

STRONG GENERATORS IN $\mathbf{D}^{\mathrm{perf}}(X)$ AND $\mathbf{D}_{\mathrm{coh}}^b(X)$

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ABSTRACT. We solve two open problems: first we prove a conjecture of Bondal and Van den Bergh, showing that the category $\mathbf{D}^{\mathrm{perf}}(X)$ is strongly generated whenever X is a quasicompact, separated scheme, admitting a cover by open affine subsets $\mathrm{Spec}(R_i)$ with each R_i of finite global dimension. We also prove that, for a noetherian scheme X of finite type over an excellent scheme of dimension ≤ 2 , the derived category $\mathbf{D}_{\mathrm{coh}}^b(X)$ is strongly generated. The known results in this direction all assumed equal characteristic, we have no such restriction.

The method is interesting in other contexts: our key lemmas turn out to give a simple proof that, if $f : X \rightarrow Y$ is a separated morphism of quasicompact, quasiseparated schemes such that $\mathbf{R}f_* : \mathbf{D}_{\mathrm{qc}}(X) \rightarrow \mathbf{D}_{\mathrm{qc}}(Y)$ takes perfect complexes to complexes of bounded-below Tor-amplitude, then f must be of finite Tor-dimension.

CONTENTS

0. Introduction	1
1. Background	14
2. Unbounded versions of Theorems 0.5 and 0.15	21
3. An object $G \in \mathbf{D}_{\mathrm{coh}}^b(X)$ generating $\mathbf{D}_{\mathrm{qc}}(X)$ in finitely many steps	24
4. Approximation in the case of quasiprojective schemes	27
5. Approximation in the general case	29
6. Proofs of Theorems 0.18 and 2.1	33
References	37

0. INTRODUCTION

Let $\mathcal{T}$ be a triangulated category and $G \in \mathcal{T}$ an object. Bondal and Van den Bergh [7, 2.2] made a string of definitions, regarding what it means for G to generate $\mathcal{T}$, and we briefly remind the reader.

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Reminder 0.1. Define the full subcategory $\langle G \rangle_1 \subset \mathcal{T}$ to consist of all direct summands of finite coproducts of suspensions of G . For integers $n \geq 1$ we inductively define subcategories $\langle G \rangle_n$: an object lies in $\langle G \rangle_{n+1}$ if it is a direct summand of an object y admitting a triangle $x \rightarrow y \rightarrow z \rightarrow$ with $x \in \langle G \rangle_1$ and $z \in \langle G \rangle_n$. The object G is a *classical generator* if $\mathcal{T} = \bigcup_{n=1}^{\infty} \langle G \rangle_n$, and a *strong generator* if there exists an n with $\mathcal{T} = \langle G \rangle_n$. The category $\mathcal{T}$ is called *regular* if a strong generator exists. Note that we are following the terminology of Orlov [41], elsewhere in the literature what we call *regular* would go by the name *strongly generated*.

Strong generators are particularly useful in triangulated categories proper over a noetherian, commutative ring R . We remind the reader of the definition of properness.

Reminder 0.2. Let R be a noetherian, commutative ring and let $\mathcal{T}$ be an R -linear triangulated category. The R -linearity of $\mathcal{T}$ means that the Hom-sets $\mathcal{T}(X, Y)$ are all R -modules, and the composition maps $\mathcal{T}(X, Y) \times \mathcal{T}(Y, Z)$ are all R -bilinear.

We say that the category $\mathcal{T}$ is *proper over R* if, for each pair of objects $X, Y \in \mathcal{T}$, the direct sum $\bigoplus_{i=-\infty}^{\infty} \mathcal{T}(X, \Sigma^i Y)$ is a finite R -module. This of course implies that $\mathcal{T}(X, \Sigma^i Y)$ vanishes for all but finitely many $i \in \mathbb{Z}$.

A key theorem that shows the usefulness of the definitions above is:

Theorem 0.3. *Let R be a noetherian, commutative ring, let $\mathcal{T}$ be a regular triangulated category proper over R , and suppose furthermore that $\mathcal{T}$ is Karoubian, meaning idempotents split. Then an R -linear functor $H : \mathcal{T} \rightarrow R\text{-Mod}$ is representable if and only if*

- (i) *H is homological.*
- (ii) *For any object $X \in \mathcal{T}$, the direct sum $\bigoplus_{i=-\infty}^{\infty} H(\Sigma^i X)$ is a finite R -module.*

When R is a field Theorem 0.3 is due to Bondal and Van den Bergh [7, Theorem 1.3], and in the generality above it may be found in Rouquier [42, Theorem 4.16 and Corollary 4.18].

Example 0.4. In view of Theorem 0.3 it becomes interesting to find examples of regular, Karoubian triangulated categories proper over a noetherian ring R . Let us begin with Karoubian: if X is a quasicompact, quasiseparated scheme then the category $\mathbf{D}^{\text{perf}}(X)$, of perfect complexes over X , is well-known to be Karoubian.

Now for examples of proper triangulated categories: if X is of finite type and separated over a noetherian ring R we have the equivalence below, which explains the terminology

$$\{X \text{ proper over } R\} \iff \{\mathbf{D}^{\text{perf}}(X) \text{ proper over } R\}.$$

If R is a field there is a simple, direct proof in Orlov [41, Proposition 3.30]. The general case may be proved by similar methods, using the curve-selection result in Lipman [28, Exercise 4.3.9]. See [29, Corollary 4.3.2] or Lemma 0.20 below for the argument.

For all the author knows there might be examples of schemes X for which $\mathbf{D}^{\text{perf}}(X)$ is proper over R , without X being of finite type and separated over R .

Now it's time to discuss examples of regular triangulated categories. This leads us to the first main theorem of the article, conjectured by Bondal and Van den Bergh (see Remark 0.7).

Theorem 0.5. *Let X be a quasicompact, separated scheme. Then $\mathbf{D}^{\text{perf}}(X)$ is regular if and only if X can be covered by open affine subschemes $\text{Spec}(R_i)$ with each R_i of finite global dimension.*

Remark 0.6. In particular: if X is a separated scheme, of finite type over a noetherian ring R of finite Krull dimension, then

$$\{X \text{ regular}\} \iff \{\mathbf{D}^{\text{perf}}(X) \text{ regular}\}.$$

Combining Theorem 0.5 with Example 0.4 tells us that, as long as X is a regular scheme proper over a noetherian ring R of finite Krull dimension, the category $\mathbf{D}^{\text{perf}}(X)$ is Karoubian, regular and proper over R . This gives lots of examples to which Theorem 0.3 applies.

Remark 0.7. We should say something about the history of Theorem 0.5. Bondal and Van den Bergh [7, Theorem 3.1.4] proved the special case where X is smooth over a field k . But in the sentence immediately following [7, Theorem 3.1.4] they go on to say: “Presumably the theorem is true under the weaker hypothesis that X is noetherian and regular”. Hence we view Theorem 0.5, in the generality proved here, as having been conjectured by Bondal and Van den Bergh.

The case where X is regular and of finite type over a field k follows from either Rouquier [42, Theorem 7.38] or Orlov [41, Theorem 3.27]. Note that all three results mentioned so far—that is [7, Theorem 3.1.4], [42, Theorem 7.38] and [41, Theorem 3.27]—assume equal characteristic.

There is an old result by Kelly [24], giving Theorem 0.5 in the special case where X is affine. Kelly's theorem is the only one prior to this article which works in mixed characteristic. But affine schemes are rarely proper over a noetherian ring R , hence Kelly's theorem does not produce interesting $\mathcal{T}$'s to which one could apply Theorem 0.3. Our theorem is the first to produce a large slew of geometric triangulated categories which are Karoubian, regular, and proper over a noetherian ring R of mixed characteristic.

The reader might be curious why one cares about results of the type described above. One of the major consequences of such theorems is

Corollary 0.8. *Let $\mathcal{T}$ be a regular, Karoubian triangulated category proper over a noetherian, commutative ring R . Let $\mathcal{S} \subset \mathcal{T}$ be a triangulated subcategory (in particular full). If the inclusion functor $I : \mathcal{S} \rightarrow \mathcal{T}$ has either a right or a left adjoint then it has both adjoints, and $\mathcal{S}$ is also Karoubian, regular and proper over R .*

For the purpose of the proof of Corollary 0.8 we recall that, if A is a class of objects in $\mathcal{T}$, then $A^\perp$ and ${}^\perp A$ are defined to be the full subcategories of $\mathcal{T}$ whose objects are

determined by the rules

$$\begin{aligned} A^\perp &= \{t \in \mathcal{T} \mid \mathcal{T}(\Sigma^i a, t) = 0 \ \forall i \in \mathbb{Z}, \forall a \in A\}, \\ {}^\perp A &= \{t \in \mathcal{T} \mid \mathcal{T}(t, \Sigma^i a) = 0 \ \forall i \in \mathbb{Z}, \forall a \in A\}. \end{aligned}$$

Proof. The properties of being Karoubian, regular and proper over R are true for $\mathcal{T}$ if and only if they are true for $\mathcal{T}^{\text{op}}$. Replacing $I : \mathcal{S} \longrightarrow \mathcal{T}$ by $I^{\text{op}} : \mathcal{S}^{\text{op}} \longrightarrow \mathcal{T}^{\text{op}}$ if necessary, we may assume that I has a right adjoint I_ρ .

The fact that $\mathcal{S}$ is proper over R follows immediately from its being a subcategory. From the existence of the right adjoint to I and [38, Proposition 9.1.18 and Corollary 9.1.14] we have that $\mathcal{S} = {}^\perp(\mathcal{S}^\perp)$. If $e : s \longrightarrow s$ is an idempotent in $\mathcal{S}$ then it splits in the Karoubian $\mathcal{T}$, that is we have an isomorphism $s \cong x \oplus y$ in $\mathcal{T}$ taking e to $1_x \oplus 0_y$. But $x \oplus y \cong s \in \mathcal{S} = {}^\perp(\mathcal{S}^\perp)$ implies that $x, y \in {}^\perp(\mathcal{S}^\perp) = \mathcal{S}$, that is $\mathcal{S}$ is Karoubian. Since [38, Remark 9.1.15 and Theorem 9.1.16] give an equivalence of $\mathcal{S}$ with the Verdier quotient $\mathcal{S} \cong \mathcal{T}/\mathcal{S}^\perp$, if G is a strong generator in $\mathcal{T}$ then the image of G is a strong generator in $\mathcal{T}/\mathcal{S}^\perp \cong \mathcal{S}$; therefore $\mathcal{S}$ is also regular.

Now let t be an object of $\mathcal{T}$. The functor $\mathcal{T}(t, I(-)) : \mathcal{S} \longrightarrow R\text{-Mod}$ is a homological functor, and because $\mathcal{T}$ is proper over R we have that, for any object $s \in \mathcal{S}$, the direct sum $\bigoplus_{i=-\infty}^\infty \mathcal{T}(t, I(\Sigma^i s))$ is a finite R -module. By Theorem 0.3, applied to the category $\mathcal{S}$, this functor is representable: there exists an object $I_\lambda(t) \in \mathcal{S}$ with

$$\mathcal{T}(t, I(s)) \cong \mathcal{T}(I_\lambda(t), s) .$$

The existence of the left adjoint to I follows formally, it takes the object $t \in \mathcal{T}$ to $I_\lambda(t) \in \mathcal{S}$. $\square$

Corollary 0.8 opens the door to the theory of semiorthogonal decompositions with admissible building blocks, we briefly remind the reader. If the inclusion $I : \mathcal{S} \longrightarrow \mathcal{T}$ has a right adjoint then [38, Remark 9.1.15] says that the inclusion $J : \mathcal{S}^\perp \longrightarrow \mathcal{T}$ has a left adjoint, giving $\mathcal{T}$ a semiorthogonal decomposition $\mathcal{T} = \langle \mathcal{S}^\perp, \mathcal{S} \rangle$. From Corollary 0.8 we learn that I and J each have both a right and a left adjoint—the subcategories $\mathcal{S}$ and $\mathcal{S}^\perp$ are both admissible. We note that until now the theory of semiorthogonal decompositions in which the pieces are admissible has been confined to equal characteristic, just because all the interesting known examples, of $\mathcal{T}$'s which are Karoubian, proper over R and regular, were in equal characteristic. The point we want to make here is that Remark 0.6 changes this—it gives a plethora of examples in mixed characteristic.

Remark 0.9. This remark is about the known constructions of examples, we are interested in producing

- (i) Karoubian, regular triangulated categories proper over a commutative, noetherian ring R .

In this remark we will assume the reader is familiar with DG enhancements of triangulated categories—a reader who doesn't feel comfortable with the theory can safely skip ahead to Remark 0.10, we will never refer back to Remark 0.9.

So far we know two ways to obtain examples of (i):

- (ii) If R is of finite Krull dimension and X is a separated, regular scheme proper over R , then $\mathcal{T} = \mathbf{D}^{\text{perf}}(X)$ is an example of (i).
- (iii) If $\mathcal{T}$ is an example of (i) then so is any admissible subcategory $\mathcal{S} \subset \mathcal{T}$, that is any triangulated subcategory $\mathcal{S}$ where the inclusion has a right or a left adjoint (and therefore both by Corollary 0.8).

There is a classical construction we haven't mentioned so far, which is based on the old theorem of Kelly [24].

- (iv) If Λ is a finite (possibly noncommutative) algebra over R and Λ is of finite global dimension, then the derived category $\mathbf{D}^b(\Lambda\text{-proj})$ is an example of (i).

As we've already noted, if the pair $\mathcal{S} \subset \mathcal{T}$ is as in (iii) then so is $\mathcal{S}^\perp \subset \mathcal{T}$. If $\mathcal{T}$ is an example of (i) then so are $\mathcal{S}$ and $\mathcal{S}^\perp$. Of course in this situation $\mathcal{T}$ is a recollement of $\mathcal{S}$ and $\mathcal{S}^\perp$, see [38, Section 9.2]. It is natural to wonder when this process can be reversed. This means: suppose we have triangulated categories $\mathcal{S}$ and $\mathcal{S}'$, both examples of (i). How can one glue them to form a new category $\mathcal{T}$, also an example of (i), with fully faithful embeddings $I : \mathcal{S} \rightarrow \mathcal{T}$ and $J : \mathcal{S}' \rightarrow \mathcal{T}$, which have right and left adjoints, and so that $J(\mathcal{S}') = I(\mathcal{S})^\perp$ and $I(\mathcal{S}) = {}^\perp J(\mathcal{S}')$? Of course there is the dumb gluing, namely $\mathcal{S} \times \mathcal{S}'$. But there are many $\mathcal{S} \subset \mathcal{T}$ where the "extension" does not split, and the question becomes how to reconstruct $\mathcal{T}$ out of $\mathcal{S}$ and $\mathcal{S}'$.

At the level of triangulated categories no one has any idea how to do this. But if we assume both $\mathcal{S}$ and $\mathcal{S}'$ have the structure of DG enhancements over the commutative, noetherian base ring R then there is a gluing procedure which we briefly recall.

Because $\mathcal{S}$ and $\mathcal{S}'$ have (strong) generators $G \in \mathcal{S}$ and $G' \in \mathcal{S}'$, we can let S and S' be the DG R -algebras $S = \text{End}(G)$ and $S' = \text{End}(G')$, where the endomorphisms are understood in the DG enhancements. We obtain equivalences $\mathcal{S} = \mathcal{H}^0(\text{Perf} - S)$ and $\mathcal{S}' = \mathcal{H}^0(\text{Perf} - S')$. Given any DG S - S' bimodule M we can form the DG matrix algebra over R

$$T = \begin{pmatrix} S & M \\ 0 & S' \end{pmatrix}$$

Then the triangulated category $\mathcal{T} = \mathcal{H}^0(\text{Perf} - T)$ is Karoubian, there are natural fully faithful functors $I : \mathcal{S} \rightarrow \mathcal{T}$ (with a right adjoint) and $J : \mathcal{S}' \rightarrow \mathcal{T}$ (with a left adjoint), we have $J(\mathcal{S}') = I(\mathcal{S})^\perp$ and $I(\mathcal{S}) = {}^\perp J(\mathcal{S}')$, and $\mathcal{T}$ is regular. If the homology of the bimodule M is finite over R then it's also true that $\mathcal{T}$ is proper over R .

The proofs of most of the statements may be found in Kuznetsov and Lunts [27]. For the proof that $\mathcal{T}$ is regular whenever $\mathcal{S}$ and $\mathcal{S}'$ are see Orlov [41, Proposition 3.20].

The construction of Kuznetsov and Lunts is very much in the spirit of noncommutative algebraic geometry, where one thinks of noncommutative schemes as triangulated categories $\mathcal{S} = \mathcal{H}^0(\text{Perf} - S)$ for DG algebras S . And, without the hypotheses of regularity and properness of $\mathcal{S}$, one does indeed expect noncommutative behaviour. Against

this background comes a lovely theorem of Orlov [41, Theorem 4.15 and Corollary 5.4], which asserts

- (v) We're still interested in examples of (i). There are the ones that come from (ii) and (iv). And then we can produce more examples by forming admissible subcategories as in (iii) and using the gluing procedure of Kuznetsov and Lunts [27] sketched above.

If the underlying commutative ring R is a perfect field then any example of (i), obtainable by a finite iteration of these steps, can also be produced much more directly: we can form it by applying (iii) once to an example as in (ii). In particular we can avoid the noncommutative gluing procedure of Kuznetsov and Lunts, and the finite-dimensional algebra examples of (iv) are all special cases of the algebro-geometric examples obtainable from (ii) and (iii).

I recommend Orlov's paper highly. The proof of the main theorem requires producing new schemes, and admissible subcategories of their categories of perfect complexes. The argument is a spectacular display of the power of semiorthogonal decompositions, coupled with the equivalences between admissible subcategories on different schemes produced in Orlov [40]. And the paper is also beautifully written: it begins with a gentle introduction and a survey of the theory, which a nonexpert like myself found very helpful. In fact the current paper was born when I was trying to understand Orlov [41].

Remark 0.10. Note the generality of Theorem 0.5: we don't even assume the schemes noetherian. The reader might wonder why we bother with this level of abstraction.

If we're willing to assume X quasi-projective, over a noetherian ring R , then the proof simplifies substantially. In fact the general case is proved by reducing to the quasi-projective situation. As it turns out the passage from the quasi-projective case to the general one is not substantially simplified by assuming the schemes noetherian—if we don't like the projectivity hypothesis, then we might as well go whole hog and prove the theorem in the generality above.

Remark 0.11. We should note that one direction in Theorem 0.5 is easy, we show

$$\left\{ \begin{array}{l} X \text{ admits a cover by } \mathrm{Spec}(R_i) \\ \text{with } R_i \text{ of finite global dimension} \end{array} \right\} \Longleftarrow \{ \mathbf{D}^{\mathrm{perf}}(X) \text{ strongly generated} \}.$$

In fact more is true: if $\mathbf{D}^{\mathrm{perf}}(X)$ is strongly generated and $U = \mathrm{Spec}(R)$ is any open affine subset of X , then R must be of finite global dimension.

We see this as follows: as U is an open subset of X , the main theorem of Thomason and Trobaugh [47] tells us that the restriction functor $j^* : \mathbf{D}^{\mathrm{perf}}(X) \rightarrow \mathbf{D}^{\mathrm{perf}}(U)$ is the idempotent completion of a Verdier quotient map. If $G \in \mathbf{D}^{\mathrm{perf}}(X)$ is a strong generator it follows that $j^*G \in \mathbf{D}^{\mathrm{perf}}(U)$ is a strong generator. But for $U = \mathrm{Spec}(R)$ we deduce that R must be of finite global dimension—see for example Rouquier [42, Theorem 7.25]. Hence it only remains to prove the direction $\implies$.

Remark 0.12. If X is noetherian the local hypothesis of Theorem 0.5 is equivalent to requiring X to be regular and of finite Krull dimension. But there are examples of nonnoetherian schemes X satisfying the hypotheses of the theorem. One source of examples is absolutely flat rings—rings for which every module is flat. From Salles [43, Proposition 3, page 702] we know that for flat modules the projective dimension and pure projective dimension agree, and hence for absolutely flat rings the global dimension (the supremum over all modules of their projective dimension) is equal to the pure global dimension (the supremum over all modules of their pure projective dimension). Now [25, Theorem 2.2] or [21, Theorems 7.47 or 11.21] tell us that for rings of cardinality $\leq \aleph_n$ the pure global dimension is $\leq n + 1$. Hence an absolutely flat ring of cardinality $\leq \aleph_n$ has global dimension $\leq n + 1$.

Concretely: if k is the field of two elements and R is the ring $R = k[x_1, x_2, x_3 \dots]/(x_1^2 - x_1, x_2^2 - x_2, x_3^2 - x_3 \dots)$ then R is a nonnoetherian absolutely flat ring of cardinality $\aleph_0$, and its global dimension is 1. Thus $X = \text{Spec}(R)$ is a nonnoetherian example where Theorem 0.5 applies.

We’ve had an extensive discussion of Theorem 0.5 and its significance, and it’s about time to move on to the other major results in the article. For the next major theorem we recall the notion of a regular alteration of a scheme.

Reminder 0.13. Let X be a noetherian scheme. A *regular alteration* of X is a proper, surjective morphism $f : Y \rightarrow X$, so that

- (i) Y is regular and finite dimensional.
- (ii) There is a dense open set $U \subset X$ over which f is finite.

In [12, 13] de Jong proves, among other things, that any scheme X , separated and of finite type over an excellent scheme S of dimension ≤ 2 , admits a regular alteration. Since de Jong’s papers Nayak [31] showed that any scheme X , separated and essentially of finite type over S , admits a localizing immersion into a scheme $\overline{X}$ separated and of finite type (even proper) over S . By restricting a regular alteration of $\overline{X}$ to $X \subset \overline{X}$ it immediately follows that any scheme X , separated and essentially of finite type over an excellent scheme S of dimension ≤ 2 , also has a regular alteration.

Observe that, if X is separated and essentially of finite type over an excellent scheme S of dimension ≤ 2 , then so is every closed subscheme of X . Therefore all closed subschemes of X admit regular alterations. Hence any X which is separated and essentially of finite type over a separated, excellent scheme S of dimension ≤ 2 satisfies

Hypothesis 0.14. *A scheme X satisfies Hypothesis 0.14 if it is noetherian, separated, and every closed subscheme of X admits a regular alteration.*

Now we are ready to state our second main result:

Theorem 0.15. *Let X be a scheme satisfying Hypothesis 0.14. Then the triangulated category $\mathbf{D}_{\text{coh}}^b(X)$, of bounded complexes of coherent sheaves on X , is regular.*

We owe the reader a summary of what was known in this direction. Of course when X is regular and finite-dimensional then the inclusion $\mathbf{D}^{\text{perf}}(X) \rightarrow \mathbf{D}_{\text{coh}}^b(X)$ is an equivalence, and Theorem 0.5 tells us that the equivalent categories $\mathbf{D}^{\text{perf}}(X) \cong \mathbf{D}_{\text{coh}}^b(X)$ are regular. By a clever extension and refinement of the argument in Bondal and Van den Bergh [7, Theorem 3.1.4], Rouquier [42, Theorem 7.38] showed that $\mathbf{D}_{\text{coh}}^b(X)$ is regular whenever X is a separated scheme of finite type over a perfect field k . The preprint by Keller and Van den Bergh [23, Proposition 5.1.2] generalized to separated schemes of finite type over arbitrary fields, but this Proposition disappeared in the passage to the published version [22]. The reader might also wish to look at Lunts [30, Theorem 6.3] for a different approach to the proof. If we specialize the result of Rouquier, extended by Keller and Van den Bergh, to the case where $X = \text{Spec}(R)$ is an affine scheme, we learn that $\mathbf{D}^b(R\text{-mod})$ is regular whenever R is of finite type over a field k . In recent years there has been interest among commutative algebraists to understand this better: the reader is referred to Aihara and Takahashi [1], Iyengar and Takahashi [19], and Bahlekeh, Hakimian, Salarian and Takahashi [3] for a sample of the literature. There is also a connection with the concept of the radius of the (abelian) category of modules over R ; see Dao and Takahashi [10, 11] and Iyengar and Takahashi [19]. The union of the known results seems to be that $\mathbf{D}^b(R\text{-mod})$ is regular if R is an equicharacteristic excellent local ring, or essentially of finite type over a field—see [19, Corollary 7.2]. In [19, Remark 7.3] it is observed that there are examples of commutative, noetherian rings for which $\mathbf{D}^b(R\text{-mod})$ is *not* regular¹.

What's really new about Theorem 0.15 is that, unlike the predecessors recalled above, it works in mixed characteristic.

We should say something about our proofs. It turns out to be convenient to work not in $\mathbf{D}^{\text{perf}}(X)$ and $\mathbf{D}_{\text{coh}}^b(X)$ but in the larger category $\mathbf{D}_{\text{qc}}(X)$; that is we switch to the unbounded derived category of cochain complexes of sheaves of $\mathcal{O}_X$ -modules with quasicoherent cohomology. There are unbounded versions of Theorems 0.5 and 0.15. We don't yet have the notation to state these theorems, they will come in §2: we will give two theorems about $\mathbf{D}_{\text{qc}}(X)$, Theorems 2.1 and 2.3, and show

$$\text{Theorem 2.1} \implies \text{Theorem 0.5}, \quad \text{Theorem 2.3} \implies \text{Theorem 0.15}.$$

and it will remain to prove Theorems 2.1 and 2.3. For now we note a couple of facts about Theorems 2.1 and 2.3.

¹In view of Theorem 0.15, we conclude that a ring R with $\mathbf{D}^b(R\text{-mod})$ not regular must be such that not every closed subscheme of $\text{Spec}(R)$ admits a regular alteration. Of course examples of noetherian rings R with $\text{Spec}(R)$ not admitting a regular alteration are known: if R is a noetherian integral domain admitting a regular alteration $f : X \rightarrow \text{Spec}(R)$, then over the generic point of $\text{Spec}(R)$ the map f is finite and faithfully flat. Hence f must be finite and faithfully flat over some nonempty open subset of $\text{Spec}(R)$, which by faithfully flat descent would have to be regular. Thus if R is a noetherian integral domain which is not J-0, then $\text{Spec}(R)$ is a noetherian scheme not admitting a regular alteration.

- (i) The hypotheses on X are identical in Theorems 2.1 and 0.5 and identical in Theorems 2.3 and 0.15. But the conclusions are about $\mathbf{D}_{\text{qc}}(X)$ instead of $\mathbf{D}^{\text{perf}}(X)$ or $\mathbf{D}_{\text{coh}}^b(X)$.
- (ii) If X is regular and satisfies Hypothesis 0.14 then Theorems 2.1 and 2.3 give the same conclusion for $\mathbf{D}_{\text{qc}}(X)$.

As the reader may have guessed, from (i) and (ii) above, the proof of Theorem 2.3 will use regular alterations to deduce the result from Theorem 2.1—Theorem 2.1 establishes Theorem 2.3 in the regular case, and alterations are about reducing to the regular case. In this field several experts have tried to use de Jong’s theorem, there is a slew of results that are known in characteristic zero using resolution of singularities and conjectured in characteristic p . Not surprisingly the experts have thought of the idea of employing alterations to prove these conjectures. What’s remarkable is that Theorem 2.3 is the first success story, and the only one so far. It is an example—in [39, proof of Theorem 4 (special case)] there is a simple proof of Theorem 2.3, deducing it from Theorem 2.1 for schemes all of whose closed subschemes have resolutions of singularities. In other words if we could use resolutions of singularities the proof simplifies substantially. The proof given here combines the use of alterations with two old theorems of Thomason’s. Thomason’s theorems fall into the area that nowadays goes by the name “support theory”—it’s the combination of regular alterations and support theory that proves Theorem 2.3.

We have said something about the key ideas that go into deducing Theorem 2.3 from Theorem 2.1, and it remains to discuss the proof of Theorem 2.1 a tiny bit. We begin with a general definition: let $\mathcal{T}$ be any triangulated category with coproducts and let G be an object. For integers $A \leq B$ we define the full subcategory $\overline{\langle G \rangle}_1^{[A,B]} \subset \mathcal{T}$ to consist of all direct summands of arbitrary coproducts of objects in the set $\{\Sigma^{-i}G, A \leq i \leq B\}$ —in other words we allow arbitrary coproducts but restrict the permitted suspensions, only suspensions in a prescribed range can occur. The inductive definition is as with $\langle G \rangle_n$, an object belongs to $\overline{\langle G \rangle}_{n+1}^{[A,B]}$ if it is a direct summand of an object y admitting a triangle $x \longrightarrow y \longrightarrow z \longrightarrow$ with $x \in \overline{\langle G \rangle}_1^{[A,B]}$ and $z \in \overline{\langle G \rangle}_n^{[A,B]}$. Our next theorem, which is a sharpening of [29, Theorem 4.1] and the first key step in the proof of Theorem 2.1, says

Theorem 0.16. *Let $m \leq n$ be integers, let X be a scheme of finite type over a noetherian ring R , and let G be a classical generator for $\mathbf{D}^{\text{perf}}(X) \subset \mathbf{D}_{\text{qc}}(X)$. Then there exist integers $N, A \leq B$ so that every object $F \in \mathbf{D}_{\text{qc}}(X)^{\leq n}$ admits a triangle $D \longrightarrow E \longrightarrow F \longrightarrow$ with $E \in \overline{\langle G \rangle}_N^{[A,B]}$ and $D \in \mathbf{D}_{\text{qc}}(X)^{< m}$.*

Remark 0.17. Note the generality: we are trying to prove Theorem 2.1 which, in the noetherian case, is an assertion about regular schemes. And I have just told the reader that the first key step is a theorem which doesn’t assume any regularity. In passing we note that it is entirely possible that some of the hypotheses on the scheme X in Theorem 0.16 are superfluous. For all the author knows the assertion might generalize to all quasicompact, quasiseparated schemes.

Theorem 0.16 is the first ingredient we need in the proof of Theorem 2.1. It enters into the proof of our next major result.

Theorem 0.18. *Let $j : U \rightarrow X$ be an open immersion of quasicompact, separated schemes. If $H \in \mathbf{D}_{\mathbf{qc}}(U)$ is a perfect complex and $G \in \mathbf{D}^{\mathrm{perf}}(X)$ a classical generator, then for any triple of integers $n, a \leq b$ there exists a triple of integers $N, A \leq B$ so that $\mathbf{R}j_* \overline{H}_n^{[a,b]} \subset \overline{G}_N^{[A,B]}$.*

In the body of the article we will proceed as follows: in §2 we state Theorems 2.1 and 2.3 and prove that Theorem 2.1 implies Theorem 0.5 and Theorem 2.3 implies Theorem 0.15. In §3 we show how to deduce Theorem 2.3 from Theorem 2.1, using regular alterations and support theory. Then §4 and §5 give the proof of Theorem 0.16—this is probably the hardest part, because we don’t want to make the simplifying assumption that X is quasiprojective. In §6 we wrap things up—first we will prove Theorem 0.18, and finally we’ll show how to use Theorem 0.18, coupled with Kelly’s old theorem about regular affine schemes, to finish off the proof of Theorem 2.1. It’s in the use of Kelly’s theorem that regularity enters.

Illustration 0.19. It’s worth noting that Theorems 0.16 and 0.18 are useful in other contexts, they have applications having nothing to do with the regularity of triangulated categories. We end the introduction with an illustration.

We next give a simple proof that a separated, finite-type morphism of noetherian schemes is perfect if and only if it is proper and of finite Tor-dimension. The original proof may be found in [29, Theorem 1.2], but the machinery developed here makes the problem a triviality and leads to sharper statements. In passing I mention that the reason I care, about different proofs of such statements, is that I’d like to generalize to morphisms of stacks—at the moment I don’t know how to do this, I haven’t yet proved useful versions of Theorems 0.16 and 0.18 valid for algebraic stacks.

Note that for simplicity we assume the schemes noetherian; this assumption can be removed just as in [29, Theorem 1.2], but without the noetherian hypothesis the statements become a little technical. In this paper we would rather not recall the definitions of pseudocoherent and quasiproper morphisms.

We should perhaps remind the reader: a morphism of schemes $f : X \rightarrow Y$ is called *perfect* if $\mathbf{R}f_* : \mathbf{D}_{\mathbf{qc}}(X) \rightarrow \mathbf{D}_{\mathbf{qc}}(Y)$ takes perfect complexes to perfect complexes. What is being asserted, in this Illustration, is that a finite-type, separated morphism of noetherian schemes is perfect if and only if it is proper and of finite Tor-dimension. One implication is classical, it’s been known since [17, Corollaire 4.8.1] that a proper morphism of finite Tor-dimension is perfect. The converse is relatively recent. Anyway: it’s customary to break this up into two bits, dealing separately with the properness and with the finite Tor-dimension.

If $f : X \rightarrow Y$ is proper, then Grothendieck’s old result [15, Théorème 3.2.1] tells us that $\mathbf{R}f_* \mathbf{D}_{\mathrm{coh}}^b(X) \subset \mathbf{D}_{\mathrm{coh}}^b(Y)$. And the sharp converse, which we prove below in

Lemma 0.20, says that if $\mathbf{R}f_*\mathbf{D}^{\text{perf}}(X) \subset \mathbf{D}_{\text{coh}}^b(Y)$ then f must be proper—you don't need to check what $\mathbf{R}f_*$ does to every complex in $\mathbf{D}_{\text{coh}}^b(X)$, it's enough to check the perfect ones. In Example 0.4 we've already mentioned a special case of this lemma, where Y is assumed affine. Lemma 0.20 isn't new, it may be found in [29, Corollary 4.3.2], but the nonnoetherian generality of the statement and proof in [29] might be confusing.

About finite Tor-dimension: in Illusie [17, Corollaire 3.4.1(a)] we can find the fact that, if $f : X \rightarrow Y$ is of finite Tor-dimension, then $\mathbf{R}f_*$ takes an object $H \in \mathbf{D}_{\text{qc}}(X)$ of bounded-below Tor-amplitude to an object $\mathbf{R}f_*H \in \mathbf{D}_{\text{qc}}(Y)$ of bounded-below Tor-amplitude. The sharp converse, see Proposition 0.21 below, asserts that if $\mathbf{R}f_*$ takes perfect complexes to complexes of bounded-below Tor-amplitude then f is of finite Tor-dimension. This result is new.

Now for the precise statements and proofs.

Lemma 0.20. *Let $f : X \rightarrow Y$ be a separated, finite-type map of noetherian schemes. If $\mathbf{R}f_*$ takes $\mathbf{D}^{\text{perf}}(X) \subset \mathbf{D}_{\text{qc}}(X)$ into $\mathbf{D}_{\text{coh}}^b(Y) \subset \mathbf{D}_{\text{qc}}(Y)$ then f is proper.*

Proof. By [28, Exercise 4.3.9] it suffices to show that $\mathbf{R}f_*$ takes $\mathbf{D}_{\text{coh}}^b(X)$ to $\mathbf{D}_{\text{coh}}^b(Y)$. Let F be an object in $\mathbf{D}_{\text{coh}}^b(X)$, we wish to show that $\mathbf{R}f_*F$ belongs to $\mathbf{D}_{\text{coh}}^b(Y)$. Shifting if necessary, we may assume $F \in \mathbf{D}_{\text{qc}}(X)^{\geq 0}$.

Choose an integer ℓ so that $\mathbf{R}f_*\mathbf{D}_{\text{qc}}(X)^{\leq 0}$ is contained in $\mathbf{D}_{\text{qc}}(Y)^{\leq \ell}$. By [29, Theorem 4.1] we may choose a triangle $D \rightarrow E \rightarrow F \rightarrow$ with E a perfect complex and $D \in \mathbf{D}_{\text{qc}}(X)^{\leq -\ell-1}$. We deduce a triangle $\mathbf{R}f_*D \rightarrow \mathbf{R}f_*E \rightarrow \mathbf{R}f_*F \rightarrow$ in $\mathbf{D}_{\text{qc}}(Y)$, and by the choice of ℓ we know that $\mathbf{R}f_*D \in \mathbf{D}_{\text{qc}}(Y)^{\leq -1}$. Hence the map $\mathbf{R}f_*E \rightarrow \mathbf{R}f_*F$ induces an isomorphism of cohomology sheaves in degrees ≥ 0 . The assumption of the Lemma gives that $\mathbf{R}f_*E$ is in $\mathbf{D}_{\text{coh}}^b(Y)$, so in degrees $i \geq 0$ the cohomology sheaves $\mathcal{H}^i(\mathbf{R}f_*E) \cong \mathcal{H}^i(\mathbf{R}f_*F)$ are coherent. But F belongs to $\mathbf{D}_{\text{qc}}(X)^{\geq 0}$, hence in degrees $i < 0$ the cohomology sheaves $\mathcal{H}^i(\mathbf{R}f_*F)$ vanish. Therefore $\mathbf{R}f_*F$ belongs to $\mathbf{D}_{\text{coh}}^b(Y)$. $\square$

Proposition 0.21. *Suppose $g : X \rightarrow Y$ is a separated morphism of quasicompact, quasiseparated schemes. If $\mathbf{R}g_*$ takes every perfect complex $F \in \mathbf{D}_{\text{qc}}(X)$ to an object $\mathbf{R}g_*F \in \mathbf{D}_{\text{qc}}(Y)$ of bounded-below Tor-amplitude, meaning Tor-amplitude in the interval $[\ell, \infty)$ for some integer ℓ which may depend on F , then g must be of finite Tor-dimension.*

Proof. First we note that the question is local in Y . Clearly it's local in Y to check that g is of finite Tor-dimension, but the condition that $\mathbf{R}g_*$ takes perfect complexes to complexes of bounded-below Tor-amplitude does not at first sight appear local. The hypothesis of the Proposition gives that $\mathbf{R}g_*$ takes perfect complexes to complexes of bounded-below Tor-amplitude. If we are given a cartesian square

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{u} & X \\ f \downarrow & & \downarrow g \\ \tilde{Y} & \xrightarrow{v} & Y \end{array}$$

with v an open immersion, we need to show that $\mathbf{R}f_*$ takes perfect complexes to complexes of bounded below Tor-amplitude.

Therefore let G be an object in $\mathbf{D}^{\text{perf}}(\tilde{X})$. Then $G \oplus \Sigma G$ has a vanishing image in $K_0(\tilde{X})$, and by the main theorem of Thomason and Trobaugh [47] there exists an object $H \in \mathbf{D}^{\text{perf}}(X)$ with $u^*H \cong G \oplus \Sigma G$. Base-change gives an isomorphism $v^*\mathbf{R}g_*H \rightarrow \mathbf{R}f_*u^*H \cong \mathbf{R}f_*G \oplus \Sigma\mathbf{R}f_*G$. Since $\mathbf{R}g_*H$ is of bounded-below Tor-amplitude so is $v^*\mathbf{R}g_*H$, and hence so is the direct summand $\mathbf{R}f_*G$.

We now know that the question is local in Y , hence we may assume Y affine, in particular separated. We are given a separated morphism $g : X \rightarrow Y$; as g and Y are separated so is X . Let $j : U \rightarrow X$ be an open immersion with U affine. Now apply Theorem 0.18 to the object $H = \mathcal{O}_U \in \mathbf{D}^{\text{perf}}(U)$ and any classical generator $G \in \mathbf{D}^{\text{perf}}(X)$ —the existence of such a G is proved in [7, Theorem 3.1.1] or [29, Theorem 4.2]. From Theorem 0.18 we have $\mathbf{R}j_*\mathcal{O}_U \in \overline{\langle G \rangle}_N^{[A,B]}$. Hence $\mathbf{R}g_*\mathbf{R}j_*\mathcal{O}_U \in \mathbf{R}g_*\overline{\langle G \rangle}_N^{[A,B]} \subset \overline{\langle \mathbf{R}g_*G \rangle}_N^{[A,B]}$. But by hypothesis the Tor-amplitude of $\mathbf{R}g_*G$ is bounded below, which gives a uniform lower bound for Tor-amplitude of the objects of $\overline{\langle \mathbf{R}g_*G \rangle}_N^{[A,B]}$, we only allow suspensions in a range. Thus $(gj)_*\mathcal{O}_U = \mathbf{R}(gj)_*\mathcal{O}_U \cong \mathbf{R}g_*\mathbf{R}j_*\mathcal{O}_U$ is of bounded-below Tor-amplitude, which means that at every point of the open affine subset $U \subset X$ the map g is of finite Tor-dimension. Since U is arbitrary g is of finite Tor-dimension. $\square$

Remark 0.22. We should end the introduction with a list of obvious open problems that follow from our results, and an account of the progress made on these problems since the article was written. But first a word of warning: to state some of the problems we assume familiarity with the notation and lemmas that appear later in the article. Hence the reader might wish to go on to the body of the article, before returning to the open problems at the end.

We promised a list—here we go.

- (i) Is the hypothesis that X is separated necessary in Theorem 0.3 (the theorem characterizing those X for which $\mathbf{D}^{\text{perf}}(X)$ is strongly generated), or in its unbounded version Theorem 2.1? Do the theorems generalize to all quasicompact, quasiseparated schemes? We should note that much of the theory developed by Bondal and Van den Bergh works in the generality of quasicompact, quasiseparated scheme, making the question natural.

There has been some progress on this since this article was written, see Jantona [20].

- (ii) What about Theorem 0.15 about the strong generation of $\mathbf{D}_{\text{coh}}^b(X)$, or its unbounded version Theorem 2.3? Is it necessary to assume that X is separated, noetherian, and that every closed subscheme has a regular alteration?

Here the beautiful article by Aoki [2] makes a giant step forward. The scheme still has to be separated and noetherian, but it suffices for it to be quasiexcellent and finite dimensional. The key idea is that, in place of combining de Jong's alterations with a couple of Thomason's old theorems as in the current article (alterations are

- not known to exist for all quasiexcellent schemes), Aoki cleverly combines Gabber's weak local uniformization theorem with the descendability techniques of Mathew.
- (iii) Assume X is a noetherian, quasiexcellent, finite dimensional separated scheme. By Aoki's theorem the category $\mathbf{D}_{\text{coh}}^b(X)$ is regular. But $\mathbf{D}_{\text{coh}}^b(X)$ is the category of compact objects in the compactly generated triangulated category $\mathbf{K}(\text{Inj } X) \cong \text{IndCoh}(X)$, see Krause [26, Proposition 2.3] or Drinfeld and Gaitsgory [14, Proposition 3.4.2]. Pursuing the parallel with $\mathbf{D}^{\text{perf}}(X)$ and the way its regularity was studied via the unbounded version Theorem 2.1, one can wonder whether the obvious analog of Theorem 2.1 holds for $\mathbf{K}(\text{Inj } X) \cong \text{IndCoh}(X)$.
 - (iv) What about the noncommutative versions, for example the categories of modules over an Azumaya algebra?
 - (v) One could look at fancier noncommutative versions, such as the noncommutative schemes of Kontsevich and Orlov. There again one can ask about strong generation of $\mathbf{D}^{\text{perf}}(X)$ and $\mathbf{D}_{\text{coh}}^b(X)$. Of course $\mathbf{D}_{\text{coh}}^b(X)$ needs to be defined. As we will see in the remainder of this Remark, this has been done. Which leads us to
 - (vi) Are the techniques developed in the article useful in contexts unrelated to strong generation? For example to the problem of defining $\mathbf{D}_{\text{coh}}^b(X)$ for a noncommutative scheme X , as in (v) above?

The answer to question (vi) is a resounding Yes, and the rest of this Remark will be devoted to saying a tiny bit about the subject that has grown out of applying the techniques developed here. For a fuller account of the theory the reader is referred to the survey article [32], or the much more detailed research articles [35, 8, 34, 33]. The order in which the research articles are listed is logical, they form a linear sequence of papers with each building on its predecessors (including the current article).

Let X be a scheme. In Definition 5.2 we establish what it means for a subcategory $\mathcal{S} \subset \mathbf{D}_{\text{qc}}^-(X)$ to be *approximable*. And then in the rest of Section 5 we produce larger and larger approximable subcategories of the category $\mathbf{D}_{\text{qc}}^-(X)$, culminating in Theorem 5.8 which tells us that, under mild conditions on X , the categories $\mathbf{D}_{\text{qc}}(X)^{\leq m}$ are approximable for every integer m ,

The way this is generalized in the sequels is the obvious: in place of $\mathbf{D}_{\text{qc}}(X)$ we allow any compactly generated triangulated category with a t -structure. More precisely: in the article [35], a compactly generated triangulated category $\mathcal{T}$ is defined to be approximable if $\mathcal{T}$ possesses a t -structure such that the categories $\mathcal{T}^{\leq m}$ satisfy the obvious generalization of what holds in $\mathbf{D}_{\text{qc}}(X)^{\leq m}$ by Theorem 5.8.

And then one proves many remarkable and useful structure theorems, which hold for all approximable triangulated categories. Among them is the fact that the t -structure used in the definition of approximability is actually close to unique: any two must be equivalent (under a suitable equivalence relation). Not only do we have (up to equivalence) a canonical t -structure, in any approximable triangulated category $\mathcal{T}$ one can define a

string of intrinsic subcategories $\mathcal{T}^c$, $\mathcal{T}^-$, $\mathcal{T}^+$, $\mathcal{T}^b$, $\mathcal{T}_c^-$ and $\mathcal{T}_b^c$, study the relations among them, and prove useful theorems about the way they interact.

Since there is a survey of the results available in [32] let me stop soon—the reader wishing to go into more detail can look at the survey. Here I only want to mention that, for X a separated, noetherian scheme and for $\mathcal{T} = \mathbf{D}_{\mathbf{qc}}(X)$, the intrinsic subcategory $\mathcal{T}_c^b \subset \mathcal{T}$ turns out to be $\mathbf{D}_{\text{coh}}^b(X) \subset \mathbf{D}_{\mathbf{qc}}(X)$. And if X is a noncommutative scheme in the sense of Kontsevich or Orlov then, under mild restrictions, the article [8] proves that the category $\mathbf{D}_{\mathbf{qc}}(X)$ is approximable. Thus starting with such a noncommutative scheme we can declare $\mathbf{D}_{\text{coh}}^b(X)$ to be $\mathbf{D}_{\mathbf{qc}}(X)_c^b$, and ask when it is strongly generated. This is the sense in which I said, in problem (v) above, that we know how to define $\mathbf{D}_{\text{coh}}^b(X)$ for noncommutative X .

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1. BACKGROUND

Reminder 1.1. Let $\mathcal{T}$ be a triangulated category; we begin by reminding the reader of some old definitions.

- (i) If $\mathcal{A}$ and $\mathcal{B}$ are two subcategories of $\mathcal{T}$, then $\mathcal{A} \star \mathcal{B}$ is the full subcategory of all objects y for which there exists a triangle $x \longrightarrow y \longrightarrow z \longrightarrow$ with $x \in \mathcal{A}$ and $z \in \mathcal{B}$.
- (ii) If $\mathcal{A}$ is a subcategory of $\mathcal{T}$, then $\text{add}(\mathcal{A})$ is the full subcategory containing all finite coproducts of objects in $\mathcal{A}$.
- (iii) If $\mathcal{A}$ is a subcategory of $\mathcal{T}$ and $\mathcal{T}$ is closed under coproducts, then $\text{Add}(\mathcal{A})$ is the full subcategory containing all (set-indexed) coproducts of objects in $\mathcal{A}$.
- (iv) If $\mathcal{A}$ is a full subcategory of $\mathcal{T}$, then $\text{smd}(\mathcal{A})$ is the full subcategory of all direct summands of objects in $\mathcal{A}$.

Remark 1.2. Reminder 1.1(i) is as in [5, 1.3.9], while Reminder 1.1(iv) is identical with [7, beginning of 2.2]. Reminder 1.1(ii) and (iii) follow the usual conventions in representation theory; in [7, beginning of 2.2] the authors adopt the (unconventional) notation that $\text{add}(\mathcal{A})$ and $\text{Add}(\mathcal{A})$ are also closed under the suspension—thus $\text{add}(\mathcal{A})$ as defined in [7] is what we would denote $\text{add}(\bigcup_{n=-\infty}^{\infty} \Sigma^n \mathcal{A})$. The definitions that follow are therefore slightly different from [7].

Recall that the octahedral lemma gives $\mathcal{A} \star (\mathcal{B} \star \mathcal{C}) = (\mathcal{A} \star \mathcal{B}) \star \mathcal{C}$. The fact that coproducts of triangles are triangles tells us that

$$\left\{ \begin{array}{l} \text{add}(\mathcal{A}) = \mathcal{A} \\ \text{add}(\mathcal{B}) = \mathcal{B} \end{array} \right\} \implies \{ \text{add}(\mathcal{A} \star \mathcal{B}) = \mathcal{A} \star \mathcal{B} \} .$$

If $\mathcal{T}$ is closed under coproducts then

$$\left\{ \begin{array}{l} \text{Add}(\mathcal{A}) = \mathcal{A} \\ \text{Add}(\mathcal{B}) = \mathcal{B} \end{array} \right\} \implies \{ \text{Add}(\mathcal{A} \star \mathcal{B}) = \mathcal{A} \star \mathcal{B} \}.$$

Note that the empty coproduct is 0, hence $0 \in \text{add}(\mathcal{A}) \subset \text{Add}(\mathcal{A})$ for any $\mathcal{A}$.

Definition 1.3. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ a subcategory. We define subcategories*

- (i) $\text{coprod}_1(\mathcal{A}) = \text{add}(\mathcal{A})$, $\text{coprod}_{n+1}(\mathcal{A}) = \text{coprod}_1(\mathcal{A}) \star \text{coprod}_n(\mathcal{A})$.
- (ii) $\text{Coproduct}_1(\mathcal{A}) = \text{Add}(\mathcal{A})$, $\text{Coproduct}_{n+1}(\mathcal{A}) = \text{Coproduct}_1(\mathcal{A}) \star \text{Coproduct}_n(\mathcal{A})$.
- (iii) $\text{coprod}(\mathcal{A}) = \bigcup_{n=1}^{\infty} \text{coprod}_n(\mathcal{A})$.
- (iv) $\text{Coproduct}(\mathcal{A})$ is the smallest strictly full subcategory of $\mathcal{T}$ containing $\mathcal{A}$ and satisfying $\text{Add}(\text{Coproduct}(\mathcal{A})) \subset \text{Coproduct}(\mathcal{A})$, $\text{Coproduct}(\mathcal{A}) \star \text{Coproduct}(\mathcal{A}) \subset \text{Coproduct}(\mathcal{A})$.

Remark 1.4. Definition 1.3 is best viewed as a useful technical refinement of Bondal and Van den Bergh [7, the paragraphs right before and right after Definition 2.2.1]. We do not allow direct summands or arbitrary suspensions, but otherwise $\text{coprod}_n(\mathcal{A})$ is the analog of $\langle \mathcal{A} \rangle_n$, $\text{Coproduct}_n(\mathcal{A})$ is the analog of $\overline{\langle \mathcal{A} \rangle}_n$, $\text{coprod}(\mathcal{A})$ is the analog of $\langle \mathcal{A} \rangle$ and $\text{Coproduct}(\mathcal{A})$ is the analog of $\overline{\langle \mathcal{A} \rangle}$. And consistency with Bondal and Van den Bergh also explains why Definition 1.3(iii) looks different from Definition 1.3(iv).

Observation 1.5. The definitions immediately give the inclusions

$$\begin{array}{ccc} \text{coprod}_n(\mathcal{A}) & \hookrightarrow & \text{coprod}(\mathcal{A}) \\ \downarrow & & \downarrow \\ \text{Coproduct}_n(\mathcal{A}) & \hookrightarrow & \text{Coproduct}(\mathcal{A}) \end{array}$$

In Remark 1.2 we noted that $0 \in \text{add}(\mathcal{A}) \subset \text{Add}(\mathcal{A})$, which we can rewrite as $0 \in \text{coprod}_1(\mathcal{A}) \subset \text{Coproduct}_1(\mathcal{A})$. If $x \in \text{coprod}_n(\mathcal{A})$ (respectively $x \in \text{Coproduct}_n(\mathcal{A})$), the triangle $0 \longrightarrow x \xrightarrow{1} x \longrightarrow$ and Definitions 1.3(ii) and (iii) tell us that $x \in \text{coprod}_{n+1}(\mathcal{A})$ (respectively $x \in \text{Coproduct}_{n+1}(\mathcal{A})$). That is

$$\text{coprod}_n(\mathcal{A}) \subset \text{coprod}_{n+1}(\mathcal{A}), \quad \text{Coproduct}_n(\mathcal{A}) \subset \text{Coproduct}_{n+1}(\mathcal{A}).$$

We have that

$$\text{add}(\text{coprod}_1(\mathcal{A})) = \text{add}(\text{add}(\mathcal{A})) = \text{coprod}_1(\mathcal{A}),$$

and similarly $\text{Add}(\text{Coproduct}_1(\mathcal{A})) = \text{Coproduct}_1(\mathcal{A})$. Induction on n gives the equalities

$$\text{add}(\text{coprod}_n(\mathcal{A})) = \text{coprod}_n(\mathcal{A}), \quad \text{Add}(\text{Coproduct}_n(\mathcal{A})) = \text{Coproduct}_n(\mathcal{A}).$$

Any finite set of objects of the increasing union $\text{coprod}(\mathcal{A}) = \bigcup_{n=1}^{\infty} \text{coprod}_n(\mathcal{A})$ must lie in $\text{coprod}_n(\mathcal{A})$ for some large n , and the fact that $\text{add}(\text{coprod}_n(\mathcal{A})) = \text{coprod}_n(\mathcal{A})$ tells us that the direct sum also lies in $\text{coprod}_n(\mathcal{A})$. Thus

$$\text{add}(\text{coprod}(\mathcal{A})) = \text{coprod}(\mathcal{A}).$$

The associativity of the $\star$ operation gives

$$\begin{aligned} \text{coprod}_m(\mathcal{A}) \star \text{coprod}_n(\mathcal{A}) &= \text{coprod}_{m+n}(\mathcal{A}), \\ \text{Coproduct}_m(\mathcal{A}) \star \text{Coproduct}_n(\mathcal{A}) &= \text{Coproduct}_{m+n}(\mathcal{A}), \end{aligned}$$

and the first of these identities tells us that $\text{coprod}(\mathcal{A}) = \bigcup_{n=1}^{\infty} \text{coprod}_n(\mathcal{A})$ satisfies

$$\text{coprod}(\mathcal{A}) \star \text{coprod}(\mathcal{A}) = \text{coprod}(\mathcal{A}).$$

We prove next the little lemma:

Lemma 1.6. *Let $\mathcal{T}$ be a triangulated category and suppose we are given subcategories $\mathcal{A}, \mathcal{C}, \mathcal{S}, \mathcal{X}, \mathcal{Z} \subset \mathcal{T}$. Assume $\text{add}(\mathcal{A}) = \mathcal{A}$ and $\text{add}(\mathcal{C}) = \mathcal{C}$. Suppose that*

- (i) *For any object $s \in \mathcal{S}$, any morphisms $s \rightarrow x$ and $s \rightarrow z$ with $x \in \mathcal{X}$ and $z \in \mathcal{Z}$ factor as*

$$s \rightarrow a \rightarrow x, \quad s \rightarrow c \rightarrow z$$

with $a \in \mathcal{A}$ and $c \in \mathcal{C}$.

- (ii) *Any morphism $d \rightarrow x$, with $x \in \mathcal{X}$ and $d \in (\Sigma^{-1}\mathcal{C}) * \mathcal{S}$, factors as $d \rightarrow a \rightarrow x$ with $a \in \mathcal{A}$.*

Then any morphism $f : s \rightarrow y$, with $s \in \mathcal{S}$ and $y \in \mathcal{X} \star \mathcal{Z}$, must factor as $s \rightarrow b \rightarrow y$ with $b \in \mathcal{A} \star \mathcal{C}$.

Proof. Because $y \in \mathcal{X} \star \mathcal{Z}$ there exists a triangle $x \rightarrow y \rightarrow z \rightarrow \Sigma x$ with $x \in \mathcal{X}$ and $z \in \mathcal{Z}$. The composite $s \rightarrow y \rightarrow z$ is a morphism from $s \in \mathcal{S}$ to $z \in \mathcal{Z}$, and by (i) it must factor as $s \rightarrow c \rightarrow z$ with $c \in \mathcal{C}$. In other words we have a diagram where the row is a triangle and the square commutes

$$\begin{array}{ccccc} & s & \longrightarrow & c & \\ & \downarrow f & & \downarrow & \\ x & \longrightarrow & y & \longrightarrow & z \longrightarrow \Sigma x \end{array}$$

This we may complete to a morphism of triangles

$$\begin{array}{ccccccc} d & \longrightarrow & s & \longrightarrow & c & \longrightarrow & \Sigma d \\ \downarrow & & \downarrow f & & \downarrow & & \downarrow \\ x & \longrightarrow & y & \longrightarrow & z & \longrightarrow & \Sigma x \end{array}$$

By (ii) the morphism $d \rightarrow x$ factors as $d \rightarrow a \rightarrow x$ with $a \in \mathcal{A}$. The composite $s \rightarrow c \rightarrow \Sigma d$ vanishes as these are two morphisms in a triangle—hence the longer composite $s \rightarrow c \rightarrow \Sigma d \rightarrow \Sigma a$ also vanishes. We obtain the following commutative

diagram

$$\begin{array}{ccccccc}
 & & s & & & & \\
 & & \searrow & & \xrightarrow{0} & & \\
 & & & c & \longrightarrow & \Sigma a & \\
 & & & \downarrow & & \downarrow & \\
 x & \longrightarrow & y & \longrightarrow & z & \longrightarrow & \Sigma x
 \end{array}$$

The commutative square on the bottom right may be extended to a morphism of triangles

$$\begin{array}{ccccccc}
 a & \longrightarrow & \tilde{b} & \longrightarrow & c & \longrightarrow & \Sigma a \\
 \downarrow & & \downarrow g & & \downarrow & & \downarrow \\
 x & \longrightarrow & y & \longrightarrow & z & \longrightarrow & \Sigma x
 \end{array}$$

with $\tilde{b} \in \mathcal{A} \star \mathcal{C}$, and the vanishing of $s \rightarrow c \rightarrow \Sigma a$ tells us that $s \rightarrow c$ must factor as $s \xrightarrow{h} \tilde{b} \rightarrow c$. We have produced an object $\tilde{b} \in \mathcal{A} \star \mathcal{C}$ and morphisms $s \xrightarrow{h} \tilde{b} \xrightarrow{g} y$. A diagram chase shows that the two composites $s \rightarrow z$ in the diagram

$$\begin{array}{ccccc}
 s & \xrightarrow{h} & \tilde{b} & & \\
 \downarrow f & \searrow g & & & \\
 x & \longrightarrow & y & \longrightarrow & z \longrightarrow \Sigma x
 \end{array}$$

are equal. It follows that $f - gh$ must factor as $s \rightarrow x \rightarrow y$. But $s \in \mathcal{S}$ and $x \in \mathcal{X}$, and (i) guarantees that $s \rightarrow x$ factors as $s \rightarrow \tilde{a} \rightarrow x$ with $\tilde{a} \in \mathcal{A}$.

The fact that $\text{add}(\mathcal{C}) = \mathcal{C}$ implies that $0 \in \mathcal{C}$, and therefore $\mathcal{A} = \mathcal{A} \star \{0\} \subset \mathcal{A} \star \mathcal{C}$. Therefore $\tilde{a} \in \mathcal{A}$ implies $\tilde{a} \in \mathcal{A} \star \mathcal{C}$. Since we also have $\text{add}(\mathcal{A}) = \mathcal{A}$ we deduce that $\text{add}(\mathcal{A} \star \mathcal{C}) = \mathcal{A} \star \mathcal{C}$, and the fact that $\tilde{a}$ and $\tilde{b}$ both lie in $\mathcal{A} \star \mathcal{C}$ means that so does $\tilde{a} \oplus \tilde{b}$. But now the map $f - gh$ factors through $\tilde{a}$ and gh factors through $\tilde{b}$, hence $f = (f - gh) + gh$ factors through $\tilde{a} \oplus \tilde{b} \in \mathcal{A} \star \mathcal{C}$. $\square$

Remark 1.7. If $\mathcal{S} \subset \mathcal{T}$ is a triangulated subcategory and contains $\mathcal{C}$, then $(\Sigma^{-1}\mathcal{C}) * \mathcal{S} \subset \mathcal{S}$ and hypothesis (ii) of Lemma 1.6 follows from hypothesis (i). In this article all the applications of Lemma 1.6 will be in situations where $\mathcal{S}$ is triangulated and contains $\mathcal{C}$.

As it has turned out, in a sequel we will need to apply Lemma 1.6 in the generality of its statement—see [34, Lemmas 2.2 and 2.3].

Lemma 1.8. *Let $\mathcal{T}$ be a triangulated category with coproducts, let $\mathcal{T}^c$ be the subcategory of compact objects in $\mathcal{T}$, and let $\mathcal{B} \subset \mathcal{T}^c$ be any subcategory. Then*

- (i) *For $x \in \text{Coproduct}_n(\mathcal{B})$ and $s \in \mathcal{T}^c$, any map $s \rightarrow x$ factors as $s \rightarrow b \rightarrow x$ with $b \in \text{coprod}_n(\mathcal{B})$.*
- (ii) *For $x \in \text{Coproduct}(\mathcal{B})$ and $s \in \mathcal{T}^c$, any map $s \rightarrow x$ factors as $s \rightarrow b \rightarrow x$ with $b \in \text{coprod}(\mathcal{B})$.*

Proof. Let us first prove (i). We begin with the case $n = 1$; any map $s \longrightarrow x$, with $s \in \mathcal{T}^c$ and $x \in \text{Coproduct}_1(\mathcal{B}) = \text{Add}(\mathcal{B})$, is a map from the compact object s to a coproduct of objects in $\mathcal{B}$, and hence factors through a finite subcoproduct. In particular it factors through an object of $\text{add}(\mathcal{B}) = \text{coprod}_1(\mathcal{B})$.

Now suppose we know the theorem for all integers up to n . Apply Lemma 1.6 with $\mathcal{S} = \mathcal{T}^c$, $\mathcal{A} = \text{coprod}_1(\mathcal{B})$, $\mathcal{C} = \text{coprod}_n(\mathcal{B})$, $\mathcal{X} = \text{Coproduct}_1(\mathcal{B})$, $\mathcal{Z} = \text{Coproduct}_n(\mathcal{B})$.

Induction tells us that the hypotheses of Lemma 1.6 are satisfied, hence any map from an object in $\mathcal{T}^c = \mathcal{S}$ to an object in $\text{Coproduct}_{n+1}(\mathcal{B}) = \mathcal{X} \star \mathcal{Z}$ factors through an object in $\text{coprod}_{n+1}(\mathcal{B}) = \mathcal{A} \star \mathcal{C}$.

It remains to prove (ii). Let $\mathcal{R}$ be the full subcategory of all objects $r \in \mathcal{T}$ so that any map $s \longrightarrow r$, with $s \in \mathcal{T}^c$, factors through an object in $\text{coprod}(\mathcal{B})$. We need to show that $\text{Coproduct}(\mathcal{B}) \subset \mathcal{R}$, and to do this we will prove three things. First:

(iii) $\mathcal{B} \subset \mathcal{R}$.

This is obvious because any map $s \longrightarrow b$, with $b \in \mathcal{B}$, factors as $s \longrightarrow b \xrightarrow{1} b$. Next

(iv) $\text{Add}(\mathcal{R}) \subset \mathcal{R}$.

Proof of (iv). Suppose we are given an object $s \in \mathcal{T}^c$, a set of objects $\{r_\lambda \in \mathcal{R}, \lambda \in \Lambda\}$, and a morphism

$$s \xrightarrow{f} \coprod_{\lambda \in \Lambda} r_\lambda.$$

Because s is compact f can be factored as

$$s \xrightarrow{\Delta} \bigoplus_{i=1}^n s \xrightarrow{\oplus_{i=1}^n f_i} \bigoplus_{i=1}^n r_i \xrightarrow{I} \coprod_{\lambda \in \Lambda} r_\lambda$$

where Δ is the diagonal map and I is the inclusion of a finite subcoproduct. Because $s \in \mathcal{T}^c$ and $r_i \in \mathcal{R}$, each map $f_i : s \longrightarrow r_i$ factors as $s \longrightarrow c_i \longrightarrow r_i$ with $c_i \in \text{coprod}(\mathcal{B})$. Hence the map f factors as

$$s \longrightarrow \bigoplus_{i=1}^n c_i \longrightarrow \coprod_{\lambda \in \Lambda} r_\lambda,$$

and $\oplus_{i=1}^n c_i$ belongs to $\text{add}(\text{coprod}(\mathcal{B})) = \text{coprod}(\mathcal{B})$. □

(v) $\mathcal{R} \star \mathcal{R} \subset \mathcal{R}$.

Proof of (v). Apply Lemma 1.6 with $\mathcal{S} = \mathcal{T}^c$, $\mathcal{A} = \mathcal{C} = \text{coprod}(\mathcal{B})$, and $\mathcal{X} = \mathcal{Z} = \mathcal{R}$. Any map $s \longrightarrow y$, with $s \in \mathcal{T}^c = \mathcal{S}$ and $y \in \mathcal{R} \star \mathcal{R} = \mathcal{X} \star \mathcal{Z}$, must factor through an object in $\mathcal{A} \star \mathcal{C} = \text{coprod}(\mathcal{B}) \star \text{coprod}(\mathcal{B}) = \text{coprod}(\mathcal{B})$. □

By definition $\text{Coproduct}(\mathcal{B})$ is the minimal subcategory of $\mathcal{T}$ satisfying (iii), (iv) and (v), hence $\text{Coproduct}(\mathcal{B}) \subset \mathcal{R}$. □

Proposition 1.9. *Let $\mathcal{T}$ be a triangulated category with coproducts, and let $\mathcal{B}$ be a subcategory of $\mathcal{T}^c$. Then*

- (i) Any compact object in $\text{Coprod}_n(\mathcal{B})$ belongs to $\text{smd}(\text{coprod}_n(\mathcal{B}))$.
- (ii) Any compact object in $\text{Coprod}(\mathcal{B})$ belongs to $\text{smd}(\text{coprod}(\mathcal{B}))$.

Proof. Let x be a compact object in $\text{Coprod}_n(\mathcal{B})$ [respectively in $\text{Coprod}(\mathcal{B})$]. The identity map $1 : x \rightarrow x$ is a morphism from the compact object x to $x \in \text{Coprod}_n(\mathcal{B})$ [respectively to $x \in \text{Coprod}(\mathcal{B})$], and Lemma 1.8(i) [respectively Lemma 1.8(ii)] tells us that $1 : x \rightarrow x$ factors through an object $b \in \text{coprod}_n(\mathcal{B})$ [respectively $b \in \text{coprod}(\mathcal{B})$]. Thus x is a direct summand of $b \in \text{coprod}_n(\mathcal{B})$ [respectively of $b \in \text{coprod}(\mathcal{B})$]. $\square$

Lemma 1.10. *Let $\mathcal{T}$ be a triangulated category with coproducts, and let $\mathcal{B}$ be an arbitrary subcategory. Then*

$$\text{Coprod}_n(\mathcal{B}) \subset \text{smd}(\text{Coprod}_n(\mathcal{B})) \subset \text{Coprod}_{2n}(\mathcal{B} \cup \Sigma\mathcal{B}) .$$

Proof. The inclusion $\text{Coprod}_n(\mathcal{B}) \subset \text{smd}(\text{Coprod}_n(\mathcal{B}))$ is obvious, we need to prove the inclusion $\text{smd}(\text{Coprod}_n(\mathcal{B})) \subset \text{Coprod}_{2n}(\mathcal{B} \cup \Sigma\mathcal{B})$. Assume therefore that x is an object of $\text{smd}(\text{Coprod}_n(\mathcal{B}))$, that is there is an object $b \in \text{Coprod}_n(\mathcal{B})$ containing x as a direct summand. But then there is an idempotent map $e : b \rightarrow b$ whose image is x and, by [38, proof of Proposition 1.6.8], x is isomorphic to the homotopy colimit of the sequence $b \xrightarrow{e} b \xrightarrow{e} b \xrightarrow{e} \dots$. In other words there is a triangle

$$\coprod_{i=0}^{\infty} b \longrightarrow x \longrightarrow \Sigma \left[\coprod_{i=0}^{\infty} b \right] \longrightarrow$$

where $\coprod_{i=0}^{\infty} b \in \text{Add}(\text{Coprod}_n(\mathcal{B})) = \text{Coprod}_n(\mathcal{B})$, and therefore x belongs to

$$\begin{aligned} \text{Coprod}_n(\mathcal{B}) \star \text{Coprod}_n(\Sigma\mathcal{B}) &\subset \text{Coprod}_n(\mathcal{B} \cup \Sigma\mathcal{B}) \star \text{Coprod}_n(\mathcal{B} \cup \Sigma\mathcal{B}) \\ &= \text{Coprod}_{2n}(\mathcal{B} \cup \Sigma\mathcal{B}) . \end{aligned}$$

$\square$

Notation 1.11. Let $\mathcal{T}$ be a triangulated category with coproducts, and let $\mathcal{B} \subset \mathcal{T}$ be a subcategory. For any pair of integers $m \leq n$ we will write

$$\mathcal{B}[m, n] = \bigcup_{i=-n}^{-m} \Sigma^i \mathcal{B} .$$

In this notation, Lemma 1.10 asserts that

$$\text{Coprod}_n(\mathcal{B}) \subset \text{smd}(\text{Coprod}_n(\mathcal{B})) \subset \text{Coprod}_{2n}(\mathcal{B}[-1, 0]) .$$

We permit $m = -\infty$ and/or $n = \infty$; for example $\mathcal{B}[m, \infty) = \bigcup_{i=-\infty}^{-m} \Sigma^i \mathcal{B}$.

In the Introduction, more precisely in the paragraph right before Theorem 0.16, we introduced the subcategories $\overline{\langle G \rangle}_N^{[A, B]}$ and then went on to express the results in terms of them. We are now in a position to compare the subcategories $\overline{\langle G \rangle}_N^{[A, B]}$ of the Introduction to the subcategories $\text{Coprod}_N(\mathcal{B}[A, B])$ of this section. To do so, we adopt the notation that when $\{G\}$ is the subcategory with just one object G , we will write $\text{Coprod}_N(G[A, B])$ for what should more accurately be denoted $\text{Coprod}_N(\{G\}[A, B])$.

Corollary 1.12. *For integers $N > 0$, $A \leq B$ the identity $\overline{\langle G \rangle}_N^{[A,B]} = \text{smd}(\text{Coproduct}_N(G[A, B]))$ always holds. We furthermore have inclusions*

$$\text{Coproduct}_N(G[A, B]) \subset \overline{\langle G \rangle}_N^{[A,B]} \subset \text{Coproduct}_{2N}(G[A-1, B]) .$$

Proof. When $N = 1$ the identity $\overline{\langle G \rangle}_1^{[A,B]} = \text{smd}(\text{Coproduct}_1(G[A, B]))$ is by the definitions of the two sides. For general N the result follows by induction, since

$$\begin{aligned} \overline{\langle G \rangle}_{N+1}^{[A,B]} &= \text{smd}[\overline{\langle G \rangle}_1^{[A,B]} \star \overline{\langle G \rangle}_N^{[A,B]}] \\ &= \text{smd}[\text{smd}(\text{Coproduct}_1(G[A, B])) \star \text{smd}(\text{Coproduct}_N(G[A, B]))] \\ &= \text{smd}[\text{Coproduct}_1(G[A, B]) \star \text{Coproduct}_N(G[A, B])] \\ &= \text{smd}[\text{Coproduct}_{N+1}(G[A, B])] , \end{aligned}$$

where the third equality is by [7, Lemma 2.2.1(ii)].

It remains to prove the “furthermore” assertion of the Corollary. Lemma 1.10 asserts that, for any $\mathcal{B}$,

$$\text{Coproduct}_N(\mathcal{B}) \subset \text{smd}(\text{Coproduct}_N(\mathcal{B})) \subset \text{Coproduct}_{2N}(\mathcal{B} \cup \Sigma \mathcal{B}) ;$$

applying this to $\mathcal{B} = \{\Sigma^i G, -B \leq i \leq -A\}$ we conclude

$$\text{Coproduct}_N(G[A, B]) \subset \overline{\langle G \rangle}_N^{[A,B]} \subset \text{Coproduct}_{2N}(G[A-1, B]) .$$

□

Remark 1.13. Corollary 1.12 tells us that, for the purpose of the statements of Theorems 0.16 and 0.18 of the Introduction, the subcategories $\overline{\langle G \rangle}_N^{[A,B]}$ and $\text{Coproduct}_N(G[A, B])$ are interchangeable—either is contained in the other up to changing the integers N, A, B . As stated, Theorems 0.16 and 0.18 are assertions that there exist integers N, A, B so that the categories $\overline{\langle G \rangle}_N^{[A,B]}$ are large enough to contain certain objects. Corollary 1.12 implies that it is equivalent for there to exist integers N', A', B' so that the categories $\text{Coproduct}_{N'}(G[A', B'])$ are large enough.

In the rest of the current article we will use only the categories $\text{Coproduct}_N(G[A, B])$, and never again mention $\overline{\langle G \rangle}_N^{[A,B]}$. We will prove Theorems 0.16 and 0.18 restated in terms of $\text{Coproduct}_N(G[A, B])$, which happen to work better in iterating the approximations that come up in the proofs.

The subcategories $\overline{\langle G \rangle}_N^{[A,B]}$ are close to the historical precursors $\overline{\langle G \rangle}_N$ considered by Bondal and Van den Bergh, and the subcategories $\text{Coproduct}_N(G[A, B])$ are a further refinement. As we have already said, back in Remark 1.4, we have introduced technical variants of the definitions of Bondal and Van den Bergh that turn out to be useful in certain proofs. But the truth is that most of the time there is little need to consider the finest version of these invariants, and in subsequent articles we will revert to using

the coarser $\overline{\langle G \rangle}_N^{[A,B]}$ in place of the finer $\text{Coproduct}_N(G[A, B])$. In the string of sequels [35, 8, 34, 33] there is exactly one point at which a proof really makes use the finer $\text{Coproduct}_N(G[A, B])$, with the cruder $\overline{\langle G \rangle}_N^{[A,B]}$ being inadequate. See [8, Lemmas 4.3, 4.4 and 4.5].

We have proved a number of general lemmas, and it is time to specialize a little. We end the section with an example.

Example 1.14. Let $\mathcal{T} = \mathbf{D}(\mathcal{A})$ for some abelian category $\mathcal{A}$ satisfying AB4 (that is coproducts exist and are exact). Let $\mathcal{B} \subset \mathcal{A}$ be a subcategory, which we view as embedded in $\mathbf{D}(\mathcal{A})$ in degree 0. If $m \leq n$ are integers and x^i , $m \leq i \leq n$ are coproducts of objects in $\mathcal{B}$, then any cochain complex of the form

$$\cdots \longrightarrow 0 \longrightarrow x^m \longrightarrow x^{m+1} \longrightarrow \cdots \longrightarrow x^{n-1} \longrightarrow x^n \longrightarrow 0 \longrightarrow \cdots$$

belongs to $\text{Coproduct}_{n-m+1}(\mathcal{B}[m, n])$. We see this by induction on $n - m$: if $n - m = 0$ the complex

$$\cdots \longrightarrow 0 \longrightarrow x^m \longrightarrow 0 \longrightarrow \cdots$$

is a coproduct of objects in $\Sigma^{-m}\mathcal{B}$, that is it belongs to $\text{Coproduct}_1(\mathcal{B}[m, m])$. The general case follows inductively, by considering the triangles

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & x^i & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & x^m & \longrightarrow & x^{m+1} & \longrightarrow & \cdots & \longrightarrow & x^{i-1} & \longrightarrow & x^i & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & x^m & \longrightarrow & x^{m+1} & \longrightarrow & \cdots & \longrightarrow & x^{i-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \end{array}$$

where the top row belongs to $\text{Coproduct}_1(\mathcal{B}[i, i])$ while the bottom row belongs to $\text{Coproduct}_{i-m}(\mathcal{B}[m, i-1])$.

2. UNBOUNDED VERSIONS OF THEOREMS 0.5 AND 0.15

As mentioned in the Introduction, in this article we will provide unbounded versions of Theorems 0.5 and 0.15. In this section we first state them, and then prove that they imply Theorems 0.5 and 0.15. First for the unbounded version of Theorem 0.5:

Theorem 2.1. *Let X be a quasicompact, separated scheme. Suppose X can be covered by open affine subschemes $\text{Spec}(R_i)$ with each R_i of finite global dimension. Then there exists an object $G \in \mathbf{D}^{\text{perf}}(X)$ and an integer $n > 0$ with $\mathbf{D}_{\text{qc}}(X) = \text{Coproduct}_n(G(-\infty, \infty))$.*

The following result is well-known, but the proof is so simple that we include it.

Lemma 2.2. *Theorem 0.5 follows from Theorem 2.1.*

Proof. Theorem 0.5 is an if and only if statement, but in Remark 0.11 we noted that one direction is easy. It sufficed to prove that, if X satisfies the hypotheses of Theorem 2.1 and Theorem 2.1 is known to be true, then $\mathbf{D}^{\text{perf}}(X)$ is regular. Let $G \in \mathbf{D}^{\text{perf}}(X)$ be the object whose existence is guaranteed by Theorem 2.1. Put $\mathcal{B} = \{\Sigma^i G, i \in \mathbb{Z}\}$. Then Theorem 2.1 tells us that $\mathbf{D}_{\text{qc}}(X) = \text{Coproduct}_n(\mathcal{B})$, and Proposition 1.9(i) gives that $\mathbf{D}^{\text{perf}}(X) = \text{smd}(\text{coprod}_n(\mathcal{B}))$. This certainly implies that G strongly generates $\mathbf{D}^{\text{perf}}(X)$. $\square$

Now for the unbounded version of Theorem 0.15:

Theorem 2.3. *Let X be a scheme satisfying Hypothesis 0.14. Then there exists an object $G \in \mathbf{D}_{\text{coh}}^b(X)$ and an integer $n > 0$ with $\mathbf{D}_{\text{qc}}(X) = \text{Coproduct}_n(G(-\infty, \infty))$.*

The fact that Theorem 0.15 follows from Theorem 2.3 is not quite so immediate, we devote the rest of the section to the proof.

Lemma 2.4. *Let $\mathcal{T}$ be a triangulated category with coproducts and a t -structure so that $\mathcal{T}^{\geq 0}$ is closed under coproducts, and let G be a bounded object of $\mathcal{T}$. There is an integer M so that*

$$\mathcal{T}^{\geq n} \cap \text{Coproduct}_N(G(-\infty, \infty)) \quad \subset \quad \text{Coproduct}_{2N}(G[n - M, \infty)) .$$

Proof. Without loss of generality we may assume $n = 0$, and replacing G by a suspension we may assume $G \in \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq a}$ for some $a \leq 0$. Let $\mathcal{I}$ be the ideal of all maps $f : x \rightarrow y$ in $\mathcal{T}$ so that any composite $\Sigma^i G \rightarrow x \xrightarrow{f} y$ vanishes, for any $i \in \mathbb{Z}$ and any map $\Sigma^i G \rightarrow x$. Now we observe

- If $w \in \mathcal{T}^{\geq m}$ is any object, there exists a triangle $w' \rightarrow w \rightarrow w'' \rightarrow$ in $\mathcal{T}$ with $w \rightarrow w''$ in $\mathcal{I}$, with $w'' \in \mathcal{T}^{\geq m+a-1}$ and with $w' \in \text{Coproduct}_1(G[m, \infty))$.

Proof of •. We let w' be the coproduct, over all nonzero maps $\Sigma^i G \rightarrow w$, of $\Sigma^i G$. Let $w' \rightarrow w$ be the obvious map, and complete to a triangle $w' \rightarrow w \rightarrow w'' \rightarrow$. Since $w \in \mathcal{T}^{\geq m}$ and $G \in \mathcal{T}^{\leq 0}$ a nonzero map $\Sigma^i G \rightarrow w$ can only happen when $i \leq -m$, giving $w' \in \text{Coproduct}_1(G[m, \infty))$, which also means $w' \in \mathcal{T}^{\geq m+a}$. The fact that $w \rightarrow w''$ belongs to the ideal $\mathcal{I}$ is immediate, and the triangle $w \rightarrow w'' \rightarrow \Sigma w'$, coupled with the fact that $w \in \mathcal{T}^{\geq m}$ and $\Sigma w' \in \mathcal{T}^{\geq m+a-1}$, gives $w'' \in \mathcal{T}^{\geq m+a-1}$ [recall that $a \leq 0$]. $\square$

Next choose $x \in \mathcal{T}^{\geq 0} \cap \text{Coproduct}_N(G(-\infty, \infty))$ and proceed inductively. Put $x_0 = x$, and let the map $x_i \rightarrow x_{i+1}$ be the morphism $x_i \rightarrow x_i''$ of •. Induction tells us that x_i belongs to $\mathcal{T}^{\geq i(a-1)}$, that the map $x_0 \rightarrow x_{i+1}$ belongs to $\mathcal{T}^{i+1}$, and that the object w_{i+1} in the triangle $w_{i+1} \rightarrow x_0 \rightarrow x_{i+1}$ belongs to $\text{Coproduct}_{i+1}(G[i(a-1), \infty))$. Since $x = x_0$ is assumed to belong to $\text{Coproduct}_N(G(-\infty, \infty))$, an easy induction on N shows that the map $x \rightarrow x_N \in \mathcal{T}^N$ must vanish. Hence the map $w_N \rightarrow x$ is a split epimorphism, with $w_N \in \text{Coproduct}_N(G[(N-1)(a-1), \infty))$. This makes x a direct summand of an object in $\text{Coproduct}_N(G[(N-1)(a-1), \infty))$, and Lemma 1.10 now tells us that x belongs to $\text{Coproduct}_{2N}(G[Na - N - a, \infty))$. $\square$

Lemma 2.5. *Let X be a noetherian scheme and G an object in $\mathbf{D}_{\text{coh}}^b(X)$. Any map $f : E \rightarrow F$, with $E \in \mathbf{D}_{\text{coh}}^b(X)$ and $F \in \text{Coproduct}_N(G[M, \infty))$, factors through an object $F' \in \text{coprod}_N(G[M, \infty))$.*

Proof. We prove this by induction on N , let us begin with the case $N = 1$. Since G is bounded it belongs to some $\mathbf{D}_{\text{qc}}(X)^{\geq a}$, and hence $\text{Coproduct}_1(G[M, \infty)) \subset \mathbf{D}_{\text{qc}}(X)^{\geq M+a}$. By [29, Theorem 4.1] there exists a triangle $C \rightarrow E \rightarrow K \rightarrow$ with C compact and $K \in \mathbf{D}_{\text{qc}}(X)^{\leq M+a-1}$. The composite $C \rightarrow E \xrightarrow{f} F$ is a map from a compact object C to a coproduct $F = \coprod_{\lambda \in \Lambda} \Sigma^{i_\lambda} G$, and hence factors through a finite subcoproduct. That is we can write $F = F' \oplus F''$, with $F' \in \text{coprod}_1(G[M, \infty))$ a finite subcoproduct, so that the square in the diagram below commutes

$$\begin{array}{ccccccc} C & \longrightarrow & E & \longrightarrow & K & \longrightarrow & \\ \downarrow & & \downarrow f & & & & \\ F' & \longrightarrow & F' \oplus F'' & \longrightarrow & F'' & \longrightarrow & \end{array}$$

The rows are triangles, hence we may complete to a morphism of triangles

$$\begin{array}{ccccccc} C & \longrightarrow & E & \longrightarrow & K & \longrightarrow & \\ \downarrow & & \downarrow f & & \downarrow & & \\ F' & \longrightarrow & F' \oplus F'' & \longrightarrow & F'' & \longrightarrow & \end{array}$$

But $K \in \mathbf{D}_{\text{qc}}(X)^{\leq M+a-1}$ and F'' is a summand of $F' \oplus F'' \in \mathbf{D}_{\text{qc}}(X)^{\geq M+a}$, and the map $K \rightarrow F''$ must vanish. Therefore the map $E \rightarrow F = F' \oplus F''$ must factor through $F' \in \text{coprod}_1(G[M, \infty))$.

We have proved the case $N = 1$ of the Lemma, and now it's time for the induction step. Suppose therefore that we know the statement up to $N \geq 1$. We apply Lemma 1.6 with $\mathcal{T} = \mathbf{D}_{\text{qc}}(X)$, and with

$$\mathcal{S} = \mathbf{D}_{\text{coh}}^b(X), \quad \mathcal{A} = \text{coprod}_1(G[M, \infty)), \quad \mathcal{C} = \text{coprod}_N(G[M, \infty)),$$

$$\mathcal{X} = \text{Coproduct}_1(G[M, \infty)), \quad \mathcal{Z} = \text{Coproduct}_N(G[M, \infty)).$$

We deduce that any map $E \rightarrow F$, with $E \in \mathcal{S} = \mathbf{D}_{\text{coh}}^b(X)$ and $F \in \mathcal{X} \star \mathcal{Z} = \text{Coproduct}_{N+1}(G[M, \infty))$, must factor through an $F' \in \mathcal{A} \star \mathcal{C} = \text{coprod}_{N+1}(G[M, \infty))$. $\square$

Lemma 2.6. *Let X be a noetherian scheme and let G be an object in $\mathbf{D}_{\text{coh}}^b(X)$. Then $\mathbf{D}_{\text{coh}}^b(X) \cap \text{Coproduct}_N(G(-\infty, \infty))$ is contained in*

$$\text{smd} \left[\text{coprod}_{2N}(G(-\infty, \infty)) \right].$$

Proof. Observe that, for the standard t -structure on $\mathbf{D}_{\text{qc}}(X)$, we have $\mathbf{D}_{\text{coh}}^b(X) = \bigcup_{n=-\infty}^0 [\mathbf{D}_{\text{coh}}^b(X) \cap \mathbf{D}_{\text{qc}}(X)^{\geq n}]$. It therefore suffices to show that, for every $n \leq 0$,

$$\mathbf{D}_{\text{coh}}^b(X) \cap \mathbf{D}_{\text{qc}}(X)^{\geq n} \cap \text{Coproduct}_N(G(-\infty, \infty)) \subset \text{smd} \left[\text{coprod}_{2N}(G(-\infty, \infty)) \right].$$

Lemma 2.4 gives us the inclusion

$$\mathbf{D}_{\text{coh}}^b(X) \cap \mathbf{D}_{\text{qc}}(X)^{\geq n} \cap \text{Coproduct}_N(G(-\infty, \infty)) \subset \mathbf{D}_{\text{coh}}^b(X) \cap \text{Coproduct}_{2N}(G[n - M, \infty))$$

for some integer M ; it suffices to show that any object $x \in \mathbf{D}_{\text{coh}}^b(X) \cap \text{Coproduct}_{2N}(G[n - M, \infty))$ belongs to $\text{smd}[\text{coprod}_{2N}(G(-\infty, \infty))]$.

To show this observe that the identity map $1 : x \rightarrow x$ is a map from $x \in \mathbf{D}_{\text{coh}}^b(X)$ to $x \in \text{Coproduct}_{2N}(G[n - M, \infty))$, and by Lemma 2.5 it must factor through an object $b \in \text{coprod}_{2N}(G[n - M, \infty))$. Thus x is a direct summand of b , which belongs to

$$\text{coprod}_{2N}(G[n - M, \infty)) \subset \text{coprod}_{2N}(G(-\infty, \infty)) .$$

□

And now it's time to finish off.

Lemma 2.7. *Theorem 0.15 follows from Theorem 2.3.*

Proof. Suppose X satisfies Hypothesis 0.14. Theorem 2.3 allows us to choose an object $G \in \mathbf{D}_{\text{coh}}^b(X)$ and an integer $N > 0$ so that $\mathbf{D}_{\text{qc}}(X) = \text{Coproduct}_N(G(-\infty, \infty))$. Therefore

$$\begin{aligned} \mathbf{D}_{\text{coh}}^b(X) &= \mathbf{D}_{\text{coh}}^b(X) \cap \mathbf{D}_{\text{qc}}(X) \\ &= \mathbf{D}_{\text{coh}}^b(X) \cap \text{Coproduct}_N(G(-\infty, \infty)) \\ &\subset \text{smd}[\text{coprod}_{2N}(G(-\infty, \infty))] \\ &\subset \langle G \rangle_{2N} , \end{aligned}$$

where the first inclusion is by Lemma 2.6 and the second inclusion is obvious. □

3. AN OBJECT $G \in \mathbf{D}_{\text{coh}}^b(X)$ GENERATING $\mathbf{D}_{\text{qc}}(X)$ IN FINITELY MANY STEPS

This section is devoted to the proof that Theorem 2.3 follows from Theorem 2.1. Let us begin with a general little lemma.

Lemma 3.1. *Let $(\mathcal{T}, \otimes, \mathbb{1})$ be a tensor triangulated category and let $H \in \mathcal{T}$ be an object. The thick tensor ideal generated by H is the union, over all objects $C \in \mathcal{T}$ and all integers $N > 0$, of the subcategories $\langle C \otimes H \rangle_N$.*

Proof. Let $\mathcal{J}$ be the thick tensor ideal generated by H . Because $\mathcal{J}$ is an ideal containing H we have $C \otimes H \in \mathcal{J}$ for every $C \in \mathcal{T}$, and because $\mathcal{J}$ is thick $\langle C \otimes H \rangle_N \subset \mathcal{J}$. Therefore

$$\bigcup_{N>0, C \in \mathcal{T}} \langle C \otimes H \rangle_N \subset \mathcal{J} .$$

We need to prove the reverse inclusion. For this it suffices to show that $\mathcal{J} = \bigcup_{N>0, C \in \mathcal{T}} \langle C \otimes H \rangle_N$ is a thick tensor ideal containing H .

Trivially $H \in \langle \mathbb{1} \otimes H \rangle_1 \subset \mathcal{J}$. Now let K be an object in $\mathcal{T}$. For any object $C \in \mathcal{T}$ and any integer $N > 0$ we have

$$K \otimes \langle C \otimes H \rangle_N \subset \langle K \otimes C \otimes H \rangle_N \subset \mathcal{J} ,$$

and hence $K \otimes \mathcal{J} \subset \mathcal{J}$. Therefore $\mathcal{T} \otimes \mathcal{J} \subset \mathcal{J}$, that is $\mathcal{J}$ is a tensor ideal. The fact that $\Sigma\langle C \otimes H \rangle_M = \langle C \otimes H \rangle_M$ tells us that $\Sigma\mathcal{J} = \mathcal{J}$, and the inclusions

$$\begin{aligned} \langle C \otimes H \rangle_M \star \langle C' \otimes H \rangle_N &\subset \langle (C \oplus C') \otimes H \rangle_M \star \langle (C \oplus C') \otimes H \rangle_N \\ &\subset \langle (C \oplus C') \otimes H \rangle_{M+N} \end{aligned}$$

imply that $\mathcal{J} \star \mathcal{J} \subset \mathcal{J}$; it therefore follows that $\mathcal{J}$ is triangulated. Finally the fact that $\text{smd}\langle C \otimes H \rangle_M = \langle C \otimes H \rangle_M$ implies that $\text{smd}(\mathcal{J}) = \mathcal{J}$. Hence $\mathcal{J}$ is a thick tensor ideal containing H . $\square$

And now we get down to business.

Proof that Theorem 2.3 follows from Theorem 2.1. Let X be a scheme satisfying Hypothesis 0.14; in particular X is noetherian. If X does not satisfy Theorem 2.3, the noetherian hypothesis allows us to choose a closed subscheme $Z \subset X$ minimal among those which do not satisfy Theorem 2.3. Replacing X by Z , we may assume that all proper closed subschemes $Z \subset X$ satisfy Theorem 2.3—it suffices to prove that so does X .

Next observe that we may assume X reduced: let $j : X_{\text{red}} \rightarrow X$ be the inclusion of the reduced part of X , let $\mathcal{J}$ be the ideal sheaf defining the subscheme $X_{\text{red}} \subset X$, and let n be an integer so that $\mathcal{J}^n = 0$. Because X is separated and noetherian [6, 6.7] tells us that the map $\mathbf{D}(\text{QCoh}(X)) \rightarrow \mathbf{D}_{\text{qc}}(X)$ is an equivalence—hence, up to replacing an object $C \in \mathbf{D}_{\text{qc}}(X)$ by an isomorph, we may assume any object $C \in \mathbf{D}_{\text{qc}}(X)$ is a complex of quasicoherent sheaves on X . But then the complex C admits a filtration

$$0 = \mathcal{J}^n C \subset \mathcal{J}^{n-1} C \subset \cdots \subset \mathcal{J} C \subset C$$

where the objects $\mathcal{J}^j C / \mathcal{J}^{j+1} C$ belong to $\mathbf{R}j_* \mathbf{D}_{\text{qc}}(X_{\text{red}})$. Thus C belongs to

$$[\mathbf{R}j_* \mathbf{D}_{\text{qc}}(X_{\text{red}})]^{*n} = \underbrace{[\mathbf{R}j_* \mathbf{D}_{\text{qc}}(X_{\text{red}})] \star [\mathbf{R}j_* \mathbf{D}_{\text{qc}}(X_{\text{red}})] \star \cdots \star [\mathbf{R}j_* \mathbf{D}_{\text{qc}}(X_{\text{red}})]}_{n \text{ copies}}$$

and it suffices to prove that $\mathbf{D}_{\text{qc}}(X_{\text{red}}) = \text{Coproduct}_{\tilde{N}}(\tilde{G}(-\infty, \infty))$ for some integer $\tilde{N} > 0$ and some object $\tilde{G} \in \mathbf{D}_{\text{coh}}^b(X_{\text{red}})$.

Now let $f : Y \rightarrow X$ be a regular alteration. Because Y is finite-dimensional, separated and regular we may apply Theorem 2.1 to Y , after all we are assuming Theorem 2.1. We may choose an object $G \in \mathbf{D}^{\text{perf}}(Y)$ and an integer N so that $\mathbf{D}_{\text{qc}}(Y) = \text{Coproduct}_N(G(-\infty, \infty))$. Hence $\mathbf{R}f_* \mathbf{D}_{\text{qc}}(Y) = \mathbf{R}f_* \text{Coproduct}_N(G(-\infty, \infty)) \subset \text{Coproduct}_N(\mathbf{R}f_* G(-\infty, \infty))$ with $\mathbf{R}f_* G \in \mathbf{D}_{\text{coh}}^b(X)$. The projection formula tells us that, for any object $C \in \mathbf{D}_{\text{qc}}(X)$, we have $C \otimes_X^{\mathbf{L}} \mathbf{R}f_* \mathcal{O}_Y \cong \mathbf{R}f_* \mathbf{L}f^* C$, and hence $\mathbf{D}_{\text{qc}}(X) \otimes_X^{\mathbf{L}} \mathbf{R}f_* \mathcal{O}_Y \subset \text{Coproduct}_N(\mathbf{R}f_* G(-\infty, \infty))$.

Now let us study the object $\mathbf{R}f_* \mathcal{O}_Y$. Since X is reduced and the map f is finite over a dense open set of X , there is a dense open subset $V \subset X$ over which f is finite and flat. Therefore the restriction of $\mathbf{R}f_* \mathcal{O}_Y$ to the open set V is a vector bundle—it is definitely a perfect complex. By Thomason's localization theorem we may choose a perfect complex

H on X , and a map $H \longrightarrow \mathbf{R}f_*\mathcal{O}_Y \oplus \Sigma\mathbf{R}f_*\mathcal{O}_Y$ inducing an isomorphism on V —see [47] or [36, statements 2.1.4 and 2.1.5]. Complete this map to a triangle

$$H \longrightarrow \mathbf{R}f_*\mathcal{O}_Y \oplus \Sigma\mathbf{R}f_*\mathcal{O}_Y \longrightarrow Q \longrightarrow$$

Then Q belongs to $\mathbf{D}_{\text{coh}}^b(X)$, and its restriction to V vanishes. By [42, Lemma 7.40] there is an inclusion of a closed subscheme $i : Z \longrightarrow X$, with image contained in $X - V$, and an object $P \in \mathbf{D}_{\text{coh}}^b(Z)$ so that $Q = \mathbf{R}i_*P$. Because Z is a proper closed subscheme of X Theorem 2.3 is true for Z , and we may choose an object $G' \in \mathbf{D}_{\text{coh}}^b(Z)$ and an integer M so that $\mathbf{D}_{\text{qc}}(Z) = \text{Coproduct}_M(G'(-\infty, \infty))$. Now let $C \in \mathbf{D}_{\text{qc}}(X)$ be arbitrary and tensor the triangle above with C ; we obtain the triangle

$$C \otimes_X^{\mathbf{L}} H \longrightarrow C \otimes_X^{\mathbf{L}} [\mathbf{R}f_*\mathcal{O}_Y \oplus \Sigma\mathbf{R}f_*\mathcal{O}_Y] \longrightarrow C \otimes_X^{\mathbf{L}} \mathbf{R}i_*P \longrightarrow$$

In the previous paragraph we saw that $C \otimes_X^{\mathbf{L}} [\mathbf{R}f_*\mathcal{O}_Y \oplus \Sigma\mathbf{R}f_*\mathcal{O}_Y]$ belongs to the category $\text{Coproduct}_N(\mathbf{R}f_*G(-\infty, \infty))$, while $C \otimes_X^{\mathbf{L}} \mathbf{R}i_*P \cong \mathbf{R}i_*[\mathbf{L}i^*C \otimes_Z^{\mathbf{L}} P]$ belongs to $\mathbf{R}i_*\text{Coproduct}_M(G'(-\infty, \infty)) \subset \text{Coproduct}_M(\mathbf{R}i_*G'(-\infty, \infty))$. The triangle tells us that $C \otimes_X^{\mathbf{L}} H$ belongs to

$$\text{Coproduct}_M(\mathbf{R}i_*G'(-\infty, \infty)) \star \text{Coproduct}_N(\mathbf{R}f_*G(-\infty, \infty))$$

which is contained in

$$\text{smd} \left[\text{Coproduct}_M((\mathbf{R}f_*G \oplus \mathbf{R}i_*G')(-\infty, \infty)) \star \text{Coproduct}_N((\mathbf{R}f_*G \oplus \mathbf{R}i_*G')(-\infty, \infty)) \right]$$

By Lemma 1.10 this is contained in $\text{Coproduct}_{2(M+N)}((\mathbf{R}f_*G \oplus \mathbf{R}i_*G')(-\infty, \infty))$. Thus

$$\mathbf{D}_{\text{qc}}(X) \otimes_X^{\mathbf{L}} H \subset \text{Coproduct}_{2(M+N)}((\mathbf{R}f_*G \oplus \mathbf{R}i_*G')(-\infty, \infty))$$

where $\mathbf{R}f_*G \oplus \mathbf{R}i_*G'$ belongs to $\mathbf{D}_{\text{coh}}^b(X)$.

Now let us study the object H . It is a compact object, and on the subset V it is isomorphic to the direct sum $\mathbf{R}f_*\mathcal{O}_Y \oplus \Sigma\mathbf{R}f_*\mathcal{O}_Y$. Since over V the map f is finite, flat and surjective, the object $\mathbf{R}f_*\mathcal{O}_Y$ restricts to a nowhere vanishing vector bundle on V . Thus the support of H contains the dense open set V , and since the support of the compact object H is closed it must be all of X . By Thomason [46, Theorem 3.15] (or by Balmer [4, Theorem 5.5]) the thick tensor ideal in $\mathbf{D}^{\text{perf}}(X)$ generated by H is all of $\mathbf{D}^{\text{perf}}(X)$. In particular the sheaf $\mathcal{O}_X$ belongs to the thick tensor ideal generated by H ; by Lemma 3.1 there exists an object $C \in \mathbf{D}^{\text{perf}}(X)$ and an integer L so that $\mathcal{O}_X \in \langle C \otimes H \rangle_L$. By Corollary 1.12 we have $\langle C \otimes H \rangle_L \subset \text{Coproduct}_{2L}((C \otimes H)(-\infty, \infty))$, and hence

$$\begin{aligned} \mathbf{D}_{\text{qc}}(X) &= \mathbf{D}_{\text{qc}}(X) \otimes_X^{\mathbf{L}} \mathcal{O}_X \\ &\subset \mathbf{D}_{\text{qc}}(X) \otimes_X^{\mathbf{L}} \text{Coproduct}_{2L}((C \otimes_X^{\mathbf{L}} H)(-\infty, \infty)) \\ &\subset \text{Coproduct}_{2L}(\mathbf{D}_{\text{qc}}(X) \otimes_X^{\mathbf{L}} C \otimes_X^{\mathbf{L}} H) \\ &\subset \text{Coproduct}_{2L}(\mathbf{D}_{\text{qc}}(X) \otimes_X^{\mathbf{L}} H) \\ &\subset \text{Coproduct}_{4L(M+N)}((\mathbf{R}f_*G \oplus \mathbf{R}i_*G')(-\infty, \infty)) \end{aligned}$$

completing the proof that Theorem 2.3 follows from Theorem 2.1. $\square$

4. APPROXIMATION IN THE CASE OF QUASIPROJECTIVE SCHEMES

We begin by reminding the reader of some standard facts.

Reminder 4.1. Let R be a ring. We have an exact sequence $0 \rightarrow R[x] \xrightarrow{x} R[x] \rightarrow R \rightarrow 0$. In other words: over the ring $R[x]$ the natural map is a quasi-isomorphism

$$\begin{array}{ccccccc} \longrightarrow & 0 & \longrightarrow & R[x] & \xrightarrow{x} & R[x] & \longrightarrow 0 \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \downarrow \\ \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow 0 \longrightarrow \end{array}$$

Let us denote the top row $\text{Cone}(R[x] \xrightarrow{x} R[x])$, and the quasi-isomorphism as $\text{Cone}(R[x] \xrightarrow{x} R[x]) \rightarrow R$. Tensoring this quasi-isomorphism (over R) with itself, $n+1$ times, we obtain a quasi-isomorphism

$$\bigotimes_{i=0}^n \text{Cone}(R[x_i] \xrightarrow{x_i} R[x_i]) \longrightarrow R$$

To express it slightly differently: over the polynomial ring $R[x_0, x_1, \dots, x_n]$, the Koszul complex on the sequence $(x_0, x_1, \dots, x_n)$ is a free resolution of the $R[x_0, x_1, \dots, x_n]$ -module R .

Under the standard correspondence, which takes graded $R[x_0, x_1, \dots, x_n]$ -modules to sheaves on $\mathbb{P}_R^n$, we deduce an exact sequence on $\mathbb{P}_R^n$

$$0 \longrightarrow \mathcal{O}(-n) \longrightarrow \mathcal{O}(-n+1)^{\oplus a_{n-1}} \longrightarrow \dots \longrightarrow \mathcal{O}(-1)^{\oplus a_1} \longrightarrow \mathcal{O}^{\oplus a_0} \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

where the a_i are suitable integers. This gives a quasi-isomorphism

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{O}(-n) & \longrightarrow & \mathcal{O}(-n+1)^{\oplus a_{n-1}} & \longrightarrow & \dots & \longrightarrow \mathcal{O}(-1)^{\oplus a_1} \longrightarrow \mathcal{O}^{\oplus a_0} \longrightarrow 0 \\ & \downarrow & & \downarrow & & & \downarrow & \downarrow \\ 0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow 0 & \longrightarrow \mathcal{O}(1) \longrightarrow 0 \end{array}$$

Now it's time to prove something new.

Lemma 4.2. Let R be a ring. In the derived category $\mathbf{D}_{\text{coh}}^b(\mathbb{P}_R^n) = \mathbf{D}_{\text{qc}}(\mathbb{P}_R^n)^c$ consider the full subcategory $\mathcal{B}$ whose set of objects is $\{\mathcal{O}(i), -n \leq i \leq 0\}$. Then every line bundle $\mathcal{O}(N)$ belongs to $\text{Coproduct}_{2(n+1)}(\mathcal{B}[-n-1, n])$.

Proof. The line bundles $\{\mathcal{O}(N), -n \leq N \leq 0\}$ all belong to $\mathcal{B} \subset \text{Coproduct}_{2(n+1)}(\mathcal{B}[-n-1, n])$, hence we only need to prove something for $N \notin [-n, 0]$. Let us begin with $N > 0$.

In Reminder 4.1 we recalled the quasi-isomorphism

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{O}(-n) & \longrightarrow & \mathcal{O}(-n+1)^{\oplus a_{n-1}} & \longrightarrow & \dots & \longrightarrow \mathcal{O}(-1)^{\oplus a_1} \longrightarrow \mathcal{O}^{\oplus a_0} \longrightarrow 0 \\ & \downarrow & & \downarrow & & & \downarrow & \downarrow \\ 0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow 0 & \longrightarrow \mathcal{O}(1) \longrightarrow 0 \end{array}$$

Hence $\mathcal{O}(N) = \mathcal{O}(1)^{\otimes N}$ is quasi-isomorphic to the N^{th} tensor power of the quasi-isomorphic complex in the top row. That is, $\mathcal{O}(N)$ is quasi-isomorphic to a complex

$$\cdots \longrightarrow \mathcal{O}(-n)^{\oplus b_n} \longrightarrow \mathcal{O}(-n+1)^{\oplus b_{n-1}} \longrightarrow \cdots \longrightarrow \mathcal{O}(-1)^{\oplus b_{-1}} \longrightarrow \mathcal{O}^{\oplus b_0} \longrightarrow 0$$

If we take the brutal truncation, that is we define the complex C to be

$$0 \longrightarrow \mathcal{O}(-n)^{\oplus b_n} \longrightarrow \mathcal{O}(-n+1)^{\oplus b_{n-1}} \longrightarrow \cdots \longrightarrow \mathcal{O}(-1)^{\oplus b_{-1}} \longrightarrow \mathcal{O}^{\oplus b_0} \longrightarrow 0$$

there are two nonzero homology sheaves: $\mathcal{H}^0(C) = \mathcal{O}(N)$, while $\mathcal{H}^{-n}(C) = \mathcal{E}$ is some vector bundle on $\mathbb{P}_R^n$. The t -structure truncation gives a triangle $\Sigma^n \mathcal{E} \longrightarrow C \longrightarrow \mathcal{O}(N) \longrightarrow \Sigma^{n+1} \mathcal{E}$, and the morphism $\mathcal{O}(N) \longrightarrow \Sigma^{n+1} \mathcal{E}$ belongs to $\text{Ext}_{\mathbb{P}_R^n}^{n+1}(\mathcal{O}(N), \mathcal{E}) = H^{n+1}(\mathcal{E}(-N)) = 0$. Therefore $C \cong \mathcal{O}(N) \oplus \Sigma^n \mathcal{E}$. By Example 1.14 the object C belongs to $\text{Coproduct}_{n+1}(\mathcal{B}[-n, 0])$, and the line bundle $\mathcal{O}(N)$, being a direct summand of C , belongs to $\text{smd}[\text{Coproduct}_{n+1}(\mathcal{B}[-n, 0])] \subset \text{Coproduct}_{2(n+1)}(\mathcal{B}[-n-1, 0])$. The last inclusion is by Lemma 1.10.

It remains to consider the case $N < -n$. In the above we produced, for every $M > 0$, a complex C of the form

$$0 \longrightarrow \mathcal{O}(-n)^{\oplus b_n} \longrightarrow \mathcal{O}(-n+1)^{\oplus b_{n-1}} \longrightarrow \cdots \longrightarrow \mathcal{O}(-1)^{\oplus b_{-1}} \longrightarrow \mathcal{O}^{\oplus b_0} \longrightarrow 0$$

which is quasi-isomorphic to $\mathcal{O}(M) \oplus \Sigma^n \mathcal{E}$ where $\mathcal{E}$ is a vector bundle. Then $\mathcal{R}\mathcal{H}\text{om}(C, \mathcal{O}(-n))$ is quasi-isomorphic to $\mathcal{O}(-n-M) \oplus \Sigma^{-n} \mathcal{R}\mathcal{H}\text{om}(\mathcal{E}, \mathcal{O}(-n))$. This makes $\mathcal{O}(-n-M)$ a direct summand of $\mathcal{R}\mathcal{H}\text{om}(C, \mathcal{O}(-n)) \in \text{Coproduct}_{n+1}(\mathcal{B}[0, n])$. Any $N < -n$ can be written as $N = -n - M$ for $M > 0$, and the line bundle $\mathcal{O}(N) = \mathcal{O}(-n-M)$ belongs to $\text{smd}[\text{Coproduct}_{n+1}(\mathcal{B}[0, n])] \subset \text{Coproduct}_{2(n+1)}(\mathcal{B}[-1, n])$. $\square$

Corollary 4.3. *Let R be a ring, X a scheme and $f : X \longrightarrow \mathbb{P}_R^n$ a morphism (a base-point free map). Let $\mathcal{L} = f^* \mathcal{O}(1)$. In the category $\mathbf{D}_{\mathbf{qc}}(X)$ define $\mathcal{B}$ to be the full subcategory with the set of objects $\{\mathcal{L}^i, -n \leq i \leq 0\}$.*

Then every line bundle $\mathcal{L}^N$ belongs to $\text{Coproduct}_{2(n+1)}(\mathcal{B}[-n-1, n])$.

Proof. Pull back the inclusion of Lemma 4.2 via $\mathbf{L}f^*$. $\square$

Now we are ready to prove Theorem 0.16 for quasiprojective X . Note that the bounds are quite explicit and simple.

Proposition 4.4. *Let the notation be as in Corollary 4.3, but assume furthermore that $\mathcal{L}$ is ample on X . Suppose F is an object of $\mathbf{D}_{\mathbf{qc}}(X)^{\leq B}$, that is $\mathcal{H}^i(F) = 0$ for all $i > B$, and let $A \leq B$ be an integer.*

Then there exists a triangle $D \longrightarrow E \longrightarrow F \longrightarrow$ in $\mathbf{D}_{\mathbf{qc}}(X)$, with $D \in \mathbf{D}_{\mathbf{qc}}(X)^{< A}$ and $E \in \text{Coproduct}_{(B-A+2)(2n+2)}(\mathcal{B}[A-n-2, B+n])$. If X is noetherian and the cohomology of F is coherent, then E may be chosen to be compact.

Proof. Let $\mathcal{C}$ be the full subcategory of the category of line bundles on X , whose set of objects is $\{\mathcal{L}^i, i \in \mathbb{Z}\}$. Because $\mathcal{L}$ is ample, Illusie [18, Lemme 2.2.8] tells us that the complex F has a resolution in terms of the $\mathcal{L}^i$: there exists a cochain complex

$$\cdots \longrightarrow x^{B-2} \longrightarrow x^{B-1} \longrightarrow x^B \longrightarrow 0 \longrightarrow \cdots$$

quasi-isomorphic to F , and where the x^i are all coproducts of objects in $\mathcal{C}$. If X is noetherian and F has coherent cohomology then Illusie [18, Proposition 2.2.9] allows us to choose all the x^i to be finite coproducts of objects of $\mathcal{C}$. Let E be the brutal truncation

$$\cdots \longrightarrow 0 \longrightarrow x^{A-1} \longrightarrow x^A \longrightarrow \cdots \longrightarrow x^{B-1} \longrightarrow x^B \longrightarrow 0 \longrightarrow \cdots$$

In the triangle $D \longrightarrow E \longrightarrow F \longrightarrow$ we have $D \in \mathbf{D}_{\text{qc}}(X)^{<A}$, and in the case where X is noetherian and the cohomology of F is coherent the object E is compact. Example 1.14 tells us that $E \in \text{Coproduct}_{B-A+2}(\mathcal{C}[A-1, B])$, while Corollary 4.3 guarantees that $\mathcal{C}[A-1, B] \subset \text{Coproduct}_{2(n+1)}(\mathcal{B}[A-n-2, B+n])$. The Proposition follows. $\square$

5. APPROXIMATION IN THE GENERAL CASE

Reminder 5.1. We recall [29, Theorem 4.1 and Theorem 4.2]; we only care here about the case where X is noetherian and separated. [29, Theorem 4.2] tells us that the category $\mathbf{D}_{\text{qc}}(X)$ has a compact generator G , while [29, Theorem 4.1] asserts that every object in $\mathbf{D}_{\text{coh}}^-(X)$ can be approximated by compacts arbitrarily well. This means: for any object $F \in \mathbf{D}_{\text{coh}}^-(X)$ and any integer m , there exists a triangle $D \longrightarrow E \longrightarrow F \longrightarrow$ with E a compact object of $\mathbf{D}_{\text{qc}}(X)$ and $D \in \mathbf{D}_{\text{coh}}^-(X)^{<m}$. If we abuse the notation slightly and write G for the category with one object G , [29, Theorem 4.2] tells us that the object E of [29, Theorem 4.1] lies in

$$\text{smd}[\text{coprod}(G(-\infty, \infty))] = \text{smd} \left[\bigcup_{n>0, A \leq B} \text{coprod}_n(G[A, B]) \right].$$

It follows that, for some integers $n > 0$ and $A \leq B$, the object E must belong to $\text{smd}[\text{coprod}_n(G[A, B])] \subset \text{Coproduct}_{2n}(G[A-1, B])$ where the inclusion is by Lemma 1.10.

This motivates the following

Definition 5.2. Let X be a noetherian, separated scheme. We say that a subcategory $\mathcal{S} \subset \mathbf{D}_{\text{qc}}^-(X)$ is approximable if there exists a compact generator $G \in \mathbf{D}_{\text{qc}}(X)$ with the property that, for every integer m , there exist integers $n, A \leq B$, depending only on $\mathcal{S}$, G and m , so that any object $F \in \mathcal{S}$ admits a triangle $D \longrightarrow E \longrightarrow F \longrightarrow$ with $D \in \mathbf{D}_{\text{qc}}(X)^{<m}$ and $E \in \text{Coproduct}_n(G[A, B])$.

Remark 5.3. If $\mathcal{S} \subset \mathbf{D}_{\text{qc}}^-(X)$ is approximable for one compact generator it is approximable for every compact generator. If G, G' are two compact generators, the fact that G' is compact while G generates tells us that for some integers n, A, B the object G'

belongs to $\mathrm{smd} [\mathrm{coprod}_n(G[A, B])] \subset \mathrm{Coproduct}_{2n}(G[A-1, B])$, and by symmetry G belongs to $\mathrm{smd} [\mathrm{coprod}_{n'}(G'[A', B'])] \subset \mathrm{Coproduct}_{2n'}(G'[A'-1, B'])$. Any G -approximation $D \rightarrow E \rightarrow F \rightarrow$ of an object in $F \in \mathcal{S}$ is also a G' -approximation, only the integers change.

Remark 5.4. Let G be a classical generator of $\mathbf{D}^{\mathrm{perf}}(X)$ [see Reminder 0.1 for the definition] and let G' be a compact generator in $\mathbf{D}_{\mathrm{qc}}(X)$. Because G' is compact it lies in $\mathbf{D}^{\mathrm{perf}}(X)$, and as G is a classical generator of $\mathbf{D}^{\mathrm{perf}}(X)$ we have $G' \in \mathrm{smd} [\mathrm{coprod}_n(G[A, B])]$ for some integers n, A, B . Hence G must be a compact generator for $\mathbf{D}_{\mathrm{qc}}(X)$. The statements in the introduction assert that classical generators of $\mathbf{D}^{\mathrm{perf}}(X)$ have some properties, and it suffices to show that compact generators of $\mathbf{D}_{\mathrm{qc}}(X)$ have these properties.

Example 5.5. Let X be a scheme over a ring R , with an ample line bundle $\mathcal{L}$. Suppose that some sections of $\mathcal{L}$ give a morphism (i.e. a base-point free map) $f : X \rightarrow \mathbb{P}_R^n$, and let $G = \bigoplus_{i=-n}^0 \mathcal{L}^i$. Clearly G is compact.

Since $\mathcal{L}^i$, $-n \leq i \leq 0$ are direct summands of G they belong to $\mathrm{smd} [\mathrm{coprod}_1(G[0, 0])] \subset \mathrm{Coproduct}_2(G[-1, 0])$. In the notation of Corollary 4.3 we have that $\mathcal{B} \subset \mathrm{Coproduct}_2(G[-1, 0])$, and the Corollary tells us that every line bundle of the form $\mathcal{L}^N$ belongs to

$$\mathrm{Coproduct}_{2(n+1)}(\mathcal{B}[-n-1, n]) \subset \mathrm{Coproduct}_{4(n+1)}(G[-n-2, n]) .$$

By [37, Example 1.10] or [44, Tag 0BQQ] the line bundles $\mathcal{L}^N$ weakly generate $\mathbf{D}_{\mathrm{qc}}(X)$, and from [37, Lemma 3.2] it follows that they generate. Since the $\mathcal{L}^n$ all belong to the subcategory generated by G , we conclude that G is a compact generator.

Proposition 4.4 now tells us that the category $\mathbf{D}_{\mathrm{qc}}(X)^{\leq B}$ is approximable for every integer B .

Lemma 5.6. *If $\mathcal{S}, \mathcal{S}'$ are approximable subcategories of $\mathbf{D}_{\mathrm{qc}}^-(X)$, then so is $\mathcal{S} \star \mathcal{S}'$.*

Proof. Let m be an integer and G a compact generator for $\mathbf{D}_{\mathrm{qc}}(X)$. Because $\mathcal{S}'$ is approximable there are integers n', A', B' so that any $F' \in \mathcal{S}'$ admits a triangle $D' \rightarrow E' \rightarrow F' \rightarrow$ with $D' \in \mathbf{D}_{\mathrm{qc}}(X)^{< m}$ and $E' \in \mathrm{Coproduct}_{n'}(G[A', B'])$.

Next by Hartshorne [16, Exercise III.4.8(c)] we may choose an integer ℓ such that $H^i(X, \mathcal{A}) = 0$ for all quasicoherent sheaves $\mathcal{A}$ and all $i > \ell$. Now G is compact, meaning a perfect complex, and hence $G^\vee = \mathcal{R}\mathcal{H}\mathrm{om}(G, \mathcal{O}_X)$ is also a perfect complex, and belongs to $\mathbf{D}_{\mathrm{qc}}(X)^{\leq a}$ for some integer a . If L belongs to $\mathbf{D}_{\mathrm{qc}}(X)^{\leq b}$ then $\mathcal{R}\mathcal{H}\mathrm{om}(G, L) = G^\vee \otimes L \in \mathbf{D}_{\mathrm{qc}}(X)^{\leq a+b}$, and $\mathrm{Hom}(G, L) = H^0[\mathcal{R}\mathcal{H}\mathrm{om}(G, L)] = 0$ whenever $a+b < -\ell$. That is $\mathrm{Hom}[G, \mathbf{D}_{\mathrm{qc}}(X)^{\leq -a-\ell-1}] = 0$, and hence

$$\mathrm{Hom}[\mathrm{Coproduct}_{n'}(G[A', B']) , \mathbf{D}_{\mathrm{qc}}(X)^{\leq A'-a-\ell-1}] = 0 .$$

Now let $\tilde{m} = \min(m, A' - a - \ell)$. Because $\mathcal{S}$ is approximable there exist integers n, A, B so that any object $F \in \mathcal{S}$ admits a triangle $D \rightarrow E \rightarrow F \rightarrow$ with $E \in \mathrm{Coproduct}_n(G[A, B])$ and $D \in \mathbf{D}_{\mathrm{qc}}(X)^{< \tilde{m}}$. So much for the construction: I assert that

every object $\tilde{F} \in \mathcal{S} \star \mathcal{S}'$ admits a triangle $\tilde{D} \longrightarrow \tilde{E} \longrightarrow \tilde{F} \longrightarrow$, with $\tilde{D} \in \mathbf{D}_{\text{qc}}(X)^{<m}$ and $\tilde{E} \in \text{Coproduct}_{n+n'}(G[\min(A, A'), \max(B, B')])$. It remains to prove the assertion.

Let $\tilde{F}$ be an object of $\mathcal{S} \star \mathcal{S}'$. Then there exists a triangle $F \longrightarrow \tilde{F} \longrightarrow F' \longrightarrow$ with $F \in \mathcal{S}$ and $F' \in \mathcal{S}'$. By the above there exist triangles $D \longrightarrow E \longrightarrow F \longrightarrow$ and $D' \longrightarrow E' \longrightarrow F' \longrightarrow$ with $E \in \text{Coproduct}_n(G[A, B])$, $E' \in \text{Coproduct}_{n'}(G[A', B'])$, $D \in \mathbf{D}_{\text{qc}}(X)^{<\tilde{m}}$ and $D' \in \mathbf{D}_{\text{qc}}(X)^{<m}$. We have a diagram

$$\begin{array}{ccccc} E' & \longrightarrow & F' & \longrightarrow & \Sigma D' \\ & & \downarrow & & \\ \Sigma E & \longrightarrow & \Sigma F & \longrightarrow & \Sigma^2 D \end{array}$$

where the rows are triangles, and the composite from top left to bottom right is a map from $E' \in \text{Coproduct}_{n'}(G[A', B'])$ to $\Sigma^2 D \in \mathbf{D}_{\text{qc}}(X)^{<\tilde{m}-2} \subset \mathbf{D}_{\text{qc}}(X)^{<A'-a-\ell-2}$. Since

$$\text{Hom}[\text{Coproduct}_{n'}(G[A', B']), \mathbf{D}_{\text{qc}}(X)^{\leq A'-a-\ell-1}] = 0$$

the map $E' \longrightarrow \Sigma^2 D$ must vanish, and there is a map $E' \longrightarrow \Sigma E$ rendering commutative the square

$$\begin{array}{ccc} E' & \longrightarrow & F' \\ \downarrow & & \downarrow \\ \Sigma E & \longrightarrow & \Sigma F \end{array}$$

Now complete this commutative square to a 3×3 diagram: we obtain a diagram where the rows and columns are triangles

$$\begin{array}{ccccccc} D & \longrightarrow & E & \longrightarrow & F & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ \tilde{D} & \longrightarrow & \tilde{E} & \longrightarrow & \tilde{F} & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ D' & \longrightarrow & E' & \longrightarrow & F' & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow & & \end{array}$$

From the triangle $E \longrightarrow \tilde{E} \longrightarrow E' \longrightarrow$ and the fact that $E \in \text{Coproduct}_n(G[A, B])$ and $E' \in \text{Coproduct}_{n'}(G[A', B'])$ we learn that $\tilde{E} \in \text{Coproduct}_{n+n'}(G[\min(A, A'), \max(B, B')])$. The triangle $D \longrightarrow \tilde{D} \longrightarrow D' \longrightarrow$ and the fact that $D \in \mathbf{D}_{\text{qc}}(X)^{<\tilde{m}} \subset \mathbf{D}_{\text{qc}}(X)^{<m}$ and $D' \in \mathbf{D}_{\text{qc}}(X)^{<m}$ give $\tilde{D} \in \mathbf{D}_{\text{qc}}(X)^{<m}$. And the triangle $\tilde{D} \longrightarrow \tilde{E} \longrightarrow \tilde{F} \longrightarrow$ is now as desired. $\square$

Lemma 5.7. *$f : X \rightarrow Y$ be a proper map of noetherian schemes, and let $\mathcal{S}$ be an approximable subcategory of $\mathbf{D}_{\text{qc}}^-(X)$. Then $\mathbf{R}f_*\mathcal{S}$ is an approximable subcategory of $\mathbf{D}_{\text{qc}}^-(Y)$.*

Proof. Choose a compact generator $G \in \mathbf{D}_{\text{qc}}(X)$. Because f is proper the object $\mathbf{R}f_*G$ has bounded-above coherent cohomology; it belongs to $\mathbf{D}_{\text{coh}}^-(Y)$ and in Remark 5.1 we learned that it is approximable. Slightly more generally: given any pair of integers $A \leq B$, the finite category with objects $\mathcal{B} = \{\Sigma^i \mathbf{R}f_*G, -B \leq i \leq -A\}$ is approximable. By Lemma 5.6 so is $\text{Coproduct}_n(\mathcal{B})$ for any n .

Choose an integer ℓ so that $\mathbf{R}f_*\mathbf{D}_{\text{qc}}(X)^{\leq 0} \subset \mathbf{D}_{\text{qc}}(Y)^{\leq \ell}$. Given an integer m , the fact that $\mathcal{S}$ is approximable in $\mathbf{D}_{\text{qc}}^-(X)$ allows us to choose integers n, A, B so that any object $F \in \mathcal{S}$ admits a triangle $D \rightarrow E \rightarrow F \rightarrow$ with $D \in \mathbf{D}_{\text{qc}}(X)^{< m-\ell}$ and $E \in \text{Coproduct}_n(G[A, B])$. By the first paragraph the category $\text{Coproduct}_n(\mathbf{R}f_*G[A, B])$ is approximable in $\mathbf{D}_{\text{qc}}^-(Y)$; fix a compact generator $H \in \mathbf{D}_{\text{qc}}(Y)$, and we may choose integers n', A', B' so that every object $F' \in \text{Coproduct}_n(\mathbf{R}f_*G[A, B])$ admits a triangle $D' \rightarrow E' \rightarrow F' \rightarrow$ with $D' \in \mathbf{D}_{\text{qc}}(Y)^{< m}$ and $E' \in \text{Coproduct}_{n'}(H[A', B'])$. We have now chosen all our integers. I assert that every object $\mathbf{R}f_*F \in \mathbf{R}f_*\mathcal{S}$ admits a triangle $\tilde{D} \rightarrow \tilde{E} \rightarrow \mathbf{R}f_*F \rightarrow$, with $\tilde{D} \in \mathbf{D}_{\text{qc}}(Y)^{< m}$ and $\tilde{E} \in \text{Coproduct}_{n'}(H[A', B'])$.

It remains to prove the assertion. By our choice of n, A, B , the object $F \in \mathcal{S}$ admits a triangle $D \rightarrow E \rightarrow F \rightarrow$ with $D \in \mathbf{D}_{\text{qc}}(X)^{< m-\ell}$ and $E \in \text{Coproduct}_n(G[A, B])$. In $\mathbf{D}_{\text{qc}}(Y)$ we deduce a triangle $\mathbf{R}f_*D \rightarrow \mathbf{R}f_*E \rightarrow \mathbf{R}f_*F \rightarrow$. The choice of the integer ℓ guarantees that $\mathbf{R}f_*D \in \mathbf{D}_{\text{qc}}(Y)^{< m}$, and we know that

$$\mathbf{R}f_*E \in \mathbf{R}f_*\text{Coproduct}_n(G[A, B]) \subset \text{Coproduct}_n(\mathbf{R}f_*G[A, B]) .$$

By our choice of the integers n', A', B' there exists a triangle $D' \rightarrow \tilde{E} \rightarrow \mathbf{R}f_*E \rightarrow$ with $D' \in \mathbf{D}_{\text{qc}}(Y)^{< m}$ and $\tilde{E} \in \text{Coproduct}_{n'}(H[A', B'])$. Now complete the composable maps $\tilde{E} \rightarrow \mathbf{R}f_*E \rightarrow \mathbf{R}f_*F$ to an octahedron; we obtain a diagram where the rows and columns are triangles

$$\begin{array}{ccccccc} D' & \longrightarrow & \tilde{D} & \longrightarrow & \mathbf{R}f_*D & \longrightarrow & \\ \parallel & & \downarrow & & \downarrow & & \\ D' & \longrightarrow & \tilde{E} & \longrightarrow & \mathbf{R}f_*E & \longrightarrow & \\ & & \downarrow & & \downarrow & & \\ & & \mathbf{R}f_*F & = & \mathbf{R}f_*F & & \\ & & \downarrow & & \downarrow & & \end{array}$$

The fact that $D', \mathbf{R}f_*D$ both belong to $\mathbf{D}_{\text{qc}}(Y)^{< m}$ says that so does $\tilde{D}$, and the triangle $\tilde{D} \rightarrow \tilde{E} \rightarrow \mathbf{R}f_*F \rightarrow$ has the desired properties. $\square$

And now for the proof of Theorem 0.16. The statement, rephrased in terms of the categories $\text{Coproduct}_N(G[A, B])$ as in Remark 1.13, is

Theorem 5.8. *Let X be a separated scheme, of finite type over a noetherian ring R . Then the category $\mathbf{D}_{\text{qc}}(X)^{\leq n}$ is approximable for any integer n .*

Proof. If the theorem were false, then the set of closed subschemes of X on which it fails would have a smallest member. We may therefore assume the theorem is true on every proper, closed subscheme of X .

By Chow's Lemma we may choose a proper morphism $f : Y \rightarrow X$, where Y is quasiprojective over R and f is an isomorphism on a dense open subset. In the triangle $Q \rightarrow \mathcal{O}_X \rightarrow \mathbf{R}f_*\mathbf{L}f^*\mathcal{O}_X \rightarrow$ we have that Q is isomorphic to a bounded complex of coherent sheaves, which vanish on the dense open set where f is an isomorphism. By [42, Lemma 7.40] there is a closed immersion $i : Z \rightarrow X$ and an object $P \in \mathbf{D}_{\text{coh}}^b(Z)$, so that $Q = \mathbf{R}i_*P$.

Now take any object in $F \in \mathbf{D}_{\text{qc}}(X)$; tensoring the triangle with F we obtain a triangle $F \otimes_X^{\mathbf{L}} \mathbf{R}i_*P \rightarrow F \rightarrow F \otimes_X^{\mathbf{L}} \mathbf{R}f_*\mathcal{O}_Y \rightarrow$, and the projection formula allows us to rewrite this as $\mathbf{R}i_*[\mathbf{L}i^*F \otimes P] \rightarrow F \rightarrow \mathbf{R}f_*\mathbf{L}f^*F \rightarrow$. If $F \in \mathbf{D}_{\text{qc}}(X)^{< m}$ then $\mathbf{L}f^*F \in \mathbf{D}_{\text{qc}}(Y)^{< m}$ and $\mathbf{L}i^*F \in \mathbf{D}_{\text{qc}}(Z)^{< m}$, and since P is bounded it lies in some $\mathbf{D}_{\text{qc}}(Z)^{\leq \ell}$ and hence $\mathbf{L}i^*F \otimes P \in \mathbf{D}_{\text{qc}}(Z)^{< m+\ell}$. The triangle therefore shows that $\mathbf{D}_{\text{qc}}(X)^{< m} \subset [\mathbf{R}i_*\mathbf{D}_{\text{qc}}(Z)^{< m+\ell}] \star [\mathbf{R}f_*\mathbf{D}_{\text{qc}}(Y)^{< m}]$.

Because Z is a proper closed subscheme of X the category $\mathbf{D}_{\text{qc}}(Z)^{< m+\ell}$ is approximable in $\mathbf{D}_{\text{qc}}^-(Z)$, while the category $\mathbf{D}_{\text{qc}}(Y)^{< m}$ is approximable in $\mathbf{D}_{\text{qc}}^-(Y)$ by Example 5.5. Lemma 5.7 tells us that $\mathbf{R}i_*\mathbf{D}_{\text{qc}}(Z)^{< m+\ell}$ and $\mathbf{R}f_*\mathbf{D}_{\text{qc}}(Y)^{< m}$ are approximable, and Lemma 5.6 permits us to conclude that $\mathbf{D}_{\text{qc}}(X)^{< m}$ is approximable. $\square$

6. PROOFS OF THEOREMS 0.18 AND 2.1

We begin with a straightforward corollary of Theorem 5.8:

Corollary 6.1. *Let $m \leq n$ be integers, let X be a scheme of finite type over a noetherian ring R , and let G be a classical generator for $\mathbf{D}^{\text{perf}}(X) \subset \mathbf{D}_{\text{qc}}(X)$. There exist integers N, A, B , depending only on G, m and n , so that if $F \in \mathbf{D}_{\text{qc}}(X)$ satisfies*

- (i) $F \in \mathbf{D}_{\text{qc}}(X)^{\leq n}$, and
- (ii) $\mathcal{R}\mathcal{H}\text{om}(F, Q) \in \mathbf{D}_{\text{qc}}(X)^{\leq -m}$ for every $Q \in \mathbf{D}_{\text{qc}}(X)^{\leq 0}$,

then F belongs to $\text{Coproduct}_N(G[A, B])$.

Proof. Choose an integer ℓ so that $H^i(X; \mathcal{M}) = 0$ for all quasicoherent sheaves $\mathcal{M}$ and all $i > \ell$, and then apply Theorem 5.8 to the integers $\min(n, m - \ell) \leq n$, the scheme X and the classical generator $G \in \mathbf{D}^{\text{perf}}(X)$. There exist integers $N, A \leq B$ so that any $F \in \mathbf{D}_{\text{qc}}(X)^{\leq n}$ admits a triangle $D \rightarrow E \rightarrow F \rightarrow$ with E in $\text{Coproduct}_N(G[A, B])$ and $D \in \mathbf{D}_{\text{qc}}(X)^{< m-\ell}$. Now assume that F also satisfies condition (ii) of the Corollary. The

map $F \longrightarrow \Sigma D$ is an element in

$$\begin{aligned} \mathrm{Hom}(F, \Sigma D) &\subset H^0[\mathcal{R}\mathcal{H}\mathrm{om}(F, \Sigma D)] \\ &\subset H^0[\mathbf{D}_{\mathbf{qc}}(X)^{<-\ell}] \\ &= 0. \end{aligned}$$

Hence the map $F \longrightarrow \Sigma D$ vanishes, and F is a direct summand of $E \in \mathrm{Coproduct}_N(G[A, B])$. Lemma 1.10 tells us that F belongs to $\mathrm{Coproduct}_{2N}(G[A-1, B])$. $\square$

In Sections 4 and 5 our schemes were mostly assumed noetherian. The next result drops this hypothesis: we're about to prove Theorem 0.18. We state its $\mathrm{Coproduct}_N(G[A, B])$ version for the reader's convenience. In the proof we will freely appeal to the fact that the categories $\mathbf{D}_{\mathbf{qc}}(X)$ contain compact generators for any quasicompact, quasiseparated scheme X —see [7, Theorem 3.1.1] or [29, Theorem 4.2].

Theorem 6.2. *Let $j : V \longrightarrow X$ be an open immersion of quasicompact, separated schemes, and let G be a compact generator for $\mathbf{D}_{\mathbf{qc}}(X)$. If H is any compact object of $\mathbf{D}_{\mathbf{qc}}(V)$, and we are given integers $n, a \leq b$, then there exist integers $N, A \leq B$ so that $\mathrm{Coproduct}_n(\mathbf{R}j_*H[a, b]) \subset \mathrm{Coproduct}_N(G[A, B])$.*

Proof. Since X is quasicompact we may write it as a finite union $X = \bigcup_{i=1}^\ell U_i$ of open affines U_i . Let $X_k = V \cup [\bigcup_{i=1}^k U_i]$, then we may factor j as the composite of the inclusions

$$U = X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \xrightarrow{j_3} \cdots \xrightarrow{j_{\ell-1}} X_{\ell-1} \xrightarrow{j_\ell} X_\ell = X$$

and it suffices to prove the theorem for each $j_i : X_{i-1} \longrightarrow X_i$. Thus we may suppose $X = U \cup V$ expresses X as the union of the quasicompact open subsets U and V , with U affine. By Thomason's localization theorem we may choose a compact object $\tilde{H} \in \mathbf{D}_{\mathbf{qc}}(X)$ and a quasiisomorphism $\mathbf{L}j^*\tilde{H} \cong H \oplus \Sigma H$; see Thomason and Trobaugh [47] or [37, statements 2.1.4 and 2.1.5]. Now consider the triangle

$$Q \longrightarrow \mathcal{O}_X \longrightarrow \mathbf{R}j_*\mathbf{L}j^*\mathcal{O}_X \longrightarrow \Sigma Q$$

If we tensor with $\tilde{H}$, and use the projection formula to obtain the first isomorphism in $[\mathbf{R}j_*\mathbf{L}j^*\mathcal{O}_X] \otimes_X^{\mathbf{L}} \tilde{H} \cong \mathbf{R}j_*\mathbf{L}j^*\tilde{H} \cong \mathbf{R}j_*(H \oplus \Sigma H)$, we obtain a triangle

$$Q \otimes_X^{\mathbf{L}} \tilde{H} \longrightarrow \tilde{H} \longrightarrow \mathbf{R}j_*H \oplus \Sigma \mathbf{R}j_*H \longrightarrow \Sigma Q \otimes_X^{\mathbf{L}} \tilde{H}$$

Because $\tilde{H}$ is compact in $\mathbf{D}_{\mathbf{qc}}(X)$ and G is a compact generator there exist integers $N, A \leq B$ with $\tilde{H} \in \mathrm{Coproduct}_N(G[A, B])$. From the triangle it suffices to show that there also exist integers $N', A' \leq B'$ with $Q \otimes_X^{\mathbf{L}} \tilde{H} \in \mathrm{Coproduct}_{N'}(G[A', B'])$.

Now Q vanishes on the open set V ; if $h : U \longrightarrow X$ is the inclusion then the natural map $Q \longrightarrow \mathbf{R}h_*\mathbf{L}h^*Q$ is an isomorphism. This makes $Q \otimes_X^{\mathbf{L}} \tilde{H} \cong [\mathbf{R}h_*\mathbf{L}h^*Q] \otimes_X^{\mathbf{L}} \tilde{H} \cong Q \otimes_X^{\mathbf{L}} [\mathbf{R}h_*\mathbf{L}h^*\tilde{H}]$. But $\mathbf{L}h^*\tilde{H}$ is a compact object in $\mathbf{D}_{\mathbf{qc}}(U)$ and $\mathcal{O}_U$ is a compact generator (this is because U is affine), and hence $\mathbf{L}h^*\tilde{H}$ must belong to $\mathrm{Coproduct}_M(\mathcal{O}_U[A'', B''])$

for some integers $M, A'' \leq B''$. Applying the functor $Q \otimes_X^{\mathbf{L}} \mathbf{R}h_*(-)$ tells us that $Q \otimes_X^{\mathbf{L}} [\mathbf{R}h_* \mathbf{L}h^* \tilde{H}]$ belongs to $\text{Coproduct}_M([Q \otimes_X^{\mathbf{L}} \mathbf{R}h_* \mathcal{O}_U][A'', B''])$, and it suffices to show that $Q \otimes_X^{\mathbf{L}} \mathbf{R}h_* \mathcal{O}_U$ belongs to $\text{Coproduct}_N(G[A, B])$ for some $N, A \leq B$. Now observe the isomorphisms $Q \otimes_X^{\mathbf{L}} \mathbf{R}h_* \mathcal{O}_U \cong Q \otimes_X^{\mathbf{L}} \mathbf{R}h_* \mathbf{L}h^* \mathcal{O}_X \cong \mathbf{R}h_* \mathbf{L}h^* Q \cong Q$; it remains to show that Q belongs to $\text{Coproduct}_N(G[A, B])$ for some $N, A \leq B$.

Now X is separated and U, V are quasicompact open subsets of X ; hence $U \cap V$ is quasicompact. It is a quasicompact open subset of the open affine subset $U = \text{Spec}(S)$. Absolute noetherian approximation, that is Thomason and Trobaugh [47, Theorem C.9] or [44, Tags 01YT and 081A], allows us to choose a scheme Y of finite type over $\mathbb{Z}$ and an affine map $f : X \rightarrow Y$, so that

- (i) Y is separated.
- (ii) There is an open affine subset $U' \subset Y$ and an open subset $V' \subset Y$ with $U' \cup V' = Y$, and so that $f^{-1}U' = U$ and $f^{-1}V' = V$. Let $h' : U' \rightarrow Y$, $j' : V' \rightarrow Y$ be the open immersions.

The affine scheme U' may be written as $U' = \text{Spec}(S')$, with S' a noetherian ring. The set $Y - V' = U' - U' \cap V'$ is a Zariski closed subset of the noetherian affine scheme U' , hence there are elements $\{g'_1, g'_2, \dots, g'_n\} \subset S'$ so that $Y - V'$ is precisely the subset $V(g'_1, g'_2, \dots, g'_n)$ of U' on which the g'_i all vanish. In the category $\mathbf{D}(S') \cong \mathbf{D}_{\text{qc}}(U')$ consider the complexes $L_i = \{0 \rightarrow S' \rightarrow S'[g_i'^{-1}] \rightarrow 0\}$, with S' in degree 0 and $S'[g_i'^{-1}]$ in degree 1. Put $Q' = \bigotimes_{i=1}^n L_i$; the natural cochain map $Q' \rightarrow S'$ may be completed to a triangle $Q' \rightarrow S' \rightarrow T \rightarrow$ in $\mathbf{D}(S')$, which we recognize in $\mathbf{D}(S') \cong \mathbf{D}_{\text{qc}}(U')$ as the canonical triangle

$$Q' \longrightarrow \mathcal{O}_{U'} \longrightarrow \mathbf{R}\alpha_* \mathcal{O}_{U' \cap V'} \longrightarrow$$

where $\alpha : U' \cap V' \rightarrow U'$ is the open immersion. Since $\mathbf{R}h'_* Q'$ is the extension by zero of Q' to all of Y we obtain a triangle in $\mathbf{D}_{\text{qc}}(Y)$

$$\mathbf{R}h'_* Q' \longrightarrow \mathcal{O}_Y \longrightarrow \mathbf{R}j'_* \mathcal{O}_{V'} \longrightarrow$$

and pulling back via $f : X \rightarrow Y$ we deduce that $Q \cong \mathbf{L}f^* \mathbf{R}h'_* Q'$. Note that Q' is the homotopy colimit over r of the complexes

$$\bigotimes_{i=1}^n \{S' \xrightarrow{g_i'^r} S'\}$$

and hence fits in a triangle

$$\prod_{r=1}^{\infty} \left[\bigotimes_{i=1}^n \{S' \xrightarrow{g_i'^r} S'\} \right] \longrightarrow Q' \longrightarrow \Sigma \prod_{r=1}^{\infty} \left[\bigotimes_{i=1}^n \{S' \xrightarrow{g_i'^r} S'\} \right] \longrightarrow$$

This means that the object $Q' \in \mathbf{D}(S')$ has a projective resolution that vanishes outside the interval $[-1, n]$. Let ℓ be the open immersion $\ell : U' \cap V' \rightarrow Y$. If F is any object in $\mathbf{D}_{\text{qc}}(Y)^{\leq 0}$, we have the standard triangle

$$F \longrightarrow \mathbf{R}h'_* \mathbf{L}h'^* F \oplus \mathbf{R}j'_* \mathbf{L}j'^* F \longrightarrow \mathbf{R}\ell_* \mathbf{L}\ell^* F \longrightarrow$$

and, applying the functor $\mathcal{R}\mathcal{H}\mathrm{om}(\mathbf{R}h'_*Q', -)$, we obtain a triangle

$$\begin{array}{ccc} \mathcal{R}\mathcal{H}\mathrm{om}(\mathbf{R}h'_*Q', F) & \longrightarrow & \mathcal{R}\mathcal{H}\mathrm{om}(\mathbf{R}h'_*Q', \mathbf{R}h'_*\mathbf{L}h'^*F) \oplus \mathcal{R}\mathcal{H}\mathrm{om}(\mathbf{R}h'_*Q', \mathbf{R}j'_*\mathbf{L}j'^*F) \\ & & \downarrow \\ & & \mathcal{R}\mathcal{H}\mathrm{om}(\mathbf{R}h'_*Q', \mathbf{R}\ell_*\mathbf{L}\ell^*F) \end{array}$$

The vanishing of the objects $\mathcal{R}\mathcal{H}\mathrm{om}(\mathbf{R}h'_*Q', \mathbf{R}j'_*\mathbf{L}j'^*F) \cong \mathbf{R}j'_*\mathcal{R}\mathcal{H}\mathrm{om}(\mathbf{L}j'^*\mathbf{R}h'_*Q', \mathbf{L}j'^*F)$ and $\mathcal{R}\mathcal{H}\mathrm{om}(\mathbf{R}h'_*Q', \mathbf{R}\ell_*\mathbf{L}\ell^*F) \cong \mathbf{R}\ell_*\mathcal{R}\mathcal{H}\mathrm{om}(\mathbf{L}\ell^*\mathbf{R}h'_*Q', \mathbf{L}\ell^*F)$ is because $\mathbf{L}j'^*\mathbf{R}h'_*Q' = 0 = \mathbf{L}\ell^*\mathbf{R}h'_*Q'$. From the triangle we learn that $\mathcal{R}\mathcal{H}\mathrm{om}(\mathbf{R}h'_*Q', F)$ is isomorphic to $\mathcal{R}\mathcal{H}\mathrm{om}(\mathbf{R}h'_*Q', \mathbf{R}h'_*\mathbf{L}h'^*F) = \mathbf{R}h'_*\mathcal{R}\mathcal{H}\mathrm{om}(\mathbf{L}h'^*\mathbf{R}h'_*Q', \mathbf{L}h'^*F) = \mathbf{R}h'_*\mathcal{R}\mathcal{H}\mathrm{om}(Q', \mathbf{L}h'^*F)$, which we can compute using the projective resolution for Q' on the affine open set U' . We deduce that $\mathcal{R}\mathcal{H}\mathrm{om}(\mathbf{R}h'_*Q', F) \in \mathbf{D}_{\mathbf{qc}}(Y)^{\leq 1}$ for every object F in $\mathbf{D}_{\mathbf{qc}}(Y)^{\leq 0}$.

Corollary 6.1 therefore applies. On the separated scheme Y , of finite type over $\mathbb{Z}$, we have that $\mathbf{R}h'_*Q'$ lies in $\mathrm{Coproduct}_K(G'[\overline{A}, \overline{B}])$ for some integers $K, \overline{A} \leq \overline{B}$ and some compact generator G' . Thus $Q = \mathbf{L}f^*\mathbf{R}h'_*Q'$ lies in $\mathbf{L}f^*\mathrm{Coproduct}_K(G'[\overline{A}, \overline{B}]) \subset \mathrm{Coproduct}_K(\mathbf{L}f^*G'[\overline{A}, \overline{B}])$. Since $\mathbf{L}f^*G'$ is compact and G is a compact generator of $\mathbf{D}_{\mathbf{qc}}(X)$, we have $\mathbf{L}f^*G' \in \mathrm{Coproduct}_{K'}(G[\alpha, \beta])$ for some integers $K', \alpha \leq \beta$, and therefore $Q \in \mathrm{Coproduct}_{KK'}(G(\alpha + \overline{A}, \beta + \overline{B}))$. $\square$

We finish the article with

Proof of Theorem 2.1. If X is affine, that is $X = \mathrm{Spec}(R)$ for some ring R of finite global dimension, then at some level the result goes back to Kelly [24]; see also Street [45]. The reader can find modern treatments in Christensen [9, Corollary 8.4] or Rouquier [42, Proposition 7.25]. More precisely: if the ring R is of global dimension $\leq M$, it is classical that $\mathbf{D}_{\mathbf{qc}}(\mathrm{Spec}(R)) \subset \mathrm{Coproduct}_{M+2}(\mathcal{O}_{\mathrm{Spec}(R)}(-\infty, \infty))$.

We treat the general case by induction on the number of open affine subsets in the cover of X . Suppose we know the theorem for all schemes which admit a cover by $\leq n$ open affines $U_i = \mathrm{Spec}(R_i)$, with each R_i of finite global dimension. Let X be a scheme admitting a cover by $n+1$ open affines $U_i = \mathrm{Spec}(R_i)$, with each R_i of finite global dimension. Put $V = \bigcup_{i=1}^n U_i$ and $U = U_{n+1}$, and let G_1 be a compact generator for $\mathbf{D}_{\mathbf{qc}}(U)$, G_2 a compact generator for $\mathbf{D}_{\mathbf{qc}}(V)$ and G a compact generator for $\mathbf{D}_{\mathbf{qc}}(X)$. Let $j_1 : U \rightarrow X$ and $j_2 : V \rightarrow X$ be the inclusions. By Theorem 6.2 there exists an integer N so that $\mathbf{R}j_{1*}G_1$ and $\mathbf{R}j_{2*}G_2$ both belong to $\mathrm{Coproduct}_N(G(-\infty, \infty))$. By induction there is an integer M so that $\mathbf{D}_{\mathbf{qc}}(U) = \mathrm{Coproduct}_M(G_1(-\infty, \infty))$ and $\mathbf{D}_{\mathbf{qc}}(V) = \mathrm{Coproduct}_M(G_2(-\infty, \infty))$. Therefore $\mathbf{R}j_{1*}\mathbf{D}_{\mathbf{qc}}(U)$ and $\mathbf{R}j_{2*}\mathbf{D}_{\mathbf{qc}}(V)$ are both contained in $\mathrm{Coproduct}_{MN}(G(-\infty, \infty))$.

But now $X = U \cup V$. Put $W = U \cap V$ and let $j : W \rightarrow U$ be the inclusion. Any object $F \in \mathbf{D}_{\mathbf{qc}}(X)$ fits in a triangle

$$\mathbf{R}j_{1*}[\mathbf{R}j_*\mathbf{L}j^*\mathbf{L}j_1^*\Sigma^{-1}F] \longrightarrow F \longrightarrow \mathbf{R}j_{1*}[\mathbf{L}j_1^*F] \oplus \mathbf{R}j_{2*}[\mathbf{L}j_2^*F] \longrightarrow$$

Thus F belongs to $[\mathbf{R}j_{1*}\mathbf{D}_{\mathrm{qc}}(U)] \star [\mathbf{R}j_{1*}\mathbf{D}_{\mathrm{qc}}(U) \oplus \mathbf{R}j_{2*}\mathbf{D}_{\mathrm{qc}}(V)]$, which is contained in $\mathrm{Coproduct}_{MN}(G(-\infty, \infty)) \star \mathrm{Coproduct}_{MN}(G(-\infty, \infty)) = \mathrm{Coproduct}_{2MN}(G(-\infty, \infty))$.

□

REFERENCES

- [1] Takuma Aihara and Ryo Takahashi, *Generators and dimensions of derived categories of modules*, Comm. Algebra **43** (2015), no. 11, 5003–5029.
- [2] Ko Aoki, *Quasiexcellence implies strong generation*, <https://arxiv.org/abs/2009.02013>.
- [3] Abdolnaser Bahlekeh, Ehsan Hakimian, Shokrollah Salarian, and Ryo Takahashi, *Annihilation of cohomology, generation of modules and finiteness of derived dimension*, Q. J. Math. **67** (2016), no. 3, 387–404.
- [4] Paul Balmer, *The spectrum of prime ideals in tensor triangulated categories*, J. Reine Angew. Math. **588** (2005), 149–168.
- [5] Alexander A. Beilinson, Joseph Bernstein, and Pierre Deligne, *Analyse et topologie sur les espaces singuliers*, Astérisque, vol. 100, Soc. Math. France, 1982 (French).
- [6] Marcel Bökstedt and Amnon Neeman, *Homotopy limits in triangulated categories*, Compositio Math. **86** (1993), 209–234.
- [7] Alexei I. Bondal and Michel Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. **3** (2003), no. 1, 1–36, 258.
- [8] Jesse Burke, Amnon Neeman, and Bregje Pauwels, *Gluing approximable triangulated categories*, <https://arxiv.org/abs/1806.05342>.
- [9] J. Daniel Christensen, *Ideals in triangulated categories: Phantoms, ghosts and skeleta*, Advances in Mathematics **136** (1998), 284–339.
- [10] Hailong Dao and Ryo Takahashi, *The radius of a subcategory of modules*, Algebra Number Theory **8** (2014), no. 1, 141–172.
- [11] ———, *The dimension of a subcategory of modules*, Forum Math. Sigma **3** (2015), e19, 31.
- [12] A. Johan de Jong, *Smoothness, semi-stability and alterations*, Inst. Hautes Études Sci. Publ. Math. (1996), no. 83, 51–93.
- [13] ———, *Families of curves and alterations*, Ann. Inst. Fourier (Grenoble) **47** (1997), no. 2, 599–621.
- [14] Vladimir Drinfeld and Dennis Gaitsgory, *On some finiteness questions for algebraic stacks*, Geom. Funct. Anal. **23** (2013), no. 1, 149–294.
- [15] Alexandre Grothendieck, *Éléments de géométrie algébrique III. Étude cohomologique des faisceaux cohérents I*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 11, 425–511.
- [16] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.
- [17] Luc Illusie, *Conditions de finitude relatives*, Théorie des intersections et théorème de Riemann-Roch, Springer-Verlag, Berlin, 1971, Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6, Exposé III), pp. 222–273. Lecture Notes in Mathematics, Vol. 225.
- [18] ———, *Existence de résolutions globales*, Théorie des intersections et théorème de Riemann-Roch, Springer-Verlag, Berlin, 1971, Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6, Exposé II), pp. 160–221. Lecture Notes in Mathematics, Vol. 225.
- [19] Srikanth B. Iyengar and Ryo Takahashi, *Annihilation of cohomology and strong generation of module categories*, Int. Math. Res. Not. IMRN (2016), no. 2, 499–535.
- [20] Victor Jatoba, *Strong generators in $\mathbf{D}^{\mathrm{perf}}(X)$ for schemes with a separator*, <https://arxiv.org/abs/1910.10676>.
- [21] C.U. Jensen and H. Lenzing, *Model theoretic algebra*, Gordon and Breach, New York, 1989.

- [22] Bernhard Keller, Daniel Murfet, and Michel Van den Bergh, *On two examples by Iyama and Yoshino*, Compos. Math. **147** (2011), no. 2, 591–612.
- [23] Bernhard Keller and Michel Van den Bergh, *On two examples by Iyama and Yoshino*, (e-print <http://arXiv.org/abs/0803.0720v1>).
- [24] G. Maxwell Kelly, *Chain maps inducing zero homology maps*, Proc. Cambridge Philos. Soc. **61** (1965), 847–854.
- [25] R. Kiełpiński and D. Simson, *On pure homological dimension*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **23** (1975), 1–6.
- [26] Henning Krause, *The stable derived category of a Noetherian scheme*, Compos. Math. **141** (2005), no. 5, 1128–1162.
- [27] Alexander G. Kuznetsov and Valery A. Lunts, *Categorical resolutions of irrational singularities*, Int. Math. Res. Not. IMRN (2015), no. 13, 4536–4625.
- [28] Joseph Lipman, *Notes on derived functors and Grothendieck duality*, Foundations of Grothendieck duality for diagrams of schemes, Lecture Notes in Mathematics, vol. 1960, Springer, Berlin, 2009, pp. 1–259.
- [29] Joseph Lipman and Amnon Neeman, *Quasi-perfect scheme maps and boundedness of the twisted inverse image functor*, Illinois J. Math. **51** (2007), 209–236.
- [30] Valery A. Lunts, *Categorical resolution of singularities*, J. Algebra **323** (2010), no. 10, 2977–3003.
- [31] Suresh Nayak, *Compactification for essentially finite-type maps*, Adv. Math. **222** (2009), no. 2, 527–546.
- [32] Amnon Neeman, *Approximable triangulated categories*, Representations of Algebras, Geometry and Physics, Contemp. Math., vol. 769, Amer. Math. Soc., Providence, RI, 2021, pp. 111–155.
- [33] ———, *The categories $\mathcal{T}^c$ and $\mathcal{T}_c^b$ determine each other*, <https://arxiv.org/abs/1806.06471>.
- [34] ———, *The category $[\mathcal{T}^c]^{\text{op}}$ as functors on $\mathcal{T}_c^b$* , <https://arxiv.org/abs/1806.05777>.
- [35] ———, *Triangulated categories with a single compact generator and a Brown representability theorem*, <https://arxiv.org/abs/1804.02240>.
- [36] ———, *The connection between the K-theory localisation theorem of Thomason, Trobaugh and Yao, and the smashing subcategories of Bousfield and Ravenel*, Ann. Sci. École Normale Supérieure **25** (1992), 547–566.
- [37] ———, *The Grothendieck duality theorem via Bousfield’s techniques and Brown representability*, Jour. Amer. Math. Soc. **9** (1996), 205–236.
- [38] ———, *Triangulated Categories*, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001.
- [39] ———, *Strong generation of some derived categories of schemes*, Research Perspectives CRM Barcelona, Trends in Mathematics, vol. 5, Springer–Verlag, 2016, pp. 121–127.
- [40] Dmitri O. Orlov, *Derived categories of coherent sheaves and equivalences between them*, Uspekhi Mat. Nauk **58** (2003), no. 3(351), 89–172, English translation Russian Math. Surveys **58** (2003), no. 3, 511–591.
- [41] ———, *Smooth and proper noncommutative schemes and gluing of DG categories*, Adv. Math. **302** (2016), 59–105.
- [42] Raphaël Rouquier, *Dimensions of triangulated categories*, J. K-Theory **1** (2008), no. 2, 193–256.
- [43] D. Salles, *Dimensions pures de modules*, Paul Dubreil and Marie–Paule Malliavin Algebra Seminar, Lec. Notes in Math., vol. 924, Springer–Verlag, 1981, pp. 423–441 (French).
- [44] The Stacks Project Authors, *stacks project*, <http://stacks.math.columbia.edu>.
- [45] Ross Street, *Homotopy classification of filtered complexes*, J. Austral. Math. Soc. **15** (1973), 298–318.
- [46] Robert W. Thomason, *The classification of triangulated subcategories*, Compositio Math. **105** (1997), 1–27.

- [47] Robert W. Thomason and Thomas F. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift (a collection of papers to honor Grothendieck's 60'th birthday), vol. 3, Birkhäuser, 1990, pp. 247–435.

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