<u>THE THEORY AND APPLICATION OF STOCHASTIC PROCESSES</u>

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PREFACE

The work presented in this thesis was done during the period October, 1953 to July, 1955.

The work is original in the sense that it was wholly done by myself. However, in the course of presenting the results it has been necessary to refer to the work of other men. Where this has been done it has always been made clear by means of a reference.

Chapter I contains no original research but is rather a review of the existing state of the subject.

In # 2.2. a theorem due to Pitman is proved so that the only original research in Chapter 2 is contained in # 2.3. where Pitman's theorem is extended.

Chapter 3 is based on a paper by Ogawara, but most of the chapter is original and consists either of extensions of Ogawara's results or of an examination of their efficiency.

Chapters 4, 5 and 6 present original results obtained by the author, while Chapter 7, where a theory due to Dr. E.G. Bowen is tested, is also wholly original.

Most of the contents of Chapters 3 and 4 were published by the author in a paper in Biometrika, Volume 42, Parts 1 and 2. Most of the contents of

Chapter 5 will appear under the author's name in Biometrika, Volume 42, Parts 3 and 4. The material making up Chapter 7 appeared as an article by the author in The Australian Journal of Physics, Volume 8, No. 2. Most of the subject matter of Chapter 6 has been communicated (by Professor P.A.P. Moran) for publication under the author's name in The Proceedings of the Cambridge Philosophical Society.

I wish to give my most sincere thanks to Professor P.A.P. Moran for suggesting the subject of this thesis to me and for his advice in the research. I have also received helpful advice from the referees of my papers, who are, however, unknown to me.

A great part of the computations required for the work presented in Chapter 7 were done for me on the C.S.&.I.R.O. electronic computer, through the help of Mr. Pearcey and Mr. Beard.

SUMMARY

THEORY AND APPLICATION OF STOCHASTIC PROCESSES

PART I

EXACT TESTS FOR CORRELATION IN TIME SERIES

Chapter I. Testing for correlation in time series.

The second section of this chapter reviews the work which has been done, mainly in recent years, on the problem of testing for serial correlation in a stationary time series. In # 1.3. the particular problem of testing for serial correlation in the residuals from a least squares regression is discussed. Finally, in # 1.4. the correlation of time series is considered.

Chapter 2. The asymptotic efficiency of tests.

If t_1 and t_2 are statistics computed from samples of size n, their asymptotic relative efficiency as test statistics of an hypothesis specified by a parameter value θ_0 is defined as:

$$E(t_1, t_2 / \theta_0) = \lim_{n \to \infty} \left\{ \frac{[\partial/\partial\theta\, \mathcal{E}(t_1)]^2}{Var(t_1)} \cdot \frac{Var(t_2)}{[\partial/\partial\theta\, \mathcal{E}(t_2)]^2} \right\}_{\theta = \theta_0}$$

Here $\mathcal{E}(t_j)$ and $Var(t_j)$ are respectively the mean and variance of the t_j. The justification for the use of this criterion rests upon a theorem due to Pitman which

is presented in # 2.2. Pitman shows that under certain regularity conditions for t_1 and t_2 with limiting normal distributions and variances or order n^{-1} the ratio of the sample sizes required for t_1 and t_2 to have the same power against alternative values of θ which differ from θ_0 by quantities of order $n^{-\frac{1}{2}}$ is in the limit given by $E(t_1, t_2 | \theta_0)$

Pitman's theorem is extended in # 2.3. to enable the asymptotic powers ~~to~~ of two variates distributed as χ^2 with p degrees of freedom to be compared. This covers also the case where it is required to compare the asymptotic powers of two tests based on multiple correlations computed from regressions on the same number of regressors.

Chapter 3. An exact test for serial correlation.

An exact test for serial correlation was derived by Ogawara by considering the conditional distribution of the x_{2t} for fixed x_{2t+1} when x_t comes from a stationary simple Markoff process. The asymptotic efficiency of this test is shown to be unity. When the hypothetical serial correlation ρ is not zero the procedure can be modified so as to obtain another exact test whose asymptotic efficiency is $(1-\rho^2)$. The procedure was also extended by Ogawara to the case of a stationary multiple Markoff process but here it is shown that the asymptotic efficiency of the estimates of the autoregression coefficients never rises above $\frac{2}{h+1}$, where h is the order of the autoregressive process (multiple Markoff process).

Chapter 4. Testing for serial correlation in the Residuals from a regression.

Ogawara's method is further extended to obtain an exact test for serial correlation in the residuals from a least squares regression. The test is shown to be asymptotically fully efficient.

Chapter 5. Exact tests for correlation between time series.

Two principal models are considered in this chapter. In the first, two processes are considered as correlated by means of a regression relation, the residual being a simple Markoff process. In the second, two linear processes, of which at least one is Markovian, are correlated by means of a correlation between the shocks, of which they are (infinite) moving averages. In each case the procedure of Chapter 4 leads to an exact test and in each case this test is for a range of circumstances of some interest in practice, asymptotically more efficient than the ordinary correlation coefficient. However, a statistic due to Quenouille is shown to be asymptotically fully efficient when the alternatives are prescribed by the second model and of fairly high efficiency when the first model is appropriate. This statistic is the partial correlation between the two series with the effects of lagged values of both removed. Though the exact distribution of Quenouille's statistic is not known it has been shown, by Quenouille, by sampling experiments, that its large sample theoretical mean and variance on the null hypothesis are also appropriate for small samples so that its treatment as an ordinary correlation will probably give a good approximation to the true significance point.

Chapter 6. Exact tests for correlation in vector processes.

The considerations of Chapters 3, 4 and 5 are extended to processes whose values at each point of time constitute a vector. The methods of correlation analysis are now replaced by those of canonical correlations. It is shown that when the order of the vector is greater than unity the test for serial independence of a vector process (including the case of a vector of residuals) obtained by Ogawara's method is no longer as powerful (asymptotically) as that which would be obtained by a straightforward canonical analysis of the relation between the vector and its lagged values. The loss of power arises from the necessity of introducing additional parameters, which follows from the (general) lack of commutativity of the matrix of regression coefficients and the covariance matrix of the residuals. Of course, in the case of the straightforward canonical analysis the distribution of the test statistic would only be known asymptotically.

A particular case of a vector process can be obtained by a redefinition of an autoregressive process. The approach of the present chapter gives no improvement (with respect to the power of the test) on that of Chapter 3 and, in general, the case of a simple Markoff process appears to stand in a unique position in relation to Ogawara's method.

PART II

APPLIED RESEARCH

Chapter 7. Testing for singularities in rainfall data.

A test of the homogeneity of daily mean rainfalls in Sydney over the period 1859 to 1952 is made. This test was suggested by the theory proposed by E.G. Bowen that showers of meteors passing into the atmosphere provided the nuclei without which certain cloud formations may not rain and, thus, since some such meteor showers are annually recurring, lead to singularities (peaks) in the seasonal rainfall pattern.

The test is made difficult by the complicated stochastic nature of the process generating the observations so that the actual significance point used may be appropriate to a more stringent test than that proposed. As the sample point does not fall in the critical region, however, there can be no doubt that the test gives a "not significant" result at the 5% level so that the test does not support Bowen's theory.

CONTENTS

		Page
PREFACE		II
SUMMARY		IV
PART I. EXACT TESTS FOR CORRELATION IN TIME SERIES		
Chapter 1.	Testing for correlation in time series	
1.1.	Introduction	1
1.2.	Testing for serial correlation in a scalar process	2
1.3.	Testing for correlation in the residuals from a regression	19
1.4.	Testing for correlation between serially correlated series	27
Chapter 2.	The asymptotic efficiency of tests	
2.1.	Introduction	35
2.2.	Pitman's theorem	35
2.3.	An extension of Pitman's theorem	40
Chapter 3.	An exact test for serial correlation	
3.1.	Simple Markoff processes	45
3.2.	Multiple Markoff processes	53
3.3.	A critique of the method	58
Chapter 4.	Testing for serial correlation in the residuals from a regression	
4.1.	An exact test against a simple Markoff alternative	60
4.2.	Examples	67

		Page
4.3.	An alternative test	72
Chapter 5.	Exact tests for correlation between time series	
5.1.	The efficiency of estimates of regression coefficients when the residual is Markovian	75
5.2.	An exact test for correlation between two series when the residual is Markovian and independent of the regressor series	80
5.3.	An exact test for correlation between two simple Markoff processes	93
5.4.	The case where the regressor process is a second order autoregressive process	101
5.5.	Exact tests for multiple correlation	105
5.6.	Exact tests for correlation when the residual is an autoregressive process of order higher than the first	109
Chapter 6.	Exact tests for correlation in vector processes	
6.1.	The canonical analysis of a vector Markoff process	110
6.2.	The asymptotic efficiency of the exact test	115
6.3.	The canonical analysis of a vector of residuals	119
6.4.	Multiple Markoff processes	124

		Page
PART II. APPLIED RESEARCH		
Chapter 7.	Testing for singularities in rainfall data	
7.1.	Meteoric dust as a cause of rainfall singularities	129
7.2.	The nature of the data and of the null hypothesis	135
7.3.	Removal of seasonal variation	146
7.4.	The test of homogeneity of means	148
7.5.	Alternative tests	150
REFERENCES		153
APPENDICES		

- - - - -