\delta)\int k_t^{pq}(\tilde{a}_t^m, q_t)dF_t^P(\tilde{a}_t^m). \tag{C.43}$$

(ii) The value function of an investor who enters the OTC round of period t with portfolio $\mathbf{a}_t$ and valuation ε is

$$V_t^I(\mathbf{a}_t, \varepsilon) = v_t^q(\mathbf{a}_t) + v_t^m(\varepsilon)a_t^m + v_t^s(\varepsilon)a_t^s + \bar{W}_t^I, \tag{C.44}$$

where

$$\begin{aligned}
v_t^q(\mathbf{a}_t) &\equiv \alpha(1-\delta)\theta\left\{u[\hat{q}_t(a_t^m)] - \varphi_t^m \hat{q}_t(a_t^m)\right\} \\
&\quad + (1-\alpha)(1-\delta)\theta\left\{u[\bar{q}_t(\mathbf{a}_t)] - \varphi_t^q \bar{q}_t(\mathbf{a}_t)\right\} \\
v_t^m(\varepsilon) &\equiv \phi_t^m + [\alpha + (1-\alpha)(1-\theta)]\delta \mathbb{I}_{\{\varepsilon_t^* < \varepsilon\}}(\varepsilon - \varepsilon_t^*) y_t \frac{1}{p_t^s} \\
&\quad + (1-\alpha)\theta \mathbb{I}_{\{p_t^{sb}\phi_t^m < 1\}} \left(\frac{1}{p_t^{sb}} - \phi_t^m\right) \\
&\quad + (1-\alpha)\delta\theta \mathbb{I}_{\{\varepsilon_t^{**} < \varepsilon\}}(\varepsilon - \varepsilon_t^{**}) y_t \frac{1}{p_t^s - \kappa p_t^{sb}\phi_t^s} \\
v_t^s(\varepsilon) &\equiv \varepsilon y_t + \phi_t^s + [\alpha + (1-\alpha)(1-\theta)]\delta \mathbb{I}_{\{\varepsilon < \varepsilon_t^*\}}(\varepsilon_t^* - \varepsilon) y_t \\
&\quad + (1-\alpha)\delta\theta\left(\phi_t^m - \frac{1}{p_t^{sb}}\right)\mathbb{I}_{\left\{1 < p_t^{sb}\phi_t^m\right\}} \kappa p_t^{sb}\phi_t^s \\
&\quad + (1-\alpha)\delta\theta(\varepsilon - \varepsilon_t^{**}) y_t \frac{\kappa p_t^{sb}\phi_t^s - \mathbb{I}_{\{\varepsilon < \varepsilon_t^{**}\}} p_t^s}{p_t^s - \kappa p_t^{sb}\phi_t^s}.
\end{aligned}$$

(iii) The value function of a producer who enters the OTC round of period t with money holdings a_t^m is

$$V_t^P(a_t^m) = \max_{q_t \in \mathbb{R}_+} \left[R^m(\varphi_t^q, \varphi_t^m) - \varrho\right] q_t + [1 + (1-\alpha)\theta i_t^{qm}]\phi_t^m a_t^m + \bar{W}_t, \tag{C.45}$$

where

$$R^m(\varphi_t^q, \varphi_t^m) = \underline{\varrho} + (1-\alpha)\theta(\varphi_t^q - \underline{\varrho})\chi(\underline{\varrho}, \varphi_t^q) + [1 - (1-\alpha)\theta](\varphi_t^m - \underline{\varrho})\chi(\underline{\varrho}, \varphi_t^m).$$

Proof. (i) The broker's value function is immediate.

(ii) With $\bar{W}_t^I$, the value function becomes

$$\begin{aligned}
V_t^I(\mathbf{a}_t, \varepsilon) &= \bar{W}_t^I + \alpha\delta\left[(\varepsilon y_t + \phi_t^s)\hat{a}_t^s(\mathbf{a}_t, \varepsilon) + \phi_t^m \hat{a}_t^m(\mathbf{a}_t, \varepsilon)\right] \\
&\quad + (1-\alpha)\delta\left[(\varepsilon y_t + \phi_t^s)\bar{a}_t^s(\mathbf{a}_t, \varepsilon) + \phi_t^m \bar{a}_t^m(\mathbf{a}_t, \varepsilon) + \bar{a}_t^b(\mathbf{a}_t, \varepsilon) - k_t(\mathbf{a}_t, \varepsilon)\right] \\
&\quad + \alpha(1-\delta)\left\{u[\hat{q}_t(a_t^m)] + \phi_t^m \hat{a}_t^m(a_t^m) + (\varepsilon y_t + \phi_t^s)a_t^s\right\} \\
&\quad + (1-\alpha)(1-\delta)\left\{u[\bar{q}_t(\mathbf{a}_t)] + \phi_t^m \bar{a}_t^m(\mathbf{a}_t) + \bar{a}_t^b(\mathbf{a}_t) + (\varepsilon y_t + \phi_t^s)a_t^s - k_t(\mathbf{a}_t)\right\}.
\end{aligned}$$

Substitute $k_t^{ij}(\mathbf{a}_t,\varepsilon)$ and $\bar{a}_t^{jb}(\mathbf{a}_t,\varepsilon)$ for $j\in\{s,q\}$, to obtain

$$\begin{aligned} V_t^I(\mathbf{a}_t,\varepsilon) &= \bar{W}_t^I + (\varepsilon y_t+\phi_t^s)a_t^s + \phi_t^m a_t^m \\ &+ [\alpha+(1-\alpha)(1-\theta)]\delta\left\{(\varepsilon y_t+\phi_t^s)[\hat{a}_t^s(\mathbf{a}_t,\varepsilon)-a_t^s] + \phi_t^m[\hat{a}_t^m(\mathbf{a}_t,\varepsilon)-a_t^m]\right\} \\ &+ (1-\alpha)\delta\theta\left\{\left(\varepsilon y_t+\phi_t^s-\frac{p_t^s}{p_t^{sb}}\right)[\bar{a}_t^s(\mathbf{a}_t,\varepsilon)-a_t^s] + \left(\phi_t^m-\frac{1}{p_t^{sb}}\right)[\bar{a}_t^m(\mathbf{a}_t,\varepsilon)-a_t^m]\right\} \\ &+ \alpha(1-\delta)\theta\left\{u[\hat{q}_t(a_t^m)] - \varphi_t^m\hat{q}_t(a_t^m)\right\} \\ &+ (1-\alpha)(1-\delta)\theta\left\{u[\bar{q}_t(\mathbf{a}_t)] + \left(\phi_t^m-\frac{1}{p_t^{qb}}\right)\bar{a}_t^m(\mathbf{a}_t) - \varphi_t^q\bar{q}_t(\mathbf{a}_t) - \left(\phi_t^m-\frac{1}{p_t^{qb}}\right)a_t^m\right\}. \end{aligned}$$

Using Lemma 2 to replace the post-trade allocations and rearrange terms to arrive at the value function in the lemma.

(iii) With $\bar{W}_t^P$, the value function becomes

$$\begin{aligned} V_t^P(a_t^m) = \bar{W}_t^P + \max_{q_t\in\mathbb{R}_+}\Big\{ &-\varrho q_t + \alpha\left[\phi_t^m\hat{a}_t^m(a_t^m,q_t) + [q_t-\hat{q}_t(a_t^m,q_t)]\underline{\varrho}\right] \\ &+ (1-\alpha)\left[\phi_t^m\bar{a}_t^m(a_t^m,q_t) + [q_t-\bar{q}_t(a_t^m,q_t)]\underline{\varrho} + \bar{a}_t^b(a_t^m,q_t) - k_t(a_t^m,q_t)\right]\Big\}. \end{aligned}$$

Substitute the allocations from Lemma 2 to obtain

$$\begin{aligned} V_t^P(a_t^m) = \bar{W}_t^P + \max_{q_t\in\mathbb{R}_+}\Big[&-\varrho q_t + \alpha\left\{\phi_t^m[a_t^m + p_t^q\chi(\underline{\varrho},\varphi_t^m)q_t] + [1-\chi(\underline{\varrho},\varphi_t^m)]\underline{\varrho}q_t\right\} \\ &+ (1-\alpha)\Big\{\frac{1}{p_t^{qb}}[a_t^m + p_t^q\chi(\underline{\varrho},\varphi_t^q)] + [1-\chi(\underline{\varrho},\varphi_t^q)]\underline{\varrho}q_t \\ &-(1-\theta)\left(i_t^{qm}\phi_t^m a_t^m + [(\varphi_t^q-\underline{\varrho})\chi(\underline{\varrho},\varphi_t^q) - (\varphi_t^m-\underline{\varrho})\chi(\underline{\varrho},\varphi_t^m)]q_t\right)\Big\}\Big]. \end{aligned}$$

Rearranging yields the value function in the lemma. ■

Lemma 8. *Consider an economy without money.*

(i) The value function of a broker at the beginning of the OTC round of period t is

$$V_t^B = \Xi_t^I + \Xi_t^P + \bar{W}_t^B, \tag{C.46}$$

where $\bar{W}_t^B \equiv \beta \mathbb{E}_t V_{t+1}^B$*, and*

$$\Xi_t^I \equiv \alpha \left[\delta \int k_t^{is}(a_t^s, \varepsilon) dH_t^I(a_t^s, \varepsilon) + (1-\delta) \int k_t^{iq}(a_t^s) dF_t^I(a_t^s) \right]$$
$$\Xi_t^P \equiv \alpha(1-\delta) \int k_t^{pq}(a_t^s) dF_t^P(a_t^s).$$

(ii) The value function of an investor who enters the OTC round of period t *with equity holding* a_t^s *and valuation* ε *is*

$$V_t^I(a_t^s, \varepsilon) = v_t^q(a_t^s) + v_t^s(\varepsilon) a_t^s + \bar{W}_t^I, \tag{C.47}$$

where

$$v_t^q(a_t^s) \equiv (1-\alpha)(1-\delta)\theta \left\{ u[\bar{q}_t(a_t^s)] - \varphi_t^q \bar{q}_t(a_t^s) \right\}$$
$$v_t^s(\varepsilon) \equiv \varepsilon y_t + \phi_t^s + (1-\alpha)\delta\theta(\varepsilon - \varepsilon_t^n) \left[\chi(\varepsilon_t^n, \varepsilon) \frac{\varphi_t^q}{\varphi_t^q - \kappa \phi_t^q} - 1 \right].$$

(iii) The value function of a producer who enters the OTC round of period t *is*

$$V_t^P = \max_{q_t \in \mathbb{R}_+} [R^n(\varphi_t^q) - \underline{\varrho}] q_t + \bar{W}_t^P, \tag{C.48}$$

where

$$R^n(\varphi_t^q) = \underline{\varrho} + (1-\alpha)\theta(\varphi_t^q - \underline{\varrho})\chi(\underline{\varrho}, \varphi_t^q).$$

Proof. (i) The broker's value function is immediate under the assumption that investors carry no money.

(ii) In a nonmonetary economy, W_t^I reduces to

$$W_t^I(a_t^s, a_t^b, k_t) = \phi_t^s a_t^s + a_t^b - k_t + \bar{W}_t^I.$$

With $\bar{W}_t^I$ and Lemma 1, we get the value function in the lemma.

(iii) In a nonmonetary economy:

$$W_t^P(a_t^g, a_t^b, k_t) = a_t^g + a_t^b - k_t + \bar{W}_t^P.$$

With $\bar{W}_t^P$ and Lemma 1, we get the value function in the lemma. ■

C.3 Euler Equations

In this section we derive the Euler equations that characterise the optimal portfolio choices in the second subperiod, in a monetary equilibrium (Lemma 9) and in a nonmonetary economy (Lemma 10).

Lemma 9. *Consider an economy with money. Let $(\tilde{a}^m_{t+1}, \tilde{a}^s_{t+1})$ denote an individual investor's portfolio choice in the second subperiod of period t. The portfolio $(\tilde{a}^m_{t+1}, \tilde{a}^s_{t+1})$ is optimal if and only if it satisfies*

$$\phi_t^m \tilde{a}^m_{t+1} - \beta \mathbb{E}_t\left[\bar{v}^q_{t+1}(\tilde{\mathbf{a}}_{t+1}) + \bar{v}^m_{t+1}\tilde{a}^m_{t+1}\right] = 0 \leq \phi_t^m - \beta\mathbb{E}_t\left[\frac{d\bar{v}^q_t(\tilde{\mathbf{a}}_{t+1})}{d\tilde{a}^m_{t+1}} + \bar{v}^m_{t+1}\right] \tag{C.49}$$

$$\phi_t^s \tilde{a}^s_{t+1} - \beta\eta \mathbb{E}_t\left[\bar{v}^q_{t+1}(\tilde{\mathbf{a}}_{t+1}) + \bar{v}^s_{t+1}\tilde{a}^s_{t+1}\right] = 0 \leq \phi_t^s - \beta\eta\mathbb{E}_t\left[\frac{d\bar{v}^q_t(\tilde{\mathbf{a}}_{t+1})}{d\tilde{a}^s_{t+1}} + \bar{v}^s_{t+1}\right], \tag{C.50}$$

where

$$\begin{aligned}
\bar{v}^m_{t+1} \equiv\ & \phi^m_{t+1} + (1-\alpha)\theta\left(\frac{1}{p^{sb}_{t+1}} - \phi^m_{t+1}\right)\mathbb{I}_{\left\{p^{sb}_{t+1}\phi^m_{t+1}<1\right\}} \\
& + [\alpha + (1-\alpha)(1-\theta)]\delta\int_{\varepsilon^*_{t+1}}^{\varepsilon_H}(\varepsilon - \varepsilon^*_{t+1})y_{t+1}\frac{1}{p^s_{t+1}}dG(\varepsilon) \\
& + (1-\alpha)\delta\theta\frac{1}{p^s_{t+1} - \kappa p^{sb}_{t+1}\phi^s_{t+1}}\int_{\varepsilon^{**}_{t+1}}^{\varepsilon_H}(\varepsilon - \varepsilon^{**}_{t+1})y_{t+1}dG(\varepsilon) \\
\bar{v}^s_{t+1} \equiv\ & \bar{\varepsilon}y_{t+1} + \phi^s_{t+1} + (1-\alpha)\delta\theta\left(\phi^m_{t+1} - \frac{1}{p^{sb}_{t+1}}\right)\mathbb{I}_{\left\{1<p^{sb}_{t+1}\phi^m_{t+1}\right\}} \\
& + [\alpha + (1-\alpha)(1-\theta)]\delta\int_{\varepsilon_L}^{\varepsilon^*_{t+1}}(\varepsilon^*_{t+1} - \varepsilon)y_{t+1}dG(\varepsilon) \\
& + (1-\alpha)\delta\theta\left[\int_{\varepsilon_L}^{\varepsilon^{**}_{t+1}}(\varepsilon^{**}_{t+1} - \varepsilon)y_{t+1}dG(\varepsilon) + \frac{\kappa p^{sb}_{t+1}\phi^s_{t+1}}{p^s_{t+1} - \kappa p^{sb}_{t+1}\phi^s_{t+1}}\int_{\varepsilon^{**}_{t+1}}^{\varepsilon_H}(\varepsilon - \varepsilon^{**}_{t+1})y_{t+1}dG(\varepsilon)\right],
\end{aligned}$$

$$\bar{v}^q_{t+1}(\tilde{\mathbf{a}}_{t+1}) \equiv \alpha(1-\delta)\theta\left\{u[\hat{q}_{t+1}(\tilde{a}^m_{t+1})] - \varphi^m_{t+1}\hat{q}_{t+1}(\tilde{a}^m_{t+1})\right\}$$
$$+ (1-\alpha)(1-\delta)\theta\left\{u[\bar{q}_{t+1}(\tilde{\mathbf{a}}_{t+1})] - \varphi^q_{t+1}\bar{q}_{t+1}(\tilde{\mathbf{a}}_{t+1})\right\},$$

and

$$\frac{d\bar{v}^q_{t+1}(\tilde{\mathbf{a}}_{t+1})}{d\tilde{a}^m_{t+1}} = \alpha(1-\delta)\theta\phi^m_{t+1}\left[\frac{u'[\hat{q}_{t+1}(\tilde{a}^m_{t+1})]}{\varphi^m_{t+1}} - 1\right]$$
$$+ (1-\alpha)(1-\delta)\theta\phi^m_{t+1}\left[\frac{u'[\bar{q}_{t+1}(\tilde{\mathbf{a}}_{t+1})]}{\varphi^m_{t+1}} - 1\right]$$
$$\frac{d\bar{v}^q_{t+1}(\tilde{\mathbf{a}}_{t+1})}{d\tilde{a}^s_{t+1}} = (1-\alpha)(1-\delta)\theta\lambda\phi^s_{t+1}\left[\frac{u'[\bar{q}_{t+1}(\tilde{\mathbf{a}}_{t+1})]}{\varphi^q_{t+1}} - 1\right].$$

Proof. The portfolio problem of an investor in the second subperiod can be written as

$$\bar{W}_t \equiv T_t + \beta\left[\bar{W}_{t+1} + (1-\eta)\bar{v}^s_{t+1}A^s\right]$$
$$+ \max_{(\tilde{a}^m_{t+1},\tilde{a}^s_{t+1})\in\mathbb{R}^2_+}\left\{-\phi^m_t\tilde{a}^m_{t+1} - \phi^s_t\tilde{a}^s_{t+1} + \beta\mathbb{E}_t\left[\bar{v}^q_{t+1}(\tilde{\mathbf{a}}_{t+1}) + \bar{v}^m_{t+1}\tilde{a}^m_{t+1} + \eta\bar{v}^s_{t+1}\tilde{a}^s_{t+1}\right]\right\},$$

where $\bar{v}^k_{t+1} \equiv \int v^k_{t+1}(\varepsilon)dG(\varepsilon)$. ■

Lemma 10. *Consider an economy with no money. Let $\tilde{a}^s_{t+1}$ denote equity holding chosen by an individual investor in the second subperiod of period t. Then $\tilde{a}^s_{t+1}$ is optimal if and only if it satisfies*

$$-\phi^s_t + \beta\eta\mathbb{E}_t\Bigg\{\bar{\varepsilon}y_{t+1} + \phi^s_{t+1} + (1-\alpha)(1-\delta)\theta\lambda\phi^s_{t+1}\left[\frac{u'[\bar{q}_{t+1}(\tilde{a}^s_{t+1})]}{\varphi^q_{t+1}} - 1\right]$$
$$+ (1-\alpha)\delta\theta\left[\int_{\varepsilon_L}^{\varepsilon^n_{t+1}}(\varepsilon^n_{t+1} - \varepsilon)y_{t+1}dG(\varepsilon) + \frac{\kappa\phi^s_{t+1}}{\varphi^q_{t+1} - \kappa\phi^s_{t+1}}\int_{\varepsilon^n_{t+1}}^{\varepsilon_H}(\varepsilon - \varepsilon^n_{t+1})y_{t+1}dG(\varepsilon)\right]\Bigg\}$$
$$\leq 0, \text{ with “=” if } \tilde{a}^s_{t+1} > 0. \tag{C.51}$$

Proof. The portfolio problem of an investor in the second subperiod can be written as

$$\max_{\tilde{a}^s_{t+1}\in\mathbb{R}_+}\Bigg[-\phi^s_t\tilde{a}^s_{t+1} + \beta\eta\mathbb{E}_t\Bigg\{(\bar{\varepsilon}y_{t+1} + \phi^s_{t+1})\tilde{a}^s_{t+1} + (1-\alpha)(1-\delta)\theta\left[u[\bar{q}_{t+1}(\tilde{a}^s_{t+1})] - \varphi^q_{t+1}\bar{q}_{t+1}(\tilde{a}^s_{t+1})\right]$$
$$+ (1-\alpha)\delta\theta\left[\int_{\varepsilon_L}^{\varepsilon^n_{t+1}}(\varepsilon^n_{t+1} - \varepsilon)y_{t+1}dG(\varepsilon) + \frac{\kappa\phi^s_{t+1}}{\varphi^q_{t+1} - \kappa\phi^s_{t+1}}\int_{\varepsilon^n_{t+1}}^{\varepsilon_H}(\varepsilon - \varepsilon^n_{t+1})y_{t+1}dG(\varepsilon)\right]\tilde{a}^s_{t+1}\Bigg\}\Bigg],$$

where $\bar{q}_{t+1}(\tilde{a}^s_{t+1}) = \min\left\{\mathrm{D}(\varphi^q_{t+1}), \frac{\lambda\phi^s_{t+1}\tilde{a}^s_{t+1}}{\varphi^q_{t+1}}\right\}$. The first-order condition yields the lemma. ∎

C.4 Market-clearing Conditions

In this section we derive the market-clearing conditions for equity and bonds in the OTC round, in a monetary economy (Lemma 11) and in a nonmonetary economy (Lemma 12).

Lemma 11. *In a monetary equilibrium, the market-clearing conditions for equity $\hat{A}^s_{It} + \bar{A}^s_{It} = \delta A^s$, goods $\hat{Q}_{It} + \bar{Q}_{It} = \hat{Q}_{Pt} + \bar{Q}_{Pt}$ and bonds $\bar{A}^b_{It} + \bar{A}^b_{Pt} = 0$*

$$0 = \alpha[1 - G(\varepsilon^*_t)]\frac{A^m_t + p^s_t A^s}{p^s_t} + (1-\alpha)[1 - G(\varepsilon^{**}_t)]\frac{A^m_t + p^s_t A^s}{p^s_t - \kappa p^{sb}_t \phi^s_t} - A^s \tag{C.52}$$

$$0 = (1-\alpha)\left\{\left\{1 - \mathbb{I}_{\{1<p^{sb}_t\phi^m_t\}} - \mathbb{I}_{\{p^{sb}_t\phi^m_t=1\}}[1 - \chi(p^{sb}_t\phi^m_t)]\right\} G(\varepsilon^{**}_t) - \frac{\kappa p^{sb}_t\phi^s_t}{p^s_t - \kappa p^{sb}_t\phi^s_t}[1 - G(\varepsilon^{**}_t)]\right\}\frac{A^m_t + p^s_t A^s}{p^{sb}_t} \tag{C.53}$$

$$0 = \min\left\{\mathrm{D}(\bar{\varphi}^m_t), \frac{\phi^m_t A^m_t}{\bar{\varphi}^m_t}\right\} - \hat{\mathrm{Q}}(\bar{\varphi}^m_t) \tag{C.54}$$

$$0 = \mathbb{I}_{\{1<p^{sb}_t\phi^m_t\}}\mathrm{D}(\bar{\varphi}^m_t) + \mathbb{I}_{\{1\geq p^{sb}_t\phi^m_t\}}\min\left\{\mathrm{D}(\bar{\varphi}^q_t), \frac{\phi^m_t A^m_t}{\bar{\varphi}^m_t} + \frac{\lambda\phi^s_t A^s}{\bar{\varphi}^q_t}\right\} - \bar{\mathrm{Q}}(\bar{\varphi}^q_t), \tag{C.55}$$

where

$$\bar{\varphi}^m_t \equiv \frac{\varrho}{1 + (1-\alpha)\theta i^{qm}_t}$$
$$\bar{\varphi}^q_t \equiv \frac{1 + i^{qm}_t}{1 + (1-\alpha)\theta i^{qm}_t}\varrho.$$

Proof. By Lemma 2, the investors' aggregate post-trade holding of equity in the OTC round of period t are

$$\bar{A}^s_{It} = (1-\alpha)\delta N_I \int \bar{a}^s_t(\mathbf{a}_t, \varepsilon) dH^I_t(\mathbf{a}_t, \varepsilon) = (1-\alpha)\delta[1 - G(\varepsilon^{**}_t)]\frac{A^m_t + p^s_t A^s}{p^s_t - \kappa p^{sb}_t\phi^s_t}$$
$$\hat{A}^s_{It} = \alpha\delta N_I \int \hat{a}^s_t(\mathbf{a}_t, \varepsilon) dH^I_t(\mathbf{a}_t, \varepsilon) = \alpha[1 - G(\varepsilon^*_t)]\frac{A^m_t + p^s_t A^s}{p^s_t}.$$

The producer's objective function is given by

$$q_t(\varphi^q_t, \varphi^m_t) = \arg\max_{q_t \in \mathbb{R}_+} [R^m(\varphi^q_t, \varphi^m_t) - \varrho]\, q_t \equiv \mathrm{Q}(\varphi^q_t, \varphi^m_t),$$

which implies that

$$\mathrm{Q}(\varphi_t^q, \varphi_t^m) \begin{cases} \to \infty & \text{if } \varrho < R^m(\varphi_t^q, \varphi_t^m) \\ \in [0, \infty) & \text{if } \varrho = R^m(\varphi_t^q, \varphi_t^m) \\ = 0 & \text{if } \varrho > R^m(\varphi_t^q, \varphi_t^m). \end{cases}$$

Notice that

$$(1 + i_t^{qm})\varphi_t^m = (1 + i_t^{qm})\phi_t^m p_t^q = \frac{1}{p_t^{qb}\phi_t^m}\phi_t^m p_t^q = \varphi_t^q.$$

Next $\varrho = R^m(\varphi_t^q, \varphi_t^m)$ since if $\varrho > R^m(\varphi_t^q, \varphi_t^m)$ then $q_t = 0$ or if $\varrho < R^m(\varphi_t^q, \varphi_t^m)$ then $q_t \to \infty$. Therefore since $\varrho = R^m(\varphi_t^q, \varphi_t^m)$ then

$$\varrho = \underline{\varrho} + (1 - \alpha)\theta(\varphi_t^q - \underline{\varrho})\chi(\underline{\varrho}, \varphi_t^q) + [1 - (1 - \alpha)\theta](\varphi_t^m - \underline{\varrho})\chi(\underline{\varrho}, \varphi_t^m).$$

From this we can observe that since $\underline{\varrho} \geq \varphi_t^q$ then $\underline{\varrho} \geq \varphi_t^q \geq \varphi_t^m$ and therefore $R^m(\varphi_t^q, \varphi_t^m) = \underline{\varrho} < \varrho$ which implies that $q_t = 0$ and the goods are never produced. Now we have,

$$\varrho = \underline{\varrho} + (1 - \alpha)\theta[(1 + i_t^{qm})\varphi_t^m - \underline{\varrho}] + [1 - (1 - \alpha)\theta](\varphi_t^m - \underline{\varrho}) \equiv R^m(\varphi_t^m),$$

which yields

$$\varphi_t^m = \bar{\varphi}_t^m \equiv \frac{\varrho}{1 + (1 - \alpha)\theta i_t^{qm}}$$
$$\varphi_t^q = \bar{\varphi}_t^q \equiv \frac{1 + i_t^{qm}}{1 + (1 - \alpha)\theta i_t^{qm}}\varrho,$$

and the investors' and producers' aggregate post-trade allocation of goods in the OTC round of period t are

$$\begin{aligned}
\bar{Q}_{It} &= (1-\alpha)(1-\delta)N_I \int \bar{q}_t(\mathbf{a}_t) dF_t^I(\mathbf{a}_t) \\
&= (1-\alpha)(1-\delta)\left[\mathbb{I}_{\{1<p_t^{qb}\phi_t^m\}}\mathrm{D}(\varphi_t^m) + \mathbb{I}_{\{1\geq p_t^{qb}\phi_t^m\}} \min\left\{\mathrm{D}(\varphi_t^q), \frac{A_t^m + \lambda p_t^{qb}\phi_t^s A^s}{p_t^q}\right\}\right] \\
\hat{Q}_{It} &= \alpha(1-\delta)N_I \int \hat{q}_t(\mathbf{a}_t) dF_t^I(\mathbf{a}_t) = \alpha(1-\delta)\min\left\{\mathrm{D}(\varphi_t^m), \frac{A_t^m}{p_t^q}\right\} \\
\bar{Q}_{Pt} &= (1-\alpha)(1-\delta)N_P \int \bar{q}_t(a_t^m, q_t) dF_t^P(a_t^m, q_t) = (1-\alpha)(1-\delta)\chi(\underline{\varrho}, \varphi_t^q)\mathrm{Q}(\varphi_t^q, \varphi_t^m) \\
\hat{Q}_{Pt} &= \alpha(1-\delta)N_P \int \hat{q}_t(a_t^m, q_t) dF_t^P(a_t^m, q_t) = \alpha(1-\delta)\chi(\underline{\varrho}, \varphi_t^m)\mathrm{Q}(\varphi_t^q, \varphi_t^m).
\end{aligned}$$

The market clearing condition in the goods market is given by $\hat{Q}_{It} + \bar{Q}_{It} = \hat{Q}_{Pt} + \bar{Q}_{Pt}$, which implies

$$\begin{aligned}
&\alpha \min\left\{\mathrm{D}(\varphi_t^m), \frac{A_t^m}{p_t^q}\right\} + (1-\alpha)\left[\mathbb{I}_{\{1<p_t^{qb}\phi_t^m\}}\mathrm{D}(\varphi_t^m) + \mathbb{I}_{\{1\geq p_t^{qb}\phi_t^m\}} \min\left\{\mathrm{D}(\varphi_t^q), \frac{A_t^m + \lambda p_t^{qb}\phi_t^s A^s}{p_t^q}\right\}\right] \\
&= \left[\alpha\chi(\underline{\varrho}, \varphi_t^m) + (1-\alpha)\chi(\underline{\varrho}, \varphi_t^q)\right]\mathrm{Q}(\varphi_t^q, \varphi_t^m),
\end{aligned}$$

and since $(1+i_t^{qm})\bar{\varphi}_t^m = \varphi_t^q$ we can write $\hat{\mathrm{Q}}(\varphi_t^m) = \chi(\underline{\varrho}, \varphi_t^m)\mathrm{Q}(\varphi_t^q, \varphi_t^m)$ and $\bar{\mathrm{Q}}(\varphi_t^q) = \chi(\underline{\varrho}, \varphi_t^q)\mathrm{Q}(\varphi_t^q, \varphi_t^m)$ therefore excess demand yields

$$\begin{aligned}
\mathrm{X}(\varphi_t^q, \varphi_t^m) = \alpha &\left[\min\left\{\mathrm{D}(\varphi_t^m), \frac{A_t^m}{p_t^q}\right\} - \hat{\mathrm{Q}}(\varphi_t^m)\right] \\
&+ (1-\alpha)\left[\mathbb{I}_{\{1<p_t^{qb}\phi_t^m\}}\mathrm{D}(\varphi_t^m) + \mathbb{I}_{\{1\geq p_t^{qb}\phi_t^m\}} \min\left\{\mathrm{D}(\varphi_t^q), \frac{A_t^m + \lambda p_t^{qb}\phi_t^s A^s}{p_t^q}\right\} - \bar{\mathrm{Q}}(\varphi_t^q)\right],
\end{aligned}$$

when $\varphi_t^m = \bar{\varphi}_t^m$ and $\varphi_t^q = \bar{\varphi}_t^q$ we let,

$$\begin{aligned}
\mathrm{X}_t^m(\bar{\varphi}_t^m) &\equiv \min\left\{\mathrm{D}(\bar{\varphi}_t^m), \frac{\phi_t^m A_t^m}{\bar{\varphi}_t^m}\right\} - \hat{\mathrm{Q}}(\bar{\varphi}_t^m) = 0 \\
\mathrm{X}_t^q(\bar{\varphi}_t^q, \bar{\varphi}_t^m) &\equiv \mathbb{I}_{\{1<p_t^{qb}\phi_t^m\}}\mathrm{D}(\bar{\varphi}_t^m) + \mathbb{I}_{\{1\geq p_t^{qb}\phi_t^m\}} \min\left\{\mathrm{D}(\bar{\varphi}_t^q), \frac{\phi_t^m A_t^m}{\bar{\varphi}_t^m} + \frac{\lambda\phi_t^s A^s}{\bar{\varphi}_t^q}\right\} - \bar{\mathrm{Q}}(\bar{\varphi}_t^q) = 0,
\end{aligned}$$

which implies that $\mathrm{X}(\bar{\varphi}_t^q, \bar{\varphi}_t^m) = 0$.

The investors' and producers aggregate post-trade holdings of bonds in the OTC round of period t are

$$
\begin{aligned}
\bar{A}^{sb}_{It} &= (1-\alpha)\delta N_I \int \bar{a}^b_t(\mathbf{a}_t,\varepsilon)dH^I_t(\mathbf{a}_t,\varepsilon) \\
&= (1-\alpha)\delta\Bigg\{ \left\{1 - \mathbb{I}_{\{1<p^{sb}_t\phi^m_t\}} - \mathbb{I}_{\{p^{sb}_t\phi^m_t=1\}}[1-\chi(1,p^{sb}_t\phi^m_t)]\right\} G(\varepsilon^{**}_t) \\
&- \frac{\kappa p^{sb}_t\phi^s_t}{p^s_t - \kappa p^{sb}_t\phi^s_t}[1-G(\varepsilon^{**}_t)]\Bigg\} \frac{A^m_t + p^s_t A^s}{p^{sb}_t} \\
\bar{A}^{qb}_{It} &= (1-\alpha)(1-\delta)N_I \int \bar{a}^b_t(\mathbf{a}_t)dF^I_t(\mathbf{a}_t) \\
&= -(1-\alpha)(1-\delta)\frac{1}{p^b_t}\Bigg\{ \bar{A}^m_{Pt} - A^m_t \\
&+ p^q_t\left[\mathbb{I}_{\{1<p^{qb}_t\phi^m_t\}}\mathrm{D}(\bar{\varphi}^m_t) + \mathbb{I}_{\{1\geq p^{qb}_t\phi^m_t\}}\min\left\{\mathrm{D}(\bar{\varphi}^q_t), \frac{\phi^m_t A^m_t}{\bar{\varphi}^m_t} + \frac{\lambda\phi^s_t A^s}{\bar{\varphi}^q_t}\right\}\right]\Bigg\} \\
\bar{A}^b_{Pt} &= (1-\alpha)(1-\delta)N_P \int \bar{a}^b_t(a^m_t, q_t(a^m_t))dF^P_t(a^m_t) \\
&= (1-\alpha)(1-\delta)\frac{1}{p^b_t}\left[A^m_t - \bar{A}^m_{It} + p^q_t\chi(\underline{\varrho},\bar{\varphi}^q_t)\mathrm{Q}(\bar{\varphi}^q_t,\bar{\varphi}^m_t)\right].
\end{aligned}
$$

The market clearing condition in the bonds market is given by $\bar{A}^b_{It} + \bar{A}^b_{Pt} = 0$,

$$
\begin{aligned}
0 &= (1-\alpha)\delta\Bigg\{ \left\{1 - \mathbb{I}_{\{1<p^b_t\phi^m_t\}} - \mathbb{I}_{\{p^b_t\phi^m_t=1\}}[1-\chi(1,p^b_t\phi^m_t)]\right\} G(\varepsilon^{**}_t) \\
&- \frac{\kappa p^b_t\phi^s_t}{p^s_t - \kappa p^b_t\phi^s_t}[1-G(\varepsilon^{**}_t)]\Bigg\} \frac{A^m_t + p^s_t A^s}{p^b_t} \\
&+ (1-\alpha)(1-\delta)\frac{1}{p^b_t}\Bigg\{ A^m_t - \bar{A}^m_{It} - \bar{A}^m_{Pt} + A^m_t \\
&- p^q_t\left[\mathbb{I}_{\{1<p^b_t\phi^m_t\}}\mathrm{D}(\bar{\varphi}^m_t) + \mathbb{I}_{\{1\geq p^b_t\phi^m_t\}}\min\left\{\mathrm{D}(\bar{\varphi}^q_t), \frac{\phi^m_t A^m_t}{\bar{\varphi}^m_t} + \frac{\lambda\phi^s_t A^s}{\bar{\varphi}^q_t}\right\} - \bar{\mathrm{Q}}(\bar{\varphi}^q_t)\right]\Bigg\}.
\end{aligned}
$$

Since $A^m_t - \bar{A}^m_{It} = \bar{A}^m_{Pt} - A^m_t$ and $\mathrm{X}^q_t(\bar{\varphi}^q_t, \bar{\varphi}^m_t) = 0$, we get the condition from the lemma. ■

Lemma 12. *In a nonmonetary equilibrium, the market-clearing condition for equity $\hat{A}^s_t + \bar{A}^s_t = \delta A^s$ and bonds $\bar{A}^b_{It} + \bar{A}^b_{Pt} = 0$ (or goods $\bar{Q}_{It} = \bar{Q}_{Pt}$) in the OTC round are:*

$$
1 = [1 - G(\varepsilon^n_t)]\frac{\bar{\phi}^s_t}{\bar{\phi}^s_t - \kappa\phi^s_t} \tag{C.56}
$$

$$
\mathrm{Q}(\hat{\varphi}^q_t) = \min\left\{\mathrm{D}(\hat{\varphi}^q_t), \frac{\kappa\phi^s_t A^s}{\hat{\varphi}^q_t}\right\}, \tag{C.57}
$$

where

$$\hat{\varphi}_t^q = \varrho + \frac{1-(1-\alpha)\theta}{(1-\alpha)\theta}(\varrho - \underline{\varrho}).$$

Proof. By Lemma 1, the investors' aggregate post-trade holdings of equity in the OTC round of period t are

$$\begin{aligned}
\bar{A}_{It}^s &= (1-\alpha)N_I \int \bar{a}_t^s(a_t^s,\varepsilon)dH_t(a_t^s,\varepsilon) = (1-\alpha)\delta \int \chi(\varepsilon_t^n,\varepsilon)\frac{\bar{\phi}_t^s}{\bar{\phi}_t^s - \kappa\phi_t^s}A^s dG(\varepsilon) \\
\hat{A}_{It}^s &= \alpha\delta N_I \int \hat{a}_t^s(a_t^s,\varepsilon)dH_t(a_t^s,\varepsilon) = \alpha\delta A^s.
\end{aligned}$$

Lemma 1 implies that individual producer's choice of production is

$$q_t(\varphi_t^q) = \arg\max_{q_t \in \mathbb{R}_+}[R^n(\varphi_t^q) - \varrho]q_t \equiv \mathrm{Q}(\varphi_t^q),$$

where

$$R^n(\varphi_t^q) = \underline{\varrho} + (1-\alpha)\theta(\varphi_t^q - \underline{\varrho})\chi(\underline{\varrho},\varphi_t^q),$$

so $R^n(\varphi_t^q) - \underline{\varrho} \leq 0$, or equivalently,

$$\varphi_t^q \leq \hat{\varphi}_t^q \equiv \varrho + \frac{1-(1-\alpha)\theta}{(1-\alpha)\theta}(\varrho - \underline{\varrho}),$$

is a necessary condition for equilibrium and implies that $\varrho < \varphi_t^q$. Hence the solution to the producer's production decision is

$$\mathrm{Q}(\varphi_t^q) \begin{cases} = 0 & \text{if } \varphi_t^q < \hat{\varphi}_t^q \\ \in [0,\infty) & \text{if } \varphi_t^q = \hat{\varphi}_t^q. \end{cases}$$

Using Lemma 1 the aggregate post-trade allocaiton of goods for agent who trade in the special goods market in the OTC goods round of period t

$$\bar{Q}_{It} = (1-\alpha)(1-\delta)\min\left\{\mathrm{D}(\varphi_t^q), \frac{\kappa\phi_t^s A^s}{\varphi_t^q}\right\}$$
$$\bar{Q}_{Pt} = (1-\alpha)(1-\delta)\chi(\underline{\varrho}, \varphi_t^q)\mathrm{Q}(\varphi_t^q),$$

and we define $\bar{\mathrm{Q}}(\varphi_t^q) = \chi(\underline{\varrho}, \varphi_t^q)\mathrm{Q}(\varphi_t^q)$, therefore the excess demand is given by

$$\mathrm{X}(\varphi_t^q) = \min\left\{\mathrm{D}(\varphi_t^q), \frac{\kappa\phi_t^s A^s}{\varphi_t^q}\right\} - \bar{\mathrm{Q}}(\varphi_t^q).$$

For $A^s > 0$, and for all $\varphi_t^q \in [0, \hat{\varphi}_t^q)$, $0 < \mathrm{X}(\hat{\varphi}_t^q)$ so equilibrium requires $\hat{\varphi}_t^q \geq \varphi_t^q$, which together with

$$\varphi_t^q \leq \hat{\varphi}^q \equiv \varrho + \frac{1-(1-\alpha)\theta}{(1-\alpha)\theta}(\varrho - \underline{\varrho}),$$

implies $\hat{\varphi}^q = \varrho + \frac{1-(1-\alpha)\theta}{(1-\alpha)\theta}(\varrho - \underline{\varrho})$ must hold in equilibrium. Therefore the market-clearing in the goods market implies $\bar{Q}_{It} = \bar{Q}_{Pt}$ which can be written as

$$\bar{\mathrm{Q}}(\hat{\varphi}^q) = \min\left\{\mathrm{D}(\hat{\varphi}^q), \frac{\kappa\phi_t^s A^s}{\hat{\varphi}^q}\right\}.$$

The aggregate post-trade holdings of bonds for agents who trade in the bond market in the OTC round of period t are

$$\bar{A}_{It}^{sb} = (1-\alpha)\delta N_I \int \bar{a}_t^b(a_t^s, \varepsilon)dH_t(a_t^s, \varepsilon) = (1-\alpha)\delta \int \bar{\phi}_t^s \left[1 - \chi(\varepsilon_t^n, \varepsilon)\frac{\bar{\phi}_t^s}{\bar{\phi}_t^s - \kappa\phi_t^s}\right] A^s dG(\varepsilon)$$
$$\bar{A}_{It}^{qb} = (1-\alpha)(1-\delta)N_I \int \bar{a}_t^b(a_t^s)dF_t^I(a_t^s) = -(1-\alpha)(1-\delta)\min\left\{\hat{\varphi}^q\mathrm{D}(\hat{\varphi}^q), \kappa\phi_t^s A^s\right\}$$
$$\bar{A}_{Pt}^{b} = (1-\alpha)(1-\delta)N_I\bar{a}_t^b(q_t) = (1-\alpha)(1-\delta)\hat{\varphi}_t^q\chi(\underline{\varrho}, \hat{\varphi}^q)\mathrm{Q}(\hat{\varphi}^q).$$

Therefore the market-clearing condition for bonds ($\bar{A}_{It}^b + \bar{A}_{Pt}^b = 0$) can be written as

$$1 = [1 - G(\varepsilon_t^n)]\frac{\bar{\phi}_t^s}{\bar{\phi}_t^s - \kappa\phi_t^s},$$

since $\mathrm{X}(\hat{\varphi}^q) = 0$. ∎

C.5 Equilibrium Conditions

In this section we state the operation definitions of monetary and nonmonetary equilibria that are used in the analysis.

C.5.1 Sequential Nonmonetary equilibrium

Definition 1. *A* (sequential) nonmonetary equilibrium *is an allocation* $\{\tilde{a}^s_{It+1}\}_{t=0}^{\infty}$ *and a sequence of prices* $\{\phi^s_t, \bar{\phi}^s_t\}_{t=0}^{\infty}$ *that satisfy the portfolio-optimality condition in Lemma 10 (with* $\tilde{a}^s_{It+1} = \tilde{A}_{It+1}$*), and the market-clearing conditions* $\tilde{A}^s_{It+1} = A^s$*.*

Definition 1 follows after recognising that all investors choose the same end-of-period portfolio that is characterised by the Euler equations derived in Lemma 10, and using the explicit version of the market-clearing condition for equity, goods and bonds, in the OTC round derived in Lemma 12. Given the equilibrium objects in Definition 1, the bargaining outcomes are immediate from Lemma 1.

According to Definition 1, a nonmonetary equilibrium can be characterised by a sequence of prices, $\{\phi^s_t, \bar{\phi}^s_t\}_{t=0}^{\infty}$ and an allocation $\{\tilde{A}^s_{It+1}\}_{t=0}^{\infty}$ that satisfy the market-clearing conditions

$$\begin{aligned} A^s &= \tilde{A}^s_{It+1} \\ 1 &= [1 - G(\varepsilon^n_t)] \frac{\bar{\phi}^s_t}{\bar{\phi}^s_t - \kappa \phi^s_t} \\ \bar{\mathrm{Q}}^n_t(\hat{\varphi}^q) &= \min \left\{ \mathrm{D}(\hat{\varphi}^q), \frac{\kappa \phi^s_t A^s}{\hat{\varphi}^q} \right\}, \end{aligned}$$

and the portfolio-optimality condition

$$\begin{aligned} & - \phi^s_t + \beta \eta \mathbb{E}_t \Bigg\{ \bar{\varepsilon} y_{t+1} + \phi^s_{t+1} + (1-\alpha)(1-\delta)\theta\lambda\phi^s_{t+1} \left[\frac{u'[\bar{\mathrm{Q}}^n_{t+1}(\hat{\varphi}^q)]}{\hat{\varphi}^q} - 1 \right] \\ & + (1-\alpha)\delta\theta \left[\int_{\varepsilon_L}^{\varepsilon^n_{t+1}} (\varepsilon^n_{t+1} - \varepsilon) y_{t+1} dG(\varepsilon) + \frac{\kappa \phi^s_{t+1}}{\bar{\phi}^s_{t+1} - \kappa \phi^s_{t+1}} \int_{\varepsilon^n_{t+1}}^{\varepsilon_H} (\varepsilon - \varepsilon^n_{t+1}) y_{t+1} dG(\varepsilon) \right] \Bigg\} \\ & \leq 0, \text{ with “=” if } \tilde{A}^s_{It+1} > 0, \end{aligned}$$

where ε_t^n is given in Lemma 1 and $\hat{\varphi}^q = \varrho + \frac{1-(1-\alpha)\theta}{(1-\alpha)\theta}(\varrho - \underline{\varrho})$.

C.5.2 Recursive Nonmonetary Equilibrium

The following result states the conditions characterising a recursive nonmonetary equilibrium (RNE). For simplicity of exposition, we henceforth maintain the following assumption

$$u(q_t) = \bar{\omega}_t q_t^{\sigma}, \text{ with } \sigma \in (0,1), \text{ and } \bar{\omega}_t \equiv (\phi_t^s)^{1-\sigma},$$

and define $\tilde{u}(\cdot)$ such that $\tilde{u}'(\mathcal{Q}_t) = u'(\mathrm{Q}_t)$ where $\mathcal{Q}_t = \mathrm{Q}_t/\phi_t^s$.

Lemma 13. *A recursive nonmonetary equilibrium is a vector* $(\varepsilon^n, \phi^s, \tilde{A}_I^s)$ *that satisfies the following conditions,*

$$0 = \tilde{A}_I^s - A^s \tag{C.58}$$

$$0 = [1 - G(\varepsilon^n)]\frac{\varepsilon^n + \phi^s}{\varepsilon^n + (1-\kappa)\phi^s} - 1 \tag{C.59}$$

$$0 = \min\left\{\tilde{\mathrm{D}}(\hat{\varphi}^q), \frac{\lambda A^s}{\hat{\varphi}^q}\right\} - \bar{\mathcal{Q}}^n(\hat{\varphi}^q), \tag{C.60}$$

and

$$\begin{aligned}\frac{1-\bar{\beta}\eta}{\bar{\beta}\eta}\phi^s &= \bar{\varepsilon} + (1-\alpha)(1-\delta)\theta\lambda\phi^s\left[\frac{\tilde{u}'[\bar{\mathcal{Q}}^n(\hat{\varphi}^q)]}{\hat{\varphi}^q} - 1\right] \\ &\quad + (1-\alpha)\delta\theta\left[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n - \varepsilon)dG(\varepsilon) + \frac{\kappa\phi^s}{\varepsilon + (1-\kappa)\phi^s}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)dG(\varepsilon)\right],\end{aligned} \tag{C.61}$$

with

$$\hat{\varphi}^q = \varrho + \frac{1-(1-\alpha)\theta}{(1-\alpha)\theta}(\varrho - \underline{\varrho}). \tag{C.62}$$

Proof. The equilibrium conditions in the statement of the lemma are obtained from Lemma 10 and Lemma 12 by using $\phi_t^s = \phi^s y_t$, $\bar{\phi}_t^s = \bar{\phi}^s y_t$, $\tilde{A}_{It}^s = \tilde{A}_I^s$ and $\varepsilon_t^n = (\bar{\phi}_t^s - \phi_t^s)\frac{1}{y_t} = \bar{\varphi}^s - \varphi^s \equiv \varepsilon^n$ ■

The first equation in the statement of Lemma 13 is the second subperiod market-clearing condition for equity. The second equation is the first subperiod market-clearing condition for equity (or bonds). The third condition is the price of the special good in terms of the bond in the first subperiod. The third condition is the investor's Euler equation for equity.

C.5.3 Sequential Monetary Equilibrium

Definition 2. *A* (sequential) monetary equilibrium *is an allocation* $\{(\tilde{a}^k_{It+1})_{k\in\{m,s\}}\}_{t=0}^{\infty}$ *and a sequence of prices,* $\{p^s_t, p^{sb}_t, p^{qb}_t, \phi^m_t, \phi^s_t\}_{t=0}^{\infty}$ *that satisfy the two optimality conditions in Lemma 9 (with* $\tilde{a}^k_{It+1} = \tilde{A}^k_{It+1}$*), and the three market-clearing condition,* $\tilde{A}^s_{It+1} = A^s$*,* $\tilde{A}^m_{It+1} = A^m_{It+1}$*,* $\tilde{A}^m_{Pt+1} = A^m_{Pt+1}$.

Definition 2 follows after recognising that all investors choose the same end-of-portfolio that is characterised by the Euler equation derived in Lemma 9, and using the explicit version of the market-clearing conditions derived in Lemma 11. The bargaining outcomes, which are part of Definition 2 are immediate from Lemma 2.

According to Definition 2, a monetary equilibrium can be characterised by a sequence of prices, $\{p^s_t, p^{sb}_t, p^{qb}_t, \phi^m_t, \phi^s_t\}_{t=0}^{\infty}$ an allocation $\{(\tilde{A}^k_{It+1})_{k\in\{m,s\}}\}_{t=0}^{\infty}$ and $\{\tilde{A}^m_{Pt+1}\}_{t=0}^{\infty}$ that satisfy the following market-clearing conditions

$$
\begin{aligned}
0 &= \tilde{A}^s_{It+1} - A^s \\
0 &= \tilde{A}^m_{It+1} - A^m_{It+1} \\
0 &= \tilde{A}^m_{Pt+1} - A^m_{Pt+1} \\
0 &= \alpha[1 - G(\varepsilon^*_t)]\frac{A^m_t + p^s_t A^s}{p^s_t} + (1-\alpha)[1 - G(\varepsilon^{**}_t)]\frac{A^m_t + p^s_t A^s}{p^s_t - \kappa p^{sb}_t \phi^s_t} - A^s \\
0 &= (1-\alpha)\left\{\left\{1 - \mathbb{I}_{\{1<p^b_t\phi^m_t\}} - \mathbb{I}_{\{p^b_t\phi^m_t=1\}}[1 - \chi(p^b_t\phi^m_t)]\right\}G(\varepsilon^{**}_t) - \frac{\kappa p^{sb}_t \phi^s_t}{p^s_t - \kappa p^{sb}_t\phi^s_t}[1 - G(\varepsilon^{**}_t)]\right\}\frac{A^m_t + p^s_t A^s}{p^{sb}_t} \\
0 &= \min\left\{\mathrm{D}(\bar{\varphi}^m_t), \frac{\phi^m_t A^m_t}{\bar{\varphi}^m_t}\right\} - \hat{\mathrm{Q}}(\bar{\varphi}^m_t) \\
0 &= \mathbb{I}_{\{1<p^{qb}_t\phi^m_t\}}\mathrm{D}(\bar{\varphi}^m_t) + \mathbb{I}_{\{1\geq p^{qb}_t\phi^m_t\}}\min\left\{\mathrm{D}(\bar{\varphi}^q_t), \frac{\phi^m_t A^m_t}{\bar{\varphi}^m_t} + \frac{\lambda\phi^s_t A^s}{\bar{\varphi}^q_t}\right\} - \bar{\mathrm{Q}}(\bar{\varphi}^q_t, \bar{\varphi}^m_t),
\end{aligned}
$$

and optimality conditions

$$
\begin{aligned}
\phi^m_t \tilde{A}^m_{It+1} - \beta\mathbb{E}_t\left[\bar{v}^q_{t+1}(\tilde{\mathbf{A}}_{It+1}) + \bar{v}^m_{t+1}\tilde{A}^m_{It+1}\right] &= 0 \leq \phi^m_t - \beta\mathbb{E}_t\left[\frac{d\bar{v}^q_t(\tilde{\mathbf{A}}_{It+1})}{d\tilde{A}^m_{It+1}} + \bar{v}^m_{t+1}\right] \\
\phi^s_t \tilde{A}^s_{It+1} - \beta\eta\mathbb{E}_t\left[\bar{v}^q_{t+1}(\tilde{\mathbf{A}}_{It+1}) + \bar{v}^s_{t+1}\tilde{A}^s_{It+1}\right] &= 0 \leq \phi^s_t - \beta\eta\mathbb{E}_t\left[\frac{d\bar{v}^q_t(\tilde{\mathbf{A}}_{It+1})}{d\tilde{A}^s_{It+1}} + \bar{v}^s_{t+1}\right],
\end{aligned}
$$

where

$$\begin{aligned}
\bar{v}^m_{t+1} \equiv\ & \phi^m_{t+1} + (1-\alpha)\theta\left(\frac{1}{p^{sb}_{t+1} - \phi^m_{t+1}}\right)\mathbb{I}_{\left\{p^{sb}_{t+1}\phi^m_{t+1}<1\right\}} \\
& + [\alpha + (1-\alpha)(1-\theta)]\delta\int_{\varepsilon^*_{t+1}}^{\varepsilon_H}(\varepsilon - \varepsilon^*_{t+1})y_{t+1}\frac{1}{p^s_{t+1}}dG(\varepsilon) \\
& + (1-\alpha)\delta\theta\left(\frac{1}{p^s_{t+1} - \kappa p^{sb}_{t+1}\phi^s_{t+1}}\right)\int_{\varepsilon^{**}_{t+1}}^{\varepsilon_H}(\varepsilon - \varepsilon^{**}_{t+1})y_{t+1}dG(\varepsilon) \\
\bar{v}^s_{t+1} \equiv\ & \bar{\varepsilon}y_{t+1} + \phi^s_{t+1} + (1-\alpha)\delta\theta\left(\phi^m_{t+1} - \frac{1}{p^{sb}_{t+1}}\right)\mathbb{I}_{\left\{1<p^{sb}_{t+1}\phi^m_{t+1}\right\}} \\
& + [\alpha + (1-\alpha)(1-\theta)]\delta\int_{\varepsilon_L}^{\varepsilon^*_{t+1}}(\varepsilon^*_{t+1} - \varepsilon)y_{t+1}dG(\varepsilon) \\
& + (1-\alpha)\delta\theta\left[\int_{\varepsilon_L}^{\varepsilon^{**}_{t+1}}(\varepsilon^{**}_{t+1} - \varepsilon)y_{t+1}dG(\varepsilon) + \frac{\kappa p^{sb}_{t+1}\phi^s_{t+1}}{p^s_{t+1} - \kappa p^{sb}_{t+1}\phi^s_{t+1}}\int_{\varepsilon^{**}_{t+1}}^{\varepsilon_H}(\varepsilon - \varepsilon^{**}_{t+1})y_{t+1}dG(\varepsilon)\right]
\end{aligned}$$

$$\begin{aligned}
\bar{v}^q_{t+1}(\tilde{\mathbf{A}}_{It+1}) \equiv\ & \alpha(1-\delta)\theta\left\{u[\hat{\mathrm{Q}}_{t+1}(\bar{\varphi}^m_{t+1}), \phi^s_{t+1}] - \bar{\varphi}^m_{t+1}\hat{\mathrm{Q}}_{t+1}(\bar{\varphi}^m_{t+1})\right\} \\
& + (1-\alpha)(1-\delta)\theta\left\{u[\bar{\mathrm{Q}}_{t+1}(\bar{\varphi}^q_{t+1}), \phi^s_{t+1}] - \bar{\varphi}^q_{t+1}\bar{\mathrm{Q}}_{t+1}(\bar{\varphi}^q_{t+1})\right\},
\end{aligned}$$

and

$$\begin{aligned}
\frac{d\bar{v}^q_{t+1}(\tilde{\mathbf{A}}_{It+1})}{d\tilde{A}^m_{t+1}} =\ & \alpha(1-\delta)\theta\phi^m_{t+1}\left[\frac{u'[\hat{\mathrm{Q}}_{t+1}(\bar{\varphi}^m_{t+1}), \phi^s_{t+1}]}{\bar{\varphi}^m_{t+1}} - 1\right] \\
& + (1-\alpha)(1-\delta)\theta\phi^m_{t+1}\left[\frac{u'[\bar{\mathrm{Q}}_{t+1}(\bar{\varphi}^m_{t+1}), \phi^s_{t+1}]}{\bar{\varphi}^m_{t+1}} - 1\right] \\
\frac{d\bar{v}^q_{t+1}(\tilde{\mathbf{A}}_{It+1})}{d\tilde{A}^s_{It+1}} =\ & (1-\alpha)(1-\delta)\theta\lambda\phi^s_{t+1}\left[\frac{u'[\bar{\mathrm{Q}}_{t+1}(\bar{\varphi}^q_{t+1}), \phi^s_{t+1}]}{\varphi^q_{t+1}} - 1\right],
\end{aligned}$$

and prices for goods

$$\begin{aligned}
\bar{\varphi}^m_t &= \frac{\varrho}{1 + (1-\alpha)\theta i^{qm}_t} \\
\bar{\varphi}^q_t &= \frac{1 + i^{qm}_t}{1 + (1-\alpha)\theta i^{qm}_t}\varrho.
\end{aligned}$$

C.5.4 Sequential Monetary Equilibrium with Credit

The following result states that the credit market would be inactive if the net nominal interest rate on bonds in the equity market, $i_t^{sm} = \frac{1}{p_t^{sb}\phi_t^m} - 1$, were negative. Similary, we specialise the analysis to the situation where $i_t^{qm} = \frac{1}{p_t^{qb}\phi_t^m} - 1$ is nonnegative, since $i_t^{qm} < 0$ entails an arbitrary opportunity inconsistent with equilibrium.

Lemma 14. *Consider a monetary equilibrium. If the bond market in the equity market is active in period t, then $p_t^{sb}\phi_t^m \leq 1$. If the bond market in the goods market is active in period t, then $p_t^{qb}\phi_t^m \leq 1$.*

Proof. In an equilibrium with $1 < p_t^{sb}\phi_t^m$ the market-clearing condition in Lemma 11 becomes

$$0 = (1-\alpha)\frac{\kappa\phi_t^s}{p_t^s - \kappa p_t^{sb}\phi_t^s}[1 - G(\varepsilon_t^{**})](A_t^m + p_t^s A^s).$$

This condition can only hold if $[1 - G(\varepsilon_t^{**})](A_t^m + p_t^s A^s) = 0$, i.e. if the bond market is inactive. The condition $1 < p_t^b\phi_t^m$ implies bond demand is nil, so the bond market can only clear with no trade.

From Lemma 6 in Appendix C, we know that if $i_t^{qm} < 0$ that $\bar{a}_t^m(a_t^m, q_t) \to \infty$ which implies that producer's in this case have an arbitrage opportunity, which is inconsistent with equilibrium. ■

According to Lemma 11 in Appendix C, a monetary equilibrium with an active bond market can be characterised by a sequence of prices, $\left\{p_t^s, p_t^{sb}, p_t^{qb}, \phi_t^m, \phi_t^s\right\}_{t=0}^{\infty}$, an allocation $\{(\tilde{A}_{It+1}^k)_{k\in\{m,s\}}\}_{t=0}^{\infty}$ and an allocation $\{\tilde{A}_{Pt+1}^m\}_{t=0}^{\infty}$ that satisfy the following market-clearing conditions

$$
\begin{aligned}
0 &= \tilde{A}^s_{It+1} - A^s \\
0 &= \tilde{A}^m_{It+1} - A^m_{It+1} \\
0 &= \tilde{A}^m_{Pt+1} - A^m_{Pt+1} \\
0 &= \alpha[1 - G(\varepsilon^*_t)]\frac{A^m_t + p^s_t A^s}{p^s_t} + (1-\alpha)[1 - G(\varepsilon^{**}_t)]\frac{A^m_t + p^s_t A^s}{p^s_t - \kappa p^{sb}_t \phi^s_t} - A^s \\
0 &= (1-\alpha)\left\{ \left[1 - \mathbb{I}_{\{p^{sb}_t \phi^m_t = 1\}}(1-\chi_{11})\right] G(\varepsilon^{**}_t) - \frac{\kappa p^{sb}_t \phi^s_t}{p^s_t - \kappa p^{sb}_t \phi^s_t}[1 - G(\varepsilon^{**}_t)] \right\} \frac{A^m_t + p^s_t A^s}{p^{sb}_t} \\
0 &= \min\left\{ \mathrm{D}(\bar{\varphi}^m_t), \frac{\phi^m_t A^m_t}{\bar{\varphi}^m_t} \right\} - \hat{\mathrm{Q}}(\bar{\varphi}^m_t) \\
0 &= \min\left\{ \mathrm{D}(\bar{\varphi}^q_t), \frac{\phi^m_t A^m_t}{\bar{\varphi}^m_t} + \frac{\lambda \phi^s_t A^s}{\bar{\varphi}^q_t} \right\} - \bar{\mathrm{Q}}(\bar{\varphi}^q_t, \bar{\varphi}^m_t),
\end{aligned}
$$

and optimality conditions

$$
\begin{aligned}
\phi^m_t \tilde{A}^m_{It+1} - \beta \mathbb{E}_t\left[\bar{v}^q_{t+1}(\tilde{\mathbf{A}}_{It+1}) + \bar{v}^m_{t+1}\tilde{A}^m_{It+1}\right] = 0 \leq \phi^m_t - \beta\mathbb{E}_t\left[\frac{d\bar{v}^q_t(\tilde{\mathbf{A}}_{It+1})}{d\tilde{A}^m_{It+1}} + \bar{v}^m_{t+1}\right] \\
\phi^s_t \tilde{A}^s_{It+1} - \beta\eta \mathbb{E}_t\left[\bar{v}^q_{t+1}(\tilde{\mathbf{A}}_{It+1}) + \bar{v}^s_{t+1}\tilde{A}^s_{It+1}\right] = 0 \leq \phi^s_t - \beta\eta\mathbb{E}_t\left[\frac{d\bar{v}^q_t(\tilde{\mathbf{A}}_{It+1})}{d\tilde{A}^s_{It+1}} + \bar{v}^s_{t+1}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{v}^m_{t+1} \equiv{}& \phi^m_{t+1} + (1-\alpha)\theta\left(\frac{1}{p^b_{t+1}} - \phi^m_{t+1}\right)\mathbb{I}_{\{p^b_{t+1}\phi^m_{t+1} < 1\}} \\
&+ [\alpha + (1-\alpha)(1-\theta)]\delta \int_{\varepsilon^*_{t+1}}^{\varepsilon_H} (\varepsilon - \varepsilon^*_{t+1}) y_{t+1} \frac{1}{p^s_{t+1}} dG(\varepsilon) \\
&+ (1-\alpha)\delta\theta \frac{1}{p^s_{t+1} - \kappa p^b_{t+1}\phi^s_{t+1}} \int_{\varepsilon^{**}_{t+1}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**}_{t+1}) y_{t+1} dG(\varepsilon) \\
\bar{v}^s_{t+1} \equiv{}& \bar{\varepsilon} y_{t+1} + \phi^s_{t+1} + (1-\alpha)\delta\theta\left(\phi^m_{t+1} - \frac{1}{p^b_{t+1}}\right)\mathbb{I}_{\{1 < p^b_{t+1}\phi^m_{t+1}\}} \\
&+ [\alpha + (1-\alpha)(1-\theta)]\delta \int_{\varepsilon_L}^{\varepsilon^*_{t+1}} (\varepsilon^*_{t+1} - \varepsilon) y_{t+1} dG(\varepsilon) \\
&+ (1-\alpha)\delta\theta\left[\int_{\varepsilon_L}^{\varepsilon^{**}_{t+1}} (\varepsilon^{**}_{t+1} - \varepsilon) y_{t+1} dG(\varepsilon) + \frac{\kappa p^b_{t+1}\phi^s_{t+1}}{p^s_{t+1} - \kappa p^b_{t+1}\phi^s_{t+1}} \int_{\varepsilon^{**}_{t+1}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**}_{t+1}) y_{t+1} dG(\varepsilon)\right] \\
\bar{v}^q_{t+1}(\tilde{\mathbf{A}}_{It+1}) \equiv{}& \alpha(1-\delta)\theta\left\{u[\hat{\mathrm{Q}}_{t+1}(\bar{\varphi}^m_{t+1}), \phi^s_{t+1}] - \bar{\varphi}^m_{t+1}\hat{\mathrm{Q}}_{t+1}(\bar{\varphi}^m_{t+1})\right\} \\
&+ (1-\alpha)(1-\delta)\theta\left\{u[\bar{\mathrm{Q}}_{t+1}(\bar{\varphi}^q_{t+1}), \phi^s_{t+1}] - \bar{\varphi}^q_{t+1}\bar{\mathrm{Q}}_{t+1}(\bar{\varphi}^q_{t+1})\right\},
\end{aligned}
$$

and

$$\frac{d\bar{v}^q_{t+1}(\tilde{\mathbf{A}}_{It+1})}{d\tilde{A}^m_{t+1}} = \alpha(1-\delta)\theta\phi^m_{t+1}\left[\frac{u'[\hat{\mathrm{Q}}_{t+1}(\bar{\varphi}^m_{t+1}), \phi^s_{t+1}]}{\bar{\varphi}^m_{t+1}} - 1\right] + (1-\alpha)(1-\delta)\theta\phi^m_{t+1}\left[\frac{u'[\bar{\mathrm{Q}}_{t+1}(\bar{\varphi}^m_{t+1}), \phi^s_{t+1}]}{\bar{\varphi}^m_{t+1}} - 1\right]$$

$$\frac{d\bar{v}^q_{t+1}(\tilde{\mathbf{A}}_{It+1})}{d\tilde{A}^s_{It+1}} = (1-\alpha)(1-\delta)\theta\lambda\phi^s_{t+1}\left[\frac{u'[\bar{\mathrm{Q}}_{t+1}(\bar{\varphi}^q_{t+1}), \phi^s_{t+1}]}{\varphi^q_{t+1}} - 1\right],$$

and prices for goods

$$\bar{\varphi}^m_t = \frac{\varrho}{1+(1-\alpha)\theta i^{qm}_t}$$

$$\bar{\varphi}^q_t = \frac{1+i^{qm}_t}{1+(1-\alpha)\theta i^{qm}_t}\varrho.$$

C.5.5 Recursive Monetary Equilibrium with Credit

The following result summarises the conditions that characterise a recursive monetary equilibrium (RME).

Lemma 15. *A recursive monetary equilibrium (with credit) is a vector* $(\varepsilon^*, \varepsilon^{**}, \phi^s, Z, A^s)$ *that satisfies*

$$0 = \left\{\alpha[1-G(\varepsilon^*)] + (1-\alpha)[1-G(\varepsilon^{**})]\frac{\varepsilon^{**}+\phi^s}{\varepsilon^{**}+(1-\kappa)\phi^s}\right\}\left(\frac{Z}{\varepsilon^*+\phi^s}+1\right) - 1 \tag{C.63}$$

$$0 = (1-\alpha)\left\{G(\varepsilon^{**}_t)\left[1-\mathbb{I}_{\{\varepsilon^*=\varepsilon^{**}\}}(1-\chi_{11})\right] - [1-G(\varepsilon^{**})]\frac{\kappa\phi^s}{\varepsilon^{**}+(1-\kappa)\phi^s}\right\}(Z+\varepsilon^*+\phi^s) \tag{C.64}$$

$$0 = \min\left\{\tilde{\mathrm{D}}(\bar{\varphi}^m), \frac{zA^s}{\bar{\varphi}^m}\right\} - \hat{\mathcal{Q}}(\bar{\varphi}^m) \tag{C.65}$$

$$0 = \min\left\{\tilde{\mathrm{D}}(\bar{\varphi}^q), \left[\frac{z}{\bar{\varphi}^m}+\frac{\lambda}{\bar{\varphi}^q}\right]A^s\right\} - \bar{\mathcal{Q}}(\bar{\varphi}^q, \bar{\varphi}^m), \tag{C.66}$$

where $\chi_{11} \in [0,1]$, $z = Z/\phi^s$ *and*

$$\begin{aligned}
i^p &= (1-\alpha)\theta\left(\frac{\varepsilon^{**}+\phi^s}{\varepsilon^*+\phi^s}-1\right) \\
&+ \alpha(1-\delta)\theta\left\{\frac{\tilde{u}'[\hat{\mathcal{Q}}(\bar{\varphi}^m)]}{\bar{\varphi}^m}-1\right\} + (1-\alpha)(1-\delta)\theta\left\{\frac{\tilde{u}'[\bar{\mathcal{Q}}(\bar{\varphi}^q,\bar{\varphi}^m)]}{\bar{\varphi}^m}-1\right\} \\
&+ [\alpha+(1-\alpha)(1-\theta)]\delta\frac{1}{\varepsilon^*+\phi^s}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)dG(\varepsilon) \\
&+ (1-\alpha)\delta\theta\frac{\varepsilon^{**}+\phi^s}{\varepsilon^*+\phi^s}\frac{1}{\varepsilon^{**}+(1-\kappa)\phi^s}\int_{\varepsilon^{**}}^{\varepsilon_H}(\varepsilon-\varepsilon^{**})dG(\varepsilon) \qquad \text{(C.67)} \\
\frac{1-\bar{\beta}\eta}{\bar{\beta}\eta}\phi^s &= \bar{\varepsilon} + (1-\alpha)(1-\delta)\theta\lambda\phi^s\left\{\frac{\tilde{u}'[\bar{\mathcal{Q}}(\bar{\varphi}^q,\bar{\varphi}^m)]}{\bar{\varphi}^q}-1\right\} \\
&+ [\alpha+(1-\alpha)(1-\theta)]\delta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)dG(\varepsilon) \\
&+ (1-\alpha)\delta\theta\left[\int_{\varepsilon_L}^{\varepsilon^{**}}(\varepsilon^{**}-\varepsilon)dG(\varepsilon) + \frac{\kappa\phi^s}{\varepsilon^{**}+(1-\kappa)\phi^s}\int_{\varepsilon^{**}}^{\varepsilon_H}(\varepsilon-\varepsilon^{**})dG(\varepsilon)\right], \qquad \text{(C.68)}
\end{aligned}$$

with

$$\bar{\varphi}^m = \frac{\varrho}{1+(1-\alpha)\theta i^{qm}} \qquad \text{(C.69)}$$

$$\bar{\varphi}^q = \frac{1+i^{qm}}{1+(1-\alpha)\theta i^{qm}}\varrho. \qquad \text{(C.70)}$$

Proof. The equilibrium conditions in the statement of the lemma are obtained by using $\phi_t^s = \phi^s y_t$, $\phi_t^m A_t^m = ZA^s y_t$ $\varepsilon_t^* = (p_t^s\phi_t^m - \phi_t^s)\frac{1}{y_t} \equiv \varepsilon^*$, $\varepsilon_t^{**} = (p_t^s/p_t^{sb} - \phi_t^s)\frac{1}{y_t} \equiv \varepsilon^{**}$, $\phi_{t+1}^s/\phi_t^s = \gamma_{t+1}$ and $\phi_t^m/\phi_{t+1}^m = \mu/\gamma_{t+1}$. ■

The first four equations in Lemma 15 are the first subperiod market-clearing condition for equity, goods and bonds. The remaining two conditions are the investor's Euler equations for money and equity, respectively.

C.6 Continuous-time Limiting Economy

In this section we derive the equlibrium conditions for the continuous-time limiting economy.

C.6.1 Equilibrium Conditions

Lemma 16. *Consider the limiting economy (as $\Delta \to 0$). A recursive nonmonetary equilibrium is a pair $(\varepsilon^n, \varphi^n)$ that satisfies*

$$1 = \frac{1 - G(\varepsilon^n)}{1 - \kappa} \tag{C.71}$$

$$\hat{\varphi}^q = \varrho + \frac{1 - (1 - \alpha)\theta}{(1 - \alpha)\theta} (\varrho - \underline{\varrho}) \tag{C.72}$$

$$\varphi^n = \frac{\bar{\varepsilon} + (1 - \alpha)\, \delta\theta \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\kappa}{1-\kappa} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)\right]}{1 - (1 - \alpha)(1 - \delta)\, \lambda \frac{\theta}{\rho} \left\{ \frac{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\hat{\varphi}^q), \lambda A^s/\hat{\varphi}^q\}]}{\hat{\varphi}^q} - 1 \right\}}. \tag{C.73}$$

Proof. From Lemma 13, if the period length is Δ, an equilibrium is a pair $(\varepsilon^n, \Phi^s(\Delta))$ that satisfies

$$1 = [1 - G(\varepsilon^n)] \frac{\varepsilon^n + \Phi^s(\Delta)}{\varepsilon^n + (1 - \kappa)\, \Phi^s(\Delta)}$$

$$\Phi^s(\Delta) = \bar{\beta}\eta \Bigg\{ \bar{\varepsilon} + \Phi^s(\Delta) + (1 - \alpha)(1 - \delta)\, \theta\lambda\Phi^s(\Delta) \left[\frac{\tilde{u}'\left[\min\{\tilde{\mathrm{D}}(\hat{\varphi}^q), \lambda A^s/\hat{\varphi}^q\}\right]}{\hat{\varphi}^q} - 1 \right]$$
$$+ (1 - \alpha)\, \delta\theta \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\kappa\Phi^s(\Delta)}{\varepsilon^n + (1 - \lambda)\, \Phi^s(\Delta)} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \right] \Bigg\}.$$

This can be written as

$$1 = [1 - G(\varepsilon^n)] \frac{\varepsilon^n\Delta + \Phi^s(\Delta)\,\Delta}{\varepsilon^n\Delta + (1 - \kappa)\, \Phi^s(\Delta)\,\Delta}$$

$$\frac{r + d - g + gd\Delta}{(1 + g\Delta)(1 - d\Delta)} \Phi^s(\Delta)\,\Delta - (1 - \alpha)(1 - \delta)\, \theta\lambda\Phi^s(\Delta)\,\Delta \left[\frac{\tilde{u}'\left[\min\{\tilde{\mathrm{D}}(\hat{\varphi}^q), \lambda A^s/\hat{\varphi}^q\}\right]}{\hat{\varphi}^q} - 1 \right]$$
$$= \bar{\varepsilon} + (1 - \alpha)\, \delta\theta \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\kappa\Phi^s(\Delta)\,\Delta}{\varepsilon^n\Delta + (1 - \kappa)\, \Phi^s(\Delta)\,\Delta} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \right].$$

Take the limit as $\Delta \to 0$ to arrive at the conditions in the statement of the lemma. ■

Lemma 17. *Consider the limiting economy (as $\Delta \to 0$). A recursive monetary equilibrium* *(with credit) is a vector* $(\varepsilon^*, \varepsilon^{**}, \varphi, \mathcal{Z})$ *that satisfies*

$$0 = \min\left\{\tilde{\mathrm{D}}(\bar{\varphi}^m), \frac{z}{\bar{\varphi}^m}\right\} - \hat{\mathcal{Q}} \tag{C.74}$$

$$0 = \min\left\{\tilde{\mathrm{D}}(\bar{\varphi}^q), \left[\frac{z}{\bar{\varphi}^m} + \frac{\lambda}{\bar{\varphi}^q}\right] A^s\right\} - \bar{\mathcal{Q}} \tag{C.75}$$

$$0 = \left\{\alpha\left[1 - G\left(\varepsilon^*\right)\right] + (1-\alpha)\left[1 - G\left(\varepsilon^{**}\right)\right]\frac{1}{1-\kappa}\right\}\left(\frac{\mathcal{Z}}{\varphi} + 1\right) - 1 \tag{C.76}$$

$$0 = G\left(\varepsilon^{**}\right)\left[1 - \mathbb{I}_{\{\varepsilon^* = \varepsilon^{**}\}}\left(1 - \chi_{11}\right)\right] - \left[1 - G\left(\varepsilon^{**}\right)\right]\frac{\kappa}{1-\kappa}, \tag{C.77}$$

where $\chi_{11} \in [0,1]$*, and*

$$\left\{\iota - \frac{\alpha(1-\delta)\theta}{\rho}\left[\frac{\tilde{u}'[\hat{\mathcal{Q}}]}{\bar{\varphi}^m} - 1\right] - \frac{(1-\alpha)(1-\delta)\theta}{\rho}\left[\frac{\tilde{u}'[\bar{\mathcal{Q}}]}{\bar{\varphi}^m} - 1\right]\right\}\varphi =$$
$$(1-\alpha)\delta\theta\left(\varepsilon^{**} - \varepsilon^*\right) + \left[\alpha + (1-\alpha)(1-\theta)\right]\delta\int_{\varepsilon^*}^{\varepsilon_H}\left(\varepsilon - \varepsilon^*\right)dG(\varepsilon)$$
$$+ (1-\alpha)\delta\theta\frac{1}{1-\kappa}\int_{\varepsilon^{**}}^{\varepsilon_H}\left(\varepsilon - \varepsilon^{**}\right)dG(\varepsilon) \tag{C.78}$$

$$\varphi = \frac{\rho}{\rho - (1-\alpha)(1-\delta)\theta\lambda\left[\frac{\tilde{u}'[\bar{\mathcal{Q}}]}{\bar{\varphi}^q} - 1\right]}\left\{\bar{\varepsilon} + \left[\alpha + (1-\alpha)(1-\theta)\right]\delta\int_{\varepsilon_L}^{\varepsilon^*}\left(\varepsilon^* - \varepsilon\right)dG(\varepsilon)\right.$$
$$\left. + (1-\alpha)\delta\theta\left[\int_{\varepsilon_L}^{\varepsilon^{**}}\left(\varepsilon^{**} - \varepsilon\right)dG(\varepsilon) + \frac{\kappa}{1-\kappa}\int_{\varepsilon^{**}}^{\varepsilon_H}\left(\varepsilon - \varepsilon^{**}\right)dG(\varepsilon)\right]\right\}, \tag{C.79}$$

with $\bar{\varphi}^m = \bar{\varphi}^q = \varrho$*.*

Proof. If the period length is Δ, the equilibrium conditions in Lemma 15 generalise to

$$0 = \left\{\alpha\left[1 - G\left(\varepsilon^*\right)\right] + (1-\alpha)\left[1 - G\left(\varepsilon^{**}\right)\right]\frac{\varepsilon^{**} + \Phi^s(\Delta)}{\varepsilon^{**} + (1-\kappa)\Phi^s(\Delta)}\right\}\left(\frac{Z(\Delta)}{\varepsilon^* + \Phi^s(\Delta)} + 1\right) - 1$$
$$0 = (1-\alpha)\left\{G\left(\varepsilon^{**}\right)\left[1 - \mathbb{I}_{\{\varepsilon^* = \varepsilon^{**}\}}\left(1 - \chi_{11}\right)\right] - \left[1 - G\left(\varepsilon^{**}\right)\right]\frac{\kappa\Phi^s(\Delta)}{\varepsilon^{**} + (1-\kappa)\Phi^s(\Delta)}\right\}\left[Z(\Delta) + \varepsilon^* + \Phi^s(\Delta)\right],$$

where $\chi_{11} \in [0,1]$, and

$$\begin{aligned}
i^p &= \alpha\left(1-\delta\right)\theta\left\{\frac{\tilde{u}'[\hat{\mathcal{Q}}]}{\bar{\varphi}^m}-1\right\}+\left(1-\alpha\right)\left(1-\delta\right)\theta\left\{\frac{\tilde{u}'[\bar{\mathcal{Q}}]}{\bar{\varphi}^m}-1\right\} \\
&+\left(1-\alpha\right)\delta\theta\frac{\varepsilon^{**}-\varepsilon^{*}}{\varepsilon^{*}+\Phi^s\left(\Delta\right)} \\
&+\left[\alpha+\left(1-\alpha\right)\left(1-\theta\right)\right]\delta\frac{1}{\varepsilon^{*}+\Phi^s\left(\Delta\right)}\int_{\varepsilon^{*}}^{\varepsilon_H}\left(\varepsilon-\varepsilon^{*}\right)dG\left(\varepsilon\right) \\
&+\left(1-\alpha\right)\delta\theta\frac{\varepsilon^{**}+\Phi^s\left(\Delta\right)}{\varepsilon^{*}+\Phi^s\left(\Delta\right)}\frac{1}{\varepsilon^{**}+\left(1-\kappa\right)\Phi^s\left(\Delta\right)}\int_{\varepsilon^{**}}^{\varepsilon_H}\left(\varepsilon-\varepsilon^{**}\right)dG\left(\varepsilon\right)
\end{aligned}$$

$$\begin{aligned}
\frac{r+d-g+dg\Delta}{\left(1+g\Delta\right)\left(1-d\Delta\right)}\Phi^s\left(\Delta\right)\Delta &= \bar{\varepsilon}+\left(1-\alpha\right)\left(1-\delta\right)\theta\lambda\Phi^s\left(\Delta\right)\left[\frac{\tilde{u}'[\bar{\mathcal{Q}}]}{\bar{\varphi}^q}-1\right] \\
&+\left[\alpha+\left(1-\alpha\right)\left(1-\theta\right)\right]\delta\int_{\varepsilon_L}^{\varepsilon^{*}}\left(\varepsilon^{*}-\varepsilon\right)dG\left(\varepsilon\right) \\
&+\left(1-\alpha\right)\delta\theta\left[\int_{\varepsilon_L}^{\varepsilon^{**}}\left(\varepsilon^{**}-\varepsilon\right)dG\left(\varepsilon\right)\right. \\
&\left.+\frac{\kappa\Phi^s\left(\Delta\right)}{\varepsilon^{**}+\left(1-\kappa\right)\Phi^s\left(\Delta\right)}\int_{\varepsilon^{**}}^{\varepsilon_H}\left(\varepsilon-\varepsilon^{**}\right)dG\left(\varepsilon\right)\right].
\end{aligned}$$

These conditions can be rewritten as

$$\begin{aligned}
0 &= \left\{\alpha\left[1-G\left(\varepsilon\right)\right]+\left(1-\alpha\right)\left[1-G\left(\varepsilon^{**}\right)\right]\frac{\varepsilon^{**}\Delta+\Phi^s\left(\Delta\right)\Delta}{\varepsilon^{**}\Delta+\left(1-\kappa\right)\Phi^s\left(\Delta\right)\Delta}\right\}\left(\frac{Z\left(\Delta\right)\Delta}{\varepsilon^{*}\Delta+\Phi^s\left(\Delta\right)\Delta}+1\right)-1 \\
0 &= \left(1-\alpha\right)\left\{G\left(\varepsilon^{**}\right)\left[1-\mathbb{I}_{\{\varepsilon^{*}=\varepsilon^{**}\}}\left(1-\chi_{11}\right)\right]\right. \\
&\left.-\left[1-G\left(\varepsilon^{**}\right)\right]\frac{\kappa\Phi^s\left(\Delta\right)\Delta}{\varepsilon^{**}\Delta+\left(1-\kappa\right)\Phi^s\left(\Delta\right)\Delta}\right\}\left[Z\left(\Delta\right)\Delta+\varepsilon^{*}\Delta+\Phi^s\left(\Delta\right)\Delta\right],
\end{aligned}$$

where $\chi_{11} \in [0,1]$, and

$$\begin{aligned}
\frac{i^p}{\Delta} &= \frac{\alpha\left(1-\delta\right)\theta\Delta}{\Delta}\left[\frac{\tilde{u}'[\hat{\mathcal{Q}}]}{\bar{\varphi}^m}-1\right]+\frac{\left(1-\alpha\right)\left(1-\delta\right)\theta\Delta}{\Delta}\left[\frac{\tilde{u}'[\bar{\mathcal{Q}}]}{\bar{\varphi}^m}-1\right] \\
&+\left(1-\alpha\right)\delta\theta\frac{\varepsilon^{**}-\varepsilon^{*}}{\varepsilon^{*}\Delta+\Phi^s\left(\Delta\right)\Delta} \\
&+\left[\alpha+\left(1-\alpha\right)\left(1-\theta\right)\right]\delta\frac{1}{\varepsilon^{*}\Delta+\Phi^s\left(\Delta\right)\Delta}\int_{\varepsilon^{*}}^{\varepsilon_H}\left(\varepsilon-\varepsilon^{*}\right)dG\left(\varepsilon\right) \\
&+\left(1-\alpha\right)\delta\theta\frac{\varepsilon^{**}\Delta+\Phi^s\left(\Delta\right)\Delta}{\varepsilon^{*}\Delta+\Phi^s\left(\Delta\right)\Delta}\frac{1}{\varepsilon^{**}\Delta+\left(1-\kappa\right)\Phi^s\left(\Delta\right)\Delta}\int_{\varepsilon^{**}}^{\varepsilon_H}\left(\varepsilon-\varepsilon^{**}\right)dG\left(\varepsilon\right)
\end{aligned}$$

$$\frac{r+d-g+dg\Delta}{(1+g\Delta)(1-d\Delta)}\Phi^s(\Delta)\Delta = \bar{\varepsilon} + (1-\alpha)(1-\delta)\theta\lambda\Phi^s(\Delta)\Delta\left[\frac{\tilde{u}'[\bar{\mathcal{Q}}]}{\bar{\varphi}^q}-1\right]$$
$$+[\alpha+(1-\alpha)(1-\theta)]\delta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)\,dG(\varepsilon)$$
$$+(1-\alpha)\delta\theta\left[\int_{\varepsilon_L}^{\varepsilon^{**}}(\varepsilon^{**}-\varepsilon)\,dG(\varepsilon)\right.$$
$$\left.+\frac{\kappa\Phi^s(\Delta)\Delta}{\varepsilon^{**}\Delta+(1-\kappa)\Phi^s(\Delta)\Delta}\int_{\varepsilon^{**}}^{\varepsilon_H}(\varepsilon-\varepsilon^{**})\,dG(\varepsilon)\right].$$

Take the limit as $\Delta \to 0$ to arrive at the conditions in the statement of the lemma. ■

C.6.2 Existence of Equilibrium

Proof of Proposition 1. The conditions in the statement of the proposition are the equilibrium conditions derived in Lemma 16. Clearly for any $\kappa \in [0,1]$ there is a unique ε^n that satisfies G, and given ε^n there is a unique φ^n. ■

Lemma 18. *In a RNE,*

$$\frac{d\varepsilon^n}{d\kappa} = \frac{1}{G'(\varepsilon^n)} > 0 \tag{C.80}$$
$$\frac{d\varphi^n}{d\kappa} = \frac{(1-\alpha)\delta\theta\frac{1}{(1-\kappa)^2}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)\,dG(\varepsilon)}{1-(1-\alpha)(1-\delta)\lambda\frac{\theta}{\rho}\left\{\frac{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\hat{\varphi}^q),\lambda A^s/\hat{\varphi}^q\}]}{\hat{\varphi}^q}-1\right\}} > 0. \tag{C.81}$$

Proof. The first result is obtained by implicitly differentiating G with respect to κ. For the second result, differentiate φ^n with respect to κ

$$\frac{d\varphi^n}{d\kappa} = \frac{(1-\alpha)\delta\theta\left[G(\varepsilon^n)\frac{d\varepsilon^n}{d\kappa}-\frac{\kappa}{1-\kappa}[1-G(\varepsilon^n)]\frac{d\varepsilon^n}{d\kappa}+\frac{1}{(1-\kappa)^2}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)\,dG(\varepsilon)\right]}{1-(1-\alpha)(1-\delta)\lambda\frac{\theta}{\rho}\left\{\frac{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\hat{\varphi}^q),\lambda A^s/\hat{\varphi}^q\}]}{\hat{\varphi}^q}-1\right\}}$$
$$= \frac{(1-\alpha)\delta\theta\frac{1}{(1-\kappa)^2}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)\,dG(\varepsilon)}{1-(1-\alpha)(1-\delta)\lambda\frac{\theta}{\rho}\left\{\frac{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\hat{\varphi}^q),\lambda A^s/\hat{\varphi}^q\}]}{\hat{\varphi}^q}-1\right\}}.$$

■

Proof of Proposition 2. The equilibrium conditions in Lemma 17 imply that there are four equations with four unknowns. The unknowns are $(\varepsilon^*,\varepsilon^{**},\varphi,\mathcal{Z})$ if $\varepsilon^*<\varepsilon^{**}$, or $(\varepsilon^*,\chi_{11},\varphi,\mathcal{Z})$ if $\varepsilon^*=\varepsilon^{**}$. We consider each in turn.

(*i*) Suppose $\varepsilon^* < \varepsilon^{**}$. In this case, the market clearing conditions in Lemma 17 implies $\varepsilon^{**} = \varepsilon^n$, where $\varepsilon^n \in [\varepsilon_L, \varepsilon_H]$ is the unique solution to $G(\varepsilon^n) = \kappa$. Now note that in this case we have

$$z = \frac{\mathcal{Z}}{\varphi} = \frac{\alpha G(\varepsilon^*)}{[1 - G(\varepsilon^*)]\alpha + 1 - \alpha}.$$

The Euler conditions imply a single equation in the unknown ε^* that can be written as $T(\varepsilon^*) = 0$, where

$$\begin{aligned}
T(x) \equiv\ & (1-\alpha)\delta\theta(\varepsilon^n - x) + [\alpha + (1-\alpha)(1-\theta)]\,\delta \int_x^{\varepsilon_H} (\varepsilon - x)\, dG(\varepsilon) \\
& + (1-\alpha)\,\delta\theta \frac{1}{1-\kappa} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \\
& - \frac{\iota - (1-\delta)\frac{\theta}{\rho\varrho}\left\{\alpha \tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), zA^s/\varrho\}] + (1-\alpha)\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), (z+\lambda)A^s/\varrho\}] - \varrho\right\}}{1 - (1-\alpha)(1-\delta)\,\lambda \frac{\theta}{\rho\varrho}\left\{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), (z+\lambda)A^s/\varrho\}] - \varrho\right\}}. \\
& \Bigg\{\bar{\varepsilon} + [\alpha + (1-\alpha)(1-\theta)]\,\delta \int_{\varepsilon_L}^{x} (x - \varepsilon)\, dG(\varepsilon) \\
& + (1-\alpha)\,\delta\theta \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\kappa}{1-\kappa}\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)\right]\Bigg\}.
\end{aligned}$$

Differentiate T and evaluate the derivative at $x = \varepsilon^*$ to obtain the following, first we write $\mathrm{L}(x) = (1-\delta)\frac{\theta}{\rho\varrho}\{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), xA^s/\varrho\}] - \varrho\}$. And notice that $\mathrm{L}'(x) \leq 0$ for all x.

$$\begin{aligned}
T'(\varepsilon^*) =\ & -(1-\alpha)\delta\theta - [\alpha + (1-\alpha)(1-\theta)]\delta[1 - G(\varepsilon^*)] \\
& - \frac{\iota - \alpha\mathrm{L}(z) - (1-\alpha)\mathrm{L}(z+\lambda)}{1 - (1-\alpha)\,\mathrm{L}(z+\lambda)} G(\varepsilon^*) \\
& + \frac{[\iota - \alpha\mathrm{L}(z) - (1-\alpha)\mathrm{L}(z+\lambda)]\,(1-\alpha)\lambda\mathrm{L}'(z+\lambda)}{1 - (1-\alpha)\,\lambda\mathrm{L}(z+\lambda)} \varphi(\varepsilon^*) \\
& - [\alpha\mathrm{L}'(z) + (1-\alpha)\mathrm{L}'(z+\lambda)]\,\varphi(\varepsilon^*).
\end{aligned}$$

Notice in order for $\varphi \geq 0$, we need $1 - (1-\alpha)\,\lambda\mathrm{L}(z+\lambda) > 0$. Therefore in order have an equilibrium, $\iota - \alpha\mathrm{L}(z) - (1-\alpha)\mathrm{L}(z+\lambda) > 0$, since ι is bounded from below (see Proposition 2), which implies

$$\iota < 1 - \alpha\mathrm{L}(z) - (1-\alpha)(1-\lambda)\mathrm{L}(z+\lambda).$$

Therefore, in order for $T'(\varepsilon^*) \leq 0$, we need,

$$\frac{\left[\iota - \alpha \mathrm{L}(z) - (1-\alpha)\mathrm{L}(z+\lambda)\right](1-\alpha)\lambda \mathrm{L}'(z+\lambda)}{1-(1-\alpha)\lambda \mathrm{L}(z+\lambda)} < \alpha \mathrm{L}'(z) + (1-\alpha)\mathrm{L}'(z+\lambda),$$

which becomes,

$$\frac{1}{\lambda} + \frac{\alpha \mathrm{L}'(z)}{(1-\alpha)\mathrm{L}'(z+\lambda)} > 1 + \frac{\iota - 1 - \alpha \mathrm{L}(z) - (1-\alpha)(1-\lambda)\mathrm{L}(z+\lambda)}{(1-\alpha)\mathrm{L}'(z+\lambda)},$$

which holds since $\iota < 1 - \alpha\mathrm{L}(z) - (1-\alpha)(1-\lambda)\mathrm{L}(z+\lambda)$. Which implies that $T(\varepsilon^*) \leq 0$. Hence, if there is a ε^* that satisfies $T(\varepsilon^*) = 0$, it is unique. Notice that $T(\varepsilon_L) \to \infty$ since as $\varepsilon^* \to \varepsilon_L$, $z \to 0$ which implies that $\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), zA^s/\varrho\}] \to \infty$. Therefore $0 < T(\varepsilon_L)$ for all ι. Therefore suppose there exists ε_L^+ and notice that,

$$\begin{aligned}
T(\varepsilon_L^+) = {} & (1-\alpha)\delta\theta(\varepsilon^n - \varepsilon_L^+) + \left[\alpha + (1-\alpha)(1-\theta)\right]\delta \int_{\varepsilon_L^+}^{\varepsilon_H} \left(\varepsilon - \varepsilon_L^+\right) dG(\varepsilon) \\
& + (1-\alpha)\delta\theta \frac{1}{1-\kappa}\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \\
& - \frac{\iota - (1-\delta)\frac{\theta}{\rho\varrho}\left\{\alpha\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), \epsilon A^s/\varrho\}] + (1-\alpha)\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), (\epsilon+\lambda)A^s/\varrho\}] - \varrho\right\}}{1-(1-\alpha)(1-\delta)\lambda\frac{\theta}{\rho\varrho}\left\{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), (\epsilon+\lambda)A^s/\varrho\}] - \varrho\right\}} \cdot \\
& \left\{\bar{\varepsilon} + \left[\alpha + (1-\alpha)(1-\theta)\right]\delta \int_{\varepsilon_L}^{\varepsilon_L^+} \left(\varepsilon_L^+ - \varepsilon\right) dG(\varepsilon) \right. \\
& \left. + (1-\alpha)\delta\theta\left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\kappa}{1-\kappa}\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)\right]\right\},
\end{aligned}$$

so $0 < T(\varepsilon_L^+)$ if and only if $\iota < \bar{\iota}(\xi)$ is defined in the statement of the proposition. Also,

$$\begin{aligned}
T(\varepsilon^n) = {} & \left[\alpha + (1-\alpha)\left(1 + \theta\frac{\kappa}{1-\kappa}\right)\right]\delta \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \\
& - \frac{\iota - (1-\delta)\frac{\theta}{\rho\varrho}\left\{\alpha\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), \hat{z}A^s/\varrho\}] + (1-\alpha)\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), (\hat{z}+\lambda)A^s/\varrho\}] - \varrho\right\}}{1-(1-\alpha)(1-\delta)\lambda\frac{\theta}{\rho\varrho}\left\{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), (\hat{z}+\lambda)A^s/\varrho\}] - \varrho\right\}} \cdot \\
& \left\{\bar{\varepsilon} + \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon) dG(\varepsilon) + (1-\alpha)\delta\theta\frac{\kappa}{1-\kappa}\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)\right\},
\end{aligned}$$

where $\tilde{z} = \frac{\alpha G(\varepsilon^n)}{[1-G(\varepsilon^n)]\alpha + 1 - \alpha}$. Since $\kappa > 0$, $T(\varepsilon^n) < 0$ if and only if $\hat{\iota}(\xi) < \iota$. Therefore, if $\hat{\iota}(\xi) < \iota < \bar{\iota}(\xi)$, then by the intermediate value theorem, there is a unique $\varepsilon^* \in (\varepsilon_L, \varepsilon^n)$ such that $T(\varepsilon^*) = 0$, where $\hat{\iota}(\xi)$ is defined in the statement of the proposition. From this expression, it is clear that $0 < \mathcal{Z} \iff \alpha > 0$ and $\varepsilon_L < \varepsilon^*$. (ii) Suppose $\varepsilon^* = \varepsilon^{**}$. In this case, our market clearing conditions and Euler equations become,

$$
\begin{aligned}
1 &= [1 - G(\varepsilon^*)]\left(\alpha + \frac{1-\alpha}{1-\kappa}\right)\left(\frac{\mathcal{Z}}{\varphi} + 1\right) \\
\chi_{11} &= \frac{\kappa}{1-\kappa}\frac{1-G(\varepsilon^*)}{G(\varepsilon^*)} \\
\iota\varphi &= \frac{\left[\alpha + (1-\alpha)\left(1-\theta+\theta\frac{1}{1-\kappa}\right)\right]\delta\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)dG(\varepsilon)}{1-(1-\alpha)(1-\delta)\lambda\frac{\theta}{\rho\varrho}\left\{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho),(z+\lambda)A^s/\varrho\}]-\varrho\right\}} \\
&\quad + (1-\delta)\frac{\theta}{\rho\varrho}\left\{\alpha\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), zA^s/\varrho\}] + (1-\alpha)\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho),(z+\lambda)A^s/\varrho\}]-\varrho\right\}\varphi \\
\varphi &= \frac{\bar{\varepsilon} + \delta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)dG(\varepsilon) + (1-\alpha)\delta\theta\frac{\kappa}{1-\kappa}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)dG(\varepsilon)}{1-(1-\alpha)(1-\delta)\lambda\frac{\theta}{\rho\varrho}\left\{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho),(\hat{z}+\lambda)A^s/\varrho\}]-\varrho\right\}}.
\end{aligned}
$$

The last two equations imply a single equation in the unknown ε^* that can be written as $\mathcal{T}(\varepsilon^*) = 0$, where

$$
\begin{aligned}
\mathcal{T}(\varepsilon^*) \equiv& \left[\alpha + (1-\alpha)\left(1-\theta+\theta\frac{1}{1-\kappa}\right)\right]\delta\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)dG(\varepsilon) \\
&- \frac{\iota - (1-\delta)\frac{\theta}{\rho\varrho}\left\{\alpha\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), zA^s/\varrho\}] + (1-\alpha)\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho),(z+\lambda)A^s/\varrho\}]-\varrho\right\}}{1-(1-\alpha)(1-\delta)\lambda\frac{\theta}{\rho\varrho}\left\{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho),(z+\lambda)A^s/\varrho\}]-\varrho\right\}}\cdot \\
&\left\{\bar{\varepsilon} + \delta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)dG(\varepsilon) + (1-\alpha)\delta\theta\frac{\kappa}{1-\kappa}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)dG(\varepsilon)\right\}.
\end{aligned}
$$

Differentiate $\mathcal{T}$ and evaluate the derivative at the ε^* that solves $\mathcal{T}(\varepsilon^*) = 0$ to obtain,

$$
\begin{aligned}
\mathcal{T}'(\varepsilon^*) =& \left[\alpha + (1-\alpha)\left(1-\theta+\theta\frac{1}{1-\kappa}\right)\right]\delta[1-G(\varepsilon^*)] \\
&- \frac{\iota - \alpha\mathrm{L}(z) - (1-\alpha)\mathrm{L}(z+\lambda)}{1-(1-\alpha)\mathrm{L}(z+\lambda)}\left[\delta G(\varepsilon^*) + (1-\alpha)\delta\theta\frac{\kappa}{1-\kappa}[1-G(\varepsilon^*)]\right] \\
&- \left\{\bar{\varepsilon} + \delta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)dG(\varepsilon) + (1-\alpha)\delta\theta\frac{\kappa}{1-\kappa}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)dG(\varepsilon)\right\}\cdot \\
&\left\{\frac{[\iota - \alpha\mathrm{L}(z) - (1-\alpha)\mathrm{L}(z+\lambda)](1-\alpha)\mathrm{L}'(z+\lambda)}{[1-(1-\alpha)\mathrm{L}(z+\lambda)]^2} - \frac{\alpha\mathrm{L}'(z) + (1-\alpha)\mathrm{L}'(z+\lambda)}{[1-(1-\alpha)\mathrm{L}(z+\lambda)]}\right\} \leq 0.
\end{aligned}
$$

Hence, if there is a ε^* that satisfies $\mathcal{T}(\varepsilon^*) = 0$, it is unique. ■

Lemma 19. *The real asset price in the RME is higher than the real asset price in the RNE, i.e.,*

(i) If $\hat{\iota}(\xi) < \iota < \bar{\iota}(\xi)$*, then*

$$0 < \frac{1}{1-(1-\alpha)(1-\delta)\lambda\frac{\theta}{\rho\varrho}\left\{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho),(z+\lambda)A^s/\varrho\}]-\varrho\right\}}\cdot$$
$$\left\{(1-\alpha)(1-\delta)\lambda\frac{\theta}{\rho}\left[\frac{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho),(z+\lambda)A^s/\varrho\}]}{\varrho}-\frac{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\hat{\varphi}^q),\lambda A^s/\hat{\varphi}^q\}]}{\hat{\varphi}^q}\right]\varphi^n\right.$$
$$\left.+[\alpha+(1-\alpha)(1-\theta)]\delta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)dG(\varepsilon)\right\}\leq\varphi-\varphi^n. \tag{C.82}$$

(ii) If $0 < \iota < \hat{\iota}(\xi)$*, then*

$$0 < \frac{1}{1-(1-\alpha)(1-\delta)\lambda\frac{\theta}{\rho\varrho}\left\{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho),(\hat{z}+\lambda)A^s/\varrho\}]-\varrho\right\}}\cdot$$
$$\left\{(1-\alpha)(1-\delta)\lambda\frac{\theta}{\rho}\left[\frac{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho),(z+\lambda)A^s/\varrho\}]}{\varrho}-\frac{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\hat{\varphi}^q),\lambda A^s/\hat{\varphi}^q\}]}{\hat{\varphi}^q}\right]\varphi^n\right.$$
$$\left.+[1-(1-\alpha)\delta\theta]\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)dG(\varepsilon)\right\}\leq\varphi-\varphi^n. \tag{C.83}$$

Proof. Subtracting φ^n from the equilibrium conditions in Proposition 2 and noting that since $\varrho < \hat{\varphi}^q$, we know,

$$\frac{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho),(z+\lambda)A^s/\varrho\}]}{\varrho}\geq\frac{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\hat{\varphi}^q),\lambda A^s/\hat{\varphi}^q\}]}{\hat{\varphi}^q}.$$

This yields the lemma. ■

Proof of Proposition 3. The proof is immediate from Lemma 19. ■

Proof of Proposition 4. We consider the two cases in turn

If $\hat{\iota}(\xi) < \iota < \bar{\iota}(\xi)$, then

$$\frac{d\varepsilon^*}{d\iota}=-\frac{\frac{\partial T(\varepsilon^*)}{\partial\iota}}{T'(\varepsilon^*)}=-\frac{-\varphi}{T'(\varepsilon^*)}<0,$$

where $T(\cdot)$ is the equilibrium map defined in the proof of Proposition 2. Then,

$$\frac{d\varphi}{d\iota} = \frac{\left\{[\alpha + (1-\alpha)(1-\theta)]\delta G(\varepsilon^*) + (1-\alpha)(1-\delta)\varphi\frac{\lambda\theta A^s}{\rho\varrho^2}\tilde{u}''\left[\min\{\tilde{\mathrm{D}}(\varrho),(z+\lambda)A^s/\varrho\}\right]\frac{dz}{d\varepsilon^*}\right\}}{1-(1-\alpha)(1-\delta)\lambda\frac{\theta}{\rho\varrho}\left\{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho),(z+\lambda)A^s/\varrho\}]-\varrho\right\}}\frac{d\varepsilon^*}{d\iota}.$$

We know that $\frac{dz}{d\varepsilon^*} > 0$ and $\frac{d\varepsilon^*}{d\iota} < 0$. Also note that $\tilde{u}''(\cdot) < 0$, therefore $\frac{d\varphi}{d\iota} < 0$ if and only if

$$\varphi < \frac{-[\alpha + (1-\alpha)(1-\theta)]\delta\rho\varrho^2 G(\varepsilon^*)}{(1-\alpha)(1-\delta)\lambda\theta A^s\tilde{u}''\left[\min\{\tilde{\mathrm{D}}(\varrho),(z+\lambda)A^s/\varrho\}\right]\frac{dz}{d\varepsilon^*}}.$$

If $0 < \iota \leq \hat{\iota}(\xi)$, then

$$\frac{d\varepsilon^*}{d\iota} = -\frac{\frac{\partial\mathcal{T}(\varepsilon^*)}{\partial\iota}}{\mathcal{T}'(\varepsilon^*)} = -\frac{-\varphi}{\mathcal{T}'(\varepsilon^*)} < 0,$$

where $\mathcal{T}(\cdot)$ is the equilibrium map defined in the proof of Proposition 2. Then,

$$\frac{d\varphi}{d\iota} = \frac{\left\{\left[G(\varepsilon^*) - (1-\alpha)\theta\frac{\kappa}{1-\kappa}[1-G(\varepsilon^*)]\right]\delta + (1-\alpha)(1-\delta)\varphi\frac{\lambda\theta A^s}{\rho\varrho^2}\tilde{u}''\left[\min\{\tilde{\mathrm{D}}(\varrho),(z+\lambda)A^s/\varrho\}\right]\frac{dz}{d\varepsilon^*}\right\}}{1-(1-\alpha)(1-\delta)\lambda\frac{\theta}{\rho\varrho}\left\{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho),(z+\lambda)A^s/\varrho\}]-\varrho\right\}}\frac{d\varepsilon^*}{d\iota}.$$

Therefore $\frac{d\varphi}{d\iota} < 0$ if and only if,

$$\varphi < \frac{-\left[G(\varepsilon^*) - (1-\alpha)\theta\frac{\kappa}{1-\kappa}[1-G(\varepsilon^*)]\right]\rho\varrho^2\delta}{(1-\alpha)(1-\delta)\lambda\theta A^s\tilde{u}''\left[\min\{\tilde{\mathrm{D}}(\varrho),(z+\lambda)A^s/\varrho\}\right]\frac{dz}{d\varepsilon^*}}.$$

Notice that

$$\begin{aligned} 0 &= \left[G(\varepsilon^n) - \frac{\kappa}{1-\kappa}[1-G(\varepsilon^n)]\right] \\ &\leq \left[G(\varepsilon^*) - \frac{\kappa}{1-\kappa}[1-G(\varepsilon^*)]\right] \\ &< G(\varepsilon^*) - (1-\alpha)\theta\frac{\kappa}{1-\kappa}[1-G(\varepsilon^*)], \end{aligned}$$

where the first inequality follows because $G(x) - \frac{\kappa}{1-\kappa}[1-G(x)]$ is increasing in x, and $\varepsilon^n \leq \varepsilon^*$ for all $0 < \iota \leq \hat{\iota}(\xi)$. This yields the lemma. ■

C.7 Cashless Limits

Proof of Proposition 5. First, notice that $\hat{\varsigma}(0;\delta) \leq \bar{\varsigma}(0;\delta)$, with "=" only if $\kappa = 0$. Assume that $\kappa > 0$, and some fixed $\iota \in (0, \bar{\iota}(0;\delta))$.

(i) For $\iota \in (\hat{\varsigma}(0;\delta), \bar{\varsigma}(0;\delta))$ and α small enough, part (i) of Proposition 2 implies the monetary equilibrium is a vector $(\varepsilon^*, \varepsilon^{**}, \varphi, \mathcal{Z})$, where

$$\varphi = \frac{\bar{\varepsilon} + (1-\alpha)\delta\theta\left[\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^* - \varepsilon)dG(\varepsilon) + \frac{\kappa}{1-\kappa}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon - \varepsilon^*)dG(\varepsilon)\right] + [\alpha + (1-\alpha)(1-\theta)]\delta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^* - \varepsilon)dG(\varepsilon)}{1 - (1-\alpha)(1-\delta)\lambda\frac{\theta}{\rho\varrho}\left\{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), (\hat{z}+\lambda)A^s/\varrho\}] - \varrho\right\}}$$

$$\mathcal{Z} = \frac{\alpha G(\varepsilon^*)}{[1 - G(\varepsilon^*)]\alpha + 1 - \alpha}\varphi,$$

where $\varepsilon^{**} = \varepsilon^n$ and ε^* is the unique $\varepsilon^* \in (\varepsilon_L, \varepsilon^n)$ that satisfies $\tilde{T}(\varepsilon^*;\alpha) = 0$, where for any $\varepsilon^* \in (\varepsilon_L, \varepsilon_H)$, $\tilde{T}(\cdot;\alpha)$ is a real valued function defined by

$$\begin{aligned}
\tilde{T}(\varepsilon^*;\alpha) \equiv{}& (1-\alpha)\delta\theta(\varepsilon^n - \varepsilon^*) + [\alpha + (1-\alpha)(1-\theta)]\,\delta\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon - \varepsilon^*)\,dG(\varepsilon) \\
&+ (1-\alpha)\,\delta\theta\frac{1}{1-\kappa}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)\,dG(\varepsilon) \\
&- \frac{\iota - (1-\delta)\frac{\theta}{\rho\varrho}\left\{\alpha\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), zA^s/\varrho\}] + (1-\alpha)\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), (z+\lambda)A^s/\varrho\}] - \varrho\right\}}{1 - (1-\alpha)(1-\delta)\lambda\frac{\theta}{\rho\varrho}\left\{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), (z+\lambda)A^s/\varrho\}] - \varrho\right\}}\cdot \\
&\left\{\bar{\varepsilon} + [\alpha + (1-\alpha)(1-\theta)]\,\delta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^* - \varepsilon)\,dG(\varepsilon)\right. \\
&\left.+ (1-\alpha)\,\delta\theta\left[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n - \varepsilon)\,dG(\varepsilon) + \frac{\kappa}{1-\kappa}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)\,dG(\varepsilon)\right]\right\}
\end{aligned}$$

As $\alpha \to 0$, the function $\tilde{T}(\cdot;\alpha)$ converges uniformly to

$$\begin{aligned}
\tilde{T}(\varepsilon^*;0) \equiv{}& \delta\theta(\varepsilon^n - \varepsilon^*) + (1-\theta)\,\delta\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon - \varepsilon^*)\,dG(\varepsilon) + \delta\theta\frac{1}{1-\kappa}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)\,dG(\varepsilon) \\
&- \frac{\iota - (1-\delta)\frac{\theta}{\rho\varrho}\left[\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), \lambda A^s/\varrho\}] - \varrho\right]}{1 - (1-\delta)\lambda\frac{\theta}{\rho\varrho}\left[\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), \lambda A^s/\varrho\}] - \varrho\right]}\cdot \\
&\left\{\bar{\varepsilon} + (1-\theta)\,\delta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^* - \varepsilon)\,dG(\varepsilon) + \delta\theta\left[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n - \varepsilon)\,dG(\varepsilon) + \frac{\kappa}{1-\kappa}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)\,dG(\varepsilon)\right]\right\}.
\end{aligned}$$

Notice that if $\tilde{u}'(q) = \sigma q^{\sigma-1}$ as we have assumed, then by l'Hopital's rule $\alpha\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), zA^s/\varrho\}] \to$

0 as $\alpha \to 0$.

(*ii*) For $\iota \in (0, \hat{\varsigma}(0;\delta)]$ and a small enough δ, part (*ii*) of Proposition 2 implies the monetary equilibrium is a vector $(\varepsilon^*, \chi, \varphi, \mathcal{Z})$ that satisfies $\chi = \frac{\kappa}{1-\kappa}\frac{1-G(\varepsilon^*)}{G(\varepsilon^*)}$,

$$\varphi = \frac{\bar{\varepsilon} + \delta \int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^* - \varepsilon)dG(\varepsilon) + (1-\alpha)\delta\theta\frac{\kappa}{1-\kappa}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon - \varepsilon^*)dG(\varepsilon)}{1-(1-\alpha)(1-\delta)\lambda\frac{\theta}{\rho\varrho}\left\{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho),(\hat{z}+\lambda)A^s/\varrho\}]-\varrho\right\}}$$

$$\mathcal{Z} = \frac{\alpha G(\varepsilon^*) + (1-\alpha)\theta\frac{\kappa}{1-\kappa}\left[1-G(\varepsilon^*)\right]}{\left[1-G(\varepsilon^*)\right]\alpha + 1 - \alpha},$$

and $\varepsilon^* = \varepsilon^{**}$, is the unique solution to $\tilde{T}(\varepsilon^*;0) = 0$, where for any $\varepsilon^* \in (\varepsilon_L, \varepsilon_H)$, $\tilde{T}(\cdot;\alpha)$ is a real valued function defined by

$$\begin{aligned}\tilde{\mathcal{T}}(\varepsilon^*;\alpha) = &\left[\alpha + (1-\alpha)\left(1+\theta\frac{\kappa}{1-\kappa}\right)\right]\delta\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)\,dG(\varepsilon) \\ &- \frac{\iota - (1-\delta)\frac{\theta}{\rho\varrho}\left\{\alpha\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho),\hat{z}A^s/\varrho\}] + (1-\alpha)\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho),(\hat{z}+\lambda)A^s/\varrho\}]-\varrho\right\}}{1-(1-\alpha)(1-\delta)\lambda\frac{\theta}{\rho\varrho}\left\{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho),(\hat{z}+\lambda)A^s/\varrho\}]-\varrho\right\}}\cdot \\ &\left\{\bar{\varepsilon} + \delta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)\,dG(\varepsilon) + (1-\alpha)\delta\theta\frac{\kappa}{1-\kappa}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)\,dG(\varepsilon)\right\}.\end{aligned}$$

As $\alpha \to 0$, the function $\tilde{\mathcal{T}}(\cdot;\alpha)$ converges uniformly to

$$\begin{aligned}\tilde{\mathcal{T}}(\varepsilon^*;0) = &\left(1+\theta\frac{\kappa}{1-\kappa}\right)\delta\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)\,dG(\varepsilon) \\ &- \frac{\iota - (1-\delta)\frac{\theta}{\rho\varrho}\left[\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho),(\hat{z}+\lambda)A^s/\varrho\}]-\varrho\right]}{1-(1-\delta)\lambda\frac{\theta}{\rho\varrho}\left\{\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho),(\hat{z}+\lambda)A^s/\varrho\}]-\varrho\right\}}\cdot \\ &\left\{\bar{\varepsilon} + \delta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)\,dG(\varepsilon) + (1-\alpha)\delta\theta\frac{\kappa}{1-\kappa}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)\,dG(\varepsilon)\right\},\end{aligned}$$

where

$$\hat{z} = \frac{G(\varepsilon^*)-\kappa}{1-G(\varepsilon^*)}.$$

■

Proof of Proposition 1 (Section B.1). (*i*) For $\iota \in (\hat{\varsigma}(0;0), \bar{\varsigma}(0;0))$ and δ small enough, part

(i) of Proposition 6 implies the cashless limit of the monetary equilibrium is a vector $(\varepsilon^*, \varepsilon^{**}, \varphi, \mathcal{Z})$, where

$$\varphi = \frac{\bar{\varepsilon} + \delta\theta \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon) dG(\varepsilon) + \frac{\kappa}{1-\kappa} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n) d(\varepsilon)\right] + (1-\theta)\delta \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon) dG(\varepsilon)}{1 - (1-\theta)\lambda \frac{\theta}{\rho\varrho} \left[\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), \lambda A^s/\varrho\}] - \varrho\right]}$$

$$\frac{\mathcal{Z}}{\varphi} \to 0$$

$$\mathcal{V} \to \delta G(\varepsilon^n) A^s,$$

where $\varepsilon^{**} = \varepsilon^n$ and $\varepsilon^* \in (\varepsilon_L, \varepsilon^n)$ that satisfies $\hat{T}(\varepsilon^*, 0; \delta)$, where for any $\varepsilon^* \in (\varepsilon_L, \varepsilon_H)$, $\hat{T}(\cdot, 0; \delta)$ is a real valued function defined by

$$\begin{aligned}\hat{T}(\varepsilon^*, 0; \delta) \equiv\ & \delta\theta(\varepsilon^n - \varepsilon^*) + (1-\theta)\,\delta \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) + \delta\theta \frac{1}{1-\kappa} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \\ & - \frac{\iota - (1-\delta)\frac{\theta}{\rho\varrho}\left[\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), \lambda A^s/\varrho\}] - \varrho\right]}{1 - (1-\delta)\,\lambda \frac{\theta}{\rho\varrho}\left[\tilde{u}'[\min\{\tilde{\mathrm{D}}(\varrho), \lambda A^s/\varrho\}] - \varrho\right]} \cdot \\ & \left\{\bar{\varepsilon} + (1-\theta)\,\delta \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) + \delta\theta \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\kappa}{1-\kappa} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)\right]\right\}.\end{aligned}$$

As $\delta \to 0$ and $\Delta > 0$ the function $\hat{T}$ becomes

$$\hat{T}(\rho^{qm}, 0; 0) \equiv -\frac{\iota - \frac{\theta}{\rho\bar{\varphi}^m(\Delta)}\left[\tilde{u}'[\min\{\tilde{\mathrm{D}}(\bar{\varphi}^m(\Delta)), \lambda A^s/\bar{\varphi}^m(\Delta)\}] - \bar{\varphi}^m(\Delta)\right]}{1 - \lambda \frac{\theta}{\rho\bar{\varphi}^q(\Delta)}\left[\tilde{u}'[\min\{\tilde{\mathrm{D}}(\bar{\varphi}^q(\Delta)), \lambda A^s/\bar{\varphi}^q(\Delta)\}] - \bar{\varphi}^q(\Delta)\right]}\bar{\varepsilon},$$

where

$$\bar{\varphi}^m(\Delta) = \frac{\varrho}{1 + \theta\Delta\rho^{qm}}$$

$$\bar{\varphi}^q(\Delta) = \frac{1 + \Delta\rho^{qm}}{1 + \theta\Delta\rho^{qm}}\varrho.$$

Let

$$\tilde{\mathrm{L}}(x) = \frac{\theta}{\rho x}\left[\tilde{u}'\left(\lambda A^s/x\right) - x\right].$$

Notice that when $\tilde{\mathrm{D}}(x) < \lambda A^s/x$ for $x = \bar{\varphi}^m(\Delta)$, that $\hat{T}(\cdot, 0; 0) = -\bar{\varepsilon}\iota$, since $\bar{\varphi}^m(\Delta) < \bar{\varphi}^q(\Delta)$.

Again, notice that if $\lambda A^s/\bar{\varphi}^m(\Delta) < \tilde{\mathrm{D}}[\bar{\varphi}^m(\Delta)] < \lambda A^s/\bar{\varphi}^q(\Delta)$, that the solution is immediate from $\hat{T}$ and equivalent to case below. Thus we only consider $\tilde{\mathrm{D}}(x) > \lambda A^s/x$. Differentiating $\hat{T}(\rho^{qm}, 0; 0)$ with respect to ρ^{qm} yields

$$\hat{T}'(\rho^{qm}, 0; 0) = \bar{\varepsilon}\frac{\left[1 - \lambda\tilde{\mathrm{L}}(\bar{\varphi}^q(\Delta))\right]\left[\tilde{\mathrm{L}}'(\bar{\varphi}^m(\Delta))\right] - \left[\iota - \tilde{\mathrm{L}}(\bar{\varphi}^m(\Delta))\right]\left[\lambda\tilde{\mathrm{L}}'(\bar{\varphi}^q(\Delta))\right]}{\left[1 - \lambda\tilde{\mathrm{L}}(\bar{\varphi}^q(\Delta))\right]^2},$$

where

$$\begin{aligned}
\tilde{\mathrm{L}}'(x) &= \frac{x\tilde{u}''[\lambda A^s/x]\frac{\lambda A^s}{x}x' - \tilde{u}'[\lambda A^s/x]x'}{x^2} \\
\bar{\varphi}^{m'}(\Delta) &= \frac{-\varrho\theta\Delta}{(1 + \theta\Delta\rho^{qm})^2} < 0 \\
\bar{\varphi}^{q'}(\Delta) &= \frac{1 - \theta\Delta}{(1 + \theta\Delta\rho^{qm})^2}\varrho > 0.
\end{aligned}$$

which implies that $\tilde{\mathrm{L}}'(\bar{\varphi}^m(\Delta)) > 0$ and $\tilde{\mathrm{L}}'(\bar{\varphi}^q(\Delta)) < 0$. Which implies that $\hat{T}(\rho^{qm}, 0; 0) \geq 0$. Within the boundary $\left[\rho_L^{qm}, \rho_H^{qm}\right]$ there exists a unique ρ^{qm} such that $\hat{T}(\rho^{qm}, 0; 0) = 0$ given by

$$\iota = (1 - \delta)\frac{\theta}{\rho\bar{\varphi}^m(\Delta)}\left[\tilde{u}'[\min\{\tilde{\mathrm{D}}(\bar{\varphi}^m(\Delta)), \lambda A^s/\bar{\varphi}^m(\Delta)\}] - \bar{\varphi}^m(\Delta)\right].$$

(ii) As $\delta \to 0$ in the continuous-time limit the asset price is constant and given in the proposition.

■

Proof of Proposition 2 (Section B.1). The price of goods in the monetary equilibrium in the discrete-time economy is given by

$$\bar{\varphi}^q(\Delta) = \frac{1 + \Delta\rho^{qm}}{1 + \theta\Delta\rho^{qm}}\varrho \equiv (1 + \Delta\tilde{\rho}^{qm})\varrho,$$

and in the nonmonetary equilibrium is given by

$$\tilde{\varphi}^q = \varrho + \frac{1 - \theta}{\theta}(\varrho - \underline{\varrho}).$$

(i) The monetary equilibrium is greater than the nonmonetary equilibrium if and only if

$$(1+\Delta\tilde{\rho}^{qm})\varrho \leq \varrho + \frac{1-\theta}{\theta}(\varrho - \underline{\varrho}).$$

Rearranging yields part (i) of the proposition.

(ii) Differentiating ρ^{qm} with respect to ι

$$\frac{d\rho^{qm}}{d\pi} = -\frac{\frac{\partial\hat{T}(\rho^{qm})}{\partial\pi}}{\hat{T}'(\rho^{qm})} = -\frac{-\frac{1}{1-\lambda\mathrm{L}(\bar{\varphi}^{q}(\Delta))}}{\hat{T}'(\rho^{qm})} > 0,$$

where $\hat{T}$ is the equilibrium map defined in part (i) of the proof of Proposition 6. Then differentiating the expression for $\Phi^{s}(\Delta)$ in part (ii) of the statement of Proposition 6,

$$\frac{d\Phi^{s}(\Delta)}{d\pi} = -\frac{\bar{\varepsilon}}{\left\{\frac{(r+d-g+dg\Delta)\Delta}{(1+g\Delta)(1-d\Delta)} + \frac{\lambda\theta\Delta}{\bar{\varphi}^{q}(\Delta)}\left[\tilde{u}'\left[\min\left\{\tilde{\mathrm{D}}(\bar{\varphi}^{q}(\Delta)), \lambda A^{s}/\bar{\varphi}^{q}(\Delta)\right\}\right] - \bar{\varphi}^{q}(\Delta)\right]\right\}^{2}}\tilde{\mathrm{L}}'(\bar{\varphi}^{q}(\Delta))\frac{d\rho^{m}}{d\pi} > 0,$$

which yields the proposition. ■

Appendix D

Chapter 3 Proofs

D.1 Portfolio Problems

The investor's second subperiod value function can be written as

$$W_t^I(\mathbf{a}_t) = \phi_t^m a_t^m + \phi_t^s a_t^s + \bar{W}_t^I, \tag{D.1}$$

with

$$\bar{W}_t^I \equiv T_t + \max_{(\tilde{a}_{t+1}^m, \tilde{a}_{t+1}^s) \in \mathbb{R}_+^2} \left\{ -\phi_t^m \tilde{a}_{t+1}^m - \phi_t^s \tilde{a}_{t+1}^s + \beta \mathbb{E}_t \sum_{i \in \{L,H\}} \pi_i V_{it+1}^I [\tilde{a}_{t+1}^m, \eta \tilde{a}_{t+1}^s + (1-\eta) A^s, \varepsilon_i] \right\}. \tag{D.2}$$

Lemma 1. *Define* $\varepsilon_t^* \equiv \frac{p\phi_t^m - \phi_t^s}{y_t}$ *and*

$$\chi(\varepsilon_t^*, \varepsilon_i) = \begin{cases} 1 & \text{if } \varepsilon_t^* < \varepsilon_i \\ \in [0,1] & \text{if } \varepsilon_t^* = \varepsilon_i \\ 0 & \text{if } \varepsilon_t^* > \varepsilon_i. \end{cases} \tag{D.3}$$

Consider an investor who enters the first subperiod of period t *with portfolio* $\mathbf{a}_t$ *and valuation*

ε_i. *The investor's post-trade portfolio,* $[\bar{a}_t^m(\mathbf{a}_t, \varepsilon_i), \bar{a}_t^s(\mathbf{a}_t, \varepsilon_i)]$, *is given by*

$$\bar{a}_t^m(\mathbf{a}_t, \varepsilon_i) = [1 - \chi(\varepsilon_t^*, \varepsilon_i)] (a_t^m + pa_t^s) \tag{D.4}$$

$$\bar{a}_t^s(\mathbf{a}_t, \varepsilon_i) = \chi(\varepsilon_t^*, \varepsilon_i)(1/p)(a_t^m + pa_t^s). \tag{D.5}$$

Proof. With W_t^I, the investor's portfolio problem can be written as

$$\max_{(\bar{a}_t^m, \bar{a}_t^s) \in \mathbb{R}_+^2} [(\varepsilon_i y_t + \phi_t^s)\bar{a}_t^s + \phi_t^m \bar{a}_t^m]$$

$$\text{s.t. } \bar{a}_t^m + p\bar{a}_t^s \leq a_t^m + pa_t^s.$$

The Lagrangian corresponding to the problem is

$$\begin{aligned} \mathcal{L} &= (\varepsilon_i y_t + \phi_t^s)\, \bar{a}_t^s + \phi_t^m \bar{a}_t^m \\ &\quad + \xi^s \left[a_t^m + pa_t^s - \bar{a}_t^m - p\bar{a}_t^s\right] + \varsigma^m \bar{a}_t^m + \varsigma^s \bar{a}_t^s, \end{aligned}$$

where $\xi^s \in \mathbb{R}_+$ is the Lagrange multiplier associated with the budget constraint, and $\varsigma^m, \varsigma^s \in \mathbb{R}_+$ are the multipliers for the nonnegativity constraints on $\bar{a}_t^m, \bar{a}_t^s$, respectively. The first-order necessary and sufficient conditions, as well as the complementary slackness conditions are

$$\begin{aligned} \varepsilon_i y_t + \phi_t^s - \xi^s p + \varsigma^s &= 0 \\ \phi_t^m - \xi^s + \varsigma^m &= 0 \\ \xi^s \left[a_t^m + pa_t^s - \bar{a}_t^m - p\bar{a}_t^s\right] &= 0 \\ \varsigma^m \bar{a}_t^m &= 0 \\ \varsigma^s \bar{a}_t^s &= 0. \end{aligned}$$

First, notice that $\xi^s > 0$ at an optimum. To see this, assume the contrary, i.e., $\xi^s = 0$. Then

$\varepsilon_i y_t + \phi_t^s = -\varsigma^s \leq 0$ which is a contradiction since $\varepsilon_i y_t + \phi_t^s > 0$. Since $\xi^s > 0$, we have

$$\bar{a}_t^m + p\bar{a}_t^s = a_t^m + pa_t^s.$$

There are three possible cases: (a) $\varsigma^s = 0 < \varsigma^m$, (b) $\varsigma^s = \varsigma^m = 0$, (c) $\varsigma^m = 0 < \varsigma^s$. In every case, we have

$$\varepsilon_i y_t + \phi_t^s + \varsigma^s = p\phi_t^m + p\varsigma^m$$

In case (a), $\varepsilon_i y_t + \phi_t^s > p\phi_t^m$ which implies ε_i must satisfy $\varepsilon_i > \varepsilon_t^*$, and yields $\bar{a}_t^m = 0$ and $\bar{a}_t^s = \frac{1}{p}(a_t^m + pa_t^s)$. In case (b), $\varepsilon_i y_t + \phi_t^s = p\phi_t^m$ which implies that $\varepsilon_i = \varepsilon_t^*$. Since $\varsigma^m = \varsigma^s = 0$, then $\bar{a}_t^m \geq 0$ and $\bar{a}_t^s \geq 0$ and satisfy $\bar{a}_t^m + p\bar{a}_t^s = a_t^s + pa_t^s$. In case (c), $\varepsilon_i y_t + \phi_t^s < p\phi_t^m$ which implies that ε_i must satisfy $\varepsilon_i < \varepsilon_t^*$, and yields $\bar{a}_t^s = 0$ and $\bar{a}_t^m = a_t^m + pa_t^s$. Collecting these cases yields the lemma. ■

D.2 Value Functions

In this section we derive the value function for brokers and investors in a monetary economy (Lemma 2).

Lemma 2. *Consider an economy with money.*

(i) The value function of a broker at the beginning of the first subperiod of period t

$$V_{jt}^B = \max_{p \in \mathcal{F}_j} [\Pi_{jt}(p)] + \Xi_{jt}^B \bar{W}_t^B, \tag{D.6}$$

for $j \in \{a, b\}$ (where 'a' denotes 'asking broker' and 'b' denotes 'bidding broker') where

$$\Pi_{at}(p) \equiv \sum_{k=1}^{\infty} \alpha_k k \left[1 - A_t(p)\right]^{k-1} \frac{1}{\kappa_t} \left(\phi_t^m p - \phi_t^m \hat{p}\right) \quad \forall p \in [\hat{p} + \kappa_t/\phi_t^m, \bar{p}]$$
$$\Pi_{bt}(p) \equiv \sum_{k=1}^{\infty} \alpha_k k \left[B_t(p)\right]^{k-1} \frac{1}{\kappa_t} \left(\phi_t^m \hat{p} - \phi_t^m p\right) \quad \forall p \in \left[\underline{p}, \hat{p} - \kappa_t/\phi_t^m\right],$$

and $\bar{W}_t^B \equiv \beta \mathbb{E}_t V_{jt+1}^B$*, where*

$$\Xi_{jt}^B \equiv \alpha_0 + \sum_{k=1}^{\infty} \alpha_k k \left\{ \mathbb{I}_{\{j=a\}} \left[1 - A_t(p)\right]^{k-1} + \mathbb{I}_{\{j=b\}} \left[B_t(p)\right]^{k-1} \right\}.$$

(ii) The value function of an investor who enters the first subperiod of period t *with portfolio* $\mathbf{a}_t$ *and valuation* ε_i *is*

$$V_t(\mathbf{a}_t, \varepsilon_i) = \int \sum_{k=1}^{\infty} \left[v_t^m(p, \varepsilon_i) a_t^m + v_t^s(p, \varepsilon_i) a_t^s\right] dG_{kt}(p, \varepsilon_i) + \Xi_t^I(\varepsilon_i) \bar{W}_t^I, \tag{D.7}$$

where

$$v_t^m(p, \varepsilon_i) \equiv \phi_t^m + \mathbb{I}_{\{\varepsilon_t^* < \varepsilon_i\}} \left(\varepsilon_i - \varepsilon_t^*\right) y_t \frac{1}{p}$$
$$v_t^s(p, \varepsilon_i) \equiv \varepsilon_i y_t + \phi_t^s + \mathbb{I}_{\{\varepsilon_i < \varepsilon_t^*\}} \left(\varepsilon_t^* - \varepsilon_i\right) y_t,$$

and

$$dG_{kt}(p, \varepsilon_i) \equiv \alpha_k k \left\{ \mathbb{I}_{\{\varepsilon_i = \varepsilon_H\}} \left[1 - A_t(p)\right]^{k-1} dA_t(p) + \mathbb{I}_{\{\varepsilon_i = \varepsilon_L\}} \left[B_t(p)\right]^{k-1} dB_t(p) \right\}$$
$$\Xi_t^I(\varepsilon_i) \equiv \alpha_0 + \int \sum_{k=1}^{\infty} dG_{kt}(p, \varepsilon_i).$$

Proof. (i) The broker's value function is immediate. (ii) With $\bar{W}_t^I$, the value function becomes

$$V_t(\mathbf{a}_t, \varepsilon_i) = \Xi_t^I(\varepsilon_i) \bar{W}_t^I + \int \sum_{k=1}^{\infty} \left[\left(\varepsilon_i y_t + \phi_t^s\right) \bar{a}_t^s(\mathbf{a}_t, \varepsilon_i) + \phi_t^m \bar{a}_t^m(\mathbf{a}_t, \varepsilon_i)\right] dG_{kt}(p, \varepsilon_i).$$

Substitute $\bar{a}_t^s(\mathbf{a}_t, \varepsilon_i)$ and $\bar{a}_t^m(\mathbf{a}_t, \varepsilon_i)$ into the value function to obtain

$$V_t(\mathbf{a}_t, \varepsilon_i) = \Xi_t^I(\varepsilon_i) \bar{W}_t^I + \int \sum_{k=1}^{\infty} \left[\left(\varepsilon_i y_t + \phi_t^s\right) \chi\left(\varepsilon_t^*, \varepsilon_i\right) (1/p) \left(a_t^m + p a_t^s\right)\right] dG_{kt}(p, \varepsilon_i)$$
$$+ \int \sum_{k=1}^{\infty} \left\{\phi_t^m \left[1 - \chi\left(\varepsilon_t^*, \varepsilon_i\right)\right] \left(a_t^m + p a_t^s\right)\right\} dG_{kt}(p, \varepsilon_i).$$

Rearranging yields

$$V_t(\mathbf{a}_t, \varepsilon_i) = \Xi_t^I(\varepsilon_i)\bar{W}_t^I + \int \sum_{k=1}^{\infty} \left[v_t^m(p, \varepsilon_i)a_t^m + v_t^s(p, \varepsilon_i)a_t^s\right] dG_{kt}(p, \varepsilon_i),$$

where

$$\begin{aligned} v_t^m(p, \varepsilon_i) &\equiv \phi_t^m + \mathbb{I}_{\{\varepsilon_t^* < \varepsilon_i\}} \left(\varepsilon_i - \varepsilon_t^*\right) y_t \frac{1}{p} \\ v_t^s(p, \varepsilon_i) &\equiv \varepsilon_i y_t + \phi_t^s + \mathbb{I}_{\{\varepsilon_i < \varepsilon_t^*\}} \left(\varepsilon_t^* - \varepsilon_i\right) y_t, \end{aligned}$$

which yields the lemma. ■

D.3 Euler Equations

In this section we derive the Euler equations that characterise the optimal portfolio choices in the second subperiod of period t, in a monetary economy (Lemma 3).

Lemma 3. *Consider an economy with money. Let* $(\tilde{a}_{It+1}^m, \tilde{a}_{It+1}^s)$ *denote an individual investor's portfolio choice in the second subperiod of period* t*. The portfolio* $(\tilde{a}_{It+1}^m, \tilde{a}_{It+1}^s)$ *is optimal if and only if it satisfies*

$$(\phi_t^m - \beta \mathbb{E}_t \bar{v}_{It+1}^m)\tilde{a}_{It+1}^m = 0 \leq \phi_t^m - \beta \mathbb{E}_t \bar{v}_{It+1}^m \tag{D.8}$$

$$(\phi_t^s - \beta\eta \mathbb{E}_t \bar{v}_{It+1}^s)\tilde{a}_{It+1}^s = 0 \leq \phi_t^s - \beta\eta \mathbb{E}_t \bar{v}_{It+1}^s, \tag{D.9}$$

where

$$\begin{aligned} \bar{v}_{It+1}^m &\equiv \phi_{t+1}^m + \pi_H \int \sum_{k=1}^{\infty} \alpha_k k \left[1 - A_{t+1}(p)\right]^{k-1} \left[\left(\varepsilon_H - \varepsilon_{t+1}^*\right) y_{t+1} \frac{1}{p}\right] dA_{t+1}(p) \\ \bar{v}_{It+1}^s &\equiv \bar{\varepsilon} y_{t+1} + \phi_{t+1}^s + \pi_L \int \sum_{k=1}^{\infty} \alpha_k k \left[B_{t+1}(p)\right]^{k-1} \left(\varepsilon_{t+1}^* - \varepsilon_L\right) y_{t+1} dB_{t+1}(p). \end{aligned}$$

Proof. The portfolio problem of an investor in the second subperiod can be written as

$$\begin{aligned}\bar{W}_t^I &\equiv T_t + \beta \mathbb{E}_t \left[\bar{W}_{t+1} + \bar{v}_{It+1}^s (1-\eta) A^s \right] \\ &+ \max_{(\tilde{a}_{t+1}^m, \tilde{a}_{t+1}^s) \in \mathbb{R}_+^2} \left[(\beta \mathbb{E}_t \bar{v}_{It+1}^m - \phi_t^m) \tilde{a}_{t+1}^m + (\beta \eta \mathbb{E}_t \bar{v}_{It+1}^s - \phi_t^s) \tilde{a}_{t+1}^s \right],\end{aligned}$$

where $\bar{v}_{It+1}^j \equiv \sum_{i \in \{L,H\}} \pi_i \int \sum_{k=1}^{\infty} v_{It+1}^j (p, \varepsilon_i) dG_{kt+1}(p, \varepsilon_i)$ for $j \in \{m, s\}$. ■

D.4 Market-clearing Condition

In this section we derive the market-clearing condition for equity in the first subperiod of period t in a monetary economy (Lemma 4).

Lemma 4. *In a monetary equilibrium, the market-clearing condition for equity, $\bar{A}_{It}^s = A^s$ in the first subperiod round of period t is*

$$0 = \pi_H \int \sum_{k=1}^{\infty} \alpha_k k \left[1 - A_t(p)\right]^{k-1} \left(\frac{A_t^m + pA^s}{p} \right) dA_t(p) - A^s. \tag{D.10}$$

Proof. By Lemma 1, the investors' aggregate post-trade holdings of equity in the first subperiod round of period t is

$$\bar{A}_{It}^s = \pi_H \int \sum_{k=1}^{\infty} \alpha_k k \left[1 - A_t(p)\right]^{k-1} \left(\frac{A_t^m + pA^s}{p} \right) dA_t(p).$$

■

D.5 Ask and Bid Distributions

In this section we derive the ask and bid distributions in the first subperiod of period t in a monetary economy (Lemma 5).

Lemma 5. *Consider an economy with money, and define the mid-quote $\hat{p} \equiv (\bar{p} + \underline{p})/2$.*[1]

[1] Note that this definition assumes the cutoff between the two distributions is set at the midpoint of the two extreme prices. While this assumption facilitates sharper analytical results, it can be relaxed without substantively altering the main conclusions.

(i) The ask distribution $A_t(p)$ faced by an investor in the first subperiod round of period t is the solution to

$$\sum_{k=1}^{\infty} \alpha_k k \left[1 - A_t(p)\right]^{k-1} = \frac{\alpha_1 \kappa_t}{\phi_t^m p - \phi_t^m \hat{p}}, \tag{D.11}$$

with support $[\hat{p} + \kappa_t/\phi_t^m, \bar{p}]$.

(ii) The bid distribution $B_t(p)$ faced by an investor in the first subperiod round of period t is the solution to

$$\sum_{k=1}^{\infty} \alpha_k k \left[B_t(p)\right]^{k-1} = \frac{\alpha_1 \kappa_t}{\phi_t^m \hat{p} - \phi_t^m p}, \tag{D.12}$$

with support $\left[\underline{p}, \hat{p} - \kappa_t/\phi_t^m\right]$.

Proof. By Lemma 2, the brokers profit is equal for all prices posted. This implies that profit generated from any price is the same as the profit generated from the monopoly price.

(i) The asking broker's profit function therefore satisfies

$$\Pi_{at}(p) \equiv \sum_{k=1}^{\infty} \alpha_k k \left[1 - A_t(p)\right]^{k-1} \frac{1}{\kappa_t} \left(\phi_t^m p - \phi_t^m \hat{p}\right) = \alpha_1 \equiv \Pi_{at}(\hat{p} + \kappa_t/\phi_t^m).$$

Rearranging yields (D.11).

(ii) The bidding broker's profit function therefore satisfies

$$\Pi_{bt}(p) \equiv \sum_{k=1}^{\infty} \alpha_k k \left[B_t(p)\right]^{k-1} \frac{1}{\kappa_t} \left(\phi_t^m \hat{p} - \phi_t^m p\right) = \alpha_1 \equiv \Pi_{bt}(\hat{p} - \kappa_t/\phi_t^m).$$

Rearranging yields (D.12). ■

D.6 Equilibrium Conditions

In this section we state the operational definitions of monetary equilibrium that are used in the analysis.

D.6.1 Sequential Monetary Equilibrium

Definition 1. *A* (sequential) monetary equilibrium *is an allocation* $\{(\tilde{a}^k_{It+1})_{k\in\{m,s\}}\}_{t=0}^{\infty}$ *and a sequence of price* $\{\phi^m_t, \phi^s_t\}_{t=0}^{\infty}$ *that satisfy the optimality conditions,* (D.5) *and* (D.6) *(with* $\tilde{a}^k_{It+1} = \tilde{A}^k_{It+1}$*), a sequence of distributions,* $\{A_t(p), B_t(p)\}_{t=0}^{\infty}$ *that satisfy* (D.8) *and* (D.9)*, and the three market-clearing conditions,* $\tilde{A}^s_{It+1} = A^s$*,* $\tilde{A}^m_{It+1} = A^m_{t+1}$ *and* (D.7)*.*

Definition 1 follows from Definition 1 from the body of Chapter 3 after recognising that all investors choose the same end-of-period portfolio that is characterised by the Euler equations derived in Lemma 3, and using the explicit version of the market clearing condition for equity in the first subperiod of period t derived in Lemma 4. Given the equilibrium objects in Definition 1, the portfolio outcomes, which are part of Definition 1 in Chapter 3 but not Definition 1 above, are immediate from Lemma 1.

According to Definition 1, a monetary equilibrium can be characterised by sequence of prices, $\{\phi^m_t, \phi^s_t\}_{t=0}^{\infty}$, distributions $\{A_t(p), B_t(p)\}_{t=0}^{\infty}$ and an allocation $\{(\tilde{A}^k_{It+1})_{k\in\{m,s\}}\}_{t=0}^{\infty}$ that satisfy the following market-clearing conditions,

$$\begin{aligned}
0 &= \tilde{A}^s_{It+1} - A^s \\
0 &= \tilde{A}^m_{It+1} - A^m_{t+1} \\
0 &= \pi_H \int \sum_{k=1}^{\infty} \alpha_k k \left[1 - A_t(p)\right]^{k-1} \left(\frac{A^m_t + pA^s}{p}\right) dA_t(p) - A^s,
\end{aligned}$$

optimality conditions,

$$\begin{aligned}
(\phi^m_t - \beta \mathbb{E}_t \bar{v}^m_{It+1})\tilde{a}^m_{It+1} &= 0 \leq \phi^m_t - \beta \mathbb{E}_t \bar{v}^m_{It+1} \\
(\phi^s_t - \beta\eta \mathbb{E}_t \bar{v}^s_{It+1})\tilde{a}^s_{It+1} &= 0 \leq \phi^s_t - \beta\eta \mathbb{E}_t \bar{v}^s_{It+1},
\end{aligned}$$

where,

$$\begin{aligned}
\bar{v}^m_{It+1} &\equiv \phi^m_{t+1} + \pi_H \int \sum_{k=1}^{\infty} \alpha_k k \left[1 - A_{t+1}(p)\right]^{k-1} \left[\left(\varepsilon_H - \varepsilon^*_{t+1}\right) y_{t+1} \frac{1}{p}\right] dA_{t+1}(p) \\
\bar{v}^s_{It+1} &\equiv \bar{\varepsilon} y_{t+1} + \phi^s_{t+1} + \pi_L \int \sum_{k=1}^{\infty} \alpha_k k \left[B_{t+1}(p)\right]^{k-1} \left(\varepsilon^*_{t+1} - \varepsilon_L\right) y_{t+1} dB_{t+1}(p),
\end{aligned}$$

and,

$$\sum_{k=1}^{\infty} \alpha_k k \left[1 - A_t(p)\right]^{k-1} = \frac{\alpha_1 \kappa_t}{\phi_t^m p - \phi_t^m \hat{p}} \quad \forall p \in [\hat{p} + \kappa_t/\phi_t^m, \bar{p}]$$
$$\sum_{k=1}^{\infty} \alpha_k k \left[B_t(p)\right]^{k-1} = \frac{\alpha_1 \kappa_t}{\phi_t^m \hat{p} - \phi_t^m p} \quad \forall p \in [\hat{p} + \kappa/\phi_t^m, \bar{p}] \, .$$

D.6.2 Recursive Monetary Equilibrium

The following result summarises the conditions that characterise a recursive monetary equilibrium (RME).

Lemma 6. *A recursive monetary equilibrium is a vector* $(\phi^s, \varphi^m, Z, A, B)$ *that satisfies*

$$0 = \pi_H \int \sum_{k=1}^{\infty} \alpha_k k \left[1 - A(p)\right]^{k-1} \left(\frac{Z}{\varphi^m p} + 1\right) dA(p) - 1, \tag{D.13}$$

and

$$\frac{\mu - \bar{\beta}}{\bar{\beta}} = \pi_H \int \sum_{k=1}^{\infty} \alpha_k k \left[1 - A(p)\right]^{k-1} \left[\frac{\varepsilon_H + \phi^s - \varphi^m p}{\varphi^m p}\right] dA(p) \tag{D.14}$$
$$\frac{1 - \bar{\beta}\eta}{\bar{\beta}\eta} \phi^s = \bar{\varepsilon} + \pi_L \int \sum_{k=1}^{\infty} \alpha_k k \left[B(p)\right]^{k-1} \left[\varphi^m p - \phi^s - \varepsilon_L\right] dB(p), \tag{D.15}$$

where

$$\sum_{k=1}^{\infty} \alpha_k k \left[1 - A(p)\right]^{k-1} = \frac{\alpha_1 \kappa}{\varphi^m p - \varphi^m \hat{p}} \quad \forall p \in [\hat{p} + \kappa_t/\phi_t^m, \bar{p}] \tag{D.16}$$
$$\sum_{k=1}^{\infty} \alpha_k k \left[B(p)\right]^{k-1} = \frac{\alpha_1 \kappa}{\varphi^m \hat{p} - \varphi^m p} \quad \forall p \in [\hat{p} + \kappa/\phi_t^m, \bar{p}] \, . \tag{D.17}$$

Proof. The equilibrium conditions in the statement of the lemma are obtained from the ones in Section A.6.2 by using $\phi_t^s = \phi^s y_t$, $p\phi_t^m \equiv p\varphi^m y_t$, $\kappa_t = \kappa y_t$, $\phi_t^m A_t^m = Z A^s y_t$ $\varepsilon_t^* = (p\phi_t^m - \phi_t^s)\frac{1}{y_t} = p\varphi^m - \phi^s$, $\phi_{t+1}^s/\phi_t^s = \gamma_{t+1}$ and $\phi_t^m/\phi_{t+1}^m = \mu/\gamma_{t+1}$. ■

The first equation in Lemma 6 is the first subperiod market-clearing condition for equity. The remaining two conditions are the investor's Euler equations for money and equity, respectively.

D.7 Poisson Distributed Quotes

In this section, we derive the ask and bid distributions when the probability of observing k quotes is Poisson distributed, i.e., α_k is Poisson, with mean $\theta \in \mathbb{R}_+$ which is the average number of quotes.

D.7.1 Ask and Bid Distributions

Lemma 7. *Consider a monetary economy where α_k is Poisson, with mean $\theta \in \mathbb{R}_+$, then:*

(i) The equilibrium number of ask quotes faced by the investor is given by

$$\lim_{N\to\infty} \sum_{k=1}^{N} \alpha_k k \left[1 - A_t(p)\right]^{k-1} \to \theta e^{-\theta A_t(p)}. \tag{D.18}$$

(ii) The equilibrium number of bid quotes faced by the investor is given by

$$\lim_{N\to\infty} \sum_{k=1}^{N} \alpha_k k \left[B_t(p)\right]^{k-1} \to \theta e^{-\theta[1-B_t(p)]}. \tag{D.19}$$

Proof. (i) With $\alpha_k = (\theta^k/k!)e^{-\theta}$, the probability that an investor meets an asking broker is

$$\begin{aligned}
\lim_{N\to\infty} \sum_{k=1}^{N} \alpha_k k \left[1 - A_t(p)\right]^{k-1} &= \sum_{k=1}^{\infty} \frac{e^{-\theta}\theta^k}{k!} k \left[1 - A_t(p)\right]^{k-1} \\
&= \theta e^{-\theta} \sum_{k=1}^{\infty} \frac{\{\theta\left[1 - A_t(p)\right]\}^{k-1}}{(k-1)!} \\
&= \theta e^{-\theta A_t(p)}.
\end{aligned}$$

(ii) Similarly, with $\alpha_k = (\theta^k/k!)e^{-\theta}$, the probability that an investor meets a bidding broker is

$$\begin{aligned}
\lim_{N\to\infty} \sum_{k=1}^{N} \alpha_k k \left[B_t(p)\right]^{k-1} &= \sum_{k=1}^{\infty} \frac{e^{-\theta}\theta^k}{k!} k \left[B_t(p)\right]^{k-1} \\
&= \theta e^{-m} \sum_{k=1}^{\infty} \frac{\left[\theta B_t(p)\right]^{k-1}}{(k-1)!} \\
&= \theta e^{-\theta[1-B_t(p)]}.
\end{aligned}$$

Thus arriving at the conditions in the statement of the lemma. ■

Proof of Proposition 1. By Lemma 2, the broker's profit is equal for all prices posted. This implies that profit generated from any price is the same as the profit generated from the monopoly price. Since we suppose that both α_k is the same for both high and low valuation investors, this implies that the spread for the ask and bid distributions are symmetric i.e., $\Gamma \equiv \hat{p} - \underline{p} = \bar{p} - \hat{p}$.

(i) The asking broker's profit function therefore satisfies

$$\begin{aligned}\Pi_{at}(p) &\equiv \lim_{N\to\infty} \sum_{k=1}^{N} \alpha_k k \left[1 - A_t(p)\right]^{k-1} \frac{\varphi^m (p - \hat{p})}{\kappa} \\ &= \theta e^{-\theta A_t(p)} \frac{\varphi^m (p - \hat{p})}{\kappa} = \theta \equiv \Pi_{at} \left(\hat{p} + \kappa/\varphi^m\right),\end{aligned}$$

when $A(\bar{p}) = 1$ which implies that $\theta = \ln (\varphi^m \Gamma/\kappa)$. Rearranging yields the ask distribution with support $[\hat{p} + \kappa/\varphi^m, \bar{p}]$. Differentiate A with respect to p to obtain

$$\frac{\partial A(p)}{\partial p} \equiv a(p) = \frac{1}{\theta(p - \hat{p})},$$

which is the probability density function of the ask prices, with support $[\hat{p} + \kappa/\varphi^m, \bar{p}]$.

(ii) The bidding broker's profit function therefore satisfies

$$\begin{aligned}\Pi_{bt}(p) &\equiv \lim_{N\to\infty} \sum_{k=1}^{N} \alpha_k k \left[B_t(p)\right]^{k-1} \frac{\varphi^m (\hat{p} - p)}{\kappa} \\ &= \theta e^{-\theta[1 - B_t(p)]} \frac{\varphi^m (\hat{p} - p)}{\kappa} = \theta \equiv \Pi_{bt} \left(\hat{p} - \kappa/\varphi^m\right),\end{aligned}$$

when $B(\underline{p}) = 0$ which implies that $\theta = \ln (\varphi^m \Gamma/\kappa)$. Rearranging yields the ask distribution with support $[\underline{p}, \hat{p} - \kappa/\varphi^m]$. Differentiate B with respect to p to obtain

$$\frac{\partial B(p)}{\partial p} \equiv b(p) = \frac{1}{\theta(\hat{p} - p)},$$

which is the probability density function of the bid prices, with support $[\underline{p}, \hat{p} - \kappa/\varphi^m]$. ■

Corollary 5. *Consider a monetary economy where α_k is Poisson with mean $\theta \in \mathbb{R}_+$, then the* *following holds:*

(i) The density of asks, $a(p)$ strictly decreases in p, is convex, and prices are right-skewed,

(ii) The density of bids, $b(p)$ strictly increases in p, is convex, and prices are left-skewed.

Proof. (i) The ask density is $a(p) = [\theta(p - \hat{p})]^{-1}$, which is strictly decreasing in p because

$$\bar{a}'(p) = -\frac{1}{\theta(p-\hat{p})^2} < 0,$$

and it is convex because

$$\bar{a}''(p) = \frac{2}{\theta(p-\hat{p})^3} > 0.$$

(ii) The bid density is $b(p) = [\theta(\hat{p} - p)]^{-1}$, which is strictly increasing in p because

$$\bar{a}'(p) = \frac{1}{\theta(\hat{p}-p)^2} > 0,$$

and it is convex because

$$\bar{a}''(p) = \frac{2}{\theta(\hat{p}-p)^3} > 0.$$

■

Corollary 6. *Consider a monetary economy where α_k is Poisson with mean $\theta \in \mathbb{R}_+$, then the following holds:*

(i) There is a unique best-ask distribution, $\bar{A}$. Moreover,

$$\bar{A}(p) = \begin{cases} 1 - \frac{\kappa}{\varphi^m(p-\hat{p})} & \text{if } p \in [\hat{p} + \kappa/\varphi^m, \bar{p}) \\ 1 & \text{if } p = \bar{p}. \end{cases} \tag{D.20}$$

The corresponding density of the best ask strictly decreases, is convex, and prices are therefore right-skewed.

(ii) There is a unique best-bid distribution, $\bar{B}$. Moreover,

$$\bar{B}(p) = \begin{cases} 0 & \text{if } p = \underline{p} \\ \frac{\kappa}{\varphi^m(\hat{p}-p)} & \text{if } p \in (\underline{p}, \hat{p} - \kappa/\varphi^m]. \end{cases} \tag{D.21}$$

The corresponding density of the best bid strictly increases, is convex, and prices are therefore left-skewed.

Proof. (i) The best-ask distribution is derived by computing the conditional best-ask distribution given k posts, then computing the weighted sum of these where the weights are the marginal probabilities of k:

$$\begin{aligned}
\bar{A}(p) &= \lim_{N\to\infty} \sum_{k=0}^{N} \alpha_k \left\{1 - [1 - A(p)]^k\right\} \\
&= \sum_{k=0}^{\infty} \frac{e^{-\theta}\theta^k}{k!} - \sum_{k=0}^{\infty} \frac{e^{-\theta}\theta^k}{k!} \left[1 - \frac{1}{\theta} \ln\left(\frac{\varphi^m p - \varphi^m \hat{p}}{\kappa}\right)\right]^k \\
&= 1 - e^{-\theta} \sum_{k=0}^{\infty} \frac{\left[\theta - \ln\left(\frac{\varphi^m p - \varphi^m \hat{p}}{\kappa}\right)\right]^k}{k!} \\
&= 1 - \frac{\kappa}{\varphi^m (p - \hat{p})}.
\end{aligned}$$

For the corresponding best-ask density, differentiate $\bar{A}(p)$ with respect to p to obtain $\partial\bar{A}(p)/\partial p \equiv \bar{a}(p) = \frac{\kappa}{\varphi^m}(p - \hat{p})^{-2}$, which is strictly decreasing in p because

$$\bar{a}'(p) = -\frac{2\kappa}{\varphi^m (p - \hat{p})^3} < 0,$$

and it is convex because

$$\bar{a}''(p) = \frac{6\kappa}{\varphi^m (p - \hat{p})^4} > 0.$$

(ii) Similarly, the best-bid distribution is derived by computing the conditional best-bid distribution given k posts, then computing the weighted sum of these where the weights are the marginal probabilities of k:

$$\begin{aligned}\bar{B}(p) &= \lim_{N\to\infty}\sum_{k=0}^{N}\alpha_k\left[B(p)\right]^k \\ &= \sum_{k=0}^{\infty}\frac{e^{-\theta}\theta^k}{k!}\left[1-\frac{1}{\theta}\ln\left(\frac{\varphi^m\hat{p}-\varphi^m p}{\kappa}\right)\right]^k \\ &= e^{-\theta}\sum_{k=0}^{\infty}\frac{\left[\theta-\ln\left(\frac{\varphi^m\hat{p}-\varphi^m p}{\kappa}\right)\right]^k}{k!} \\ &= \frac{\kappa}{\varphi^m\left(\hat{p}-p\right)}.\end{aligned}$$

For the corresponding best-bid density, differentiate $\bar{B}(p)$ with respect to p to obtain $\partial\bar{B}(p)/\partial p \equiv \bar{b}(p) = \frac{\kappa}{\varphi^m}\left(\hat{p}-p\right)^{-2}$, which is strictly increasing in p because

$$\bar{b}'(p) = \frac{2\kappa}{\varphi^m\left(\hat{p}-p\right)^3} > 0,$$

and it is convex because

$$\bar{b}''(p) = \frac{6\kappa}{\varphi^m\left(\hat{p}-p\right)^4} > 0.$$

■

Corollary 2A. *Consider a monetary economy where α_k is Poisson with mean $\theta \in \mathbb{R}_+$, then the following holds:*

(i) The Value-at-Risk (quantile function) of the best-ask distribution, $VaR_a \equiv \mathcal{Q}_a$, and is given by

$$\mathcal{Q}_a(z) = \begin{cases}\hat{p}+\frac{\kappa}{\varphi^m(1-z)} & \text{for } z\in\left[0,1-\frac{\kappa}{\varphi^m(\bar{p}-\hat{p})}\right) \\ \bar{p} & \text{for } z=\left[1-\frac{\kappa}{\varphi^m(\bar{p}-\hat{p})},1\right].\end{cases} \tag{D.22}$$

(ii) The Value-at-Risk (quantile function) of the best-bid distribution, $VaR_b \equiv \mathcal{Q}_b$, and is given

by

$$\mathcal{Q}_b(z) = \begin{cases} \underline{p} & \text{for } z \in \left[0, \frac{\kappa}{\varphi^m(\hat{p}-\underline{p})}\right) \\ \hat{p} - \frac{\kappa}{\varphi^m z} & \text{for } z \in \left[\frac{\kappa}{\varphi^m(\hat{p}-\underline{p})}, 1\right]. \end{cases} \tag{D.23}$$

■

Proof. (i) From Corollary 2, the best-ask distribution is defined on the interval $p \in [\hat{p}+\kappa/\varphi^m, \bar{p}]$, we have:

$$\bar{A}(p) = 1 - \frac{\kappa}{\varphi^m(p-\hat{p})} \quad \forall p \in [\hat{p}+\kappa/\varphi^m, \bar{p}),$$

and $\bar{A}(\bar{p}) = 1$. We solve for p in terms of z where

$$z = 1 - \frac{\kappa}{\varphi^m(p-\hat{p})} \implies p = \hat{p} + \frac{\kappa}{\varphi^m(1-z)},$$

when $p = \hat{p} + \kappa/\varphi^m$, $z = 0$ and when $p = \bar{p}$, $z = 1 - \frac{\kappa}{\varphi^m(\bar{p}-\hat{p})}$. Thus the quantile function $\mathcal{Q}_a(z) = \inf\{p : \bar{A}(p) \geq z\}$ is given in the statement of the Corollary.

(ii) Similarly, from Corollary 2, the best-bid distribution is defined on the interval $p \in [\underline{p}, \hat{p} - \kappa/\varphi^m]$, we have:

$$\bar{B}(p) = \frac{\kappa}{\varphi^m(\hat{p}-p)} \quad \forall p \in (\underline{p}, \hat{p}-\kappa/\varphi^m],$$

and $\bar{B}(\underline{p}) = 0$. We solve for p in terms of z where

$$z = \frac{\kappa}{\varphi^m(\hat{p}-p)} \implies p = \hat{p} - \frac{\kappa}{\varphi^m z},$$

when $p = \underline{p}$, $z = \frac{\kappa}{\varphi^m(\hat{p}-\underline{p})}$ and when $p = \hat{p} - \kappa/\varphi^m$, $z = 1$. Thus the quantile function $\mathcal{Q}_b(z) = \inf\{p : \bar{B}(p) \geq z\}$ is given in the statement of the Corollary. ■

D.7.2 Equilibrium Conditions

In this section, we derive the equilibrium conditions when the probability of observing k quotes is Poisson distributed, i.e., α_k is Poisson with mean $\theta \in \mathbb{R}_+$ which is the average number of quotes.

Lemma 8. *Consider a monetary economy where α_k is Poisson with mean $\theta \in \mathbb{R}_+$. A recursive monetary equilibrium is a vector (ϕ^s, φ^m, Z) that satisfies*

$$1 = \pi_H \int \theta e^{-\theta A(p)} \left(\frac{Z}{\varphi^m p} + 1 \right) dA(p), \tag{D.24}$$

and

$$\frac{1-\bar{\beta}\eta}{\bar{\beta}\eta}\phi^s = \bar{\varepsilon} + \pi_L \int \theta e^{-\theta[1-B(p)]} \left(\varphi^m p - \phi^s - \varepsilon_L \right) dB(p) \tag{D.25}$$

$$\frac{\mu - \bar{\beta}}{\bar{\beta}} = \pi_H \int \theta e^{-\theta A(p)} \left(\frac{\varepsilon_H + \phi^s - \varphi^m p}{\varphi^m p} \right) dA(p), \tag{D.26}$$

where

$$A(p) = \frac{1}{\theta} \ln \left[\frac{\varphi^m (p - \hat{p})}{\kappa} \right] \quad \forall p \in [\hat{p} + \kappa/\varphi^m, \bar{p}] \tag{D.27}$$

$$B(p) = 1 - \frac{1}{\theta} \ln \left[\frac{\varphi^m (\hat{p} - p)}{\kappa} \right] \quad \forall p \in [\underline{p}, \hat{p} - \kappa/\varphi^m]. \tag{D.28}$$

Proof. The equilibrium conditions in the statement of the lemma are obtained from Lemma 6, 7 and Proposition 1. ■

Lemma 9. *Consider a monetary economy where α_k is Poisson with mean $\theta \in \mathbb{R}_+$. A recursive monetary equilibrium is a vector (ϕ^s, φ^m, Z) that satisfies*

$$0 = \pi_L \left[1 - e^{-\theta} \right] - \pi_H Z \left\{ \int_{\hat{p}+\frac{\kappa}{\varphi^m}}^{\bar{p}} \frac{\kappa}{p \left[\varphi^m (p - \hat{p}) \right]^2} dp \right\}, \tag{D.29}$$

and

$$\phi^s = \frac{\bar{\beta}_H \eta}{1-\bar{\beta}_H\eta}\left\{\varepsilon_H + \frac{\pi_L}{1-\pi_L}\left[\varphi^m \hat{p} - \frac{\kappa\theta}{1-e^{-\theta}}\right]\right\} \tag{D.30}$$

$$\frac{\mu-\bar{\beta}_L}{\bar{\beta}_L} = \frac{\varepsilon_H + \phi^s}{1-e^{-\theta}}\int_{\hat{p}+\frac{\kappa}{\varphi^m}}^{\bar{p}} \frac{\kappa}{p\left[\varphi^m(p-\hat{p})\right]^2}dp. \tag{D.31}$$

Proof. The equilibrium conditions are (D.24)-(D.28), as reported in Lemma 8. These are three equations in three unknowns. The unknowns are (ϕ^s, φ^m, Z). We consider equation (D.25) first:

$$\begin{aligned}\theta e^{-\theta[1-B(p)]} &= \frac{\kappa\theta}{\varphi^m(\hat{p}-p)}\\ \frac{\partial B(p)}{\partial p} &= \frac{1}{\theta(\hat{p}-p)}.\end{aligned}$$

Thus, (D.25) becomes

$$\frac{1-\bar{\beta}_H\eta}{\bar{\beta}_H\eta}\phi^s = \varepsilon_H + \frac{\pi_L}{\pi_H}\int \frac{\kappa p}{(\hat{p}-p)^2}dp,$$

which is equivalent to integrating, p, over the best-bid distribution, $\bar{B}$, noting that we truncate the distribution over the support $[\underline{p}, \hat{p}-\kappa/\varphi^m]$. Therefore, (D.25) becomes:

$$\frac{1-\bar{\beta}_H\eta}{\bar{\beta}_H\eta}\phi^s = \varepsilon_H + \frac{\varphi^m\pi_L}{\pi_H\left[1-\bar{B}(\underline{p})\right]}\int_{\underline{p}}^{\hat{p}-\frac{\kappa}{\varphi^m}} p d\bar{B}(p).$$

Using the distribution of the best-bid, $\bar{B}$ in Corollary 2 and applying integration by parts yields:

$$\begin{aligned}\frac{1-\bar{\beta}_H\eta}{\bar{\beta}_H\eta}\phi^s &= \varepsilon_H + \frac{\varphi^m\pi_L}{\pi_H\left[1-\bar{B}\left(\underline{p}\right)\right]}\left\{-p\left[1-\bar{B}(p)\right]\Big|_{\underline{p}}^{\hat{p}-\frac{\kappa}{\varphi^m}} + \int_{\underline{p}}^{\hat{p}-\frac{\kappa}{\varphi^m}}\left[1-\bar{B}(p)\right]dp\right\}\\ &= \varepsilon_H + \frac{\varphi^m\pi_L}{\pi_H\left[1-\bar{B}(\underline{p})\right]}\left[-\underline{p}\bar{B}\left(\underline{p}\right) + \hat{p} - \frac{\kappa}{\varphi^m} - \int_{\underline{p}}^{\hat{p}-\frac{\kappa}{\varphi^m}}\bar{B}(p)dp\right]\\ &= \varepsilon_H + \frac{\varphi^m\pi_L}{\pi_H}\left[\frac{\hat{p}-\frac{\kappa}{\varphi^m}-\underline{p}}{1-\bar{B}\left(\underline{p}\right)} + \underline{p} - \frac{1}{1-\bar{B}\left(\underline{p}\right)}\int_{\underline{p}}^{\hat{p}-\frac{\kappa}{\varphi^m}}\bar{B}(p)dp\right],\end{aligned}$$

with $1-\bar{B}\left(\underline{p}\right)=1-\frac{\kappa}{\varphi^m\hat{p}-\varphi^m\underline{p}}=1-e^{-\theta}$ and

$$\int_{\underline{p}}^{\hat{p}-\frac{\kappa}{\varphi^m}}\bar{B}(p)dp=\frac{\kappa}{\varphi^m}\int_{\underline{p}}^{\hat{p}-\frac{\kappa}{\varphi^m}}\frac{1}{\hat{p}-p}dp=-\frac{\kappa}{\varphi^m}\ln\left[\frac{\hat{p}-\left(\hat{p}-\frac{\kappa}{\varphi^m}\right)}{\hat{p}-\underline{p}}\right]=\frac{\kappa\theta}{\varphi^m},$$

notice that since $\frac{\kappa}{\varphi^m\left(\hat{p}-\underline{p}\right)}=e^{-\theta}$, we have $\frac{\kappa}{\varphi^m}=e^{-\theta}\left(\hat{p}-\underline{p}\right)$. Therefore we can write

$$\begin{aligned}\frac{1-\bar{\beta}_H\eta}{\bar{\beta}_H\eta}\phi^s&=\varepsilon_H+\frac{\varphi^m\pi_L}{1-\pi_L}\left\{\frac{\hat{p}-\underline{p}-e^{-\theta}\left(\hat{p}-\underline{p}\right)}{1-e^{-\theta}}+\underline{p}-\frac{\kappa\theta}{\varphi^m\left[1-e^{-\theta}\right]}\right\}\\&=\varepsilon_H+\frac{\pi_L}{1-\pi_L}\left[\varphi^m\hat{p}-\frac{\kappa\theta}{1-e^{-\theta}}\right],\end{aligned}$$

rearranging yields equation (D.30). Moving to equations (D.24) and (D.26) notice that:

$$\begin{aligned}\theta e^{-\theta A(p)}&=\frac{\kappa\theta}{\varphi^m(p-\hat{p})}\\\frac{\partial A(p)}{\partial p}&=\frac{1}{\theta(p-\hat{p})},\end{aligned}$$

thus (D.24) and (D.26) become, respectively,

$$\begin{aligned}1&=\frac{\pi_H Z}{\varphi^m}\int\frac{\kappa}{\varphi^m p\left(p-\hat{p}\right)^2}dp+\pi_H\int\frac{\kappa}{\varphi^m\left(p-\hat{p}\right)^2}dp\\\frac{\mu-\bar{\beta}}{\bar{\beta}}&=\frac{\pi_H\left(\varepsilon_H+\phi^s\right)}{\varphi^m}\int\frac{\kappa}{\varphi^m p\left(p-\hat{p}\right)^2}dp-\pi_H\int\frac{\kappa}{\varphi^m\left(p-\hat{p}\right)^2}dp,\end{aligned}$$

which is equivalent to integrating, p, over the best-ask distribution, $\bar{A}$, noting that we truncate the distribution over the support $[\hat{p}+\kappa/\varphi^m,\bar{p}]$. Therefore,

$$\int\frac{\kappa}{\varphi^m(p-\hat{p})^2}dp\equiv\frac{1}{\bar{A}(\bar{p})}\int_{\hat{p}+\frac{\kappa}{\varphi^m}}^{\bar{p}}d\bar{A}(p)=1,$$

with $\bar{A}(\bar{p})=1-\frac{\kappa}{\varphi^m\bar{p}-\varphi^m\hat{p}}=1-e^{-\theta}$. Rearranging yields the lemma. ∎

D.7.3 Existence of Equilibrium

Proof of Proposition 2. The equilibrium conditions are (D.29)-(D.31), as reported in Lemma 9. There are three equations in three unknowns. The unknowns are (ϕ^s,φ^m,Z). Combined, (D.29), (D.30) and (D.31) imply a single equation in the unknown φ^m that can be written as $T(\varphi^m) = 0$, where

$$T(x) \equiv \frac{1}{1-\bar{\beta}_H\eta}\left[\varepsilon_H + \bar{\beta}_H\eta\left(x\hat{p} - \frac{\kappa\theta(x)}{1-e^{-\theta(x)}}\right)\right] - \frac{\mu - \bar{\beta}_L}{\bar{\beta}_L}Z(x),$$

where $\bar{\pi} = \pi_L/\pi_H$ and,

$$Z(x) = \bar{\pi}\left[1 - e^{-\theta(x)}\right][I(x)]^{-1}, \quad \text{and} \quad I(x) = \int_{\hat{p}+\frac{\kappa}{x}}^{\bar{p}} \frac{1}{xp}d\bar{A}(p,x) = \int_{\hat{p}+\frac{\kappa}{x}}^{\bar{p}} \frac{\kappa}{p\left[x\left(p-\hat{p}\right)\right]^2}dp.$$

Notice that $x_L = \kappa/\Gamma$ is the lower bound of the support of x, since $\theta(x_L) = 0$. Using L'Hospital's rule, we can compute the limit of $T(x)$ as x approaches x_L, Note that $J(x) = (x^2/\kappa)I(x)$ and $\theta(x) = 1/x$:

$$\begin{aligned}
\lim_{x\to x_L} T(x) &= \lim_{x\to x_L}\frac{1}{1-\bar{\beta}_H\eta}\left[\varepsilon_H + \bar{\beta}_H\eta\left(x\hat{p} - \frac{\kappa\theta(x)}{1-e^{-\theta(x)}}\right)\right] - \lim_{x\to x_L}\frac{\mu-\bar{\beta}_L}{\bar{\beta}_L}\bar{\pi}\frac{x^2\left[1-e^{-\theta(x)}\right]}{\kappa J(x)} \\
&= \lim_{x\to x_L}\frac{1}{1-\bar{\beta}_H\eta}\left[\varepsilon_H + \bar{\beta}_H\eta\left(x\hat{p} - \frac{\kappa}{e^{-\theta(x)}}\right)\right] - \lim_{x\to x_L}\frac{\mu-\bar{\beta}_L}{\bar{\beta}_L}\bar{\pi}\frac{xe^{-\theta(x)} + 2x\left[1-e^{-\theta(x)}\right]}{\kappa J'(x)} \\
&= \frac{1}{1-\bar{\beta}_H\eta}\left[\varepsilon_H + \bar{\beta}_H\eta\left(x_L\hat{p} - \kappa\right)\right] - \bar{\pi}\frac{\mu-\bar{\beta}_L}{\bar{\beta}_L}\frac{x_L}{\kappa J'(x_L)}.
\end{aligned}$$

Using Leibnitz' rule, we can write $J'(x) = [\kappa\left(\hat{p} + \frac{\kappa}{x}\right)]^{-1}$, thus we have

$$\begin{aligned}
\lim_{x\to x_L} T(x) &= \frac{1}{1-\bar{\beta}_H\eta}\left[\varepsilon_H + \bar{\beta}_H\eta\left(x_L\hat{p} - \kappa\right)\right] - \frac{\mu-\bar{\beta}_L}{\bar{\beta}_L}\frac{\bar{\pi}x_L}{\kappa J'(x_L)} \\
&= \frac{1}{1-\bar{\beta}_H\eta}\left[\varepsilon_H + \bar{\beta}_H\eta\frac{p\kappa}{\Gamma}\right] - \frac{\mu-\bar{\beta}_L}{\bar{\beta}_L}\frac{\bar{\pi}\kappa\bar{p}}{\Gamma},
\end{aligned}$$

so $\lim_{x\to x_L} T(x) > 0$ if and only if $\mu < \bar{\mu}$, where $\bar{\mu}$ is defined in the statement of the proposition. Now, compute the limit of $T(x)$ as $x \to \infty$:

$$\lim_{x\to\infty} T(x) = \lim_{x\to\infty}\frac{1}{1-\bar{\beta}_H\eta}\left[\varepsilon_H + \bar{\beta}_H\eta\left(x\hat{p} - \frac{\kappa\theta(x)}{1-e^{-\theta(x)}}\right)\right] - \lim_{x\to\infty}\frac{\mu-\bar{\beta}_L}{\bar{\beta}_L}\bar{\pi}\frac{x^2\left[1-e^{-\theta(x)}\right]}{\kappa J(x)}.$$

Analyse each term of $T(x)$ separately in the limit $x \to \infty$. Since $\theta(x) = \ln x + \ln[\Gamma/\kappa]$ it implies that $\theta(x) = \ln x + O(1)$ as $x \to \infty$. In particular $\theta(x) \to \infty$ and $e^{-\theta(x)} \to 0$. Hence, $1 - e^{-\theta(x)} \to 1$

as $x \to \infty$, and since $\theta(x) \to \infty$, we also have

$$\frac{\theta(x)}{1 - e^{-\theta(x)}} \sim \theta(x) \sim \ln x \quad \text{as } x \to \infty.$$

Now focusing on:

$$\begin{aligned} A(x) &:= \frac{1}{1 - \bar{\beta}_H \eta} \left[\varepsilon_H + \bar{\beta}_H \eta \left(x\hat{p} - \frac{\kappa \theta(x)}{1 - e^{-\theta(x)}} \right) \right] \\ &\sim \frac{1}{1 - \bar{\beta}_H \eta} \left[\varepsilon_H + \bar{\beta}_H \eta \left(x\hat{p} - \kappa \ln x \right) \right] \quad \text{as } x \to \infty. \end{aligned}$$

Therefore, $A(x) = \Theta(x)$; that is, it grows linearly in x, with a logarithmic correction. Now define C and note that since $1 - e^{-\theta(x)} \to 1$ and $J(x) \to c > 0$ as $x \to \infty$, where c is a constant, that we have

$$\begin{aligned} C(x) &:= \frac{\mu - \bar{\beta}_L}{\bar{\beta}_L} \bar{\pi} \frac{x^2 \left[1 - e^{-\theta(x)} \right]}{\kappa J(x)} \\ &\sim \frac{\mu - \bar{\beta}_L}{\bar{\beta}_L} \cdot \frac{\bar{\pi}}{\kappa c} x^2 \quad \text{as } x \to \infty. \end{aligned}$$

Hence, $C(x) = \Theta(x^2)$ and the second term in $T(x)$ dominates the first in magnitude, and appears with a negative sign. Thus $\lim_{x \to \infty} T(x) \to -\infty$. Therefore, there must exist an x_H such that $T(x_H) < 0$, which occurs when $\bar{\beta} \leq \mu$. Now write

$$\tilde{T}(x) = DG(x) - EZ(x), \quad D = \frac{\bar{\beta}_H \eta}{1 - \bar{\beta}_H \eta} > 0, \quad E = \frac{\mu - \bar{\beta}_L}{\bar{\beta}_L} > 0,$$

where,

$$G(x) = x\hat{p} - \frac{\kappa \theta(x)}{1 - e^{-\theta(x)}},$$

and Z is defined above, hence $\tilde{T}'(x) = T'(x)$. Compute $G'(x)$ and evaluate $\theta'(x) = 1/x$, set

$$\begin{aligned} G'(x) &= \hat{p} - \frac{\kappa}{x} \frac{1 - e^{-\theta(x)} - \theta(x) e^{-\theta(x)}}{\left[1 - e^{-\theta(x)} \right]^2} \\ &= \hat{p} - \frac{\kappa}{x} h \left[\theta(x) \right], \end{aligned}$$

where $h(\theta) = \frac{1-e^{-\theta}-\theta e^{-\theta}}{\left[1-e^{-\theta}\right]^2}$. Notice that $h(\theta) < 1$ for all $\theta > 0$, since $e^{-\theta}\left(\theta + e^{-\theta} - 1\right) > 0$ for all $\theta > 0$. Hence $G'(x) > \hat{p} - \kappa/x > \underline{p} > 0$ for all $x > x_L = \kappa/\Gamma$. Compute $Z'(x)$, we have

$$Z(x) = \bar{\pi}\frac{\left[1 - e^{-\theta(x)}\right]}{I(x)} = \bar{\pi}\frac{x^2\left[1 - e^{-\theta(x)}\right]}{\kappa J(x)}.$$

Differentiate Z with respect to x to obtain

$$Z'(x) = \frac{\bar{\pi}x}{\kappa\left[J(x)\right]^2}\left\{J(x)\left[2 - e^{-\theta(x)}\right] - \frac{x\left[1 - e^{-\theta(x)}\right]}{\kappa\left(\hat{p} + \frac{\kappa}{x}\right)}\right\},$$

write $Z'(x) = \frac{\bar{\pi}x}{\kappa[J(x)]^2}f(x)$, where

$$f(x) = J(x)\left[2 - e^{-\theta(x)}\right] - \frac{x\left[1 - e^{-\theta(x)}\right]}{\kappa\left(\hat{p} + \frac{\kappa}{x}\right)}.$$

Differentiating $f(x)$ with respect to x yields

$$\begin{aligned} f'(x) &= \frac{J(x)e^{-\theta(x)}}{x} + \frac{x\left[2 - e^{-\theta(x)}\right]}{\kappa\left(\hat{p}x + \kappa\right)} - \frac{x\left[2 - e^{-\theta(x)}\right]}{\kappa\left(\hat{p}x + \kappa\right)} + \frac{\kappa\hat{p}x^2\left[1 - e^{-\theta(x)}\right]}{\left[\kappa\left(\hat{p}x + \kappa\right)\right]^2} \\ &= \frac{J(x)e^{-\theta(x)}}{x} + \frac{\kappa\hat{p}x^2\left[1 - e^{-\theta(x)}\right]}{\left[\kappa\left(\hat{p}x + \kappa\right)\right]^2} > 0. \end{aligned}$$

Now take the limit of $f(x)$ as $x \to x_L$:

$$\begin{aligned} \lim_{x\to x_L} f(x) &= \lim_{x\to x_L} J(x)\left[2 - e^{-\theta(x)}\right] - \lim_{x\to x_L}\frac{x\left[1 - e^{-\theta(x)}\right]}{\kappa\left(\hat{p} + \frac{\kappa}{x}\right)} \\ &\sim \lim_{x\to x_L} J(x)\left[2 - e^{-\theta(x)}\right] + \lim_{x\to x_L}\frac{x^2}{\kappa^2} \quad \text{as } x \to x_L \\ &\sim \frac{x_L^2}{\kappa^2} > 0 \quad \text{as } x \to x_L. \end{aligned}$$

Therefore $f(x) > 0$ for all $x > x_L$. Since $f'(x) > 0$, we have $f(x) > f(x_L) > 0$ for all $x > x_L$. Hence, $Z'(x) > 0$ for all $x > x_L$ and further $T'(x) < 0$ for all $x > x_L$. Thus for $\bar{\beta} < \mu < \bar{\mu}$, there exists a unique φ^m that satisfies $T(\varphi^m) = 0$, and $\varphi^m \in (\varphi_L^m, \varphi_H^m)$. ■

Proof of Proposition 3. For part (i), we consider the following

$$\frac{\partial \varphi^m}{\partial \mu} = -\frac{\frac{\partial T(\varphi^m)}{\partial \mu}}{T'(\varphi^m)} = -\frac{-Z(\varphi^m)}{T'(\varphi^m)} < 0,$$

where $T(\cdot)$ is the equilibrium map defined in the proof of Proposition 2, and $Z(\cdot)$ is defined in Lemma 9. For part (ii) differentiating the expression for ϕ^s in the statement of Proposition 2,

$$\frac{\partial \phi^s}{\partial \mu} = \frac{\bar{\pi}\bar{\beta}_H \eta}{1-\bar{\beta}_H \eta} \left\{ \hat{p} - \frac{\kappa}{\varphi^m} \cdot \frac{1-e^{-\theta}-\theta e^{-\theta}}{\left[1-e^{-\theta}\right]^2} \right\} \frac{\partial \varphi^m}{\partial \mu} < 0,$$

since,

$$\hat{p} - \frac{\kappa}{\varphi^m} \cdot \frac{1-e^{-\theta}-\theta e^{-\theta}}{\left[1-e^{-\theta}\right]^2} = \hat{p} - \frac{\kappa}{x} h\left(\theta\right),$$

where, again, $h(\theta) = \frac{1-e^{-\theta}-\theta e^{-\theta}}{\left(1-e^{-\theta}\right)^2} < 1$ for all $\theta > 0$, since $e^{-\theta}\left(\theta + e^{-\theta} - 1\right) > 0$ for all $\theta > 0$. Hence $\hat{p} - (\kappa/x)h(\theta) > \hat{p} - \kappa/x > \underline{p} > 0$ for all $x > x_L = \kappa/\Gamma$. ■

Proof of Proposition 4. Recalling the definition of $Z(\cdot)$ from the proof of Proposition 2, and noting that $\partial\theta/\partial\kappa = -1/\kappa < 0$. Differentiating Z with respect to κ yields

$$\begin{aligned}
\frac{\partial Z(x)}{\partial \kappa} &= -\frac{\bar{\pi}x^2}{\kappa^2\left[J(x)\right]^2} \left\{ J(x)e^{-\theta(x)} + \left[1-e^{-\theta(x)}\right]\left[\frac{\partial J(x)}{\partial \kappa} + J(x)\right]\right\} \\
&= -\frac{\bar{\pi}x^2}{\kappa^2\left[J(x)\right]^2} \left\{ J(x) + \left[1-e^{-\theta(x)}\right]\kappa\frac{\partial J(x)}{\partial \kappa}\right\}.
\end{aligned}$$

Using Leibnitz' rule, we have $\frac{\partial J(x)}{\partial \kappa} = -\frac{x^2}{\kappa^2(\hat{p}x+\kappa)}$, we can write

$$\begin{aligned}
f(x) &= J(x) + \left[1-e^{-\theta(x)}\right]\kappa\frac{\partial J(x)}{\partial \kappa} \\
&= J(x) - \frac{x^2\left[1-e^{-\theta(x)}\right]}{\kappa\left(\hat{p}x+\kappa\right)}.
\end{aligned}$$

Now differentiating with respect to κ

$$\begin{aligned}\frac{\partial f(x)}{\partial \kappa} &= -\frac{x^2}{\kappa^2(\hat{p}x+\kappa)} + \frac{e^{-\theta(x)}x^2}{\kappa^2(\hat{p}x+\kappa)} + \frac{x^2\left[1-e^{-\theta(x)}\right](\hat{p}x+2\kappa)}{\left[\kappa(\hat{p}x+\kappa)\right]^2} \\ &= \frac{\kappa x^4\left[1-e^{-\theta(x)}\right]}{\left[\kappa^2(\hat{p}x+\kappa)\right]^2} > 0,\end{aligned}$$

with "=" as $x \to x_L$. Notice also that $\lim_{x\to x_L} f(x) \to 0$. Therefore $f(x) > 0$ for all $x > x_L$. Since $f'(x) > 0$, we have $f(x) > f(x_L) = 0$ for all $x > x_L$. Thus, $\partial Z/\partial\kappa < 0$. For (i) implicit differentiation of $T(\varphi^m) = 0$ implies

$$\frac{\partial \varphi^m}{\partial \kappa} = -\frac{\frac{\partial T(\varphi^m)}{\partial \kappa}}{T'(\varphi^m)} = -\frac{-\frac{\kappa\bar{\beta}_H\eta}{1-\bar{\beta}_H\eta}\left[\frac{1-e^{-\theta}-\theta e^{-\theta}}{\left(1-e^{-\theta}\right)^2}\right]\frac{\partial\theta}{\partial\kappa} - \bar{\pi}\left(\frac{\mu-\bar{\beta}_L}{\bar{\beta}_L}\right)\frac{\partial Z}{\partial\kappa}}{T'(\varphi^m)} > 0,$$

where $T(\cdot)$ is the equilibrium map defined in the proof of Proposition 2, $T'(\cdot)$ is the derivative with respect to φ^m found in the proof of Proposition 2. For part (ii), differentiate the expression for ϕ^s with respect to κ in the statement of Proposition 2,

$$\frac{\partial \phi^s}{\partial \kappa} = \frac{\bar{\pi}\bar{\beta}_H\eta}{1-\bar{\beta}_H\eta}\left\{\hat{p}\frac{\partial\varphi^m}{\partial\kappa} - \left[\kappa h(\theta)\frac{\partial\theta}{\partial\kappa} + \frac{\theta}{1-e^{-\theta}}\right]\right\},$$

where $h(\theta) = \frac{1-e^{-\theta}-\theta e^{-\theta}}{\left[1-e^{-\theta}\right]^2} < 1$ for $\theta > 0$. Now, given $\theta = \ln(\varphi^m\Gamma/\kappa)$, differentiating yields

$$\begin{aligned}\frac{\partial\theta}{\partial\kappa} &= \frac{\kappa}{\varphi^m\Gamma}\left[\frac{\kappa\Gamma\frac{\partial\varphi^m}{\partial\kappa} - \varphi^m\Gamma}{\kappa^2}\right] \\ &= \frac{1}{\varphi^m}\frac{\partial\varphi^m}{\partial\kappa} - \frac{1}{\kappa}.\end{aligned}$$

Now write,

$$\begin{aligned}D &\equiv \hat{p}\frac{\partial\varphi^m}{\partial\kappa} - \left[\kappa h(\theta)\frac{\partial\theta}{\partial\kappa} + \frac{\theta}{1-e^{-\theta}}\right] \\ &= \hat{p}\frac{\partial\varphi^m}{\partial\kappa} - \kappa h(\theta)\left[\frac{1}{\varphi^m}\frac{\partial\varphi^m}{\partial\kappa} - \frac{1}{\kappa}\right] - \frac{\theta}{1-e^{-\theta}} \\ &= \left\{\hat{p} - \frac{\kappa}{\varphi^m}h(\theta)\right\}\frac{\partial\varphi^m}{\partial\kappa} + h(\theta) - \frac{\theta}{1-e^{-\theta}},\end{aligned}$$

again notice that $h(\theta) < \frac{\theta}{1-e^{-\theta}}$ since $1 - e^{-\theta} < \theta$, thus simplifying further, we have

$$\begin{aligned} D &= \left\{\hat{p} - \frac{\kappa}{\varphi^m} h(\theta)\right\} \frac{\partial \varphi^m}{\partial \kappa} + h(\theta) - \frac{\theta}{1-e^{-\theta}} \\ &= \frac{\varphi^m}{\kappa} \left\{ \left[\hat{p} - \frac{\kappa}{\varphi^m} h(\theta)\right] \left(\frac{\kappa}{\varphi^m}\frac{\partial \varphi^m}{\partial \kappa} - 1\right) + \hat{p} - \frac{\kappa\theta}{\varphi^m(1-e^{-\theta})} \right\} \\ &> \frac{\varphi^m}{\kappa} \left\{ \left[\hat{p} - \frac{\kappa\theta}{\varphi^m(1-e^{-\theta})}\right] \left(\frac{\kappa}{\varphi^m}\frac{\partial \varphi^m}{\partial \kappa} - 1\right) + \hat{p} - \frac{\kappa\theta}{\varphi^m(1-e^{-\theta})} \right\} \\ &= \left[\hat{p} - \frac{\kappa\theta}{\varphi^m(1-e^{-\theta})}\right] \frac{\partial \varphi^m}{\partial \kappa} > 0. \end{aligned}$$

Therefore $D > 0$, which implies $\partial \phi^s / \partial \kappa > 0$. ■

Proof of Proposition 5. For part (i) recall $\theta'(\varphi^m) = 1/\varphi^m > 0$.

Therefore $\partial\theta/\partial\mu = \theta'(\varphi^m)\partial\varphi^m/\partial\mu < 0$. For part (ii), Assumption 2 guarantees the result.

■

Proof of Proposition 6. We compute the derivatives of the best-ask distribution from Corollary 2 with respect to μ and κ. For part (i), differentiate $\bar{A}$ with respect to μ to obtain

$$\frac{\partial \bar{A}(p)}{\partial \mu} = \frac{\kappa(p-\hat{p})}{[\varphi^m(p-\hat{p})]^2} \cdot \frac{\partial \varphi^m}{\partial \mu} < 0$$

Similarly, for part (ii), differentiate $\bar{A}$ with respect to κ to obtain

$$\frac{\partial \bar{A}(p)}{\partial \kappa} = \frac{1}{\kappa\varphi^m(p-\hat{p})} \left(\frac{1}{\varphi^m}\frac{\partial \varphi^m}{\partial \kappa} - \frac{1}{\kappa}\right).$$

By Assumption 2, $\partial\bar{A}(p)/\partial\kappa < 0$, which delivers the proposition. ■

Proof of Proposition 6A. We compute the derivatives of the ask distribution from Proposition 1 with respect to μ and κ. For part (i), differentiate A with respect to μ to obtain

$$\begin{aligned} \frac{\partial A(p)}{\partial \mu} &= \frac{1}{\theta}\left\{\frac{1}{\varphi^m} - \frac{1}{\varphi^m\theta}\ln\left[\frac{\varphi^m(p-\hat{p})}{\kappa}\right]\right\}\frac{\partial \varphi^m}{\partial \mu} \\ &= \frac{1}{\theta}\left[\left(\frac{1}{\varphi^m} - \frac{1}{\varphi^m}\right) - \frac{1}{\varphi^m\theta}\ln\left(\frac{p-\hat{p}}{\Gamma}\right)\right]\frac{\partial \varphi^m}{\partial \mu} < 0, \end{aligned}$$

since we can write

$$\ln\left[\frac{\varphi^m(p-\hat{p})}{\kappa}\right] = \ln\left(\frac{p-\hat{p}}{\Gamma}\cdot\frac{\varphi^m\Gamma}{\kappa}\right) = \ln\left(\frac{p-\hat{p}}{\Gamma}\right) + \theta,$$

and since $p - \hat{p} < \Gamma$ for all $p \in [\hat{p} + \kappa/\varphi^m, \bar{p}]$ then $\ln[(p-\hat{p})/\Gamma] < 0$. Similarly, for part (ii), differentiate A with respect to κ to obtain

$$\frac{\partial A(p)}{\partial\kappa} = \frac{1}{\theta}\left\{\frac{\kappa}{\varphi^m(p-\hat{p})}\left[\frac{\kappa(p-\hat{p})\frac{\partial\varphi^m}{\partial\kappa} - \varphi^m(p-\hat{p})}{\kappa^2}\right]\right\} - \frac{1}{\theta^2}\frac{\partial\theta}{\partial\kappa}\ln\left[\frac{\varphi^m(p-\hat{p})}{\kappa}\right].$$

Simplifying, and noting

$$\ln\left[\frac{\varphi^m(p-\hat{p})}{\kappa}\right] = \ln\left(\frac{p-\hat{p}}{\Gamma}\cdot\frac{\varphi^m\Gamma}{\kappa}\right) = \ln\left(\frac{p-\hat{p}}{\Gamma}\right) + \theta,$$

we get

$$\begin{aligned}\frac{\partial A(p)}{\partial\kappa} &= \frac{1}{\varphi^m\theta}\frac{\partial\varphi^m}{\partial\kappa} - \frac{1}{\kappa\theta} - \frac{1}{\theta^2}\left(\frac{1}{\varphi^m}\frac{\partial\varphi^m}{\partial\kappa} - \frac{1}{\kappa}\right)\ln\left[\frac{\varphi^m(p-\hat{p})}{\kappa}\right]\\ &= -\frac{1}{\theta^2}\left(\frac{1}{\varphi^m}\frac{\partial\varphi^m}{\partial\kappa} - \frac{1}{\kappa}\right)\ln\left(\frac{p-\hat{p}}{\Gamma}\right).\end{aligned}$$

Again, since $p - \hat{p} < \Gamma$ for all $p \in [\hat{p} + \kappa/\varphi^m, \bar{p}]$ then $\ln[(p-\hat{p})/\Gamma] < 0$, and by Assumption 2, we have $\partial A(p)/\partial\kappa < 0$, which delivers the proposition. ■

Proof of Proposition 7. We compute the derivatives of the best-bid distribution from Corollary 2 with respect to μ and κ. For part (i), differentiate $\bar{B}$ with respect to μ to obtain

$$\frac{\partial\bar{B}(p)}{\partial\mu} = -\frac{\kappa(\hat{p}-p)}{[\varphi^m(\hat{p}-p)]^2}\cdot\frac{\partial\varphi^m}{\partial\mu} > 0$$

Similarly, for part (ii), differentiate $\bar{B}$ with respect to κ to obtain

$$\frac{\partial\bar{B}(p)}{\partial\kappa} = -\frac{\kappa}{\varphi^m(p-\hat{p})}\left(\frac{1}{\varphi^m}\frac{\partial\varphi^m}{\partial\kappa} - \frac{1}{\kappa}\right).$$

By Assumption 2, $\partial\bar{B}(p)/\partial\kappa > 0$, which delivers the proposition. ■

Proof of Proposition 7A. We compute the derivatives of the bid distribution from Proposition 1 with respect to μ and κ. For part (i), differentiate B with respect to μ to obtain

$$\begin{aligned}\frac{\partial B(p)}{\partial \mu} &= -\frac{1}{\theta}\left\{\frac{1}{\varphi^m} - \frac{1}{\varphi^m\theta}\ln\left[\frac{\varphi^m(\hat{p}-p)}{\kappa}\right]\right\}\frac{\partial \varphi^m}{\partial \mu} \\ &= -\frac{1}{\theta}\left[\left(\frac{1}{\varphi^m} - \frac{1}{\varphi^m}\right) - \frac{1}{\varphi^m\theta}\ln\left(\frac{\hat{p}-p}{\Gamma}\right)\right]\frac{\partial \varphi^m}{\partial \mu} > 0,\end{aligned}$$

since we can write

$$\ln\left[\frac{\varphi^m(\hat{p}-p)}{\kappa}\right] = \ln\left(\frac{\hat{p}-p}{\Gamma}\cdot\frac{\varphi^m\Gamma}{\kappa}\right) = \ln\left(\frac{\hat{p}-p}{\Gamma}\right) + \theta,$$

and since $\hat{p} - p < \Gamma$ then $\ln\left[(\hat{p}-p)/\Gamma\right] < 0$. Thus, we have $\partial B(p)/\partial\mu > 0$. Similarly, for part (ii), differentiate B with respect to κ to obtain

$$\frac{\partial B(p)}{\partial \kappa} = \frac{1}{\theta^2}\frac{\partial\theta}{\partial\kappa}\ln\left[\frac{\varphi^m(\hat{p}-p)}{\kappa}\right] - \frac{1}{\theta}\left\{\frac{\kappa}{\varphi^m(\hat{p}-p)}\left[\frac{\kappa(\hat{p}-p)\frac{\partial\varphi^m}{\partial\kappa} - \varphi^m(\hat{p}-p)}{\kappa^2}\right]\right\},$$

simplifying, and noting

$$\ln\left[\frac{\varphi^m(p-\hat{p})}{\kappa}\right] = \ln\left(\frac{\hat{p}-p}{\Gamma}\cdot\frac{\varphi^m\Gamma}{\kappa}\right) = \ln\left(\frac{\hat{p}-p}{\Gamma}\right) + \theta,$$

obtains

$$\begin{aligned}\frac{\partial B(p)}{\partial \kappa} &= -\frac{1}{\varphi^m\theta}\frac{\partial\varphi^m}{\partial\kappa} + \frac{1}{\kappa\theta} + \frac{1}{\theta^2}\left(\frac{1}{\varphi^m}\frac{\partial\varphi^m}{\partial\kappa} - \frac{1}{\kappa}\right)\ln\left[\frac{\varphi^m(p-\hat{p})}{\kappa}\right] \\ &= \frac{1}{\theta^2}\left(\frac{1}{\varphi^m}\frac{\partial\varphi^m}{\partial\kappa} - \frac{1}{\kappa}\right)\ln\left(\frac{p-\hat{p}}{\Gamma}\right).\end{aligned}$$

Again, since $p - \hat{p} < \Gamma$ for all $p \in [\hat{p} + \kappa/\varphi^m, \bar{p}]$ then $\ln\left[(p-\hat{p})/\Gamma\right] < 0$, and by Assumption 2, we have $\partial B(p)/\partial\kappa > 0$, which delivers the proposition. ■

Proof of Proposition 8. (i) With Corollary 2, the mean of p, $\mathcal{M}_a \equiv \mathbb{E}_a p$ with respect to the best-ask distribution becomes

$$
\begin{aligned}
\mathcal{M}_a &= \frac{1}{\bar{A}(\bar{p})}\int_{\hat{p}+\frac{\kappa}{\varphi^m}}^{\bar{p}} p d\bar{A}(p) \\
&= \frac{1}{\bar{A}(\bar{p})}\left[\bar{p}\bar{A}(\bar{p}) - \int_{\hat{p}+\frac{\kappa}{\varphi^m}}^{\bar{p}} \bar{A}(p)dp\right] \\
&= \frac{1}{\bar{A}(\bar{p})}\left\{\bar{p}\left(1-e^{-\theta}\right) - \left[\bar{p} - \left(\hat{p}+\frac{\kappa}{\varphi^m}\right)\right] + \frac{\kappa}{\varphi^m}\int_{\hat{p}+\frac{\kappa}{\varphi^m}}^{\bar{p}} \frac{1}{p-\hat{p}}dp\right\} \\
&= \frac{1}{1-e^{-\theta}}\left\{\bar{p}\left(1-e^{-\theta}\right) - \left(\bar{p}-\hat{p}\right)\left(1-e^{-\theta}\right) + \frac{\kappa}{\varphi^m}\int_{\hat{p}+\frac{\kappa}{\varphi^m}}^{\bar{p}} \frac{1}{p-\hat{p}}dp\right\} \\
&= \hat{p} + \frac{\kappa\theta}{\varphi^m\left(1-e^{-\theta}\right)},
\end{aligned}
$$

where we have $\bar{A}(\bar{p}) = 1 - \frac{\kappa}{\varphi^m\bar{p}-\varphi^m\hat{p}} = 1-e^{-\theta}$. To obtain the variance $\mathcal{V}_a \equiv \mathrm{Var}_a(p)$, use the fact that $\mathrm{Var}_a(p) = \mathbb{E}_a p^2 - \left(\mathbb{E}_a p\right)^2$. Compute $\mathbb{E}_a p^2$,

$$
\begin{aligned}
\mathbb{E}_a p^2 &= \frac{1}{\bar{A}(\bar{p})}\int_{\hat{p}+\frac{\kappa}{\varphi^m}}^{\bar{p}} p^2 d\bar{A}(p) \\
&= \frac{1}{\bar{A}(\bar{p})}\int_{\hat{p}+\frac{\kappa}{\varphi^m}}^{\bar{p}} \frac{\kappa p^2}{\varphi^m\left(p-\hat{p}\right)^2}dp.
\end{aligned}
$$

Again, using the distribution of the best ask $\bar{A}(p)$ in Corollary 2 and applying integration by substitution, we let $u = p - \hat{p}$, which implies $dp = du$ and yields

$$\begin{aligned}
\mathbb{E}_a p^2 &= \frac{\kappa}{\varphi^m \bar{A}(\bar{p})} \int_{\frac{\kappa}{\varphi^m}}^{\bar{p}-\hat{p}} \frac{(u+\hat{p})^2}{u^2} du \\
&= \frac{\kappa}{\varphi^m \bar{A}(\bar{p})} \int_{\frac{\kappa}{\varphi^m}}^{\bar{p}-\hat{p}} \left[1 + \frac{2\hat{p}}{u} + \frac{\hat{p}^2}{u^2}\right] du \\
&= \frac{\kappa}{\varphi^m \bar{A}(\bar{p})} \left[u + 2\hat{p}\ln u - \frac{\hat{p}^2}{u}\right]_{\frac{\kappa}{\varphi^m}}^{\bar{p}-\hat{p}} \\
&= \frac{\kappa}{\varphi^m \bar{A}(\bar{p})} \left\{2\hat{p}\ln\left[\frac{\varphi^m(\bar{p}-\hat{p})}{\kappa}\right] + (\bar{p}-\hat{p}) - \frac{\kappa}{\varphi^m} - \frac{\hat{p}^2}{(\bar{p}-\hat{p})} + \frac{\varphi^m \hat{p}^2}{\kappa}\right\} \\
&= \frac{\kappa}{\varphi^m \bar{A}(\bar{p})} \left\{2\hat{p}\theta + (\bar{p}-\hat{p})\left[1 - \frac{\kappa}{\varphi^m(\bar{p}-\hat{p})}\right] + \frac{\varphi^m \hat{p}^2}{\kappa}\left[1 - \frac{\kappa}{\varphi^m(\bar{p}-\hat{p})}\right]\right\} \\
&= \frac{\kappa}{\varphi^m \bar{A}(\bar{p})} \left\{2\hat{p}\theta + \left(\frac{\varphi^m \hat{p}^2}{\kappa} + \bar{p} - \hat{p}\right)\left[1 - \frac{\kappa}{\varphi^m(\bar{p}-\hat{p})}\right]\right\} \\
&= \frac{\kappa}{\varphi^m}\left[\frac{\hat{p}^2\varphi^m}{\kappa} + (\bar{p}-\hat{p})\right] + \frac{2\hat{p}\kappa\theta}{\varphi^m\left[1-e^{-\theta}\right]}.
\end{aligned}$$

Therefore the variance becomes

$$\begin{aligned}
\mathcal{V}_a &= \frac{\kappa}{\varphi^m}\left[\frac{\hat{p}^2\varphi^m}{\kappa} + (\bar{p}-\hat{p})\right] + \frac{2\hat{p}\kappa\theta}{\varphi^m(1-e^{-\theta})} - \left[\hat{p} + \frac{\kappa\theta}{\varphi^m(1-e^{-\theta})}\right]^2 \\
&= \frac{\kappa}{\varphi^m}\left[\frac{\hat{p}^2\varphi^m}{\kappa} + (\bar{p}-\hat{p})\right] - \hat{p}^2 - \left[\frac{\kappa\theta}{\varphi^m(1-e^{-\theta})}\right]^2 \\
&= \frac{\kappa(\bar{p}-\hat{p})}{\varphi^m} - \left[\frac{\kappa\theta}{\varphi^m(1-e^{-\theta})}\right]^2 \\
&= \frac{\kappa^2}{(\varphi^m)^2}\left[e^\theta - \left(\frac{\theta}{1-e^{-\theta}}\right)^2\right].
\end{aligned}$$

Differentiate $\mathcal{M}_a$ with respect to μ to obtain

$$\frac{\partial \mathcal{M}_a}{\partial \mu} = \frac{\kappa}{(\varphi^m)^2}\left[\frac{1-e^{-\theta}-\theta}{(1-e^{-\theta})^2}\right]\frac{\partial \varphi^m}{\partial \mu} > 0,$$

since $1 - e^{-\theta} < \theta$ for all $\theta > 0$, which is true from all $\varphi^m > \varphi^m_L$, and $\partial\varphi^m/\partial\mu < 0$. Now differentiate $\mathcal{M}_a$ with respect to κ to obtain

$$\begin{aligned}\frac{\partial \mathcal{M}_a}{\partial \kappa} &= \frac{\kappa}{\varphi^m\left(1-e^{-\theta}\right)}\left\{\frac{\partial \theta}{\partial \kappa} - \theta\left[\frac{1}{\varphi^m}\frac{\partial \varphi^m}{\partial \kappa} - \frac{1}{\kappa} + \frac{e^{-\theta}}{\left(1-e^{-\theta}\right)}\frac{\partial \theta}{\partial \kappa}\right]\right\} \\ &= \frac{\kappa}{\varphi^m}\left[\frac{1-e^{-\theta}-\theta}{\left(1-e^{-\theta}\right)^2}\right]\frac{\partial \theta}{\partial \kappa} > 0,\end{aligned}$$

since $1-e^{-\theta} < \theta$ for all $\theta > 0$, which is true from all $\varphi^m > \varphi^m_L$, and $\partial\theta/\partial\kappa < 0$. For part (ii) differentiate $\mathcal{V}_a$ with respect to μ to obtain

$$\begin{aligned}\frac{\partial \mathcal{V}_a}{\partial \mu} &= \frac{\kappa^2}{\left(\varphi^m\right)^2}\left\{\frac{e^\theta}{\varphi^m} - \frac{2}{\varphi^m}\left(\frac{\theta}{1-e^{-\theta}}\right)\left[\frac{1-e^{-\theta}-\theta e^{-\theta}}{\left(1-e^{-\theta}\right)^2}\right]\right\}\frac{\partial \varphi^m}{\partial \mu} - \frac{2\kappa^2}{\left(\varphi^m\right)^3}\left\{e^\theta - \left[\frac{\theta}{1-e^{-\theta}}\right]^2\right\}\frac{\partial \varphi^m}{\partial \mu} \\ &= \frac{\kappa^2}{\left(\varphi^m\right)^3}\left[2\left(\frac{\theta}{1-e^{-\theta}}\right)^2 - 2\left(\frac{\theta}{1-e^{-\theta}}\right)h\left(\theta\right) - e^\theta\right]\frac{\partial \varphi^m}{\partial \mu},\end{aligned}$$

where $h(\theta) = \frac{1-e^{-\theta}-\theta e^{-\theta}}{\left(1-e^{-\theta}\right)^2} < 1$ for all $\theta > 0$. Define

$$\begin{aligned}g(\theta) &= 2\left(\frac{\theta}{1-e^{-\theta}}\right)^2 - 2\left(\frac{\theta}{1-e^{-\theta}}\right)\left[\frac{1-e^{-\theta}-\theta e^{-\theta}}{\left(1-e^{-\theta}\right)^2}\right] - e^\theta \\ &= \frac{2\theta^2 - 2\theta + 2\theta e^{-\theta}}{(1-e^{-\theta})^3} - e^\theta \\ &= \frac{1}{(1-e^{-\theta})^3}\left[2\theta^2 - 2\theta + 2\theta e^{-\theta} - e^\theta(1-e^{-\theta})^3\right] \\ &< \frac{\theta}{(1-e^{-\theta})^3}\left[2(\theta - 1 + e^{-\theta}) - e^\theta\theta^2\right],\end{aligned}$$

since $1-e^{-\theta} < \theta$ for all $\theta > 0$. Define $\tilde{g}(\theta) = 2(\theta - 1 + e^{-\theta}) - e^\theta\theta^2$. Differentiate, $\tilde{g}$ to obtain

$$\tilde{g}'(\theta) = 2(1-e^{-\theta}) - e^\theta(2\theta + \theta^2) < 0.$$

Therefore, since $\tilde{g}(0) = 0$ and $\tilde{g}$ is a decreasing function, it follows that $\tilde{g}(\theta) \leq 0$ for all $\theta \geq 0$, which implies that $g(\theta) < 0$ for all $\theta > 0$. Therefore we have $\partial\mathcal{V}_a/\partial\mu > 0$. Now differentiate $\mathcal{V}_a$ with respect to κ to obtain

$$\begin{aligned}\frac{\partial \mathcal{V}_a}{\partial \kappa} &= \frac{\kappa^2}{\left(\varphi^m\right)^2}\left\{e^\theta - 2\left(\frac{\theta}{1-e^{-\theta}}\right)\left[\frac{1-e^{-\theta}-\theta e^{-\theta}}{\left(1-e^{-\theta}\right)^2}\right]\right\}\frac{\partial \theta}{\partial \kappa} - \frac{2\kappa^2}{\left(\varphi^m\right)^2}\left[e^\theta - \left(\frac{\theta}{1-e^{-\theta}}\right)^2\right]\frac{\partial \theta}{\partial \kappa} \\ &= \frac{\kappa^2}{\left(\varphi^m\right)^2}\left[2\left(\frac{\theta}{1-e^{-\theta}}\right)^2 - 2\left(\frac{\theta}{1-e^{-\theta}}\right)h\left(\theta\right) - e^\theta\right]\frac{\partial \theta}{\partial \kappa},\end{aligned}$$

where $h(\theta) = \frac{1-e^{-\theta}-\theta e^{-\theta}}{\left(1-e^{-\theta}\right)^2} < 1$ for all $\theta > 0$. Again, recognise

$$\begin{aligned} g(\theta) &= 2\left(\frac{\theta}{1-e^{-\theta}}\right)^2 - 2\left(\frac{\theta}{1-e^{-\theta}}\right)\left[\frac{1-e^{-\theta}-\theta e^{-\theta}}{(1-e^{-\theta})^2}\right] - e^{\theta} \\ &= \frac{2\theta^2 - 2\theta + 2\theta e^{-\theta}}{(1-e^{-\theta})^3} - e^{\theta} \\ &= \frac{1}{(1-e^{-\theta})^3}\left[2\theta^2 - 2\theta + 2\theta e^{-\theta} - e^{\theta}(1-e^{-\theta})^3\right] \\ &< \frac{\theta}{(1-e^{-\theta})^3}\left[2(\theta - 1 + e^{-\theta}) - e^{\theta}\theta^2\right], \end{aligned}$$

since $1 - e^{-\theta} < \theta$ for all $\theta > 0$. Define $\tilde{g}(\theta) = 2(\theta - 1 + e^{-\theta}) - e^{\theta}\theta^2$. Differentiate, $\tilde{g}$ to obtain

$$\tilde{g}'(\theta) = 2(1 - e^{-\theta}) - e^{\theta}(2\theta + \theta^2) < 0.$$

Therefore, since $\tilde{g}(0) = 0$ and $\tilde{g}$ is a decreasing function, it follows that $\tilde{g}(\theta) \leq 0$ for all $\theta \geq 0$, which implies that $g(\theta) < 0$ for all $\theta > 0$. Therefore by Assumption 2, we have $\partial\mathcal{V}_a/\partial\kappa > 0$. ■

Proof of Proposition 9. (i) With Corollary 2, the mean of p, $\mathcal{M}_b \equiv \mathbb{E}_b p$ with respect to the best-bid distribution becomes

$$\begin{aligned} \mathcal{M}_b &= \frac{1}{1-\bar{B}(\underline{p})}\int_{\underline{p}}^{\hat{p}-\frac{\kappa}{\varphi^m}} p d\bar{B}(p) \\ &= \frac{1}{1-\bar{B}(\underline{p})}\left[\hat{p} - \frac{\kappa}{\varphi^m} - \underline{p}\bar{B}\left(\underline{p}\right) - \int_{\underline{p}}^{\hat{p}-\frac{\kappa}{\varphi^m}} \bar{B}(p)dp\right] \\ &= \hat{p} - \frac{\kappa\theta}{\varphi^m\left[1-e^{-\theta}\right]}. \end{aligned}$$

The details of the last step can be found in the proof of Lemma 9. To obtain the variance $\mathcal{V}_b \equiv \mathrm{Var}_b(p)$, use the fact that $\mathrm{Var}_b(p) = \mathbb{E}_b p^2 - (\mathbb{E}_b p)^2$. Compute $\mathbb{E}_b p^2$,

$$\begin{aligned} \mathbb{E}_b p^2 &= \frac{1}{1-\bar{B}(\underline{p})}\int_{\underline{p}}^{\hat{p}-\frac{\kappa}{\varphi^m}} p^2 d\bar{B}(p) \\ &= \frac{1}{1-\bar{B}(\underline{p})}\int_{\underline{p}}^{\hat{p}-\frac{\kappa}{\varphi^m}} \frac{\kappa p^2}{\varphi^m\left(\hat{p}-p\right)^2}dp. \end{aligned}$$

Again, using the distribution of the best bid $\bar{B}(p)$ in Corollary 2 and applying integration by substitution, we let $u = \hat{p} - p$, which implies $dp = -du$ and yields

$$\begin{aligned}
\mathbb{E}_b p^2 &= \frac{\kappa}{\varphi^m \left[1 - \bar{B}(\underline{p})\right]} \int_{\frac{\kappa}{\varphi^m}}^{\hat{p}-\underline{p}} \frac{(\hat{p}-u)^2}{u^2} du \\
&= \frac{\kappa}{\varphi^m \left[1 - \bar{B}(\underline{p})\right]} \int_{\frac{\kappa}{\varphi^m}}^{\hat{p}-\underline{p}} \left[\frac{\hat{p}^2}{u^2} - \frac{2\hat{p}}{u} + 1\right] du \\
&= \frac{\kappa}{\varphi^m \left[1 - \bar{B}(\underline{p})\right]} \left[-\frac{\hat{p}^2}{u} - 2\hat{p}\ln(u) + u\right] \Bigg|_{\frac{\kappa}{\varphi^m}}^{\hat{p}-\underline{p}} \\
&= \frac{\kappa}{\varphi^m \left[1 - \bar{B}(\underline{p})\right]} \left\{-2\hat{p}\ln\left[\frac{\varphi^m(\hat{p}-\underline{p})}{\kappa}\right] - \frac{\hat{p}^2}{(\hat{p}-\underline{p})} + \frac{\hat{p}^2\varphi^m}{\kappa} - \frac{\kappa}{\varphi^m} + (\hat{p}-\underline{p})\right\} \\
&= \frac{\kappa}{\varphi^m \left[1 - \bar{B}(\underline{p})\right]} \left\{-2\hat{p}\theta + \frac{\hat{p}^2\varphi^m}{\kappa}\left[1 - \frac{\kappa}{\varphi^m(\hat{p}-\underline{p})}\right] + (\hat{p}-\underline{p})\left[1 - \frac{\kappa}{\varphi^m(\hat{p}-\underline{p})}\right]\right\} \\
&= \frac{\kappa}{\varphi^m \left[1 - \bar{B}(\underline{p})\right]} \left\{-2\hat{p}\theta + \left[\frac{\hat{p}^2\varphi^m}{\kappa} + (\hat{p}-\underline{p})\right]\left(1 - e^{-\theta}\right)\right\} \\
&= \frac{\kappa}{\varphi^m}\left[\frac{\hat{p}^2\varphi^m}{\kappa} + (\hat{p}-\underline{p})\right] - \frac{2\hat{p}\kappa\theta}{\varphi^m\left(1-e^{-\theta}\right)}.
\end{aligned}$$

Therefore the variance becomes

$$\begin{aligned}
\mathcal{V}_b &= \frac{\kappa}{\varphi^m}\left[\frac{\hat{p}^2\varphi^m}{\kappa} + (\hat{p}-\underline{p})\right] - \frac{2\hat{p}\kappa\theta}{\varphi^m\left(1-e^{-\theta}\right)} - \left[\hat{p} - \frac{\kappa\theta}{\varphi^m\left(1-e^{-\theta}\right)}\right]^2 \\
&= \frac{\kappa}{\varphi^m}\left[\frac{\hat{p}^2\varphi^m}{\kappa} + (\hat{p}-\underline{p})\right] - \hat{p}^2 - \left[\frac{\kappa\theta}{\varphi^m\left(1-e^{-\theta}\right)}\right]^2 \\
&= \frac{\kappa(\hat{p}-\underline{p})}{\varphi^m} - \left[\frac{\kappa\theta}{\varphi^m\left(1-e^{-\theta}\right)}\right]^2 \\
&= \frac{\kappa^2}{(\varphi^m)^2}\left[e^\theta - \left(\frac{\theta}{1-e^{-\theta}}\right)^2\right].
\end{aligned}$$

Differentiate $\mathcal{M}_b$ with respect to μ to obtain

$$\frac{\partial \mathcal{M}_b}{\partial \mu} = -\frac{\kappa}{(\varphi^m)^2}\left[\frac{1-e^{-\theta}-\theta}{\left(1-e^{-\theta}\right)^2}\right]\frac{\partial \varphi^m}{\partial \mu} < 0$$

since $1-e^{-m} < m$ for all $m > 0$, which is true from all $\varphi^m > \varphi^m_L$, and $\partial\varphi^m/\partial\mu < 0$. Differentiate

$\mathcal{M}_b$ with respect to κ to obtain

$$\begin{aligned}\frac{\partial \mathcal{M}_b}{\partial \kappa} &= -\frac{\kappa}{\varphi^m \left(1-e^{-\theta}\right)} \left\{ \frac{\partial \theta}{\partial \kappa} - \theta \left[\frac{1}{\varphi^m} \frac{\partial \varphi^m}{\partial \kappa} - \frac{1}{\kappa} + \frac{e^{-\theta}}{\left(1-e^{-\theta}\right)} \frac{\partial \theta}{\partial \kappa} \right] \right\} \\ &= -\frac{\kappa}{\varphi^m} \left[\frac{1-e^{-\theta}-\theta}{\left(1-e^{-\theta}\right)^2} \right] \frac{\partial \theta}{\partial \kappa} < 0,\end{aligned}$$

since $\theta > 1 - e^{-\theta}$ and $\partial\theta/\partial\kappa < 0$. For part (*ii*) notice that $\mathrm{Var}_a(p) = \mathrm{Var}_b(p)$, therefore $\partial\mathcal{V}_a/\partial\mu = \partial\mathcal{V}_b/\partial\mu$ and $\partial\mathcal{V}_a/\partial\kappa = \partial\mathcal{V}_b/\partial\kappa$. ■

Corollary 7. *If α_k is Poisson with mean $\theta \in \mathbb{R}_+$ and given Assumption 2, then in the recursive monetary equilibrium, the coefficient-of-variation of the best-bid distribution CV_b is increasing in both μ and κ.*

Proof. Immediate from Proposition 9. ■

Proof of Proposition 10. Given the expected value of p with respect to the best-ask distribution and the best-bid distribution respectively,

$$\mathcal{S}^m = |\mathcal{M}_a - \mathcal{M}_b| = \frac{2\kappa\theta}{\varphi^m \left(1-e^{-\theta}\right)}.$$

For part (*i*) differentiate $\mathcal{S}^m$ with respect to μ to obtain

$$\frac{\partial \mathcal{S}^m}{\partial \mu} = \frac{2\kappa}{\left(\varphi^m\right)^2} \left[\frac{1-e^{-\theta}-\theta}{\left(1-e^{-\theta}\right)^2} \right] \frac{\partial \varphi^m}{\partial \mu} > 0,$$

since $\theta > 1 - e^{-\theta}$ and $\partial\varphi^m/\partial\mu < 0$. For part (*ii*) differentiate $\mathcal{S}^m$ with respect to κ to obtain

$$\frac{\partial \mathcal{S}^m}{\partial \kappa} = \frac{2\kappa}{\left(\varphi^m\right)^2} \left[\frac{1-e^{-\theta}-\theta}{\left(1-e^{-\theta}\right)^2} \right] \frac{\partial \theta}{\partial \kappa} > 0,$$

since $\theta > 1 - e^{-\theta}$ and $\partial\theta/\partial\kappa < 0$. ■

Corollary 8. *If α_k is Poisson with mean $\theta \in \mathbb{R}_+$ and given Assumption 2, then in the recursive monetary equilibrium, the real bid-ask spread $\mathcal{S}$: (i) decreases with inflation, i.e., $\partial\mathcal{S}/\partial\mu < 0$; and (ii) increases with the posting cost, i.e., $\partial\mathcal{S}/\partial\kappa > 0$.*

Proof. (i) Recognise that $\mathcal{S} \equiv \varphi^m \mathcal{S}^m = \frac{2\kappa\theta}{1-e^{-\theta}}$ Differentiate $\mathcal{S}$ with respect to μ to obtain

$$\frac{\partial \mathcal{S}}{\partial \mu} = \frac{2\kappa}{(1-e^{-\theta})^2}\left(1 - e^{-\theta} - \theta e^{-\theta}\right)\frac{\partial \theta}{\partial \mu} < 0.$$

Since for $\theta > 0$, we set $f(\theta) = 1 - (1+\theta)e^{-\theta}$; and $f(0) = 0$ and $f'(\theta) = \theta e^{-\theta} > 0$, so $f(\theta) > 0$, i.e. $1 - e^{-\theta} - \theta e^{-\theta} > 0$. For part (ii) differentiate $\mathcal{S}$ with respect to κ to obtain

$$\begin{aligned}
\frac{\partial \mathcal{S}}{\partial \kappa} &= \frac{2\left\{\left(1-e^{-\theta}\right)\left[\kappa \frac{\partial \theta}{\partial \kappa} + \theta\right] - \kappa\theta\left[e^{-\theta}\frac{\partial \theta}{\partial \kappa}\right]\right\}}{\left(1-e^{-\theta}\right)^2} \\
&= \frac{2\kappa}{1-e^{-\theta}}\left\{\frac{\partial \theta}{\partial \kappa} - \theta\left[-\frac{1}{\kappa} + \frac{e^{-\theta}}{1-e^{-\theta}}\frac{\partial \theta}{\partial \kappa}\right]\right\} \\
&= \frac{2\kappa}{1-e^{-\theta}}\left\{\frac{\partial \theta}{\partial \kappa} - \theta\left[\frac{1}{\varphi^m}\frac{\partial \varphi^m}{\partial \mu} - \frac{1}{\kappa} - \frac{1}{\varphi^m}\frac{\partial \varphi^m}{\partial \mu} + \frac{e^{-\theta}}{1-e^{-\theta}}\frac{\partial \theta}{\partial \kappa}\right]\right\} \\
&= \frac{2\kappa}{1-e^{-\theta}}\left\{\frac{\partial \theta}{\partial \kappa} + \frac{\theta}{\varphi^m}\frac{\partial \varphi^m}{\partial \kappa} - \theta\left[\frac{\partial \theta}{\partial \kappa} + \frac{e^{-\theta}}{1-e^{-\theta}}\frac{\partial \theta}{\partial \kappa}\right]\right\} \\
&= \frac{2\kappa}{1-e^{-\theta}}\left(1 - \theta - \frac{\theta e^{-\theta}}{1-e^{-\theta}}\right)\frac{\partial \theta}{\partial \kappa} + \frac{2\kappa\theta}{\varphi^m(1-e^{-\theta})}\frac{\partial \varphi^m}{\partial \kappa} \\
&= \frac{2\kappa}{\left(1-e^{-\theta}\right)^2}\left(1 - \theta - e^{-\theta}\right)\frac{\partial \theta}{\partial \kappa} + \frac{2\kappa\theta}{\varphi^m(1-e^{-\theta})}\frac{\partial \varphi^m}{\partial \kappa} > 0.
\end{aligned}$$

Since $\theta > 1 - e^{-\theta}$ for all $\theta > 0$ and $\partial\theta/\partial\kappa < 0$. ■

Proof of Proposition 11. The proportion of investors that receive no quotes is given by α_0, which corresponds to the flash crash probability, therefore the probability of a flash crash is given by $\mathcal{F} \equiv \alpha_0 = e^{-\theta}$. For part (i) and (ii) differentiate with respect to μ and κ, respectively, to obtain

$$\frac{\partial \mathcal{F}}{\partial \mu} = -e^{-\theta}\frac{\partial \theta}{\partial \mu} > 0 \quad \text{and} \quad \frac{\partial \mathcal{F}}{\partial \kappa} = -e^{-\theta}\frac{\partial \theta}{\partial \kappa} > 0,$$

since $\partial\theta/\partial\mu < 0$ and $\partial\theta/\partial\kappa < 0$. ■

Proof of Proposition 11. From condition (D.30) in Lemma 9, and where $\hat{\phi}^s \equiv \left[\bar{\beta}\eta/(1-\bar{\beta}\eta)\right]\bar{\varepsilon}$, the speculative premium is defined as

$$\begin{aligned}\mathcal{P} &= \phi^s - \hat{\phi}^s \\ &= \frac{\bar{\beta}_H \eta}{1-\bar{\beta}_H \eta}\left\{\varepsilon_H + \frac{\pi_L}{1-\pi_L}\left[\varphi^m \hat{p} - \frac{\kappa\theta}{1-e^{-\theta}}\right]\right\} - \frac{\bar{\beta}\eta}{1-\bar{\beta}\eta}\bar{\varepsilon}\end{aligned}$$

For part (i) differentiate $\mathcal{P}$ with respect to μ to obtain

$$\frac{\partial \mathcal{P}}{\partial \mu} = \frac{\bar{\pi}\bar{\beta}_H \eta}{1-\bar{\beta}_H \eta}\left\{\hat{p} - \frac{\kappa}{\varphi^m}\cdot\frac{1-e^{-\theta}-\theta e^{-\theta}}{\left[1-e^{-\theta}\right]^2}\right\}\frac{\partial \varphi^m}{\partial \mu} < 0,$$

since,

$$\hat{p} - \frac{\kappa}{\varphi^m}\cdot\frac{1-e^{-\theta}-\theta e^{-\theta}}{\left[1-e^{-\theta}\right]^2} = \hat{p} - \frac{\kappa}{x}h(\theta),$$

where, again, $h(\theta) = \frac{1-e^{-\theta}-\theta e^{-\theta}}{\left(1-e^{-\theta}\right)^2} < 1$ for all $\theta > 0$, since $e^{-\theta}\left(\theta + e^{-\theta} - 1\right) > 0$ for all $\theta > 0$. Hence $\hat{p} - (\kappa/x)h(\theta) > \hat{p} - \kappa/x > \underline{p} > 0$ for all $x > x_L = \kappa/\Gamma$. For (ii) differentiate $\mathcal{P}$ with respect to κ to obtain

$$\frac{\partial \mathcal{P}}{\partial \kappa} = \frac{\bar{\pi}\bar{\beta}_H \eta}{1-\bar{\beta}_H \eta}\left\{\hat{p}\frac{\partial \varphi^m}{\partial \kappa} - \left[\kappa h(\theta)\frac{\partial \theta}{\partial \kappa} + \frac{\theta}{1-e^{-\theta}}\right]\right\},$$

where $h(\theta) = \frac{1-e^{-\theta}-\theta e^{-\theta}}{\left[1-e^{-\theta}\right]^2} < 1$ for $\theta > 0$. Now, given $\theta = \ln(\varphi^m \Gamma/\kappa)$, differentiating yields

$$\begin{aligned}\frac{\partial \theta}{\partial \kappa} &= \frac{\kappa}{\varphi^m \Gamma}\left[\frac{\kappa\Gamma\frac{\partial \varphi^m}{\partial \kappa} - \varphi^m \Gamma}{\kappa^2}\right] \\ &= \frac{1}{\varphi^m}\frac{\partial \varphi^m}{\partial \kappa} - \frac{1}{\kappa}.\end{aligned}$$

Now write,

$$\begin{aligned}D &\equiv \hat{p}\frac{\partial \varphi^m}{\partial \kappa} - \left[\kappa h(\theta)\frac{\partial \theta}{\partial \kappa} + \frac{\theta}{1-e^{-\theta}}\right] \\ &= \hat{p}\frac{\partial \varphi^m}{\partial \kappa} - \kappa h(\theta)\left[\frac{1}{\varphi^m}\frac{\partial \varphi^m}{\partial \kappa} - \frac{1}{\kappa}\right] - \frac{\theta}{1-e^{-\theta}} \\ &= \left\{\hat{p} - \frac{\kappa}{\varphi^m}h(\theta)\right\}\frac{\partial \varphi^m}{\partial \kappa} + h(\theta) - \frac{\theta}{1-e^{-\theta}},\end{aligned}$$

again notice that $h(\theta) < \frac{\theta}{1-e^{-\theta}}$ since $1 - e^{-\theta} < \theta$, thus simplifying further, we have

$$\begin{aligned}
D &= \left\{\hat{p} - \frac{\kappa}{\varphi^m} h(\theta)\right\} \frac{\partial \varphi^m}{\partial \kappa} + h(\theta) - \frac{\theta}{1 - e^{-\theta}} \\
&= \frac{\varphi^m}{\kappa} \left\{ \left[\hat{p} - \frac{\kappa}{\varphi^m} h(\theta)\right] \left(\frac{\kappa}{\varphi^m} \frac{\partial \varphi^m}{\partial \kappa} - 1\right) + \hat{p} - \frac{\kappa\theta}{\varphi^m(1 - e^{-\theta})} \right\} \\
&> \frac{\varphi^m}{\kappa} \left\{ \left[\hat{p} - \frac{\kappa\theta}{\varphi^m(1 - e^{-\theta})}\right] \left(\frac{\kappa}{\varphi^m} \frac{\partial \varphi^m}{\partial \kappa} - 1\right) + \hat{p} - \frac{\kappa\theta}{\varphi^m(1 - e^{-\theta})} \right\} \\
&= \left[\hat{p} - \frac{\kappa\theta}{\varphi^m(1 - e^{-\theta})}\right] \frac{\partial \varphi^m}{\partial \kappa} > 0.
\end{aligned}$$

Therefore $D > 0$, which implies $\partial \mathcal{P}/\partial \kappa > 0$. ■

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