Drinfeld Modules, Explicit Class Field Theory and Λ-Structures

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Declaration

The work in this thesis is my own except where otherwise stated.

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Abstract

This thesis translates some Λ-geometric results from number fields, [1] and [2] in particular, to function fields. There are two major results in this thesis.

For each set P of almost all primes of $\mathbb{F}_q[t]$, we use R_P to denote the ring corresponding to an affine open subset of $\mathbb{P}^1_{\mathbb{F}_q}$, i.e., the closed points of $\mathrm{Spec}(R_P)$ are the elements in P. Let Q be another set of almost all primes of $\mathbb{F}_q[t]$. We give an equivalent condition for a finite étale Λ-scheme S over $\mathrm{Spec}(\mathbb{F}_q(t))$ to have a Q-$\Lambda(P)$-model, by which we mean a finite flat and reduced scheme over $\mathrm{Spec}(R_Q)$ together with a family of commuting Frobenius lifts for each place in P. We show that such an S has a Q-$\Lambda(P)$-model if and only if the action of $\bigoplus_P \mathbb{N}$ on the absolute Galois group of $\mathbb{F}_q(t)$ on $\mathrm{Hom}(\mathrm{Spec}(\overline{\mathbb{F}_q(t)}), S)$ factors through the monoid $\mathbb{N}^{P\setminus Q} \times \left(\prod_{v\in P\cap Q} \mathcal{O}_v^\circ \times \prod_{v\notin P\cap Q} \mathcal{O}_v^* \times \hat{\mathbb{Z}}\right) \Big/ \mathbb{F}_q^*$.

Secondly, we show that given a global function field K, the maximal abelian extension of K is Λ-geometric. In other words, we present a scheme X of finite type over a Dedekind domain A, with K being the fraction field of A, such that the maximal abelian extension of K is contained in the $\mathfrak{f}$-periodic locus of X as $\mathfrak{f}$ runs over all ideals of A.

Contents

Chapter 0

Introduction

This thesis consists of two parts.

The motivation of the first part comes from generalizing the results on Λ-integral models of finite étale algebras over $\mathbb{Q}$ in [1] to results on Λ-models of finite étale algebras over $\mathbb{F}_q(t)$. Let E be an algebra, we say E admits a Λ-structure if E admits of a family of commuting Frobenius lifts. In other words, there exist ring endomorphisms of E, denoted as $\psi_p : E \to E$, one for each rational prime number p such that the following properties are satisfied:

1. $\psi_p(x) - x^p \in pE$;
2. $\psi_p(\psi_q(x)) = \psi_q(\psi_p(x))$ for all p, q.

From now on, assume E is a $\mathbb{Q}$-algebra. Note that condition 1 is trivial in this case. We say E has an integral Λ-model A of E if there is a finite flat algebra over $\mathbb{Z}$ such that $A \otimes_{\mathbb{Z}} \mathbb{Q} = E$ and A admits of a family of commuting Frobenius lifts. In [1], Borger and de Smit showed a necessary and sufficient condition for E to have an integral Λ-model. We may view E as a finite discrete set $S = \operatorname{Hom}(E, \bar{\mathbb{Q}})$ with a continuous action of the absolute Galois group $G_{\mathbb{Q}}$. Let N' be the commutative monoid $\{1, 2, \dots\}$ under multiplication equipped with the discrete topology. Since every element in N' can be written uniquely as a product of rational primes, a finite étale algebra with a family of commuting Frobenius lifts can be viewed as a finite set with a continuous action of $G_{\mathbb{Q}} \times N'$. In other words, we have the following continuous map

$$G_{\mathbb{Q}} \times N' \longrightarrow \operatorname{Map}(S, S)$$

There is a natural map $G_{\mathbb{Q}} \times N' \to \hat{\mathbb{Z}}^{\circ}$, where $\hat{\mathbb{Z}}^{\circ}$ denotes the profinite integers viewed as a multiplicative monoid, defined to be given by the cyclotomic character $G_{\mathbb{Q}} \to \hat{\mathbb{Z}}^{*} \subset \hat{\mathbb{Z}}^{\circ}$ on the first factor and the natural inclusion on the second factor.

Theorem 0.0.1. *([1], Theorem 0.1) E has an integral Λ-model if and only if the action of $G_{\mathbb{Q}} \times N'$ on S factors (necessarily uniquely) through $\hat{\mathbb{Z}}^{\circ}$ (by the natural map defined above); in more precise terms, if and only if there is a continuous monoid map $\hat{\mathbb{Z}}^{\circ} \to Map(S,S)$ so that the following diagram commutes:*

$$\begin{tikzcd}
G_{\mathbb{Q}} \times N' \arrow[r] \arrow[dr] & \hat{\mathbb{Z}}^{\circ} \arrow[d] \\
& Map(S,S)
\end{tikzcd}$$

0.0.2. We wish to obtain a similar result for $\mathbb{F}_q(t)$. More precisely, we say an $\mathbb{F}_q(t)$-algebra is a $(\mathbb{F}_q[t]\text{-})\Lambda$-ring if it has a commuting family of Frobenius ring endomorphism ψ_π, one for each monic irreducible polynomial $\pi \in \mathbb{F}_q[t]$. Let E be a finite étale algebra over $\mathbb{F}_q(t)$. We say E has an integral $(\mathbb{F}_q[t]\text{-})\Lambda$-model A if A is finite flat and reduced over $\mathbb{F}_q[t]$ and A has a commuting family of Frobenius lifts with $A \otimes \mathbb{F}_q(t) = E$.

We seek a necessary and sufficient condition for an integral Λ-model to exist. By a similar argument for $\mathbb{Q}$, since every monic polynomial over $\mathbb{F}_q$ factors uniquely as product of irreducible monic polynomails, it is clear that we can view a finite étale algebra E over $\mathbb{F}_q(t)$ with a commuting Frobenius family of ring endomorphsim as the following continuous map

$$G_{\mathbb{F}_q(t)} \times \mathrm{Id}(\mathbb{F}_q[t]) \longrightarrow \mathrm{Map}(S,S)$$

where $G_{\mathbb{F}_q(t)}$ is the Galois group of $\mathbb{F}_q(t)$ with respect to an algebraic closure $\overline{\mathbb{F}_q(t)}$ and $\mathrm{Id}(\mathbb{F}_q[t])$ denotes the monoid of non-zero ideals of $\mathbb{F}_q[t]$ under multiplication equipped with the discrete topology, which is the same as the monoid of monic polynomials over $\mathbb{F}_q$.

If we follow the framework of $\mathbb{Q}$ and try to apply it to $\mathbb{F}_q(t)$, we would seek a monoid M such that E has an integral model if and only if the action above factors through the action of M. The main result of the first chapter is to show that $M = \hat{\mathbb{F}_q[t]}^{\circ} \times \hat{\mathbb{Z}} \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])$. In other words, we have

Theorem 0.0.3. *A finite étale algebra E over $\mathbb{F}_q(t)$ has an integral $\mathbb{F}_q[t]$-Λ-model if and only if the action of $G_{\mathbb{F}_q(t)} \times Id(\mathbb{F}_q[t])$ on $S = Hom(E, \overline{\mathbb{F}_q(t)})$ factors (necessarily unique) through $\hat{\mathbb{F}_q[t]}^{o} \times \hat{\mathbb{Z}} \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])$ (by the map defined below* *in 0.0.4); in more precise terms, if and only if there is a continuous monoid map* $\widehat{\mathbb{F}_q[t]}^{o} \times \hat{\mathbb{Z}} \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]]) \to \mathit{Map}(S, S)$ *so that the following diagram commutes:*

$$\begin{tikzcd}
\mathit{Id}(\mathbb{F}_q[t]) \times G_{\mathbb{F}_q(t)} \arrow[r] \arrow[dr] & \widehat{\mathbb{F}_q[t]}^{o} \times \hat{\mathbb{Z}} \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]]) \arrow[d] \\
 & \mathit{Map}(S, S)
\end{tikzcd}$$

0.0.4. The map from $\mathrm{Id}(\mathbb{F}_q[t]) \times G_{\mathbb{F}_q(t)}$ to $\widehat{\mathbb{F}_q[t]}^{o} \times \hat{\mathbb{Z}} \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])$ is defined as the following: On the first factor, take an element in $\mathrm{Id}(\mathbb{F}_q[t])$, it is generated by a unique monic polynomial $g(t) \in \mathbb{F}_q[t]$. Then the map sends $g(t)$ to $(g(t), \deg(g(t)), (1/t)^{\deg(g)}g(t))$ canonically. On the second factor, we first send $G_{\mathbb{F}_q(t)}$ to its abelianization $G^{ab}_{\mathbb{F}_q(t)}$, which is isomorphic to $\widehat{\mathbb{F}_q[t]}^{*} \times \hat{\mathbb{Z}} \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])$ (see 1.1.7), we then send it to $\widehat{\mathbb{F}_q[t]}^{o} \times \hat{\mathbb{Z}} \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])$ as a natural inclusion.

The first chapter consists of a proof of theorem 0.0.3. We first give a brief summary of abelian extensions over $\mathbb{F}_q[t]$, using the theory of Carlitz modules. We then show the condition in theorem 0.0.3 is both sufficient (section 1.2) and necessary (section 1.3).

In the final section, we replace the ring $\mathbb{F}_q[t]$ with other rings R and define a generalized version of theorem 0.0.3. More precisely, let P, Q be sets containing all but finitely many primes of $\mathbb{F}_q(t)$. We define a ring R_Q such that $\mathrm{Spec}(R_Q) = Q$ and we say a finite étale algebra E over K has an R_Q-$\Lambda(P)$-model A_Q if A_Q is a finite flat R_Q-algebra and it has a commuting family of Frobenius lifts for each prime in P. We find a different monoid for each P, Q, such that a finite étale algebra E has an R_Q-$\Lambda(P)$-model if and only if the action of the Galois group and the ideals factor through such a monoid.

The motivation of the second part of this thesis comes from explicit class field theory, more precisely, from Borger and de Smit's paper ([2]) and their work on explicit class field theory using Λ-geometry. In [2], they let F be a number field and B be a Dedekind domain with finite residue fields. Let P be a set of maximal ideals of B and let Id_P denote all the ideals generated by maximal ideals of P. Furthermore, assume X is a separated flat B-scheme and let $\mathrm{End}_B(X)$ denote the monoid of B-scheme endomorphisms of X.

Definition 0.0.5. A $\Lambda_{B,P}$-scheme is a scheme X together with a family of maps $\psi_{\mathfrak{p}} \in \mathrm{End}(X)$, one for each $\mathfrak{p} \in P$, such that the induced endomorphism on the fibre $X \times_{\mathrm{Spec}(B)} \mathrm{Spec}(B/\mathfrak{p})$ is the Frobenius endomorphism, i.e., the homomorphism induced by $x \mapsto x^{|B/\mathfrak{p}|} \bmod \mathfrak{p}$. Furthermore, for all $\mathfrak{p}, \mathfrak{q} \in P$, we require $\psi_{\mathfrak{p}} \circ \psi_{\mathfrak{q}} = \psi_{\mathfrak{p}} \circ \psi_{\mathfrak{q}}$.

We will often drop the subscript of B or P or both when there is no ambiguity. For now, let us assume P consists of all non-zero prime ideals of B.

Definition 0.0.6. Let $\mathfrak{f}$ be a cycle of F (recall this means a formal product of places of F), then the cycles $\mathfrak{a}, \mathfrak{b}$ are $\mathfrak{f}$-equivalent, denoted as $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}$, if

1. $\mathfrak{a}$ and $\mathfrak{b}$ have the same greatest common divisor $\mathfrak{d}$ (a cycle of F) with $\mathfrak{f}$;
2. The Artin symbols $\mathfrak{a}\mathfrak{d}^{-1}$ and $\mathfrak{b}\mathfrak{d}^{-1}$ are the same on the ray class field of conductor $\mathfrak{f}\mathfrak{d}^{-1}$.

It is clear that $\sim_{\mathfrak{f}}$ is an equivalence relation on Id_P and we define the quotient $\mathrm{Id}_P/\sim_{\mathfrak{f}}$ the ray class monoid of conductor $\mathfrak{f}$ (one checks the quotient inherits a unique monoid structure from Id_P) and denote it by $\mathrm{Cl}_P(\mathfrak{f})$.

Definition 0.0.7. Let $\mathfrak{f}$ be a cycle of F. The $\mathfrak{f}$-periodic locus of X, denoted as $X(\mathfrak{f})$, is defined as

$$X(\mathfrak{f}) = \bigcap_{\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}} X(\psi_{\mathfrak{a}} = \psi_{\mathfrak{b}})$$

where $\mathfrak{a}, \mathfrak{b}$ are cycles of K supported at P, $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}$ denotes the equivalence in the ray class monoid in definition 0.0.6 and the scheme $X(\psi_{\mathfrak{a}} = \psi_{\mathfrak{b}})$ denotes the equalizer of the maps $\psi_{\mathfrak{a}}$, $\psi_{\mathfrak{b}} : X \to X$. In other words, for R an B-algebra, the R points of $X(\mathfrak{f})$ is defined as

$$X(\mathfrak{f})(R) = \{x \in X(R) \mid \psi_{\mathfrak{a}}(x) = \psi_{\mathfrak{b}}(x) \text{ for all } \mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}\}$$

For $x \in X(R)$, if there exists $\mathfrak{f}$ such that $x \in X(\mathfrak{f})$, we say x is in the $\mathfrak{f}$-periodic locus of $X(R)$.

Since X is separated, $X(\mathfrak{f})$ is defined by equalizers so $X(\mathfrak{f})$ is a closed subscheme of X. The relationship between the $\mathfrak{f}$-periodic loci and explicit class field theory can be summarized in the following theorem.

Theorem 0.0.8. *([2], Theorem 8.4) Let X be a flat and separated $\Lambda_{B,P}$-scheme as before. Further assume X is of finite type over B and P contains all but finitely many primes in B, and let $X(\mathfrak{f})_{fl}$ denote the maximal B-flat subscheme of $X(\mathfrak{f})$. Then $X(\mathfrak{f})_{fl} = \mathrm{Spec}(C)$ where C is a finitely generated B-algebra, and $F \otimes_B C$ is a finite product of abelian extensions of F of conductor dividing $\mathfrak{f}$.*

We say an abelian extension L/F is Λ-geometric if there exists a Λ-scheme X such that L is generated by adjoining the $\mathfrak{f}$-periodic locus for some $\mathfrak{f}$, i.e., $L \subseteq F(X(\mathfrak{f})(\overline{F}))$. Given the theorem above, it is natural to ask if the maximal abelian extension of a given a number field F is Λ-geometric. In other words, is it possible to find an X such that the periodic loci over all cycles generates the maximal abelian extension $F^{\mathrm{ab}}$of K? I.e., does there exist X such that

$$F^{\mathrm{ab}} = \bigcup_{\mathfrak{f}} F(X(\mathfrak{f})(\overline{F}))$$

where $\overline{F}$ is an algebraic closure of F?

Such a Λ-scheme were constructed for the case where $F = \mathbb{Q}$ or an imaginary quadratic field and the maximal abelian extension over these fields are all shown to be Λ-geometric, see [2] and [10]. These cases were all done by translating existing theory including the Kronecker-Weber theorem and the theory of CM elliptic curves into this Λ-geometry framework.

The purpose of this chapter is to give an analogue and solve this in a function field setting. In other words, we replace the number field F by the function field K, the Dedekind domain B by A (see the definition of K and A in the beginning of chapter 2) and the class groups of number fields by the class groups we introduce in definition 2.3.1, 2.3.2 and 2.3.5. With these adjustments, we want to answer the following question:

Given a function field K, is the maximal abelian extension of K Λ-geometric?

We will show the answer to the question above is yes and we will show this by constructing such a Λ-scheme by translating the existing theory of Drinfeld modules into our Λ-geometric framework.

In chapter 2, we first fix notation in section 2.1. We then give some useful results from class field theory and define what we mean by ray class fields over a function field K in sections 2.2 and 2.3. We finish chapter 2 by giving the definitions of various terminologies from algebraic geometry, such as sheaves with respect to a Grothendieck topology and fibered categories.

In chapter 3, we give a summary of the theory of Drinfeld modules. The main result from this section is to show the moduli space of rank 1 Drinfeld modules with level-I structures is representable by the ring $A_{K_I}[I^{-1}]$, using the notation in chapter 2.

Finaly, in chapter 4, we show how one naturally constructs a Λ-scheme using the theory of Drinfeld modules. This is done by constructing the space of universal rank 1 Drinfeld modules and we show this is isomorphic to Spec($A_H[x]$). We then show that the periodic loci of this Λ-scheme generates ray class fields of K totally split at ∞.

Chapter 1

Algebras over $\mathbb{F}_q(t)$ with Λ model

We prove theorem 0.0.3 in this chapter. In the next section, we first give a brief overview on the abelian extensions over $\mathbb{F}_q(t)$ and define the following map on the second factor in theorem 0.0.3

$$G_{\mathbb{F}_q(t)} \to \hat{\mathbb{F}_q[t]}^{o} \times \hat{\mathbb{Z}} \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])$$

We then follow the structure of [1] and we show the sufficiency and necessity of the theorem 0.0.3 in the following sections.

1.1 Abelian Extensions over $\mathbb{F}_q(t)$

Firstly, let us follow the notation introduced by Carlitz in [3] and include some relevant results of Carlitz polynomials over $\mathbb{F}_q[t]$ in [12]. We define $\omega_1(x) := x$, $\omega_t(x) := x^p + tx$ and define inductively $\omega_{t^n}(x) = \omega_t(\omega_{t^{n-1}}(x))$. We can extend this definition linearly to define the Carlitz polynomial for all $g(t) \in \mathbb{F}_q[t]$, i.e. for $g(t) = a_n t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0$, $a_i \in \mathbb{F}_q$, we have

$$\omega_{g(t)}(x) := a_n \omega_{t^n}(x) + a_{n-1}\omega_{t^{n-1}}(x) + \ldots + a_1\omega_t(x) + a_0 x$$

It is clear that these polynomials are additive, meaning $\omega_{g(t)}(x + y) = \omega_{g(t)}(x) + \omega_{g(t)}(y)$ for all $g(t) \in \mathbb{F}_q[t]$. This follows from the fact that p^{th}-power map is additive in a ring of characteristic p.

1.1.1. Let K_g denote the splitting field of $\omega_{g(t)}(x)$. Theorem 2.3 of [12] shows that we have an isomorphism $\mathrm{Gal}(K_g/\mathbb{F}_q(t)) \cong (\mathbb{F}_q[t]/g(t))^*$ and the isomorphism is given by the following. Let $f(t)$ be another polynomial over $\mathbb{F}_q$, one checks that

$$\omega_{g(t)}(\omega_{f(t)}(x)) = \omega_{gf(t)}(x) = \omega_{f(t)}(\omega_{g(t)}(x)) \qquad (1.1.2)$$

In particular, let α be a root of $\omega_{g(t)}(x)$, we have

$$\omega_{g(t)}(\omega_{f(t)}(\alpha)) = \omega_{f(t)}(\omega_{g(t)}(\alpha)) = \omega_{f(t)}(0) = 0$$

Let Ω_g denote the set of roots of $\omega_g(x)$. It is clear that if $\alpha, \beta \in \Omega_g$ then $\omega_g(\alpha+\beta) = 0$. Moreover, it is clear that if $f_1(t) = f_2(t) + h(t) \times g(t)$ for some $h(t) \in \mathbb{F}_q[t]$ then $\omega_{f_1}(\alpha) = \omega_{f_2}(\alpha)$ for all $\alpha \in \Omega_g$. Thus, we can define a map $\mathbb{F}_q[t]/g(t) \times \Omega_g \to \Omega_g$ by $(f, \alpha) \mapsto \omega_f(\alpha)$. Therefore, we can view Ω_g as an $(\mathbb{F}_q[t]/g(t))$-module.

In fact, from theorem 1.6 of [12], Ω_g is an $(\mathbb{F}_q[t]/g(t))$-torsor. In particular, there exists a generator in Ω_g as a $(\mathbb{F}_q[t]/g(t))$-module. We call this generator a primitive root of $\omega_{g(t)}(x)$ and we denote it as α_g.

Furthermore, the map $f(t) : \alpha_g \mapsto \omega_{f(t)}(\alpha_g)$ defines an automorphism of K_g for all $f(t)$ coprime with $g(t)$. This gives a group homomorphism $(\mathbb{F}_q[t]/g(t))^* \to \mathrm{Gal}(K_g/\mathbb{F}_q(t))$ and theorem 2.3 of [12] shows this is an isomorphism.

Lemma 1.1.3. *Let $g(t), h(t) \in \mathbb{F}_q[t]$ be two coprime polynomials. Let α_g, α_h be a primitive root of $\omega_{g(t)}(x)$ and $\omega_{h(t)}(x)$ as above. Then the field $K_{gh} = \mathbb{F}_q(t, \alpha_g, \alpha_h)$.*

Proof. It is clear that both K_g and K_h are contained in K_{gh} since by equation (1.1.2), both K_g and K_h divides K_{gh}, so $K_{gh} \supseteq \mathbb{F}_q(t, \alpha_g, \alpha_h)$.

To show equality, we show that we have an isomorphism of $(\mathbb{F}_q[t]/gh(t))$-modules $\Omega_{gh} \cong \Omega_g \oplus \Omega_h$, where the isomorphism from the right hand side to the left hand side is given by addition, i.e., $(\beta_g, \beta_h) \in \Omega_g \oplus \Omega_h$ is mapped to $\beta_g + \beta_h$. Note that here Ω_g is viewed as a $(\mathbb{F}_q[t]/gh(t))$-modules through the natural map $(\mathbb{F}_q[t]/gh(t)) \to (\mathbb{F}_q[t]/g(t))$ and similarly for Ω_h. We show the isomorphism by showing that for some polynomial $i(t), j(t) \in \mathbb{F}_q[t]$, $\omega_{i(t)}(\alpha_g) + \omega_{j(t)}(\alpha_h) = \alpha_{gh}$. If this is true, then by writing each $\beta \in \Omega_{gh}$ as $\omega_k(\alpha_{gh})$ (this is always possible because α_{gh} is a generator), we have

$$\beta = \omega_k(\alpha_{gh}) = \omega_{k(t)}(\omega_{i(t)}(\alpha_g) + \omega_{j(t)}(\alpha_h) = \omega_{ki(t)}(\alpha_g) + \omega_{kj(t)}(\alpha_g)$$

and hence $\Omega_{gh} \subseteq \Omega_g \oplus \Omega_h$. Since they both have the same size, they must be equal.

It remains to show there exist $i(t), j(t) \in \mathbb{F}_q[t]$ such that $\omega_{i(t)}(\alpha_g) + \omega_{j(t)}(\alpha_h) = \alpha_{gh}$. Since $g(t)$ and $h(t)$ are coprime, there exists $f_1(t), f_2(t)$ such that $f_1(t)g(t) + f_2(t)h(t) = 1$ and hence $\omega_{f_1}(\omega_{g(t)}(\alpha_{gh})) + \omega_{f_2}(\omega_{h(t)}(\alpha_{gh}) = \alpha_{gh}$. One observes that $\omega_{g(t)}(\alpha_{gh}) \in \Omega_h$, since $\omega_{h(t)}(\omega_{g(t)}(\alpha_{gh})) = \omega_{hg(t)}(\alpha_{gh}) = 0$, so there exists $f_1' \in \mathbb{F}_q[t]$

such that $\omega_{g(t)}(\alpha_{gh}) = \omega_{f_1'(t)}(\omega_h)$. Similarly, there exists $f_2' \in \mathbb{F}_q[t]$ such that $\omega_{h(t)}(\alpha_{gh}) = \omega_{f_2'(t)}(\omega_g)$. Therefore,

$$\alpha_{gh} = \omega_{f_1}(\omega_{g(t)}(\alpha_{gh})) + \omega_{f_2}(\omega_{h(t)}(\alpha_{gh}) = \omega_{f_1}(\omega_{f_1'(t)}(\omega_h)) + \omega_{f_2}(\omega_{f_2'(t)}(\omega_g))$$

One sees that we let $i(t) = f_2 f_2'(t)$ and $j(t) = f_1 f_1'(t)$ and we are done. □

Recall the definition of an $\mathbb{F}_q[t]$-Λ-ring and an $\mathbb{F}_q[t]$-Λ-model in 0.0.2.

Lemma 1.1.4. *(Properties of $\mathbb{F}_q[t,x]/\omega_{g(t)}(x)$)*
The set $S = Hom(\mathbb{F}_q[t,x]/\omega_{g(t)}(x), \overline{\mathbb{F}_q(t)})$ is isomorphic to $(\mathbb{F}_q[t]/g(t))$ as a multiplicative monoid, which we denote as $(\mathbb{F}_q[t]/g(t))^\circ$.

Furthermore, $\mathbb{F}_q[t,x]/\omega_{g(t)}(x)$ is a $\mathbb{F}_q[t]$-Λ-model of $\mathbb{F}_q(t,\alpha_g)$.

Proof. A map from $\mathbb{F}_q[t,x]/\omega_{g(t)}(x)$ to $\overline{\mathbb{F}_q(t)}$ is map x to a root of $\omega_{g(t)}(x)$, which are all of the form $[\overline{f}(t)](\alpha_g)$, $\overline{f}(t) \in (\mathbb{F}_q[t]/g(t))^\circ$. This shows the first statement.

Let $R = \mathbb{F}_q[t,x]/\omega_{g(t)}(x)$. It is clear that $\mathbb{F}_q(t) \otimes R = \mathbb{F}_q(t,\alpha_g)$, so it is enough to show that R has a Frobenius lift for all primes $\pi(t) \in \mathbb{F}_q[t]$. But corollary 2.5 of [12] showed that $\omega_{\pi(t)}(x) \equiv p^{deg(\pi)} \mod \pi(t)$, hence the Frobenius lift condition is also satisfied. □

There are other abelian extensions over $\mathbb{F}_q(t)$, for example, one can simply take $\mathbb{F}_{q^n}(t)$ over $\mathbb{F}_q(t)$. It is clear that the extension $\mathbb{F}_{q^n}(t)$ over $\mathbb{F}_q(t)$ is unramified at every prime, including the prime at infinity, by which we mean the valuation corresponding to the degree over t for each polynomial $f(t) \in \mathbb{F}_q[t]$, which we denote as ∞. In addition to that, Theorem 3.2 of [12] shows that for any irreducible monic polynomial $\pi(t) \in \mathbb{F}_q[t]$ and $n \in \mathbb{N}$, let $g(t) = \pi(t)^n$. The infinite prime has ramification index $p-1$ in the extension $K_g/\mathbb{F}_q(t)$. This means we have not yet constructed extensions that are wildly ramified at ∞.

1.1.5. We follow definition 5.1 of [12] to construct extensions wildly ramified at ∞. Firstly, denote $u = 1/t$ and define Carlitz polynomials over $\mathbb{F}_q[u]$, i.e., $[u](x) = x^p + ux$ and extend by composition and linearity just as what we did before. Let K_{m+1}, $m \geq 1$, denote the splitting field of $[u^{m+1}](x)$ over $F = \mathbb{F}_q(t) = \mathbb{F}_q(u)$. By repeating a almost identical argument given in 1.1.1, one shows $\mathrm{Gal}(K_{m+1}/F) \cong (\mathbb{F}_q[u]/u^{m+1})^*$. Let L_m denote the fixed field of $\mathbb{F}_q^*$.

Proposition 1.1.6. *(Properties of L_m)*

1. *(Galois group) Gal($L_m/\mathbb{F}_q(t)$) $\cong (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])/(\frac{1}{t^{m+1}})$*

2. *(Ramification) The prime ∞ is totally ramified and all finite primes are unramified in L_m.*

3. *(Artin symbol) Let $\pi(t) = t^r + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0$ be a prime in $\mathbb{F}_q[t]$, then the Artin symbol $(L_m/\mathbb{F}_q(t), \pi) = (1/t)^r \pi(t)$ mod $1/t^m \in Gal(L_m/\mathbb{F}_q(t))$.*

Proof. 1. is clear since the Galois group $\mathrm{Gal}(K_m/\mathbb{F}_q(t)) \cong (\mathbb{F}_q[u]/u^{m+1})^*$, which can be shown by a similar argument given in 1.1.1. L_m is the fixed field of $\mathbb{F}_q^*$ and clearly, for each positive integer m, $(\mathbb{F}_q[u]/u^{m+1})^* \cong \mathbb{F}_q^* \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])/(\frac{1}{t^{m+1}})$.

Let $u = 1/t$, then the ramification of ∞ in L_m over $\mathbb{F}_q(t)$ is the same as the ramification of u in L_m over $\mathbb{F}_q(u)$. By [12, proposition 2.2], the only primes ramified in K_{m+1} over $\mathbb{F}_q[u]$ is u and $1/u$ (i.e. t). Furthermore, u is totally ramified in K_{m+1}, hence also totally ramified in L_m. For the ramification of t, we use [12, theorem 3.2] to show that in K_{m+1} over $\mathbb{F}_q[u]$, there are p^m primes above t, each has ramification index $p-1$. By comparing these numbers with the degrees of the extensions, we conclude that t splits completely in L_m and each prime above t is totally ramified in K_{m+1}/L_m. This shows 2.

Moving on to 3., for $\pi(t) \neq t$, the assertion is true since if $\pi'(u) = (1/t)^r \pi(t)$, then both $\pi(t), \pi'(u)$ generates the same ideal in $\mathbb{F}_q[t^{\pm 1}]$ and the Artin symbol of $\pi'(u)$ in K_{m+1} is $[a\pi'(u)]$ as before, where $a \in \mathbb{F}_q^*$ such that $\pi'(u)$ is monic. Hence the Artin symbol of $\pi(t)$ in L_m is the restriction of $[a\pi'(u)]$ in L_m if we view L_m as a subfield of K_{m+1}. But L_m is the fixed field of $\mathbb{F}_q^*$ and $\pi'(u)$ has constant coefficient 1 since $\pi(t)$ is monic, so the Artin symbol of $\pi(t)$ in L_m is $[\pi'(u)]$. Finally, we have established t splits completely in L_m, so the Artin symbol must be 1. □

Finally, we conclude this section by stating the main theorem in [12], which classifies all abelian extensions of $\mathbb{F}_q(t)$.

Theorem 1.1.7. *(Carlitz-Hayes Theorem)*
Let K_g and L_m be the field extensions over $\mathbb{F}_q(t)$ defined in 1.1.1 and 1.1.5. Then the extensions $\mathbb{F}_{p^n}(t)$, K_g and L_m over $\mathbb{F}_q(t)$ have pairwise trivial intersection (i.e. the intersection of any two of them is $\mathbb{F}_q(t)$), so the Galois group $Gal(K_g\mathbb{F}_{p^n}(t)L_m/\mathbb{F}_q(t))$ is naturally isomorphic to the direct product $(\mathbb{F}_q[t]/g(t))^* \times C_n \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])/(\frac{1}{t^{m+1}})$.

Furthermore, any abelian extension $F/\mathbb{F}_q(t)$ is contained in $K_g\mathbb{F}_{p^n}(t)L_m$, for some $g(t) \in \mathbb{F}_q[t]$, $n, m \in \mathbb{N}$. Therefore,

$$G^{ab}_{\mathbb{F}_q(t)} \cong \widehat{\mathbb{F}_q[t]}^* \times \hat{\mathbb{Z}} \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])$$

where $G^{ab}_{\mathbb{F}_q(t)}$ *denotes the absolute abelian Galois group of* $\mathbb{F}_q(t)$, $\widehat{\mathbb{F}_q[t]}^*$ *denotes the inverse limit* $\varprojlim_{g(t)\in Id(\mathbb{F}_q[t])}(\mathbb{F}_q[t]/g(t))^*$, $\hat{\mathbb{Z}}$ *denotes the profinite integers, as the inverse limit of cyclic groups and* $(1+\frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])$ *denotes the power series over* $\mathbb{F}_q$ *in* $1/t$ *with constant coefficient* 1.

A proof of the theorem above can be found by combining proposition 5.2 and theorem 7.1 in [12].

Remark 1.1.8. Lemma 1.1.4 states that the multiplicative monoid $(\mathbb{F}_q[t]/g(t))^{\circ}$ naturally associates with the Λ-ring $\mathbb{F}_q[t,x]/\omega_{g(t)}(x)$. For all $g(t)\in\mathbb{F}_q[t]$, $n,m\in\mathbb{N}$, we extend this association to monoid of the form

$$(\mathbb{F}_q[t]/g(t))^*\times C_n\times(1+\frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])/(\frac{1}{t^{m+1}}).$$

We associate it with the following Λ-ring

$$\mathbb{F}_q[t,x]/\omega_{g(t)}(x)\otimes_{\mathbb{F}_q(t)}\mathbb{F}_{q^n}(t)L_m.$$

We note that the ring above is a Λ-ring. Indeed, the tensor product of a Λ-ring is naturally a Λ-ring, so it is enough to show that $\mathbb{F}_q[t,x]/\omega_{g(t)}(x)$, $\mathbb{F}_{q^n}$ and L_m are all Λ-rings. But this is clear from lemma 1.1.4 and 1.1.6, and $\mathbb{F}_{q^n}(t)$ is an everywhere unramified extension of $\mathbb{F}_q(t)$.

Remark 1.1.9. Let $F/\mathbb{F}_q(t)$ be an abelian extension and by the theorem above there exist $g(t)\in\mathbb{F}_q[t]$, $n,m\in\mathbb{Z}$ such that $F\subseteq K_g\mathbb{F}_{p^n}(t)L_m$. Suppose $\pi(t)$ is finite prime that is ramified in $F/\mathbb{F}_q(t)$. Then $\pi(t)\mid g(t)$ since the only way to obtain ramification at π in the absolute abelian Galois group is to have K_{π^a}, for some $a\in\mathbb{N}$.

Conversely, suppose F is contained in $K_g\mathbb{F}_{p^n}(t)L_m$ for some $g\in\mathbb{F}_q[t]$. Let $\pi'(t)$ be a prime dividing g but unramified in F. Let $g'(t)$ be the polynomial obtained by dividing $g(t)$ by as many powers of $\pi'(t)$ as $g(t)$ contains, i.e., $g(t)=g'(t)\pi'(t)^c$ for some $c\in\mathbb{N}$ and $g'(t)$ and $\pi'(t)$ are coprime. We write $K_g=\mathbb{F}_q(t,\alpha_{g'},\alpha_{\pi'^c})$, where $\alpha_{g'},\alpha_{\pi'^c}$ are the generators for $K_{g'}$ and $K_{\pi'^c}$ as in 1.1.1 and we can write them as generators of the field due to lemma 1.1.3.

Now we observe that for all $b\in\mathbb{N}$, the field $F\cap K_{\pi'^b}$ is a subfield of $K_{\pi'^b}$ that is unramified at all finite primes of $\mathbb{F}_q[t]$, so it must be $\mathbb{F}_{p^m}(t)$ for some $m\mid n$. Therefore, $F\subseteq K_{g'}\mathbb{F}_{p^n}(t)L_m$.

In conclusion, for $F\subseteq K_g\mathbb{F}_{p^n}(t)L_m$, we can always choose $g(t)$ such that $g(t)$ is only divisible by the ramified primes in F.

1.2 Sufficient Condition

In this section, we show the condition given in theorem 0.0.3 is a sufficient condition for the existence of a $\mathbb{F}_q[t]$-Λ-integral model in corollary 1.2.3 below.

1.2.1. Let $\mathrm{Id}(\mathbb{F}_q[t])$ denote the monoid of non-zero ideals in $\mathbb{F}_q[t]$. There is a one-to-one correspondence between monic polynomials $g(t) \in \mathbb{F}_q[t]$ and non-zero ideals $I_g := (g(t)) \in \mathrm{Id}(\mathbb{F}_q[t])$. For the rest of this chapter, we slightly abuse notation and say $g(t) \in \mathrm{Id}(\mathbb{F}_q[t])$ to mean the corresponding ideal $I_g \in \mathrm{Id}(\mathbb{F}_q[t])$.

Recall the definition of an $\mathbb{F}_q[t]$-Λ-ring and an $\mathbb{F}_q[t]$-Λ-model in 0.0.2.

Proposition 1.2.2. *Let $g(t) \in Id(\mathbb{F}_q[t])$, $n, m \in \mathbb{N}$ and let K be the corresponding finite étale $\mathbb{F}_q(t)$-Λ-ring of the monoid $(\mathbb{F}_q[t]/g(t)\mathbb{F}_q[t])^\circ \times C_n \times (1+\frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])/(\frac{1}{t^m})$ (see 1.1.8 for the explicit form of K). Then K has a $\mathbb{F}_q[t]$-Λ-model.*

Proof. It is clear that the tensor product of two torsion free Λ-rings is naturally a Λ-ring, so it is enough to show that the corresponding finite étale $\mathbb{F}_q(t)$-Λ-ring of each term in the direct product above, i.e., the $\mathbb{F}_q(t)$-Λ-ring corresponding to $(\mathbb{F}_q[t]/g(t)\mathbb{F}_q[t])^\circ$, C_n and $(1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])/(\frac{1}{t^m})$, has a $\mathbb{F}_q[t]$-Λ-model.

Let K_1 denote the finite étale $\mathbb{F}_q(t)\Lambda$-ring corresponding to $(\mathbb{F}_q[t]/g(t)\mathbb{F}_q[t])^\circ$. From 1.1.4 we see that $K_1 = \mathbb{F}_q(t)[x]/(\omega_{g(t)}(x))$ and it has a $\mathbb{F}_q[t]$-Λ-model, namely $\mathbb{F}_q[t,x]/(\omega_{g(t)}(x))$, with $\psi_\pi(x) = \omega_\pi(x)$.

Similarly, let K_2 and K_3 be the finite étale $\mathbb{F}_q(t)\Lambda$-ring corresponding to C_n and $(1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])/(\frac{1}{t^m})$ respectively. These field extensions are unramified at the all the finite primes. Hence we can take the integral closure over $\mathbb{F}_q[t]$ (i.e. the ring of integers in the function field setting) as the $\mathbb{F}_q[t]$-Λ-models, with ψ_π defined to be the Artin symbol of π. This concludes the proof. □

Corollary 1.2.3. *Let $g(t), n, m$ as above, let S be a finite $(\mathbb{F}_q[t]/g(t)\mathbb{F}_q[t])^\circ \times C_n \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])/(\frac{1}{t^m})$-set, K the corresponding finite étale $\mathbb{F}_q(t)\Lambda$-ring. Then K has a $\mathbb{F}_q[t]$-Λ-model.*

Proof. We denote the monoid $(\mathbb{F}_q[t]/g(t)\mathbb{F}_q[t])^\circ \times C_n \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])/(\frac{1}{t^m})$ by $M(g(t), n, m)$. Let Y be a finite set such that $\coprod_Y M(g(t), n, m) \to S$ is a surjective map of $M(g(t), n, m)$-sets. Such a set Y exists as one can simply take $Y = S$. Let L be the corresponding finite étale $\mathbb{F}_q(t)\Lambda$-ring of $M(g(t), n, m)$. Then K is a sub-Λ-ring of L^Y, which has a $\mathbb{F}_q[t]$-Λ-model R^Y by 1.2.2. Therefore, K has a $\mathbb{F}_q[t]$-Λ-model, namely $K \cap R^Y$. □

1.3 Necessary Condition

In this section, we show the condition given in theorem 0.0.3 is a necessary condition for the existence of a $\mathbb{F}_q[t]$-Λ-integral model.

Following the same notation as the proof of corollary 1.2.3, we denote the monoid $(\mathbb{F}_q[t]/g(t)\mathbb{F}_q[t])^{\circ} \times C_n \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])/(\frac{1}{t^m})$ by $M(g(t), n, m)$. We are also going to write $M^*(g(t), n, m)$ for the group $(\mathbb{F}_q[t]/g(t)\mathbb{F}_q[t])^* \times C_n \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])/(\frac{1}{t^m})$.

Recall the definition of an $\mathbb{F}_q[t]$-Λ-ring and an $\mathbb{F}_q[t]$-Λ-model in 0.0.2.

Proposition 1.3.1. *Let K be a finite étale $\mathbb{F}_q(t)\Lambda$-ring with a $\mathbb{F}_q[t]$-Λ-model A. There exists $g(t) \in \mathbb{F}_q[t]$, $n, m \in \mathbb{Z}$, where $g(t)$ is divisible only by primes that ramify in A, such that the action of $G_{\mathbb{F}_q(t)}$ on a finite set S factors through $G_{\mathbb{F}_q(t)} \to M^*(g(t), n, m)$. If a prime $\pi(t)$ in $\mathbb{F}_q[t]$ is unramified in A, then the prime ideals $\pi(t) \in Id(\mathbb{F}_q[t])$ and*

$$(\pi(t) \ mod \ g(t), r \ mod \ n, \frac{1}{t^r}\pi(t) \ mod \ \frac{1}{t^m}) \in M^*(g(t), n, m)$$

where $r = deg(\pi(t))$, act the same way on S.

Proof. Define N to be the fixed field of the kernel of $G_{\mathbb{F}_q(t)} \to Map(S, S)$. N is Galois over $\mathbb{F}_q(t)$ and denote G' to be the Galois group $\mathrm{Gal}(N/\mathbb{F}_q(t))$. Let $\mathcal{O}_N$ denote the ring of integers of N.

From the Tchebotarev density theorem [19, theorem 9.13], for any $g \in G'$, there exists infinitely many irreducible polynomials $\pi(t) \in \mathbb{F}_q[t]$ and primes $\mathfrak{P}$ of N above $\pi(t)$ such that $g(x) \equiv x^{q^{\deg(\pi)}} \bmod \mathfrak{P}$ for all $x \in \mathcal{O}_N$. Since there are only finitely many primes ramified in A and infinitely many $\mathfrak{P}$ satisfying the equivalence, we may take an unramified $\pi(t)$.

Now observe that for any $s \in S = \mathrm{Hom}(A, \mathcal{O}_N)$, the maps $s \circ \psi_\pi$ and $g \circ s$ from A to $\mathcal{O}_N$ are equal. To see this, note that for π unramified in A, there is an injection from $\mathrm{Hom}(A, \mathcal{O}_N)$ to $\mathrm{Hom}(A, \mathcal{O}_N/\mathfrak{P})$. Hence, to show the equality of the two maps $s \circ \psi_\pi$ and $g \circ s$, it is enough to show their composition with the natural quotient map $\mathcal{O}_N \to \mathcal{O}_N/\mathfrak{P}$ are equal. But this is immediate from the choice of $\mathfrak{P}$ and the image of all $a \in A$ under each map is $s(a)^{deg(\pi)} \bmod \mathfrak{P}$.

Therefore, the image of $G_{\mathbb{F}_q(t)}$ is contained in the image of $\mathrm{Id}(\mathbb{F}_q[t]$ and it follows that the G' is abelian. By theorem 1.1.7 (Carlitz-Hayes theorem), there exists $g(t) \in \mathbb{F}_q[t]$, $n, m \in \mathbb{Z}$ such that N is contained is contained in the fixed field of $(\mathbb{F}_q[t]/g(t)\mathbb{F}_q[t])^* \times C_n \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])/(\frac{1}{t})^m$, where we can choose $g(t)$ to be divisible only by the finite primes ramify in N by remark 1.1.9. Hence we can choose $g(t)$ such that $g(t)$ is only divisible by the finite primes ramify in A since N is the common Galois closure of all the components in $A \otimes \mathbb{F}_q(t)$. Finally, the last statement follows from the fact that a prime $\pi(t) \nmid g(t)$ is unramified and the Artin symbol of such prime is $(\pi(t) \bmod g(t), r \bmod n, \frac{1}{t^r}\pi(t) \bmod \frac{1}{t^m})$, where $r = deg(\pi)$. □

Proposition 1.3.2. *Let Y be a finite discrete set such that there exist a continuous action of $Id(\mathbb{F}_q[t]) \times \hat{\mathbb{F}_q[t]}^* \times \hat{\mathbb{Z}} \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])$ on Y. Then the action factors through a continuous action of $\hat{\mathbb{F}_q[t]}^\circ \times \hat{\mathbb{Z}} \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])$ if the action has the following two properties:*

1. *all but finitely many irreducible polynomials $\pi(t)$ act as automorphisms on Y.*

2. *for all $f(t) \in Id(\mathbb{F}_q[t])$, there exists $g_f(t) \in Id(\mathbb{F}_q[t])$, $m_f, n_f \in \mathbb{N}$ such that the action of $\hat{\mathbb{F}_q[t]}^*$ on $f(t)Y$ factors through $M^*(g_f(t), n, m)$, and for each $\sigma(t) \in Id(\mathbb{F}_q[t])$, $r = deg(\sigma)$ with $\sigma(t)f(t)Y = f(t)Y$, we have $\sigma(t)$ is relatively prime to $g_f(t)$ and the actions of*

$$(\sigma(t) \ mod \ g_f(t), r \ mod \ n_f, \frac{1}{t^r}\sigma(t) \ mod \ \frac{1}{t_f^m}) \in M^*(g_f(t), n, m)$$

and $\sigma(t) \in Id(\mathbb{F}[t])$ on $f(t)Y$ are the same

Proof. Suppose 1. and 2. holds. For each $\pi[t] \in \mathbb{F}_p[t]$ irreducible polynomial, let a_π be the smallest integer $a \geq 0$ such that $\pi(t)^a Y = \pi^{a+1} Y$. Since almost all and possibly except finitely many π's act as automorphisms, $\rho_0(t) = \prod_\pi \pi^{a_\pi}$ is a finitely product and a polynomial. Note that for any $\mu \in \mathbb{F}_p[t]$, $\mu Y = \gcd(\mu, \rho_0)Y$.

Let $\rho(t)$ be any polynomial divisible by $fg_f(t)$, n divisible by all n_f and $m \geq m_f$ for all $f \mid \rho_0(t)$. Let us show that the action of $\mathrm{Id}(\mathbb{F}_q[t]) \times \hat{\mathbb{F}_q[t]}^* \times \hat{\mathbb{Z}} \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])$ on Y, a finite discrete set, factors through $(\mathbb{F}_q[t]/\rho(t)\mathbb{F}_q[t])^\circ \times C_n \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])/(\frac{1}{t^m})$. To do this, we will show directly that any two elements $(f_1(t), \tau_1(t), r_1, w_1(1/t)), (f_2(t), \tau_2(t), r_2, w_2(1/t)) \in \mathrm{Id}(\mathbb{F}_q[t]) \times (\mathbb{F}_q[t]/g_f(t)\mathbb{F}_q[t])^* \times C_n \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])/(\frac{1}{t^m})$, satisfying the following relations, act the same way on Y.

$$f_1(t)\tau_1(t) \equiv f_2(t)\tau_2(t) \bmod \rho(t) \tag{1.3.3}$$

$$deg(f_1) + r_1 \equiv deg(f_2) + r_2 \bmod n \tag{1.3.4}$$

$$(1/t)^{deg(f_1)} f_1(t) w_1(1/t) \equiv (1/t)^{deg(f_2)} f_2(t) w_2(1/t) \bmod \frac{1}{t^m} \tag{1.3.5}$$

As $\rho_0(t) \mid \rho(t)$, equivalence (1.3.3) shows that f_1 and f_2 has the same divisor f with ρ_0, so $f_1 Y = f = f_2 Y$. Let $f_i = f_i' f$ for each $i = 1, 2$, so we have $f_i'(fY) = fY$. Using the second property in the proposition, we must have f_i' and g_f are coprime, hence $f_i' \in (\mathbb{F}_q[t]/g_f(t)\mathbb{F}_q[t])^*$ and the action of f_i' on fY is

$$(f_i'(t) \bmod g_f(t), deg(f_i') \bmod n_f, \frac{1}{t^s} f_i'(t) \bmod \frac{1}{t^{m_f}})$$

Since $fg_f \mid \rho$ and equation (1.3.3) shows that

$$f_1'(t)f(t)\tau_1(t) \equiv f_2'(t)f(t)\tau_2(t) \bmod fg_f(t),$$

hence $f_1'(t)\tau_1(t) \equiv f_2'(t)\tau_2(t) \bmod g_f(t)$. Similarly, since $n_f \mid n$, equation (1.3.4) shows that $deg(f) + deg(f_1') + r_1 \equiv deg(f) + deg(f_2') + r_2 \bmod n_f$, so $deg(f_1') + r_1 \equiv deg(f_2') + r_2 \bmod n_f$. Finally, since $m \geq m_f$, from equation (1.3.5), we have

$$(\frac{1}{t})^{deg(f)} f(t) (\frac{1}{t})^{deg(f_1')} f_1'(t) w_1(\frac{1}{t}) \equiv (\frac{1}{t})^{deg(f)} f(t) (\frac{1}{t})^{deg(f_2')} f_2'(t) w_2(\frac{1}{t}) \bmod \frac{1}{t^{m_f}},$$

so $(1/t)^{deg(f_1')} f_1'(1/t) w_1(1/t) \equiv (1/t)^{deg(f_2')} f_2'(1/t) w_2(1/t) \bmod 1/t^{m_f}$. Therefore, $(f_1'(t), \tau_1(t), r_1, w_1(1/t))$ and $(f_2'(t), \tau_2(t), r_2, w_2(1/t))$ act the same way on fY. Finally, composing with $(f(t), 0, 1, 1)$ shows that the two elements

$$(f_1(t), \tau_1(t), r_1, w_1(1/t)), (f_2(t), \tau_2(t), r_2, w_2(1/t))$$

act the same way on Y. □

Given the proposition above, to prove the necessity in theorem 0.0.3, it is therefore enough to show the two conditions are satisfied for $Y = S$. Firstly, there are only finitely many ramified primes in a given $\mathbb{F}_p[t]$-Λ-ring A, so condition 1 is met.

Secondly, given $f(t) \in \mathbb{F}_p[t]$, consider the sub-$\mathbb{F}_q[t]$-Λ-ring $\psi_f(A)$ corresponding to the $\mathrm{Id}(\mathbb{F}_q[t]) \times G_{\mathbb{F}_q(t)}$-set $f(t)S$ of S. Applying proposition 1.3.1 to $\psi_f(A)$ gives $g_f(t), n_f, m_f$ such that the action of the Galois group factors through the group $M^*(g_f(t), n_f, m_f)$. By, [1, 2.1], ψ_π is an automorphism if and only if π is unramified in A. Therefore, if $h(t) \in \mathrm{Id}(\mathbb{F}_q[t]$ such that $h(t)f(t)S = f(t)S$, then $h(t)$ is a product of primes unramified in $\psi_f(A)$. Using proposition 1.3.1 gives us condition 2.

1.4 More general Λ-models

In this section, we generalize the result we found in the previous section. More precisely, we prove theorem 1.4.6.

1.4.1. We fix the following notation.

$$\begin{aligned}
K &:= \mathbb{F}_q(t), \\
\bar{K} &:= \text{A fixed algebraic closure of } K, \\
G_K &:= \text{The absolute Galois group of } K \text{ relative to } \bar{K}, \\
\mathbb{P}^1_{\mathbb{F}_q} &:= \text{The projective line over } \mathbb{F}_q, \\
\infty &:= \text{The closed point in } \mathbb{P}^1_{\mathbb{F}_q} \setminus \operatorname{Spec}(\mathbb{F}_q[t]), \\
\Pi &:= \{\text{places of } K\}, \text{ finite or infinite}, \\
\operatorname{Div}^+(K) &:= \bigoplus_{\mathfrak{p}} \mathbb{N}, \text{ where } \mathfrak{p} \text{ runs over elements of } \Pi \text{ (the effective divisors of } K), \\
P, Q, U, V, W &: \text{denotes subsets of } \Pi \text{ excluding at most finitely many places}, \\
X_P &:= \text{The subscheme of } \mathbb{P}^1_{\mathbb{F}_q} \text{ such that the closed points are } P, \\
\operatorname{Div}^+_P(K) &:= \bigoplus_{\mathfrak{p}} \mathbb{N}, \text{ where } \mathfrak{p} \text{ runs over elements of } P, \\
\mathcal{O}_v &:= \text{The completion of } \mathbb{F}_q[t] \text{ with respect to a valuation } v, \\
\mathcal{O}_v^* &:= \text{The group of invertible elements in } \mathcal{O}_v, \\
\mathcal{O}_v^\circ &:= \text{The set } \mathcal{O}_v \text{ viewed as a multiplicative monoid}, \\
P \subsetneq \Pi, R_P &:= \text{A ring over } \mathbb{F}_q \text{ such that } \operatorname{Spec}(R_P) = X_P
\end{aligned}$$

Definition 1.4.2. Let $P, Q \subseteq \Pi$ and S be a finite étale scheme over $\operatorname{Spec}(K)$. Recall the definition of a $\Lambda_{B,P}$-scheme in 0.0.5. If $Q \neq \Pi$, we say S has a Q-$\Lambda(P)$-model, or a Λ^P_Q-model, if there exists a finite flat and reduced $\Lambda_{R_Q,P}$-scheme E, such that the pullback $E \times_{\mathbb{P}^1_{\mathbb{F}_q}} \operatorname{Spec}(K) \cong S$.

If $Q = \Pi$, we say S has a Λ^P_Q-model if there exists an open affine cover $\{U_i \to \mathbb{P}^1_{\mathbb{F}_q}\}$, where the closed points of U_i are Q_i, such that $S \times_{\mathbb{P}^1_{\mathbb{F}_q}} U_i$ has a $\Lambda^P_{Q_i}$-model.

Definition 1.4.3. For any $P, Q \subseteq \Pi$,

$$M^P_Q := \bigoplus_{P \setminus Q} \mathbb{N} \times \left(\prod_{v \in P \cap Q} \mathcal{O}_v^\circ \times \prod_{v \notin P \cap Q} \mathcal{O}_v^* \times \hat{\mathbb{Z}} \right) \Big/ \mathbb{F}_q^*$$

where v is a finite or infinite prime of $\mathbb{F}_q(t)$ and $\mathbb{F}_q^*$ is embedded diagonally into each $\mathcal{O}_v^*$ and $\mathcal{O}_v^\circ$.

Definition 1.4.4. Let $P, Q \subseteq \Pi$ be sets of almost all places (i.e., they can at most exclude finitely many primes). We define a map

$$\operatorname{Div}^+_P(K) \times G_K \to M^P_Q$$

for all sets P, Q. By passing the image to the quotient of $\mathbb{F}_q^*$, it is enough to define the following maps

$$a : \mathrm{Div}_P^+(K) \to \bigoplus_{P\setminus Q} \mathbb{N} \times \prod_{v\in P\cap Q} \mathcal{O}_v^\circ \times \prod_{v\notin P\cap Q} \mathcal{O}_v^* \times \hat{\mathbb{Z}}$$

and

$$b : G_K \to \bigoplus_{P\setminus Q} \mathbb{N} \times \prod_{v\in P\cap Q} \mathcal{O}_v^\circ \times \prod_{v\notin P\cap Q} \mathcal{O}_v^* \times \hat{\mathbb{Z}}$$

We first define the map a. Let $\mathfrak{p} \in P$, fix the following notation.

$$\mathrm{Div}_P^+(K) \ni 1_{\mathfrak{p}} := (0, \cdots, 0, 1, 0, \cdots), \quad \text{where 1 appears in the } \mathfrak{p}\text{-th position;}$$

$$\mathfrak{p} \in V \subseteq P,\ e_{\mathfrak{p}} := \text{the } \mathfrak{p}\text{-th standard basis vector of } \textstyle\bigoplus_V \mathbb{N}$$

$$\pi_{\mathfrak{p}}(t) := \begin{cases} \text{the unique monic generator of the prime} \\ \quad \text{ideal in } \mathbb{F}_q[t] \text{ corresponding to } \mathfrak{p}, & \text{if } \mathfrak{p} \neq \infty \\ \quad t^{-1}, & \text{otherwise} \end{cases}$$

$$r_{\mathfrak{p}} := \begin{cases} \text{the degree of } \pi_{\mathfrak{p}}(t) \text{ as a polynomial in } \mathbb{F}_q[t], & \text{if } \mathfrak{p} \neq \infty \\ -1, & \text{otherwise} \end{cases}$$

$$v \in \Pi,\ f_{\mathfrak{p},v} := \begin{cases} t^{-r_{\mathfrak{p}}}\pi_{\mathfrak{p}}(t) & \text{if } v = \infty \text{ and } \mathfrak{p} \neq \infty, \text{ or } v = 0 \text{ and } \mathfrak{p} = \infty, \\ \pi_{\mathfrak{p}}(t) & \text{otherwise} \end{cases}$$

We remark that in particular, $f_{\infty,0} = f_{0,\infty} = 1$. We define the map a as follows.

$$\begin{aligned} a : \mathrm{Div}_P^+(K) &\to \bigoplus_{P\setminus Q} \mathbb{N} \times \prod_{v\in P\cap Q} \mathcal{O}_v^\circ \times \prod_{v\notin P\cap Q} \mathcal{O}_v^* \times \hat{\mathbb{Z}} \\ 1_{\mathfrak{p}} &\mapsto \left(e_{\mathfrak{p}}, \prod_{v\in P\cap Q} f_{\mathfrak{p},v}, \prod_{v\notin P\cap Q} f_{\mathfrak{p},v}, |r_{\mathfrak{p}}| \right) \end{aligned}$$

We remark $\pi_{\mathfrak{p}}(t)$ is invertible in $\mathcal{O}_v$ for all $v \notin P$ because $\mathfrak{p} \in P$ so $\pi_{\mathfrak{p}}(t)$ must be invertible in those complete rings.

For the map b, we send G_K to its abelianization and write G_K^{ab} explicitly using theorem 1.1.7. By viewing $\pi_{\mathfrak{p}}(t) \in \mathbb{F}_q[t]$ and t^{-1} as a uniformizer of each $\mathcal{O}_{\mathfrak{p}}$, we can identify $\prod_{v\neq\infty} \mathcal{O}_v^*$ as $\widehat{\mathbb{F}_q[t]}^*$ and $1 + t^{-1}\mathbb{F}_q[[t^{-1}]]$ as a subgroup of $\mathcal{O}_\infty^*$. Thus, we can define b as the following composition of maps.

$$G_K \to \widehat{\mathbb{F}_q[t]}^* \times (1+\frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]]) \times \hat{\mathbb{Z}} \hookrightarrow \prod_{v\in\Pi} \mathcal{O}_v^* \times \hat{\mathbb{Z}} \hookrightarrow \bigoplus_{P\setminus Q} \mathbb{N} \times \prod_{v\in P\cap Q} \mathcal{O}_v^\circ \times \prod_{v\notin P\cap Q} \mathcal{O}_v^* \times \hat{\mathbb{Z}}$$

Remark 1.4.5. The map above is dependent on the choice of a uniformizer $\pi_{\mathfrak{p}}$ for each prime $\mathfrak{p} \in \Pi$, but we are not going to investigate this dependence further.

The main result in this section is the following theorem.

Theorem 1.4.6. *Let K, Π, P and Q be defined in 1.4.1. Let M_Q^P be the monoid defined in 1.4.3. A Λ-scheme S over Spec(K) has a Λ_Q^P-model if and only if there exist a continuous monoid map $M_Q^P \to Map(\bar{S}, \bar{S})$, where $\bar{S} = S(\bar{K})$, such that the following diagram commutes.*

$$\begin{tikzcd}
Div_P^+(K) \times G_K \arrow[r] \arrow[dr] & M_Q^P \arrow[d] \\
 & Map(\bar{S}, \bar{S})
\end{tikzcd}$$

where the map $Div_P^+(K) \times G_K \to M_Q^P$ is defined in 1.4.4.

We give the proof of the theorem above in 1.4.20.

1.4.7. For example, for $P = Q = \Pi \setminus \infty$, our previous section shows the theorem holds. Indeed. we can identify an isomorphism $\mathcal{O}_\infty^* \cong \mathbb{F}_q^* \times (1 + t^{-1}\mathbb{F}_q[[t^{-1}]]$ by viewing t^{-1} as a uniformizer. Furthermore, we see the map we defined in 1.4.4 agrees with the map defined in 0.0.4.

We give a few more specific examples.

Proposition 1.4.8. *Let P' be a subset of $\Pi \setminus \{\infty\}$ (containing all but finitely many primes). Theorem 1.4.6 holds for $P = Q = P'$. In other words, a Λ-scheme S has a $\Lambda_{P'}^{P'}$-model if and only if there exist a continuous monoid map $M_{P'}^{P'} \to Map(\bar{S}, \bar{S})$, where $\bar{S} = S(\bar{K})$, such that the following diagram commutes.*

$$\begin{tikzcd}
Div_{P'}^+(K) \times G_K \arrow[r] \arrow[dr] & M_{P'}^{P'} \arrow[d] \\
 & Map(\bar{S}, \bar{S})
\end{tikzcd}$$

where the map $Div_{P'}^+(K) \times G_{\mathbb{F}_q(t)} \to M_{P'}^{P'}$ is described in 1.4.4.

Proof. For sufficiency, we follow the notation in 1.2.1 and follow a similar proof shown in 1.2.2 and 1.2.3. We take a finite quotient of the monoid $M_{P'}^{P'}$ that has the following form

$$(\mathbb{F}_q[t]/g(t))^\circ \times (\mathbb{F}_q[t]/f(t))^* \times C_n \times (1 + \frac{1}{t}\mathbb{F}_q[[\frac{1}{t}]])/(\frac{1}{t^m})$$

where $f(t)$ is not divisible by any primes in P' and $g(t)$ is only divisible by the primes in P'. It is enough to show the K-algebra corresponding to each term in the product has a P'-$\Lambda(P')$-model.

By 1.2.2, we see that the algebra corresponds to the first, third and last term has a $\Pi \setminus \{\infty\}$-$\Lambda(\Pi \setminus \{\infty\})$-model, so it must also have a $\Pi \setminus \{\infty\}$-$\Lambda(P')$-model (by forgetting the Λ-structure corresponding to the primes not in P') and hence a P'–$\Lambda(P')$-model by 1.4.12 shown below.

Moreover, since the primes in P' are not ramified in the K-algebra corresponding to the second term by [12, proposition 2.2], the ring of $R_{P'}$-integral elements inside the K-algebra together with the ring map $\psi_{\mathfrak{p}}(x) = \omega_\pi(x)$ defines a P'-$\Lambda(P')$-model, where ω_π is the Carlitz polynomial defined in the beginning of section 1.1. Finally, we can give a similar corollary as 1.2.3 to conclude the sufficiency.

For necessity, we view $\mathrm{Div}^+_{P'}(K) \times G_K$ as a submonoid of $\mathrm{Div}^+_{\Pi\setminus\{\infty\}}(K) \times G_K$. We see that, since P' contains all but finitely many primes, 1.3.1 and 1.3.2 applies to the action of $\mathrm{Div}^+_{P'}(K) \times G_K$ on S, so the action of $\mathrm{Div}^+_{P'}(K) \times G_K$ on S must factor through the image of $\mathrm{Div}^+_{P'}(K) \times G_K$ under the map $\mathrm{Div}^+_{\Pi\setminus\{\infty\}}(K) \times G_K \to M^{\Pi\setminus\{\infty\}}_{\Pi\setminus\{\infty\}}$ described in 1.4.7.

Therefore, we proceed by looking at the image under the map above. We see that, since the copy of $\mathbb{N}$ in the $\mathfrak{p}$-th position is missing for each $\mathfrak{p} \in \Pi \setminus P'$, the image in the $\mathcal{O}_{\mathfrak{p}}$ component must be invertible, i.e., must be contained in $\mathcal{O}^*_{\mathfrak{p}}$. Furthermore, the image of G_K in the $\mathcal{O}_{\mathfrak{p}}$ component is the whole $\mathcal{O}^*_{\mathfrak{p}}$. Thus, we conclude that the action of $\mathrm{Div}^+_{P'}(K) \times G_K$ on S must factor through the monoid

$$\left(\prod_{v \in P'} \mathcal{O}^\circ_v \times \prod_{v \notin P'} \mathcal{O}^*_v \times \hat{\mathbb{Z}} \right) \Big/ \mathbb{F}_q^*$$

This is the monoid given in 1.4.3 with $P = Q = P'$ and we are done. □

Proposition 1.4.9. *Let* 0 *denote the prime correspond to the prime ideal* $(t) \subseteq \mathbb{F}_q[t]$ *and let* $P' = \Pi \setminus \{0\}$. *Theorem 1.4.6 holds for* $P = Q = P'$. *In other words, a* Λ*-scheme* S *has a* $\Lambda^{P'}_{P'}$*-model if and only if there exist a continuous monoid map*

$M_{P'}^{P'} \to Map(\bar{S}, \bar{S})$, where $\bar{S} = S(\bar{K})$, such that the following diagram commutes.

$$\begin{tikzcd}
Div_{P'}^{+}(K) \times G_K \arrow[r] \arrow[dr] & M_{P'}^{P'} \arrow[d] \\
& Map(\bar{S}, \bar{S})
\end{tikzcd}$$

where the map $Div_{P'}^{+}(K) \times G_{\mathbb{F}_q(t)} \to M_{P'}^{P'}$ is described in 1.4.4.

Proof. By reversing the roles of the primes 0 and ∞ and looking at the ring $\mathbb{F}_q[t^{-1}]$ instead of $\mathbb{F}_q[t]$, we repeat everything we have done in section 1.1 to 1.3 and similarly define a map $\mathrm{Div}_{P'}^{+}(K) \times G_K \to M_{P'}^{P'}$ that is induced by a map that sends G_K to its abelianization, which is isomorphic to the following group by 1.1.7 and by reversing the roles of t and t^{-1}.

$$G_K \xrightarrow{\sim} \widehat{\mathbb{F}_q[t^{-1}]}^{*} \times \hat{\mathbb{Z}} \times (1 + t\mathbb{F}_q[[t]])$$

On the second factor, the reversed 0.0.4, by which we mean the same definition but with the roles of 0 and ∞ reversed, tells us to send $\mathrm{Div}_{P'}$ to $M_{P'}^{P'}$ by the map induced by the following.

$$\begin{aligned}
\mathbb{N}_{\mathfrak{p}} &\to \mathcal{O}_0^{\circ} \times \prod_{\mathfrak{p} \neq 0} \mathcal{O}_{\mathfrak{p}}^{\circ} \times \hat{\mathbb{Z}} \\
\mathfrak{p} &\mapsto \pi(t), (\pi'(t^{-1}), \pi'(t^{-1}), \pi'(t^{-1}), \cdots, \pi'(t^{-1})), \deg(\pi'(t^{-1}))
\end{aligned}$$

where $\mathfrak{p} \in P'$ (i.e., $\mathfrak{p} \neq 0$), $\pi(t)$ is once again the unique monic generator of the prime ideal of $\mathbb{F}_q[t]$ corresponding to $\mathfrak{p}$, and $\pi'(t^{-1})$ is the unique monic generator of the prime ideal of $\mathbb{F}_q[t^{-1}]$ corresponding to $\mathfrak{p}$. One sees that, by homogenizing $\pi(t)$ to a function on $\mathbb{P}^1_{\mathbb{F}_q}$ then dehomogenizing it to an element in $\mathbb{F}_q[t^{-1}]$, we see that $\pi(t)$ and $\pi'(t)$ are related by $\pi'(t^{-1}) = \lambda^{-1} t^{-r} \pi(t)$, where $\lambda \in \mathbb{F}_q^*$ is the constant coefficient of $\pi(t)$ and $r = \deg(\pi) = \deg(\pi')$. Section 1.1 to 1.3 tells us that S has a P'-$\Lambda(P')$-model if and only if the map $\mathrm{Div}_{P'}^{+}(K) \times G_K \to \mathrm{Map}(S, S)$ factors through the map $\mathrm{Div}_{P'}^{+}(K) \times G_K \to M_{P'}^{P'}$ described above.

It remains to show that the map S has a P'-$\Lambda(P')$-model if and only if the map $\mathrm{Div}_{P'}^{+}(K) \times G_K \to \mathrm{Map}(S, S)$ factors through the map $a : \mathrm{Div}_{P'}^{+}(K) \times G_K \to M_{P'}^{P'}$ described in 1.4.4. However, by the above, we described a map $b : \mathrm{Div}_{P'}^{+}(K) \times G_K \to M_{P'}^{P'}$, which the map $\mathrm{Div}_{P'}^{+}(K) \times G_K \to \mathrm{Map}(\bar{S}, \bar{S})$ factors through it if and only if S has a P'-$\Lambda(P')$-model, so it is enough to give an isomorphism

$M_{P'}^{P'} \to M_{P'}^{P'}$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Div}_{P'}^{+}(K) \times G_K \arrow[r, "a"] \arrow[dr, "b"'] & M_{P'}^{P'} \arrow[d, "\sim"] \\
& M_{P'}^{P'}
\end{tikzcd}$$

We note that 1.3.1 tells us the image of G_K is contained in the image of $\mathrm{Div}_{P'}^{+}(K)$, so it is enough to give a map that between the monoids such that the image of $\mathrm{Div}_{P'}^{+}(K)$ commutes in the diagram above. However, this is clear since the image of $\mathbb{N}_\infty$ agrees on both maps, and since for all $\mathfrak{p} \neq 0, \infty$, we see the $\mathcal{O}_\infty$ and $\mathcal{O}_0$ components agree. Furthermore, for the $\mathcal{O}_v$ components, $v \neq 0, \infty$, the corresponding generator of prime ideals in the two rings are related by $\pi'(t^{-1}) = \lambda^{-1}t^{-r}\pi(t)$. It is clear that, since $t \in \mathcal{O}_v^*$, the map $\mathcal{O}_v \to \mathcal{O}_v$ defined by $x \mapsto \lambda^{-1}t^{-r}x$ for each $v \neq 0, \infty$ induces an isomorphism $M_{P'}^{P'} \to M_{P'}^{P'}$ such that the diagram above commutes. □

Corollary 1.4.10. *Let P' be a subset of $\Pi \setminus \{0\}$ (containing all but finitely many primes). Theorem 1.4.6 holds for $P = Q = P'$.*

Proof. One repeats the same proof 1.4.8 but with the roles of 0 and ∞ reversed as we have shown in 1.4.9 □

1.4.11. (Base changing Λ-models) Let $W \subseteq U \subseteq V \subseteq \Pi$ (recall U, V, W contains almost all primes as we defined in 1.4.1). We show how to construct a U-$\Lambda(W)$-model of a Λ-scheme S if S has a V-$\Lambda(W)$-model, and vice versa.

The case where $U = V$ is obvious, so let us assume $U \subsetneq V$. Let E be a finite étale $\mathbb{F}_q(t)$-scheme with a V-$\Lambda(W)$-model, i.e. there is a finite flat reduced scheme S_V over $\mathrm{Spec}(R_V)$ such that $\mathrm{Spec}(\mathbb{F}_q(t)) \times_{\mathbb{F}_q[t]} S_V \cong E$. Since we assumed $U \subsetneq V \subseteq \Pi$, there exists a ring R_U such that $\mathrm{Spec}(R_U)$=U. Moreover, the fiber $S_U = \mathrm{Spec}(R_U) \times_{\mathbb{F}_q[t]} S_V$ is represented by a finite flat and reduced algebra over R_U and $\mathbb{F}_q(t) \otimes A_U \cong E$. Furthermore, both S_V and $\mathrm{Spec}(R_U)$ have a $\Lambda(W)$-structure, the former by assumption and the latter by the fact that $W \subseteq U$. Since the tensor product of Λ-rings is naturally a Λ-ring, S_U is a U-$\Lambda(W)$-model of E.

On the other hand, now suppose we have a finite étale $\mathbb{F}_q(t)$ scheme L with a U-$\Lambda(W)$-model, represented by the ring B_U. Let $V' \subseteq V$ such that $X_{V'}$ is an open affine subset of X_V and let $L' = L \times_{\mathrm{Spec}(\mathbb{F}_q[t])} X_{V'}$ be an open affine subset of L. Let $\mathcal{O}_{L',R_{V'}}$ denote the set of elements in the algebra representing L' that are integral $R_{V'}$. One sees that $B_{V'} = B_U \cap \mathcal{O}_{L',R_{V'}}$ is a finite flat and reduced algebra over $R_{V'}$. Furthermore, we see that $B_{V'}$ has a $\Lambda(W \cap V')$-structure, since both $B_{V'}$ and $R_{V'}$ has one. This works for all affine subsets of L containing U and this construction gives us a V-$\Lambda(W)$-model of L.

Corollary 1.4.12. *Let $W \subseteq U \subseteq V \subseteq \Pi$. Theorem 1.4.6 holds for $P = W, Q = U$ if and only if the theorem holds for $P = W$ and $Q = V$.*

Proof. Suppose the theorem holds for $P = W, Q = U$. Let S be a Λ-scheme with a U-$\Lambda(W)$-model and let $\bar{S} = S(\bar{K})$. This is equivalent to the existence of a unique map $M_U^W \to \mathrm{Map}(\bar{S}, \bar{S})$ such that the map $\mathrm{Div}_W^+(K) \times G_K \to \mathrm{Map}(\bar{S}, \bar{S})$ factors through it by assumption. By 1.4.11, the is equivalent to having a V-$\Lambda(W)$-model. We note that since $W \subseteq U \subseteq V \subseteq \Pi$, $M_U^W = M_V^W$ by definition 1.4.3, so we have shown that the map $\mathrm{Div}_W^+(K) \times G_K \to \mathrm{Map}(\bar{S}, \bar{S})$ factors through $M_U^W = M_V^W \to \mathrm{Map}(\bar{S}, \bar{S})$ if and only if it has a V-$\Lambda(W)$-model, so the theorem holds for $P = W, Q = U$.

The converse also holds by a similar argument. □

Proposition 1.4.13. *Let $U \subseteq V \subseteq \Pi$ and $W \subseteq \Pi$ such that $(V \setminus U) \cap W = \emptyset$. Theorem 1.4.6 holds for $P = U$ and $Q = W$ if and only if the theorem also holds for $P = V$ and $Q = W$.*

Proof. Suppose the theorem holds for $P = U$ and $Q = W$. Let S be a finite étale $\mathbb{F}_q(t)$ scheme with a W-$\Lambda(V)$-model X and let $\bar{S} = S(\bar{K})$. It is clear that X is also a W-$\Lambda(U)$-model, since one may simply forget the Frobenius lift for all $\mathfrak{p} \in V \setminus U$. This forgets the action of $\bigoplus_{V \setminus U} \mathbb{N}$ on $\bar{S}$, so we conclude that if S has a W-$\Lambda(V)$-model then the action of $\mathrm{Div}_V^+(K) \times G_K$ factors through $M_W^U \times \bigoplus_{V \setminus U} \mathbb{N}$. Since $U \subseteq V$ and $(V \setminus U) \cap W = \emptyset$, $V \setminus W = V \setminus U \sqcup U \setminus W$ where $\sqcup$ denotes a disjoint union, $M_W^U \times \bigoplus_{V \setminus U} \mathbb{N} = M_W^V$.

Conversely, suppose the action of $\mathrm{Div}_V^+(K) \times G_K$ factors through $M_W^V = M_W^U \times \bigoplus_{V \setminus U} \mathbb{N}$. This is equivalent to, by assumption, S having a W-$\Lambda(U)$-model Y together with an action of $\bigoplus_{V \setminus U} \mathbb{N}$ on it. We claim this is enough to show that Y is a W-$\Lambda(V)$-model of S. Indeed, the case $U = V$ is trivial. Otherwise, an action of $\bigoplus_{V \setminus U} \mathbb{N}$ on $\bar{S}$ induces an action on Y, which gives a family of commuting maps, one for each $\mathfrak{p} \in V \setminus U$. We note that $\mathfrak{p}$ corresponds to a unit ideal for any $\mathfrak{p} \in V \setminus U$ in any affine open subset of Y, say $\mathrm{Spec}(A)$. Thus, any map $A \to A$ is a Frobenius lift of $\mathfrak{p}$ and the maps defined using that action of $\bigoplus_{V \setminus U} \mathbb{N}$ on Y together with the $\Lambda(U)$-structure on Y gives Y a $\Lambda(V)$-structure.

Now suppose the theorem holds for $P = V$ and $Q = W$. Let S' be a finite étale $\mathbb{F}_q(t)$ scheme with a W-$\Lambda(U)$-model X' and let $\bar{S}' = S'(\bar{K})$. By a similar argument as the paragraph above, X' together with an action of $\bigoplus_{V\setminus U}\mathbb{N}$ on X' shows X' is a W-$\Lambda(V)$-model of S, so the action of $\mathrm{Div}_V^+(K)\times G_K$ factors through M_W^V. Forgetting the action of $\bigoplus_{V\setminus U}\mathbb{N}$ again shows if S' has a W-$\Lambda(U)$-model then the action of $\mathrm{Div}_U^+(K)\times G_K$ factors through M_W^U.

Conversely, suppose the action of $\mathrm{Div}_U^+(K)\times G_K$ on $\mathrm{Map}(\bar{S}', \bar{S}')$ factors through M_W^U. This implies the action of $\bigoplus_{v\setminus U}\mathbb{N}\times\mathrm{Div}_U^+(K)\times G_K$ on $\mathrm{Map}(\bar{S}', \bar{S}')$ factors through $M_W^U\times\bigoplus_{v\setminus U}\mathbb{N}$, so S' has W-$\Lambda(V)$-model by assumption. Forgetting the Frobenius lifts for each prime in $V\setminus U$ gives us a W-$\Lambda(U)$-model. □

1.4.14. Let $U\subseteq V\subseteq W\subseteq\Pi$. There exists a map

$$M_U^W\to M_V^W$$

as follows. We have a map

$$\bigoplus_{W\setminus U}\mathbb{N}\times\prod_{v\in U}\mathcal{O}_v^\circ\times\prod_{v\notin U}\mathcal{O}_v^*\times\hat{\mathbb{Z}}\to\bigoplus_{W\setminus V}\mathbb{N}\times\prod_{v\in V}\mathcal{O}_v^\circ\times\prod_{v\notin V}\mathcal{O}_v^*\times\hat{\mathbb{Z}}$$

induced by the identity map $\mathcal{O}_v\to\mathcal{O}_v$ for all $\mathcal{O}_v^*$ and $\mathcal{O}_v^\circ$ components by matching labels (which induces an inclusion map $\mathcal{O}_v^*\hookrightarrow\mathcal{O}_v^\circ$ on some components), the identity map $\mathbb{N}\to\mathbb{N}$ in the $\mathfrak{p}$-th position for all $\mathfrak{p}\in W\setminus V$, and the map

$$\begin{aligned}\mathbb{N}_\mathfrak{q}&\to\mathcal{O}_\mathfrak{q}^\circ\\ n&\mapsto\pi^n\end{aligned}$$

where $\mathbb{N}_\mathfrak{q}$ denotes a copy of $\mathbb{N}$ in the $\mathfrak{q}$-position in the product, π is the unique monic generator of the ideal in $\mathbb{F}_q[t]$ corresponding to $\mathfrak{q}$ if $\mathfrak{q}\neq\infty$ and $\pi=t^{-1}$ if $\mathfrak{q}=\infty$, for all $\mathfrak{q}\in V\setminus U$.

The map above induces a map $M_U^W\to M_V^W$ by passing to the quotient of $\mathbb{F}_q^*$.

Lemma 1.4.15. *Let $U_1\subseteq U_2\subseteq U_3\subseteq W\subseteq\Pi$. Let $f_{i,j}: M_{U_i}^W\to M_{U_j}^W$ be the map defined in 1.4.14 for each $1\le i<j\le 3$. Then $f_{1,3}=f_{2,3}\circ f_{1,2}$.*

Proof. We follow 1.4.14 and pick representative before taking the quotient of $\mathbb{F}_q^*$. We go through each $\mathfrak{p}\in\Pi$ and look at each component labelled by each $\mathfrak{p}$.

We first note that all f's map the $\mathcal{O}_v^*$ and $\mathcal{O}_v^\circ$ components to $\mathcal{O}_v$ under the identity map, so the claim is clear on these components. It remains to show the claim of each copy of $\mathbb{N}$ in the $\bigoplus_{W\setminus U_i}\mathbb{N}$ part of the monoid.

Let $\mathrm{id} : \mathbb{N} \to \mathbb{N}$ be the identity map on $\mathbb{N}$ and let $r : \mathbb{N} \to \mathcal{O}_v^\circ$ be the map defined in the end of 1.4.14. We consider the following three cases of v.

Case 1: If $v \in W \setminus U_3$, then each $f_{i,j}$ sends elements in $\mathbb{N}$ in the v component to itself under the identity. Clearly, $\mathrm{id} = \mathrm{id} \circ \mathrm{id}$.

Case 2: If $v \in U_3 \backslash U_2$, then $f_{2,3}$ and $f_{1,3}$ send elements in $\mathbb{N}$ in the v component to $\mathcal{O}_v$ via the map r, but $f_{1,2}$ is the map id. Clearly, $r = r \circ \mathrm{id}$.

Case 3: If $v \in U_2 \backslash U_1$, then $f_{1,2}$ and $f_{1,3}$ sends elements in $\mathbb{N}$ in the v component to $\mathcal{O}_v$ via the map r, but there are no $\mathbb{N}$ components labelled with v in $M_{U_2}^W$ so $f_{2,3}$ is on the v components is the identity map on $\mathrm{id}_v : \mathcal{O}_v \to \mathcal{O}_v$. Clearly, $r = \mathrm{id}_v \circ r$.

□

Lemma 1.4.16. *Let $P, Q \subseteq \Pi$. The following diagram is a pushout diagram,*

$$\begin{tikzcd}
M_{P\cap Q}^{P\cup Q} \arrow[r] \arrow[d] & M_P^{P\cup Q} \arrow[d] \\
M_Q^{P\cup Q} \arrow[r] & M_{P\cup Q}^{P\cup Q}
\end{tikzcd}$$

where the maps are defined in 1.4.14.

Proof. Recall each monoid above is a quotient of a product of monoids, we first show this is a pushout diagram before passing to the quotient of $\mathbb{F}_q^*$. We show this by checking the universal property of pushout for each component.

For each $\mathcal{O}_\mathfrak{p}^*$ or $\mathcal{O}_\mathfrak{p}^\circ$ term, the diagram above is just a square induced by the identity map $\mathcal{O}_\mathfrak{p} \to \mathcal{O}_\mathfrak{p}$ (which induces a map $\mathcal{O}_\mathfrak{p}^* \hookrightarrow \mathcal{O}_\mathfrak{p}^\circ$ on some components), so it is enough to show that, for $\mathfrak{p} \in P \setminus Q$, the following diagram

$$\begin{tikzcd}
\mathbb{N} \arrow[r] \arrow[d] & \mathbb{N} \arrow[d] \\
\mathcal{O}_\mathfrak{p}^\circ \arrow[r] & \mathcal{O}_\mathfrak{p}^\circ
\end{tikzcd}$$

where the top and bottom maps are the identity map, and the vertical maps are the map shown in the second part of 1.4.14. However, it is clear that the square above is a pushout diagram. □

1.4.17. Let $U \subseteq V \subseteq W \subseteq \Pi$. Suppose theorem 1.4.6 holds for U-$\Lambda(W)$-models and for V-$\Lambda(W)$-models, so there exists maps $M_U^W \to \mathrm{Map}(\bar{S}, \bar{S})$ and

$M_V^W \to \mathrm{Map}(\bar{S}, \bar{S})$ such that the map $\mathrm{Div}_W^+(K) \times G_K \to \mathrm{Map}(\bar{S}, \bar{S})$ factors through such maps.

Let S be a finite étale scheme over K with a U-$\Lambda(W)$-model, hence also a V-$\Lambda(W)$-model by 1.4.12, we have the following diagram.

$$\begin{tikzcd}
\mathrm{Div}_W^+(K) \times G_K \arrow[rr, "a"] \arrow[dr, "b"] \arrow[ddr] & & M_U^W \arrow[dl, "c"'] \arrow[ddl] \\
& M_V^W \arrow[d] & \\
& \mathrm{Map}(\bar{S}, \bar{S}) &
\end{tikzcd}$$

where the maps a, b are defined in 1.4.4, the map c is defined in 1.4.14, and the rest of the maps are the same maps as in theorem 1.4.6.

Proposition 1.4.18. *The above diagram in 1.4.17 commutes*

Proof. We see that the bottom left triangle and the big, outer triangle commutes since we assumed theorem 1.4.6 to hold for our subsets of Π. Thus, it is enough to show the commutativity of the top triangle involving the maps a, b, c.

Once again, we do this by picking a representative of elements in M_U^W and M_V^W before taking the quotient of $\mathbb{F}_q^*$. Since for all $\mathfrak{p} \in U$, the map c is induced by the identity map $\mathcal{O}_\mathfrak{p}^\circ \to \mathcal{O}_\mathfrak{p}^\circ$. For $\mathfrak{p} = \infty$, the map c is induced by the identity maps $\mathcal{O}_\infty^* \to \mathcal{O}_\infty^*$ and $\mathbb{N} \to \mathbb{N}$.

Finally, for the components in the 0-position, the maps a and b are both sending G_K to its abelianization via the isomorphism given in 1.1.7. Furthermore, the $\mathbb{N}$ component in the 0-th position in Div_K^+ and in M_U^Π are both sent to $t \in \mathcal{O}_0^\circ$ in M_V^Π. Therefore, by passing to the quotient of $\mathbb{F}_q^*$, we have shown that the top triangle commutes, hence the whole diagram commutes. □

Proposition 1.4.19. *Let U, V be subsets of $W \subseteq \Pi$ such that $U \cup V = W$ (and U, V contains all but finitely many primes). Suppose theorem 1.4.6 holds for the cases where $P = Q = U$ and $P = Q = V$ and $P = Q = (U \cap V)$, then theorem 1.4.6 holds for $P = Q = W$.*

Proof. Suppose X is a W-$\Lambda(W)$-model of S. This is equivalent to saying that $X_U = X \times_{\mathrm{Spec}(\mathbb{F}_q[t])} \mathrm{Spec}(R_U)$ is a U-$\Lambda(W)$-model and X_V, similarly defined, is a

V-$\Lambda(W)$-model, and their pullbacks to $\mathrm{Spec}(R_{U\cap V})$ agrees as a $(V \cap U)$-$\Lambda(W)$-model of S.

However, by assumption, the existence of a V-$\Lambda(V)$-model is equivalent to the existence of a map $M_V^V \to \mathrm{Map}(\bar{S}, \bar{S})$ such that the map $\mathrm{Div}^+(K) \times G_K \to \mathrm{Map}(\bar{S}, \bar{S})$ factors through it. Furthermore, applying 1.4.13 shows the existence of a V-$\Lambda(W)$-model is equivalent to the existence of a map $M_V^W \to \mathrm{Map}(\bar{S}, \bar{S})$ such that the map $\mathrm{Div}^+(K) \times G_K \to \mathrm{Map}(\bar{S}, \bar{S})$ factors through it.

Similarly, the existence of a U-$\Lambda(U)$-model and $U \cap V$-$\Lambda(U \cap V)$-model is equivalent to the existence of a map $M_U^U \to \mathrm{Map}(\bar{S}, \bar{S})$ and a map $M_{U\cap V}^{U\cap V} \to \mathrm{Map}(\bar{S}, \bar{S})$ such that the map $\mathrm{Div}^+(K) \times G_K \to \mathrm{Map}(\bar{S}, \bar{S})$ factors through them. Applying 1.4.13 again gives the equivalence of a existence of a U-$\Lambda(W)$-model, respectively a $U \cap V$-$\Lambda(W)$-model, to a map $M_U^W \to \mathrm{Map}(\bar{S}, \bar{S})$, respectively a map $M_{U\cap V}^W \to \mathrm{Map}(\bar{S}, \bar{S})$, such that the map $\mathrm{Div}^+(K) \times G_K \to \mathrm{Map}(\bar{S}, \bar{S})$ factors through it.

To summarize, S has a W-$\Lambda(W)$-model if and only if there exists maps from the monoids to $\mathrm{Map}(\bar{S}, \bar{S})$ in the following diagram, where the other maps are defined in 1.4.4 and 1.4.14.

$$\begin{tikzcd}
\mathrm{Div}^+(K) \times G_K \arrow[dr] \arrow[drr] \arrow[ddr] & & \\
& M_{U\cap V}^W \arrow[r] \arrow[d] \arrow[dr] & M_V^W \arrow[d] \\
& M_U^W \arrow[r] & \mathrm{Map}(S, S)
\end{tikzcd}$$

But 1.4.16 shows the bottom right square of the diagram is equivalent to a map $M_W^W \to \mathrm{Map}(\bar{S}, \bar{S})$. Furthermore, this diagram commutes by 1.4.18, so the diagram gives a map $\mathrm{Div}^+(K) \times G_K \to M_W^W$ by following either paths of the following diagram.

$$\begin{tikzcd}
\mathrm{Div}^+(K) \times G_K \arrow[r] & M_{U\cap V}^W \arrow[r] \arrow[d] & M_V^W \arrow[d] \\
& M_U^W \arrow[r] & M_W^W
\end{tikzcd}$$

Finally, we check this map $\mathrm{Div}^+(K) \times G_K \to M_W^W$ agrees with the map we defined in 1.4.4. It is enough to check if the following diagram commutes,

$$\begin{tikzcd}
\mathrm{Div}^+(K) \times G_K \arrow[r] \arrow[dr] & M_{U\cap V}^W \arrow[d] \\
 & M_W^W
\end{tikzcd}$$

where the maps going away from $\mathrm{Div}^+(K) \times G_K$ are the maps defined in 1.4.4 and the vertical map is defined as the composition of maps $M_{U\cap V}^W \to M_V^W \to M_W^W$ (or $M_{U\cap V}^W \to M_U^W \to M_W^W$) defined in 1.4.14, which by 1.4.15 is also the map defined in 1.4.14. However, the diagram above is exactly the triangle with maps labelled a, b, c in 1.4.18, which commutes for all subsets of Π satisfying the containment condition in 1.4.18 and we have $(U \cap V) \subseteq W \subseteq W$. □

1.4.20. *Proof of theorem 1.4.6.* We claim it is enough to consider the case $P = Q$. Indeed, for any set V, U, $V = V \setminus Q \sqcup V \cap Q$, where $\sqcup$ denotes the disjoint union, we see that the theorem holds for $P = V$ and $Q = U$ if and only if the theorem holds for $P = V \cap U$ and $Q = U$ by 1.4.13. Furthermore, by 1.4.12, the theorem holds for $P = V \cap U$ and $Q = U$ if and only if the theorem holds for $P = Q = V \cap U$.

We now show the theorem holds for $P = Q = W$ for all $W \subseteq \Pi$. We use 1.4.19 to show this. In other words, for each W, it is enough to give two sets $U, V \subseteq W$ with $U \cup V = W$ such that the theorem holds for $P = Q = U$, $P = Q = V$ and $P = Q = V \cap U$. Let $V := W \setminus \{\infty\}$ and $U := W \setminus \{0\}$, it is clear that $U, V \subseteq W$ and $U \cup V = W$. Since $(V \cap U) \subseteq V \subseteq \Pi \setminus \{\infty\}$, we see that the theorem holds for $P = Q = V$ and $P = Q = V \cap U$ by 1.4.8. Similarly, $U \subseteq \Pi \setminus \{0\}$, the theorem holds for $P = Q = U$ by 1.4.10. □

Chapter 2

Background

2.1 Definition and Notation

We follow most of the set up and definitions in [13]. Let K be a global function field of over a field of constants $\mathbb{F}_p$, with the largest finite field contained in K being $\mathbb{F}_q \cong \mathbb{F}_{p^m}$. We use the symbol ∞ to denote a place of K which is fixed throughout this chapter.

We use the word "prime" and "place" interchangeably, and for any field extension L/K, we use phrases such as "$\mathfrak{p}$ is a finite prime of L" to mean $\mathfrak{p} \nmid \infty$. We reserve the letter $\mathfrak{p}$ or π to denote a place of K, which usually excludes ∞. We define $K_\mathfrak{p}$ to be the field K completed with respect to the topology defined at $\mathfrak{p}$, i.e., by picking a uniformizer t, $K_\mathfrak{p} \cong \mathbb{F}_{p^{md_\mathfrak{p}}}((t))$ and we refer $\mathbb{F}_{p^{md_\mathfrak{p}}}$ as the residue field of $\mathfrak{p}$ and $d_\mathfrak{p}$ the residue degree. We have a function $v_\mathfrak{p} : K^* \to \mathbb{Z}$ known as the valuation and we almost always assume such a map is normalized, i.e., the map $v_\mathfrak{p}$ is surjective and hence $v_\mathfrak{p}(t) = 1$. We often drop the subscript and just refer to the valuation as v if there is no ambiguity of which place $\mathfrak{p}$ we are working with.

There is a notion of absolute value for a fixed valuation. We define

$$|\cdot|_v : K_\mathfrak{p} \to \mathbb{Q}$$

to be a function such that $|0|_v = 0$ and $|x|_v = q^{-d_\mathfrak{p} v(x)}$ for all $x \neq 0$. We define the ring $\mathcal{O}_\mathfrak{p} \subseteq K_\mathfrak{p}$ as the set of elements $\mathcal{O}_\mathfrak{p} = \{x \in K_\mathfrak{p} \mid |x|_v \leq 1\}$ and the units $\mathcal{O}_\mathfrak{p}^* = \{x \in K_\mathfrak{p} \mid |x|_v = 1\}$. We often denote the absolute value $|\cdot|_{v_\mathfrak{p}}$ as $|\cdot|_\mathfrak{p}$.

We use $\mathcal{C}$ to denote the completion of the algebraic closure of K_∞. The letter $\mathcal{C}$ is chosen since $\mathcal{C}$ is both algebraically closed and complete so one can think of $\mathcal{C}$ as the function field version of the complex numbers.

2.1.1. We define a function $\deg : K \to \mathbb{Z}$ such that $\deg(x) = -d_\infty v_\infty(x)$. We also define $A = A_\infty \subset K$ to be the ring of elements that have no poles away from ∞, so every element $a \in A$ has $\deg(a) \in \mathbb{Z}_{\geq 0}$ and $\deg(a) = \dim_{\mathbb{F}_q}(A/(a))$. For a field L/K, we define A_L to be the ring of integral elements in L over A. Since an extension L/K throughout this chapter is almost always an algebraic extension of K, we identify L as a subfield of $\mathcal{C}$ so $A \subseteq A_L \subseteq A_{\mathcal{C}}$.

2.2 Containment of Abelian Extensions

We give a sufficient condition for an abelian extension to contain another. We follow Chapter VII, section 13 of [17] very closely, with some minor adaptation and changes from number fields to function fields, we include the results here for the convenience of the reader.

Definition 2.2.1. Let S, T be two sets of places of K. We say $S \preceq T$ if $s \in S \Rightarrow s \in T$ $\underline{\text{except for finitely many } s \in S}$. We say $\underline{S \text{ contains almost all primes}}$ if $S \preceq \{\text{All places of } K\}$.

Definition 2.2.2. Let L/K be a finite abelian extension of K. We denote $P(L/K)$ to be the set of totally splitting primes of K in L. We denote $P_{L/K}(\sigma)$, $\sigma \in \mathrm{Gal}(L/K)$, to be the set of unramified primes $\mathfrak{p}$ of K such that σ is the Frobenius element of $\mathfrak{p}$.

Lemma 2.2.3. *Let $K \subseteq L \subseteq N$ be a tower of fields such that L/K and N/K are finite abelian extensions. Let $G = Gal(N/K)$ and $H = Gal(N/L)$, then*

$$P(L/K) = \bigcup_{\sigma \in H} P_{N/K}(\sigma) \tag{2.2.4}$$

Proof. Let $\mathfrak{p}$ be a prime in K. Let $\mathfrak{p}$ be unramified in the extension L/K and let $\sigma_{\mathfrak{p}}^{L/K} \in \mathrm{Gal}(L/K)$ denote the Artin symbol of $\mathfrak{p}$ in L/K. Following the proof in Chapter VII, lemma 13.5 of [17], the result is clear since in this finite abelian setting, since

$$\begin{aligned} \mathfrak{p} \in P(L/K) &\Leftrightarrow \mathfrak{p} \text{ splits completely in } L/K \\ &\Leftrightarrow \sigma_{\mathfrak{p}}^{L/K} = 1 \in \mathrm{Gal}(L/K) = G/H \\ &\Leftrightarrow \sigma_{\mathfrak{p}}^{N/K} \in H \\ &\Leftrightarrow \mathfrak{p} \in \bigcup_{\sigma \in H} P_{N/K}(\sigma) \end{aligned}$$

□

Proposition 2.2.5. *Let L/K, F/K be finite abelian extensions of K. Then*

$$P(F/K) \succeq P(L/K) \Leftrightarrow F \subseteq L \tag{2.2.6}$$

where $P(F/K) \succeq P(L/K)$ is defined in definition 2.2.1.

Proof. The $\Leftarrow$ direction is clear, so let us follow Chapter VII, Proposition 13.9 of [17] to prove the $\Rightarrow$ implication in (2.2.6). Let N/K be an abelian extension over K containing both F and L, Let $G = \mathrm{Gal}(N/K)$, $H = \mathrm{Gal}(N/F)$ and $H' = \mathrm{Gal}(N/L)$. Using equation (2.2.4), we have

$$P(L/K) = \bigcup_{\sigma \in H'} P_{N/K}(\sigma) \preceq P(F/K) = \bigcup_{\sigma \in H} P_{N/K}(\sigma)$$

Let $\sigma' \in H'$ and let $\mathfrak{p} \in P_{N/K}(\sigma')$. Such a prime always exists since the Tchebotarev Density Theorem for function fields (see Theorem 9.13 of [19]) indicated that there are infinitely many primes of K in $P_{N/K}(\sigma')$, so there are still infinitely many choices even if we excluded finitely many. By the containment above, there exists some $\sigma \in H$ such that $\mathfrak{p} \in P_{N/K}(\sigma)$. However, since N/K is an abelian extension, σ' must equal to σ as they are both Frobenius elements of $\mathfrak{p}$ and Frobenius elements are unique. Therefore, $\sigma' \in H$ for all $\sigma' \in H'$ and hence $H' \subseteq H$ and $L \supseteq F$ as required. □

Corollary 2.2.7. *Let L/K, F/K be finite abelian extensions of K. For any π a place of K, let $\sigma_\pi^{L/K}$ and $\sigma_\pi^{F/K}$ denote the Frobenius element at π in the extensions L/K and F/K. Then F is a subfield of L if and only if there exists a group homomorphism*

$$\theta : Gal(L/K) \to Gal(F/K)$$

such that $\theta(\sigma_\pi^{L/K}) = \sigma_\pi^{F/K}$ for all but finitely many places π of K.

Proof. Suppose F is a subfield of L, then it is easy to see that the restriction of the Frobenius element in L is the Frobenius element in any of its subfields, including F. This result can be found in many class field theory textbooks, such as proposition 1.6 in [5]. This shows the forward implication.

For the converse, we have a group homomorphism $\theta : \mathrm{Gal}(L/K) \to \mathrm{Gal}(F/K)$ with Frobenius elements agreeing in both Galois groups for almost all places. Therefore, for almost all primes $\mathfrak{p}$, we have:

$$\begin{aligned} \mathfrak{p} \text{ in } K \text{ splits completely in } L &\Leftrightarrow \sigma_{\mathfrak{p}}^{L/K} = 1 \in \mathrm{Gal}(L/K) \\ &\Rightarrow \theta(\sigma_{\mathfrak{p}}^{L/K}) = 1 \in \mathrm{Gal}(F/K) \\ &\Leftrightarrow \mathfrak{p} \text{ in } K \text{ splits completely in } F \end{aligned}$$

Hence, we see that $P(F/K) \succeq P(L/K)$ and by proposition 2.2.5 we must have $F \subseteq L$. □

2.3 Ray Class Fields

We give three definitions of the ray class group for function fields and we show that each definition is canonically isomorphic to the other definitions. We use these definition interchangeably in the upcoming sections without further elaboration.

We say $\mathfrak{f}$ is a cycle on K if $\mathfrak{f}$ is a (finite) formal product of primes of K and we almost always assume that it is supported away from ∞, i.e. $\mathfrak{f} = \prod_{\mathfrak{p} \neq \infty} \mathfrak{p}^{n_\mathfrak{p}}$, $n_\mathfrak{p} \geq 0$ for all $\mathfrak{p}$. We will write the index $n_\mathfrak{p}$ as $\mathrm{ord}_\mathfrak{p}(\mathfrak{f})$. Since any ideal $0 \neq I \subseteq A$ can be written as a unique product of prime ideals of A, we can view any non-zero ideal I as a cycle by viewing I as a product of prime ideals. Similarly, we can view any cycle supported away from ∞ as a non-zero ideal of A.

For an element $x \in K^*$, we slightly abuse notation and also denote its $\mathfrak{p}$-adic order ($v_\mathfrak{p}(x)$ in the previous section) as $\mathrm{ord}_\mathfrak{p}(x)$, i.e., if we pick a uniformizer $t_\mathfrak{p}$ of $K_\mathfrak{p}$ and view x as an element of $K_\mathfrak{p}$, we can write x as a power series over the residue field of $\mathfrak{p}$ in $t_\mathfrak{p}$, i.e., $x = \sum_{i \leq \mathrm{ord}_\mathfrak{p}(x)} a_i t_\mathfrak{p}^i$ with $a_{\mathrm{ord}_\mathfrak{p}(x)} \neq 0$. Note that if $x \in A$ so that $\mathrm{ord}_\mathfrak{p}(x) \geq 0$ for all $\mathfrak{p} \neq \infty$, then the two definitions agree if we view x as a finite product of primes.

Everything we show here is based on James Borger's unpublished notes on ray class groups, with changes and adaption from number fields to function fields.

Definition 2.3.1. Let $\mathfrak{f}$ be a cycle on K supported away from our chosen place ∞, $\mathfrak{p}$ a place of K. We fix the following notation:

$$\begin{aligned} \mathcal{I}_\mathfrak{f} &= \{\text{the group of fractional ideals of } A \text{ coprime to } \mathfrak{f}, \infty\} \\ K(\mathfrak{f}) &= \{x \in K^* \mid \mathrm{ord}_\mathfrak{p}(x-1) \geq \mathrm{ord}_\mathfrak{p}(\mathfrak{f}) \text{ for all } \mathfrak{p} \text{ dividing } \mathfrak{f}\} \\ \mathcal{P}_\mathfrak{f} &= \{\text{the image of the map } K(\mathfrak{f}) \to \mathcal{I}_\mathfrak{f}\} \end{aligned}$$

The ray class group of conductor $\mathfrak{f}$ totally splitting at ∞, denoted as $\mathrm{Cl}_1(\mathfrak{f})$, is defined to be the quotient $\mathcal{I}_\mathfrak{f}/\mathcal{P}_\mathfrak{f}$.

Definition 2.3.2. Let $\mathfrak{f}$ be a cycle on K supported away from our chosen place

∞, $\mathfrak{p}$ a place of K. Fix the following notation:

$$\begin{aligned}
\mathbf{A}_K^* &= \text{the idele group} \prod_{\mathfrak{p}}{}' K_{\mathfrak{p}}^* = \{x = (\cdots, x_{\mathfrak{p}}, \cdots)\} \\
&\quad \text{where } x_{\mathfrak{p}} \in K_{\mathfrak{p}}^* \text{ and for all but finitely many } \mathfrak{p},\ x_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}^* \\
U_n(K_{\mathfrak{p}}) &= \{x \in \mathcal{O}_K^* \mid \operatorname{ord}_{\mathfrak{p}}(x-1) \geq n\} \text{ where } n \geq 0 \text{ and } \mathfrak{p} \neq \infty \\
U_n(K_\infty) &= K_\infty^* \, \forall n \geq 0 \\
U_{\mathfrak{f}} &= \{x \in \mathbf{A}_K^* \mid x_{\mathfrak{p}} \in U_{\operatorname{ord}_{\mathfrak{p}}(\mathfrak{f})}(K_{\mathfrak{p}}) \text{ for all } \mathfrak{p} \text{ (including } \mathfrak{p} = \infty)\}
\end{aligned}$$

The ray class group of conductor $\mathfrak{f}$ totally splitting at ∞, denoted as $\mathrm{Cl}_2(\mathfrak{f})$, is defined to be the cokernel of map $K^* \to \mathbf{A}_K^*/U_{\mathfrak{f}}$ induced by the diagonal embedding $K^* \to K_{\mathfrak{p}}^*$ for all $\mathfrak{p}$.

Proposition 2.3.3. *Definitions 2.3.1 and 2.3.2 agree. More precisely, let* $v_{\mathfrak{p}} : K_{\mathfrak{p}}^* \to \mathbb{Z}$ *be a surjective valuation for each prime* $\mathfrak{p}$ *(including* $\mathfrak{p} = \infty$*), then the following map*

$$\begin{aligned}
\mathbf{A}_K^* &\to \{\textit{ideals of } A\} \\
(\cdots, x_{\mathfrak{p}}, \cdots) &\mapsto \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(x_{\mathfrak{p}})}
\end{aligned}$$

induces an isomorphism $Cl_2(\mathfrak{f}) \to Cl_1(\mathfrak{f})$ *for all cycles* $\mathfrak{f}$ *on* K *supported away from* ∞.

Proof. We first make the following definition;

$$U_{\mathfrak{f}}' = \{x \in \mathbf{A}_K^* \mid x_{\mathfrak{p}} \in U_{\operatorname{ord}_{\mathfrak{p}}(\mathfrak{f})}(K_{\mathfrak{p}}) \text{ for all } \mathfrak{p} \mid \mathfrak{f}\}$$

We have the following diagram:

$$\begin{tikzcd}
\mathbb{F}_q^* \arrow[r] \arrow[d, "\text{diag}"] & K(\mathfrak{f}) \arrow[r] \arrow[d, "\text{diag}"] & \mathcal{P}_{\mathfrak{f}} \arrow[d] \\
U_{\mathfrak{f}} \arrow[r] \arrow[d] & U_{\mathfrak{f}}' \arrow[r, "\gamma"] \arrow[d, "\theta"] & \mathcal{I}_{\mathfrak{f}} \arrow[d] \\
U_{\mathfrak{f}}/(A^* \cap K(\mathfrak{f})) \arrow[r] & \mathbf{A}_K^*/K^* \arrow[r, dashed, "\phi"] & \mathrm{Cl}_1(\mathfrak{f})
\end{tikzcd}$$

The map θ is the cokernel map $K^* \to \mathbf{A}_K^*/U_{\mathfrak{f}}$ induced by the diagonal embedding $K^* \to K_{\mathfrak{p}}^*$ for all $\mathfrak{p}$ introduced in definition 2.3.2. The map γ is the map defined in the proposition, i.e., it is defined by sending an element $x \in U'_{\mathfrak{f}}$ to the ideal $\prod_{\mathfrak{p} \nmid \mathfrak{f}\infty} \mathfrak{p}^{v_{\mathfrak{p}}(x_{\mathfrak{p}})}$ where $v_{\mathfrak{p}} : K_{\mathfrak{p}}^* \to \mathbb{Z}$ is a surjective valuation. Note that the product above is finite, since there are all but finitely many $x_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}^*$.

We first show the middle column is fully exact and we start by constructing a surjective map θ. We then show the top and middle rows are fully exact and conclude that there is an isomorphism between $\mathrm{Cl}_1(\mathfrak{f})$ and $\mathrm{Cl}_2(\mathfrak{f})$.

The map θ is surjective: We start by taking any $y \in \mathbf{A}_K^*$ and we show that there exists $k \in K^*$ such that $y - k \in U'_{\mathfrak{f}}$. In other words, we will find some $k \in K^*$ such that for all $\mathfrak{p} \mid \mathfrak{f}$, $\mathrm{ord}_{\mathfrak{p}}(y_{\mathfrak{p}} - k - 1) \geq \mathrm{ord}_{\mathfrak{p}}(\mathfrak{f})$, or equivalently, $|y_{\mathfrak{p}} - k - 1|_{\mathfrak{p}} \leq q_{\mathfrak{p}}^{-\mathrm{ord}_{\mathfrak{p}}(\mathfrak{f})}$, where $q_{\mathfrak{p}}$ is the size of the residue field of $K_{\mathfrak{p}}$. However, this is true by the following result.

Theorem 2.3.4. *Let v_0 be a fixed valuation of the global field K. Suppose we are given the following:*

1. *a finite set S of valuations $v \neq v_0$;*
2. *elements $\alpha_v \in K_v$ for all $v \in S$;*
3. *$\epsilon > 0$.*

Then, there exists $\beta \in K$ such that $|\beta - \alpha_v|_v < \epsilon$ for all $v \in S$ and $|\beta|_v \leq 1$ for all $v \notin S$, $v \neq v_0$.

The proof of the statement above can be found in the proof of the strong approximation theorem in the beginning of section 15 of [4]. If we let $v_0 = \infty$, $S = \{\text{primes dividing } \mathfrak{f}\}$, $\alpha_v = y_{\mathfrak{p}}$ and $\epsilon = \min_{\mathfrak{p}|\mathfrak{f}}(q_{\mathfrak{p}}^{-\mathrm{ord}_{\mathfrak{p}}(\mathfrak{f})})$, then the theorem ensures the existence of $k \in K$ such that $|y_{\mathfrak{p}} - k - 1|_{\mathfrak{p}} \leq q_{\mathfrak{p}}^{-\mathrm{ord}_{\mathfrak{p}}(\mathfrak{f})}$ for all $\mathfrak{p} \neq \infty$ and hence the map θ is surjective.

The kernel of θ is $K(\mathfrak{f})$: $y \in \mathrm{Ker}(\theta)$ if and only if there exists $k \in K^*$ such that $\mathrm{diag}(k) = y$ and $y_{\mathfrak{p}} \in U_{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{f})}(K_{\mathfrak{p}})$ for all $\mathfrak{p} \mid \mathfrak{f}$, which is equivalent to $k \in K(\mathfrak{f})$.

Since the middle column and the right column are both exact, the exists a unique map ϕ such that the bottom right hand square of the diagram commutes.

Top row is exact: The map $K(\mathfrak{f}) \to \mathcal{P}_{\mathfrak{f}}$ is surjective by definition. Furthermore, k sends to $1 \in \mathcal{P}_{\mathfrak{f}}$ if and only if k has zeroes and poles only at places dividing $\mathfrak{f}\infty$. However, for all $\mathfrak{p} \mid \mathfrak{f}$ we have $\mathrm{ord}_{\mathfrak{p}}(k - 1) \geq \mathrm{ord}_{\mathfrak{p}}(\mathfrak{f})$, so k has no zeroes or poles at $\mathfrak{p} \mid \mathfrak{f}$ either. Therefore, k must be a global unit and hence in $\mathbb{F}_q^*$

Middle row is exact: The map $U'_{\mathfrak{f}} \to \mathcal{I}_{\mathfrak{f}}$ is clearly surjective, for all $I \in \mathcal{I}_{\mathfrak{f}}$ one takes $\prod'_{\mathfrak{p}} U_{\mathrm{ord}_{\mathfrak{p}}(I)}(K_{\mathfrak{p}}^*)$ to be its pre-image. Let x be in the kernel, then for all

$\mathfrak{p} \nmid \mathfrak{f}$, $x_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}^*$, which is equivalent to $x_{\mathfrak{p}} \in U_{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{f})}(K_{\mathfrak{p}})$ since $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{f}) = 1$. Hence $x \in U_{\mathfrak{f}}$.

Since the middle column, the right column, the top row and the middle row are all fully exact, the bottom row must be fully exact by the snake lemma. Hence, we have the isomorphism

$$\mathrm{Cl}_2(\mathfrak{f}) \cong \mathbf{A}_K^*/U_{\mathfrak{f}}K^* \xrightarrow{\sim} \mathrm{Cl}_1(\mathfrak{f})$$

□

Definition 2.3.5. Let $\mathfrak{f}$ be a cycle on K supported away from ∞. For any A-module M, an $\mathfrak{f}$-trivialization of M is defined to be an isomorphism

$$\iota : A/\mathfrak{f} \otimes_A M \xrightarrow{\quad\sim\quad} A/\mathfrak{f}$$

Trivialized invertible A-modules form a category. A morphism from (M_1, ι_1) to (M_2, ι_2) is defined to be an A-module homomorphism $\theta : M_1 \to M_2$ such that the endomorphism of $A/\mathfrak{f}$ induced by the trivialization is the identity. We make the following notation:

$$\begin{aligned} \mathcal{PIC}_{\mathfrak{f}}(A) &= \text{the category of } \mathfrak{f}\text{-trivialized invertible } A\text{-modules} \\ \mathrm{Pic}_{\mathfrak{f}}(A) &= \text{the set of isomorphism classes of objects of } \mathcal{PIC}_{\mathfrak{f}} \end{aligned}$$

The ray class group of conductor $\mathfrak{f}$ totally splitting at ∞, denoted as $\mathrm{Cl}_3(\mathfrak{f})$, is defined to be $\mathrm{Pic}_{\mathfrak{f}}(A)$.

Proposition 2.3.6. *Definitions 2.3.1 and 2.3.5 agree. More precisely, the map $\beta : \mathcal{I}_{\mathfrak{f}} \to Pic_{\mathfrak{f}}(A)$, defined in the first paragraph of the proof, induces an isomorphism $Cl_1(\mathfrak{f}) \to Cl_3(\mathfrak{f})$ for all cycles $\mathfrak{f}$ on K supported away from ∞.*

Proof. We define a homomorphism $\beta : \mathcal{I}_{\mathfrak{f}} \to \mathrm{Pic}_{\mathfrak{f}}(A)$ as the following. Take a fractional ideal $J \in \mathcal{I}_{\mathfrak{f}}$, since it is supported away from $\mathfrak{f}$, its image in $K_{\mathfrak{p}}$ for all $\mathfrak{p} \mid \mathfrak{f}$ is contained in $\mathcal{O}_{\mathfrak{p}}$. Therefore, there is a canonical $\mathfrak{f}$-trivialization whose $\mathfrak{p}$-part is given by

$$J \to \mathcal{O}_{\mathfrak{p}} \to \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{f})}$$

Note that since we can always find $j \in J$ such that $j \equiv 1 \bmod \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{f})}$, the map $\mathcal{O}_{\mathfrak{p}}/\mathfrak{f} \otimes_A J \to A/\mathfrak{f}$ induced by the map above is an isomorphism. We show that β is surjective and the kernel is $\mathcal{P}_{\mathfrak{f}}$.

Surjectivity: Let M be an $\mathfrak{f}$-trivialized invertible A-module. Since M is an invertible A-module, $K \otimes_A M \cong K$, so we have an isomorphism between M and some fractional ideal J. Let $m \in M$ be an element mapping to $(1, \cdots, 1)$ under the trivialization

$$M \longrightarrow A/\mathfrak{f} \xrightarrow{\sim} \prod_{\mathfrak{p}} A/\mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{f})}$$

Let $x \in J$ be the element corresponding to m via the isomorphism between M and J. We claim that the canonical trivialization of $x^{-1}J$ is isomorphic to M. Indeed, we have the following diagram:

$$\begin{tikzcd}
A/\mathfrak{f} \otimes_A M \arrow[r] \arrow[d] & A/\mathfrak{f} \arrow[d] \\
A/\mathfrak{f} \otimes_A x^{-1}J \arrow[r] & A/\mathfrak{f}
\end{tikzcd}$$

The horizontal maps are all isomorphisms and the left map is also an isomorphism induced by the isomorphism between M and $x^{-1}J$, so the right map has to be an isomorphism. Hence, to show if it is the identity map, we just need to check if it sends 1 to 1 and by chasing the maps in the diagram above, we get

$$\begin{tikzcd}
A/\mathfrak{f} \arrow[r] & A/\mathfrak{f} \otimes_A M \arrow[r] & A/\mathfrak{f} \otimes_A x^{-1}J \arrow[r] & A/\mathfrak{f} \\
1 \arrow[r, mapsto] & m \arrow[r, mapsto] & 1 \arrow[r, mapsto] & 1
\end{tikzcd}$$

Kernel: Suppose $J \in \mathcal{I}_{\mathfrak{f}}$ has canonical trivialization isomorphic to the identity. Then, we must have an isomorphism of A-modules $J \xrightarrow{\sim} A$ and let $x \in J$ be the pre-image of 1 and it is clear that $J = (x)$. Since x is the pre-image of 1, the induced map $A/\mathfrak{f} \otimes_A (x) \to A/\mathfrak{f}$ sends x (or $1 \otimes x$ to be more precise) to 1. Therefore, the map sends $x-1$ (or $1 \otimes x - 1 \otimes 1$) to 0 and hence $\mathrm{ord}_{\mathfrak{p}}(x-1) \geq \mathrm{ord}_{\mathfrak{p}}(\mathfrak{f})$ for all $\mathfrak{p}$ and so $x \in K(\mathfrak{f})$. □

Since the definitions we gave are all canonically isomorphic, via the isomorphisms shown in 2.3.3 and 2.3.6, we often drop the subscript numbering and just denote the ray class group as $\mathrm{Cl}(\mathfrak{f})$.

Proposition 2.3.7. *Let $\mathfrak{f}$ be a cycle on K supported away from ∞. The group Cl($\mathfrak{f}$) is finite and there exists a canonical field L with Galois group Gal(L/K) $\cong$ Cl($\mathfrak{f}$). Furthermore, a prime $\mathfrak{p}$ is ramified in the extension L/K if and only if $\mathfrak{p} \mid \mathfrak{f}$. Moreover, ∞ splits completely in L/K.*

Proof. Using the second definition of the ray class group $\mathrm{Cl}_2(\mathfrak{f})$, we first show the existence of the field using part (D) in the main theorems of abelian extensions stated in section 5 of [23], which states that every open subgroup of finite index N in $\mathbf{A}_K^*/K^*$ corresponds to a unique abelian extension L/K such that $\mathrm{Gal}(L/K) \cong (\mathbf{A}_K^*/K^*)/N$. Therefore, to show the first part of the proposition, we just need to show $U_\mathfrak{f}$ is open and is of finite index.

We follow the construction in section 16 of [4] for the topology of $\mathbf{A}_K^*/K^*$. The topology of $\mathbf{A}_K^*/K^*$ is induced by the restricted product topology of the $K_\mathfrak{p}^*$ with respect to the units $\mathcal{O}_\mathfrak{p}^*$. In other words, a set $S' \subseteq \mathbf{A}_K^*/K^*$ is open if a pre-image S in $\mathbf{A}_K^*$ has $\mathfrak{p}$-th component equals to an open subgroup of $K_\mathfrak{p}^*$ for all $\mathfrak{p}$ and for all but finitely $\mathfrak{p}$ the $\mathfrak{p}$-th component equals $\mathcal{O}_\mathfrak{p}^*$. However, recalling that $U_\mathfrak{f} = \{x \in \mathbf{A}_K^* \mid x_\mathfrak{p} \in U_{\mathrm{ord}_\mathfrak{p}(\mathfrak{f})}(K_\mathfrak{p}) \text{ for all } \mathfrak{p}\}$. Therefore, for all $\mathfrak{p} \nmid \mathfrak{f}\infty$, the $\mathfrak{p}$-th component of $U_\mathfrak{f}$ equals to $\mathcal{O}_\mathfrak{p}^*$ and hence $U_\mathfrak{f}$ is an open subgroup of $\mathbf{A}_K^*/K^*$.

We now show the index of $U_\mathfrak{f}$ is finite. A proof of the fact that $Cl_2(1)$ is finite can be found in the last theorem in section 17 of [4]. The proof involves the topology of the ideles and the fact that a compact discrete group is finite, which we are not going to get into detail here.

For a general $\mathfrak{f}$, we first compute the kernel of the map $\mathbb{A}_K^*/U_\mathfrak{f} \to \mathbb{A}_K^*/U_1$ induced by the inclusion map $1 + t_\mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(\mathfrak{f})}\mathcal{O}_\mathfrak{p} \to \mathcal{O}_\mathfrak{p}^*$. One of the group isomorphism theorems says if one has a chain of groups $N_1 \subseteq N_2 \subseteq G$ such that N_1, N_2 are normal subgroups of G, then there exists an isomorphism $(G/N_1)/(N_2/N_1) \cong G/N_2$. Therefore, the kernel of $\mathbb{A}_K^*/U_\mathfrak{f} \to \mathbb{A}_K^*/U_1$ is isomorphic to $U_\mathfrak{f}/U_1 \cong \prod_{\mathfrak{p}|\mathfrak{f}} \mathcal{O}_\mathfrak{p}^*/(1 + t_\mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(\mathfrak{f})}\mathcal{O}_\mathfrak{p})$.

We continue by working at each $\mathfrak{p}$ separately. One observes the cosets of the quotient $\mathcal{O}_\mathfrak{p}^*/(1 + t_\mathfrak{p}^n\mathcal{O}_\mathfrak{p})$ for all n can be represented by an element of the form $a_0 + a_1 t_\mathfrak{p} + \cdots + a_{n-1}t_\mathfrak{p}^{n-1}$ where each a_i is an element of the residue field, with $a_0 \neq 0$. Hence, there is an isomorphism between $\mathcal{O}_\mathfrak{p}^*/(1 + t_\mathfrak{p}^n\mathcal{O}_\mathfrak{p})$ and $(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)^*$. Furthermore, the natural inclusion $A \hookrightarrow \mathcal{O}_\mathfrak{p}$ induces an isomorphism $A/\mathfrak{p}^n \cong \mathcal{O}_\mathfrak{p}/\mathfrak{p}^n$. Therefore, the kernel of the map $\mathbb{A}_K^*/U_\mathfrak{f} \to \mathbb{A}_K^*/U_1$ is isomorphic to $\prod_{\mathfrak{p}|\mathfrak{f}} \mathcal{O}_\mathfrak{p}^*/(1 + t_\mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(\mathfrak{f})}\mathcal{O}_\mathfrak{p}) \cong \prod_{\mathfrak{p}|\mathfrak{f}}(A/\mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(\mathfrak{f})})^* \cong (A/\mathfrak{f})^*$.

We have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{F}_q^* \arrow[r] \arrow[d] & \mathrm{div}(K^*) \arrow[r] \arrow[d] & \mathrm{div}(K^*) \arrow[d] \\
(A/\mathfrak{f})^* \arrow[r] \arrow[d] & \mathbb{A}_K^*/U_{\mathfrak{f}} \arrow[r] \arrow[d] & \mathbb{A}_K^*/U_1 \arrow[d] \\
(A/\mathfrak{f})^*/\mathbb{F}_q^* \arrow[r] & \mathrm{Cl}_2(\mathfrak{f}) \arrow[r] & \mathrm{Cl}_2(1)
\end{tikzcd}$$

We have already established that the middle row is exact. For the top row, we used $\mathrm{div}(K^*)$ to denote the image of K^* in $\mathbb{A}_K^*$, i.e., $\mathrm{div}(k) = \prod_{\mathfrak{p}} t_{\mathfrak{p}}^{\mathrm{ord}_{\mathfrak{p}}(k)}$ for all $k \in K^*$. The kernel of $\mathrm{div}(K^*) \to \mathrm{div}(K^*)$ must be the elements in K^* that have no zeroes (or poles) anywhere, hence the kernel must be contained in $\mathbb{F}_q^*$. It is clear that, by definition, the middle and the right columns are exact, hence the bottom row must also be exact. Therefore, $\mathrm{Cl}(\mathfrak{f})$ is an extension of a finite group by another finite group, hence finite.

For the second part of the proposition, note that for all $\mathfrak{p}$ prime in K, we have the following commutative diagram:

$$\begin{tikzcd}
K_{\mathfrak{p}}^* \arrow[r] \arrow[d] & \mathrm{Gal}(K_{\mathfrak{p}}^{\mathrm{ab}}/K_{\mathfrak{p}}) \arrow[d] \\
\mathbf{A}_K^*/K^* \arrow[r] \arrow[d] & \mathrm{Gal}(K^{\mathrm{ab}}/K) \arrow[d] \\
\mathbf{A}_K^*/K^*U_{\mathfrak{f}} \arrow[r] & \mathrm{Gal}(L/K)
\end{tikzcd}$$

where the top map $K_{\mathfrak{p}}^* \to \mathrm{Gal}(K_{\mathfrak{p}}^{\mathrm{ab}}/K_{\mathfrak{p}})$ which we denote as θ, which is the same map θ introduced in section 2.2 in [20] known as the reciprocity map. The middle map $\mathbf{A}_K^*/K^* \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is the global Artin map. The definition of the global Artin map, and the fact that the top square commute is a consequence of global class field theory and the map $\mathbf{A}_K^* \to G$ (where G is a complete commutative topological group) is unique if the map satisfies some properties. For more detail, see the section titled "*Relation Between Global and Local Artin Maps*" in section 6 of [23]. Finally, the bottom map is an isomorphism by the definition of L.

Let I_K denote the inertia subgroup of $\mathrm{Gal}(K_{\mathfrak{p}}^{\mathrm{ab}}/K_{\mathfrak{p}})$. By theorem 3a in section 2 of [20], the map θ gives an isomorphism between $\mathcal{O}_{\mathfrak{p}}^*$ and I_K, hence a prime $\mathfrak{p}$ is unramified if and only if the image of $\mathcal{O}_{\mathfrak{p}}^*$ on the left column is trivial. But this happens if and only if $U_{\mathfrak{f}}$ contains $\mathcal{O}_{\mathfrak{p}}^*$, which happens if and only if $\mathfrak{p} \nmid \mathfrak{f}$.

Finally, for the place ∞, note that every element in $K_{\mathfrak{p}}^*$ maps to the trivial element in $\mathrm{Gal}(L/K)$. Therefore, the Frobenius element at ∞ is trivial and hence ∞ splits completely in L/K. □

Definition 2.3.8. We define the ray class field of conductor $\mathfrak{f}$ totally splitting at ∞ to be the field corresponding to the ray class group of conductor $\mathfrak{f}$ and we denote such a field by $K_{\mathfrak{f}}$. We almost always assume our fields to be totally splitting at ∞, so we often drop the "totally splitting at ∞" at the end of our description and call $K_{\mathfrak{f}}$ the ray class field of conductor $\mathfrak{f}$.

In the case where $\mathfrak{f} = 1$, we usually refer to K_1 as H or H_∞ and we call this the Hilbert class field totally splits at ∞ or just the Hilbert class field.

Corollary 2.3.9. *$\mathrm{Gal}(K_{\mathfrak{f}}/H) \cong (A/\mathfrak{f})^*/\mathbb{F}_q^*$*

Proof. By Galois theory, $\mathrm{Gal}(K_{\mathfrak{f}}/H)$ is the kernel of the map $\mathrm{Gal}(K_{\mathfrak{f}}/K) \to \mathrm{Gal}(H/K)$, but this is clear in the first diagram of the proof of proposition 2.3.7. □

Proposition 2.3.10. *The Frobenius lift of a prime ideal co-prime to $\mathfrak{f}$ in $\mathrm{Gal}(K_{\mathfrak{f}}/K)$ is the image in $\mathrm{Cl}_1(\mathfrak{f})$ under the canonical map*

$$\{\textit{Ideals of } A \textit{ prime to } \mathfrak{f}\} \longrightarrow \mathrm{Cl}_1(\mathfrak{f}) \cong \mathrm{Gal}(K_{\mathfrak{f}}/K)$$

Proof. Let L denote the field $K_{\mathfrak{f}}$, $\mathfrak{p}$ be a prime ideal co-prime to $\mathfrak{f}$ and let $L_{\mathfrak{p}}$ be the local field containing L obtained by localizing at some prime above $\mathfrak{p}$. In order to show that $\sigma \in \mathrm{Gal}(L/K)$ is the Frobenius element at the prime $\mathfrak{p}$, we start with the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Cl}_1(\mathfrak{f}) \arrow[d] & \mathrm{Cl}_2(\mathfrak{f}) \arrow[l] \\
\{\text{Ideals prime to } \mathfrak{f}\} \arrow[ur] \arrow[r] & \mathrm{Gal}(L/K) & \mathbf{A}_K^* \arrow[l] \arrow[u] \arrow[dl] \\
 & \prod_{\mathfrak{p}} \mathrm{Gal}(L_{\mathfrak{p}}/K_{\mathfrak{p}}) \arrow[u] &
\end{tikzcd}$$

where the maps involving $\mathrm{Cl}_1(\mathfrak{f})$ and $\mathrm{Cl}_2(\mathfrak{f})$ are defined in their definitions 2.3.1 and 2.3.2. The map $\mathbf{A}_K^* \to \mathrm{Gal}(L/K)$ is the global reciprocity map shown in the main theorems of abelian extensions stated in section 5 of [23]. The map $\mathbf{A}_K^* \to \prod_{\mathfrak{p}} \mathrm{Gal}(L_{\mathfrak{p}}/K_{\mathfrak{p}})$ is the map induced by $K_{\mathfrak{p}}^* \to \mathrm{Gal}(L_{\mathfrak{p}}/K_{\mathfrak{p}})$ which was denoted as θ and it is map introduced in section 2.2 in [20] known as the reciprocity map. The map $\prod_{\mathfrak{p}} \mathrm{Gal}(L_{\mathfrak{p}}/K_{\mathfrak{p}}) \to \mathrm{Gal}(L/K)$ is defined by taking the product of the image of $\mathrm{Gal}(L_{\mathfrak{p}}/K_{\mathfrak{p}}) \to \mathrm{Gal}(L/K)$ for each $\mathfrak{p}$. Note that this is a finite product because almost all entries are in $\mathcal{O}_{\mathfrak{p}}^*$ and by theorem 3a in section 2 of [20], any $\mathfrak{p}$ prime to $\mathfrak{f}$ must send $\mathcal{O}_{\mathfrak{p}}^*$ to 1. The fact that bottom triangle of the diagram commute is once again a consequence of global class field theory and the uniqueness of the map $\mathbf{A}_K^* \to G$ (where G is a complete commutative topological group) if some conditions are satisfied. We once again reference "*Relation Between Global and Local Artin Maps*" in section 6 of [23].

Let $[\mathfrak{p}]$ denote the image of $\mathfrak{p}$ in $\mathrm{Cl}_1(\mathfrak{f})$, we pick a lift of $[\mathfrak{p}]$ in $\mathbf{A}_K^*$, for example the element $(\cdots, 1, 1, t, 1, 1, \cdots)$ where the only non-1 entry is at the $\mathfrak{p}$-th position and t is a uniformizer of $K_{\mathfrak{p}}$. The map $\mathrm{Cl}_2(\mathfrak{f}) \to \mathrm{Cl}_1(\mathfrak{f})$ is induced by $U_{\mathfrak{f}}' \to \mathcal{I}_{\mathfrak{f}}$ by the diagram in the proof of 2.3.3 so the image of $(\cdots, 1, 1, t, 1, 1, \cdots)$ in $\mathrm{Cl}_1(\mathfrak{f})$ is $[\mathfrak{p}]$. Furthermore, the image of $(\cdots, 1, 1, t, 1, 1, \cdots)$ under the projection map onto $K_{\mathfrak{p}}^*$ is clearly just t.

To identify the image of t in $\mathrm{Gal}(L_{\mathfrak{p}}/K_{\mathfrak{p}})$, we use the following result, which is listed as proposition 2 of section 2.5 in [20].

Proposition 2.3.11. *Let $L_{\mathfrak{p}}/K_{\mathfrak{p}}$ be an unramified extension (of local fields) of degree n and let $\sigma_{\mathfrak{p}} \in \mathrm{Gal}(L_{\mathfrak{p}}/K_{\mathfrak{p}})$ be the Frobenius element. Let $\alpha \in K_{\mathfrak{p}}^*$ and let $v(\alpha)$ be its normalized valuation (i.e., $v : K^* \to \mathbb{Z}$ is a surjective homomorphism). Then $\theta(\alpha) = \sigma_{\mathfrak{p}}^{v(\alpha)}$.*

Since t is a uniformizer, it is clear that $v(t) = 1$ so $\theta(t) = \sigma_{\mathfrak{p}}$. Finally, we remark that any two choices of lift of $[\mathfrak{p}]$ in $\mathbf{A}_K^*$ differ by an element in $K^* U_{\mathfrak{f}}$, which maps to $1 \in \mathrm{Gal}(L/K)$ by definition of L. Therefore, the image of $[\mathfrak{p}]$ in $\mathrm{Gal}(L/K)$ equals the Frobenius element at $\mathfrak{p}$. $\square$

2.4 Grothendieck Topologies and Sheaves

We follow the notation of sheaves and schemes in section 1.4 of [18] and the notation of Grothendieck topologies in section 2.3 of [24]. Let $\mathcal{D}$ be a category and we denote an object D in $\mathcal{D}$ as $D \in \mathrm{obj}(\mathcal{D})$, or sometimes slightly abusively,

$D \in \mathcal{D}$. We consider functors $\mathcal{F}$ of the form

$$\mathcal{F} : \mathcal{D}^{\mathrm{op}} \to (\mathrm{Sets})$$

where (Sets) denotes the category of sets. We call this type of functor a presheaf, or more precisely, we say $\mathcal{F}$ is a presheaf on $\mathcal{D}$ if $\mathcal{F}$ is a functor from $\mathcal{D}^{\mathrm{op}} \to (\mathrm{Sets})$. A morphism of between presheaves is a morphism of functors, also known as natural transformation.

For every $D \in \mathrm{obj}(\mathcal{D})$, we can define a presheaf on $\mathcal{D}$ via the Yoneda imbedding. More precisely, for every D, we have

$$\begin{aligned} h_D : \mathcal{D}^{\mathrm{op}} &\to (\mathrm{Sets}) \\ C &\mapsto \mathrm{Hom}_{\mathcal{D}}(C, D) \end{aligned}$$

Conversely, a functor $\mathcal{F}$ is called representable or represented by D if there exists some object $D \in \mathrm{obj}(\mathcal{D})$ such that $\mathcal{F}(C) = \mathrm{Hom}_{\mathcal{D}}(C, D)$. We sometimes abuse notation and call a representable functor by the object it is represented by. In other words, we say $D \in \mathrm{obj}(\mathcal{D})$ is a presheaf on $\mathcal{D}$ to mean the presheaf h_D.

2.4.1. Let $\mathcal{F}$ be a presheaf and h_D be a representable presheaf on $\mathcal{D}$. The Yoneda lemma ([22], tag 001P) states that there is a natural bijection between morphisms $h_D \to \mathcal{F}$ and $\mathcal{F}(D)$. This natural isomorphism is induced by sending $d : h_D \to \mathcal{F}$ to $d_D(\mathrm{id}_D) \in \mathcal{F}(D)$, where d_X represents the morphism $d_X : h_D(X) \to \mathcal{F}(X)$ for each $X \in \mathcal{D}$ and id_D denotes the identity map $\mathrm{id}_D : D \to D$. In particular, to specify a morphism between the representable functors $h_D \to h_C$, it is enough to specify an element in $h_C(D) = \mathrm{Hom}_{\mathcal{D}}(D, C)$.

From now on, we always assume that fiber products exist in our category. Let $C, D_1, D_2 \in \mathrm{obj}(\mathcal{D})$, we denote the two project maps as pr_1^* and pr_2^*. I.e., we have

$$\mathrm{pr}_1^* : D_1 \times_C D_2 \to D_1$$

and similarly for pr_2^*.

Definition 2.4.2. ([24], 2.24) Let $\mathcal{D}$ be a category. A Grothendieck topology on $\mathcal{D}$ is, for each object $D \in \mathrm{obj}(\mathcal{D})$, a collection of sets of arrows $\{D_i \to D\}$, where i ranges over some indexing set. Such a set is called a covering of D. The collection of coverings satisfies the following properties.

1. Let $C \to D$ is an isomorphism, then $\{C \to D\}$ is a covering of D.

2. If $\{D_i \to D\}$ is a covering and $C \to D$ is an arrow in $\mathcal{D}$. Then $\{C \times_D D_i \to C\}$ is a covering of C.
3. If $\{D_i \to D\}$ is a covering of D and $\{D_{i,j} \to D_i\}$ is a covering of D_i. Then $\{D_{i,j} \to D\}$, where the morphisms are defined by the composition $\{D_{i,j} \to D_i \to D_i\}$, is a covering of D.

A category with a Grothendieck topology is called a site, in other words, we say $\mathcal{D}$ is a site to mean $\mathcal{D}$ is a category with some Grothendieck topology on it.

Definition 2.4.3. ([18], 2.2.2) Let $\mathcal{D}$ be a site and let $\mathcal{F} : \mathcal{D}^{\mathrm{op}} \to (\mathrm{Sets})$ be a presheaf on $\mathcal{D}$.

1. We say $\mathcal{F}$ is separated if for all $D \in \mathcal{D}$ and all covering $\{D_i \to D\}$, the map
$$\mathcal{F}(D) \to \prod_i \mathcal{F}(D_i)$$
is injective.
2. We say $\mathcal{F}$ is a sheaf if for all $D \in \mathcal{D}$ and all covering $\{D_i \to D\}$, the sequence
$$\mathcal{F}(D) \to \prod_i \mathcal{F}(D_i) \rightrightarrows \prod_{i,j} \mathcal{F}(D_i \times_D D_j)$$
is exact, where the two maps on the right are induced by the two projections $\mathrm{pr}_1 : D_i \times_D D_j \to D_i$ and $\mathrm{pr}_2 : D_i \times_D D_j \to D_j$.

Proposition 2.4.4. *([18], theorem 2.2.4) Let $\mathcal{D}$ be a site. The inclusion*
$$(\textit{sheaves on } \mathcal{D}) \hookrightarrow (\textit{presheaves on } \mathcal{D})$$
has a left adjoint $\mathcal{F} \mapsto \mathcal{F}^a$.

Proof. We give the construction of the left adjoint and we omit some of the justification. For a complete proof, one can follow the proof given by Olsson in the source cited above.

We show both of the inclusions
$$(\text{separated presheaves on } \mathcal{D}) \hookrightarrow (\text{presheaves on } \mathcal{D}) \tag{2.4.5}$$
$$(\text{sheaves on } \mathcal{D}) \hookrightarrow (\text{separated presheaves on } \mathcal{D}) \tag{2.4.6}$$
have left adjoint, then the composition of them gives what we want.

We first construct the left adjoint in (2.4.5). Let $\mathcal{F}$ be a presheaf on $\mathcal{D}$. For $D \in \mathcal{D}$, we define an equivalence relation on $\mathcal{F}(D)$ where $a \sim b$, $a, b \in \mathcal{F}(D)$, if there exists a cover $\{D_i \to D\}$ such that the images of a and b agree under the map $\mathcal{F} \to \prod_i \mathcal{F}(D_i)$. Let $\mathcal{F}^s$ be defined with D-points $\mathcal{F}^s(D) = \mathcal{F}(D)/\sim$. One checks that $\mathcal{F}^s$ is a presheaf on $\mathcal{D}$ and that if $\mathcal{G}$ is a separated presheaf on $\mathcal{D}$, any map $\mathcal{F} \to \mathcal{G}$ factors (necessarily uniquely) through $\mathcal{F}^s$. Thus $\mathcal{F}^s$ is the left adjoint of the map in (2.4.5).

Now we construct the left adjoint in (2.4.6). Let $\mathcal{F}$ be a separated presheaf on $\mathcal{D}$ and let $U \in \mathcal{D}$, we first define a functor $\mathcal{F}'$ which associates U to the set of pairs

$$(\{U_i \to U\}_{i \in \mathcal{I}}, \{a_i\})$$

where $\{U_i \to U\}$, i ranges through an indexing set $\mathcal{I}$, is a covering of U in the site $\mathcal{D}$ and an element

$$\{a_i\} \in \mathrm{Eq}(\prod_{i \in \mathcal{I}} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in \mathcal{I}} \mathcal{F}(U_i \times_U U_j))$$

where $\mathrm{Eq}(-)$ denotes the equalizer of the maps. We define an equivalence relation on $\mathcal{F}'(U)$ by defining two pairs

$$(\{U_i \to U\}_{i \in \mathcal{I}}, \{a_i\}) \sim (\{V_j \to U\}_{j \in \mathcal{J}}, \{b_j\})$$

are equivalent if for all $i \in \mathcal{I}, j \in \mathcal{J}$, a_i and b_j have the same image in $\mathcal{F}'(U_i \times_U V_j)$. The U-point of the functor $\mathcal{F}^a$ is the equivalence classes of $\mathcal{F}'$, i.e.,

$$\mathcal{F}^a(U) = \mathcal{F}'(U)/\sim$$

One checks $\mathcal{F}^a$ is a sheaf on $\mathcal{D}$ and any map $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf factors (necessarily uniquely) through the map $\mathcal{F} \to \mathcal{F}^a$ defined by $a \to (\{\mathrm{id} : U \to U\}, \{a\})$. Hence $\mathcal{F}^a$ is the left adjoint of (2.4.6). □

We further define what we mean by a scheme in this functorial approach. Almost all of the definitions related to schemes for the rest of this section can be found in section 1.4 of [18], unless stated otherwise.

Definition 2.4.7. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on $\mathcal{D}$. We say that f is relatively representable if for every $D \in \mathcal{D}$ and every $g : h_D \to \mathcal{G}$, the presheaf $h_D \times_{\mathcal{G}} \mathcal{F}$ is representable.

We remark that the presheaf $h_D \times_{\mathcal{G}} \mathcal{F}$ is defined by $C \mapsto h_D(C) \times_{\mathcal{G}(C)} \mathcal{F}(C)$ where the fiber product is taken in (Sets).

From now on, $\mathcal{D}^{\mathrm{op}}$ will be the category of algebras over a ring R, which we denote as (Alg_R). An affine schemes S is a representable functor, so it is represented by some R-algebra, say T, and the affine scheme $S = h_T$ is denoted as $\mathrm{Spec}(T)$.

Definition 2.4.8. A morphism of functors $f : \mathcal{F} \to \mathcal{G}$ is an affine open (or closed) imbedding if

1. f is relatively representable, in the sense of definition 2.4.7;
2. For all $X \in \mathcal{D}$ and $g : h_D \to \mathcal{G}$, the presheaf $h_D \times_{\mathcal{G}} \mathcal{F} \to h_D$ is an open (or closed) imbedding.

As a remark, the first condition says that the presheaf $h_D \times_{\mathcal{G}} \mathcal{F}$ is representable, so the second condition $h_D \times_{\mathcal{G}} \mathcal{F} \to h_D$ is morphism of affine schemes, which is equivalent to a morphism of rings by 2.4.1, so it makes sense to say if the morphism is an open (or closed) imbedding.

We say that a presheaf $\mathcal{F}$ over $\mathcal{D}$ is a big Zariski sheaf if $\mathcal{F}$ is a sheaf (see 2.4.3) in the global Zariski topology. In other words, a covering $\{U_i \to U\}$ in $\mathcal{D}$ is a collection of open affine subschemes $U_i \hookrightarrow U$ that covers the whole topological space U.

Definition 2.4.9. A morphism sheaves $g : \mathcal{F} \to \mathcal{G}$ over $\mathcal{D}$ is called surjective if for any $U \in \mathcal{D}$ and $u \in \mathcal{G}(U)$ there exists a Zariski covering $\{U_i \to U\}$ such that $u_i \in \mathcal{G}(U_i)$ is in the image of $\mathcal{F}(U_i)$ for every i.

Definition 2.4.10. A functor $\mathcal{F} : (\mathrm{Alg}_R) \to (\mathrm{Sets})$ is called a separated R-scheme (also called a separated S-scheme if we let $S = \mathrm{Spec}(R)$) if the following holds

1. $\mathcal{F}$ is a big Zariski sheaf;
2. The diagonal morphism $\Delta : \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is an affine closed imbedding, in the sense of 2.4.8;
3. There exists a $X_i \in (\mathrm{Alg}_R)^{\mathrm{op}}$ and morphisms $\nu_i : h_{X_i} \to \mathcal{F}$ such that each ν_i is an affine open imbedding and the map of big Zariski sheaves $\coprod_i h_{X_i} \to \mathcal{F}$ is surjective, in the sense of 2.4.9.

Remark 2.4.11. An R-scheme (or S-scheme if $S = \mathrm{Spec}(R)$) can be defined similarly if we replace condition 2 in definition 2.4.10 with the following:

Let $\mathcal{F}^2 = \mathcal{F} \times \mathcal{F}$, then $\Delta : \mathcal{F} \to \mathcal{F}^2$ has the following property: for any $U \in (\mathrm{Alg}_R)^{\mathrm{op}}$ and $h_U \to \mathcal{F}^2$, the fiber product $\mathcal{F} \times_{\mathcal{F}^2} h_U$ is a separated U-scheme.

Remark 2.4.12. In proposition 1.4.11 in [18], Olsson proves that there is an equivalence of categories between separated R-schemes, defined using the traditional point-set topology method (such as the method shown in chapter 2, section 4 of [11]), and functors that satisfy criteria 1-3 in definition 2.4.10.

Definition 2.4.13. ([24], section 1.2) Let $\mathcal{D}$ be a category and let $X \in \mathcal{D}$. We define the comma category of $\mathcal{D}$ over X, denoted as $(\mathcal{D}/X)$, to be the category defined as the following. An object in $\mathcal{D}/X$ is of the form (U, u) where $U \in \mathcal{D}$ and $u : U \to X$ is a morphism in $\mathcal{D}$. A morphism $(U, u) \to (U', u')$ in $\mathcal{D}/X$ is a morphism $f : U \to U'$ in $\mathcal{D}$ such that $u = u' \circ f$. In other words, a morphism in $\mathcal{D}/X$ is a morphism f such that the following diagram commutes.

$$\begin{tikzcd}
U \arrow[rr, "f"] \arrow[dr, "u"'] & & U' \arrow[dl, "u'"] \\
& X &
\end{tikzcd}$$

Schemes form a category, denoted as (Sch) (where morphisms are morphisms of functors). The category of schemes over a base scheme S, or just S-schemes in short, denoted as $(\mathrm{Sch}_{/S})$ or (S-Sch), is the comma category over (Sch) defined above. We also sometimes denote the category of S-schemes as $(\mathrm{Sch}_{/R})$ or (R-Sch) if the base scheme $S = \mathrm{Spec}(R)$ is an affine scheme.

Definition 2.4.14. ([24], definition 2.57) A site or a topology $\mathcal{T}$ on a category $\mathcal{D}$ is called subcanonical if every representable functor $\mathcal{F} : \mathcal{D}^{\mathrm{op}} \to (\mathrm{Sets})$ is a sheaf with respect to $\mathcal{T}$.

Example 2.4.15. We give some examples of a Grothendieck topology on $(\mathrm{Sch}_{/S})$, without justification. Firstly, the Zariski topology on $(\mathrm{Sch}_{/S})$ is an example, where coverings are jointly surjective (recall the definition of surjectivity in 2.4.9) open subsets of a given scheme. Another example of Grothendieck topology over $(\mathrm{Sch}_{/S})$ are the fppf topology, where coverings are jointly surjective flat morphisms locally of finite presentation.

A morphism of schemes $S' \to S$ is étale if it is formally smooth, formally unramified and locally of finite presentation. Another example of a Grothendieck topology is the finite étale toplogy on $(\mathrm{Sch}_{/S})$, where covering are finite, étale and jointly surjective morphisms.

Remark 2.4.16. Each of the topology on $(\mathrm{Sch}_{/S})$ shown in 2.4.15 are usually defined as the big or global sites. The big topology refers to the site such that objects are any scheme U over S, without requiring any properties on the map $U \to S$.

On the other hand, we can define the small topology, which is the subcategory of $(\mathrm{Sch}_{/S})$ such that objects are S-schemes V such that the map $V \to S$ has the property defined in the topology, e.g., we require $V \to S$ to be finite étale (not necessarily surjective) in the small finite étale site.

We always talk about the big sites in the following chapters, unless stated otherwise.

We end the section by stating that all the sites we mentioned in 2.4.15, i.e., the fppf, étale and Zariski topology on $(\mathrm{Alg}_R)^{\mathrm{op}}$ are all subcanonical sites. In other words, all affine schemes are Zariski sheaves, fppf-sheaves and étale-sheaves. A proof of this can be found in theorem 2.55 in [24], which is done by refining all the topologies mentioned above to the fpqc topology, then proving directly that the fpqc topology is subcanonical. Since sheaves are defined using local conditions, this fact is equivalent to all schemes are sheaves with respect to these topologies.

2.5 Fibered Category and Stack

We follow chapter 3 of [24] and give a brief summary on fibered categories and stacks. We start by defining a pseudo-functor on a category $\mathcal{D}$ and a fibered category over $\mathcal{D}$. We define of the category of objects with descent data over $\mathcal{D}$ and finally we define a stack to be a pseudo functor with an equivalence of category with the category of objects with descent data. Almost all the definitions we exhibit in this section can be found in section 3.1 of [24], unless stated otherwise.

Definition 2.5.1. A (contravariant) pseudo-functor, or a (contravariant) lax 2-functor Φ on a category $\mathcal{D}$ is a following data:

1. A category ΦU for each object $U \in \mathrm{obj}(\mathcal{D})$;
2. A functor $f^* : \Phi V \to \Phi U$ for each arrow $U \xrightarrow{f} V$, $U, V \in \mathrm{obj}(\mathcal{D})$;
3. A natural isomorphism
$$\epsilon_U : \mathrm{id}_U^* \xrightarrow{\sim} \mathrm{id}_{\Phi U}$$
between the two functors $\Phi U \to \Phi U$ for each object $U \in \mathrm{obj}(\mathcal{D})$;
4. A natural isomorphism
$$\alpha_{f,g} : f^* g^* \xrightarrow{\sim} (gf)^* : \Phi W \to \Phi U$$

for each pair of morphisms $U \xrightarrow{f} V \xrightarrow{g} W$ in $\mathcal{D}$,

such that the following conditions are satisfied:

(a) For all, $U \xrightarrow{f} V$, $\eta \in \text{obj}(\Phi V)$, we have the following equality of morphisms

$$\alpha_{\text{id}_U, f}(\eta) = \epsilon_U(f^*\eta) : id_U^* f^* \eta \to f^* \eta$$
$$\alpha_{f, \text{id}_V}(\eta) = f^*\epsilon_V(\eta) : f^* id_V^* \eta \to f^* \eta$$

(b) For all triple of morphisms $U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} T$, $\theta \in \text{obj}(\Phi T)$, we have the following commutative diagram

$$\begin{tikzcd}
f^*g^*h^*\theta \arrow[r, "\alpha_{f,g}(h^*\theta)"] \arrow[d, "f^*\alpha_{g,h}(\theta)"] & (gf)^*h^*\theta \arrow[d, "\alpha_{gf,h}(\theta)"] \\
f^*(hg)^*\theta \arrow[r, "\alpha_{f,hg}(\theta)"] & (hgf)^*\theta
\end{tikzcd}$$

A pseudo functor is closely related to a fibered category. A category over a category $\mathcal{D}$ is a pair $(\mathcal{F}, P_{\mathcal{F}})$ where $\mathcal{F}$ is a category together with a functor $P_{\mathcal{F}} : \mathcal{F} \to \mathcal{D}$. A fibered category $\mathcal{F}$ over $\mathcal{D}$ is a category over $\mathcal{D}$ such that we can pull-back objects in $\mathcal{F}$ along any morphisms in $\mathcal{D}$. More precisely, we have the following definitions.

Definition 2.5.2. A morphism $F_1 \xrightarrow{\alpha} F_2$ in $\mathcal{F}$ is Cartesian if for any $F_1' \xrightarrow{\alpha'} F_2$ and any morphsim $P_{\mathcal{F}}F_1' \xrightarrow{b} P_{\mathcal{F}}F_1$ with $P_{\mathcal{F}}\alpha' = P_{\mathcal{F}}\alpha \circ b$, there exists a unique morphism $F_1' \xrightarrow{\beta} F_1$ such that $P_{\mathcal{F}}\beta = b$ and $\alpha\beta = \alpha'$.

For clarity, we express this in diagrams. The morphism $F_1 \xrightarrow{\alpha} F_2$ in $\mathcal{F}$ is Cartesian if given morphsims α' and b such that the following diagram commutes,

$$\begin{tikzcd}
 & & F_1' \arrow[dll, "\alpha'"'] \arrow[dd] \\
F_2 \arrow[dd] & F_1 \arrow[l, "\alpha"] \arrow[dd] & \\
 & & P_{\mathcal{F}}F_1' \arrow[dl, "b"'] \\
P_{\mathcal{F}}F_2 & P_{\mathcal{F}}F_1 \arrow[l] &
\end{tikzcd}$$

then there exists a unique $F_1' \xrightarrow{\beta} F_1$, making the whole diagram below commute

$$\begin{tikzcd}
 & & F_1' \arrow[dll, "\alpha'"'] \arrow[dl, "\beta"] \arrow[dd] \\
F_2 \arrow[dd] & F_1 \arrow[l, "\alpha"] \arrow[dd] & \\
 & & P_{\mathcal{F}}F_1' \arrow[dl, "b"'] \\
P_{\mathcal{F}}F_2 & P_{\mathcal{F}}F_1 \arrow[l] &
\end{tikzcd}$$

Definition 2.5.3. A category $\mathcal{F}$ over $\mathcal{D}$ is called a fibered category over $\mathcal{D}$ if any arrow $D_1 \xrightarrow{d} D_2$ in $\mathcal{D}$ and an object $F_2 \xrightarrow{P_{\mathcal{F}}} D_2$, then there is a Cartesian morphism $F_1 \xrightarrow{f} F_2$ such that $P_{\mathcal{F}}f = d$.

Once again, we express the definition above in commutative diagrams for clarity. Given the following diagram,

$$\begin{tikzcd}
F_2 \arrow[d, "P_{\mathcal{F}}"] & \\
D_2 & D_1 \arrow[l, "d"']
\end{tikzcd}$$

there exists $F_1 \in \text{obj}(\mathcal{F})$ together with maps such that the following diagram commutes

$$\begin{tikzcd}
F_2 \arrow[d, "P_{\mathcal{F}}"] & F_1 \arrow[l, "f"'] \arrow[d, "P_{\mathcal{F}}"] \\
D_2 & D_1 \arrow[l, "d"']
\end{tikzcd}$$

We introduce the definition of a cleavage to explain the relationship between a pseudo-functor and fibered category.

Definition 2.5.4. A cleavage $\mathcal{K}$ of a fibered category $P_{\mathcal{F}} : \mathcal{F} \to \mathcal{D}$ is a collection of arrows in $\mathcal{F}$ such that for each morphism $D_1 \xrightarrow{d} D_2$ in $\mathcal{D}$ and $F_2 \xrightarrow{P_{\mathcal{F}}} D_2$, there exists a unique arrow $f \in \mathcal{K}$ sending F_1 to F_2 and $P_{\mathcal{F}}f = d$.

In other words, a cleavage is a choice of pull-backs along any arrow in $\mathcal{D}$. Assuming the axiom of choice, every fibered category admits a cleavage.

Theorem 2.5.5. *Let $\mathcal{D}$ be a category. A pseudo-functor on $\mathcal{D}$ is equivalent to a fibered category over $\mathcal{D}$ with a choice of cleavage.*

2.5.6. The proof of the theorem above can be found in proposition 3.11 and in section 3.1.3 in [24]. The proof of the theorem above says the following: let $P_{\mathcal{F}} : \mathcal{F} \to \mathcal{D}$ be a map of fibered category over $\mathcal{D}$, then the map

$$\Phi(D) = \{F \in \mathrm{obj}(\mathcal{F}) \mid P_{\mathcal{F}}(F) = D\}$$

defines a pseudo-functor on $\mathcal{D}$, where the morphisms of $\Phi(D)$ are morphisms in $f \in \mathcal{F}$ such that $P_{\mathcal{F}} f = \mathrm{id}_D$. The proof of the theorem says that this is an equivalence. In particular, given a pseudo-functor Φ on $\mathcal{D}$, the following category

$$\mathcal{F} = \{(D, \xi_D \mid D \in \mathrm{obj}(\mathcal{D}), \xi_D \in \Phi(D)\}$$

together with the forgetful map $P_{\mathcal{F}} : \mathcal{F} \to \mathcal{D}$ sending $(D, \Phi(D)) \mapsto D$ defines a fibered category over $\mathcal{D}$ with a choice of cleavage. Moreover, note that since Cartesian arrows are unique up to unique isomorphism, a choice of cleavage is also unique up to unique isomorphism. This means a fibered category defines a unique pseudo-functor up to unique isomorphism.

We proceed by defining objects with descent data for a given pseudo-functor. Technically, objects with descent data are defined over fibered categories and not over pseudo functors, but by the remark in 2.5.6 above, they define each other uniquely up to unique isomorphism. We remain somewhat loose about this and often use both terms interchangeably, i.e., when we say "$\mathcal{F}$ is fibered catergoy", we imply that there is pseudo functor, usually denoted as Φ, associated with $\mathcal{F}$ and vice versa. All definitions shown below can be found in section 4.1.2 and 4.1.3 in [24].

2.5.7. Fix a site $\mathcal{D}$ and Φ a pseudo functor on $\mathcal{D}$. Let $\mathcal{U} = \{U_i \to U\}$ be a cover of U in $\mathcal{D}$. We define the map pr_1 between covers to be the canonical map induced by the projection to the first factor, for example

$$\mathrm{pr}_1 : U_i \times_U U_j \to U_i$$

and likewise the map $\mathrm{pr}_{1,2}$ would be the projection from a triple fiber product to the first and second factor, and so on. Since Φ is contravariant, this induces a map

$$\mathrm{pr}_1^* : \Phi(U_i) \to \Phi(U_i \times_U U_j)$$

We use multiple numbers in the subscript when we have a fiber product of more than two objects in $\mathcal{D}$ and we project onto more than one factors. For example, $\mathrm{pr}_{1,3}^*$ would be the map corresponding to the map

$$\mathrm{pr}_{1,3} : U_i \times_U U_j \times_U U_k \to U_i \times U_k$$

Definition 2.5.8. An <u>object with descent data</u> is a collection data $(\{\xi_i\}, \{\phi_{i,j}\})$ where $\xi_i \in \mathrm{obj}(\Phi(U_i))$ and $\phi_{i,j}$ are isomorphisms in the category $\Phi(U_i \times_U U_j)$

$$\phi_{i,j} : \mathrm{pr}_2^*\xi_j \xrightarrow{\sim} \mathrm{pr}_1^*\xi_i$$

such that the following condition is satisfied:

$$\mathrm{pr}_{1,3}^*\phi_{i,k} = \mathrm{pr}_{1,2}^*\phi_{i,j} \circ \mathrm{pr}_{2,3}^*\phi_{j,k} : \mathrm{pr}_3^*\xi_k \to \mathrm{pr}_1^*\xi_i$$

Using the same notation as above, we say two objects $x_i \in \mathrm{obj}(\Phi(U_i))$ and $x_j \in \mathrm{obj}(\Phi(U_j))$ are locally isomorphic if there is a common cover of V over U_i and U_j such that x_i and x_j are isomorphic as objects in $\Phi(V)$. The last condition in the definition above says the following: if we have a local isomorphism between any two of the three objects $\xi_i \in \mathrm{obj}(\Phi(U_i))$, $\xi_j \in \mathrm{obj}(\Phi(U_j))$ and $\xi_k \in \mathrm{obj}(\Phi(U_k))$, then upon passing to a finer cover, all the isomorphisms commute with each other. In other words, the isomorphism between ξ_k to ξ_i is given by the map from ξ_k to ξ_j, then from ξ_j to ξ_i.

Definition 2.5.9. An <u>arrow between objects with descent data</u>

$$\{\alpha_i\} : (\{\xi_i\}, \{\phi_{i,j}\}) \to (\{\eta_i\}, \{\psi_{i,j}\})$$

is a collection of arrows, one from each $\Phi(U_i)$

$$\alpha_i : \xi_i \to \eta_i$$

such that the α's interact with all the pr's, ϕ's and ψ's as expected. More precisely, for all i and j, we have the following commutative diagram,

$$\begin{tikzcd}
\mathrm{pr}_2^*\xi_j \arrow[r, "\mathrm{pr}_2^*\alpha_j"] \arrow[d, "\phi_{i,j}"] & \mathrm{pr}_2^*\eta_j \arrow[d, "\psi_{i,j}"] \\
\mathrm{pr}_1^*\xi_1 \arrow[r, "\mathrm{pr}_1^*\alpha_i"] & \mathrm{pr}_1^*\eta_i
\end{tikzcd}$$

2.5.10. Using the two definitions above, we have defined the category of objects with descent data. We denote this as $\Phi(\{U_i \to U\})$ or as $\Phi(\mathcal{U})$ for a covering $\mathcal{U} = \{U_i \to U\}$. There is a functor $\Phi(U) \to \Phi(\{U_i \to U\})$ induced by $\sigma_i : U_i \to U$, so a morphism between two objects $\xi \xrightarrow{\alpha} \eta$ in $\Phi(U)$ induces a morphism $\sigma_i^*\xi \xrightarrow{\sigma_i^*\alpha} \sigma_i^*\eta$ and one checks that each of these $\sigma_i^*\alpha$'s satisfies the properties of a morphism between objects with descent data.

We give the definition of a stack.

Definition 2.5.11. Fix a site $\mathcal{D}$ and Φ a pseudo functor on $\mathcal{D}$.

1. Φ is a prestack over $\mathcal{D}$ if the functor $\Phi(U) \to \Phi(\{U_i \to U\})$ is fully faithful for all covering $\{U_i \to U\}$.

2. Φ is a stack over $\mathcal{D}$ if the functor $\Phi(U) \to \Phi(\{U_i \to U\})$ is an equivalence of categories for all covering $\{U_i \to U\}$.

Since pseudo functors are close related to fibered categories (2.5.6), we also call a fibered category $\mathcal{F}$ over $\mathcal{D}$ a prestack or a stack if the associated pseudo functor is a prestack or a stack.

Recall that a category is called a groupoid if every morphism in such category is an isomorphism. We say a fibered category $P_{\mathcal{F}} : \mathcal{F} \to \mathcal{D}$ is a fibered category in groupoids if for all $D \in \mathcal{D}$, the category $\Phi(D) = \{F \in \mathrm{obj}(\mathcal{F}) \mid P_{\mathcal{F}}(F) = D\}$ defined in 2.5.6 is a groupoid. We exhibit an alternative definition of a stack if $P_{\mathcal{F}} : \mathcal{F} \to \mathcal{D}$ is a fibered category in groupoids, shown in definition 4.1 in [6]. We show the version found in definition 4.6.1 in [18].

Definition 2.5.12. ([18], 3.4.7) Let $P_{\mathcal{F}} : \mathcal{F} \to \mathcal{D}$ be a fibered category and let Φ be a corresponding pseudo functor (which is unique up to unique isomorphism). Let $X \in \mathcal{D}$ and let $x, x' \in \Phi(X)$, and we recall the definition of the comma category $\mathcal{D}/X$ in definition 2.4.13.

Let $u : U \to X$ be an object in $\mathcal{D}/X$. Since $\mathcal{F}$ is a fibered category over $\mathcal{D}$, there exist an object $f_x \in \Phi(U)$ and a morphism $\alpha_x : f_x \to U$ (the object and the morphism are both unique up to unique isomorphism) such that the following diagram is Cartesian.

$$\begin{tikzcd}
x \arrow[d, "P_{\mathcal{F}}"] & f_x \arrow[l, "\alpha_x"'] \arrow[d, "P_{\mathcal{F}}"] \\
X & U \arrow[l, "u"']
\end{tikzcd}$$

Similarly, we define $\alpha_{x'} : f_{x'} \to U$ for $x' \in \Phi(X)$. We define the following presheaf

$$\begin{aligned}
\underline{\mathrm{Isom}}(x, x') : (\mathcal{D}/X)^{\mathrm{op}} &\to \mathrm{Set} \\
U &\mapsto \mathrm{Isom}_{\Phi(U)}(f_x, f_{x'})
\end{aligned}$$

By the source cited in the definition above, this is a preasheaf on $\mathcal{D}/X$.

Definition 2.5.13. ([18], definition 4.6.1) Let $P_{\mathcal{F}} : \mathcal{F} \to \mathcal{D}$ be a fibered category and let Φ be a corresponding pseudo functor. $\mathcal{F}$ is a stack if and only if the following holds.

1. For all $X \in \mathcal{D}$ and for all $x, x' \in \Phi(X)$, the presheaf $\underline{\mathrm{Isom}}(x, x')$ in definition 2.5.12 is a sheaf;
2. The functor $\Phi(U) \to \Phi(\{U_i \to U\})$ is essentailly surjective.

We skip of the proof of the fact that the definition given in 2.5.11 agrees with the one given in 2.5.13 above. The idea is that the condition 1 in 2.5.13 is equivalent to condition 1 in 2.5.11 in the case of groupoids. Moreover, a fully faithful essentially surjective map between two categories is an equivalence of categories, which shows the two definitions agree with each other.

Example 2.5.14. We give a basic example of a stack. Fix a base affine scheme $T = \mathrm{Spec}(R)$ and for each affine scheme S over T, the collection of all affine schemes over S form a category. Let $\Phi(S)$ be the groupoid of affine schemes over S (recall the groupoid of a category is the subcategory with all morphisms are isomorphisms). We show this is a stack with respect to any subcanonical site on T-schemes such that the coverings maps are faithfully flat, such as fpqc, fppf, étale etc.

We first show Φ is a pseudo-functor on the category of T-schemes. By 2.5.6, this is equivalent to show that for all $S \in T$-Sch, a relatively affine scheme X over S and a cover $\{S_i \to S\}$, there exists a unique relatively affine scheme X_i over S_i for each i such that the following diagram is Cartesian.

$$\begin{tikzcd}
(S, X) & (S_i, X_i) \arrow[l] \arrow[d] \\
S \arrow[from=u] & S_i \arrow[l]
\end{tikzcd}$$

where the vertical maps are the projection maps and the top map is given by maps of schemes $X_i \to X$ and $S_i \to S$. Since all affine schemes are quasi-compact, we may assume the cover $\{S_i\}$ is a finite family and by forming the disjoint union, we may assume the cover contains only one affine scheme. It is clear that there exists an X_i because we can take $X_i = X \times_S S_i$.

We show this fibered category is a stack by checking the conditions in definition 2.5.13. Condition 1 is clear since an isomorphism between affine schemes in given by an invertible element in the corresponding ring, so the subcanonical condition implies the isomorphism presheaf is a sheaf.

Moving on to condition 2, since we assumed our maps are faithfully flat, the essentially surjectivity follows from faithfully flat descent, see theorem 4.2.1 in [24] for more detail.

Chapter 3

Drinfeld Modules

In this chapter, we follow our previous notation (most notably, the definition of the ring A in 2.1.1) and give a brief review of the theory of Drinfeld modules.

In the first section, we give the definition of Drinfeld modules. We then focus on rank 1 Drinfeld modules and show the category of rank 1 Drinfeld modules forms a stack over $\mathrm{Spec}(A)$. Secondly, we define level structures on rank 1 Drinfeld modules. In the third section, we define the action of ideals on the category of rank 1 Drinfeld modules over fields, with or without level structure. Finally, on the last section, we show the functor that inputs A-schemes and outputs rank 1 Drinfeld modules over S with a level-I structure is representable by the ring $A_{K_I}[I^{-1}]$, the A-integral elements in the field K_I (recall from 2.3.8 this is the ray class field of conductor I totally splitting at ∞) with elements in the ideal I inverted.

3.1 Definition

We follow the definitions in the first two chapters of [14] to define a rank d Drinfeld A-module over a scheme S.

Let R be a ring of characteristic p. The endomorphism ring of $\mathbb{G}_{a,R} = \mathrm{Spec}(R[t])$, denoted as $\mathrm{End}(\mathbb{G}_{a,R})$ can be identified with the following noncommutative ring.

Definition 3.1.1. Let R be a ring of characteristic p. The left twisted polynomial ring $R[\tau]$ is the non-commutative ring defined as the following. An element $f(\tau) \in R[\tau]$ is of the form $f(\tau) = \sum_{i\in\mathbb{N}} a_i\tau^i$, $a_i \in R$ and $a_i = 0$ for all $i > n$ for some $n \in \mathbb{N}$. Let $g(\tau) = \sum_{j\in\mathbb{N}} b_j\tau^j$ be another element in $R[\tau]$. The polynomial

$f(\tau) + g(\tau)$ is given by the usual addition of polynomials

$$f(\tau) + g(\tau) = \sum_{i \in \mathbb{N}} (a_i + b_i)\tau^i$$

The multiplication is given by $\tau \cdot u = u^p \cdot \tau$ for all $u \in R$, so the product of monomials $u\tau^i, s\tau^j \in R[\tau]$ is

$$u\tau^i \cdot s\tau^j = u\tau^{i-1}(\tau \cdot s)\tau^j = u\tau^{i-1}(s^p\tau)\tau^j = \cdots = us^{p^j}\tau^{i+j}$$

and we extend this multiplication linearly for elements in $R[\tau]$. In other words, the polynomial $f(\tau) \cdot g(\tau)$ is given by

$$f(\tau) \cdot g(\tau) = \sum_{k \in \mathbb{N}} \sum_{i+j=k} a_i\tau^i(b_j\tau^j) = \sum_{k \in \mathbb{N}} \sum_{i+j=k} a_i b_j^{p^j}\tau^k$$

An element in $R[\tau]$ is called an additive polynomial over R.

We always mean the non-commutative ring above when we use the symbol τ in $R[\tau]$, even though the notation is very similar to the commutative polynomial ring $R[t]$.

Remark 3.1.2. We say a polynomial $f(t) \in R[t]$ is additive if $f(t)$ is in the image of the map that sends $\tau \to t^p$, i.e., $f(t)$ is additive if and only if all non-zero coefficient of t appear in powers divisible by p. Note that for a left twisted polynomial $f(\tau) = \sum_{i \in \mathbb{N}} f_i\tau^i$, the associated (regular) polynomial is $f(t) = \sum_{i \in \mathbb{N}} f_i t^{p^i}$, i.e., it is not obtained by formally substituting τ with t. One sees that checking $f(t)$ is additive is equivalent to checking $f(x+y) = f(x) + f(y)$ for all $x, y \in R$.

For an additive polynomial f, we use the variable t to view f as a regular polynomial with regular polynomial multiplication, and we use the variable τ to view f as a left twisted polynomial with the unusual multiplication defined in 3.1.1. We use both variables for f interchangeably, through the association described above, without further explanation.

We say an additive polynomial $g(\tau) \in R[\tau]$ has a certain property, e.g., monic, separable etc, if and only if the corresponding polynomial $g(x)$ has that same property.

Note that a map between additive group schemes $\mathrm{Spec}(R[t]) \to \mathrm{Spec}(R[s])$ is given by a map $s \mapsto f(t)$ for some additive polynomial $f(\tau) \in R[\tau]$, so we are slightly abusing notation by denoting both the additive polynomial and an element in $\mathrm{End}(\mathbb{G}_{a,R})$ by $f(\tau)$. If $f \in \mathrm{End}(\mathbb{G}_{a,R})$, we use f^* or $f^*(\tau)$ to denote a map between rings when we need to be completely clear. Similarly, for $g(\tau) : R[s] \to R[t]$, we use $\mathrm{Spec}(g)$ to denote the corresponding map of schemes $\mathrm{Spec}(g) : \mathrm{Spec}(R[t]) \to \mathrm{Spec}(R[s])$.

Remark 3.1.3. Let $\rho(\tau) = \sum r_i\tau^i \in R[\tau]$, we view R as an $R[\tau]$-module by defining an action $\rho(\alpha) = \sum r_i\alpha^{p^i}$ for all $\alpha \in R$. Then $\rho(\alpha) = 0$ if and only if α is a root of the polynomial $\sum r_i x^{p^i} \in R[x]$. Therefore, the roots of a twisted polynomial in $R[\tau]$ coincide with the roots of the polynomial in $R[x]$, with the identification $\tau \to x^p$.

Definition 3.1.4. ([14], 1.2.1) Let S be an A-scheme, a Drinfeld A-module of rank d over S is a pair (E, ϕ), where E is a commutative group scheme over S and ϕ is a ring homomorphism $\phi : A \to \mathrm{End}(E)$ such that

1. E is locally (with respect to the Zariski topology on S) isomorphic to $\mathbb{G}_{a,S}$, the additive group scheme over S.

2. For all affine Zariski open subset of $U = \mathrm{Spec}(R)$ of S, isomorphism ψ : $E_U \xrightarrow{\sim} \mathbb{G}_{a,U}$, where E_U denotes the fiber product $E \times_S U$, and for all $a \in A$,

$$\psi \circ \phi(a) \circ \psi^{-1} = \sum_n \alpha_n(a)\tau^n \in R[\tau]$$

   such that the $\alpha_{d\cdot\deg(a)}(a) \in R^\times$ and $\alpha_n(a)$ is nilpotent in R for all $n > d \cdot \deg(a)$. (Recall the definition of deg in 2.1.1)

Definition 3.1.5. Let E be a Drinfeld module over S, then by definition 3.1.4 there exists open affine Zariski open subsets U of S such that $E_U \xrightarrow{\sim} \mathbb{G}_{a,U}$. We call such open subsets "a set of local coordinates" or just "coordinates" on E.

Let E' be another Drinfeld modules over S. An isogeny from (E, ϕ) to (E', ϕ') is locally a non-zero homomorphism of A-module schemes. In other words, there exists the same set of coordinates on E and E' such that if we choose some ψ, ψ' and fix coordinates (E, ϕ, ψ) and (E', ϕ', ψ') over $U = \mathrm{Spec}(R)$, i.e., $\phi, \phi' : A \to R[\tau]$, there exists an isogeny from (E, ϕ, ψ) to (E', ϕ', ψ') if there exists a non-zero left twisted polynomial $u(\tau) \in R[\tau]$ such that, for all $a \in A$, $u \circ \phi(a) = \phi'(a) \circ u$.

Remark 3.1.6. A Drinfeld A-module is called standard if we replace the last part of condition 2 in the above by "$\alpha_n(a) = 0$ for all $n > d \cdot \deg(a)$". By remark 1.2.2 in [14], every Drinfeld A-module is isomorphic, in the sense of isogeny defined in 3.1.5, to a standard Drinfeld A-module of the same rank.

Let (E, ϕ) be a Drinfeld module over S. We call ϕ an "A-module structure on E". We often drop the A-module structure on E and just say"E is a Drinfeld module" to implicitly mean that (E, ϕ) is a Drinfeld module for some A-module structure ϕ on E.

Definition 3.1.7. ([14], 1.2.3) Let R be an A-algebra, we have a map $D_R : R[\tau] \to R$ defined as

$$D_R(\sum_n \alpha_n(a)\tau^n) = \alpha_0(a) \tag{3.1.8}$$

Let (E, ϕ) be a Drinfeld module over S. We have a map

$$D : \mathrm{End}(E) \to H^0(S, O_S)$$

such that, for every affine open subset $U = \mathrm{Spec}(R)$ of S over which there exists a coordinate $\psi : E_U \xrightarrow{\sim} \mathbb{G}_{a,U}$, the map D is given by the map D_R in (3.1.8). The characteristic of a Drinfeld module is the map $D \circ \phi : A \to H^0(S, O_S)$. We will often view the characteristic as a map of schemes, so we say "the characteristic $\theta : S \to \mathrm{Spec}(A)$" to mean the corresponding map of rings induced by θ is the map $D \circ \phi$ as above.

3.1.9. Note that this map θ above is independent of the choice of coordinates on E, since two different choices of coordinates induces different maps $\phi_1, \phi_2 : A \to R[\tau]$, which for all $a \in R$ are related by $r\phi_1(a)r^{-1} = \phi_2(a)$ for some $r \in R[\tau]^*$. One sees that if we let $\phi_1(a) = \sum_n \alpha_n \tau^n$ and $\phi_2(a) = \sum_n \beta_n \tau^n$,

$$\beta_0 = D_R(\phi_2) = D_R(r\phi_1 r^{-1}) = r\alpha_0 r^{-1} = \alpha_0$$

so θ is independent of the choice of coordinates. Furthermore, it is conventional to assume that $\theta : A \to R$ agrees with the A-algebra structure of R, otherwise we will be looking at a different orbit of Drinfeld modules under isogeny (i.e., two Drinfeld modules are in the same orbit if there exists an isogeny taking one to the other).

3.1.10. We will mostly be working with the case where S is an affine scheme, so for $S = \mathrm{Spec}(R)$, the characteristic is the same as an algebra map $\theta : A \to R$. If the kernel of θ is the zero ideal, for example when $R = K$ or $R = \mathcal{C}$, we say such Drinfeld module has generic characteristic.

3.1.11. Let (E, ϕ) be a Drinfeld module over S. We say "we fix coordinates on (E, ϕ)" or "(E, ϕ, ψ) is a coordinatized Drinfeld module over S" to mean we choose an isomorphism $\psi : E_U \xrightarrow{\sim} \mathbb{G}_{a,U}$ for some affine open subset U of S. We will also slightly abuse notation and often refer to the map $\psi \circ \phi \circ \psi^{-1}$ as just ϕ after choosing a coordinate. We will also often just say $\phi : A \to R[\tau]$ to imply that we pick some coordinates on E in some affine open subset $U = \mathrm{Spec}(R)$.

3.1.12. Note that the definition above is independent of the choice of coordinate, i.e., if there exists an isogeny in one choice of coordinates, then there exist isogenies for all other choices of coordinates. Indeed, let (E, ϕ) be a Drinfeld module over an A-algebra R and let $E \to \operatorname{Spec}(R[x])$ and $E \to \operatorname{Spec}(R[y])$ be two coordinates on E, then there exist $r \in R[\tau]^*$ such that the map $R[x] \to R[y]$ defined $x \to r^{-1}y$ coincides with the coordinate maps. So $u \in R[\tau] = \operatorname{End}(R[x])$ becomes $rur^{-1} \in R[\tau] = \operatorname{End}(R[y])$ for all elements $u \in R[\tau]$. So if there exist $u \in R[\tau]$ such that $u \circ \phi(a) = \phi'(a) \circ u$ then one sees that $(rur^{-1})(r\phi(a)r^{-1}) = (r\phi'(a)r^{-1})(rur^{-1})$.

The Drinfeld modules of a fixed rank forms a category, where the morphisms are isogenies. It is clear if there is an isogeny between ϕ and ϕ', the two Drinfeld modules must have the same rank by a simple degree argument.

From now on, we will only be working with rank 1 Drinfeld modules, so for the rest of our discussion, we say (E, ϕ) is a Drinfeld module to mean it is a rank 1 Drinfeld A-module.

Definition 3.1.13. For an A-scheme S, we a define a category

$$\mathcal{M}_A^1(S) = \{\text{The groupoid of rank 1 Drinfeld } A\text{-modules } (E, \phi) \text{ over } S\}$$

Recall the groupoid of a category $\mathcal{D}$ is a subcategory of $\mathcal{D}$ with the same objects but we forget all the non-isomorphisms.

3.1.14. (Base change) Let $S' \to S$ be a morphism of A-schemes, we see that if E is a Drinfeld module over S, the $E' = E \times_S S'$ is a Drinfeld module over S'. Indeed, let $\{U_i \to S\}$ be a coordinate on E, one sees that $\{U_i \times_S S' \to S'\}$ is a coordinate on E'.

Proposition 3.1.15. *The map*

$$\begin{aligned} \mathcal{M}_A^1 : (\textit{Affine } A\textit{-Schemes})^{op} &\to (\textit{Groupoids}) \\ S &\mapsto (\textit{rank 1 Drinfeld modules over } S) \end{aligned}$$

is a pseudo functor (2.5.1) on the category of affine A-schemes.

Proof. There are two ways to prove this. Firstly, we can following definition 2.5.1 and give the four pieces of data, then show the commutative conditions in detail.

The second approach uses the remark in 2.5.6. Let $\mathcal{F}$ be the category of triples (S, E, ϕ) where S is an affine A-scheme and (E, ϕ) is a rank 1 Drinfeld module over S, and morphisms $(S', E', \phi') \to (S, E, \phi)$ are defined by a morphism of affine A-schemes $S' \to S$ and $(E', \phi') \to (E, \phi)$ is a morphism of Drinfeld modules induced by the morphism of schemes as in 3.1.14. Let $P_{\mathcal{F}} : \mathcal{F} \to$ (Affine A-Schemes) be the forgetful map that sends $(S, E, \phi) \mapsto S$. Then by 2.5.6, it is enough to show $(\mathcal{F}, P_{\mathcal{F}})$ is a fibered category, with a choice of cleavage, over (Affine A-Schemes). We now verify the Cartesian diagram in definition 2.5.3, which follows immediately from the fact that the fiber product of schemes on each coordinate on E is Cartesian, and we are done. ☐

Proposition 3.1.16. *The pseudo functor $\mathcal{M}^1_A$ is a stack, with respect to any subcanonical site such that the coverings maps are faithfully flat (see 2.4.15 for some examples), over the category of affine A-schemes (see 2.5.11 for the definition of a stack).*

Proof. Fix S an affine A-scheme. We have already shown $\mathcal{M}^1_A$ is pseudo functor. Since $\mathcal{M}^1_A(S)$ is a groupoid by definition, we use the two criteria in 2.5.13 to check if $\mathcal{M}^1_A$ is a stack.

Firstly, let E, E' be two Drinfeld modules over S where S is an affine A-scheme and let Y be an affine scheme over S. By 2.4.3, We need to show the isomorphism presheaf, $\underline{\text{Isom}}$(E, E') is a sheaf, i.e., we need to show for all covering $\{Z \to Y\}$ (since Y is affine, it is quasi-compact so we may assume any covering is finite, and by taking the disjoint union if necessary we may assume the cover consists of one member), the following sequence is exact

$$\underline{\text{Isom}}(\text{E, E'})(Y) \to \underline{\text{Isom}}(\text{E, E'})(Z) \rightrightarrows \underline{\text{Isom}}(\text{E, E'})(Z \times_Y Z)$$

The Y-points of isomorphism presheaf $\underline{\text{Isom}}$(E, E')(Y) by definition is the set of isomorphism between the Drinfeld modules $E_Y = E \times_S Y$ and $E'_Y = E' \times_S Y$. There exists a Zariski open covering of E_Y and E'_Y such that there exists a coordinate on them. By refinement if necessary, we may assume these open coverings are the same, and an isomorphism $E_Y \xrightarrow{\sim} E'_Y$ is a collection of invertible elements on the corresponding ring, one for each member of the common affine covering. Since we are working over a subcanonical site, it is clear that on an affine covering of E_Y and E'_Y, Spec(R), and a covering $\{\text{Spec}(R') \to \text{Spec}(R)\}$, the following sequence is exact.

$$\mathbb{G}_m(\text{Spec}(R)) \to \mathbb{G}_m(\text{Spec}(R')) \rightrightarrows \mathbb{G}_m(\text{Spec}(R' \otimes_R R'))$$

So the isomorphism presheaf is a sheaf. Thus, criterion 1 holds.

For the second criterion, we need to show for any covering $\{S_i \to S\}$, the map between $\mathcal{M}^1_A(S) \to \mathcal{M}^1_A(\{S_i \to S\})$ is essentially surjective. Once again, by the quasi-compactness of S, we may assume the cover has a single member S'.

In 2.5.14, we showed that there is an equivalence of categories between affine schemes over S (denoted as Aff_S) and affine schemes over S' with descent data for all covering $\{S' \to S\}$ (denoted as $\mathrm{Aff}_{\{S' \to S\}}$). Therefore, there must be an equivalence of categories between the group objects of the two categories above.

We claim that we can likewise define (left) A-module objects in these categories. Indeed, we can define an A-module object to be a group object M together with a map $\cdot_M : A \times M \to M$ such that the map $\cdot_M$ obeys all the usual A-module axioms in the form of a series of commutative diagrams in this categorical setting. This is similar to the definition of a (left) group object action on an object in diagram (1) of chapter V, section 6 of [15]. For example, the property that $(a+b)m = am + bm$ for all $a, b \in A$ and m in the module can be expressed by requiring the following diagram commutes, where A_1, A_2 are two different copies of A, $+_A$ is the addition in the ring A and μ_M is the group operation of the group object M.

$$\begin{tikzcd}
A_1 \times A_2 \times M \arrow[r, "+_A"] \arrow[dd, "\sim"] & A \times M \arrow[dr, "\cdot_M"] & \\
& & M \\
(A_1 \times M) \times (A_2 \times M) \arrow[r, "(\cdot_M, \cdot_M)"] & M \times M \arrow[ur, "\mu_M"'] &
\end{tikzcd}$$

Since the categories Aff_S and $\mathrm{Aff}_{\{S' \to S\}}$ are equivalent, there is an equivalence of categories between the A-module objects of the two categories.

Fix an object E' in $\mathcal{M}^1_A(\{S_i \to S\})$, this object is an affine S'-scheme with an A-module structure with descent data, together with a local isomorphism with $\mathbb{G}_a$ such that the A-mod structure acts by condition 2 in definition 3.1.4. By the equivalence of A-module categories above, there must be an affine S-scheme E with an A-module structure mapping to E'. Moreover, the local isomorphism between E' and $\mathbb{G}_a$ also gives a local isomorphism from E to $\mathbb{G}_a$. Therefore, the map $\mathcal{M}^1_A(S) \to \mathcal{M}^1_A(\{S_i \to S\})$ is essentially surjective and we are done. □

Remark 3.1.17. Proposition 3.1.16 above also shows $\mathcal{M}^1_A$ extends uniquely up to unique isomorphism to a pseudo-functor over all A-schemes, not only affine A-schemes.

Definition 3.1.18. For an A-scheme S, we a define a presheaf

$$\begin{aligned} \mathcal{M}_A^{1,\mathrm{sc}} : (\text{Affine } A\text{-Schemes})^{\mathrm{op}} &\to (\text{Sets}) \\ S &\mapsto (\text{rank 1 standard coordinatized} \\ &\qquad \text{Drinfeld } A\text{-modules over } S) \end{aligned}$$

where the meaning of "standard" is defined in 3.1.6.

Proposition 3.1.19. *The presheaf $\mathcal{M}_A^{1,sc}$ defined in 3.1.18 is representable.*

Proof. Let $A = \mathbb{F}_q[x_1, \cdots x_n]/I$ for some ideal I of the polynomial ring. Recall the definition of deg in 2.1.1. Let

$$\mathcal{A} := A[a_i^{(i)\pm 1}, a_j^{(i)} \mid i = 1, \cdots, n, j = 1, \cdots, \deg(x_i) - 1]$$

Let

$$\phi^{(A)}(x_i) := \sum_{j=1}^{\deg(x_i)} a_j^{(i)} x^{p^j} + x_i \tag{3.1.20}$$

Let $\mathbb{I}$ be an ideal of $\mathcal{A}$ generated by I and $\phi^{(A)}(x_i)$, more precisely, a polynomial $f(x_1, \cdots, x_n) \in I \Rightarrow f(\phi^{(A)}(x_1), \cdots, \phi^{(A)}(x_n)) \in \mathbb{I}$. Let $\mathcal{R} := \mathcal{A}/\mathbb{I}$, we see that $\mathcal{M}_A^{1,\mathrm{sc}}$ is represented by $\mathcal{R}$.

Indeed, let $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{R})$ be a morphism. Let $r_j^{(i)} \in R$ be the image of $a_j^{(i)}$ for all i, j. Then the following additive polynomials

$$\phi^{(R)}(x_i) = \sum_{j=1}^{\deg(x_i)} r_j^{(i)} \tau^j + x_i \tag{3.1.21}$$

defines a standard A-module structure on $\mathrm{Spec}(R[x])$. Conversely, it is clear that every standard Drinfeld A-module of the form $(\mathrm{Spec}(R[x]), \phi^{(R)})$ is defined by specifying the coefficients of the additive polynomials $\phi^{(R)}(x_i)$ for each x_i. Therefore, every standard coordinatized Drinfeld A-module defines a morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{R})$ by $a_j^{(i)} \mapsto r_j^{(i)}$. □

Definition 3.1.22. We denote the A-algebra representing the functor $\mathcal{M}_A^{1,\mathrm{sc}}$ by $\mathcal{R}$. We always view $\mathrm{Spec}(\mathcal{R})$ as a functor that parameterizes rank 1 standard coordinatized Drinfeld A-module, i.e., there is always an implicit isomorphism between $\mathrm{Spec}(\mathcal{R})$ and $\mathcal{M}_A^{1,\mathrm{sc}}$.

We call the scheme $\mathrm{Spec}(\mathcal{R}[x])$ the universal rank 1 standard coordinatized Drinfeld A-module, denoted as $\mathcal{E}^{sc}$. We give $\mathrm{Spec}(\mathcal{R}[x])$ this name since every rank 1 standard coordinatized Drinfeld A-module over a ring R is a pull-back of $\mathrm{Spec}(\mathcal{R}[x])$ along the map $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{R})$. I.e., given E a rank 1 standard coordinatized Drinfeld A-module over a ring R, the following diagram is Cartesian.

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & \mathrm{Spec}(\mathcal{R}[x]) \arrow[d] \\
R \arrow[r] & \mathrm{Spec}(\mathcal{R})
\end{tikzcd} \tag{3.1.23}$$

3.2 Division Points

Let I be a non-zero proper ideal of A. We follow Laumon's definition of I-division points of a Drinfeld module in the first chapter in [14] and we exhibit some relevant properties and consequences. The main result of this section is that, there exists local coordinates such that the ideal defining the I-division points is generated by a single polynomial. We pick a generator of the I-division points in 3.3.8 and call such generator $\phi_I^*(x)$. We later show $\phi_I^*(x)$ is an additive polynomial, so we sometimes denote it as $\phi_I^*(\tau)$ or just ϕ_I by following the notation in 3.1.2.

Definition 3.2.1. (Division points) Let I be an ideal of A. The scheme of I-division points for (E, ϕ) is the subscheme

$$E_I = \bigcap_{a \in I} \mathrm{Ker}(E \xrightarrow{\phi(a)} E) \subseteq E$$

Remark 3.2.2. If I is a principal ideal generated by $\alpha \in I$. It is clear that

$$\begin{aligned}
& x \in E_I \\
\Leftrightarrow\, & x \in \bigcap_{a \in I} \mathrm{Ker}(E \xrightarrow{\phi(a)} E) \\
\Leftrightarrow\, & x \in \bigcap_{b \in A} \mathrm{Ker}(E \xrightarrow{\phi(b)\phi(\alpha)} E) \\
\Leftrightarrow\, & x \in \mathrm{Ker}(E \xrightarrow{\phi(\alpha)} E) \\
\Leftrightarrow\, & \phi(a)(x) = 0
\end{aligned}$$

Lemma 3.2.3. *Let I, J be non-zero proper ideals of A and let E_I and E_J be the I and J-division points defined in 3.2.1.*

1. *If $J \subseteq I$, then $E_I \subseteq E_J$.*

2. *If I is generated by $a_1, a_2, \cdots a_n \in A$. Then $E_I = E_{(a_1)} \cap E_{(a_2)} \cap \cdots \cap E_{(a_n)}$, where $E_{(a_i)}$ is the (a_i)-division points for the principal ideals generated by a_i for each i.*

3. *For two coprime elements $a, b \in A$, $E_{(ab)} = E_{(a)} \oplus E_{(b)}$, where we view the division points as A-modules.*

Proof. Statement 1 is obvious because $J \subseteq I$ means E_I is the intersection of fewer elements, so it contains E_J.

For statement 2, $(a_i) \subseteq I$ for each i, thus $E_I \subseteq E_{(a_1)} \cap E_{(a_2)} \cap \cdots \cap E_{(a_n)}$ by statement 1. On the other hand,

$$\begin{aligned} x \in E_{(a_1)} \cap \cdots \cap E_{(a_n)} &\Rightarrow \phi(a_1)(x) = \phi(a_2)(x) = \cdots = \phi(a_n)(x) = 0 \\ &\Rightarrow \phi(l_1 a_1)(x) = \cdots = \phi(l_n a_n)(x) = 0, \forall l_i \in A \\ &\Rightarrow \phi(\sum_{i=1}^{n} l_i a_i)(x) = 0, \forall l_i \in A \\ &\Rightarrow \phi(c)(x) = 0, \forall c \in I \\ &\Rightarrow x \in E_I \end{aligned}$$

Hence, $E_I \subseteq E_{(a_1)} \cap E_{(a_2)} \cap \cdots \cap E_{(a_n)} \subseteq E_I$. This shows statement 2.

Finally, for statement 3, it is clear that $E_{(a)}, E_{(b)} \subseteq E_{(ab)}$ by statement 1. We show $E_{(a)} \cap E_{(b)} = 0$, but this is clear from statement 2 because a, b are coprime is equivalent to saying $(a, b) = 1$ and E_A is clearly just $\{0\}$. Therefore, $E_{(a)} \oplus E_{(b)} \subseteq E_{(ab)}$. Compare the degrees of the polynomial on both sides shows that they must equal. □

Proposition 3.2.4. *Let S be an A-scheme and let (E, ϕ) be a Drinfeld A-module over S. The scheme of I-division points E_I is a finite flat scheme over S.*

Proof. Let us assume S is affine and $S = \text{Spec}(R)$. The claim is clear if I is a principal ideal, since E_I is a free module by 3.2.2. For a general ideal I, suppose I is generated by two elements $I = (a_1, a_2)$. We choose another ideal $J = (b_1, b_2)$ such that IJ is principal and each b_j is coprime with each a_i.

We claim that $E_{IJ} = E_I \oplus E_J$ as A-modules. By the second part of 3.2.3,

$E_I = E_{(a_1)} \cap E_{(a_2)}$ and $E_J = E_{(b_1)} \cap E_{(b_2)}$. Therefore,

$$\begin{aligned} E_I \oplus E_J &= (E_{(a_1)} \cap E_{(a_2)}) \oplus (E_{(b_1)} \cap E_{(b_2)}) \\ &= \bigcap_{\{i,j\}=\{1,2\}} E_{(a_i)} \oplus E_{(b_j)} \\ &= \bigcap_{\{i,j\}=\{1,2\}} E_{(a_i b_j)} \\ &= E_{IJ} \end{aligned}$$

Thus, as S-schemes, $E_{IJ} = E_I \times E_J$. Let R_I, R_J and R_{IJ} be the R-algebras representing the corresponding division points. Now, one see that the projection $E_I \times E_J \to E_I$ admits a zero section $E_I \to E_I \times E_J$ sending $e \mapsto (e, 0)$, thus $R_{IJ} = R_I \oplus_R M$ for some R-module M. R_{IJ} is clearly finite flat over R since IJ is principal, so R_I must also be finite flat over R and we are done. $\square$

Proposition 3.2.5. *Let E be a rank 1 Drinfeld A-module over an affine A-scheme S and let E_I be the close subscheme defined in 3.2.1. The exists a Zariski covering $\{Spec(T_i) \to S\}$ such that E has a coordinate and the ideal defining E_I in each coordinate is a principal ideal in each $T_i[x]$.*

Proof. We choose a coordinate on E so we may assume $E = \mathrm{Spec}(R[x])$. Since $E_I \to E$ is a close embedding, the corresponding map of rings is of the form $R[x] \to R[x]/\mathbb{I} = B$ for some ideal $\mathbb{I} \subseteq R[x]$.

Let $\bar{x} \in B$ be the image of x under the close embedding shown above. We claim that the ideal $\mathbb{I}$ is generated by the characteristic polynomial of $\mu_{\bar{x}}$, the linear map defined by the multiplication by $\bar{x}$. If the claim holds, then $\mathbb{I}$ is generated by a single polynomial with coefficient in R and we are done.

By proposition 3.2.4, $E_I \to S$ is finite flat. Furthermore, E is locally isomorphic to $\mathbb{G}_{a,S}$ and E_I is a close subscheme of E, so $E_I \to S$ is locally of finite presentation. Thus, by [22, tag 02K9], $E_I \to S$ is finite, flat, and locally of finite presentation if and only if it is locally free. In other words, there exists a cover $\{R \to R_i\}$ such that $B_i = B \otimes_R R_i \cong R_i[x]/\mathbb{I}$ is a free module over R_i and the characteristic polynomial of the multiplication of the image of $\bar{x} \otimes_R 1 \in B_i$ generates $\mathbb{I}$.

Since $\mathbb{I}$ is generated by the characteristic polynomial of the image of $\mu_{\bar{x}}$ over a cover $\{R \to R_i\}$, $\mathbb{I}$ is generated by the characteristic polynomial of $\mu_{\bar{x}}$. $\square$

Since E_I is locally generated by a single polynomial, we denote a generator of E_I by $\phi_I^*(x)$. Note that for now this is only well defined up to a unit of the base ring for now, we give a precise definition of $\phi_I^*(x)$ without this ambiguity in definition below. Following to notation in 3.1.2, the map of schemes induced by $\phi_I^*(x)$ as denoted as $\phi_I : E \to E$.

Lemma 3.2.6. *Locally on E such that $\phi_I^*(x)$ is defined, let ϕ_I denote the corresponding map of schemes. Then*

$$E_I = Ker(E \xrightarrow{\phi_I} E).$$

Proof. Suppose R is the ring which $\phi_I^*(x)$ is defined over. Directly from the definition, $R[x]/(\phi_I^*(x))$ is the ring representing the close subscheme E_I and it is clear that this ring is the cokernel of the map $R[x] \xrightarrow{\phi_I^*} R[x]$. $\square$

Proposition 3.2.7. *Suppose E, E' are two coordinatized Drinfeld modules over an A-scheme S.*

Let $Spec(R)$ be an affine open subset of S on which both E and E' have coordinates and both $\phi_I^*(x), {\phi'_I}^*(x) \in R[x]$ are defined. Let $f^* \in R[\tau]$ such that

$$f : E \times_S Spec(R) \to E' \times_S Spec(R)$$

induces a map between two coordinatized Drinfeld modules over $Spec(R)$. Then for a fixed ${\phi'_I}^*(x)$, there exists a unique $\phi_I^*(x)$ such that $\phi_I^*(x) \circ f^* = f^* \circ {\phi'_I}^*(x)$.

Proof. Fix ${\phi'_I}^*(x)$ and choose any $\phi_I^*(x)$. Since f induces a map of Drinfeld A-modules, we must have $\phi'(a) \circ f = f \circ \phi(a)$ for all $a \in A$.

We express $\operatorname{Ker}(E \xrightarrow{f\circ\phi_I} E')$ in another way. Note that

$$\begin{aligned}
\operatorname{Ker}(E \xrightarrow{f\circ\phi_I} E') &= \operatorname{Ker}(E \xrightarrow{\phi_I} E \xrightarrow{f} E') \\
&= \bigcap_{a\in I} \operatorname{Ker}(E \xrightarrow{\phi(a)} E \xrightarrow{f} E') \\
&= \bigcap_{a\in I} \operatorname{Ker}(E \xrightarrow{f\circ\phi(a)} E') \\
&= \bigcap_{a\in I} \operatorname{Ker}(E \xrightarrow{\phi'(a)\circ f} E') \\
&= \bigcap_{a\in I} \operatorname{Ker}(E \xrightarrow{f} E' \xrightarrow{\phi'(a)} E') \\
&= \operatorname{Ker}(E \xrightarrow{f} E' \xrightarrow{\phi'_I} E') \\
&= \operatorname{Ker}(E \xrightarrow{\phi'_I\circ f} E')
\end{aligned}$$

Therefore, the corresponding polynomial $\phi_I^*(x) \circ f^*$ and $f^* \circ {\phi'_I}^*(x)$ generates the same ideal in $R[x]$, so there exists a unique $u \in R^*$ such that $u\phi_I^*(x) \circ f^* = f^* \circ {\phi'_I}^*(x)$. We replace the choice of $\phi_I^*(x)$ in the beginning by $u\phi_I^*(x)$ if necessary and we are done. □

3.3 Level Structure

We once again follow Laumon's approach in the first chapter in [14], which is summarized in the previous section, to define level-I structures for Drinfeld modules for each non-zero ideal $I \subsetneq A$.

3.3.1. Let $V(I)$ denote the set of prime ideals containing I. We only consider level-I structures of Drinfeld modules over S such that the image of the characteristic $\theta : S \to \mathrm{Spec}(A)$ does not intersect with $V(I)$. In other words, we only consider level-I structure for Drinfeld modules over $A[I^{-1}]$-schemes and the characteristic is the map $\theta : S \to \mathrm{Spec}(A[I^{-1}])$.

Definition 3.3.2. If the image of the characteristic θ does not intersect with $V(I)$, so $\theta : S \to \mathrm{Spec}(A[I^{-1}])$, then a level-$I$ structure is an isomorphism $\iota : (I^{-1}/A)^d_S \xrightarrow{\sim} E_I$, where $(I^{-1}/A)^d_S$ denotes the constant A-module scheme $\coprod_{(I^{-1}/A)^d} S$ over S.

We say (E, ϕ, ι) is a Drinfeld module with level-I structure if the pair (E, ϕ) is a Drinfeld module and ι is a level-I structure as above. We often leave the A-module structure on E implicit and just say (E, ι) is a Drinfeld module with level-I structure. Similarly, we say (E, ϕ, ψ, ι) is a coordinatized Drinfeld module if in addition, ψ is a map $\psi : E \to \mathbb{G}_{a,S}$.

We state the following result without proof.

Proposition 3.3.3. *([14], proposition 1.3.2) Let (E, ϕ) be a Drinfeld module over an A-scheme S. Let $V(I)$ denote the prime ideals of A dividing I. Then the restriction of (E, ϕ) to $S \setminus \theta^{-1}(V(I))$ admits level-I structure locally for the étale topology on $S \setminus \theta^{-1}(V(I))$.*

We remark that proposition 3.3.3 says we can always put a level-I structure on a Drinfeld module over some scheme S if we invert the primes dividing I and base change to some étale cover of S.

3.3.4. If $U = \mathrm{Spec}(R)$ is an affine scheme over $\mathrm{Spec}(A[I^{-1}])$, then $\phi_I^*(x) \in R[x]$ can be chosen to be

$$\phi_I^*(x) = \prod_{a \in (I^{-1}/A)_U(U')} (x - \iota(a)) \in R[x]$$

where U' is an étale cover of U such that there exists an isomorphism

$$\iota : (I^{-1}/A)_U(U') \xrightarrow{\sim} E_I(U')$$

Such an isomorphism exists by 3.3.3. We remark that $\phi_I^*(x)$ indeed has coefficients in the $A[I^{-1}]$-algebra R by proposition 3.2.5.

3.3.5. ($\phi_I^{sc*}(x) \in \mathcal{R}[x]$) Recall the definition the universal rank 1 standard coordinatized Drinfeld A-module $\mathcal{E}^{sc} = \mathrm{Spec}(\mathcal{R}[x])$ and the ring $\mathcal{R}$ in 3.1.22. We define a generator of $\mathcal{E}_I^{sc}$, which we denote as $\phi_I^{sc*}(x) \in \mathcal{R}[x]$, without the ambiguity of a choice of a unit in $\mathcal{R}$.

We first define ${\phi'_I}^*(x) \in (\mathcal{R}[I^{-1}])[x]$ to be the polynomial defined in 3.3.4. We observe that $\mathcal{R} \to \mathcal{R}[I^{-1}] := \mathcal{R} \otimes_A A[I^{-1}]$ is an injection, so there is a unique polynomial in $\mathcal{R}[x]$ such that the image equals to ${\phi'_I}^*(x) \in (\mathcal{R}[I^{-1}])[x]$. We fix this unique polynomial as our definition of $\phi_I^{sc*}(x)$ over $\mathcal{E}^{sc}$.

3.3.6. Let E be a coordinatized Drinfeld module over S. By 3.1.6, there exists an isomorphism $f : E \to E'$ where E' is a standard coordinatized Drinfeld module over S. Thus, there exists a unique map $f' : E' \to \mathcal{E}^{sc} \times_{\mathrm{Spec}(A)} S$. By 3.2.7, locally for some coordinate on E, there is a unique polynomial $\phi_I^*(x)$ generating E_I defined by the image of $\phi_I^{sc*}(x)$ under the fiber product and the map $f' \circ f$.

Lemma 3.3.7. *The unique polynomial $\phi_I^*(x)$ defined above is independent of the choice of the map f and the standard coordinatized Drinfeld module E'.*

Proof. Let $f : E \to E'$ and $g : E \to E''$ be two maps from E to two standard coordinatized Drinfeld modules E', E''. Let f', g' be the uniquely determined maps from E' and E'' to $\mathcal{E}^{sc}$. We need to show that a polynomial $\phi_I^*(x)$ satisfy $\phi_I^*(x) \circ f^* \circ f'^* = f'^* \circ f^* \circ \phi_I^{sc*}$ if and only if $\phi_I^*(x) \circ g^* \circ g'^* = g'^* \circ g^* \circ \phi_I^{sc*}$. We note that there exists an isomorphism $h : E' \to E''$ such that the following is a commutative diagram of coordinatized Drinfeld modules.

$$\begin{tikzcd}
& E' \arrow[dd, "h"] \arrow[dr, "f'"] & \\
E \arrow[ur, "f"] \arrow[dr, "g"'] & & \mathcal{E}^{sc} \\
& E'' \arrow[ur, "g'"'] &
\end{tikzcd}$$

Therefore, $\phi_I^*(x) \circ g^* \circ g'^* = g'^* \circ g^* \circ \phi_I^{sc*}$ if and only if $\phi_I^*(x) \circ f^* \circ h^* \circ g'^* = g'^* \circ h^* \circ f^* \circ \phi_I^{sc*}$, if and only if $\phi_I^*(x) \circ f^* \circ h^* \circ h^{-1*} \circ f'^* = f'^* \circ h^{-1*} \circ h^* \circ f^* \circ \phi_I^{sc*}$, if and only if $\phi_I^*(x) \circ f^* \circ f'^* = f'^* \circ f^* \circ \phi_I^{sc*}$. ☐

Definition 3.3.8. Let E be a rank 1 Drinfeld A-module over S and let I be a non-zero proper ideal of A. Let $\operatorname{Spec}(R)$ be an affine open subset of S such that $E_I \times_S \operatorname{Spec}(R)$ is a principal ideal. We define $\phi_I^*(x)$ to be the polynomial defined in 3.3.6 (which is unique by lemma 3.3.7).

Lemma 3.3.9. *Let $g : E \to E'$ be a map between coordinatized Drinfeld modules and let $\phi_I^*(x)$ and ${\phi'_I}^*(x)$ be the polynomial (locally) defined in 3.3.6. Then $\phi_I^*(x) \circ g^* = g^* \circ {\phi'_I}^*(x)$.*

Proof. There exist maps $f : E \to \mathcal{E}^{sc}$ and $f' : E' \to \mathcal{E}^{sc}$ such that they commute with g, i.e., $f = f' \circ g$. Let ${\phi'_I}^*(x)$ be the polynomial (locally) defined in 3.3.6, by 3.2.7 there exists a unique polynomial $h^*(x)$ defining E_I such that $h^*(x) \circ g^* = g^* \circ {\phi'_I}^*(x)$. This implies $g^* \circ {\phi'_I}^*(x) \circ f'^* = h^*(x) \circ g^* \circ f'^* = h^*(x) \circ f^*$. Moreover, by the definition of ${\phi'_I}^*$, $g^* \circ {\phi'_I}^*(x) \circ f'^* = g^* \circ f'^* \circ \phi_I^{sc*}(x) = f^* \circ \phi_I^{sc*}(x)$.

Therefore, $h^*(x) \circ f^* = f^* \circ \phi_I^{sc*}(x)$. Since there is a unique generator of E_I satisfying this equation, we conclude that $h^*(x) = \phi_I^*(x)$. ☐

Lemma 3.3.10. *Using the notation in 3.3.8, the polynomial $\phi_I^*(x)$ is an additive polynomial.*

Proof. By the construction of ϕ_I^* shown in 3.3.6, it is enough to check if $\phi_I^*(x) \in \mathcal{R}[I^{-1}][x]$ is additive. Let $f : \phi_I^*(x)$. We use the form given if 3.3.4 and check if $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathcal{R}[I^{-1}]$. We observe that, since $G := E_I(U')$ is an additive group, for all $g \in G$, $f(x+g) = f(x)$. In other words, the y-polynomial $f(x+y) - f(x)$ has roots containing G. Therefore,

$$f(x+y) - f(x) = f(y)t(x,y)$$

for some polynomial $t(x,y) \in \mathcal{R}[I^{-1}][x,y]$. we see that the left hand side of the equation above has y-degree equals $n = |G|$, which is the degree of $f(y)$, so $t(x,y) = t(x) \in \mathcal{R}[I^{-1}][x]$ is a polynomial in x.

Now let ax^d be the highest degree term in $t(x)$, so there must be an $ay^n x^d$ term on the left hand side, but the highest y-degree term is y^n. Therefore, we conclude that $a = 1$ and $d = 0$ and we are done. ☐

Remark 3.3.11. From now on, we start to follow 3.1.2 again and sometimes denote both the map between schemes and the map between algebras as just ϕ_I. We continue to use the notation ϕ_I^* and $\operatorname{Spec}(\phi_I)$ when we need to be clear.

Definition 3.3.12. (Isogeny between Drinfeld modules with level structures) Let (E_1, ϕ_1, ι_1) and (E_2, ϕ_2, ι_2) be two Drinfeld modules over S. An isogeny u from (E_1, ϕ_1, ι_1) to (E_2, ϕ_2, ι_2) is an isogeny from (E_1, ϕ_1) to (E_2, ϕ_2) as Drinfeld modules (see 3.1.5) which is also consistent with the level structure, i.e., we have $\iota_2 = u \circ \iota_1$.

Therefore, the Drinfeld modules with level structure form a category where morphisms are isogenies and we can make the following definition similar to definition 3.1.13.

Definition 3.3.13. For an $A[I^{-1}]$-algebra R, we define a pseudo functor $\mathcal{M}_I^1$ on $A[I^{-1}]$-algebras. The R-points are defined as

$$\begin{aligned}\mathcal{M}_I^1(R) = \{&\text{The groupoid of rank 1 Drinfeld } A\text{-modules} \\ &\text{with level } I \text{ structure } (E, \phi, \iota) \text{ over } R\}\end{aligned}$$

Definition 3.3.14. Let (E, ϕ, ψ) be a coordinatized rank 1 Drinfeld A-module over a field L. Let $\phi_I(x)$ be the polynomial defined in 3.3.8 (note: we are following remark 3.3.11 here). We remark that I^{-1}/A is a free A/I-module of rank 1, hence the roots of $\phi_I(x)$ can be viewed as an A/I-torsor. A root of $\phi_I(x)$ is <u>primitive</u> if it is a generator of all roots as an A/I-module.

In other words, given a coordinate on E, specifying a level-I structure of ϕ in a given coordinate is the same as specifying a primitive root of $\phi_I(x)$.

We state the following results.

Proposition 3.3.15. *([14], theorem 1.4.1 and 1.5.1) Let $I \neq A$ be a non-zero ideal. Then $\mathcal{M}_I^1$ is representable by an affine scheme of finite type of over $\mathbb{F}_p$. Furthermore, the map of affine schemes $\mathcal{M}_I^1 \to$ Spec$(A[I^{-1}])$ is a smooth morphism of relative dimension* 0.

Proposition 3.3.16. *([7], proposition 7.1) Let $\mathfrak{p}$ be a prime ideal of A and (E, ϕ) be a rank 1 Drinfeld A-module over a local field $K_\mathfrak{p}$. There exists an extension of local fields $L_\mathfrak{P}/K_\mathfrak{p}$, where $\mathfrak{P}$ is a prime above $\mathfrak{p}$, and $L_\mathfrak{P} \supseteq \mathcal{O}_\mathfrak{P} = \{x \in L_\mathfrak{P} \mid |x|_\mathfrak{P} \leq 1\}$, such that by base changing (E, ϕ) as a rank 1 Drinfeld A-module over $L_\mathfrak{P}$, it is isomorphic (over $L_\mathfrak{P}$) to a rank 1 Drinfeld A-module over $\mathcal{O}_\mathfrak{P}$.*

The result above is also known as "every rank 1 Drinfeld A-module over K has potentially good reduction". This is very similar to elliptic curves with complex multiplication. The result that every elliptic curve with complex multiplication has potentially good reduction can be deduced from the result that shows that the j-invariant is an algebraic integer, which is proven in chapter 2, theorem 6.1 in [21].

We also remark that, following the notation in proposition 3.3.16, given a level-I structure on a Drinfeld module (E_L, ϕ_L) over $L_{\mathfrak{P}}$, the isomorphism with a Drinfeld module $(E_{\mathcal{O}_{\mathfrak{P}}}, \phi_{\mathcal{O}_{\mathfrak{P}}})$ over $\mathcal{O}_{\mathfrak{P}}$, given by a change of coordinates given in proposition 3.3.16, gives a level-I structure on such a Drinfeld module over $\mathcal{O}_{\mathfrak{P}}$.

Proposition 3.3.17. *The map of affine schemes* $\mathcal{M}_I^1 \to Spec(A[I^{-1}])$ *is a finite morphism.*

Proof. A morphism is finite if and only if it is quasi-finite and proper. The quasi-finiteness is clear and by proposition 3.3.15, it is enough to use the valuative criterion to show the morphism $\mathcal{M}_I^1 \to \mathrm{Spec}(A[I^{-1}])$ is proper. In other words, let $\mathfrak{p} \neq \infty$, $\mathfrak{p} \nmid I$ be a finite prime of K not dividing I and let $K_{\mathfrak{p}}$ and $\mathcal{O}_{\mathfrak{p}}$ be the local field and the discrete valuation ring defined by the valuation associated with $\mathfrak{p}$ (see the beginning of chapter 2 for more detail). We show that for every $\mathfrak{p}$ and a commutative diagram of solid arrows as below, there exists a unique map, indicated by the dotted arrow, making the diagram commute.

$$\begin{tikzcd}
\mathrm{Spec}(K_{\mathfrak{p}}) \arrow[r] \arrow[d] & \mathcal{M}_I^1 \arrow[d] \\
\mathrm{Spec}(\mathcal{O}_{\mathfrak{p}}) \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}(A[I^{-1}])
\end{tikzcd}$$

Since $\mathcal{M}_I^1 \to \mathrm{Spec}(A[I^{-1}])$ is a map of affine schemes by proposition 3.3.15, such a map is unique if it exists, so it is enough to show the existence of such a map.

By viewing a Drinfeld module over a ring R as a morphism $\mathrm{Spec}(R) \to \mathcal{M}_I^1$, given the diagram of solid arrows above, proposition 3.3.16 with its following remark shows there exists an extension of local fields $L_{\mathfrak{P}}/K_{\mathfrak{p}}$, where $\mathfrak{P}$ is a prime above $\mathfrak{p}$, and $L_{\mathfrak{P}} \supseteq \mathcal{O}_{\mathfrak{P}} = \{x \in L_{\mathfrak{P}} \mid |x|_{\mathfrak{P}} \leq 1\}$, such that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Spec}(L_{\mathfrak{P}}) \arrow[r] \arrow[d] & \mathrm{Spec}(K_{\mathfrak{p}}) \arrow[r] \arrow[d] & \mathcal{M}_I^1 \arrow[d] \\
\mathrm{Spec}(\mathcal{O}_{\mathfrak{P}}) \arrow[r] \arrow[urr] & \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}}) \arrow[r] & \mathrm{Spec}(A[I^{-1}])
\end{tikzcd} \tag{3.3.18}$$

We show we can find a map from $\operatorname{Spec}(\mathcal{O}_{\mathfrak{p}}) \to \operatorname{Spec}(A[I^{-1}])$ given the diagram above, but this is clear from the following result.

Lemma 3.3.19. *([22], Tag 0ARH) Let S be a scheme. Let $f : X \to Y$ be a separated morphism of algebraic spaces over S. Let B is a discrete valuation ring with field of fractions F, $F' \supseteq F$ is an arbitrary field extension of F and B' is a valuation ring dominating B. Suppose given a diagram of solid arrows*

$$\begin{tikzcd}
\mathit{Spec}(F') \arrow[r] \arrow[d] & \mathit{Spec}(F) \arrow[r] \arrow[d] & X \arrow[d] \\
\mathit{Spec}(B') \arrow[r] \arrow[urr] & \mathit{Spec}(B) \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

(3.3.20)

then the map indicated by the dotted arrow $\mathit{Spec}(B) \to X$ exists.

We apply the lemma above by matching the objects in diagram (3.3.18) with the objects in the same position in diagram (3.3.20). Note that we can apply this lemma by substituting $X = \mathcal{M}_I^1$ since proposition 3.3.15 tells us $\mathcal{M}_I^1$ is an affine scheme. By the lemma above, there is a map $\operatorname{Spec}(\mathcal{O}_{\mathfrak{p}}) \to \mathcal{M}_I^1$ and we are done. □

3.4 Action of Ideals on Drinfeld Modules

Instead of starting with two Drinfeld modules and asking if there is an isogeny between them, one may start with a Drinfeld module E and an element $u \in F[\tau]$ in each coordinate and wish to define a new Drinfeld module (E', ϕ') such that on each coordinate and for all $a \in A$, $u \circ \phi(a) = \phi'(a) \circ u$. We construct such a Drinfeld module (E', ϕ') for each $u = \phi_I(\tau) \in F[\tau]$ define in 3.3.8. We denote this Drinfeld module as $(I * E, I * \phi)$.

We construct this new Drinfeld module by first picking a system of coordinates on E. Since the A-module structure on E is given locally, we may assume this system of coordinates consists of only one element. We follow the notation in 3.1.11 and denote our coordinatized Drinfeld module as (E, ϕ, ψ), where ψ is an isomorphism of schemes $\psi : E \xrightarrow{\sim} \mathbb{G}_{a,S}$. We construct the coordinatized Drinfeld module $(I * E, I * \phi, \psi')$ with respect to this choice of coordinates. We then show that this construction does not depend on the choice of the coordinates ψ in the sense that two choice of coordinates on (E, ϕ) gives a canonical way to identify the different coordinates in $(I*E, I*\phi)$. We then extend this to Drinfeld modules with level structure.

Recall the definition of $\phi_I^*(x)$ in definition 3.3.8.

Definition 3.4.1. Let F be an A-algebra, $S := \mathrm{Spec}(R[x])$, $\mathrm{id} : S \to S$ be the identity map on S and let $\phi : A \to F[\tau]$ such that (S, ϕ, id) is a coordinatized Drinfeld module. Then the I-isogenized Drinfeld module of (S, ϕ, id) or the I-isogeny of (S, ϕ, id) is a coordinatized Drinfeld module $(S, I*\phi, \mathrm{id})$ and the A-module structure $I*\phi$ is given by the relation

$$\phi_I \phi(a) = (I*\phi)(a)\phi_I$$

for all $a \in A$. We show this indeed defines a coordinatized Drinfeld A-module in 3.4.2 below. We have a canonical map of Drinfeld modules $(S, \phi, \mathrm{id}) \to (S, I*\phi, \mathrm{id})$ induced by the map of schemes $\phi_I : S \to S$ given by $x \mapsto \phi_I^*(x)$.

Proposition 3.4.2. *With the same notation as in definition 3.4.1 as above, the triple $(S, I*\phi, id)$ defines a coordinatized Drinfeld module. Furthermore, the coordinatized Drinfeld module $(S, I*\phi, id)$ has the same rank as (S, ϕ, id).*

Proof. For clarity, let us change notation and use the variable y in the image. More precisely, ϕ_I is the map $\phi_I : S \to S'$, where $S' := \mathrm{Spec}(R[y])$, given by $y \mapsto \phi_I(x)$.

We first see that the map $(I*\phi)_a := (I*\phi)(a)$ for all A satisfying

$$\phi_I \phi(a) = (I*\phi)_a \phi_I$$

preserves the additive group structure on S', i.e., $(I*\phi)_a(y_1+y_2) = (I*\phi)_a(y_1) + (I*\phi)_a(y_2)$ for all $y_1, y_2 \in F[y]$. The claim is clear since we can write y_1, y_2 as polynomials in $\phi_I^*(x)$ and we can deduce this using the additivity of ϕ_I (by 3.3.10) and $\phi(a)$ for all $a \in A$.

Therefore, for all $a \in A$, $(I*\phi)_a$ defines an element in $\mathrm{End}(\mathbb{G}_{a,F})$. Finally, one checks condition 2 in definition 3.1.4 by comparing the terms of the same degree in τ on both sides. Therefore, $I*\phi$ defines a Drinfeld A-module structure on S'. Comparing the degrees on both sides also show that the Drinfeld modules (S, ϕ) and $(S', I*\phi)$ have the same rank.

□

Proposition 3.4.3. *Let $f : E'' \to E'$ be an isogeny of coordinatized Drinfeld A-modules. Let $I*E''$ and $I*E'$ be the I-isogenized Drinfeld modules of E'' and*

E' defined by following the construction given in 3.4.1. Then the map f defines an isogeny $f : I * E'' \to I * E'$ such that the following diagram is a commutative diagram of Drinfeld modules.

$$\begin{tikzcd}
E'' \arrow[r, "\phi''_I"] \arrow[d, "f"] & I * E'' \arrow[d, "f"] \\
E' \arrow[r, "\phi'_I"] & I * E'
\end{tikzcd} \tag{3.4.4}$$

Proof. It is enough to show that the Drinfeld A-module structure on $I * E'$ agrees by following the two paths from $E'' \to I * E'$ in the diagram. However, this is clear since 3.3.9 shows $\phi''_I \circ f = f \circ {\phi'}_I$. $\square$

3.4.5. Let (E, ϕ) be a (not coordinatized) rank 1 Drinfeld A-module over an A-algebra F. Let $S = \operatorname{Spec}(F[x])$ and $\psi : E \to S$ be an isomorphism. Let $\phi'(a) = \psi^{-1}\phi(a)\psi$ for all $a \in A$ so that (S, ϕ', id) is a coordinatized Drinfeld module isomorphic to (E, ϕ, ψ).

By definition 3.4.1, we have a coordinatized Drinfeld module $(S, I * \phi', \mathrm{id})$. The isomorphism ψ^{-1} together with $(S, I * \phi', \mathrm{id})$ defines a coordinatized Drinfeld module, which we denote as $(I * E, I * \phi, \psi)$. We define $(I * E, I * \phi)$ to be the Drinfeld module which we forget the coordinate ψ.

In other words, given (E, ϕ), we define $(I * E, I * \phi)$ by choosing a coordinate on it and reducing it to our previous definition given in 3.4.1. We show in the proposition below that, given two choices of coordinates, there exists a canonical way to identify the Drinfeld modules constructed with these choices of coordinates.

Proposition 3.4.6. *Using the same notation as 3.4.5, let $\psi_1, \psi_2 : E \to S$ be two coordinates on E. For $i = 1, 2$, let ϕ_i denote the I-isogeny $\phi_i : (E, \phi, \psi_i) \to (I * E, I * \phi, \psi_i)$. Then the map $\psi_2 \circ \psi_1^{-1} : S \to S$ induces an isomorphism $(I * E, I * \phi, \psi_1) \xrightarrow{\sim} (I * E, I * \phi, \psi_2)$ such that $\psi_2 \circ \psi_1^{-1} \circ \phi_1 = \phi_2 \circ \psi_2 \circ \psi_1^{-1}$.*

Proof. Let $E'' = (I * E, I * \phi, \psi_1)$, $E' = (I * E, I * \phi, \psi_2)$ and let $f = \psi_2 \circ \psi_1^{-1}$. Apply 3.4.3 and we are done. $\square$

Definition 3.4.7. (Isogeny of Drinfeld modules) Let (E, ϕ) be a rank 1 Drinfeld A-module over an A-algebra F. We define the I-isogenized Drinfeld module of (E, ϕ), which we denote as $(I * E, I * \phi)$, or just $I * E$, to be the Drinfeld module we constructed in 3.4.5.

Remark 3.4.8. Following the notation in 3.4.5, we have a map between Drinfeld modules

$$\psi^{-1}\phi'_I\psi : E \to I * E$$

which we show is canonically unique up to the choice of coordinate ψ in proposition 3.4.6. Hence, we will from now on, slightly abuse notation and just refer to the map $\psi^{-1}\phi'_I\psi$ as just ϕ_I.

Lemma 3.4.9. *Let E be a Drinfeld module over Spec(R) and let $I = (w)$ be a non-zero principal ideal of A. If we fix coordinates on E, then there exists a unit $u \in R^*$ such that the induced map on E satisfies $\phi_I =$ Spec$(u) \circ \phi(w)$. Furthermore, $I * \phi$, the A-module structure on $I * E$ with respect to the coordinates we fixed, is given by*

$$(I * \phi)(a) = \mathit{Spec}(u) \circ \phi(a) \circ \mathit{Spec}(u^{-1})$$

for all $a \in A$.

Proof. By 3.2.2 and the definition of ϕ_I in 3.3.8, it is clear that there exists a unit $u \in R^*$ such that $\phi_I^* = u \circ \phi^*(w)$. Furthermore, by 3.2.2 once again, there exists an equality of subschemes

$$\mathrm{Ker}(E \xrightarrow{\phi_I} E) = \mathrm{Ker}(E \xrightarrow{\phi_w} E) =: E_w$$

Therefore, we have the following left exact diagram of A-module schemes

$$\begin{tikzcd}
0 \arrow[r] & E_I \arrow[r] \arrow[d, "="] & E \arrow[r, "\phi_I"] \arrow[d, "\mathrm{Spec}(u)"] & E \arrow[d, "="] \\
0 \arrow[r] & E_w \arrow[r] & E \arrow[r, "\phi(w)"] & E
\end{tikzcd} \tag{3.4.10}$$

Therefore, u must induce an isomorphism of A-module schemes, thus for all $a \in A$, $\mathrm{Spec}(u) \circ \phi(a) = \phi(a) \circ \mathrm{Spec}(u)$. In particular, taking $a = w$, $\phi_I = \phi(w) \circ \mathrm{Spec}(u) = \mathrm{Spec}(u) \circ \phi(w)$ and thus giving our first statement.

The second statement follows from the first statement. Write ϕ_a and $(I * \phi)_a$ for $\phi(a)$ and $(I * \phi)(a)$. Then for all $a \in A$, $(I * \phi)_a$ is defined as the unique polynomial satisfying

$$\phi_I\phi_a = (I * \phi)_a\phi_I$$

But by the first statement,

$$\begin{aligned}\phi_I\phi_a &= \mathrm{Spec}(u)\phi_w\phi_a\\ &= \mathrm{Spec}(u)\phi_{wa}\\ &= \mathrm{Spec}(u)\phi_a\phi_w\\ &= \mathrm{Spec}(u)\phi_a\mathrm{Spec}(u^{-1})\mathrm{Spec}(u)\phi_w\\ &= \mathrm{Spec}(u)\phi_a\mathrm{Spec}(u^{-1})\phi_I\end{aligned}$$

Therefore, $(I * \phi)_a = \mathrm{Spec}(u)\phi_a\mathrm{Spec}(u^{-1})$ by uniqueness. □

Lemma 3.4.11. *([9], lemma 4.9.2) Let (E, ϕ) be a Drinfeld module over an A-scheme S and let I, J be ideals of A. Then*

1. $(I * \phi)_J \circ \phi_I = \phi_{IJ}$,

2. $IJ * \phi = I * (J * \phi)$

Proof. Locally on S, ϕ_I is given by a polynomial defined in 3.3.8. Recall from the same definition that the polynomial corresponding to the map ϕ_I is defined by the polynomial in the ring representing the universal rank 1 standard coordinatized Drinfeld A-module. Therefore, it is enough to show this for the case where $E = \mathcal{E}^{sc}$.

Moreover, as explained in 3.3.5, $\mathcal{R}$ the ring representing $\mathcal{E}^{sc}$ is L-torsion free for all ideal L of A, i.e., $\mathcal{R} \to \mathcal{R}[L^{-1}] := \mathcal{R} \otimes_A A[L^{-1}]$ is an injection. Therefore, we can replace $\mathcal{E}^{sc}$ by $\mathcal{E}^{sc} \times_{\mathrm{Spec}(A)} \mathrm{Spec}A[L^{-1}]$ for some ideal L of A.

Fix ideals I, J given in the lemma, let $L = (I \cap J)$. We observe that for all schemese S over $A[L^{-1}]$ and $e \in E(S)$,

$$\begin{aligned}(I * \phi)_J\phi_I(e) = 0 &\Leftrightarrow \forall j \in J, (I * \phi)_j\phi_I(e) = 0\\ &\Leftrightarrow \forall j \in J, \phi_I\phi_j(e) = 0\\ &\Leftrightarrow \forall j \in J, i \in I, \phi_i\phi_j = \phi_{ij}(e) = 0\\ &\Leftrightarrow \phi_{IJ}(e) = 0\end{aligned}$$

Since S is a scheme over $\mathrm{Spec}(A[(I \cap J)^{-1}])$, the roots of all the polynomials above are distinct by 3.3.3. Furthermore, the polynomials inducing $(I * \phi)_J\phi_I$ and ϕ_{IJ} are both monic by 3.3.4, so they must equal.

For 2, we have the following diagram.

$$\begin{tikzcd}
E \arrow[r, "\phi_J"] \arrow[d, "\phi_{IJ}"] & J * E \arrow[d, "(J * \phi)_I"] \\
IJ * E \arrow[r] & I * (J * E)
\end{tikzcd}$$

If we pick coordinates on E, then by 1, the bottom map must be an equality of coordinatized Drinfeld modules and we are done. □

3.4.12. We extend the action of ideals on Drinfeld modules with level structure. Let I be an ideal of A prime to the characteristic of an A-algebra F and let (E, ϕ, ι) be a Drinfeld module over F with level-I structure. We define the Drinfeld module with level-I structure $(J * E, J * \phi, J * \iota)$ where $J * E$ and $J * \phi$ are as above. For the level structure, the map of schemes $\phi_J : E \to J * E$ defined in remark 3.4.8 induces an isomorphism $\phi_J : E_I \to (J * E)_I$, where E_I and $(J * E)_I$ are the I-division points of E and $J * E$ as in definition 3.2.1. Hence it is natural to define the level structure $(J * \iota) = \phi_J \circ \iota$.

Remark 3.4.13. Given (E, ϕ, ι) a Drinfeld module over a field F with level-I structure and an ideal J coprime with I, we can give a more concrete way to describe $(J * E, J * \phi, J * \iota)$ by picking a coordinate on E and ϕ becomes a map $\phi : A \to F[\tau]$ under this choice of coordinate.

By 3.3.14 and identifying the roots of ϕ_I as a A/I, a Drinfeld module with a level-I structure is equivalent to a triple (E, ϕ, λ) where λ is a primitive root of ϕ_I. Let J be an ideal of A prime to the characteristic of F and $V(I)$, the map of schemes $\phi_J : E \to J{*}E$ defined in remark 3.4.8 sends $x \mapsto \phi_I(x)$. Therefore, given a coordinate, the $J*-$ action is given by sending $(E, \phi, \lambda) \mapsto (J{*}E, J{*}\phi, \phi_J(\lambda))$.

Definition 3.4.14. Let (E, ϕ) be a rank d Drinfeld A-module over F. We define $\mathrm{End}(E, \phi)$ to be the subring of $\mathrm{End}(E)$ which consists of elements that commute with $\phi(a)$ for all $a \in A$.

In other words, a non-zero element in $\mathrm{End}(E, \phi)$ is an isogeny from (E, ϕ) to itself as an A-module scheme. It is clear that $\mathrm{End}(E, \phi)$ is an A-module via ϕ. We state the following result without proof.

Proposition 3.4.15. *([14], 2.1.4) Let (E, ϕ) be a rank d Drinfeld A-module over a field L. Then End(E, ϕ) is a projective A-module of rank $\leq d^2$.*

Corollary 3.4.16. *Let (E, ϕ) be a rank 1 Drinfeld A-module over a field L. Then End$(E, \phi) = A$ and Aut$(E, \phi) = A^* = \mathbb{F}_q^*$.*

Proposition 3.4.17. *Let (E, ϕ) be a rank 1 Drinfeld A-module over an A-scheme S. Then End$(E, \phi) = A$ and Aut$(E, \phi) = A^* = \mathbb{F}_q^*$.*

Proof. We note that for Drinfeld module E over $\mathrm{Spec}(R)$ and for all ring homomorphism $R \to R'$, we have a map

$$\mathrm{End}(E, \phi) \to \mathrm{End}(E \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R'), \phi \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R'))$$

defined by the pull-back.

We first show this for the case where S is the spectrum of a local ring R. Let $\mathfrak{m}$ be the unique maximal ideal of R, $k := R/\mathfrak{m}$ and let (E_0, ϕ_0) denote the Drinfeld module obtained by taking the fiber $E_0 := E \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$. We have the following commutative diagram.

$$\begin{tikzcd}
A \arrow[rd, "\phi_0"] \arrow[d, "\phi"'] & \\
\mathrm{End}(E, \phi) \arrow[r] & \mathrm{End}(E_0, \phi_0)
\end{tikzcd} \tag{3.4.18}$$

By 3.4.15, ϕ_0 induces an isomorphism $A \to \mathrm{End}(E_0, \phi_0)$, so it is enough to show the map $\mathrm{End}(E, \phi) \to \mathrm{End}(E_0, \phi_0)$, which is induced by reduction, is an injection. Suppose $0 \neq f \in \mathrm{End}(E, \phi) \subseteq \mathrm{End}(E) \cong R[\tau]$ is sent to 0 by the reduction (quotient) map. Using proposition 5.1 of [7], which we state in proposition 3.4.19 below, since $f \neq 0$, we must be in the case where the leading coefficient of $f(\tau)$ is a unit in R. However, this contradicts the fact that $f(\tau)$ reduces to 0 under the reduction map. Therefore, the unique pre-image of 0 under the map $\mathrm{End}(E, \phi) \to \mathrm{End}(E_0, \phi_0)$ is 0, showing this map is injective and we are done for the case where R is a local ring.

For the general case, we use one of the results proven in [22, Tag 00EN]. Let R be a ring and M be an R-module. Let $M_{\mathfrak{p}}$ denote the localization of M by a prime ideal $\mathfrak{p}$ of R. For an element $x \in M$, the following are equivalent:

1. $x = 0$;
2. x maps to 0 under all localization maps $M \to M_{\mathfrak{p}}$ for all prime ideal $\mathfrak{p}$ of R;
3. x maps to 0 under all localization maps $M \to M_{\mathfrak{m}}$ for all maximal ideal $\mathfrak{m}$ of R.

Now we show the unlabelled map in (3.4.18) is an isomorphism for all E over any base A-scheme S by again, showing it is an injection. By considering an affine cover of S it is enough to show this for the case where $S = \mathrm{Spec}(R)$ where R is an A algebra and not necessarily local.

Suppose $f \in \mathrm{End}(E, \phi)$ maps to 0 under the unlabelled map and we wish to show $f = 0$. But by the result above and by letting $M = \mathrm{End}(E, \phi)$, it is enough to show f maps to 0 under all localization maps. Therefore, we reduce to the previous case where R is a local ring and we are done. □

We state the following result from Drinfeld to complete the proof above.

Proposition 3.4.19. *([7], 5.1) Let B be a ring of characteristic p with Spec(B) connected. Let $f_1, f_2 \in B[\tau]$, $f_j = \sum_{i=0}^{d_j} a_{i,j}\tau^i$, $d_1 > 0$, $a_{d_j,j}$ invertible for $j = 1, 2$. Let $h \in B[\tau]$ and $hf_1 = f_2h$.*

1. *If $d_1 \neq d_2$, then $h = 0$;*
2. *If $d_1 = d_2$ and $h \neq 0$, the h has the form $\sum_{i=0}^{d_3} h_i\tau^i$ where h_{d_3} is invertible.*

3.5 Drinfeld Modules over $\mathcal{C}$

In this section, we first briefly explain how one may use analytic methods to obtain results relating to Drinfeld modules, then we show the functor $\mathcal{M}_I^1$ is represented by $A_{K_I}[I^{-1}]$ (recall the definition in the beginning of chapter 2). In the first part, We mostly follow Hayes' summary in [13] to give an overview on finding the $\mathcal{C}$ points of $\mathcal{M}_I^1$ and we will state most results without proof. For a more detail explanation, one can consult chapters 2 to 4 in [9].

We put coordinates on all our Drinfeld modules in this section, so throughout this section, (E, ϕ, ψ) is a coordinatized Drinfeld module over $\mathcal{C}$ and from this point of view to construct a Drinfeld module, it is enough to construct a map $\phi : A \to \mathcal{C}[\tau]$ with the desired properties in definition 3.1.4, so we slightly abuse notation and just say "ϕ is a Drinfeld module".

Definition 3.5.1. A lattice is a discrete A-submodule Γ such that $K \otimes_A \Gamma$ has finite K-dimension, which we call the rank of Γ and denote it as r_Γ.

Remark 3.5.2. Since $\mathcal{C}$ is an infinite dimensional K_∞-vector space, there is a lattice of every positive rank.

We will focus on the case where the rank is 1. We remark that a rank 1 lattice is isomorphic to a fractional ideal of A.

Definition 3.5.3. The exponential function associated to a lattice Γ, $e_\Gamma(z) : \mathcal{C} \to \mathcal{C}$, is defined by the infinite product

$$e_\Gamma(z) = z \prod_{0 \neq \gamma \in \Gamma} \left(1 - \frac{z}{\gamma}\right)$$

Note that the infinite sum $\sum_{0 \neq \gamma \in \Gamma} \frac{1}{\gamma}$ converges absolutely in the topology in $\mathcal{C}$ so the infinite product also converges for all $z \in \mathcal{C}$.

Theorem 3.5.4. *([13], theorem 8.5) The function $e_\Gamma(z)$ is an entire function on $\mathcal{C}$ with the following properties:*

1. *$e_\Gamma(z)$ is a surjective $\mathbb{F}_q$-linear endomorphism of $\mathcal{C}$;*

2. *$e_\Gamma(z)$ is periodic with Γ as a group of periods.*

Definition 3.5.5. Let $\Gamma' \supseteq \Gamma$ be two lattices with finite index, we define

$$P(\Gamma'/\Gamma; t) = t \prod_{0 \neq \sigma \in e_\Gamma(\Gamma')} \left(1 - \frac{t}{\sigma}\right)$$

Note that $P(\Gamma'/\Gamma; t)$ is a $\mathbb{F}_q$-linear polynomial of degree $|\Gamma'/\Gamma|$.

Proposition 3.5.6. *([13], chapter 8) Let Γ be a lattice, $x \in A$. Let $\phi_x^\Gamma(t) := xP(x^{-1}\Gamma/\Gamma; t)$, then:*

1. *Every lattice defines a Drinfeld module over $\mathcal{C}$, via the map $\phi^\Gamma : x \mapsto \phi_x^\Gamma$, with the identification $t^p \mapsto \tau$.*

2. *The rank of the Drinfeld module ϕ^Γ is the same as the rank of the lattice r_Γ.*

3. *Every coordinatized Drinfeld module ϕ over $\mathcal{C}$ has a uniquely determined lattice Γ such that $\phi^\Gamma = \phi$.*

We label this uniquely defined lattice attached to ϕ as Γ_ϕ. We once again emphasise the uniqueness in part 3 in the proposition above only holds for coordinatized Drinfeld module and hence ϕ is viewed as map between algebras $\phi : A \to \mathcal{C}[\tau]$.

Theorem 3.5.7. *([13], theorem 8.14) Let I be a non-zero ideal of A and Γ a lattice. Let $\phi' = I * \phi^\Gamma$ (recall the $*$ notation in definition 3.4.7), then*

$$\Gamma_{\phi'} = D(\phi_I^\Gamma) \cdot I^{-1}\Gamma$$

where D is the map $D(\sum_n \alpha_n(a)\tau^n) = \alpha_0$ as in equation (3.1.8).

Proof. A proof can be found in [9], corollary 4.9.5. □

Corollary 3.5.8. *The action of non-zero fractional ideals of A on the isomorphism classes of rank 1 Drinfeld modules over $\mathcal{C}$ (through $*$) factors through $Cl_1(1)$(see definition 2.3.1). This action makes the set of isomorphism classes of rank 1 Drinfeld modules over $\mathcal{C}$ a $Cl_1(1)$-torsor.*

Proof. We follow the steps in chapter 9, [13]. We denote the set of rank 1 lattice in $\mathcal{C}$ as $\mathrm{Lat}_1(\mathcal{C})$ and define the an equivalence relation on $\mathrm{Lat}_1(\mathcal{C})$ by $\Gamma \sim \Gamma'$ if $\Gamma' = c\Gamma$ for some $c \in \mathcal{C}^\times$. Let $\mathcal{L}_1$ be the quotient of $\mathrm{Lat}_1(\mathcal{C})/\sim$.

The action of non-zero fractional ideals of A on $\mathrm{Lat}_1(\mathcal{C})$ defined by $I : \Gamma \to I^{-1}\Gamma$ gives a well defined, transitive and faithful action of $\mathrm{Cl}_1(1)$ on $\mathcal{L}_1$. Finally, one checks $c \in \mathcal{C}$ is an isomorphism from ϕ^Γ to $\phi^{\Gamma'}$ if and only if $\Gamma' = c\Gamma$ (see section 4.9 in [9]), hence theorem 3.5.7 gives us a bijection between $\mathcal{L}_1$ and the set of isomorphism classes of rank 1 Drinfeld modules over $\mathcal{C}$ and the group action of $\mathrm{Cl}_1(1)$ agrees on both sets. □

Definition 3.5.9. We say $L \subseteq \mathcal{C}$ is a field of definition of ϕ a Drinfeld module over $\mathcal{C}$ or ϕ is defined over L if there exists an isomorphism in $\mathcal{C}$ from ϕ to ϕ' such that $\phi' \in L[\tau]$.

3.5.10. Proposition 10.2 in [13] shows K_∞ is a field of definition for every rank 1 Drinfeld module over $\mathcal{C}$. Theorem 10.3 in the same article also shows there exists a finite extension of K such that it is a field of definition of every rank 1 Drinfeld module over $\mathcal{C}$, which is also contained in every field of definition. We call this the smallest field of definition.

Theorem 3.5.11. *Let $I \subsetneq A$ be a non-zero ideal, the functor $\mathcal{M}_I^1 \times_{Spec(A)} Spec(K)$ is represented by the K_I, the ray class field over K of conductor I, totally splitting at ∞.*

Proof. Following 3.4.12, we have an action of ideals prime to I acting on $\mathcal{M}_I^1(\mathcal{C})$. By the identification between lattices and Drinfeld modules in corollary 3.5.6, we view the action of ideals of A on Drinfeld module as a transitive action of fractional ideals on lattices. The stabilizer of this action on fractional ideals is clearly contained in {Principal Fractional Ideals of A}, since each principal fractional ideal defines an isomorphism by corollary 3.5.8.

Furthermore, following remark 3.4.13, let (E, ϕ, λ) be a rank 1 Drinfeld A-module over $\mathcal{C}$ with level-I structure and λ is a primitive root of ϕ_I.

A principal ideal $(a) \subseteq A$ fixes the level structure if and only if $\phi_{(a)}(\lambda) = u\lambda$ for some $u \in \mathbb{F}_q^*$ by 3.4.16. We pick another generator of a' of (a) such that $\phi_{(a)} = u\phi_{a'}$, hence $\phi_a(\lambda) - u\lambda = u(\phi_{a'}(\lambda) - \lambda) = u(\phi_{a'-1}(\lambda)) = 0$, so we must have $a' \equiv 1 \bmod I$ since λ is a primitive root. Therefore, there is a one to one correspondence between $\mathcal{M}_I^1(\mathcal{C})$ and the following group

$$\{\text{Fraction ideals}\}/\{\text{Principal Fractional Ideals } (a) \mid a \equiv 1 \bmod I\}$$

The group above can clearly be identified as $\mathrm{Cl}_1(I)$. Hence, $\mathcal{M}_I^1(\mathcal{C})$ is a $\mathrm{Cl}_1(I)$-torsor. In fact, in stead of $\mathcal{C}$, one can work with any field L such that L is a finite extension of K containing all the roots of ϕ_I and is a field of definition (see 3.5.10) of ϕ. Since each equivalence class in $\mathcal{M}_I^1(\mathcal{C})$ can be represented by a Drinfeld module over L, so $\mathcal{M}_I^1(\mathcal{C}) = \mathcal{M}_I^1(L)$. This holds for any field F such that $L \subseteq F \subseteq \mathcal{C}$. Therefore, for some finite Galois extension L/K, the functor $(\mathcal{M}_I^1 \times_{\mathrm{Spec}(A)} \mathrm{Spec}(K)) \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ is isomorpic to $\coprod_{\mathrm{Cl}_1(I)} \mathrm{Spec}(L)$ and hence the functor $\mathcal{M}_I^1 \times_{\mathrm{Spec}(A)} \mathrm{Spec}(K)$ is, in the fppf topology, locally isomorphic to $\coprod_{\mathrm{Cl}_1(I)} \mathrm{Spec}(K)$.

Therefore, the functor $\mathcal{M}_I^1 \times_{\mathrm{Spec}(A)} \mathrm{Spec}(K)$ is representable by a finite étale algebra over K, i.e., a finite product of finite Galois extension of K. Let us call this algebra F_I and note that F_I is a field since the action of $\mathrm{Cl}_1(I)$ is transitive.

Recall K_I is the ray class field over K of conductor I, totally splitting at ∞ (we gave many different different definitions of this in chapter 2, but we will be mostly using the definition in 2.3.1). Let π be a finite place of K not dividing $I\infty$ and let $\sigma_\pi \in \mathrm{Gal}(K_I/K)$ be the Frobenius element over π. We show the fields K_I and F_I are equal by using corollary 2.2.7 and showing there exists an isomorphism $\mathrm{Gal}(K_I/K) \to \mathrm{Cl}_1(I)$ such that all but finitely many Frobenius elements are preserved. We start by identifying $\mathrm{Gal}(K_I/K)$ with $\mathrm{Cl}_1(I)$ using definition 2.3.1, then the Frobenius element of π is the image of $\pi \in \mathrm{Cl}_1(I)$ by proposition 2.3.10. We show in proposition 3.5.12 below that the identity map $\mathrm{Cl}_1(I) \to \mathrm{Cl}_1(I)$ defined by $\pi \mapsto \pi * -$ is a map that preserves almost all Frobenius elements, i.e., the map $\pi * -$ is a Frobenius elements on $\mathcal{M}_I^1$.

Since the identity map preserves all primes not dividing $I\infty$, we conclude the fields F_I and K_I must be equal by corollary 2.2.7 and we are done. □

Proposition 3.5.12. *Let π be a prime ideal not dividing I. The map $E \mapsto \pi * E$ defines a Frobenius automorphism on $\mathcal{M}_I^1$.*

Proof. Pick a rank 1 Drinfeld A-module (E, ϕ) over K and fix some coordinate on it. We know that by fixing coordinates and viewing ϕ as a map $\phi : A \to K[\tau]$

the action of $\pi * -$ on (E, ϕ) is induced by the map of rings $\mathrm{Spec}(K[x]) \to \mathrm{Spec}(K[x])$ defined by $x \mapsto \phi_\pi(x)$. In this point of view, showing $\pi * -$ acts as the Frobenius element is the same as showing $x \mapsto \phi_\pi(x)$ satisfy the Frobenius condition. However, this is true due to the corollary 5.9 of [13] which says the following:

3.5.13. Let (E, ϕ) be a rank 1 Drinfeld module over a field with characteristic $\pi \neq 0$. Let $\mathfrak{b} = \pi^e$, $e \geq 1$. Then $\phi_{\mathfrak{b}} = x^{p^{deg(\mathfrak{b})}}$.

In other words, taking $e = 1$, ϕ_π acts as the Frobenius modulo π on $\mathcal{M}_I^1$ and we are done. □

Corollary 3.5.14. *Let $I \neq A$ be a non-zero ideal of A. The scheme $\mathcal{M}_I^1$ is represented by $A_{K_I}[I^{-1}]$, where K_I is the ray class field over K of conductor I, totally splitting at ∞.*

Proof. Proposition 3.3.15 shows $\mathcal{M}_I^1$ is an affine scheme over $A[I^{-1}]$. Suppose B is the $A[I^{-1}]$-algebra representing $\mathcal{M}_I^1$, i.e., $\mathcal{M}_I^1 = \mathrm{Spec}(B)$. Proposition 3.3.15 shows that the map of affine schemes

$$\mathcal{M}_I^1 \to \mathrm{Spec}(A[I^{-1}])$$

is a smooth morphism of curves over $\mathbb{F}_q$, so the ring B must be a regular algebra over $A[I^{-1}]$. Since the functor $\mathcal{M}_I^1 \times_{\mathrm{Spec}(A[I^{-1}])} \mathrm{Spec}(K)$ is represented by K_I, the ring B must equal to $A_{K_I}[I^{-1}][J^{-1}]$ for some ideal $J \subseteq A_{K_I}$. Furthermore, proposition 3.3.17 shows the morphism $\mathcal{M}_I^1 \to \mathrm{Spec}(A[I^{-1}])$ above is finite and hence $\mathcal{M}_I^1 = \mathrm{Spec}(A_{K_I}[I^{-1}])$. □

3.5.15. For each ideal J coprime with I, we give another point of view on the Drinfeld module with level-I structure $(J * E, J * \phi, J * \iota) \in \mathcal{M}_I^1(R)$ for a given $(E, \phi, \iota) \in \mathcal{M}_I^1(R)$ for any A-algebra R.

The functor $\mathcal{M}_I^1 \times_{\mathrm{Spec}(A[I^{-1}])} \mathrm{Spec}(K) \cong \mathrm{Spec}(K_I)$ in theorem 3.5.11 and in the proof of this theorem, it is shown that for every prime π of A not dividing I, the automorphism

$$\pi * - : \mathcal{M}_I^1 \times_{\mathrm{Spec}(A[I^{-1}])} \mathrm{Spec}(K) \to \mathcal{M}_I^1 \times_{\mathrm{Spec}(A[I^{-1}])} \mathrm{Spec}(K)$$

is induced by $\sigma_\pi \in \mathrm{Gal}(K_I/K)$ the Frobenius element at the prime π. By corollary 3.5.14, $\mathcal{M}_I^1 \cong \mathrm{Spec}(A_{K_I}[I^{-1}])$, one see that the any Galois group element $\sigma \in \mathrm{Gal}(K_I/K)$ is a map $\sigma : A_{K_I}[I^{-1}] \to A_{K_I}[I^{-1}]$ since I is an ideal in A so $\sigma(I) = I$.

We therefore define for every prime π of A not dividing I, the automorphism $\pi * - : \mathcal{M}_I^1 \to \mathcal{M}_I^1$ to be the map induced by σ_π.

For a general ideal J coprime with I, we factorise J into product of prime ideals and form a Galois group element σ_J by taking the product of all Frobenius elements in $\mathrm{Gal}(K_I/K)$ and we define the automorphism $J * - : \mathcal{M}_I^1 \to \mathcal{M}_I^1$ to be the map induced by σ_J . Note that this multiplicative definition is consistent with the $J * -$ action on $\mathcal{M}_I^1 \times_{\mathrm{Spec}(A[I^{-1}])} \mathrm{Spec}(K)$ by lemma 3.4.11.

Another way to define $J * -$ this is to view a rank 1 Drinfeld module over an $A[I^{-1}]$-scheme S with level-I structure as a morphism $e : S \to \mathcal{M}_I^1$. Then $(J * E, J * \phi, J * \iota)$ is the defined as the Drinfeld module with level-I structure corresponding to the composition $S \xrightarrow{e} \mathcal{M}_I^1 \xrightarrow{\sigma_J} \mathcal{M}_I^1$.

We state the following theorem. Recall the definition of the Hilbert class field H in 2.3.8 and a field of definition in 3.5.9.

Theorem 3.5.16. *H is a field of definition.*

A proof of this can be found in the first part of theorem 15.6 in [13]. The proof involves introducing the notion of sign functions and sign normalized Drinfeld modules. In fact, H is known as the minimal field of definition, meaning every other field of definition contains H. A proof of this fact can be found is the same theorem cited above.

Corollary 3.5.17. *Let (E, ϕ, ψ) be a coordinatized Drinfeld A-modules over $\mathcal{C}$, so ϕ is viewed as a map $\phi : A \to \mathcal{C}[\tau]$. Let λ be a primitive root of ϕ_I (recall this means the generator of the roots of ϕ_I as an A/I-module, see definition 3.3.14), then $K_I = H(\lambda^{q-1})$.*

Proof. It is clear that both K_I and $H(\lambda^{q-1})$ contain H and are contained in $H(\lambda)$. The group $\mathrm{Gal}(H(\lambda)/H)$ is isomorphic to $(A/I)^*$ and it acts by sending $\lambda \to \phi_a(\lambda)$ for all $a \in A$. One see that, by viewing $\mathrm{Spec}(K_I)$ as $\mathcal{M}_I^1 \times_{\mathrm{Spec}(A[I^{-1}])} \mathrm{Spec}(K)$, both of these fields are the invariant subfields fixed by the field automorphisms $\lambda \to \mu\lambda$, $\mu \in \mathbb{F}_q^*$. ☐

Remark 3.5.18. The statement in corollary 3.5.17 is independent on the choice of coordinatized Drinfeld modules. I.e., let (E, ϕ, ψ), (E', ϕ', ψ') be two rank 1 coordinatized Drinfeld A-modules, λ, λ' be primitive roots of ϕ_I, ϕ'_I. Then $K_I = H(\lambda^{q-1}) = H(\lambda'^{q-1})$.

Chapter 4

Abelian Extensions over K

We follow the same notation in chapters 2 and 3. Let K be a global function field and let A be a subring of K as described in the beginning of chapter 2. Let P be the set of all prime ideals of A. We say (E, ϕ) is a Drinfeld module to mean it is a rank 1 Drinfeld A-module as defined in chapter 3.

This chapter explains how we naturally construct a $\Lambda_{A,P}$-scheme (see definition 0.0.5), which we refer as just Λ-scheme throughout this chapter, using the theory of Drinfeld modules. More precisely, we first show there is a natural map from the stack $\mathcal{M}_A^1$ to $\mathrm{Spec}(A_H)$, which we call the coarse map. We then construct the space of the universal rank 1 Drinfeld modules $\mathcal{E}$ and use the coarse map to define a map from $\mathcal{E}$ to an affine scheme $\mathcal{L}$ that is isomorphic to $\mathrm{Spec}(A_H[x])$. Finally. we show that the periodic loci of $\mathcal{L}$ generates the maximal abelian extension of K totally split at ∞, and hence by considering the constructions for another choice of a place of K, say ∞', the maximal abelian extension of K is Λ-geometric.

4.1 Coarse Space of $\mathcal{M}_A^1$

The major result in this section is to show the coarse space of the stack $\mathcal{M}_A^1$ is isomorphic to $\mathrm{Spec}(A_H)$. This is a known result which can be deduce from theorem 1 of [7] by Drinfeld, in which he shows the group of finite ideles $\mathbb{A}_K^*$ acts on $\varprojlim \mathcal{M}_I^1$ (note that in [7] $\mathcal{M}_I^1$ are schemes over $\mathrm{Spec}(A)$) and this action is consistent with class field theory. From this, using a similar result like lemma 4.1.8 or otherwise, one can determine the coarse space of $\mathcal{M}_A^1$. We compute the coarse space using a similar method, with some change in notation to match our notation in the previous chapters.

4.1.1. Let $I_1 \subseteq I_2$ be two non-zero ideals of A and let S_{I_1} be a scheme over $\mathrm{Spec}(A[I_1^{-1}])$. There is a canonical map $\alpha_{I_1,I_2} : \mathcal{M}^1_{I_1}(S_{I_1}) \to \mathcal{M}^1_{I_2}(S_{I_1})$ induced by the inclusion $I_2^{-1} \hookrightarrow I_1^{-1}$ and this map is just the forgetful map that forgets the level structure if $I_2 = A$.

4.1.2. In theorem 3.5.11, we showed that for all non-zero proper ideals I of A, there exists an isomorphism between $\mathcal{M}^1_I$ and $\mathrm{Spec}(A_H[I^{-1}])$. In fact, by the lemma below, there exists an isomorphism

$$\xi_I : \mathcal{M}^1_I \to \mathrm{Spec}(A_H[I^{-1}])$$

such that this is compatible with the inclusion map and all the α's we defined above. In other words, we require the following diagram to commute for all non-zero proper ideals $J \subseteq I$.

$$\begin{tikzcd}
\mathcal{M}^1_J \arrow[r, "\xi_J"] \arrow[d, "\alpha_{J,I}"] & \mathrm{Spec}(A_H[J^{-1}]) \arrow[d, "\mathrm{pr}_{J,I}"] \\
\mathcal{M}^1_I \arrow[r, "\xi_I"] & \mathrm{Spec}(A_H[I^{-1}])
\end{tikzcd}$$

where $\mathrm{pr}_{J,I}$ denotes the map induced by the inclusion map $A_H[I^{-1}] \hookrightarrow A_H[J^{-1}]$.

Lemma 4.1.3. *There exists a collection* $\{\xi_I\}$ *for each non-zero proper ideal* $I \subsetneq A$ *such that the diagram in 4.1.2 commutes for all* $J \subseteq I$.

Proof. Fix an algebraic closure $\bar{K}/K$ and let E be a rank 1 Drinfeld A-module over $\bar{K}$. By 3.3.3, we can put a level-I structure on E for all I. Furthermore, we can require all the level structures we chose are compatible. I.e., for all $J \subseteq I$, we choose level-I and level-J structure ι_I, ι_J on E such that $\alpha_{J,I} : (E, \phi, \iota_J) \mapsto (E, \phi, \iota_I)$. By the moduli space interpretation, this is equivalent to the following commutative diagram,

$$\begin{tikzcd}
\mathrm{Spec}(\bar{K}) \arrow[r, "e_J"] \arrow[dr, "e_I"] & \mathcal{M}^1_J \arrow[d, "\alpha_{J,I}"] \\
 & \mathcal{M}^1_I
\end{tikzcd}$$

where e_I, e_J denotes the Drinfeld modules $(E, \phi, \iota_I), (E, \phi, \iota_J)$ over $\bar{K}$.

Since the ray class fields can be identified as subfields of the algebraic closure we fixed in the beginning, for all ideals $0 \neq I \neq A$, there exists a map $\mathrm{inc}_I : \mathrm{Spec}(\bar{K}) \to \mathrm{Spec}(A_{K_I}[I^{-1}])$ induced by the subring inclusion $A_{K_I}[I^{-1}] \hookrightarrow K_I \hookrightarrow \bar{K}$. By 3.5.11, $\mathcal{M}_I^1$ and $\mathrm{Spec}(A_{K_I}[I^{-1}])$ are isomorphic, and since $\bar{K}$ is algebraically closed, there exists an isomorphism $\xi_I : \mathcal{M}_I^1 \to \mathrm{Spec}(A_{K_I}[I^{-1}])$ such that it commutes with e_I and inc_I. I.e., the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{Spec}(\bar{K}) \arrow[d, "e_I"] \arrow[dr, "\mathrm{inc}_I"] & \\
\mathcal{M}_I^1 \arrow[r, "\xi_I"] & \mathrm{Spec}(A_{K_I}[I^{-1}])
\end{tikzcd}$$

Now we show the diagram in 4.1.2 is commutative with our choice of ξ_I for all I. Since any ring homomorphism $A_{K_J}[J^{-1}] \to \bar{K}$ is necessarily a monomorphism, any map of schemes $\mathrm{Spec}(\bar{K}) \to \mathrm{Spec}(A_H[J^{-1}])$ must be an epimorphism (in the category of affine schemes), thus it is enough to show that $\xi_I \alpha_{J,I} e_J = \mathrm{pr}_{J,I} \xi_J e_J$ in the diagram below.

$$\begin{tikzcd}
\mathrm{Spec}(\bar{K}) \arrow[d, "e_J"] \arrow[dr, "\mathrm{inc}_J"] & \\
\mathcal{M}_J^1 \arrow[r, "\xi_J"] \arrow[d, "\alpha_{J,I}"] & \mathrm{Spec}(A_H[J^{-1}]) \arrow[d, "\mathrm{pr}_{J,I}"] \\
\mathcal{M}_I^1 \arrow[r, "\xi_I"] & \mathrm{Spec}(A_H[I^{-1}])
\end{tikzcd}$$

By construction, we have

$$\mathrm{pr}_{J,I} \xi_J e_J = \mathrm{pr}_{J,I} \mathrm{inc}_J = \mathrm{inc}_I$$

where the last equality follows from the fact that the composition of two subring inclusions is again a subring inclusion. On the other hand, $\xi_I \alpha_{J,I} e_J = \xi_I e_I$ by the compatibility of level structures. Finally, one sees that $\xi_I e_I = \mathrm{inc}_I$ by definition and we are done. □

4.1.4. Recall in 3.5.15 that for all non-zero proper ideals J of A coprime to I, we have a map $J * - : \mathcal{M}_I^1 \to \mathcal{M}_I^1$. In the proof of theorem 3.5.11, this action of ideals coprime with I factors through its image in the ray class group $\mathrm{Cl}(I)$. Let

$\mathcal{G}$ denote the Galois group of maximal abelian extension of K totally splitting at ∞ over K, i.e., $\mathcal{G}$ is the following projective limit of finite Galois groups

$$\mathcal{G} = \varprojlim_I \mathrm{Cl}(I)$$

where I runs over all ideals of A and the map $\mathrm{Cl}(J) \to \mathrm{Cl}(I)$ for all ideals I dividing J is the map of Galois groups $\mathrm{Gal}(K_J/K) \to \mathrm{Gal}(K_I/K)$. We have an action of $\mathcal{G}$ on $\mathcal{M}_I^1$ through the quotient $\mathcal{G} \to \mathrm{Cl}(I)$.

4.1.5. In the proof of proposition 2.3.7, we can view $(A/I)^*/\mathbb{F}_q^*$ as a subgroup of $\mathrm{Cl}(I)$. Furthermore, $(A/I)^*/\mathbb{F}_q^*$ is the kernel of the following map

$$\mathrm{Cl}(I) \to \mathrm{Cl}(1)$$

Thus, the action of the ray class group $\mathrm{Cl}(I)$ on $\mathcal{M}_I^1$ by $*$ induces an action of $(A/I)^*/\mathbb{F}_q^*$ on $\mathcal{M}_I^1$. Let G be the group

$$G := \left(\prod_{\mathfrak{p}} \varprojlim_n (A/\mathfrak{p}^n)^*\right) \Big/ \mathbb{F}_q^*$$

where $\mathfrak{p}$ runs over all prime ideals of A. By viewing $(A/\mathfrak{p}^n)^*/\mathbb{F}_q^*$ as a subgroup of $\mathrm{Cl}(\mathfrak{p}^n)$, we can view G as a subgroup of $\mathcal{G}$ and hence by 4.1.4 we have an action of G on $\mathcal{M}_I^1$.

Intuitively, the action of G on M_I^1 is an action that fixes the Drinfeld module and gives another level structure to E. This is stated more precisely in the proposition below.

Proposition 4.1.6. *Let G be the group together with the action on $\mathcal{M}_I^1$ defined in 4.1.5. Let S be an A-scheme and let $(E, \phi, \iota), (E', \phi', \iota') \in \mathcal{M}_I^1(S)$. There exists $g \in G$ such that*

$$g : (E, \phi, \iota) \mapsto (E', \phi', \iota')$$

if and only if there exists an isomorphism of Drinfeld modules $(E, \phi) \xrightarrow{\sim} (E', \phi')$.

Proof. Suppose $g \in G$, since the group G acts on $\mathcal{M}_I^1$ through a finite quotient, it is enough to show the forward direction for all $g \in (A/I)^*/\mathbb{F}_q^* \subseteq \mathrm{Cl}(I)$.

Let E be a Drinfeld module with level structure as stated in the proposition. Since the group $(A/I)^*/\mathbb{F}_q^*$ is identified as the kernel of $\mathrm{Cl}(I) \to \mathrm{Cl}(1)$, the action of g is given by a principal ideal $\mathfrak{a} = (a) \in A$.

By 3.4.9, there exists $u \in R^*$ such that $(\mathfrak{a} * \phi)_b = \mathrm{Spec}(u) \circ \phi_b \circ \mathrm{Spec}(u^{-1})$ for all $b \in A$. Moreover, by the left exact diagram given in 3.4.10, u defines an

A-module isomorphism on E, so we must have $u \in \mathbb{F}_q^*$ by 3.4.17. This shows the forward direction.

We show the converse by a counting argument, more precisely, we count the number of possible level structure one can put on a Drinfeld module. Since $\mathcal{M}_I^1$ is representable by the ring $A_{K_I}[I^{-1}]$, it is locally a $\coprod_{\mathrm{Cl}(I)} \mathrm{Spec}(A[I^{-1}])$-torsor, and so the action of $(A/I)^*/\mathbb{F}_q^* \subseteq \mathrm{Cl}(I)$ is free. Thus, it is enough to show that the number of possible level-I structures one can put on E equal the size of the group $(A/I)^*/\mathbb{F}_q^*$.

Recall that a level structure is an isomorphism $\iota : I^{-1}/A \xrightarrow{\sim} E_I$, one sees that, by viewing I^{-1}/A as a rank 1 A/I-module, any $\acute{g} \in (A/I)^*$ defines a level structure by pre-composition, i.e., $\iota' = \iota \circ \acute{g}$ is also a level structure on E. Moreover, it is clear that for a fixed ι, we can find such a group element $\acute{g}$ for any other level-I structure.

Finally, by 3.4.17 once again, an element $A^* = \mathbb{F}_q^*$ defines an automorphism of E, so two level structures are considered equal if they differ by an element of A^*. Thus, the possible number of level-I structure we can put on E is $|(A/I)^*/\mathbb{F}_q^*|$ and we are done. □

We define quotient of an object X by a group object G as the categorical quotient of X by G given in definition 0.5 of [16]. More precisely, we define G-invariant maps and G-quotients as the following.

4.1.7. Let G be a group object in a category $\mathcal{D}$ with fiber products. Let G act on X for some object in a category X, so there are two maps $G \times X \to X$ called the action map (denoted as α) and the projection map (denoted as ρ). We say the map between objects $f : X \to Y$ is G-invariant if the following maps are commutative:

$$G \times X \rightrightarrows X \xrightarrow{f} Y$$

i.e., the composition of maps $f \circ \rho = f \circ \alpha$. An object Q together with a map $X \to Q$ is called the quotient of X by G if the map is G-invariant and furthermore, for all Y and G-invariant maps $X \to Y$, there exists a unique morphism $Q \to Y$ such that the following diagram commutes

$$\begin{tikzcd} & Q \arrow[dr] & \\ X \arrow[ur] \arrow[rr] & & Y \end{tikzcd}$$

Note that the quotient of X by G is unique up to unique isomorphism if it exists. We usually denote the object of the quotient by X/G and the quotient map by quo : $X \to X/G$. We sometimes leave the map "quo" implicit and just say "X/G is the quotient of X by G" to imply there is a quotient map "quo" associated with X/G.

Lemma 4.1.8. *Fix a base scheme S. Let G be a finite group acting on a ring R and let R^G denote the G-invariant subring of R. Let G_S be the constant group scheme $\coprod_G S$ and let $X = Spec(R)$, $Y = Spec(R^G)$ be affine S-schemes with the map $f : X \to Y$ induced by the algebra map $f^* : R^G \to R$.*

Suppose f is an étale covering. Furthermore, let α and ρ be the action map and the projection map $G_S \times X \to X$ as defined in 4.1.7 and suppose the map $(\alpha, \rho) : G_S \times X \to X \times X$ induces an isomorphism between $G_S \times X$ and $X \times_Y X$. Then the étale sheaf theoretic quotient of X by G_S, which means the quotient in the category of étale sheaves over S, is Y together with the map $f : X \to Y$.

Proof. By the proof of chapter 1, section 2, theorem 1.1 of [16], $f : X \to Y$ is the quotient in the category of schemes over S. Let X/G denote the quotient of X by G_S in the category of étale sheaves over S. Since schemes are étale sheaves, there exists a unique map $X/G \to Y$ and we want to show this map is an isomorphism.

We observe that the map $X \to X/G$ can be expressed as a coequalizer and $G_S \times X \rightrightarrows X \to X/G$. This is because the universal property of coequalizer is satisfied by definition 4.1.7. On the other hand, since f is an étale covering, $X \times_Y X \rightrightarrows X \to Y$ is also a coequalizer diagram. This follows from the fact that every epimorphism is effective in the category of étale shaves, see theorem IV.7.8 in [15]. Therefore, the map $X/G \to Y$ is an isomorphism if both $X \times_Y X$ and $G_S \times X$ defines the same equivalence relation on X. The latter condition is equivalent to saying the image of $(\alpha, \rho) : G_S \times X \to X \times X$ agrees with the image of the inclusion map $X \times_Y X \hookrightarrow X \times X$, which is true by assumption. Therefore, the map $X/G \to Y$ is an isomorphism. □

Corollary 4.1.9. *Let $S = Spec(A[I^{-1}])$, G be the group defined in 4.1.5 and let $G_S = \coprod_G S$. The étale sheaf theoretic quotient $\mathcal{M}_I^1/G_S$ is $Spec(A_H[I^{-1}])$ together with the map induced by the inclusion map $f_I^* : A_H[I^{-1}] \hookrightarrow A_{K_I}[I^{-1}]$.*

Proof. Although G is not a finite group, the action of G on $\mathcal{M}_I^1$ factors through $U = \mathrm{Gal}(K_I/H) \cong (A/I)^*/\mathbb{F}_q^*$, so we can apply the result in lemma 4.1.8 with the finite group U. Following the notation in 4.1.8, if we take $X = \mathcal{M}_I^1 \cong \mathrm{Spec}(A_{K_I}[I^{-1}])$ and $Y = \mathrm{Spec}(A_H[I^{-1}])$, then $A_H[I^{-1}]$ is the U-invariant subring of $A_{K_I}[I^{-1}]$. Moveover, the map $f_I : X \to Y$ is a finite étale covering.

Furthermore, we see that $(\alpha,\rho) : U_S \times X \xrightarrow{\sim} X \times_Y X$. Indeed, the ring representing the affine scheme $U_S \times X$ is the ring $(\prod_U A[I^{-1}]) \otimes_{A[I^{-1}]} A_{K_I}[I^{-1}]$, which is canonically isomorphic to $\prod_U A_{K_I}[I^{-1}]$. On the other hand, the ring representing $X \times_Y X$ is $A_{K_I}[I^{-1}] \otimes_{A_H[I^{-1}]} A_{K_I}[I^{-1}]$. One sees that, since the fields K_I/H is a finite Galois extension and $A_{K_I}[I^{-1}]$ is finite étale, the map of rings associated with (α,ρ) sends $a \otimes 1 \mapsto \prod_{g\in U} g(a)$, $1 \otimes b \mapsto \prod_{g\in U} b$ for all $a, b \in A_{K_I}[I^{-1}]$ and this map is an isomorphism.

Hence, all hypotheses are satisfied in lemma 4.1.8 and we are done.

$\square$

Remark 4.1.10. Let G be a finite group. Suppose the category $\mathcal{D}$ in 4.1.7 has finite coproducts and a terminal object S (for example, the category of S-sheaves for some scheme S), then it is natural to identify G with the group object $\coprod_G S$. We often abuse notation and just say $G \in \mathcal{D}$ rather than $\coprod_G S \in \mathcal{D}$ in this case. In particular, we often denote the quotient $X/\coprod_G S$ as X/G.

4.1.11. We fix a quotient map quo : $\mathcal{M}_I^1 \to \mathcal{M}_I^1/G$ and an isomorphism δ_I : $\mathcal{M}_I^1/G \to \mathrm{Spec}(A_H[I^{-1}])$ for each I such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{M}_I^1 \arrow[r, "\mathrm{quo}"] \arrow[d, "\xi_I"] & \mathcal{M}_I^1/G \arrow[d, "\delta_I"] \\
\mathrm{Spec}(A_{K_I}[I^{-1}]) \arrow[r, "f_I"] & \mathrm{Spec}(A_H[I^{-1}])
\end{tikzcd}$$

where ξ_I is defined in 4.1.2 and f_I^* is the inclusion map defined in 4.1.9.

4.1.12. We define a presheaf on (A-schemes)

$$\begin{aligned}
|\mathcal{M}_A^1| : (A\text{-schemes})^{\mathrm{op}} &\to (\mathit{Sets}) \\
S &\mapsto \{\text{Isomorphism classes of rank 1} \\
&\qquad\quad \text{Drinfeld modules over } S\}
\end{aligned}$$

Note that we can make this definition because $\mathcal{M}_A^1(S)$ is a groupoid. We define the coarse space of $\mathcal{M}_A^1$ to be $M_A^1 = |\mathcal{M}_A^1|^a$, the sheafification of $|\mathcal{M}_A^1|$ in the site of étale sheaves over Spec(A) (see 2.4.4). We define the coarse map $\mathfrak{m} : \mathcal{M}_A^1 \to M_A^1$ to be the composition of the map

$$\mathcal{M}_A^1 \to |\mathcal{M}_A^1| \to |\mathcal{M}_A^1|^a =: M_A^1$$

4.1.13. We note $\pi * - : \mathcal{M}^1_A \to \mathcal{M}^1_A$ induces a map $M^1_A \to M^1_A$. Indeed, the map $\pi * -$ extends to a map $\mathcal{M}^1_A \to M^1_A$ by post-composition. It is clear that $\pi * -$ factors through the map $\mathcal{M}^1_A \to |\mathcal{M}^1_A|$, so $\pi * -$ induces a map $|\mathcal{M}^1_A| \to M^1_A$, which induces a map $M^1_A \to M^1_A$ by the universal property of sheafification.

4.1.14. Let I be a non-zero proper ideal and let

$$|\mathcal{M}^1_A|[I^{-1}] := |\mathcal{M}^1_A| \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A[I^{-1}]).$$

We construct a map $\beta_I : |\mathcal{M}^1_A|[I^{-1}] \to \mathrm{Spec}(A_H[I^{-1}])$ using the quotient map $\mathrm{quo} : \mathcal{M}^1_I \to \mathcal{M}^1_I/G$ and the isomorphism $\delta_I : \mathcal{M}^1_I/G \to \mathrm{Spec}(A_H[I^{-1}])$ defined in 4.1.11. We show this map β_I realizes $\mathrm{Spec}(A_H[I^{-1}])$ as the sheafification of $|\mathcal{M}^1_A|[I^{-1}]$ in proposition 4.1.17 below.

Let $\mathcal{M}^1_A[I^{-1}] = \mathcal{M}^1_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A[I^{-1}])$, it is clear that every map $\mathcal{M}^1_A[I^{-1}] \to \mathcal{F}$, where $\mathcal{F}$ is a sheaf, factors through $\mathcal{M}^1_A[I^{-1}] \to |\mathcal{M}^1_A|[I^{-1}]$ uniquely, so it is enough to construct a map $\beta'_I : \mathcal{M}^1_A[I^{-1}] \to \mathrm{Spec}(A_H[I^{-1}])$.

In 4.1.1, we have a map $\alpha_{I,A}$, which is the map that forgets the level structure. This induces a map $\mathcal{M}^1_I \to \mathcal{M}^1_A[I^{-1}]$. We slightly abuse notation and also denote this map by $\alpha_{I,A}$ for the rest of this section.

In 4.1.9, we showed that quotient of $\mathcal{M}^1_I$ by G, in the category of étale sheaves over $\mathrm{Spec}(A[I^{-1}])$, is isomorphic to $\mathrm{Spec}(A_H[I^{-1}])$ together with the map induced by the G-invariant subring map $f_I : (A_{K_I}[I^{-1}])^G = A_H[I^{-1}] \hookrightarrow A_{K_I}[I^{-1}]$, where the isomorphism is given and denoted as δ_I in 4.1.11.

We construct β'_I by following a similar approach to the proof of 4.1.8. We have two columns of maps as the following.

$$\begin{tikzcd}
\mathcal{M}^1_I \times_{\mathcal{M}^1_A[I^{-1}]} \mathcal{M}^1_I \arrow[d, shift left] \arrow[d, shift right] & G \times \mathcal{M}^1_I \arrow[d, shift left] \arrow[d, shift right] \\
\mathcal{M}^1_I \arrow[r, equal] \arrow[d] & \mathcal{M}^1_I \arrow[d] \\
\mathcal{M}^1_A[I^{-1}] & \mathcal{M}^1_I/G
\end{tikzcd}$$

We show in 4.1.16 that, the existence of a map $\mathcal{M}^1_I \times_{\mathcal{M}^1_A[I^{-1}]} \mathcal{M}^1_I \to G \times \mathcal{M}^1_I$, such that the following diagram commutes, defines a map $\mathcal{M}^1_A[I^{-1}] \to \mathcal{M}^1_I/G$, which defines a map to $\mathrm{Spec}(A_H[I^{-1}])$ through δ_I.

$$\begin{tikzcd}
\mathcal{M}_I^1 \times_{\mathcal{M}_A^1[I^{-1}]} \mathcal{M}_I^1 \arrow[r, hook] \arrow[d] & \mathcal{M}_I^1 \times \mathcal{M}_I^1 \\
G \times \mathcal{M}_I^1 \arrow[ur, "(\alpha,\rho)"'] &
\end{tikzcd} \tag{4.1.15}$$

Therefore, to construct a map $\beta_I : |\mathcal{M}_A^1|[I^{-1}] \to \mathrm{Spec}(A_H[I^{-1}])$, it is enough to construct a map $\mathcal{M}_I^1 \times_{\mathcal{M}_A^1[I^{-1}]} \mathcal{M}_I^1 \to G \times \mathcal{M}_I^1$ such that diagram 4.1.15 commutes. An element in $\mathcal{M}_I^1 \times_{\mathcal{M}_A^1[I^{-1}]} \mathcal{M}_I^1$ is a pair of isomorphic Drinfeld modules $(E,\phi),(E',\phi')$ with potentially different level-I structures on them. By fixing an isomorphism between them, the pair can be written as (E,ι_1,ι_2), where ι_1,ι_2 are two level structures on E. The image of (E,ι_1,ι_2) in $\mathcal{M}_I^1 \times \mathcal{M}_I^1$ is (E,ι_1,E,ι_2).

By 4.1.6, there exists $g_{2,1} \in G$ such that $g_{2,1} : (E,\iota_2) \mapsto (E,\iota_1)$. We define a map $\mathcal{M}_I^1 \times_{\mathcal{M}_A^1[I^{-1}]} \mathcal{M}_I^1 \to G_{\mathrm{Spec}(A[I^{-1}])} \times \mathcal{M}_I^1$ by $(E,\iota_1,\iota_2) \mapsto (g_{2,1},E,\iota_2)$. One sees that $(\alpha,\rho) : (g_{2,1},E,\iota_2) \mapsto (E,\iota_1,E,\iota_2)$, so the map we constructed makes the diagram in 4.1.15 commutative.

By composition with the isomorphism $\delta_I : \mathcal{M}_I^1/G \to \mathrm{Spec}(A_H)[I^{-1}]$, we have constructed a map $\beta_I' : \mathcal{M}_A^1[I^{-1}] \to \mathrm{Spec}(A_H[I^{-1}])$ and hence a map $\beta_I : |\mathcal{M}_A^1|[I^{-1}] \to \mathrm{Spec}(A_H[I^{-1}])$ as we claimed.

Proposition 4.1.16. *A map*

$$\mathcal{M}_I^1 \times_{\mathcal{M}_A^1[I^{-1}]} \mathcal{M}_I^1 \to G \times \mathcal{M}_I^1$$

induces a map

$$\mathcal{M}_A^1[I^{-1}] \to \mathcal{M}_I^1/G$$

Proof. Fix an $A[I^{-1}]$-scheme S and let $E \in \mathcal{M}_A^1(S)$. Since both $\mathcal{M}_A^1[I^{-1}]$ and $\mathcal{M}_I^1/G$ are stacks in the étale topology, it is enough to work over an étale cover of S. Let S' be a cover of S such that $E' = E \times_S S'$ admits a level-I structure (exists by 3.3.3) and let (E',ι) be a preimage of E' along $\alpha_{I,A}$.

We claim that the map that sends E' to the image of (E',ι) in $\mathcal{M}_I^1/G$ gives a well-defined map $\mathcal{M}_A^1[I^{-1}] \to \mathcal{M}_I^1/G$. Indeed, it is enough to show that this map is independent of the choice the level-I structure on E'. Suppose there are two level-I structure on E', this gives a point in $\mathcal{M}_I^1 \times_{\mathcal{M}_A^1[I^{-1}]} \mathcal{M}_I^1$. Thus, given the commutative diagram in (4.1.15), we see that such a point is in the equivalence relation defined by $G \times \mathcal{M}_I^1$ so the image of (E',ι) in $\mathcal{M}_I^1/G$ is independent of the choice of ι. □

Proposition 4.1.17. *The map β_I defined in 4.1.14 realizes Spec$(A_H[I^{-1}])$ as the sheafification of $|\mathcal{M}^1_A|[I^{-1}]$.*

Proof. Let $\beta'_I : \mathcal{M}^1_A[I^{-1}] \to \mathrm{Spec}(A_H[I^{-1}])$ be the map induced by β_I as shown in 4.1.14. We show the map β'_I obeys the universal property of sheafification. In other words, given a sheaf $\mathcal{F}$ and a map $\gamma_I : \mathcal{M}^1_A[I^{-1}] \to \mathcal{F}$, we need to show there exists a unique map $\gamma'_I : \mathrm{Spec}(A_H[I^{-1}]) \to \mathcal{F}$ (indicated by a dotted arrow below) such that the following diagram commutes.

$$\begin{tikzcd}
 & \mathrm{Spec}(A_H[I^{-1}]) \arrow[d, dashed, "\gamma'_I"] \\
\mathcal{M}^1_A[I^{-1}] \arrow[ur, "\beta_I"] \arrow[r, "\gamma_I"] & \mathcal{F}
\end{tikzcd} \tag{4.1.18}$$

Given a map $\gamma_I : \mathcal{M}^1_A[I^{-1}] \to \mathcal{F}$, there exists a map $\mathcal{M}^1_I \to \mathcal{F}$ by precomposition with $\alpha_{I,A}$. Furthermore, this composition of maps is G-invariant since the map $\alpha_{I,A} : \mathcal{M}^1_I \to |\mathcal{M}^1_A|[I^{-1}]$ is G-invariant by 4.1.6. Therefore, by the universal property of G-quotient, there exists a unique map γ'_I such that the square in the following diagram commutes.

$$\begin{tikzcd}
\mathcal{M}^1_I \arrow[r, "\delta_I \circ \text{quo}"] \arrow[d, "\alpha_{I,A}"'] & \mathrm{Spec}(A_H[I^{-1}]) \arrow[d, "\gamma'_I"] \\
|\mathcal{M}^1_A|[I^{-1}] \arrow[ur, "\beta_I"] \arrow[r, "\gamma_I"] & \mathcal{F}
\end{tikzcd}$$

Furthermore, the upper left triangle is the diagram above commutes by construction. Therefore, the lower right triangle must also commute and we are done. □

Proposition 4.1.19. *Let β_I be the map introduced in 4.1.17. The collection of β_I for all non-zero proper ideal I extends to a (necessarily unique) map*

$$\beta : |\mathcal{M}^1_A| \to \mathit{Spec}(A_H)$$

Proof. Let $\mathcal{M}^1_A[I^{-1}] = \mathcal{M}^1_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A[I^{-1}])$. Given β_I, we have a map

$$\beta'_I : \mathcal{M}^1_A[I^{-1}] \to \mathrm{Spec}(A_H[I^{-1}])$$

defined by precomposing with the map $\mathcal{M}^1_A[I^{-1}] \to |\mathcal{M}^1_A|[I^{-1}]$ (see 4.1.12) and taking the fiber product. Once again, it is clear that a map from $\mathcal{M}^1_A$ to a presheaf over Spec(A) factors uniquely through the map $\mathcal{M}^1_A \to |\mathcal{M}^1_A|$, so it is enough to show that, given a collection of β'_I, we can define a map $\beta' : \mathcal{M}^1_A \to \mathrm{Spec}(A_H)$.

Since $\mathcal{M}^1_A$ is a stack, for each A-scheme S, we can replace $\mathcal{M}^1_A(S)$ with the category of descend data (recall the definition in 2.5.8) with respect to the Zariski topology, $\mathcal{M}^1_A(\{S[I^{-1}] \to S\})$ where I ranges through all non-zero proper ideals of A. We repeat the process for Spec$(A_H)(S)$ and we show the collection of β'_I defines a morphism between these two categories of descend data. In other words, following definition 2.5.9, we need to show that for all ideals $J \subseteq I$, a Drinfeld module E and its image $E^{(I)}$, $E^{(J)}$ in $\mathcal{M}^1_A(S[I^{-1}])$ $\mathcal{M}^1_A(S[J^{-1}])$, the following diagram commutes.

$$\begin{tikzcd}
E^{(I)} \arrow[r, "\beta'_I"] \arrow[d, "\sim"] & \beta'_I E^{(I)} \arrow[d, "="] \\
\mathrm{pr}(E^{(J)}) \arrow[r, "\mathrm{pr}\beta'_J"] & \mathrm{pr}(\beta'_J E^{(J)})
\end{tikzcd}$$

where pr denote the map induced by the A-scheme map pr : $S[J^{-1}] \to S[I^{-1}]$ on each corresponding fibered category. Note that the isomorphism on each column follows from the definition of descend data and the right column is an equality since Spec$(A_H[J^{-1}])$ is a sheaf. We observe that the commutativity of the diagram above is equivalent to the commutativity of the diagram below.

$$\begin{tikzcd}
\mathcal{M}^1_A[J^{-1}] \arrow[r, "\beta'_J"] \arrow[d, "\mathrm{pr}"] & \mathrm{Spec}(A_H[J^{-1}]) \arrow[d, "\mathrm{pr}"] \\
\mathcal{M}^1_A[I^{-1}] \arrow[r, "\beta'_I"] & \mathrm{Spec}(A_H[I^{-1}])
\end{tikzcd}$$

We show commutativity of the diagram by expanding the diagram on the left hand side.

$$\begin{tikzcd}
\mathcal{M}^1_J \arrow[r, "\mathrm{pr}^*\alpha_{J,I}"] \arrow[d, "\alpha_{J,I}"] & \mathcal{M}^1_A[J^{-1}] \arrow[r, "\beta'_J"] \arrow[d, "\mathrm{pr}"] & \mathrm{Spec}(A_H[J^{-1}]) \arrow[d, "\mathrm{pr}"] \\
\mathcal{M}^1_I \arrow[r, "\alpha_{I,A}"] & \mathcal{M}^1_A[I^{-1}] \arrow[r, "\beta'_I"] & \mathrm{Spec}(A_H[I^{-1}])
\end{tikzcd}$$

where $\mathrm{pr}^*\alpha_{J,I}$ denotes the map defined by the pull back of $\alpha_{J,I}$ along pr.

It is immediate from the definition that the left square commutes, meaning that there is a canonical isomorphism between the two objects obtained by following the different paths. Therefore, it is enough to show the big rectangle commutes.

Recall the following isomorphisms introduced in 4.1.11 for all I:

$$\xi_I : \mathcal{M}^1_I \to \mathrm{Spec}(A_{K_I}[I^{-1}])$$

$$\delta_I : \mathcal{M}^1_I/G \to \mathrm{Spec}(A_H[I^{-1}])$$

such that the diagrams in 4.1.2 and 4.1.11 commute. Under these isomorphisms, the quotient map $\mathrm{quo} : \mathcal{M}^1_I \to \mathcal{M}^1_I/G$ becomes the map

$$f_I : \mathrm{Spec}(A_{K_I}[I^{-1}]) \to \mathrm{Spec}(A_H[I^{-1}]),$$

which is defined by an inclusion of A-algebras $f_I^* : A_H[I^{-1}] \to A_{K_I}[I^{-1}]$. Similarly, the map f_J^* is also an inclusion, so the map $\beta'_J \circ \mathrm{pr}^*\alpha_{I,A}$ in the top row of the diagram is defined by an inclusion. Finally, the vertical maps are defined by maps of rings $A_H[I^{-1}] \hookrightarrow A_H[J^{-1}]$ and $A_{K_I}[I^{-1}] \hookrightarrow A_{K_I}[J^{-1}]$. Since both maps mapping $A_H[I^{-1}] \to A_{K_I}[J^{-1}]$ are inclusions, it is clear that the big rectangle commutes and we are done. □

Theorem 4.1.20. *The map $\beta : |\mathcal{M}^1_A| \to$ Spec(A_H) defined in 4.1.19 realizes Spec(A_H) as the sheafification of $|\mathcal{M}^1_A|$.*

Proof. First of all, we keep the all notation introduced in 4.1.17 and 4.1.19, especially the ones in diagram (4.1.18). In addition, we define pr_I to be a map induced by $S[I^{-1}] \to S$ on each fibered category.

To show the theorem holds, it is enough to show the following: Given a map $\gamma : \mathcal{M}^1_A \to \mathcal{F}$ where $\mathcal{F}$ is an étale sheaf over Spec(A), for all ideas $J \subseteq I$, the triangle on the right marked with dotted lines in following diagram commutes.

$$\begin{tikzcd}
\mathcal{M}^1_J \arrow[r, "\alpha_{J,A}"] \arrow[d, "\alpha_{J,I}"'] & \mathcal{M}^1_A[J^{-1}] \arrow[rr, "\beta_J"] \arrow[d, "\mathrm{pr}"] & & \mathrm{Spec}(A_H[J^{-1}]) \arrow[dl, dashed, "\mathrm{pr}"] \arrow[dd, dashed, "\gamma'_J"] \\
\mathcal{M}^1_I \arrow[r, "\alpha_{I,A}"] & \mathcal{M}^1_A[I^{-1}] \arrow[r, "\beta_I"] \arrow[d, "\mathrm{pr}_I"] & \mathrm{Spec}(A_H[I^{-1}]) \arrow[dr, dashed, "\gamma'_I"] & \\
& \mathcal{M}^1_A \arrow[rr, "\gamma"] & & \mathcal{F}
\end{tikzcd}$$

We remark that the whole diagram other than the dotted triangle is commutative, which can either be shown directly from definition or in the proof of 4.1.17 or 4.1.19. Indeed, it is clear that the small square on the left commutes. Furthermore, the top trapezium in the diagram commutes by the proof of 4.1.19 and the bottom trapezium commutes because by definition, γ'_I is the unique map such that $\mathrm{pr}_I \circ \mathrm{pr}_I^* \gamma = \gamma'_I \circ \beta_I$ (here $\mathrm{pr}_I : \mathcal{F}[I^{-1}] \to \mathcal{F}$ and $\mathrm{pr}_I^*\gamma$ denotes the pull-back). Similarly, the big square with β_J as its top edge and γ as its bottom edge commutes by the definition of the map γ'_J.

Since the map $\beta_J \circ \alpha_{J,A} : \mathcal{M}_J^1 \to \mathrm{Spec}(A_H[J^{-1}])$ is a G-quotient, it is a coequalizer and thus an epimorphism. So it is enough to show the precomposition of $\beta_J \circ \alpha_{J,A}$ with the maps in the dotted line commutes. It remains to show the commutativity of the dotted arrows by diagram chasing. We have

$$\begin{aligned}
&\gamma'_J \circ \beta_J \circ \alpha_{J,A} \\
=&\gamma \circ \mathrm{pr}_I \circ \mathrm{pr} \circ \alpha_{J,A} && \text{(commutativity of the big square)} \\
=&\gamma'_I \circ \beta_I \circ \mathrm{pr} \circ \alpha_{J,A} && \text{(commutativity of the bottom trapezium)} \\
=&\gamma'_I \circ \mathrm{pr} \circ \beta_J \circ \alpha_{J,A} && \text{(commutativity of the top trapezium)}
\end{aligned}$$

□

Corollary 4.1.21. *Let π be a prime and let $\mathfrak{m} : \mathcal{M}_A^1 \to M_A^1$ denote the coarse map defined in 4.1.12. The map $M_A^1 \to M_A^1$ induced by $E \mapsto \pi * E$ (exists by 4.1.13) is the unique Frobenius lift at the prime π.*

Proof. Let I be an ideal of A coprime with π. We have the forgetful map of level structure $\mathcal{M}_I^1 \to \mathcal{M}_A^1$ and the coarse map $\mathcal{M}_A^1 \to M_A^1$. These two maps compose and define a map of affine schemes $\mathcal{M}_I^1 \to M_A^1$. By 4.1.20 and the uniqueness of sheafification, the coarse map $\mathfrak{m}$ is given by the map $\beta : \mathcal{M}_A^1 \to \mathrm{Spec}(A_H)$, thus the map of affine schemes can be viewed as an inclusion of A-algebras $A_H \to A_{K_I}[I^{-1}]$. We see in proposition 3.5.12 that $\pi * - : \mathcal{M}_I^1 \to \mathcal{M}_I^1$ is the Frobenius automorphism of $\mathcal{M}_I^1$, so $\pi * -$ also defines a Frobenius automorphism of M_A^1 by viewing A_H as a subring of $A_{K_I}[I^{-1}]$. From class field theory, the Frobenius lift of a prime ideal π of A in A_H is unique, so $\pi * -$ must be the unique one. □

4.2 Periodic Loci and Abelian Extensions

The aim is to present a scheme, of finite type, such that the periodic loci generates ray class fields of K. The case for elliptic curves done in [10], the idea of such a scheme is obtained by forming the quotient of the universal elliptic curve over the moduli stack of CM elliptic curves. We will follow the same idea by first defining the universal rank 1 Drinfeld modules and construct our scheme from there.

4.2.1. We have a canonical isomorphism of A-algebras

$$\begin{aligned} A_H \otimes_A A[x] &\to A_H[x] \\ a \otimes_A b &\mapsto ab \end{aligned}$$

So $\operatorname{Spec}(A_H[x]) \cong \operatorname{Spec}(A_H) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A[x])$ and the isomorphism is given by the map above. We always refer to this canonical isomorphism whenever we say "$\operatorname{Spec}(A_H[x])$ is isomorphic to $\operatorname{Spec}(A_H) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A[x])$". We often say $(e, y) \in \operatorname{Spec}(A_H[x])$ for some $e \in \operatorname{Spec}(A_H)$ and $y \in \operatorname{Spec}(A[x])$ using this isomorphism.

We also have the canonical projections

$$\operatorname{pr}_{A_H} : \operatorname{Spec}(A_H[x]) \to \operatorname{Spec}(A_H)$$

corresponding to the map of rings induced by induced by $a \mapsto a \otimes_A 1$.

4.2.2. Recall the definition of the ring $\mathcal{R}$ and the rank 1 universal standard coordinatized Drinfeld A-module $\mathcal{E}^{sc}$ in 3.1.22. There exists a map $\operatorname{fgt} : \operatorname{Spec}(\mathcal{R}) \to \mathcal{M}_A^1$ which is defined by forgetting the coordinates.

Following 4.2.1, we may write $\mathcal{E}^{sc} = \operatorname{Spec}(\mathcal{R}) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A[x])$ and we have the canonical projection map $\operatorname{pr}_{\mathcal{R}}$ similarly defined.

We define a map $\mathfrak{c}' : \mathcal{E}^{sc} \to \operatorname{Spec}(A_H[x])$ as follows. Let $(E, y) \in \mathcal{E}^{sc}$ using the fiber product described above by

$$\mathfrak{c}' : (E, y) \mapsto (\beta(\operatorname{fgt}(E)), y^{q-1})$$

where β is the coarse map defined in 4.1.19. In summary, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{E}^{sc} \arrow[r, "\mathfrak{c}'"] \arrow[d, "\operatorname{pr}_{\mathcal{R}}"] & \operatorname{Spec}(A_H[x]) \arrow[d, "\operatorname{pr}_{A_H}"] \\
\mathcal{R} \arrow[r, "\beta \circ \operatorname{fgt}"] & \operatorname{Spec}(A_H)
\end{tikzcd}$$

Definition 4.2.3. The universal rank 1 Drinfeld A-module $\mathcal{E}$, is a stack over $\mathcal{M}_A^1$ such that the S-points of $\mathcal{E}$ is defined as

$$\mathcal{E}(S) = \{(E(S), y) \mid E \in \mathcal{M}_A^1, y \in E(S)\}$$

We have a forgetful map $\mathcal{E}^{sc} \to \mathcal{E}$ defined by the pull-back of the forgetful map $\mathrm{fgt} : \mathrm{Spec}(\mathcal{R}) \to \mathcal{M}_A^1$, which we slightly abuse notation and also call fgt.

Lemma 4.2.4. *The map $\mathfrak{c}' : \mathcal{E}^{sc} \to Spec(A_H[x])$ introduced in 4.2.2 factors through the forgetful map $fgt : \mathcal{E}^{sc} \to \mathcal{E}$.*

Proof. It is enough to show that, for a Drinfeld module (E, ϕ), the image under $\mathfrak{c}'$ is independent of the choice of standard coordinates we put on (E, ϕ). Write an element in $\mathcal{E}$ as a pair (E, y), it is clear that the first component E in the pair is independent of choices of coordinates by definition. Moreover, by 3.4.17 any two such choices differ by an element in $\mathbb{F}_q^*$, i.e., if the two pre-images of (E, y) under fgt are (E_1, y_1) and (E_2, y_2), then there exists $\mu \in \mathbb{F}_q^*$ such that $y_2 = \mu y_1$. It is clear that the $(q-1)$-power map is fixed under any choice of μ. □

4.2.5. We denote the map shown in 4.2.4 as $\mathfrak{c}$, i.e., $\mathfrak{c}' = \mathfrak{c} \circ \mathrm{fgt}$. Thus, following same notation in 4.2.2, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\mathfrak{c}"] \arrow[d, "\mathrm{pr}_{\mathcal{M}_A^1}"] & \mathrm{Spec}(A_H[x]) \arrow[d, "\mathrm{pr}_{A_H}"] \\
\mathcal{M}_A^1 \arrow[r, "\beta"] & \mathrm{Spec}(A_H)
\end{tikzcd}$$

4.2.6. By 4.2.4, the maps $\mathfrak{c}$ and $\mathfrak{c}'$ defines each other. Therefore, every time we work with the map $\mathfrak{c}$, we almost always work with $\mathfrak{c}'$ instead and assume the Drinfeld module has a standard coordinate on it without further explanation.

The map $x \mapsto x^{q-1}$ is invariant under replacing x by μx for all $\mu \in \mathbb{F}_q^*$. Informally, one may think of the map $\mathfrak{c}$ as the coarse map together with the "quotient by A^*" map.

We make the following definition for Drinfeld modules with coordinates and the universal Drinfeld modules with coordinates.

Let P denote the set of prime ideals of A and let $\Lambda = \Lambda_P = \Lambda(P)$. Let X be a scheme, we say a family of maps $\psi_{\mathfrak{p}} \in \mathrm{End}(X)$, one for each $\mathfrak{p} \in P$, defines a Λ-structure on a scheme X if X together with $\{\psi_{\mathfrak{p}}\}$ is a $\Lambda_{A,P}$-scheme defined in 0.0.5.

We proceed with the following idea: the action of ideals of A should define a Λ-structure on $\mathcal{E}$ and this Λ-structure should extend to a Λ-structure on $\mathrm{Spec}(A_H[x])$. We are not going to explain what a Λ-structure on a stack is, but we show that the action of ideals of A defines a Λ-structure on $\mathrm{Spec}(A_H[x])$.

Lemma 4.2.7. *Let $Spec(A_H[x]) \cong Spec(A_H) \times_{Spec(A)} Spec(A[x])$ be the isomorphism defined in 4.2.1. Let $(e, y^{q-1}) \in Spec(A_H) \times_{Spec(A)} Spec(A[x])$ be a point in $Spec(A_H[x])$. Let π be a prime ideal of A, the action*

$$\psi_\pi((e, y^{q-1})) = (\phi_\pi(e), \phi_\pi(y)^{q-1})$$

where the map $e \mapsto \phi_\pi(e)$ is the map $\phi_I : M_A^1 \to M_A^1$ defined in corollary 4.1.21 and $\phi_\pi(y)$ is the evaluation of the additive polynomial ϕ_π at y, defines a Λ-structure on $Spec(A_H[x])$. In other words, the map ψ_π lifts the Frobenius map at the prime π.

Proof. By identifying $\mathrm{Spec}(A_H[x]) \cong \mathrm{Spec}(A_H) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A[x])$, it is enough to show the map ψ_π acts as the Frobenius lift on each component. More precisely, we will show the restriction maps $\phi_\pi|_{\mathrm{Spec}(A_H)} : \mathrm{Spec}(A_H) \to \mathrm{Spec}(A_H)$ and $\phi_\pi|_{\mathrm{Spec}(A[x])} : \mathrm{Spec}(A[x]) \to \mathrm{Spec}(A[x])$ are both lifts of the Frobenius at the prime π.

The map $\phi_\pi|_{\mathrm{Spec}(A_H)} : \mathrm{Spec}(A_H) \to \mathrm{Spec}(A_H)$ given by $\psi_\pi(\beta(E)) = \beta(\pi * E)$ defines a Λ-structure on $\mathrm{Spec}(A_H[x])$ by 4.1.21. It remains to show that the map $\phi_\pi|_{\mathrm{Spec}(A[x])} : \mathrm{Spec}(A[x]) \to \mathrm{Spec}(A[x])$ defined as $x^{q-1} \to \phi_\pi(x)^{q-1}$ is a Frobenius lift. Firstly, we note the expression $\phi_\pi(x)^{q-1}$ is indeed a polynomial in x^{q-1} since for some $a \in A^*$

$$\phi_\pi(x)^{q-1} = (ax(x^{q^{deg(\pi)}-1} + \cdots + a_1 x^{q-1} + a_0))^{q-1} \tag{4.2.8}$$

and each exponent of x in the bracket on the right hand side is of the form $q^n - 1$ for some natural number n and hence the term in the bracket is a polynomial in x^{q-1}.

Furthermore, the congruence condition

$$(x^{q-1})^{q^{deg(\pi)}} \equiv \phi_\pi(x)^{q-1} \bmod \pi$$

is clear by 3.5.13. □

We proceed by defining what we meant by a periodic locus in precise terms by following definitions in chapters 6 to 8 in [2]. Let X be an A-scheme with a Λ_P-structure, where P be a set of places of A.

4.2.9. We define the Frobenius lift at ∞ of $M_A^1 \cong \mathrm{Spec}(A_H)$ to be the identity map. We make this convention because H is a field that is totally splitting at ∞ in the sense of class field theory, so defining $\psi_\infty = \mathrm{id}$ makes the most sense. Similarly, we define the Frobenius lift at ∞ of $\mathrm{Spec}(A_H)[x]$ to be the identity map.

We recall the definition of $\mathfrak{f}$-equivalence in definition 0.0.6. Two cycles $\mathfrak{a}$ and $\mathfrak{b}$ (recall this means a formal finite product of places of K) are $\mathfrak{f}$-equivalent if they both share the same greatest common divisor with $\mathfrak{f}$ and the image of them after dividing the greatest common divisor agrees in $\mathrm{Cl}(\mathfrak{f}')$, where $\mathfrak{f}'$ is the cycle obtained by dividing $\mathfrak{f}$ with the greatest common divisor. Note that the ray class groups $\mathrm{Cl}(\mathfrak{f}')$ are defined to be totally splitting at ∞ as in definition 2.3.8.

Proposition 4.2.10. *Let $\mathfrak{a}, \mathfrak{b}$ be cycles of K and assume that ∞ does not divide $\mathfrak{a}$ or $\mathfrak{b}$. Then $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}$ if and only if $\mathfrak{a} = x\mathfrak{b}$ for some element $x \in 1 + \mathfrak{f}\mathfrak{b}^{-1}$.*

Proof. Since ∞ does not divide $\mathfrak{a}$ or $\mathfrak{b}$, so we can view $\mathfrak{a}$ and $\mathfrak{b}$ as ideals of A. Following the proof in proposition 4.3 of [2] and unpacking the definitions and results in 2.3.1, 2.3.10 and 0.0.6, $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}$ if and only if there exists $x \in K^*$ satisfying

1. $\mathfrak{a}\mathfrak{d}^{-1} = x\mathfrak{b}\mathfrak{d}^{-1}$;
2. For each place π of K, if $1 \leq n_\pi := \mathrm{ord}_\pi(\mathfrak{f}\mathfrak{d}^{-1})$, then $x \equiv 1 \bmod \pi^{n_\pi}$.

where $\mathfrak{d} = \gcd(\mathfrak{a}, \mathfrak{f}) = \gcd(\mathfrak{b}, \mathfrak{f})$ and π denotes all non-zero prime ideals of A. Note that condition 1 is the same as $\mathfrak{a} = x\mathfrak{b}$, so we only need to show that, under condition 1, condition 2 is the same as $x \in 1 + \mathfrak{f}\mathfrak{b}^{-1}$. Let us denote $m_\pi = \mathrm{ord}_\pi(\mathfrak{f}\mathfrak{b}^{-1})$ where π is a place and we want to show that $x \equiv 1 \bmod \pi^{m_\pi}$. For π such that $\mathrm{ord}_\pi(\mathfrak{f}) \geq \mathrm{ord}_\pi(\mathfrak{b})$, $n_\pi = m_\pi$ so condition 2 holds in this case.

For the remaining case where $\mathrm{ord}_\pi(\mathfrak{f}) < \mathrm{ord}_\pi(\mathfrak{b})$, we must have $n_\pi = 0$ so condition 2 does not apply. Therefore, it suffices to show that $x \equiv 1 \bmod \pi^{m_\pi}$. Since $\gcd(\mathfrak{a}, \mathfrak{f}) = \gcd(\mathfrak{b}, \mathfrak{f})$, $\mathrm{ord}_\pi(\mathfrak{f}) \leq \mathrm{ord}_\pi(\mathfrak{a})$ and hence

$$\begin{aligned}\mathrm{ord}_\pi(x-1) &\geq \min(\mathrm{ord}_\pi(x), 0) = \min(\mathrm{ord}_\pi(\mathfrak{a}\mathfrak{b}^{-1}), 0)\\ &\geq \min(\mathrm{ord}_\pi(\mathfrak{f}\mathfrak{b}^{-1}), 0) = \min(m_\pi, 0) = m_\pi\end{aligned}$$

□

Recall the definition of the periodic locus in definition 0.0.7. For $\mathfrak{f}$ a cycle of K, X a Λ-scheme,

$$X(\mathfrak{f}) = \bigcap_{\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}} X(\psi_{\mathfrak{a}} = \psi_{\mathfrak{b}})$$

4.2.11. Let us from now on refer $\mathrm{Spec}(A_H[x])$ as $\mathcal{L}$, an affine line over $\mathrm{Spec}(A_H)$. We will always view $\mathcal{L}$ through the isomorphism $\mathrm{Spec}(A_H[x]) \cong \mathrm{Spec}(A_H) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A[x])$ defined in 4.2.1, so we will always represent an element of $\mathcal{L}$ as $(e, y) \in \mathcal{L}$ for some $e \in \mathrm{Spec}(A_H)$ and $y \in \mathrm{Spec}(A[x])$.

4.2.12. Let $\mathrm{pr}_{A_H} : \mathcal{L} \to \mathrm{Spec}(A_H)$ be the map defined in 4.2.1. We define a section of pr_{A_H}, denoted as x_0, by choosing the point $x = 0$. I.e.,

$$\begin{aligned} x_0 : \mathrm{Spec}(A_H) &\to \mathcal{L} \\ e &\mapsto (e, 0) \end{aligned}$$

Definition 4.2.13. Let $\mathcal{L}$ be the scheme in 4.2.11 as before. We define $\mathcal{L}[\mathfrak{f}]$ to be the subscheme of $\mathcal{L}$ containing elements in the image of x_0 defined in 4.2.12 under the map $\psi_{\mathfrak{f}}$. In other words, using the notation in 4.2.1, $\mathcal{L}[\mathfrak{f}]$ consists of pairs of the form (e, c) where $c \mapsto 0$ under $\psi_{\mathfrak{f}}|_{\mathrm{Spec}(A[x])}$.

Remark 4.2.14. It is clear that $\mathcal{L}[\mathfrak{f}]$ is a closed subscheme of $\mathcal{L}$, because it is the pre-image of $\{0\} \times \mathrm{Spec}(A_H)$ under $\psi_{\mathfrak{f}}$.

It is also clear that if $\mathfrak{f}' \mid \mathfrak{f}$, then $\mathcal{L}[\mathfrak{f}'](R) \subseteq \mathcal{L}[\mathfrak{f}](R)$ and $\mathcal{L}(\mathfrak{f}')(R) \subseteq \mathcal{L}(\mathfrak{f})(R)$.

Remark 4.2.15. The torsion loci are only defined for cycles supported away from ∞. For a general cycle, we will write $\mathfrak{f} = \mathfrak{f}_{\mathrm{fin}}\infty^n$ such that $\infty \nmid \mathfrak{f}_{\mathrm{fin}}$ and conventionally define $\mathcal{L}[\mathfrak{f}] = \mathcal{L}[\mathfrak{f}_{\mathrm{fin}}]$. We make this convention by following 4.2.9.

Proposition 4.2.16. *The field over K generated by the torsion loci of $\mathcal{L}$, i.e., the field*

$$\bigcup_{\mathfrak{f}} K(\mathcal{L}[\mathfrak{f}](\mathcal{C}))$$

is the maximal abelian extension of K, totally splitting at ∞.

Note that $K(\mathcal{L}[\mathfrak{f}](\mathcal{C}))$ means the smallest subfield F of $\mathcal{C}$ containing K such that F contains all the $\mathcal{C}$-points of $\mathcal{L}[\mathfrak{f}]$. In other words, F is the smallest field such that every morphism $\mathrm{Spec}(\mathcal{C}) \to \mathcal{L}[\mathfrak{f}]$ factors through a morphism $\mathrm{Spec}(F) \to \mathcal{L}[\mathfrak{f}]$ which commutes with the map $\mathrm{Spec}(\mathcal{C}) \to \mathrm{Spec}(F)$.

Proof. Fix a cycle $\mathfrak{f}$ which we may assume to be coprime with ∞ by remark 4.2.15. We will show that

$$K(\mathcal{L}[\mathfrak{f}](\mathcal{C})) = K_{\mathfrak{f}}$$

We view $\mathcal{L} \cong \mathrm{Spec}(A[x]) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_H)$ as a fiber product and $\mathcal{L}[\mathfrak{f}]$ is the pre-image of $\{0\} \times \mathrm{Spec}(A_H)$ under the map $\psi_{\mathfrak{f}} : \mathcal{L} \to \mathcal{L}$ as shown in remark

4.2.14. Note that $\psi_{\mathfrak{f}}$ acts on each component of $\mathcal{L}$ so to find the pre-image of $\{0\} \times \mathrm{Spec}(A_H)$, it is enough to find the pre-image of $\{0\}$ and $\mathrm{Spec}(A_H)$ in each of the component. The restriction $\psi_{\mathfrak{f}} : \mathrm{Spec}(A_H) \to \mathrm{Spec}(A_H)$ acts as the Frobenius map by corollary 4.1.21, in particular, the $\mathcal{C}$-points of $\mathrm{Spec}(A_H)$ is a bijection under $\psi_{\mathfrak{f}}$. Therefore,

$$K(\mathcal{L}[\mathfrak{f}](\mathcal{C})) = K(A_H, \bigcup c) = H(\bigcup c)$$

where the union runs over all c on all Drinfeld modules (E, ϕ) over $\mathcal{C}$ such that $\phi(c^{1/(q-1)})^{q-1} = 0$.

Let $\alpha^{q-1} = c$ for some $\alpha \in \mathcal{C}$. This is equivalent to $\phi_{\mathfrak{f}}(\alpha) = 0$ for each ϕ such that (E, ϕ) is a rank 1 Drinfeld A-module.

Fix a Drinfeld modules (E, ϕ) and let α be a primitive root of $\phi_{\mathfrak{f}}$, then we claim that the field $H(\bigcup c)$ is the field $H(\alpha^{q-1})$. It is clear that $H(\bigcup c) \supseteq H(\alpha^{q-1})$. For the reverse inclusion, it is enough to show that every c is contained in $H(\alpha^{q-1})$. For each c, there exists an element $\omega \in \mathcal{C}$ such that $c = \omega^{q-1}$ and for some Drinfeld module (E', ϕ') together with a coordinate so that ϕ' can be viewed as a map $\phi' : A \to \mathcal{C}[\tau]$ and $\phi'_{\mathfrak{f}}(\omega)^{q-1} = 0$. Let α' be a primitive root of $\phi'_{\mathfrak{f}}$. Then $\omega' = \phi'_a(\alpha')$ for some $a \in A$ and hence $\omega'^{q-1} = \phi'_a(\alpha')^{q-1}$ which we showed in equation (4.2.8) is a polynomial in α'^{q-1}. Therefore, for all c, we have

$$H(c) = H(\omega^{q-1}) \subseteq H(\alpha'^{q-1}) = H(\alpha^{q-1})$$

where the last equality comes from remark 3.5.18. Now we are done since proposition 3.5.17 shows $H(\alpha^{q-1})$ is the ray class field of conductor $\mathfrak{f}$. □

4.2.17. Let (E, ϕ) be a Drinfeld module over $\mathcal{C}$, and we fix coordinates on E so let us assume $\phi : A \to \mathcal{C}[\tau]$. Recall that for each ideal I of A, we have an additive polynomial ϕ_I (see 3.3.8). Since E is a Drinfeld module over $\mathcal{C}$, $\phi_I(x)$ splits as a product as shown in 3.3.4.

Let $\mathcal{C}[[\tau]]$ denote the ring of formal power series in τ as in section 1 of [8], i.e., elements in $\mathcal{C}[[\tau]]$ is of the form $\sum_{i \in \mathbb{N}} a_i \tau^i$ where $a_i \in \mathcal{C}$. For all non-zero additive polynomial $f(\tau) \in \mathcal{C}[\tau]$, $1/f$ exists in $\mathcal{C}[[\tau]]$.

For a non-zero proper ideal I of A, we wish to define $\phi_{I^{-1}} \in \mathcal{C}[[\tau]]$. It is tempting to define $\phi_{I^{-1}} = \phi_I^{-1}$, but this is not the correct definition as the following relation, stated in lemma 3.4.11, holds for ideals I, J of A:

$$\phi_{IJ} = (J * \phi)_I \phi_J \tag{4.2.18}$$

Therefore, it is more natural to give the following definition of $\phi_{I^{-1}}$ that follows the relation above.

Definition 4.2.19. Let I be an ideal of A and hence I^{-1} is a fractional ideal. The Drinfeld module $I^{-1} * \phi$ is defined by $J * \phi$ where J is an ideal of A such that J is the inverse of I in the class group $\mathrm{Cl}_1(1)$. Then $\phi_{I^{-1}} \in \mathcal{C}[[\tau]]$ is defined as

$$\phi_{I^{-1}} := (J * \phi)_I^{-1}$$

This definition of $\phi_{I^{-1}}$ is independent of the choice of J, since the $*$ action on $\mathcal{M}_A^1(\mathcal{C})$ makes it a $\mathrm{Cl}_1(1)$-torsor by corollary 3.5.8.

For a general fractional ideal T of A, we decompose T into prime ideals and inverses of prime ideals and repetitively apply equation (4.2.18) to define ϕ_T. We will show in lemma 4.2.21 that $\phi_T(\tau)$ defined this way is independent of the order in which we apply equation (4.2.18).

Remark 4.2.20. The definition above is based on the relation

$$1 = \phi_1 = \phi_{II^{-1}} = (I * \phi)_{I^{-1}} \phi_I$$

So by replacing ϕ with $I^{-1} * \phi$ and noting that for all ideals I, J, we have $I * (J * \phi) = IJ * \phi$ (stated in lemma 3.4.11), we have our definition of $\phi_{I^{-1}}$. Alternatively, we have

$$1 = \phi_1 = \phi_{II^{-1}} = (I^{-1} * \phi)_I \phi_{I^{-1}}$$

which also gives our definition in a more direct way.

Lemma 4.2.21. *Let T be a fractional ideal of A. Then $\phi_T(\tau) \in \mathcal{C}[[\tau]]$ is independent of the order in which we apply equation (4.2.18).*

In other words, lemma 4.2.21 shows that ϕ_T is well-defined.

Proof. Let (E, ϕ, γ) be a coordinatized Drinfeld module over $\mathcal{C}$ and T be a factional ideal of A. Suppose $T = I^m J^n$, for some fractional ideals I, J of A and we want to compute ϕ_T using definition 4.2.19. On one hand, $\phi_T = (J^n * \phi)_{I^m} \phi_{J^n}$ and on the other hand, $\phi_T = (I^m * \phi)_{J^n} \phi_{I^m}$. Therefore, ϕ_T is well defined if the following equation is true.

$$(J^n * \phi)_{I^m} \phi_{J^n} = (I^m * \phi)_{J^n} \phi_{I^m} \tag{4.2.22}$$

Since definition 4.2.19 defines ϕ_T through its ideal part and its purely fractional part (a fractional ideal is purely fractional if it is an inverse of an ideal), it is enough to consider the special case where I, J are ideals of A and m, n takes values 1 or -1.

The case $m = n = 1$ is the case where both I and J are ideals of A and it is true due to lemma 3.4.11. Let us show it for the case $n \neq m$. By symmetry, we assume $n = -1$ and $m = 1$ so the left hand side of equation (4.2.22) becomes

$$(J^{-1} * \phi)_I \phi_{J^{-1}} = (J^{-1} * \phi)_I (J^{-1} * \phi)_J^{-1}$$

and the right hand side becomes

$$(I * \phi)_{J^{-1}} \phi_I = (I(J^{-1}) * \phi)_J^{-1} \phi_I$$

The two expressions above are equal if and only if the following equality holds

$$(I(J^{-1}) * \phi)_J (J^{-1} * \phi)_I = \phi_I (J^{-1} * \phi)_J$$

However, the equality is clear since both sides are equal to $(J^{-1} * \phi)_{IJ}$. Similarly, for the case $n = m = -1$, one follows the same procedures and see equation (4.2.22) holds for this case if and only if the following equation holds,

$$\phi_{(IJ)^{-1}} = ((IJ)^{-1} * \phi)_I^{-1} (J^{-1} * \phi)_J^{-1} = ((IJ)^{-1} * \phi)_J^{-1} (I^{-1} * \phi)_I^{-1}$$

By taking inverses of both sides, the equation above holds if and only if the equality

$$(J^{-1} * \phi)_J ((IJ)^{-1} * \phi)_I = (J^{-1} * \phi)_I ((IJ)^{-1} * \phi)_J$$

holds, but this is true since both sides are equal to $((IJ)^{-1} * \phi)_{IJ}$. Note that by taking inverses, we have also shown that $\phi_{(IJ)^{-1}} = ((IJ)^{-1} * \phi)_{IJ}^{-1}$. □

Lemma 4.2.23. *Let I, J be fractional ideals of A with $J \subseteq I$, then there exists an integral ideal Q such that $J = QI$.*

Furthermore, the additive polynomial $(I * \phi)_Q = f(\tau) \in \mathcal{C}[\tau]$ (has finite degree in τ) satisfies

$$f(\tau)\phi_I = \phi_J$$

Proof. The fraction ideals I and J can be decomposed uniquely into primes and inverses of primes and $J \subseteq I$ if and only if $\mathrm{ord}_{\mathfrak{p}}(J) \geq \mathrm{ord}_{\mathfrak{p}}(I)$ for all primes $\mathfrak{p}$.

Let $Q = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(J) - \mathrm{ord}_{\mathfrak{p}}(I)}$, where $\mathfrak{p}$ runs over all the maximal ideals of A. Then it is clear that $J = QI$ and every exponent in the product defining Q is positive with almost all of them equal to 0, so Q is an integral ideal. The rest of the statement is clear from definition 4.2.19. □

Proposition 4.2.24. *For all cycles $\mathfrak{f}$ of K, we have*

$$\mathcal{L}(\mathfrak{f})(\mathcal{C}) = \mathcal{L}[\mathfrak{f}](\mathcal{C})$$

Proof. Since we are working over $\mathcal{C}$, every local ismorphism class of rank 1 Drinfeld module is an isomorphism class of rank 1 Drinfeld module over $\mathcal{C}$. Hence we denote points in $\mathcal{L}(\mathcal{C})$ as (E, c), where (E, ϕ) is a coordinatized Drinfeld module over $\mathcal{C}$ for some $\phi : A \to \mathcal{C}[\tau]$.

First of all, we have show in 4.2.9 that ψ_∞ is the identity map on $\mathcal{L}$. We therefore assume that all ideals and fractional ideals we deal with are supported away from ∞.

Let us first show that $\mathcal{L}(\mathfrak{f})(\mathcal{C}) \subseteq \mathcal{L}[\mathfrak{f}](\mathcal{C})$. Let $(1 + a)$ be a principal ideal of A, then, since $1 \sim_1 (1 + a)$, we have $\mathfrak{f} \sim_{\mathfrak{f}} (1 + a)\mathfrak{f}$.

Let $(E, c^{q-1}) \in \mathcal{L}(\mathcal{C})$ be in the $\mathfrak{f}$-periodic locus of $\mathcal{L}(\mathcal{C})$. Since $\mathcal{C}$ is algebraically closed it is harmless to assume our point on the Drinfeld module $E(\mathcal{C})$ is a $(q-1)$st power, so by the above we must have

$$\begin{aligned}(\mathfrak{f} * E(\mathcal{C}), \phi_{\mathfrak{f}}(c)^{q-1}) &= \psi_{\mathfrak{f}}(E(\mathcal{C}), c^{q-1}) \\ &= \psi_{(1+a)}(\psi_{\mathfrak{f}}(E(\mathcal{C}), c^{q-1})) \\ &= (((1 + a) * \mathfrak{f} * E)(\mathcal{C}), \phi_{(1+a)}(\phi_{\mathfrak{f}}(c))^{q-1})\end{aligned}$$

In particular, we have $\phi_{\mathfrak{f}}(c)^{q-1} = \phi_{(1+a)}(\phi_{\mathfrak{f}}(c))^{q-1}$, so $\phi_{\mathfrak{f}}(c) = \lambda\phi_{(1+a)}(\phi_{\mathfrak{f}}(c))$ for some $\lambda \in \mathbb{F}_q^*$. By multiplying $1 + a$ by λ^{-1} if necessary, we may assume $\lambda = 1$. Therefore, we have $\phi_{\mathfrak{f}}(c) = \phi_{(1+a)}(\phi_{\mathfrak{f}}(c))$ and hence $\phi_a(\phi_{\mathfrak{f}}(c)) = 0$. This holds for all a and if one begins with two principal ideals $(1 + a)$ and $(1 + a')$ such that a and a' are coprime, it is clear that we must have $\phi_{\mathfrak{f}}(c) = 0$ by 3.2.3.

Conversely, let (E, c) be in the $\mathfrak{f}$-torsion, we want to show that $(E, c) \in \mathcal{L}(\mathfrak{f})$. Let $\mathfrak{a}, \mathfrak{b}$ be two $\mathfrak{f}$ equivalent cycles. By 0.0.6, this implies $\mathfrak{a} = x\mathfrak{b}$ for some $x \in K^*$, satisfying $x \in 1 + \mathfrak{f}\mathfrak{b}^{-1}$. Hence, it is clear that, for $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}$, we have $\mathfrak{m}(\mathfrak{a} * E)(\mathcal{C}) = \mathfrak{m}(\mathfrak{b} * E)(\mathcal{C})$ (recall $\mathfrak{m}$ is the coarse map defined in 4.1.12) since the two cycles differ by some $x \in K^*$ which implies the image is the same in the coarse space $M_1(\mathcal{C})$ (isomorphic to Cl(1)).

Let c be a point on a Drinfeld module (E, ϕ) over $\mathcal{C}$ such that (E, c) is a $\mathfrak{f}$-torsion point. We now show that it is in the periodic locus. We follow definition 4.2.19 and extend our Drinfeld module ϕ over $\mathcal{C}$ to a map from K to the left twisted power series $\mathcal{C}[[\tau]]$. We will, as before, slightly abuse notation and still refer the map $K \to \mathcal{C}[[\tau]]$ as ϕ. We have

$$\begin{aligned}
(\phi_{\mathfrak{a}} - \phi_{\mathfrak{b}})(c) &= (\phi_{x\mathfrak{b}} - \phi_{\mathfrak{b}})(c) \\
&= ((\mathfrak{b} * \phi)_x \phi_{\mathfrak{b}} - \phi_{\mathfrak{b}})(c) \\
&= ((\mathfrak{b} * \phi)_x - (\mathfrak{b} * \phi)_1)(\phi_{\mathfrak{b}}(c)) \\
&= (\mathfrak{b} * \phi)_{x-1}(\phi_{\mathfrak{b}}(c)) \\
&= f(\tau)(\mathfrak{b} * \phi)_{\mathfrak{f}\mathfrak{b}^{-1}}(\phi_{\mathfrak{b}}(c)) \\
&= f(\tau)\phi_{\mathfrak{f}}(c) \\
&= 0
\end{aligned}$$

where we have repetitively used the fractional ideal version of equation (4.2.18), which is well-defined by lemma 4.2.21. $f(\tau) \in \mathcal{C}[\tau]$ is an element satisfying the equality $\phi_{\mathfrak{b}(x-1)} = f(\tau)\phi_{\mathfrak{f}}$. Such $f(\tau)$ exists because $x - 1 \in \mathfrak{f}\mathfrak{b}^{-1}$, in other words, $(x-1) \subseteq \mathfrak{f}\mathfrak{b}^{-1}$ so we can apply lemma 4.2.23. Hence, for c in the torsion locus, $\phi_{\mathfrak{a}}(c) = \phi_{\mathfrak{b}}(c)$, so it is also in the periodic locus and we are done. □

Remark 4.2.25. It is not true that for every Λ-scheme X, the two subschemes $X[\mathfrak{f}]$ and $X(\mathfrak{f})$ agree with each other. In general, it is very hard to describe $X(\mathfrak{f})$.

Corollary 4.2.26. *The field generated by the periodic loci of $\mathcal{L}$ over all cycles, i.e. the field*

$$\bigcup_{\mathfrak{f}} K(\mathcal{L}(\mathfrak{f})(\mathcal{C}))$$

where $\mathfrak{f}$ runs over all cycles of K, is the maximal abelian extension of K, totally splitting at ∞.

Proof. We have shown, in proposition 4.2.24, that the periodic locus $\mathcal{L}(\mathfrak{f})(\mathcal{C})$ is the same as the torsion locus $\mathcal{L}[\mathfrak{f}](\mathcal{C})$, so the claim is equivalent to the statement of torsion locus generating abelian extensions shown in proposition 4.2.16. □

Our construction of the A-scheme $\mathcal{L}$ is based on the choice of ∞ at the beginning where $A = A_\infty$ and we will denote $\mathcal{C}$ and $\mathcal{L}$ as $\mathcal{C}_\infty$ and $\mathcal{L}_\infty$ to emphasize this arbitrary choice. One can choose another point ∞' and repeat everything we have done so far and obtain $\mathcal{L}_{\infty'}$.

Corollary 4.2.27. *Let $\infty \neq \infty'$ be two arbitrarily chosen places of K. Let $\overline{K}_s$ denote the maximal separable extension of K and let $\mathcal{X} = \mathcal{L}_\infty \amalg \mathcal{L}_{\infty'}$, then the field*

$$\bigcup_{\mathfrak{f}} K(\mathcal{X}(\mathfrak{f})(\overline{K}_s))$$

is the maximal abelian extension of K.

Proof. Since adjoining $\mathfrak{f}$-periodic loci give separable and algebraic extensions over K, we have $K(\mathcal{L}_\infty(\mathfrak{f})(\mathcal{C}_\infty)) = K(\mathcal{L}_\infty(\mathfrak{f})(\overline{K}_s))$ by viewing $\overline{K}_s$ as a subfield of $\mathcal{C}_\infty$ and similarly for $\mathcal{C}_{\infty'}$. Then

$$\bigcup_{\mathfrak{f}} K(\mathcal{X}(\mathfrak{f})(\overline{K}_s)) = \bigcup_{\mathfrak{f}} K((\mathcal{L}_\infty \amalg \mathcal{L}_{\infty'})(\mathfrak{f})(\overline{K}_s)) = \bigcup_{\mathfrak{f}} K(\mathcal{L}_\infty(\mathfrak{f})(\mathcal{C}_\infty) \amalg \mathcal{L}_{\infty'}(\mathfrak{f})(\mathcal{C}_{\infty'}))$$

which the last field is the composition of maximal abelian extensions totally splitting at two different primes, so it must be the maximal abelian extension. ☐

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