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T-Duality: Topology Change from H-Flux

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Abstract: T-duality acts on circle bundles by exchanging the first Chern class with the fiberwise integral of the H-flux, as we motivate using E_8 and also using S-duality. We present known and new examples including NS5-branes, nilmanifolds, lens spaces, both circle bundles over $\mathbb{R}P^n$, and the $AdS^5 \times S^5$ to $AdS^5 \times \mathbb{C}P^2 \times S^1$ with background H-flux of Duff, Lü and Pope. When T-duality leads to M-theory on a non-*spin* manifold the gravitino partition function continues to exist due to the background flux, however the known quantization condition for G_4 receives a correction. In a more general context, we use *correspondence spaces* to implement isomorphisms on the twisted K-theories and twisted cohomology theories and to study the corresponding Grothendieck-Riemann-Roch theorem. Interestingly, in the case of decomposable twists, both twisted theories admit fusion products and so are naturally rings.

1. Introduction

T-duality is a generalization of the $R \to 1/R$ invariance of string theory compactified on a circle of radius R. The local transformation rules of the low energy effective fields under T-duality, known as the Buscher rules [1] (see also, e.g., [2–4]), have been known for some time, but global issues, in particular in the presence of NS 3-form H-flux, have remained obscure. It is known, however, through many examples in the literature [5–8], that the general case involves a change in the topology of the manifold. However no systematic method has been developed for determining the topology change. In this paper we will propose a formula for the topology change under T-duality, and we will show that it yields the desired isomorphism both in the context of twisted cohomology as well as twisted K-theory. We conjecture that the duality holds, however, in the full string theory as well.

To simplify the discussion we will restrict ourselves in this paper to T-duality in one direction only, *i.e.* T-dualizing on a circle S^1. A more general case with a d-dimensional torus can be obtained by successive dualizations so long as the integral of H over each

2-subtorus vanishes. If this integral does not vanish, then after T-dualizing about one circle the other circle no longer exists. We will relate the obstruction to T-duality to a particular type of failure of the 2-torus to lift to F-theory.[1] In integral cohomology the story is the same, as the integral of H inhabits $H^1(M, \mathbb{Z})$ which cannot have a torsion piece because of the Universal Coefficient Theorem.

First, consider the case where spacetime E is a product manifold $M \times S^1$ and the NS 3-form H is trivial in $H^3(E, \mathbb{Z})$, *i.e.* we can write $H = dB$ globally. Similarly, for the T-dual we have $\hat{H} = d\hat{B}$. In this case, upon T-dualizing on S^1, the Buscher rules on the RR fields can be conveniently encoded in the formula [9]

$$\hat{G} = \int_{S^1} e^{\mathcal{F} - B + \hat{B}}\, G\,, \tag{1.1}$$

where G is the total (gauge invariant) RR fieldstrength, $G = \sum_p G_{p+2}$ ($p = 0, 2, 4, \ldots, 8$ for type IIA and $p = -1, 1, \ldots, 7$ for type IIB), and $\mathcal{F} = d\theta \wedge d\hat{\theta}$ is the curvature of the Poincaré linebundle $\mathcal{P}$ on $S^1 \times \hat{S}^1$, so that $e^{\mathcal{F}} = ch(\mathcal{P})$ is the Chern character of $\mathcal{P}$. The right-hand side of (1.1) is interpreted as a (closed) form on $M \times S^1 \times \hat{S}^1$, and integrated along S^1 to yield a form on the T-dual space $\hat{E} = M \times \hat{S}^1$.[2]

The RR field G is d_H-closed, where $d_H = d - H\wedge$ is the H-twisted differential, and it follows that its T-dual $\hat{G}$ is $d_{\hat{H}}$-closed. This is just the supergravity Bianchi identity. Gauge invariance is implemented through $\delta C = e^B d\alpha$, where the gauge potential C is related to G by $G = e^B d(e^{-B} C) = d_H C$. Thus, we can interpret (1.1) as an isomorphism

$$T_* : H^\bullet(M \times S^1, H) \xrightarrow{\cong} H^{\bullet+1}(M \times \hat{S}^1, \hat{H}). \tag{1.2}$$

Of course, since in this case $H = dB$ globally, the twisted cohomology $H^\bullet(E, H)$ is canonically isomorphic to the usual cohomology $H^\bullet(E)$, by noting that $d(e^{-B} G) = e^{-B} d_H G$.

The discussion above can be lifted to K-theory, [9] (see also [10–13]), and thus to the classification of D-branes on $M \times S^1$ and $M \times \hat{S}^1$, by using the correspondence

$$\begin{tikzcd} & M \times S^1 \times \hat{S}^1 \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \\ M \times S^1 & & M \times \hat{S}^1 \end{tikzcd} \tag{1.3}$$

This gives rise to an isomorphism of K-theories

$$T_! : K^\bullet(M \times S^1) \xrightarrow{\cong} K^{\bullet+1}(M \times \hat{S}^1) \tag{1.4}$$

[1] However there are torii that do not lift to F-theory on which we may T-dualize, for example, a 2-torus that supports G_3 flux. S-dualizing, the obstruction to T-duality on a torus with H-flux is the controversial obstruction to S-duality in the presence of G_1 flux.

[2] Strictly speaking, the various forms entering (1.1) are the pull-backs of forms to the correspondence space $M \times S^1 \times \hat{S}^1$.

by

$$T_! = \hat{p}_! \left(p^!(\,\cdot\,) \otimes \mathcal{P}\right). \tag{1.5}$$

It is well-known that the application of T-duality is not restricted to product manifolds $M \times S^1$, but can also be applied locally in the case of S^1-fibrations over M [14], and moreover, can be generalized to situations with nontrivial NS 3-form flux H. While in this more general case, strictly speaking, (1.1) does not make sense since neither the Poincaré bundle, nor B, are defined globally, it does appear that in some sense the equation still makes sense locally as it does give rise to the correct Buscher rules even in this more general setting.

In this paper we investigate the more general case where E is an oriented S^1-bundle over M

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \big\downarrow{\scriptstyle \pi} \\ & & M \end{array} \tag{1.6}$$

characterized by its first Chern class $c_1(E) \in H^2(M,\mathbb{Z})$, in the presence of (possibly nontrivial) H-flux $H \in H^3(E,\mathbb{Z})$.[3] We will argue that the T-dual of E is again an oriented S^1-bundle over M, denoted by $\hat{E}$, [4]

$$\begin{array}{ccc} \hat{S}^1 & \longrightarrow & \hat{E} \\ & & \big\downarrow{\scriptstyle \hat{\pi}} \\ & & M \end{array} \tag{1.7}$$

supporting H-flux $\hat{H} \in H^3(\hat{E},\mathbb{Z})$, such that

$$c_1(\hat{E}) = \pi_* H\,, \qquad c_1(E) = \hat{\pi}_* \hat{H}\,, \tag{1.8}$$

where $\pi_* : H^k(E,\mathbb{Z}) \to H^{k-1}(M,\mathbb{Z})$, and similarly $\hat{\pi}_*$, denote the pushforward maps.[5]

Mathematically, the reason for the duality (1.8) can be understood as follows: For an oriented S^k-bundle E, we have a long exact sequence in cohomology called the *Gysin sequence* (cf. [15, Prop. 14.33]). In particular, for an oriented S^1 bundle with first Chern class $c_1(E) = F \in H^2(M,\mathbb{Z})$, we have

$$\begin{aligned} \ldots \longrightarrow H^k(M,\mathbb{Z}) \xrightarrow{\pi^*} H^k(E,\mathbb{Z}) &\xrightarrow{\pi_*} H^{k-1}(M,\mathbb{Z}) \\ &\xrightarrow{F\cup} H^{k+1}(M,\mathbb{Z}) \longrightarrow \ldots . \end{aligned}$$

Consider the $k=3$ segment of this sequence. It shows that to any H-flux $H \in H^3(E,\mathbb{Z})$ we have an associated element $\hat{F} = \pi_* H \in H^2(M,\mathbb{Z})$, and that, moreover, $F \cup \hat{F} = 0$

[3] To simplify the notations we will use the same notation for a cohomology class $[H]$, or for a representative H, throughout this paper. It should be clear which is meant from the context.

[4] Throughout this paper the notation $\hat{E}$ will refer to the T-dual of the bundle E, and *not* to the dual bundle in the usual sense.

[5] At the level of de Rham cohomology, the pushforward maps π_* and $\hat{\pi}_*$ are simply the integrations along the S^1-fibers of E and $\hat{E}$, respectively.

in $H^4(M,\mathbb{Z})$. Now, let $\hat{E}$ be the S^1-bundle associated to $\hat{F}$. Reversing the roles of E and $\hat{E}$ in the Gysin sequence, we see that since $F \cup \hat{F} = \hat{F} \cup F = 0$, there exists an $\hat{H} \in H^3(\hat{E},\mathbb{Z})$ such that $\hat{\pi}_* \hat{H} = F$, where $\hat{H}$ is unique up to an element of $\pi^* H^3(M,\mathbb{Z})$. The transformation $(E, H) \to (\hat{E}, \hat{H})$, for a particular choice of $\hat{H}$, is precisely what can be identified with T-duality. The ambiguity in $\hat{H}$, up to an element in $\pi^* H^3(M,\mathbb{Z})$, is fixed by requiring that T-duality should act trivially on $\pi^* H^3(M,\mathbb{Z})$, *i.e.* T-duality should not affect H-flux which is completely supported on M. Since H and $\hat{H}$ live on different spaces, in order to compare them we have to pull them back to the correspondence space. The correspondence space in this more general setting is the fibered product $E \times_M \hat{E} = \{(x,\hat{x}) \in E \times \hat{E} \mid \pi(x) = \hat{\pi}(\hat{x})\}$, which is both an $\hat{S}^1$-bundle over E, as well as an S^1-bundle over $\hat{E}$.

Before we continue, let us observe that in the case of a 2-dimensional base manifold M, the Gysin sequence immediately gives an isomorphism between $H^3(E,\mathbb{Z})$ and $H^2(M,\mathbb{Z})$, *i.e.* between Dixmier-Douady classes on E and line bundles on M. This correspondence is used for example in [16, Sect. 4.3] to give an explicit construction of a PU-bundle (with given decomposable DD class) over E from a linebundle over M. As a particular concrete example, note that S^3 can be considered as an S^1-bundle over S^2 by means of the Hopf fibration. By (1.8) its T-dual, in the absence of H-flux, is $S^2 \times S^1$ supported by 1 unit of H-flux. This example was studied in [5], but the observation that the H-flux on the $S^2 \times S^1$ side is nontrivial was apparently missed.

In order to discuss the generalization of (1.1) we have to choose specific representatives of the cohomology classes. In particular, upon choosing connections A and $\hat{A}$, on the S^1-bundles E and $\hat{E}$, respectively, the isomorphism T_* that generalizes (1.1) is now given by

$$\hat{G} = \int_{S^1} e^{A\wedge\hat{A}}\, G\,, \tag{1.9}$$

where the right-hand side is a form on $E \times_M \hat{E}$, and the integration is along the S^1-fiber of E.[6] In terms of A, $\hat{A}$, and their curvatures $F = dA$, $\hat{F} = d\hat{A}$, we can write (see Sect. 3.1 for more details)

$$H = A \wedge \hat{F} - \Omega\,, \tag{1.10}$$

for some $\Omega \in \Omega^3(M)$, while the T-dual $\hat{H}$ is given by

$$\hat{H} = F \wedge \hat{A} - \Omega\,. \tag{1.11}$$

Locally, we have $A = d\theta + \hat{\pi}_* \hat{B}$, $\hat{A} = d\hat{\theta} + \pi_* B$. Equations (1.8) are easily checked. We note that

$$d(A \wedge \hat{A}) = -H + \hat{H}\,, \tag{1.12}$$

so that (1.9) indeed maps d_H-closed forms to $d_{\hat{H}}$-closed forms.

We recall that the RR fields G are determined by the twisted K-theory classes Q via the twisted Chern map [19–23]

$$G = ch_H(Q)\sqrt{\widehat{A}(TE)}\,, \tag{1.13}$$

where $\widehat{A}$ is the A-roof genus.

[6] Strictly speaking, the various forms entering (1.9) and beyond are the pullbacks of forms on living on E and $\hat{E}$ to $E \times_M \hat{E}$.

The discussion above can be lifted to K-theory and, in this more general setting, T-duality gives an isomorphism of the twisted K-theories of E and $\hat{E}$, descending to an isomorphism between the twisted cohomologies of E and $\hat{E}$, as expressed in the following commutative diagram (see Theorem 3.6)

$$\begin{array}{ccc} K^\bullet(E, H) & \xrightarrow{T_!} & K^{\bullet+1}(\hat{E}, \hat{H}) \\ {\scriptstyle ch_H}\downarrow & & \downarrow{\scriptstyle ch_{\hat{H}}} \\ H^\bullet(E, H) & \xrightarrow{T_*} & H^{\bullet+1}(\hat{E}, \hat{H}) \end{array} \tag{1.14}$$

Several of the constructions used in the definition of T-duality on twisted K-theory are adapted from [17, 18].

The rationale for the normalization in (1.13) by $\sqrt{\widehat{A}(TE)}$ is fairly standard. A special case of the cup product pairing (3.19) followed by the standard index pairing of elements of K-theory with the Dirac operator, explains the upper horizontal arrows in the diagram,

$$\begin{array}{ccccc} K^\bullet(E, H) \times K^\bullet(E, -H) & \longrightarrow & K^0(E) & \xrightarrow{\text{index}} & \mathbb{Z} \\ {\scriptstyle ch_H \times ch_{-H}}\downarrow & & \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ||} \\ H^\bullet(E, H) \times H^\bullet(E, -H) & \longrightarrow & H^{even}(E) & \xrightarrow{\int_E \widehat{A}(TE)\wedge} & \mathbb{Z} \end{array} \tag{1.15}$$

The bottom horizontal arrows are cup product in twisted cohomology (3.7) followed by cup product by $\widehat{A}(TE)$ and by integration. By the Atiyah-Singer index theorem, the diagram (1.15) commutes. Therefore the normalization in (1.13) makes the pairings in twisted K-theory and twisted cohomology isometric.

The twisted K-theory isomorphism is the geometric analogue of results of Raeburn and Rosenberg [24] who studied spaces with an $\mathbb{R}$-action in terms of crossed products of C^*-algebras of the type $A \times_\alpha \mathbb{R}$, such that the spectrum of $A \times_\alpha \mathbb{R}$ is precisely the circle bundle E in the discussion above. The isomorphism in the upper horizontal arrow in (1.14) is then a direct consequence of the Connes-Thom isomorphism [25] of the K-theory of these crossed C^*-algebras.

The paper is organized as follows. In Sect. 2 we provide some physical intuition and motivation for our conjectured description of T-duality although we restrict attention to the special case in which H is only nontrivial on one side of the duality. In Sect. 2.1 we see how T-duality and Eq. (1.8) arise in the E_8 gauge bundle formalism of M-theory, and in Sect. 2.2 we provide a physical derivation from S-duality for the case in which H is proportional to G_3. Both approaches illustrate the connection between the fibered product $E \times_M \hat{E}$ and F-theory. The full derivation of the isomorphism and the corresponding maps appears in the more mathematical Sect. 3.

In Sect. 4 we will provide a number of examples of this correspondence, including T-duality transverse to an NS5-brane and T-duality of circle bundles over Riemann surfaces which include the nilmanifolds, lens spaces and also $AdS^3 \times S^3 \times T^4$ with its $\mathbb{Z}_n$ quotients. An example with torsion H-flux, the circle bundles over $\mathbb{R}\mathrm{P}^2$, will also be treated. In Sect. 5 we consider circle bundles over $\mathbb{R}\mathrm{P}^n$. As these examples may be 4-dimensional or higher, we will not be able to compute K-groups simply by using the Atiyah-Hirzebruch spectral sequence as in the previous section, but also we need to solve an extension problem. However T-duality will relate these bundles to bundles in which the extension problem is trivial, and so T-duality may be used to solve the extension problem in our original bundles and thus to calculate the twisted K-groups of circle bundles over $\mathbb{RP}^n$.

In Sect. 6 we will consider the T-duality between $AdS^5 \times S^5$ and $AdS^5 \times \mathbb{C}\mathrm{P}^2 \times S^1$ with H-flux, and its $\mathbb{Z}_n$ quotients [4]. These are interesting because the right-hand side is not *spin*, and so one might expect a gravitino anomaly. However there is no gravitino anomaly before the T-duality. We show that in this case and in general, as a result of the $\psi H \psi$ coupling in the type-II supergravity action, the nontrivial H-flux precisely forces the gravitino anomalies to match before and after the T-duality.[7] We will see that the global anomalies before and after the T-duality agree because they are determined by the topology of the fibered product. In the example, this leads to an anomaly on both sides precisely when n is even. On the other hand both sides are consistent when n is odd, the IIB side because spacetime is *spin* and the IIA side because a 9-dimensional analog of the quantum Hall effect in the dimensionally reduced theory means that the low energy modes of the gravitinos behave like bosons. As the M-theory lift is not *spin*, the usual formula for G_4 flux quantization [27] does not make sense, however the global gravitino anomaly allows a new condition to be found in the torus-bundle case. Finally, in Sect. 7, we present some of the many remaining open problems.

2. Physical Motivation

2.1. T-Duality from E_8. The T-duality discussed in the introduction is a consequence of a conjecture [28] made in the context of the E_8 gauge bundle formalism [27, 29–32]. In this formalism, M-theory's 4-form fieldstrength G_4 is interpreted as the characteristic class of an E_8 bundle P over the $11d$ bulk Y^{11}. Consider the case in which Y^{11} is a $T^2 = S^1_M \times S^1_{\mathrm{IIA}}$ torus bundle over the 9-manifold M^9. Dimensionally reducing out the M-theory circle S^1_M we obtain [33] an LE_8 bundle P' over the 10-dimensional circle bundle E, whose based part is characterized by a 3-form $H = \int_{S^1_M} G_4$. LE_8 is the loopgroup of E_8. Reducing on the other circle yields an LLE_8 bundle $\hat{E}$ whose based part is characterized by a two-form

$$F = \int_{S^1_{\mathrm{IIA}}} H. \tag{2.1}$$

In fact the based part of LLE_8 is homotopic to the circle S^1_{IIB}, and so F is just the curvature of a circle bundle. The above discussion is summarized by the following equation:

$$\left\{\begin{array}{ccc} E_8 & \to & P \\ & & \downarrow \\ S^1_M & \to & Y^{11} \\ & & \downarrow \\ S^1_{\mathrm{IIA}} & \to & E \\ & & \downarrow \\ & & M^9 \end{array}\right\} \longrightarrow \left\{\begin{array}{ccc} LE_8 & \to & P' \\ & & \downarrow \\ S^1_{\mathrm{IIA}} & \to & E \\ & & \downarrow \\ & & M^9 \end{array}\right\} \longrightarrow \left\{\begin{array}{ccc} LLE_8 \sim S^1_{\mathrm{IIB}} & \to & \hat{E} \\ & & \downarrow \\ & & M^9 \end{array}\right\} \tag{2.2}$$

[7] The global gravitino anomaly in question is the ill-definedness of the partition function that appears when an uncharged fermion is placed on a non-*spin* manifold, not the $(4d + 2)$-dimensional chiral anomaly discussed in, for example. Ref. [26].

The conjecture in Ref. [28] is that the fiber of this circle bundle S^1_{IIB} is the T-dual circle which appears in IIB. As desired, the first Chern class of this bundle is precisely the H-flux in IIA integrated over the fiber S^1_{IIA} as seen in Eq. (2.1). In this note we further claim that the first Chern class of the S^1_{IIA} bundle, the spacetime on the type IIA side, is the integral over S^1_{IIB} of the H-flux on the type IIB side (1.8).

2.2. T-duality from S-duality. An alternate approach to the T-duality relation (1.8), is via the F-theory [34] lift of this story, where T-duality will simply be a choice of projection map. This approach is similar to that of Ref. [5] where it was shown that the sigma models on E and $\hat{E}$ may both be obtained from a sigma model on $E \times_M \hat{E}$ by integrating out different variables. Their argument, like the one in this section, only applies to the case in which H is nonvanishing on one side of the duality, and the normalization is unclear. However it may be possible to generalize their argument to the case in which H and $\hat{H}$ are both nontrivial (or even to higher-dimensional torii).

Recall from Eq. (2.2) that the bosonic data of M-theory is encoded in an LLE_8 bundle over M^9. To arrive at type IIB string theory we considered only the based part of this loop group which is homotopy equivalent to the circle S^1_{IIB}, but in fact [35] the loop groups are free and trivially centrally extended. Thus we find that

$$\pi_1(LLE_8) = \mathbb{Z}^3, \tag{2.3}$$

where the three circles are S^1_M, S^1_{IIA} and S^1_{IIB}. These circles are all fibered over M^9, with Chern classes that in type IIA we name G_2, $c_1(E)$ and $c_1(\hat{E}) = \int_{S^1_{\mathrm{IIA}}} H$ respectively. The total space of the fibered product of these three circle bundles over M^9 is twelve-dimensional, and this $12d$ perspective is called F-theory.

The total space of F-theory is an S^1_M bundle over the fibered product $E \times_M \hat{E}$ and also a torus bundle over $\hat{E}$, the spacetime of type IIB. This torus is generated by the circles S^1_M and S^1_{IIA}. Interchanging these two circles (with a minus sign) is called S-duality in type IIB and is called a 9-11 flip in type IIA. Therefore we have the commuting diagram:

IIA IIB

$$\begin{tikzcd}
c_1(E) = a \arrow[r, leftrightarrow, "\text{T-Duality}"] \arrow[d, leftrightarrow, "\text{9-11 Flip}"] & H = a \cup b \arrow[d, leftrightarrow, "\text{S-Duality}"] \\
G_2 = a \arrow[r, leftrightarrow, "\text{T-Duality}"] & G_3 = a \cup b
\end{tikzcd}$$

relating the two IIA and two IIB configurations described above.

This diagram will allow us to perform T-duality from IIB to IIA in two ways, by proceeding left directly, or by performing an S-duality followed by a T-duality followed by a 9-11 flip. We will start in type IIB on $M^9 \times S^1_{\mathrm{IIB}}$ with

$$H = a \cup b \in H^2(M^9) \otimes H^1(S^1_{\mathrm{IIB}}) \tag{2.4}$$

and no G_3 flux. Performing an S-duality leaves $G_3 = a \cup b$ and H now vanishes.[8] Now that there is no H flux, we may perform a T-duality along S^1_{IIB} without changing the 10-dimensional topology. After T-duality we find type IIA string theory on $M^9 \times S^1$ with $G_2 = \int_{S^1_{\mathrm{IIB}}} G_3 = a$. The M-theory circle S^1_M is nontrivially fibered over M^9 with Chern class equal to $G_2 = a$. The 9-11 flip interchanges the M-theory circle S^1_M with the IIA circle S^1_{IIA} and so leaves $G_2 = 0$ and a 10-dimensional spacetime E which is a S^1_{IIA} circle bundle over M^9 with first Chern class

$$c_1(E) = a = \int_{S^1_{\mathrm{IIB}}} H \tag{2.5}$$

as desired, where H is the original H-flux in type IIB.

3. T-duality Isomorphism in Twisted K-theory and Twisted Cohomology: The Case of Circle Bundles

3.1. The setup. We elaborate here on the setup in the introduction. Suppose that M is a compact connected manifold and E be a principal circle bundle over M with projection map π and H a closed, integral 3-form on E having the property that $\pi_*(H)$ is a closed integral 2-form on M. [For clarity of exposition we mostly use the language of differential forms, but the discussion can easily be formulated in terms of integer cohomology (i.e. Čech cohomology) classes, and the results hold in those cases as well. In particular, the case where H is a torsion class is covered by our theorems (see Sect. 3.3)]. Then we know by the classification of circle bundles that there is a circle bundle $\hat{E}$ over M with projection map $\hat{\pi}$ and with first Chern class $c_1(\hat{E}) = \pi_*(H)$. $\hat{E}$ will be referred to as the *T-dual* of E, which is not to be confused with the dual bundle to E. We define the *correspondence space* of E and $\hat{E}$ to be the fibered product $E \times_M \hat{E}$, since it implements T-duality in generalized cohomology theories such as K-theory, cohomology and their twisted analogues. Correspondence spaces also occur in other parts of mathematical physics, such as twistor theory and noncommutative geometry. We have the following commutative diagram:

$$\begin{tikzcd} & E \times_M \hat{E} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \\ E \arrow[dr, "\pi"'] & & \hat{E} \arrow[dl, "\hat{\pi}"] \\ & M & \end{tikzcd} \tag{3.1}$$

[8] Had we allowed G_3 flux proportional to H we could still have arranged this by performing a different $SL(2, \mathbb{Z})$ transformation on S^1_M and S^1_{IIA}.

Note that the correspondence space $E \times_M \hat{E}$ is a circle bundle over E with first Chern class $\pi^*(c_1(\hat{E}))$, and it is also a circle bundle over $\hat{E}$ with first Chern class $\hat{\pi}^*(c_1(E))$, by the commutativity of the diagram above, (3.1). If $\hat{E} = E$ or if $\hat{E} = M \times S^1$, then the correspondence space $E \times_M \hat{E}$ is diffeomorphic to $E \times S^1$.

Let $A \in \Omega^1(E)$ and $\hat{A} \in \Omega^1(\hat{E})$ be connection one forms on E and $\hat{E}$ respectively, and denote their curvatures in $H^2(M)$ by $F = dA$ and $\hat{F} = d\hat{A} = \pi_* H$, respectively. The connections A and $\hat{A}$ are normalized such that $\pi_* A = 1 = \hat{\pi}_* \hat{A}$. Let $H \in \Omega^3(E)$ be the given closed integral 3-form on E as above. We will now argue, as mentioned in the introduction, that there exists a 3-form $\Omega \in \Omega^3(M)$ such that

$$H = A \wedge \pi^* \hat{F} - \pi^* \Omega \qquad \in \Omega^3(E). \tag{3.2}$$

Consider the Gysin sequence associated to the S^1-bundle E (at the level of de Rham cohomology)

$$\begin{aligned} \dots \longrightarrow H^k(M) \xrightarrow{\pi^*} H^k(E) &\xrightarrow{\pi_*} H^{k-1}(M) \\ &\xrightarrow{F\wedge} H^{k+1}(M) \longrightarrow \dots . \end{aligned}$$

The $k = 3$ segment of the Gysin sequence shows that $F \wedge \hat{F} = 0$ in $H^4(M)$. Therefore $F \wedge \hat{F} = d\alpha$ with $\alpha \in \Omega^3(M)$. Thus, $A \wedge \pi^* \hat{F} - \pi^* \alpha$ is a closed 3-form in $\Omega^3(E)$, i.e. an element of $H^3(E)$. Consider $H - (A \wedge \pi^* \hat{F} - \pi^* \alpha) \in H^3(E)$. Clearly, $\pi_*(H - (A \wedge \pi^* \hat{F} - \pi^* \alpha)) = 0$, since $\pi_* \circ \pi^* = 0$ and $\pi_* A = 1$. Hence we conclude that $H - (A \wedge \pi^* \hat{F} - \pi^* \alpha) = \pi^*(\beta + d\gamma)$, for some $\beta \in H^2(M)$ and $\gamma \in \Omega^2(M)$. Putting $\Omega = \alpha - \beta - d\gamma$ proves (3.2). Now define $\hat{H} \in \Omega^3(\hat{E})$ by

$$\hat{H} = \hat{\pi}^* F \wedge \hat{A} - \hat{\pi}^* \Omega \qquad \in \Omega^3(\hat{E}). \tag{3.3}$$

It easily follows that $\hat{H}$ is closed, i.e. defines an element in $H^3(\hat{E})$ and that $F = c_1(E) = \hat{\pi}_* \hat{H}$ in $H^2(M)$. I.e., to summarize, we find the relations

$$\pi_* H = c_1(\hat{E}), \quad \hat{\pi}_* \hat{H} = c_1(E) \qquad \in H^2(M). \tag{3.4}$$

Note that if we define

$$\mathcal{B} = p^* A \wedge \hat{p}^* \hat{A} \quad \in \Omega^2(E \times_M \hat{E}) \tag{3.5}$$

then it follows that

$$d\mathcal{B} = d(p^* A \wedge \hat{p}^* \hat{A}) = -p^* H + \hat{p}^* \hat{H} \tag{3.6}$$

by virtue of the commutativity of the diagram (3.1), and so the pullbacks of the two H-fluxes are cohomologous on the correspondence space $E \times_M \hat{E}$.

3.2. T-duality in twisted cohomology. Here we will prove T-duality in twisted cohomology. Recall that twisted cohomology $H^\bullet(M, H)$ is by definition the $\mathbb{Z}_2$-graded cohomology of the complex $(\Omega^\bullet(M), d_H)$, with differential $d_H = d - H\wedge$. Nilpotency $d_H^2 = 0$ follows from the fact that H is a closed 3-form on M. Twisted cohomology has been studied in detail in the papers [22, 23].

The basic functorial properties of twisted cohomology are as follows:

1. (Normalization) If $H = 0$ then $H^\bullet(M, H) = H^\bullet(M)$.
2. (Module property) $H^\bullet(M, H)$ is a module over $H^{\text{even}}(M)$.
3. (Cup product) There is a cup product homomorphism
$$H^p(M, H) \otimes H^q(M, H') \to H^{p+q}(M, H + H')\,. \tag{3.7}$$
4. (Naturality) If $f\colon N \to M$ is a continuous map, then there is a homomorphism
$$f^* : H^\bullet(M, H) \to H^\bullet(N, f^*H).$$
5. (Pushforward) If $f\colon N \to M$ is a smooth map which is oriented, that is $TN \oplus f^*TM$ is an oriented vector bundle, then there is a homomorphism
$$f_* : H^\bullet(N, f^*H) \to H^{\bullet+d}(M, H),$$
where $d = \dim M - \dim N$.

Properties 1 to 4 were detailed in [22] and [23]. The pushforward Property 5 is established in a manner formally similar to the analogous property for twisted K-theory that will be discussed below and so its proof will be omitted for sake of brevity.

We have homomorphisms
$$p^* : H^\bullet(E, H) \to H^\bullet(E \times_M \hat{E}, p^*H), \tag{3.8}$$
$$e^{\mathcal{B}} : H^\bullet(E \times_M \hat{E}, p^*H) \to H^\bullet(E \times_M \hat{E}, \hat{p}^*\hat{H}), \tag{3.9}$$
and
$$\hat{p}_* : H^\bullet(E \times_M \hat{E}, \hat{p}^*\hat{H}) \to H^{\bullet+1}(\hat{E}, \hat{H}). \tag{3.10}$$
The composition of the maps
$$T_* := \hat{p}_* \circ e^{\mathcal{B}} \circ p^* : H^\bullet(E, H) \to H^{\bullet+1}(\hat{E}, \hat{H}) \tag{3.11}$$
is called T-duality. The situation is completely symmetric and the inverse map is
$$T_*^{-1} := p_* \circ e^{-\mathcal{B}} \circ \hat{p}^* : H^\bullet(\hat{E}, \hat{H}) \to H^{\bullet+1}(E, H). \tag{3.12}$$
To summarize, we have,

Theorem 3.1. *In the situation described above, T-duality in twisted cohomology*
$$T_* \;:\; H^\bullet(E, H) \to H^{\bullet+1}(\hat{E}, \hat{H})\,,$$
is an isomorphism.

On the correspondence space, we can express the isomorphism as

$$\hat{G} = T_*(G) = \hat{p}_*(e^{\mathcal{B}} \wedge p^*G)\,, \tag{3.13}$$

where we notice that since $d\mathcal{B} = -p^*H + \hat{p}^*\hat{H}$, we have $d(e^{\mathcal{B}}) = (-p^*H + \hat{p}^*\hat{H}) \wedge e^{\mathcal{B}}$. So

$$d_{\hat{H}}\hat{G} = \hat{p}_*(e^{\mathcal{B}} \wedge p^* d_H G). \tag{3.14}$$

It follows that G is d_H-closed if and only if $\hat{G}$ is $d_{\hat{H}}$-closed. Moreover the formula can be inverted,

$$G = T_*^{-1}(\hat{G}) = p_*(e^{-\mathcal{B}} \wedge \hat{p}^*\hat{G}), \tag{3.15}$$

proving the assertion.

We next describe special cases. The first case that we will consider is when E, $\hat{E}$ are trivial bundles and $H = 0$. This case was discussed in [9] (see also [10–12]). Explicitly, $E = M \times S^1$ and $\hat{E} = M \times \hat{S}^1$, and the connections on the respective trivial bundles are $A = d\theta$ and $\hat{A} = d\hat{\theta}$. $\mathcal{B} = d\theta \wedge d\hat{\theta}$ is the first Chern class of the Poincaré line bundle $\mathcal{P}$ over $S^1 \times \hat{S}^1$, and $\Lambda_{\mathcal{B}}$ is given by the exterior product with $e^{\mathcal{B}}$, which is equal to the Chern character of the Poincaré bundle $ch(\mathcal{P})$. In this case, the T-duality reduces to an isomorphism,

$$T_* : H^\bullet(M \times S^1) \to H^{\bullet+1}(M \times \hat{S}^1). \tag{3.16}$$

Now let $E = M \times S^1$ be the trivial circle bundle and let

$$H = F \wedge d\theta \in H^2(M) \otimes H^1(S^1) \cong H^3(M \times S^1, \mathbb{Z}) \tag{3.17}$$

be a decomposable class on $M \times S^1$ such that $p^*H = d\hat{A} \wedge d\theta \in \Omega^3(\hat{E} \times S^1)$. Then by (3.3) and (3.4), we must have $\hat{p}^*\hat{H} = 0$ and $\mathcal{B} = \hat{A} \wedge d\theta$ and the first Chern class $c_1(\hat{E}) = \pi_* H \in H^2(M, \mathbb{Z})$.

So T-duality in this case yields an isomorphism $T_* : H^\bullet(M \times S^1, H) \to H^{\bullet+1}(\hat{E})$. What is remarkable in this case is that twisted cohomology *does not* have a canonical ring structure in general, but in this case, one can use the T-duality isomorphism to define the *fusion product* on $H^\bullet(M \times S^1, H)$. We will generalize this as follows.

Theorem 3.2. *Let X be a compact connected manifold, and let $H \in H^3(X, \mathbb{Z})$ be a decomposable class. Then there is a fusion product on twisted cohomology $H^\bullet(X, H)$, making it into a ring.*

To prove this, we notice that a decomposable class H yields a continuous map $F = (F_1, F_2) : X \to BS^1 \times S^1$, where BS^1 is the classifying space of S^1. But we have argued before that the T-dual of $BS^1 \times S^1$ is the total space of the universal circle bundle $ES^1 \to BS^1$. So we can pullback the diagram (3.1) to see that in this case, T-duality yields an isomorphism

$$T_* \ : \ H^\bullet(X, H) \to H^{\bullet+1}(\hat{E})\,, \tag{3.18}$$

that determines the fusion product on twisted cohomology. Here $c_1(\hat{E}) = kF_1^* c_1(ES^1)$, where $[F_2]$ is k times the generator.

3.3. T-duality in twisted K-theory. The generalization of this duality to twisted K-theory has been known for some time [24]. In this section we will give a geometric description of the isomorphism along the lines of the description of the isomorphism of twisted cohomology described above. We will then see that these two isomorphisms are related by the Chern map.

We first recall the definition of twisted K-theory, cf. [36, 37]. It is a well known fact that the unitary group U of an infinite dimensional Hilbert space is contractible in the norm topology, therefore the projective unitary group $PU = U/U(1)$ is an Eilenberg-Maclane space $K(\mathbb{Z}, 2)$. This in turn implies that the classifying space BPU of principal PU bundles is $K(\mathbb{Z}, 3)$. Thus we see that $H^3(X, \mathbb{Z}) = [X, BPU]$, where the right-hand side denotes homotopy classes of maps between the two spaces. Another well known fact is that PU is the automorphism group of the algebra of compact operators on the Hilbert space. So given a closed 3-form H on X, it determines an algebra bundle $\mathcal{E}_H$ up to isomorphism: a particular choice will be assumed. This is equivalent to a particular choice of the associated principal PU-bundle P_H with Dixmier-Douady invariant $[H]$. The twisted K-theory is by definition the K-theory of the noncommutative algebra of continuous sections of the algebra bundle $\mathcal{E}_H$. A geometric description of objects in twisted K-theory is given in [22].

The basic properties of twisted K-theory are as follows:

1. (Normalization) If $H = 0$ then $K^\bullet(M, H) = K^\bullet(M)$.
2. (Module property) $K^\bullet(M, H)$ is a module over $K^0(M)$.
3. (Cup product) There is a cup product homomorphism
$$K^p(M, H) \otimes K^q(M, H') \to K^{p+q}(M, H + H'). \tag{3.19}$$
4. (Naturality) If $f\colon N \to M$ is a continuous map, then there is a homomorphism
$$f^! : K^\bullet(M, H) \to K^\bullet(N, f^*H).$$
5. (Pushforward) Let $f : N \longrightarrow M$ be a smooth map between compact manifolds which is K-oriented, that is $TN \oplus f^*TM$ is a $spin^{\mathbb{C}}$ vector bundle over N. Then there is a homomorphism
$$f_! \ : \ K^\bullet(N, f^*H) \to K^{\bullet+d}(M, H)\,, \tag{3.20}$$
where $d = \dim M - \dim N$.

Properties 1, 3 and 4 were detailed in [22], and Property 2 in [23]. The pushforward Property 5 will be discussed in Sect. 3.4, since it is central to our construction of T-duality.

Using the naturality Property 4, we have the homomorphism,
$$p^! : K^j(E, H) \to K^j(E \times_M \hat{E}, p^*H)\,. \tag{3.21}$$

Observe that the principal PU-bundles P_{p^*H} and $P_{\hat{p}^*\hat{H}}$ are canonically isomorphic to p^*P_H and $\hat{p}^*P_{\hat{H}}$, respectively. Since $-p^*H + \hat{p}^*\hat{H} = d\mathcal{B}$, we conclude that P_{p^*H} and $P_{\hat{p}^*\hat{H}}$ are isomorphic.

We digress to discuss automorphisms of twisted K-theory. First recall that tensoring by any line bundle on E is an automorphism of K-theory, $K^\bullet(E)$ (for example, tensoring by the Poincaré line bundle on the torus). By the module Property 2 of twisted K-theory, we see that tensoring by any line bundle on E is also an automorphism of twisted K-theory, $K^\bullet(E, H)$. However, any line bundle on P_H also gives rise to an automorphism of twisted K-theory as explained next. The first fact that is needed is that stably equivalent bundle gerbes (i.e. tensoring by a trivial gerbe) define the same twisted K-theory, cf. [22]. The next fact is that any line bundle on P_H determines a trivial bundle gerbe, which when tensored with the lifting bundle gerbe of P_H, defines a bundle gerbe that is stably equivalent to the lifting bundle gerbe of P_H.

Next we recall the homomorphism $\psi : PU \times PU \to PU$ that is not the group multiplication, but is defined as follows. Choose an isomorphism of the infinite dimensional Hilbert spaces $\phi : H \otimes H \to H$. This induces an isomorphism $\phi : B(H) \times B(H) \to B(H)$ defined by $\phi(A, B)(v) = \phi A \otimes B(\phi^{-1}(v))$. This restricts to a homomorphism $\phi : U \times U \to U$, where U denotes the unitary operators, such that $\phi(U(1) \times U(1)) \subset U(1)$. Therefore we get the induced homomorphism on the quotient $\psi : PU \times PU \to PU$.

Let $\lambda : P_H \to E$ be the principal PU-bundle over E with curving f and 3-curvature H. That is $df = \lambda^* H$. We also make similar choices $\hat\lambda : P_{-\hat H} \to \hat E$ with curving $-\hat f$ and 3-curvature $-\hat H$ satisfying $d(-\hat f) = -\hat\lambda^* \hat H$. Then on the correspondence space $E \times_M \hat E$, we can form the trivial bundle gerbe $\tilde\lambda : P = (p^* P_H \times \hat p^* P_{-\hat H}) \times_\psi PU \to E \times_M \hat E$ which has curving $f - \hat f$ and 3-curvature $H - \hat H$ (which is equal to $-d\mathcal{B}$). We have simplified the notation by omitting some of the pullback maps, since it is clear on which space the differential forms live. Since by definition, $\pi_* A = 1$ and $\hat\pi_* \hat A = 1$, we see that $\mathcal{B}$ is an integral 2-form. Since H and $\hat H$ are integral 3-forms, we can choose f and $\hat f$ to be integral 2-forms. Observe that the following identity holds:

$$d(f - \hat f) = \tilde\lambda^*(H - \hat H) = d(-\tilde\lambda^* \mathcal{B}) . \tag{3.22}$$

It follows that $\tilde\lambda^* \mathcal{B} + f - \hat f \in \Omega^2(P)$ is a closed 2-form on the trivial gerbe P that has integral periods, and therefore determines a line bundle $\mathcal{L} \to P$ over the trivial bundle gerbe P, with curvature $\mathcal{B} + f - \hat f$ and first Chern class $c_1(\mathcal{L}) = [\mathcal{B} + f - \hat f]$. By the discussion above, tensoring by the trivial bundle gerbe determined by this line bundle $\mathcal{L}$ induces the following isomorphism in twisted K-theory:

$$\Lambda_{\mathcal{B}} : K^j(E \times_M \hat E, p^* H) \to K^j(E \times_M \hat E, \hat p^* \hat H) . \tag{3.23}$$

Using the pushforward Property 5, we have a homomorphism,

$$\hat p_! : K^j(E \times_M \hat E, \hat p^* \hat H) \to K^{j+1}(\hat E, \hat H) . \tag{3.24}$$

The composition of the maps

$$T_! := \hat p_! \circ \Lambda_{\mathcal{B}} \circ p^! : K^j(E, H) \to K^{j+1}(\hat E, \hat H) \tag{3.25}$$

is the T-duality in twisted K-theory. The situation is completely symmetric and the inverse map is

$$T_!^{-1} := p_! \circ \Lambda_{-\mathcal{B}} \circ \hat p^! : K^j(\hat E, \hat H) \longrightarrow K^{j+1}(E, H) . \tag{3.26}$$

To summarize, we have

Theorem 3.3. *In the situation described above, T-duality in twisted K-theory,*

$$T_! \; : \; K^\bullet(E, H) \longrightarrow K^{\bullet+1}(\hat E, \hat H)$$

is an isomorphism.

The special cases discussed above in the context of twisted cohomology are virtually identical in the case of twisted K-theory. In particular, in the decomposable case we find a ring structure (cf. [38]).

Theorem 3.4. *Let X be a compact connected manifold, and let $H \in H^3(X, \mathbb{Z})$ be a decomposable class. Then there is a fusion product on twisted K-theory $K^j(X, H)$, making it into a ring.*

3.4. The pushforward map. In this section we define the pushforward of a K-oriented map in twisted K-theory, *i.e.* Property 5. We shall see in this section that this is essentially the topological index in [17, 18], and we will follow the construction given there.

$$\begin{tikzcd}
 & N(E\times\mathbb{R}^{2N}/Z)\cong U \arrow[dr, hook, "i_1"] & \\
Z \arrow[ur, hook, "j_{1!}"] \arrow[rr, hook, "j"] \arrow[d, "p"'] & & E\times\mathbb{R}^{2N} \arrow[d, "p_1"] \\
E \arrow[urr, hook, "i"] \arrow[rr, "="] & & E
\end{tikzcd} \tag{3.27}$$

For the discussion below, we will make use of the commutative diagram above, which we now explain. Given a fiber bundle $p : Z \to E$ where the projection map p is K-oriented, there is an embedding $i : Z \hookrightarrow E \times \mathbb{R}^{2N}$ that commutes with the projection map p, cf. [39]. Let $i : E \hookrightarrow E \times \mathbb{R}^{2N}$ be the zero section embedding and $p_1 : E \times \mathbb{R}^{2N} \to E$ the projection map to the first factor. Now the total space Z embeds as the zero section of the normal bundle to the embedding j, *i.e.* $j_1 : Z \hookrightarrow N(E \times \mathbb{R}^{2N}/Z)$. The normal bundle $N(E \times \mathbb{R}^{2N}/Z)$ is diffeomorphic to a tubular neighborhood U of the image of the correspondence space in $E \times \mathbb{R}^{2N}$. Finally, $i_1 : U \hookrightarrow E \times \mathbb{R}^{2N}$ is the inclusion map.

Lemma 3.5. *There is a canonical isomorphism*

$$i_! \ : \ K^\bullet(E, H) \cong K_c^\bullet(E \times \mathbb{R}^{2N}, p_1^* H)$$

that is determined by Bott periodicity.

Proof. Recall that $K_c^\bullet(E \times \mathbb{R}^{2N}, p_1^* H) = K_\bullet(C_0(E \times \mathbb{R}^{2N}, \mathcal{E}_{p_1^* H}))$. Now there is a canonical isomorphism $\mathcal{E}_{p_1^* H} \cong p_1^* \mathcal{E}_H$, which induces a canonical isomorphism $C_0(E \times \mathbb{R}^{2N}, \mathcal{E}_{p_1^* H}) \cong C(E, \mathcal{E}_H) \widehat{\otimes} C_0(\mathbb{R}^{2N})$. Thus, $K_c^\bullet(E \times \mathbb{R}^{2N}, p_1^* H) \cong K_\bullet(C(E, \mathcal{E}_H) \otimes C_0(\mathbb{R}^{2N}))$. Bott periodicity asserts that $K_\bullet(C(E, \mathcal{E}_H) \otimes C_0(\mathbb{R}^{2N})) \cong K^\bullet(E, H)$, proving the lemma.

Our goal is to next define $j_! : K^\bullet(Z, p^*H) \longrightarrow K_c^\bullet(E \times \mathbb{R}^{2N}, p_1^*H)$. To do this, we first consider

$$\begin{aligned} j_{1!} \ : \ K^\bullet(Z, p^*H) &\longrightarrow K_c^\bullet(N(E \times \mathbb{R}^{2N}/Z), \pi_1^*H), \\ \xi &\longmapsto \pi_1^*\xi \otimes (\pi_1^*S^+, \pi_1^*S^-, c(v)), \end{aligned} \tag{3.28}$$

where $\pi_1 : N(E \times \mathbb{R}^{2N}/Z) \to Z$ is the projection and $(\pi^*S^+, \pi^*S^-, c(v))$ is the usual Thom class of the complex vector bundle $N(E \times \mathbb{R}^{2N}/Z)$. On the right-hand side we have used the module Property 2. The Thom isomorphism in this context, cf. [18], asserts that $j_{1!}$ is an isomorphism. Now, $N(E \times \mathbb{R}^{2N}/Z)$ is diffeomorphic to a tubular neighborhood U of the image of Z in $E \times \mathbb{R}^{2N}$: let $\Phi : U \longrightarrow N(E \times \mathbb{R}^{2N}/Z)$ denote this diffeomorphism. We have

$$\Phi^! \circ j_{1!} \ : \ K_c^\bullet(Z, p^*H) \longrightarrow K_c^\bullet(U, \Phi^*\pi_1^*H)\,.$$

The inclusion of the open set U in $E \times R^{2N}$ induces a map $K_c^\bullet(U, \Phi^*\pi_1^*H) \longrightarrow K_c^\bullet(E \times R^{2N}, p_1^*H)$. The composition of these maps defines the Gysin map. In particular we get the Gysin map in twisted K-theory,

$$j_! \ : \ K^\bullet(Z, p^*H) \longrightarrow K_c^\bullet(E \times \mathbb{R}^{2N}, p_1^*H)\,,$$

where $j_! = i_1 \circ \Phi^! \circ j_{1!}$. Now define the *pushforward*

$$p_! = i_!^{-1} \circ j_! : K_c^\bullet(Z, p^*H) \longrightarrow K^\bullet(E, H)\,,$$

where we apply Lemma 3.5 to see that the inverse $j_!^{-1}$ exists.

This defines the pushforward for submersions and immersions. The general case can be deduced in the standard manner. Let $f : N \to M$ be a smooth map that is K-oriented. Then f can be canonically factorized into an embedding followed by a submersion as follows. Consider the graph embedding $i_f : N \hookrightarrow N \times M$ defined by $i_f(n) = (n, f(n))$, which is K-oriented since f is K-oriented, and the submersion $p_2 : N \times M \to M$, which is also K-oriented for the same reasons. Then we already know how to define the homomorphisms

$$i_{f!} : K^\bullet(N, f^*H) \to K^\bullet(N \times M, p_1^*H)\,,$$

and also

$$p_{2!} : K^\bullet(N \times M, p_1^*H) \to K^\bullet(M, H)\,.$$

Define the pushforward of a general K-oriented map as

$$f_! = p_{2!} \circ i_{f!}\,. \tag{3.29}$$

3.5. T-duality and twisted Grothendieck-Riemann-Roch formulae. We will first recall the twisted Chern character $ch_H : K^\bullet(E,H) \to H^\bullet(E,H)$ and then compute the twisted Chern character of the T-dual of an element in twisted K-theory. Since for dimension reasons $\mathrm{Todd}(T^{vert}E) = 1 = \mathrm{Todd}(T^{vert}\hat{E})$, this yields the following,

Theorem 3.6. *In the notation of Sect. 3, there is a commutative diagram,*

$$\begin{array}{ccc} K^\bullet(E,H) & \xrightarrow{T_!} & K^{\bullet+1}(\hat{E},\hat{H}) \\ {\scriptstyle ch_H}\big\downarrow & & \big\downarrow{\scriptstyle ch_{\hat{H}}} \\ H^\bullet(E,H) & \xrightarrow{T_*} & H^{\bullet+1}(\hat{E},\hat{H}). \end{array} \tag{3.30}$$

The Grothendieck-Riemann-Roch formula in this context expresses this commutativity,

$$ch_{\hat{H}}(T_!(Q)) = T_*(ch_H(Q)) \tag{3.31}$$

for all $Q \in K^\bullet(E,H)$.

Equation. (3.31) can be re-expressed as

$$ch_{\hat{H}}(T_!(Q)) = \hat{p}_*(e^{\mathcal{B}} \wedge ch_H(Q)). \tag{3.32}$$

We begin by recalling that in [22] a homomorphism $ch_H : K^0(E,H) \to H^{even}(E,H)$ was constructed with the following properties:

1) ch_H is natural with respect to pullbacks,

2) ch_H respects the $K^0(E)$-module structure of $K^0(E,H)$,

3) ch_H reduces to the ordinary Chern character in the untwisted case when $H = 0$.

It was proposed that ch_H was the Chern character for twisted K-theory. We give a heuristic construction of ch_H here, referring to [22] and [23] for details.

Let $\lambda : P_H \to E$ be a principal PU bundle with given gerbe connection, and curving to be explained below. Let $\mathcal{E}_i \to P$ be U_{tr}-modules for the lifting bundle gerbe $L \to {P_H}^{[2]}$, where U_{tr} denotes the unitary operators of the form identity plus trace class - then $[\mathcal{E}_1] - [\mathcal{E}_0] \in K^0(E,H)$. That is, there is an action of L on $\mathcal{E}_i$ via an isomorphism $\psi : \pi_1^*\mathcal{E}_i \otimes L \to \pi_2^*\mathcal{E}_i$. We suppose that L comes equipped with a bundle gerbe connection ∇_L and a choice of curving f such that the associated 3-curvature is H, a closed, integral 3-form on E representing the image, in real cohomology, of the Dixmier-Douady class of P_H. Since the ordinary Chern character ch is multiplicative, we have

$$\pi_1^*(ch(\mathcal{E}_1) - ch(\mathcal{E}_0))ch(L) = \pi_2^*(ch(\mathcal{E}_1) - ch(\mathcal{E}_0)). \tag{3.33}$$

It turns out that this equation holds on the level of differential forms. Then $ch(L)$ is represented by the curvature 2-form F_L of the bundle gerbe connection ∇_L on L. A choice of a curving for ∇_L is a 2-form f on P_H such that $F_L = \delta(f) = \pi_1^* f - \pi_2^* f$ and f has the property that $df = \lambda^* H$. It follows that $ch(L)$ is represented by $\exp(F_L) = \exp(\pi_1^* f - \pi_2^* f) = \exp(-\pi_2^* f)\exp(\pi_1^* f)$. Therefore we can rearrange Eq. (3.33) above to get

$$\pi_1^* \exp(f)(ch(\mathcal{E}_1) - ch(\mathcal{E}_0)) = \pi_2^* \exp(f)(ch(\mathcal{E}_1) - ch(\mathcal{E}_0)). \tag{3.34}$$

Since we are assuming that Eq. (3.34) holds at the level of differential forms, this implies that the differential form $\exp(f)(ch(\mathcal{E}_1) - ch(\mathcal{E}_0))$ descends to a differential form on E which is clearly closed with respect to the twisted differential $d - H$, and is the Chern-Weil representative of the twisted Chern character. That is, $\lambda^* ch_H(\mathcal{E}_1 - \mathcal{E}_0) = \exp(f)(ch(\mathcal{E}_1) - ch(\mathcal{E}_0))$. We will use the simplified notation,

$$\lambda^* ch_H(Q) = e^f ch(Q), \qquad Q \in K^0(E, H). \tag{3.35}$$

In Sect. 5, [23], a similar formula was obtained for the odd twisted Chern character,

$$\lambda^* ch_H(Q) = e^f ch(Q), \qquad Q \in K^1(E, H). \tag{3.36}$$

We next study the Grothendieck-Riemann-Roch formula in twisted K-theory, following the computation of the Chern character of the topological index in [17, 18]. Let $\tau : \mathcal{Q} \longrightarrow E$ be a $spin^{\mathbb{C}}$ vector bundle over E and $i : E \longrightarrow \mathcal{Q}$ the zero section embedding. Let P_H be the principal PU-bundle over E: then for $\xi \in K^\bullet(E, H)$, we compute,

$$\begin{aligned} ch_{\tau^* H}(i_! \xi) &= ch_{\tau^* H}(i_! 1 \otimes \tau^* \xi) \\ &= ch(i_! 1) \cup ch_{\tau^* H}(\pi^* \xi), \end{aligned}$$

where we have used the fact that the Chern character respects the $K^0(E)$-module structure. The standard Riemann-Roch formula asserts that

$$ch(i_! 1) = i_* \mathrm{Todd}(\mathcal{Q})^{-1} = i_* 1 \cup \tau^* \mathrm{Todd}(\mathcal{Q})^{-1}.$$

Therefore we obtain the following Riemann-Roch formula for linear embeddings in twisted K-theory,

$$ch_{\tau^* H}(i_! \xi) = i_* \left\{ \mathrm{Todd}(\mathcal{Q})^{-1} \cup ch_H(\xi) \right\}. \tag{3.37}$$

We will refer to the commutative diagram (3.27) in what follows. Now $p_! = i_!^{-1} \circ j_!$, therefore for $\Xi \in K^\bullet(Z, p^* H)$,

$$ch_H(p_! \Xi) = ch_H(i_!^{-1} \circ j_! \Xi).$$

By the Riemann-Roch formula for linear embeddings in twisted K-theory, cf. (3.37),

$$ch_{p_1^* H}(i_! \Xi) = i_* ch_H(\Xi),$$

since $p_1 : E \times \mathbb{R}^{2N} \longrightarrow E$ is a trivial bundle. Since $p_{1*} i_* 1 = (-1)^n$, it follows that for $\ell \in K_c^\bullet(E \times \mathbb{R}^{2N}, p_1^* H)$, one has

$$ch_H(i_!^{-1} \ell) = (-1)^n p_{1*} ch_{p_1^* H}(\ell).$$

Therefore

$$ch_H(i_!^{-1} \circ j_! \, \Xi) = (-1)^n p_{1*} ch_{p_1^* H}(j_! \, \Xi). \tag{3.38}$$

By the Riemann-Roch formula for linear embeddings in twisted K-theory (3.37),

$$ch_{p_1^* H}(j_! \Xi) = j_* \left\{ \mathrm{Todd}(N)^{-1} \cup ch_{p^* H}(\Xi) \right\}, \tag{3.39}$$

where $N = N(E \times \mathbb{R}^{2N}/Z)$ is the complex normal bundle to the embedding $j : Z \longrightarrow E \times \mathbb{R}^{2N}$. Therefore $\mathrm{Todd}(N)^{-1} = \mathrm{Todd}(T(Z/E))$ and (3.39) becomes

$$ch_{p_1^* H}(j_! \Xi) = j_* \left\{\mathrm{Todd}(T(Z/E)) \cup ch_{p^* H}(\Xi)\right\}.$$

Therefore (3.38) becomes

$$\begin{aligned} ch_H(i_!^{-1} \circ j_! \Xi) &= (-1)^n p_{1*} j_* \left\{\mathrm{Todd}(T(Z/E)) \cup ch_{p^* H}(\Xi)\right\} \\ &= (-1)^n p_* \left\{\mathrm{Todd}(T(Z/E)) \cup ch_{p^* H}(\Xi)\right\} \end{aligned} \tag{3.40}$$

since $p_* = p_{1*} j_*$. Therefore

$$ch_H(p_! \Xi) = (-1)^n p_* \left\{\mathrm{Todd}(T(Z/E)) \cup ch_{p^* H}(\Xi)\right\}, \tag{3.41}$$

proving the Grothendieck-Riemann-Roch for K-oriented submersions. For a general K-oriented smooth map $f : N \to M$, we have seen that it can be factorized as $f = p_2 \circ i_f$, where $i_f : N \to N \times M$ is the graph embedding, and $p_2 : N \times M \to M$ is the submersion given by projection onto the second factor. Since $f_! = p_{2!} \circ i_{f!}$ and using the fact that we have obtained the Grothendieck-Riemann-Roch theorem for immersions and submersions in twisted K-theory, we can deduce it in the general case to get,

$$ch_H(f_! \Xi) = (-1)^n f_* \left\{\mathrm{Todd}(TN/f^* TM) \cup ch_{f^* H}(\Xi)\right\} . \tag{3.42}$$

The pullbacks and tensor products commute with the Chern map by the functoriality of the characteristic class, and so we need only verify that the pushforward commutes. In this case N is the correspondence space and M is $\hat{E}$ and so TN/f^*TM is one-dimensional, too small to have a nontrivial Todd class. Equation (3.42) then reduces to (3.31) up to a sign which may be absorbed into the definition of the K-theory pushforward map. We can apply this now to the commutative diagram (3.1) to deduce the formula (3.31) in Theorem 3.6.

Using (3.35), (3.36) and simplifying the notation, we compute,

$$\begin{aligned} ch_{\hat{H}}(T_!(Q)) &= ch_{\hat{H}}(\hat{p}_!(\mathcal{L} \otimes Q)) \\ &= \hat{p}_*(ch_{\hat{H}}(\mathcal{L} \otimes Q)) \\ &= \hat{p}_*(e^{\hat{f}} ch(\mathcal{L} \otimes Q)) \\ &= \hat{p}_*(e^{\hat{f}} e^{c_1(\mathcal{L})} ch(Q)) \\ &= \hat{p}_*(e^{\hat{f}} e^{\mathcal{B}+f-\hat{f}} ch(Q)) \\ &= \hat{p}_*(e^{\mathcal{B}} e^{f} ch(Q)) \\ &= \hat{p}_*(e^{\mathcal{B}} ch_H(Q)) = T_*(ch_H(Q)), \end{aligned} \tag{3.43}$$

proving Theorem 3.6.

It is possible to refine Theorem 3.6 to an equality on the level of differential forms, using the method in [40] - this will be done elsewhere.

4. 3-Dimensional Examples

4.1. Circle bundles over the 2-torus. Our first example is a slight generalization of a well-known example related to the Scherk-Schwarz compactification of string theory on $M^7 \times T^3$ (see, e.g., [7, 8]). Consider the 3-dimensional manifold E, a so-called nilmanifold, with metric

$$g = dx^2 + dy^2 + (dz + jx\,dy)^2\,, \tag{4.1}$$

and H-flux

$$H = k\,dx \wedge dy \wedge dz\,, \tag{4.2}$$

where the coordinates (x, y, z) are subject to the identifications

$$(x, y, z) \sim (x, y+1, z) \sim (x, y, z+1) \sim (x+1, y, z-jy)\,. \tag{4.3}$$

We can think of E as an S^1-bundle over $T^2 = \{(x, y)\}$ by

$$(x, y, z) \sim (x+1, y, z-jy)\,. \tag{4.4}$$

The S^1-bundle has a connection $A = dz + jx\,dy$, with first Chern class $c_1(E) = dA = j\,dx \wedge dy$, and

$$\int_E H = k\,, \qquad \int_M c_1(E) = j\,. \tag{4.5}$$

Let $\kappa = \partial/\partial z$ denote the Killing vector field associated with the circle action, *i.e.* $\mathcal{L}_\kappa g = 0 = \mathcal{L}_\kappa H$. Consider the coordinate patch $x \in (0, 1)$. We choose a gauge in which

$$B = kx\,dy \wedge dz\,, \tag{4.6}$$

so that $\mathcal{L}_\kappa B = 0$, and we can apply the Buscher rules [1] (see, e.g., App. A in [8] for a concise summary of these rules). We find a T-dual metric and B-field given by

$$\begin{aligned} \hat{g} &= dx^2 + dy^2 + (d\hat{z} + kx\,dy)^2\,, \\ \hat{B} &= jx\,dy \wedge d\hat{z}\,. \end{aligned} \tag{4.7}$$

I.e., the T-dual corresponds again to an S^1-bundle over T^2, this time with H-flux related to the initial configuration by the interchange $j \leftrightarrow k$, in accordance with Eq. (1.8). Note, moreover, that

$$A \wedge \hat{A} = dz \wedge d\hat{z} - kx\,dy \wedge dz + jx\,dy \wedge d\hat{z} = dz \wedge d\hat{z} - B + \hat{B}\,, \tag{4.8}$$

so that locally Eq. (1.9) does indeed agree with Eq. (1.1). For a discussion of the isomorphism of K-theories we refer to the next section, where the more general case of circle bundles over a Riemann surface is discussed.

Note that this particular example clearly illustrates the possible obstruction to T-dualizing over a two-torus (cf. the discussion in [8]). Upon starting with a three-torus (the case $j = 0$ in the above), with k units of H-flux and three commuting circle actions, T-dualizing over one circle leaves us with a circle bundle (the nilmanifold) with only one (global) S^1-action left, the circle action on the dual S^1.

4.2. Circle bundles on a Riemann surface. In this section we will find the twisted K-groups of circle bundles over 2-manifolds and their T-duals and show that K^0 of each space is related to K^1 of its dual. This class of examples will be seen to include the familiar examples of NS5-branes, 3-dimensional lens spaces and nilmanifolds. The K-groups in the examples of this section (but not the next) will be uniquely determined by the Atiyah-Hirzebruch spectral sequence [36, 41]. In fact it will suffice to consider only the first differential

$$d_3 = Sq^3 + H \tag{4.9}$$

of the sequence. Furthermore the Sq^3 term will be trivial, although it would be interesting to test this correspondence in an example in which the Sq^3 term is nontrivial. Thus the K-classes will consist of cohomology classes whose cup product with the NS fieldstrength H vanishes quotiented by those classes that are themselves cup products of classes by H. Explicitly, if $H^{even}(E,\mathbb{Z})$ and $H^{odd}(E,\mathbb{Z})$ are the even and odd cohomology classes of the manifold E with integer coefficients, then the twisted K-groups are

$$K^0(E,H) = \frac{\ker(H\cup : H^{even} \to H^{odd})}{H \cup H^{odd}(E,\mathbb{Z})}, \quad K^1(E,H) = \frac{\ker(H\cup : H^{odd} \to H^{even})}{H \cup H^{even}(E,\mathbb{Z})}. \tag{4.10}$$

More precisely, this procedure only yields the associated graded algebras of the twisted K-theory, to find the actual K-groups from these one must in general solve an extension problem. That is to say, torsion classes in $H^p(E,H)$ may mix with classes in H^{p+2}, yielding the wrong answer. However H^p only has torsion classes for $p \geq 2$ and H^{p+2} is only nontrivial for manifolds of dimension $d \geq p+2$. Thus the associated graded algebras only differ from the K-groups for manifolds of dimension $d \geq p+2 \geq 4$. In this section we will consider only 3-dimensional examples and so will not need to concern ourselves with the extension problem. In the next section we will.

Circle bundles E over a manifold M are entirely classified by their first Chern class

$$c_1(E) = F \in H^2(M,\mathbb{Z}), \tag{4.11}$$

where F is the curvature of the bundle and $H^2(M,\mathbb{Z})$ is the manifold's second cohomology group with integer coefficients.[9] In the case of an orientable 2-manifold, like the 2-sphere or a more general genus g Riemann surface, $H^2(M,\mathbb{Z}) = \mathbb{Z}$ and so topologically circle bundles are classified by an integer j.

If the circle bundle is the trivial bundle $j = 0$, then the cohomology of the total space E of the bundle is given by the Künneth formula

$$H^0(E,\mathbb{Z}) = \mathbb{Z}, \qquad H^1(E,\mathbb{Z}) = \mathbb{Z}^{2g+1}, \quad H^2(E,\mathbb{Z}) = \mathbb{Z}^{2g+1}, \quad H^3(E,\mathbb{Z}) = \mathbb{Z}. \tag{4.12}$$

A quick application of the Meyer-Vietoris sequence shows that if the Chern class is equal to $j \neq 0$ then the cohomology of E is

$$H^0(E,\mathbb{Z}) = \mathbb{Z}, \qquad H^1(E,\mathbb{Z}) = \mathbb{Z}^{2g}, \quad H^2(E,\mathbb{Z}) = \mathbb{Z}^{2g} \oplus \mathbb{Z}_j, \quad H^3(E,\mathbb{Z}) = \mathbb{Z}. \tag{4.13}$$

[9] Factors of 2π will be systematically absorbed into curvatures to make all quantities integral.

The $\mathbb{Z}^{2g}$'s will not play any important role in what follows, and so the reader may choose to ignore them and consider only the 2-sphere case, $g = 0$.

The H-flux inhabits $H^3(E, \mathbb{Z}) = \mathbb{Z}$ and so the possible flux is classified by another integer k. We will always choose a basis for H^2 and H^3 such that j and k are nonnegative. The cup product with an element of H^3 increases the dimension of a cocycle by 3, so it is only nontrivial on 0-cocycles, which it maps to 3-cocycles: $H^0 \to kH^3$. If $k = 0$ then $H = 0$ and so everything is in the kernel of $d_3 = H\cup$. The image of $H\cup$ in this case is trivial, and so the untwisted K-theory is simply the cohomology

$$K^0(E, H = 0) = H^0(E, \mathbb{Z}) \oplus H^2(E, \mathbb{Z}) = \begin{cases} \mathbb{Z}^{2g+2} & \text{if } j = 0\,, \\ \mathbb{Z}^{2g+1} \oplus \mathbb{Z}_j & \text{if } j \neq 0\,, \end{cases}$$
$$K^1(E, H = 0) = H^1(E, \mathbb{Z}) \oplus H^3(E, \mathbb{Z}) = \begin{cases} \mathbb{Z}^{2g+2} & \text{if } j = 0\,, \\ \mathbb{Z}^{2g+1} & \text{if } j \neq 0\,. \end{cases} \quad (4.14)$$

If $k \neq 0$ then the kernel of $H\cup$ consists of all cocycles of dimension greater than 0. The image consists of all 3-cocycles that are multiples of k, that is, the image is $kH^3(E, \mathbb{Z}) = k\mathbb{Z}$. The quotient of the kernel by the image yields the K-groups

$$K^0(E, H = k) = H^2(E, \mathbb{Z}) = \begin{cases} \mathbb{Z}^{2g+1} & \text{if } j = 0\,, \\ \mathbb{Z}^{2g} \oplus \mathbb{Z}_j & \text{if } j \neq 0\,, \end{cases}$$
$$K^1(E, H = k) = H^1(E, \mathbb{Z}) \oplus H^3(E, \mathbb{Z})/kH^3(E, \mathbb{Z})$$
$$= \begin{cases} \mathbb{Z}^{2g+1} \oplus \mathbb{Z}_k & \text{if } j = 0\,, \\ \mathbb{Z}^{2g} \oplus \mathbb{Z}_k & \text{if } j \neq 0\,. \end{cases} \quad (4.15)$$

According to Eq. (1.8) T-duality is the interchange of j and k. In every case above this results in the twisted K-groups $K^0(E, H)$ and $K^1(E, H)$ being interchanged, which corresponds to the fact that RR fieldstrengths are classified by $K^0(E, H)$ in type IIA string theory and by $K^1(E, H)$ in IIB. This means that one can find the new RR fieldstrengths from the old ones by applying the isomorphism between the two K-groups.[10] In this example it is quite straightforward, one simply interchanges the $\mathbb{Z}^{2g}$ between H^1 and H^2 and the rest of the cohomology groups are swapped $H^0 \leftrightarrow H^1,\ H^2 \leftrightarrow H^3$.

4.3. Comparison with the literature. Several subcases of this class of examples have been studied in the literature. For example, consider type II string theory on $\mathbb{R}^9 \times S^1$ with a stack of k NS5-branes at the same point in a transverse $\mathbb{R}^3 \times S^1$. Consider a 2-sphere $S^2 \subset \mathbb{R}^3$ such that $S^2 \times S^1$ links the stack once. The generalization to an arbitrary Riemann surface is straightforward. The integral of H over $S^2 \times S^1$ follows from Gauss' law

$$\int_{S^2 \times S^1} H = k\,. \quad (4.16)$$

The circle is trivially fibered over S^2 and so, in the above notation, the first Chern class j vanishes.

T-duality interchanges j and k, which means that the T-dual configuration has no H-flux, so that the NS5-branes have disappeared. Instead the circle bundle is now nontrivially fibered, with a first Chern class of k over each Riemann surface that links (once)

[10] The general prescription for computing the dual fieldstrengths is given in Sect. 3.

the place where the stack was. This configuration is a charge k Kaluza-Klein monopole solution, which is known to be T-dual to k NS5-branes that do not wrap the dualized circle (see, e.g., [42] and references therein).

If we restrict to a linking 2-sphere, we obtain an isomorphism of the twisted K-theories of lens spaces $L(1, p) = S^3/\mathbb{Z}_p$,

$$K^i(L(1, j), H = k) \cong K^{i+1}(L(1, k), H = j)\,. \tag{4.17}$$

We recall that $L(1, p) = S^3/\mathbb{Z}_p$ is the total space of the circle bundle over the 2-sphere with Chern class equal to p times the generator of $H^2(S^2, \mathbb{Z}) \cong \mathbb{Z}$. Note that $L(1, 1) = S^3$ and $L(1, 0) = S^2 \times S^1$.

In the case of a single NS5-brane, $j = 1$, the total space of the circle bundle over the linking 2-sphere is a 3-sphere, the group manifold of $SU(2)$. Thus we obtain an isomorphism

$$K^i(SU(2), H = k) \cong K^{i+1}(L(1, k), H = 1)\,, \tag{4.18}$$

between the K-theory of $SU(2)$, twisted by $H = k \in H^3(S^3, \mathbb{Z})$ and the (parity shifted) K-theory of the lens space $L(1, k)$ twisted by only one unit.

The special case of string theory on a 7-manifold crossed with the 3-torus T^3 with k units of H-flux on the T^3 was considered in Sect. 4.1. This is a trivial circle bundle over T^2, and so $g = 1$ and $j = 0$. Using Eq. (1.8), T-duality along any circle yields a circle bundle over T^2 with Chern class k and no H-flux. The total space of this bundle, in agreement with the literature, is just the k^{th} nilmanifold.

An example along the lines of that in Ref. [6] is IIB on $AdS^3 \times S^3 \times T^4$ with N units of G_3-flux supported on the S^3. The 3-sphere is a circle bundle over S^2 with Chern class $j = 1$ and one may T-dualize this fiber. The Chern class is converted into H-flux, and because we began with no H-flux the resulting bundle is trivial. This leaves type IIA on $AdS^3 \times S^2 \times S^1 \times T^4$. There is now one unit of H-flux supported on the $S^2 \times S^1$, as a result of the Chern class of the original bundle. The isomorphism of K-groups exchanged H^2 and H^3 and so the G_3-flux becomes G_2-flux. Thus we find

$$\int_{S^2\times S^1} H = 1\,, \qquad \int_{S^2} G_2 = N\,. \tag{4.19}$$

The large N duality to a $2d$ conformal field theory is much more mysterious in this framework, even the R-symmetry is nontrivially encoded in the geometry.

4.4. Bundles over $\mathbb{R}P^2$. In this section we will consider T-dualities of the two circle bundles over $\mathbb{R}\mathrm{P}^2$. To obtain the rest of the nonorientable 3-manifolds which are circle bundles, one needs only connect sum the $\mathbb{R}\mathrm{P}^2$ with a Riemann surface, which, as above will add factors of $\mathbb{Z}^{2g}$ which will play no role. However the nonorientable cases are more difficult to adapt to string theory because we cannot make a consistent background for type II by simply (topologically) crossing them with a 7-manifold, as the total space will continue to be nonorientable. To make a consistent string theory background from this example one has several choices. For example, one may consider an orientifold projection, or one may consider a topology which is only locally this example crossed with a 7-manifold. In the first case, complex twisted K-theory will no longer be the K-theory which classifies fluxes and branes. In the second, the relevant complex K-theory will not simply be the tensor of the K-theory that we find below with that of the 7-manifold.

So in either case, adapting the results below to classify fluxes in a string background is less trivial than for the other examples of this note. However this example does illustrate that the twisted K-theory isomorphism appears to work when H is torsion and also for nonorientable manifolds (although, strictly speaking, in the discussion up to now we have assumed the S^1-bundle to be orientable).

To classify bundles on $\mathbb{R}\mathrm{P}^2$, we must first know its $\mathbb{Z}$-valued cohomology:

$$H^0(\mathbb{R}\mathrm{P}^2,\mathbb{Z})=\mathbb{Z}\,,\qquad H^1(\mathbb{R}\mathrm{P}^2,\mathbb{Z})=0\,,\qquad H^2(\mathbb{R}\mathrm{P}^2,\mathbb{Z})=\mathbb{Z}_2\,. \tag{4.20}$$

T-duality interchanges the Chern class with the H-flux. If both of them are zero then it takes the trivial bundle with no H-flux to itself. It interchanges K^0 and K^1, which is consistent with the fact that they are isomorphic.

We next consider the trivial bundle with 1 unit of H-flux. The cup product of this H-flux with $k\in H^0(\mathbb{R}\mathrm{P}^2\times S^1)=\mathbb{Z}$ is $k\in H^3(\mathbb{R}\mathrm{P}^2\times S^1)=\mathbb{Z}_2$ and so is zero if k is even and one if k is odd. Thus the subset of H^0 that is in the kernel of $H\cup$ consists of the even integers $2\mathbb{Z}\cong\mathbb{Z}$ which are isomorphic to the integers. The rest of the cohomology is automatically in the kernel. The image consists of H^3, and so the quotient of the kernel by the image is

$$\begin{aligned}K^0(\mathbb{R}\mathrm{P}^2\times S^1,H=1)&=2H^0\oplus H^2=\mathbb{Z}\oplus\mathbb{Z}_2,\\ K^1(\mathbb{R}\mathrm{P}^2\times S^1,H=1)&=H^1\oplus H^3/H^3=\mathbb{Z}.\end{aligned} \tag{4.21}$$

The T-dual is obtained by interchanging the Chern class of the bundle, which is zero, with H, which is one.

The result is the nontrivial bundle with no H-flux. A simple construction of this nontrivial bundle is as follows. It is the nontrivial S^2 bundle over S^1. That is to say, begin with the $3d$ cylinder $S^2\times I$, where I is the interval. Glue the S^2's at the two ends of the cylinder together by attaching each point on the S^2 to its antipodal point $(x,0)\sim(-x,1)$, as one would construct the Klein bottle in the case of a $2d$ cylinder. To see that the resulting space is E, an S^1 bundle over $\mathbb{R}\mathrm{P}^2$, notice that there is an S^1 action given by moving along the circle which we constructed by gluing together the two ends of the interval. If one begins at $(x,0)$, one arrives later at $(x,1)\sim(-x,0)$ and later at $(-x,1)\sim(x,0)$ once again. Thus the space of orbits of this circle action is just the 2-sphere with x and $-x$ identified. As desired, this is $\mathbb{R}\mathrm{P}^2$. The projection map $E\to\mathbb{R}\mathrm{P}^2$ identifies each orbit with the corresponding point in $\mathbb{R}\mathrm{P}^2$.

We find the homology of E analogously to the case of the $2d$ Klein bottle. The circle generates $H_1(E,\mathbb{Z})=\mathbb{Z}$. The two-sphere is the generator $x\in H_2$, but it gets identified with its mirror image, and so $x\sim -x$ because the antipodal map negates the orientation of even dimensional spheres. This yields the relation $2x=0$ and so $H_2(E,\mathbb{Z})=\mathbb{Z}_2$. The space is not orientable and so the top homology class vanishes $H_3(E,\mathbb{Z})=0$. The universal coefficient theorem allows us to find the cohomology of E,

$$H^0(E,\mathbb{Z})=\mathbb{Z},\qquad H^1(E,\mathbb{Z})=\mathbb{Z},\qquad H^2(E,\mathbb{Z})=0,\qquad H^3(E,\mathbb{Z})=\mathbb{Z}_2\,. \tag{4.22}$$

The T-dual of the trivial bundle with H-flux is E with no flux, and so the twisted K-theory is the untwisted K-theory

$$K^0(E)=H^0(E,\mathbb{Z})\oplus H^2(E,\mathbb{Z})=\mathbb{Z}\,,\quad K^1(E)=H^1(E,\mathbb{Z})\oplus H^3(E,\mathbb{Z})=\mathbb{Z}\oplus\mathbb{Z}_2\,. \tag{4.23}$$

As desired, K^0 and K^1 are the same as the K-groups K^1 and K^0 of the T-dual in Eq. (4.21).

There is one more case. The nontrivial bundle E may support one unit of H-flux. Taking the cohomology with respect to the cup product by H proceeds identically to the case of the trivial bundle discussed above, and we find

$$K^0(E, H = 1) = 2H^0 \oplus H^2 = \mathbb{Z}, \quad K^1(E, H = 1) = H^1 \oplus H^3/H^3 = \mathbb{Z}. \tag{4.24}$$

These are the same K-groups as those found in (4.21) except that $H^2(E, \mathbb{Z}) = 0 \neq H^2(\mathbb{R}\mathrm{P}^2 \times S^1, \mathbb{Z}) = \mathbb{Z}_2$ and so K^0 does not contain a $\mathbb{Z}_2$-factor here. This is crucial, as it means that $K^0(E, H = 1) = K^1(E, H = 1)$. This configuration is self-dual under T-duality, interchanging K^0 and K^1.

5. Application: Circle Bundles over $\mathbb{R}\mathrm{P}^n$

In general calculating the twisted K-theory of high-dimensional manifolds is quite difficult as many of the differentials of the Atiyah-Hirzebruch spectral sequence for twisted K-theory are not known. Except for the H-term in d_3 used above, these differentials d_{2k+1} take *even* or *odd* cohomology classes to the torsion part of *odd* or *even* cohomologies. As we will see, the odd cohomology classes of $\mathbb{R}\mathrm{P}^n$ do not contain any torsion, and so no differentials have an image in odd cohomology. Furthermore the only odd cohomology class that is nonvanishing is the top-dimensional one, which is automatically annihilated by all differentials, and so all odd dimensional cohomology is in the kernel of the differentials. Thus, except for the $H\cup$ term used above, all of the differentials act trivially on the cohomology of $\mathbb{R}\mathrm{P}^n$. No extra complication is introduced by crossing with a circle, and the nontrivial circle bundle is in fact even simpler. The result is that all K-groups in this subsection can be found by taking the elements of the cohomology that are annihilated by H and quotienting by those that are cup products with H, just as in the three-dimensional case.

As explained above, an additional complication arises in the case of manifolds of dimension greater than 3. The spectral sequence does not necessarily yield the desired twisted K-groups, but only an associated graded algebra. To find the K-groups, in general one must then solve an extension problem. We will see that in this set of examples T-duality maps bundles with a nontrivial extension problem to bundles with a trivial extension problem, and so T-duality will provide the extension problem's solution.

It will prove to be convenient to treat the case of odd and even n separately. For example, the $\mathbb{R}\mathrm{P}^{2m+1}$'s are orientable and the $\mathbb{R}\mathrm{P}^{2m}$'s are not. It is therefore the odd n cases that are directly applicable to consistent type II string theory compactifications. The nontrivial integral cohomology groups are

$$\begin{aligned} &H^0(\mathbb{R}\mathrm{P}^n, \mathbb{Z}) = \mathbb{Z}, \qquad H^{2p}(\mathbb{R}\mathrm{P}^n, \mathbb{Z}) = \mathbb{Z}_2\,, \quad H^{2m+1}(\mathbb{R}\mathrm{P}^{2m+1}, \mathbb{Z}) = \mathbb{Z}\,, \\ &\qquad\qquad p = 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor . \end{aligned} \tag{5.1}$$

The cohomology of the trivial circle bundle is similarly

$$\begin{aligned} &H^0(\mathbb{R}\mathrm{P}^n \times S^1, \mathbb{Z}) = H^1(\mathbb{R}\mathrm{P}^n \times S^1, \mathbb{Z}) = \mathbb{Z}, \quad H^q(\mathbb{R}\mathrm{P}^n \times S^1, \mathbb{Z}) = \mathbb{Z}_2, \\ &q = 2, \dots, n-1\,, \quad H^{2m}(\mathbb{R}\mathrm{P}^{2m} \times S^1, \mathbb{Z}) = H^{2m+1}(\mathbb{R}\mathrm{P}^{2m} \times S^1, \mathbb{Z}) = \mathbb{Z}_2\,, \\ &H^{2m+1}(\mathbb{R}\mathrm{P}^{2m+1} \times S^1, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2, \quad H^{2m+2}(\mathbb{R}\mathrm{P}^{2m+1} \times S^1, \mathbb{Z}) = \mathbb{Z}, \end{aligned} \tag{5.2}$$

where we have assumed that $n > 1$, thus losing the case of $\mathbb{R}\mathrm{P}^1$ in which no nontrivial fibrations are possible.

Possible twists are elements of the third cohomology group

$$H^3(\mathbb{R}\mathrm{P}^n \times S^1, \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } n = 3\,, \\ \mathbb{Z}_2 & \text{if } n \neq 3\,. \end{cases}$$

The extra $\mathbb{Z}$ in the special case of $\mathbb{R}\mathrm{P}^3$ consists of classes in $H^3(\mathbb{R}\mathrm{P}^3) = \mathbb{Z}$, and not in $H^2(\mathbb{R}\mathrm{P}^3, \mathbb{Z}) \otimes H^1(S^1, \mathbb{Z}) = \mathbb{Z}_2$. Therefore when integrated over the circle H-twists in this $\mathbb{Z}$ are trivial, and do not change the topology of the T-dual manifold. Of course, it is possible that H is the sum of such a class with the nontrivial element of the $\mathbb{Z}_2$, that is $H = (k, 1)$. In this case it will be a critical consistency check of our conjecture that the T-dual manifold also have a subgroup $\mathbb{Z} \subset H^3(E, \mathbb{Z})$ so that there may be a T-dual flux $\hat{H} = (k, 0)$. We will see that the cohomology of the T-dual does in fact have such a subgroup.

We begin again with the case of vanishing H-flux. In this case the K-theory is simply the cohomology

$$\begin{aligned} K^0(\mathbb{R}\mathrm{P}^{2m} \times S^1) &= \bigoplus_p H^{2p} = \mathbb{Z} \oplus \mathbb{Z}_2^m\,, \\ K^1(\mathbb{R}\mathrm{P}^{2m} \times S^1) &= \bigoplus_p H^{2p+1} = \mathbb{Z} \oplus \mathbb{Z}_2^m\,, \\ K^0(\mathbb{R}\mathrm{P}^{2m+1} \times S^1) &= \bigoplus_p H^{2p} = \mathbb{Z}^2 \oplus \mathbb{Z}_2^m\,, \\ K^1(\mathbb{R}\mathrm{P}^{2m+1} \times S^1) &= \bigoplus_p H^{2p+1} = \mathbb{Z}^2 \oplus \mathbb{Z}_2^m\,. \end{aligned} \tag{5.3}$$

As the Thom isomorphism or equivalently here the Künneth theorem guarantees, in each case $K^0 \cong K^1$ and so T-duality on the circle simply acts by interchanging classes in these two K-groups. As a check on these results, one may recall that $\mathbb{R}\mathrm{P}^{2m+1}$ is a circle bundle over $\mathbb{C}\mathrm{P}^m$ with two units of Chern class and one may T-dualize about that circle. This yields $\mathbb{C}\mathrm{P}^m \times S^1$ with $H = 2$. The twisted K-theory of $\mathbb{R}\mathrm{P}^{2m+1} \times S^1$ is then just the cohomology of $\mathbb{C}\mathrm{P}^m \times T^2$, which consists of $\mathbb{Z}^2$ for each group, quotiented by H. A quick calculation shows that these twisted K-groups agree with their T-duals in Eq. (5.3).

If we turn on nontrivial H-flux in the $\mathbb{Z}_2 \subset H^3$ then the twisted K-theory will be the kernel of $H\cup$ quotiented by its image. This flux cups nontrivially on even cohomology groups, taking each to the Z_2 torsion part of the odd group three dimensions higher. In particular all torsion odd cohomology groups are in the image and so are quotiented out of the K-theory. Only even elements of the even-dimensional cohomology groups are in the kernel, which means only the zero elements of the torsion groups, and $2\mathbb{Z} \cong \mathbb{Z}$ in H^0. All of H^{2m-1} and H^{2m} is in the kernel for dimensional reasons. In sum, the twisted K-theory is

$$\begin{aligned} &K^0(\mathbb{R}\mathrm{P}^{2m} \times S^1, H = 1) \overset{?}{=} 2H^0 \oplus H^{2m} = \mathbb{Z} \oplus \mathbb{Z}_2\,, \\ &K^1(\mathbb{R}\mathrm{P}^{2m} \times S^1, H = 1) \overset{?}{=} 2H^1 = \mathbb{Z}\,, \\ &K^0(\mathbb{R}\mathrm{P}^{2m+1} \times S^1, H = 1) \overset{?}{=} 2H^0 \oplus H^{2m} \oplus H^{2m+2} = \mathbb{Z}^2 \oplus \mathbb{Z}_2\,, \\ &K^1(\mathbb{R}\mathrm{P}^{2m+1} \times S^1, H = 1) \overset{?}{=} 2H^1 \oplus H^{2m+1} = \mathbb{Z}^2\,. \end{aligned} \tag{5.4}$$

The question marks indicate that there is a nontrivial extension problem to solve here, which will be solved later by imposing our T-duality conjecture and also argued from the explicit construction of our isomorphism. As noted above, in the case of $\mathbb{R}\mathrm{P}^3 \times S^1$, one may also add m units of nontorsion H-flux. In this case the $\mathbb{Z}^2$'s above are replaced by $\mathbb{Z}_m$'s.

The T-dual is the nontrivial circle bundle E_n over $\mathbb{R}\mathrm{P}^n$, which as above is an S^n-bundle over the circle made from $S^n \times I$ via the gluing $(x, 0) \sim (-x, 1)$. Notice however that in the case of odd $n = 2m + 1$ the map $x \mapsto -x$ is homotopic to the identity, and so for odd n the T-dual space is $S^1 \times S^{2m+1}$. The cohomology is found as in the $\mathbb{R}\mathrm{P}^2$ case for n even and by Künneth for n odd to be

$$\begin{aligned} &H^0(E_n, \mathbb{Z}) = H^1(E_n, \mathbb{Z}) = \mathbb{Z}\,, \qquad H^{2m}(E_{2m+1}, \mathbb{Z}) = \mathbb{Z}\,, \\ &H^{2m+1}(E_{2m}, \mathbb{Z}) = \mathbb{Z}_2\,, \qquad H^{2m+1}(E_{2m+1}, \mathbb{Z}) = \mathbb{Z}. \end{aligned} \tag{5.5}$$

This allows for H-flux only in the cases of $\mathbb{R}\mathrm{P}^2$, treated above, and also $\mathbb{R}\mathrm{P}^3$. $H^3(\mathbb{R}\mathrm{P}^3, \mathbb{Z}) = \mathbb{Z}$ and so the H-flux may assume any integer value, which is reassuring as the T-dual also allowed for an extra integer in the definition of the H-flux. These two integers must agree.

Thus we need consider only the case of vanishing H-flux, and so the K-groups are just the cohomology groups

$$\begin{aligned} &K^0(E_{2m}) = H^0 = \mathbb{Z}\,, \qquad K^1(E_{2m}) = H^1 \oplus H^{2m+1} = \mathbb{Z} \oplus \mathbb{Z}_2\,, \\ &K^0(E_{2m+1}) = H^0 \oplus H^{2m} = \mathbb{Z}^2\,, \qquad K^1(E_{2m+1}) = H^1 \oplus H^{2m+1} = \mathbb{Z}^2\,. \end{aligned} \tag{5.6}$$

These groups are all consistent with their T-duals as calculated in Eq. (5.4), except for

$$K^1(E_{2m+1}) = \mathbb{Z}^2 \quad \neq \quad \mathbb{Z}^2 \oplus \mathbb{Z}_2 = K^0(\mathbb{R}\mathrm{P}^n \times S^1, H = 1)\,. \tag{5.7}$$

From this we infer that the associated graded algebra and the K-group are in fact different in this case. The relevant extension problem is

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow K^1(E_{2m+1}) \longrightarrow \mathbb{Z}_2 \longrightarrow 0, \tag{5.8}$$

which admits $\mathbb{Z}^2$ as a solution as well as $\mathbb{Z}^2 \oplus \mathbb{Z}_2$, the solution which we assumed above. Our T-duality conjecture appears to predict that the desired solution is $\mathbb{Z}^2$.

This solution to the extension problem can be inferred topologically from our construction of the isomorphism of twisted K-groups. The fibered product of our two circle bundles is $S^{2m+1} \times S^1 \times \hat{S}^1$ and it fits into the following commutative diagram:

$$\begin{tikzcd} & S^{2m+1}\times S^1\times \hat{S}^1 \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \\ \mathbb{R}\mathrm{P}^{2m+1}\times S^1 \arrow[dr, "\pi"'] & & S^{2m+1}\times \hat{S}^1 \arrow[dl, "\hat{\pi}"] \\ & \mathbb{R}\mathrm{P}^{2m+1} & \end{tikzcd} \tag{5.9}$$

Recall that the top cohomology group of our trivial $\mathbb{R}\mathrm{P}^{2m+1}$ bundle is $H^{2m+2}(\mathbb{R}\mathrm{P}^{2m+1}\times S^1,\mathbb{Z})=\mathbb{Z}$. This is the Poincaré dual of a point x. The key realization is that the preimage of this point $p^{-1}(x)$ is a circle which wraps $\hat{S}^1$ *twice*. This is because the projection map p projects to the orbits of a circle which simultaneously wraps $\hat{S}^1$ and acts on the S^{2m+1} via a nonvanishing vectorfield scaled such that after wrapping $\hat{S}^1$ once, one arrives at the antipodal point in S^{2m+1}. Thus the orbit only closes after wrapping $\hat{S}^1$ a second time.

Our isomorphism, acting now on integral homology, takes x to $\hat{p}p^{-1}(x)$, which again wraps $\hat{S}^1$ twice. The tensoring with the Poincaré bundle is trivial because $p^{-1}(x)$ does not wrap S^1. In sum, we have found that

$$T_*: H_0(\mathbb{R}\mathrm{P}^{2m+1}\times S^1,\mathbb{Z})=\mathbb{Z} \quad\longrightarrow\quad H_1(S^{2m+1}\times\hat{S}^1,\mathbb{Z})=\mathbb{Z}\ :\ 1\mapsto 2\,. \tag{5.10}$$

This means that the class $1\in H_0(\mathbb{R}\mathrm{P}^{2m+1}\times S^1,\mathbb{Z})$ actually corresponds to the class 2 in twisted K-theory, which is only consistent if the solution to the extension problem (5.8) is given by

$$K^1(E_{2m+1})=\mathbb{Z}^2\,. \tag{5.11}$$

6. Anomalies

6.1. Quotients of $AdS^5\times S^5$. A more nontrivial check of our conjecture (1.8) comes in its application to circle bundles on $\mathbb{C}\mathrm{P}^2$. We have $H^2(\mathbb{C}\mathrm{P}^2)=\mathbb{Z}$, and so again circle bundles are parametrized by a single integer j. The total space of such a bundle is the lens space $L(2,j)$, i.e. the nonsingular quotient $E=S^5/\mathbb{Z}_j$, when $j\neq 0$, and $E=\mathbb{C}\mathrm{P}^2\times S^1$ when $j=0$. The nonvanishing integral cohomology groups are (see, e.g., [15])

$$H^{0\le p\le 5}(\mathbb{C}\mathrm{P}^2\times S^1)=\mathbb{Z}\,, \tag{6.1}$$
$$H^0(L(2,j),\mathbb{Z})=H^5(L(2,j),\mathbb{Z})=\mathbb{Z}\,,\qquad H^2(L(2,j),\mathbb{Z})=H^4(L(2,j),\mathbb{Z})=\mathbb{Z}_j\,.$$

Thus H-flux is only possible for the trivial bundle $j=0$, as the nontrivial bundles have trivial third cohomology. In the case of the trivial bundle, the cup product with the H-flux maps H^0 to H^3 and H^2 to H^4 while H^1, H^3 and H^5 are all in $ker(d_3 = H\cup)$. The next differential in the spectral sequence, d_5, may act nontrivially on the cohomology ring,[11] but is trivial on the kernel of d_3 and so does not affect the twisted K-theory of $\mathbb{C}P^2 \times S^1$.

T-duality relates the trivial bundle with $H = j$ to the bundle with first Chern class j and no flux. The twisted K-theory of the former is

$$\begin{aligned} K^0(\mathbb{C}P^2 \times S^1, H = j) &= H^4(\mathbb{C}P^2 \times S^1, \mathbb{Z}) = \mathbb{Z}\,, \\ K^1(\mathbb{C}P^2 \times S^1, H = j) &= H^1 \oplus H^3 \oplus H^5/(jH^3 \oplus jH^5) = \mathbb{Z} \oplus \mathbb{Z}_j{}^2\,. \end{aligned} \tag{6.2}$$

In the latter case H vanishes and so the K-groups are just the cohomology groups

$$\begin{aligned} K^0(L(2, j)) &= H^0 \oplus H^2 \oplus H^4 = \mathbb{Z} \oplus \mathbb{Z}_j^2\,, \\ K^1(L(2, j)) &= H^1 \oplus H^3 \oplus H^5 = \mathbb{Z}\,. \end{aligned} \tag{6.3}$$

And so we see that cases (6.2) and (6.3) differ by the exchange of K^0 and K^1 as desired.

Of course such T-dualities are interesting because IIB string theory on $AdS^5 \times S^5$ is comparably well understood. This $j = 1$ example of the above T-duality was first studied in Ref. [6] where it was observed that the spacetime on the IIA side is not $spin$, making the duality quite nontrivial.

The resulting RR fluxes are easily computed. If we start with N units of G_5-flux supported on $L(2, j)$ in IIB, then in IIA there will be N units of G_4-flux supported on $\mathbb{C}P^2$ and j units of H-flux supported on $H^2(\mathbb{C}P^2, \mathbb{Z}) \otimes H^1(S^1, \mathbb{Z})$.

6.2. Gravitino anomalies before and after. One might worry that type IIA string theory (and also its M-theory lift) on a non-$spin$ manifold is inconsistent, because the gravitino requires a spin structure to exist. There is no such anomaly on the IIB side, whose space-time is the $spin$ manifold $AdS^5 \times S^5$, thus it is a critical check of this duality that the anomaly be cancelled on the IIA side.

The authors of Ref. [6] have shown that the anomaly is in fact cancelled. This cancellation is a result of the $11d$ supergravity coupling

$$\mathcal{L}_{11d} \supset \overline{\Psi} G_4 \Psi \tag{6.4}$$

of the gravitino Ψ to the 4-form fieldstrength G_4. Dimensionally reducing away the M-theory circle and S^1_{IIA} one finds, among other terms, the 9-dimensional coupling

$$\mathcal{L}_{9d} \supset \overline{\Psi} F_2 \Psi \tag{6.5}$$

identifying the gravitino as a fermion charged under a $U(1)$ gauge symmetry. The anomaly should be independent of the high energy physics such as the massive KK-modes which are uncharged under this $U(1)$.

Such a fermion may be consistent on a manifold M that is not $spin$, but is merely $spin^{\mathbb{C}}$ if the second Stiefel-Whitney class $w_2(M)$ is equal to twice the fermions charge Q multiplied by the Chern class of the bundle

$$w_2(TM) = 2Qc_1(E)\,. \tag{6.6}$$

[11] Whether it does depends on an ill-defined division by 2 in Ref. [44].

The right-hand side of this equation is naturally an element of $H^2(M)$ with integral coefficients. The left-hand side of course is an element of cohomology with $U(1)$ coefficients, but due to the $spin^{\mathbb{C}}$ condition it also lifts to integral cohomology. To find $c_1(E)$, recall that, according to the E_8 interpretation, the fibers of the $U(1)$ bundle are just the circle S^1_{IIB} that appears in type IIB. Thus the Chern class is $j = 1$, more precisely, it is the generator of $H^2(\mathbb{C}\mathrm{P}^2, \mathbb{Z})$. The second Stiefel-Whitney class of the IIA spacetime is the same class, and so the anomaly cancellation condition (6.6) is only satisfied if Q is half-integral.

In Ref. [6] it was concluded that Q is in fact half-integral and so the anomaly vanishes on the IIA side. To see this, perform a gauge transformation by an angle of 2π. This contributes a phase to the gravitino's wavefunction

$$\Psi \longrightarrow e^{2\pi Q}\Psi\,. \tag{6.7}$$

To calculate this phase, we look at the IIB side. This is a rotation of the IIB circle over 2π, and so corresponds to transporting the gravitino around the circle. If we chose the supersymmetric spin structure on the circle then the gravitino's phase acquires a -1, and so Q is half-integral as required. It is interesting that the matching of anomalies required us to choose the supersymmetric spin structure on the circle about which we T-dualized; if we had not then the result may not have been IIA but possibly type-0 [43].

6.3. The gravitino anomaly in the general case. We have found that the H-field arising from our T-duality cancels the gravitino anomaly on the IIA side, so that the IIA theory is consistent. It is a critical test of our proposal (1.8) that the two sides of the duality be consistent and inconsistent at the same time. That is, the gravitino anomalies must match on the two sides in general. To see that they do, we extend the above argument to the general case. We will begin with the case in which there is no H-flux on the IIB side, and so a trivial bundle in IIA.

As there is no H-flux on the IIB side, the gravitino anomaly is entirely determined by the second Stiefel-Whitney class of the S^1_{IIB} bundle E,

$$\mathrm{Anomaly}_{\mathrm{IIB}} = w_2(TE) = w_2(M^9) + w_2(E) = w_2(M^9) + c_1(E)\,, \text{ mod } 2, \tag{6.8}$$

where $w_2(TE) \subset H^2(E)$ is the Stiefel-Whitney class of the tangent space to E, whereas $w_2(E) \subset H^2(M^9)$ and $c_1(E) \subset H^2(M^9)$ are characteristic classes of the S^1 bundle over the base M^9. In the case of $L(2, j) = S^5/\mathbb{Z}_j$ this anomaly is $1 + j$ and so the IIB side is anomalous when j is even.

To compute the anomaly on the IIA side we will first dimensionally reduce away the trivially fibered circle S^1_{IIA}. If our T-duality conjecture is correct this will be IIB reduced to 9-dimensions and so the anomalies will match. To check that it does, notice that the anomaly for a $U(1)$ charged fermion in 9 dimensions is given by (6.6),

$$\mathrm{Anomaly}_{\mathrm{IIA}} = w_2(M^9) + 2Qc_1(E) \text{ mod } 2, \tag{6.9}$$

where E is now interpreted as our gauge bundle, although the E_8 description tells us that it is the same E as we encountered on the IIB side. If we again take the supersymmetric spin structure then by the same argument we conclude that Q is half-integral and so the anomalies (6.9) and (6.8) as computed in type IIA and IIB agree.

To extend this argument further, to the general case in which there is H-flux before and the T-duality, one need only observe that the total anomaly in both cases is the second

Stiefel-Whitney class of the sum of the two circle bundles. Thus they are both the sum of $w_2(M^9)$ plus w_2 of the two circle bundles, where the fact that Q is half-integral for the chosen spin structure has been used to rewrite one Chern class as a Stiefel-Whitney class. As both anomalies are given by the same formula, they agree. It is suggestive (mysterious) to rewrite the anomaly as w_2 of the fibered product. One may then include the IIA coupling of the gravitino to G_2 to conclude that the total anomaly is w_2 of the F-theory 12-manifold.

6.4. The G_4 quantization condition in M-theory. In Ref. [27] Witten showed that when the spacetime Y^{11} is *spin* the M-theory four-form obeys the twisted flux quantization condition which we heuristically write as

$$G_4 = \tfrac{1}{4} p_1(TY^{11}) \bmod 2. \tag{6.10}$$

For an interpretation of these divisions, we refer the reader to the original paper. While the first Pontrjagin class p_1 of the tangent space may always be canonically divided by two, the division by 4 in (6.10) is canonical because Y^{11} is *spin*.

As explained in [6], G_2 vanishes in the above example of IIA on $AdS^5 \times \mathbb{C}\mathrm{P}^2 \times S^1$ and so the M-theory topology is $AdS^5 \times \mathbb{C}\mathrm{P}^2 \times T^2$, which is not *spin*. Therefore the above divison by two may not exist. In fact, the G_4 flux in this example is a cup product of the generator of $H^2(T^2, \mathbb{Z}) = \mathbb{Z}$ and so does not satisfy the twisted quantization condition (6.10).

Instead we see that when M-theory is compactified on a 2-torus T^2 the shifted quantization condition is

$$\int_{T^2} G_4 = w_2(TM^9) \bmod 2\,. \tag{6.11}$$

This equation may well generalize to 2-torus bundles, and possibly the 2-torus may be replaced by any 2-manifold. When the spacetime is not of such a form, perhaps the anomaly-cancellation used above cannot work and so the 11-dimensional spacetime must be *spin*, and thus Eq. (6.10) is applicable. Nonetheless, it may be interesting to find a single formula that works in all of the cases.

7. Concluding Remarks

We have conjectured that any orientable circle bundle is T-dual to another circle bundle, where the Chern class of each bundle is the integral of the T-dual H-flux over the dual circle. As evidence, we have provided physical motivation in a number of special cases and have seen that this definition of T-duality always leads to the desired isomorphism of the twisted K-theories with a shift in dimension by one.

However to be certain that this isomorphism of twisted K-theory is a duality of the full string theory in the most general cases one requires more powerful methods. The most obvious choice is the σ-model on $E \times_M \hat{E}$ program of [2] and later [5]. This approach has been used to find that nontrivial bundles are dual to a singular B-flux. Thus it may be possible to compute the corresponding H-flux and verify that it obeys our conjecture. This calculation would then need to be extended to the case in which H is nontrivial both before and after the T-duality.

This approach may allow a number of other open problems to be tackled directly. An obvious one is the generalization of the results of this paper to higher-dimensional torii. The obstruction that we conjecture exists when the integral of H over a subtorus is nontrivial may be visible directly in such an approach. In the present approach, the obstruction is mysterious because the S-dual to an obstructed T-duality is the T-duality of a 2-torus supporting G_3-flux. Such a T-duality is perfectly legitimate, and leads to G_1-flux. Thus one may suspect that the forbidden T-duality of a 2-torus with H-flux yields the S-dual of a configuration with G_1-flux. The S-duals of such configurations have been described extensively in the literature, but unfortunately the descriptions tend not to agree. One common feature among papers that claim that such a duality makes sense is that the dilaton ceases to be globally defined, which may explain why we have difficulty understanding such a theory.

Another obvious generalization is that we may allow our circle fibers to degenerate. This would then include examples such as mirror symmetry. While the σ-model approach may be promising here, the traditional approach to this subject [45, 46] suggests that a linear sigma model which flows in the IR to the conformal theory may provide a much more practical tool for this and the previous generalizations.

The generalization to the equivariant case appears to be straightforward, and we hope to come back to this in future work. More intriguing is the extension from $U(1)$-bundles to non-abelian bundles yielding nonabelian dualities. Such bundles are also treated in [24] although the results are much more limited.

There are a number of more tangentially related applications and open problems. As mentioned above, the shifted quantization condition of G_4 in the non-*spin* case is still unknown in general, although in the Riemann surface bundle case the contribution above may be the entire condition. The T-duality above between $\mathbb{R}\mathrm{P}^3 \times S^1$ with H-flux and $S^3 \times S^1$ may be dimensionally reduced to a 7-dimensional duality of gauged supergravities. This may relate an $SU(2)$ symmetry to an $SO(3)$ symmetry with a $\mathbb{Z}_2$ Wilson line activated.

Perhaps the most mysterious aspect of this realization of T-duality is the connection to F-theory. Consistency seemed to require that the 12-manifold be *spin*, as if it were inhabited by fermions despite the lack of a single-time $12d$ SUSY algebra. More significantly, the σ-model approach introduces an auxilliary dimension as an intermediate step, and that step seems to be a kind of σ-model on the fibered product, which is F-theory compactified on the M-theory circle. Could this mean that F-theory is a theory after all?

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References

1. Buscher, T.: A symmetry of the string background field equations. Phys. Lett. **B194**, 59–62 (1987); Buscher, T.: Path integral derivation of quantum duality in nonlinear sigma models. Phys. Lett. **B201**, 466–472 (1988)
2. Roček, M., Verlinde, E.: Duality, quotients, and currents. Nucl. Phys. **373**, 630–646 (1992)
3. Álvarez, E., Álvarez-Gaumé, L., Lozano, Y.: An introduction to T-duality in string theory. Nucl. Phys. Proc. Suppl. **41**, 1–20 (1995)
4. Bergshoeff, E., Hull, C.M., Ortin, T.: Dualty in the type-II superstring effective action. Nucl. Phys. **B451**, 547–578 (1995)
5. Álvarez, E., Álvarez-Gaumé, L., Barbón, J.L.F., Lozano, Y.: Some global aspects of duality in string theory. Nucl. Phys. **B415**, 71–100 (1994)
6. Duff, M.J. , Lü, H., Pope, C.N.: $\mathrm{AdS}_5 \times S^5$ untwisted. Nucl. Phys. **B532**, 181–209 (1998)

7. Gurrieri, S., Louis, J., Micu, A., Waldram, D.: Mirror symmetry in generalized Calabi-Yau compactifications. Nucl. Phys. **B654**, 61–113 (2003)
8. Kachru, S., Schulz, M., Tripathy, P., Trivedi, S.: New supersymmetric string compactifications. J. High Energy Phys. **03**, 061 (2003)
9. Hori, K.: D-branes, T-duality, and index theory. Adv. Theor. Math. Phys. **3**, 281–342 (1999)
10. Gukov, S.: K-theory, reality, and orientifolds. Commun. Math. Phys. **210**, 621–639 (2000)
11. Sharpe, E.R.: D-branes, derived categories, and Grothendieck groups. Nucl. Phys. **B561**, 433–450 (1999)
12. Olsen, K., Szabo, R.J.: Constructing D-Branes from K-Theory. Adv. Theor. Math. Phys. **3**, 889–1025 (1999)
13. Moore, G., Saulina, N.: T-duality, and the K-theoretic partition function of TypeIIA superstring theory. Nud. Phys. **B670**, 27–89 (2003)
14. Strominger, A., Yau, S.-T., Zaslow, E.: Mirror symmetry is T-duality. Nucl. Phys. **B479**, 243–259 (1996)
15. Bott, R., Tu, L.: *Differential forms in algebraic topology.* Graduate Texts in Mathematics **82**, New York: Springer Verlag, 1982
16. Brylinski, J.-L.: *Loop spaces, characteristic classes and geometric quantization*. Prog. Math. **107**, Boston: Birkhäuser Boston, 1993
17. Mathai, V., Melrose, R.B., Singer, I.M.: *The index of projective families of elliptic operators*. http://arXiv.org/abs/math.DG/0206002, 2002
18. Mathai, V., Melrose, R.B., Singer, I.M.: *Work in progress*
19. Minasian, R., Moore, G.: *K-theory and Ramond-Ramond charge*. J. High Energy Phys. **11**, 002 (1997)
20. Witten, E.: D-Branes and K-Theory. J. High Energy Phys. **12**, 019 (1998)
21. Moore, G., Witten, E.: Self duality, Ramond-Ramond fields, and K-theory. J. High Energy Phys. **05**, 032 (2000)
22. Bouwknegt, P., Carey, A., Mathai, V., Murray, M., Stevenson, D.: Twisted K-theory and K-theory of bundle gerbes. Commun. Math. Phys. **228**, 17–45 (2002)
23. Mathai, V., Stevenson, D.: Chern Character in Twisted K-Theory: Equivariant and Holomorphic Cases. Commun. Math. Phys. **236**, 161–186 (2003)
24. Raeburn, I., Rosenberg, J.: Crossed products of continuous-trace C^*-algebras by smooth actions. Trans. Am. Math. Soc. **305**, 1–45 (1988)
25. Connes, A.: An analogue of the Thom isomorphism for crossed products of a C^* algebra by an action of $\mathbb{R}$. Adv. Math. **39**, 31–55 (1981)
26. Álvarez-Gaumé, L., Ginsparg, P.: The Structure of Gauge and Gravitational Anomalies. Ann. Phys. **161**, 423 (1985) Erratum-ibid. **171**, 233 (1986)
27. Witten, E.: On Flux Quantization in M-Theory and the Effective Action. J. Geom. Phys. **22**, 1–13 (1997)
28. Evslin, J.: Twisted K-Theory from Monodromies. J. High Energy Phys. **05**, 030 (2003)
29. Diaconescu, E., Moore, G., Witten, E.: E_8 Gauge Theory, and a Derivation of K-Theory from M-Theory. Adv. Theor. Math. **6**, 1031–1134 (2003)
30. Gomez, C., Manjarin, J. J.: *Dyons, K-theory and M-theory*. http://arXiv.org/abs/hep-th/0111169, 2001
31. Adams, A., Evslin, J.: The Loop Group of E_8 and K-Theory from 11d. J. High Energy Phys. **02**, 029 (2003)
32. Morrison, D. R.: *Half $K3$ surfaces*. Talk at Strings 2002, Cambridge. http://www.damtp.cam.ac.uk/strings02/avt/morrison/
33. Hořava, P.: Unpublished
34. Vafa, C.: Evidence for F-Theory. Nucl. Phys. **B469**, 403–418 (1996)
35. Adams, A., Evslin, J., Varadarajan, U.: To appear
36. Rosenberg, J.: Continuous trace C^*-algebras from the bundle theoretic point of view. J. Aust. Math. Soc. **A47**, 368 (1989)
37. Bouwknegt, P., Mathai, V.: D-branes, B-fields and twisted K-theory. J. High Energy Phys. **03**, 007 (2000)
38. Freed, D., Hopkins, M., Telemann, C.: Unpublished; Freed, D.S.: *The Verlinde algebra is twisted equivariant K-theory*. Turkish J. Math. **25**, 159–167 (2001)
39. Atiyah, M. F., Singer, I. M.: The index of elliptic operators, IV. Ann. of Math. (2) **93**, 119–138 (1971)
40. Mathai, V., Quillen, D. G.: Superconnections, Thom classes and equivariant differential forms. Topology **25**(1), 85–110 (1986)
41. Maldacena, J., Moore, G., Seiberg, N.: D-Brane Instantons and K-Theory Charges. J. High Energy Phys. **11**, 062 (2001)
42. Tong, D.: NS5-branes, T-duality and worldsheet fermions. J. High Energy Phys. **07**, 013 (2002)

43. David, J., Gutperle, M., Headrick, M., Minwalla, S.: Closed String Tachyon Condensation on Twisted Circles. J. High Energy Phys. **04**, 041 (2002)
44. Evslin, J., Varadarajan, U.: K-Theory and S-Duality: Starting over from Square 3. J. High Energy Phys. **03**, 026 (2003)
45. Witten, E.: Phases of $N = 2$ Theories in 2 Dimensions. Nucl. Phys. **B403**, 159–222 (1993)
46. Hori, K., Vafa, C.: *Mirror Symmetry*. http://arXiv.org/abs/hep-th/0002222, 2000

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