SOME USES OF TYPE THEORY IN THE ANALYSIS OF LANGUAGE

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F O R E W O R D

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February 1974.

John Passmore

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"At its very origins, formal logic was used by Aristotle and the Stoics to appraise ordinary arguments; it has been used whenever it has flourished; it still is so used by distinguished modern logicians like Prior and Quine.

The 'ordinary language' philosophers, who want to keep the estate they claim strictly preserved against the poaching of formal logicians, are, I think, people with a vested interest in confusion."

P.T. Geach [23] p.x.

"It has for fifteen years been possible for at least one philosopher (myself) to maintain that philosophy, at this stage in history, has as its proper theoretical framework set theory with individuals and the possible addition of empirical predicates."

R.M. Montague [47] p.185.

"I am aware of being even more casual than usual with that fetish of second-rate logicians, quotes and use and mention. My appeal here is to the principle that one is to be considered innocent until one has been found guilty of an actual confusion caused by a failure to tell use from mention."

K.J.J. Hintikka [31] p.vii.

ABSTRACT

In this monograph, Church's formulation of the simple theory of types is applied to two areas of current concern in logic, namely the theory of predicate modifiers and the logic of intensional discourse. In such a formulation, the theory of predicate modifiers becomes both more general and more ramified than in previous formulations, and the notation and theory provided for intensional logics has considerable generality and utility in philosophical applications. A number of problems are posed throughout the monograph: in these cases the system used allows formal presentation of philosophical problems and should aid in their solution.

SOME USES OF TYPE THEORY IN THE ANALYSIS OF LANGUAGE

0. INTRODUCTION

In this monograph we apply the theory of types to two broad areas of current concern in the logical analysis of discourse, namely to the theory of predicate modifiers and to intensional logics including in particular quantified modal logics. We use Church's formulation [8] of the simple theory of types, together with the set-theoretic semantics of Henkin [27]: Church's formulation makes use of his notion of functional abstraction, and this has advantages for our purposes over the kind of restricted set abstraction used in systems such as that of Leblanc and Meyer [41]. Although we presuppose some familiarity with the basics of [8] and [27] (leaving aside the development of number theory in the former and the classic completeness proof, using general models, in the latter), we will give an informal outline of our syntax and semantics in this introduction. This will allow us to introduce a few quite inessential abbreviations of our own, and will slightly increase the monograph's self-containment coefficient.

There are two basic type symbols, o and ι : truth-values have type o and individuals have type ι . Type symbols in general are defined recursively as follows:

1. o and ι are type symbols
2. If α and β are type symbols, so is $(\alpha\beta)$.

Functions from (entities of) type β to (entities of) type α have type $(\alpha\beta)$: thus e.g. $(o\iota)$ is the type of functions from individuals to truth-values, that is of predicates of individuals. Type symbols appear always as subscripts to variables and constants in wffs: for any type symbol α we have the following denumerably many variables

$$f_\alpha, g_\alpha, h_\alpha, m_\alpha, n_\alpha, x_\alpha, y_\alpha, z_\alpha, f_\alpha^{(1)}, g_\alpha^{(1)}, \cdots ;$$

we have the two constants

$$\mathcal{N}_{(oo)} \ , \ \mathcal{A}_{((oo)o)}$$

of fixed type, and for any type symbol α we have the constants

$$\Pi_{(o(o\alpha))} \ , \ \iota_{(\alpha(o\alpha))} .$$

Our stock of primitives is completed by the improper symbols λ, (and).

Every wff has an associated type: wffs and their types are defined by the following recursive definition

1. Any variable v_α is a wff, and has type α.
2. Any constant is a wff, and has the type of its subscript.

(Thus for example $f^{(2)}_{((o(oo))o)}$ is a wff of type $((o(oo))o)$ and $\Pi_{(o(o(\iota\iota)))}$ is a wff of type $(o(o(\iota\iota)))$.)

3. If $A_{(\alpha\beta)}$ is a wff of type $(\alpha\beta)$ and B_β is a wff of type β, then $(A_{(\alpha\beta)} B_\beta)$ is a wff of type α.
4. If A_α is a wff of type α and v_β is a variable of type β, then $(\lambda v_\beta A_\alpha)$ is a wff of type $(\alpha\beta)$.

As an example of the way in which these rules work, consider just the propositional calculus, where only o is the basic type symbol, where the only variables are of type o, and where the only constants are $\mathcal{N}_{(oo)}$ and $\mathcal{A}_{((oo)o)}$. We write $\mathcal{D}_o = \{T, F\}$ for the domain of truth values: every variable v_o (and in fact every wff of type o) takes a value from this domain. By formation rule 1, every variable v_o is a wff, and this matches the usual clause that a propositional variable standing alone is a wff. The constant $\mathcal{N}_{(oo)}$ denotes a function from $\mathcal{D}_o$ to $\mathcal{D}_o$, viz. the negation function whose value is F for argument T and

T for argument F. By formation rule 2, $N_{(oo)}$ is a wff, and this has no counterpart in ordinary propositional calculus, since $\sim$ standing alone is not a wff: more significantly, by rule 3, $(N_{(oo)} x_o)$ is a wff of type o, and in general $(N_{(oo)} A_o)$ is a wff of type o if A_o is such a wff. This matches the ordinary rule that if α is a wff then so is $\sim\alpha$, and for readability and familiarity we introduce the abbreviation

$$(\sim A_o) \qquad \text{for} \qquad (N_{(oo)} A_o) \quad .$$

In order to explain the constant $A_{((oo)o)}$, temporarily let $T_{(oo)}$ be the function from $\mathcal{D}_o$ to $\mathcal{D}_o$ whose value is always T, and let $I_{(oo)}$ be the function from $\mathcal{D}_o$ to $\mathcal{D}_o$ whose value is the same as its argument. Then $A_{((oo)o)}$ is the function from $\mathcal{D}_o$ to $\mathcal{D}_{(oo)}$ (where $\mathcal{D}_{(oo)}$ is the set of all functions from $\mathcal{D}_o$ to $\mathcal{D}_o$) whose value is $T_{(oo)}$ for argument T and $I_{(oo)}$ for argument F. It can be seen then that the value of $((A_{((oo)o)} x_o) y_o)$ is the ordinary inclusive disjunction of the values of x_o and y_o, and thus we may use the abbreviation

$$(A_o \vee B_o) \qquad \text{for} \qquad ((A_{((oo)o)} x_o) y_o) \quad .$$

The "trick" whereby disjunction, a dyadic function, is defined solely in terms of monadic functions, is used throughout Church's formulation of type theory: it depends in effect upon the set-theoretic equivalence between

$$A^{(B \times C)} \qquad \text{and} \qquad (A^B)^C \quad ,$$

allowing Cartesian products to be replaced by equivalent (but not identical) constructions in terms of insertion sets.

As well as the abbreviations $(\sim A_o)$ and $(A_o \vee B_o)$, we will use $(A_o \& B_o)$, $(A_o \supset B_o)$ and $(A_o \equiv B_o)$ in the expected way. With all such abbreviations a certain loss in explicitness is sustained, since the appropriate type symbols are absent from $\sim, \vee, \&, \ldots$, and the way in which formation rule 3 gives wffs whose interpretation involves functional application (of the denotatum of $A_{(\alpha\beta)}$ to that of B_β) is obscured by writing these propositional operators in infix position.

Turning to first-order quantification theory, we now allow variables of type ι, as individual variables taking their values from a (non-empty) domain D_ι. We also have variables of types $(o\iota)$, $((o\iota)\iota)$, $(((o\iota)\iota)\iota)$, ...: these serve as variables for 1-adic, 2-adic, 3-adic,... relations, given the "trick" used for $A_{((oo)o)}$, extended so that $((o\iota)\iota)$ can be the type of a dyadic relation between individuals, etc. The constant $\Pi_{(o(o\iota))}$ is now admitted, and serves as a universal quantifier: formally $\Pi_{(o(o\iota))}$ denotes the function from $D_{(o\iota)}$ to D_o whose value is T iff its argument is the function from D_ι to D_o whose value is T for all arguments. In order to introduce a quantification that looks like ordinary universal quantification, with a bound variable etc., we have to explain the improper symbol λ and the semantics corresponding to formation rule 4.

Informally, λ is an operator of functional abstraction, and $(\lambda v_\beta A_\alpha)$ denotes the functional relationship between v_β and A_α. For example, if both D_β and D_α were the set of all natural numbers, and if something like x^2+2x were well-formed, then $\lambda x(x^2+2x)$ would denote the functional relationship between x and x^2+2x, that is the function of squaring and then adding itself twice. A clear account of the use of the operator in this kind of case is given in Suppes's [67] § 11.3. Returning to our formalism, consider

$$\lambda x_\iota ((f_{(o\iota)} x_\iota) \,\&\, (g_{(o\iota)} x_\iota)) \quad :$$

by rule 4, this has type $(o\iota)$, and thus is an expression for a monadic predicate. This predicate is the functional relationship between x_ι and $((f_{(o\iota)} x_\iota) \,\&\, (g_{(o\iota)} x_\iota))$: this "functional relationship" is the function whose value, for a given value of the variable x_ι in D_ι, is T iff both $(f_{(o\iota)} x_\iota)$ and $(g_{(o\iota)} x_\iota)$ have value T. In short, $\lambda x_\iota ((f_{(o\iota)} x_\iota) \,\&\, (g_{(o\iota)} x_\iota))$ denotes the intersection of the two subsets of D_ι denoted by $f_{(o\iota)}$ and $g_{(o\iota)}$: in this case the λ operator is very like the operator ^ of Principia Mathematica.

We are now in a position to explain the abbreviation

$$(x_\iota)A_o \qquad \text{for} \qquad (\Pi_{(o(o\iota))} (\lambda x_\iota A_o)) \quad .$$

We see that $(\lambda x_\iota A_o)$ has type $(o\iota)$, and thence $\Pi_{(o(o\iota))} (\lambda x_\iota A_o)$ has type o : it has value T iff $(\lambda x_\iota A_o)$ has value T for all arguments from D_ι, that is, roughly, iff A_o is true for all values of x_ι. Given the abbreviation $(x_\iota)A_o$, we can obviously also use

$$(\exists x_\iota)A_o \qquad \text{for} \qquad (\sim(x_\iota)(\sim A_o)) \quad .$$

The type of the quantifier $\Pi_{(o(o\iota))}$ makes it clear that quantifiers are not functions from truth-values to truth-values, despite the superficial impression one might gain from the formation rules of ordinary quantification theory. Rather, a quantifier is a property of (or: a subset of) subsets of individuals: in particular, the universal quantifier is that property of subsets which holds when and only when the subset is the whole domain of individuals (and the existential quantifier is the property of being non-null).

It remains to give some explanation of the constants $\iota_{(\alpha(o\alpha))}$, in particular of $\iota_{(\iota(o\iota))}$. In any model, $\iota_{(\iota(o\iota))}$ denotes a function whose value, for a given subset of $\mathcal{D}_\iota$ as argument, is a member of that subset unless the subset is null, when the value is an arbitrary member of $\mathcal{D}_\iota$. If the subset happens to be a unit set, then $\iota_{(\iota(o\iota))}$ acts as a definite description operator : however, we will quite often trade on its choice-operator properties as well. To gain a variable-binding operator, we proceed in a similar fashion to the definition of (x_ι) from $\Pi_{(o(o\iota))}$, and use the abbreviation

$$(\imath x_\iota) A_o \qquad \text{for} \qquad \iota_{(\iota(o\iota))}(\lambda x_\iota A_o)\ .$$

It is clear that quantification (and the choice-operator) is available for any type, and not just ι as it appears in the last two paragraphs. If we have higher-order quantification, we can expect to define identity (for any type) and this is done by the two abbreviations

$$Q_{((o\alpha)\alpha)} \qquad \text{for} \qquad (\lambda x_\alpha(\lambda y_\alpha(f_{(o\alpha)})((f_{(o\alpha)}x_\alpha)\supset(f_{(o\alpha)}y_\alpha))))$$

and

$$(A_\alpha = B_\alpha) \qquad \text{for} \qquad ((Q_{((o\alpha)\alpha)}A_\alpha)B_\alpha)\ .$$

As is shown by Henkin [28] and Andrews [1], a more economical basis for the theory can be had by taking $Q_{((o\alpha)\alpha)}$ as primitive : however the definitions of the other constants are then slightly difficult to grasp, and we have preferred the more intuitive but less economical basis that we have set out informally.

We now set out a number of conventions and abbreviations designed to reduce the number of parentheses in wffs, and to reduce the length of wffs in general. Firstly, within type symbols and within wffs, parentheses will be omitted according to a convention of association to the left. Thus $Q_{o\alpha\alpha}$ abbreviates $Q_{((o\alpha)\alpha)}$, but $\Pi_{o(o\iota)}$ (and not $\Pi_{oo\iota}$) abbreviates $\Pi_{(o(o\iota))}$. And $f_{o\iota\iota}x_\iota y_\iota$ abbreviates $((f_{o\iota\iota}x_\iota)y_\iota)$, and $x_o \& y_o \supset z_o$ abbreviates $((x_o \& y_o) \supset z_o)$. Especially where wffs of type o are involved, we may use Church's single-dot abbreviation as in [11] p.75, so that

$x_o \& y_o \supset . z_o \supset y_o \supset . z_o \supset x_o$ abbreviates

$((x_o \& y_o) \supset ((z_o \supset y_o) \supset (z_o \supset x_o)))$ (which itself abbreviates some lengthy wff using $x_o, y_o, z_o, N_{oo}, A_{ooo}$ and parentheses only).

In contrast to the general convention of association to the left, we use $\lambda x_\alpha \lambda y_\alpha A_\beta$ as an abbreviation for $(\lambda x_\alpha(\lambda y_\alpha A_\beta))$. Thence $(\lambda x_\alpha \lambda y_\alpha A_\beta)m_\alpha n_\alpha$ is an abbreviation for $(((\lambda x_\alpha(\lambda y_\alpha A_\beta))m_\alpha)n_\alpha)$, which is a wff of type β. Some authors further abbreviate $\lambda x_\alpha \lambda y_\alpha A_\beta$ to $\lambda x_\alpha y_\alpha A_\beta$, but we do not follow this practice, if only because $\lambda x_\alpha y_\alpha$ by itself is well-formed of type $\alpha\alpha$, and so $\lambda x_\alpha y_\alpha A_\beta$ looks to be ill-formed.

We will often need to refer to the type $o\iota\ldots\iota$, with j iotas, of a j-adic predicate of individuals, and in general to the type $o\alpha\ldots\alpha$, with j alphas, of a j-adic predicate of entities of type α. To facilitate such reference, we define

$$\alpha^1 = \alpha$$
$$\alpha^{n+1} = \alpha^n\alpha$$

deliberately omitting parentheses. Then the type symbol $o\iota^j$ is $o\iota\ldots\iota$, i.e. $(\ldots((o\iota)\iota)\ldots\iota)$ after parentheses have been inserted according to association to the left. We then define the type symbol π_j by

$$\pi_j =_{df} (o\iota^j)$$

so that π_j is the type of a j-adic predicate of individuals.

Wffs like $f_{\pi_j} x_\iota^{(1)} \dots x_\iota^{(j)}$ can then be expected to occur frequently, and we will further abbreviate these to $f_{\pi_j} x_\iota^{((j))}$, where

$$x_\iota^{((1))} = x_\iota^{(1)}$$

$$x_\iota^{((n+1))} = x_\iota^{((n))} x_\iota^{(n+1)} ,$$

again with deliberate omission of parentheses. From this point, we can expect wffs like

$$(x_\iota^{(1)})(x_\iota^{(2)}) \dots (x_\iota^{(j)})(f_{\pi_j} x_\iota^{((j))} \supset g_{\pi_j} x_\iota^{((j))}) ,$$

and to abbreviate the quantifier prefix we will write

$$(x_\iota^{((j))})(f_{\pi_j} x_\iota^{((j))} \supset g_{\pi_j} x_\iota^{((j))}) .$$

Similarly multiple abstraction like

$$\lambda x_\iota^{(1)} \lambda x_\iota^{(2)} \dots \lambda x_\iota^{(j)} A_o$$

will occur, and will be abbreviated as

$$\lambda x_\iota^{((j))} A_o .$$

The abbreviations of the last two paragraphs must be treated with caution, since non-type symbols and non-wffs are being abbreviated, and since their functioning correctly depends upon other abbreviations and conventions, particularly concerning restoration of parentheses by association to the left. However without these abbreviations we would find ourselves writing very long wffs, with many occurrences of ..., and the practical utility of the abbreviations compensates for their somewhat non-standard nature.

1. PREDICATE MODIFIERS AND TYPE THEORY

Partly as a reaction to Davidson's analysis in [19] of action sentences in terms of events and their predicates, and partly as a natural part of a program to generalize and enrich existing systems of logic, there have recently appeared a number of papers concerned with the theory of predicate modifiers. Some such papers are those of Parsons [52], Clark [12], Montague [48] and [49], Lewis [44], Malinas and Rennie [45] and Rennie [57]. Parsons's paper criticizes earlier proposals of Reichenbach for dealing with predicate modifiers by way of second-order predicate calculus, and, without filling in many details, sketches a notation that takes predicate modifiers seriously as functions from predicates to predicates; the latter portion of Clark's paper gives an analysis of some of the different kinds of properties predicate modifiers might have, in terms of the various relations that may obtain between the predicate (before modification) and the modified predicate; the treatment of modifiers in Montague's papers is part of a more general approach to syntax and semantics - in both papers an explicit intensionalized semantics is given for many different syntactical parts of English including predicate modifiers, and in [48] some attention is also given to some properties of predicate modifiers in terms of the semantical system in which they are set; Lewis relates the syntax of modifiers to categorial grammars in the style of Ajdukiewicz and, again as part of a more general semantical system, gives explicit semantics (of the possible-worlds type) for modifiers along with many other categories of expression; Malinas and Rennie generalize modifiers so that prepositions can be treated as separate elements of prepositional phrases used as modifiers - and (independently of Clark or Montague) discuss different kinds of properties of modifiers; Rennie provides a completeness proof for a simple theory of modifiers based on ordinary first-order quantification theory, and discusses (largely from the point of view of a set-theoretic semantics) some more complicated types of modifiers added to a first-order extensional basis.

We should note that at least one author, viz. Lewis, is able to distinguish between adjectives and adverbs since he is able to distinguish (albeit on inadequate grounds, as we shall argue in §2.5), between different kinds of predicative expressions; we also note that most authors argue that the theory of predicate modifiers should be geared to an intensional logic rather than an extensional one, since it appears that an inadequate theory will result if modifiers are treated, semantically, simply as functions on the extensions of predicates.

To balance the weight of papers in favour of setting up a formal logic of predicate modifiers, there are at least the papers of Lakoff [38] and Harman [25], each of which propose arguments against treating predicate modifiers as functions from predicates to predicates. We do not enter into a detailed refutation of these arguments here: Lakoff's argument depends on the example "Sam sliced all the bagels carefully", the ambiguity of which is claimed by Lakoff not to be expressible in predicate modifier notation. However Lakoff's own linguistic analysis of the ambiguity seems to confuse CAREFUL IN with CAREFUL TO, and in any event the two different senses of the example <u>can</u> be expressed, in our predicate modifier notation, by

$$(x_\iota)(\mathrm{bagel}_{o\iota}x_\iota \supset \mathrm{care}_{(o\iota)^2}(sl_{o\iota\iota}\,\mathrm{sam}_\iota)x_\iota)$$

(each bagel was carefully sliced by Sam) and

$$\mathrm{care}_{(o\iota)^2}(\lambda y_\iota(x_\iota)(\mathrm{bagel}_{o\iota}x_\iota \supset sl_{o\iota\iota}y_\iota x_\iota))\mathrm{sam}_\iota$$

(Sam has the property of carefully slicing all the bagels). Harman's whole paper is based on some canons about logical theory: most of these, e.g. "rules of logical implication correspond to axiom schemata with an infinite number of instances", "The list of axioms must be finite", "rules of logical implication should be kept as close as possible to the rules of ordinary (first order) quantificational logic", are either false (the first two quoted) or question begging (the third quoted), and so any arguments of detail in Harman are largely rendered ineffective by his dependence on these canons.

Thus we find no serious bar, in the arguments presented by Lakoff or Harman, to proceeding with our investigation of the logic of predicate modifiers. Nor need we be prevented, by the various other caveats mentioned, from investigating the structure of predicate modification as an extension of ordinary extensional quantification theory with no distinction between kinds of predicative expressions. Many of the notions and problems that arise in a theory with a more discriminatory basis arise already in the theory about to be presented, and most of the material to be presented can be readily taken over in an intensionalized system. We will find that we need no further mechanism than that presented in the Introduction in order to give a more detailed and comprehensive analysis of predicate modification other than has hitherto been available: in itself this fact provides sufficient justification for not complicating our logical basis <u>at this stage</u> of development.

1.1. Types of modifiers

We begin by assigning types to the various (syntactical) kinds of modifiers introduced in [45] and [57]. With the type-theoretical basis, <u>any</u> wff of the appropriate type can be treated as a modifier, not just variables of that type or variables introduced specifically for modifiers. (Note the extended sense of "wff" in type theory : wffs may be of any type, not just o.)

The simplest kind of modifier is a modifier involving no individuals and modifying a monadic predicate to yield a monadic predicate. The "red" in "is a red rose" and the "briskly" in "walks briskly" could be treated as modifiers of this kind. Formally, it is clear that their type is π_1^2 : then the sentence "Jack walks briskly" would be symbolized as

$$b_{\pi_1^2} w_{\pi_1} j_\iota$$

(when symbolizing from English we will use any letters, often those with some mnemonic value, and not just the variables listed in our formal syntax). In this symbolization, $b_{\pi_1^2} w_{\pi_1}$ is a wff of type π_1, say W_{π_1}, corresponding to the compound predicate "walks briskly". Then $W_{\pi_1} j_\iota$ is a wff of type o, corresponding to the sentence under analysis.

Next we may have modifiers like those of type π_1^2 except that they modify k-adic predicates rather than just monadic predicates. The adverbs "easily" and "strictly" in "St. George easily beat Canterbury" and "6 is strictly between 5 and 7" could be treated as such modifiers for dyadic and triadic predicates respectively. The general type of such modifiers is π_k^2, and the two examples could be symbolized as

$$e_{\pi_2^2} b_{\pi_2} s_\iota c_\iota$$

and

$$st_{\pi_3^2} bet_{\pi_3} 6_\iota 5_\iota 7_\iota$$

respectively. As with out first example, the convention of association to the left ensures that firstly a relation of the appropriate type ($e_{\pi_2^2} b_{\pi_2}$ of type π_2 , $st_{\pi_3^2} bet_{\pi_3}$ of type π_3) is formed and then applied to its arguments.

When a number j of individuals are involved in the modification, we say that the modifier is j-ary (reserving the affix "-adic" for the adicity of the predicate being modified). In general, we may have a j-ary modifier modifying k-adic predicates : the type of such modifiers is $\pi_k^2\iota^j$. For example, the "in" in "Marlon Brando blanked Maria Schneider in The Last Tango in Paris" can be treated as a 1-ary modifier modifying a dyadic relation, and, assuming films as well as people to have type ι, the sentence may be symbolized as

$$i_{\pi_2^2\iota}\, l_\iota\, bl_{\pi_2}\, br_\iota\, sch_\iota$$

in which $i_{\pi_2^2\iota} l_\iota$ is a wff of type π_2^2 , formed from the modifier $i_{\pi_2^2\iota}$ (for "in") and the individual l_ι (for Last Tango...). Then $i_{\pi_2^2\iota} l_\iota$ modifies bl_{π_2}, as before, to give a wff of type π_2, and this is then applied to the arguments br_ι and sch_ι.

Each of the modifiers to date has modified a single predicate : more generally we may consider r-place modifiers, which take r predicates and form a single predicate from them. Such modifiers go under the generic name "polyadic" : we may have j-ary r-place polyadic modifiers modifying k-adic predicates, and the type of these is $(\pi_k^{r+1}\iota^j)$. A simple example of a 0-ary 2-place modifier modifying dyadic relations is the "and" in "John caught and kissed Mary" : so construed, the sentence may be symbolized as

$$\text{and}_{\pi_2^3}\, c_{\pi_2}\, k_{\pi_2}\, j_\iota\, m_\iota$$

in which $\text{and}_{\pi_2^3} c_{\pi_2}$ is a wff of type π_2^2, and

and $and_{\pi_2^3} c_{\pi_2} k_{\pi_2}$ is a wff of type π_2 .

Even more generally, we may assign a type to heteradic modifiers, where the adicity of the result of modification may be different from that of the predicates modified. Using the terminology of [57], we assign the type $(\pi_{\mu(k_1,\ldots,k_r)}\pi_{k_1}\ldots\pi_{k_r}\iota^j)$ to j-ary μ-resultant r-place heteradic modifiers. A simple example of such a modifier is implicit in English when a transitive verb is converted into an intransitive verb, essentially by existential quantification of the object to the verb. Thus the transitive "hunts" in "Hemingway hunts hyenas" may be converted to the intransitive "hunts" in "Hemingway hunts" by such a modifier. This modifier is a 0-ary predecessor-resultant 1-place heteradic modifier, and if we symbolize it by $conv_{\pi_1\pi_2}$ then "Hemingway hunts" might be symbolized by

$$conv_{\pi_1\pi_2}\, hunt_{\pi_2}\, hem_\iota$$

where $hunt_{\pi_2}$ is the formal version of the transitive verb.

Finally, we might consider modifier modifiers. If α is the type of any of the modifiers given (including, indeed, those about to be defined!) then a j-ary modifier modifier of modifiers of type α will have type $(\alpha^2\iota^j)$. For instance, the "very" in 'Gödel's is a very long proof' can be treated as a modifier modifier, modifying modifiers of type π_1^2, and the sentence can be symbolized as

$$v_{(\pi_1^2)^2}\, \ell_{\pi_1^2}\, p_{\pi_1}\, g_\iota$$

in which, as usual, association to the left is assumed, and thus $v_{(\pi_1^2)^2}\ell_{\pi_1^2}$ has type π_1^2 , $v_{(\pi_1^2)^2}\ell_{\pi_1^2}p_{\pi_1}$ has type π_1 , and the whole wff has type o . Modifier modifiers may also be polyadic or heteradic : e.g, a reasonable construal of "Boxing Day is a warm to hot day" would treat "to" as a 2-place polyadic modifier modifier, and the sentence would be symbolized as

$$\text{to}_{(\pi_1{}^2)^3}\ \text{warm}_{\pi_1{}^2}\ \text{hot}_{\pi_1{}^2}\ \text{day}_{\pi_1}\ b_\iota \quad .$$

In summary, the modifiers considered and their types are given in the table:

Modifier	Type
0-ary modifier modifying monadic predicates	$\pi_1{}^2$
0-ary modifier modifying k-adic predicates	$\pi_k{}^2$
j-ary modifier modifying k-adic predicates	$(\pi_k{}^2\iota^j)$
j-ary r-place polyadic modifiers modifying k-adic predicates	$(\pi_k{}^{r+1}\iota^j)$
j-ary μ-resultant r-place heteradic modifier	$(\pi_{\mu(k_1,\ldots,k_r)}\pi_{k_1}\ldots\pi_{k_r}\iota^j)$
j-ary modifier modifier modifying modifiers of type α	$(\alpha^2\iota^j)$

The basic result of our use of the type-theoretic notation so far is that once a type is assigned to a modifier, its syntax and set-theoretic semantics is thereby determined. Moreover, although our abbreviations may tend to obscure the fact, such a syntax and semantics has been available at least since Henkin's [27], and some later papers could well have availed themselves of this syntax and semantics.

1.2 Some constant modifiers

Within the theory of types, we have seen that we have adequate resources for expressing modifiers of various kinds : we also have adequate resources for the definition of various constant modifiers, using just the (logical) constants introduced into the theory so far. For instance we can define a constant modifier for negation by:

$$N_{\pi_k^2} =_{df} (\imath f_{\pi_k^2})(g_{\pi_k})(y_\iota^{((k))})(f_{\pi_k^2} g_{\pi_k} y_\iota^{((k))} \equiv (\sim g_{\pi_k} y_\iota^{((k))})) \quad .$$

This definition can be shortened somewhat if we observe that

$$N_{\pi_k^2} = \lambda g_{\pi_k} \lambda y_\iota^{((k))} (\sim g_{\pi_k} y_\iota^{((k))}) \quad ,$$

whereby the latter equality could be used as the defining formula. This formula makes it clear that we are defining predicate negation ($N_{\pi_k^2}$) directly in terms of sentence negation $(\sim)$: a different logical basis, e.g. a many-valued logic, would allow different kinds of predicate negation to be defined.

The possibility of the second form of definition of $N_{\pi_k^2}$ depends upon the fact that in the first form $(\imath f_{\pi_k^2})$ is acting as a definite description operator : the modifier $f_{\pi_k^2}$ is fully determined by the wff in the scope of $(\imath f_{\pi_k^2})$. This situation can be contrasted with the case of the constant modifier for contrariety, whose definition does not have a shortened form and in which we trade on the fact that $\imath$ can function as a choice-operator:

$$C_{\pi_k^2} =_{df} (\imath f_{\pi_k^2})(g_{\pi_k})(y_\iota^{((k))})(f_{\pi_k^2} g_{\pi_k} y_\iota^{((k))} \supset (\sim g_{\pi_k} y_\iota^{((k))})) \quad .$$

We have already mentioned that "and" can act as a polyadic predicate modifier, of type π_k^3 : a suitable definition is

$$K_{\pi_k^3} =_{df} \lambda g_{\pi_k} \lambda h_{\pi_k} \lambda y_\iota^{((k))} (g_{\pi_k} y_\iota^{((k))} \;\&\; h_{\pi_k} y_\iota^{((k))}) \quad .$$

Predicate modifiers for disjunction, etc., could be defined similarly.

For existential quantification in the u-th place, a generalization of the heteradic modifier $\mathrm{conv}_{\pi_1\pi_2}$ introduced by way of example in §1.1, we have the definition

$$S^{u}_{\pi_k \pi_{k+1}} =_{df} \lambda g_{\pi_{k+1}} \lambda y_\iota^{(1)} \ldots \lambda y_\iota^{(u-1)} \lambda y_\iota^{(u+1)} \ldots \lambda y_\iota^{(k+1)} (\exists y_\iota^{(u)})(g_{\pi_{k+1}} y_\iota^{((k+1))})$$

for $1 < u < k+1$, with trivial variants for the cases $1 = u$, $u = k+1$.

We might also mention the identity modifier :

$$I_{\pi_k^2} =_{df} \lambda f_{\pi_k} f_{\pi_k} \quad .$$

Especially when $I_{\pi_k^2}$ is considered, it is clear that some of these constant modifiers could be described by, or related to, various combinators in combinatory logic. For example the deferment combinator β (see e.g. Curry [18]) could be used to describe the definition of $N_{\pi_k^2}$ in terms of N_{oo} .

Since we do not propose to set up any calculus of modifiers per se, nor to eliminate variables from our formulae, we do not investigate in detail the relations between modifiers and combinatory logic.

1.3. Quine's constant modifiers

In the system adumbrated in [55], Quine has six constant predicate modifiers : he outlines a proof to the effect that with these modifiers and the identity predicate all closed wffs of first-order quantification theory can be expressed. In type-theoretical terms each of Quine's operators is actually infinitely many operators, since each can modify predicates of any adicity : for example his first operator, complementation, is simply our $N_{\pi_k^2}$ and the adicity k is free in $N_{\pi_k^2}$. His other operators are:

intersection

$$\cap_{\pi_{max(k,\ell)}\pi_k\pi_\ell} =_{df} \lambda h_{\pi_\ell} \lambda g_{\pi_k} \lambda y_\iota^{((max(k,\ell)))}(g_{\pi_k} y_\iota^{((k))} \,\&\, h_{\pi_k} y_\iota^{((\ell))})$$

of which our $K_{\pi_k^3}$ is a special case;

major permutation

$$P_{\pi_k^2} =_{df} \lambda g_{\pi_k} \lambda y_\iota^{((k))}(g_{\pi_k} y_\iota^{(k)} y_\iota^{((k-1))}) \qquad ,$$

minor permutation

$$p_{\pi_k^2} =_{df} \lambda g_{\pi_k} \lambda y_\iota^{((k))}(g_{\pi_k} y_\iota^{(2)} y_\iota^{(1)} y_\iota^{(3)} \cdots y_\iota^{(k)}) \quad ,$$

padding

$$Invnun_{\pi_{k+1}\pi_k} =_{df} \lambda g_{\pi_k} \lambda y_\iota^{((k+1))}(g_{\pi_k} y_\iota^{(2)} \cdots y_\iota^{(k+1)}) \quad ,$$

and

cropping

$$Nun_{\pi_k\pi_{k+1}} =_{df} S^1_{\pi_k\pi_{k+1}} \quad .$$

(Quine actually uses the Hebrew letter nun for cropping, and an upside-down nun for padding, hence our names here).

In order to sketch the way in which closed wffs of first-order quantification theory are expressed in Quine's "predicate-functor logic", we have to suppose that the free variables k and ℓ in these definitions can take the value 0, and that $\pi_0 = o$. That is, some of the predicate-functors, such as complementation and intersection, will become "sentence-functors" such as N_{o^2} and K_{o^3} in this case. We also have to suppose that the identity predicate Q_{π_2} is available as a primitive predicate constant. Now let us take the wff

$$(x)((y)Fxyy \supset (\exists z)Fxzz)$$

and convert it firstly into type-theoretic notation

$$(x_\iota)((y_\iota)F_{\pi_3} x_\iota y_\iota y_\iota \supset (\exists z_\iota)F_{\pi_3} x_\iota z_\iota z_\iota) \qquad ,$$

as a preliminary to converting it into our version of Quine's notation.

Our first task is to achieve the effect of identifying the variables y_ι in $F_{\pi_3} x_\iota y_\iota y_\iota$ and z_ι in $F_{\pi_3} x_\iota z_\iota z_\iota$. Concentrating on the former, we have

$$F_{\pi_3} x_\iota y_\iota y_\iota \equiv . \ p_{\pi_3^2} F_{\pi_3} y_\iota x_\iota y_\iota$$
$$\equiv . \ P_{\pi_3^2} (p_{\pi_3^2} F_{\pi_3}) y_\iota y_\iota x_\iota$$

Then by an extension of the principle $Fxx \equiv (\exists y)(x = y \,\&\, Fxy)$ we have, letting $P_{\pi_3^2} (p_{\pi_3^2} F_{\pi_3}) = Z_{\pi_3}$,

$$Z_{\pi_3} y_\iota y_\iota x_\iota \equiv . \ Nun_{\pi_2 \pi_3} (\cap_{\pi_3 \pi_3 \pi_2} Q_{\pi_2} Z_{\pi_3}) y_\iota x_\iota \quad .$$

Then $(y_\iota) Z_{\pi_3} y_\iota y_\iota x_\iota \equiv . \ N_{\pi_1^2} (Nun_{\pi_1 \pi_2} (N_{\pi_2^2} (Nun_{\pi_2 \pi_3} (\cap_{\pi_3 \pi_3 \pi_2} Q_{\pi_2} Z_{\pi_3})))) x_\iota$;

similarly

$$(\exists z_\iota) Z_{\pi_3} z_\iota z_\iota x_\iota \equiv . \ Nun_{\pi_1 \pi_2} (Nun_{\pi_2 \pi_3} (\cap_{\pi_3 \pi_3 \pi_2} Q_{\pi_2} Z_{\pi_3})) x_\iota \quad .$$

Then the final $\supset$ and (x_ι) are translated, and we end up with $N_{0^2} (Nun_{0 \pi_1} (\cap_{\pi_1^3} (N_{\pi_1^2} (Nun_{\pi_1 \pi_2} (N_{\pi_2^2} (Nun_{\pi_2 \pi_3} (\cap_{\pi_3 \pi_3 \pi_2} Q_{\pi_2} (P_{\pi_3^2} (p_{\pi_3^2} F_{\pi_3}))))))) (N_{\pi_1^2} (Nun_{\pi_1 \pi_2} (Nun_{\pi_2 \pi_3} (\cap_{\pi_3 \pi_3 \pi_2} Q_{\pi_2} (P_{\pi_3^2} (p_{\pi_3^2} F_{\pi_3})))))))$,

which is not quite so transparently valid as the wff we started with.

We have four main criticisms of Quine's system as a system of "predicate-functors". Firstly, as we have already noted, each of Quine's constants requires indexing when represented explicitly in type theory: to this extent the elegance and economy of Quine's system is somewhat superficial. In the presence of $P_{\pi_k^2}$ and $p_{\pi_k^2}$, which in combination can effect any permutation of subject letters, Quine needs only the ability to quantify in some fixed position; thus his operator $Nun_{\pi_k \pi_{k+1}}$ is a special case of our $S^u_{\pi_k \pi_{k+1}}$, and he avoids the necessity for the index u . But all the other indices cannot be dispensed with in this way, and would re-appear either explicitly or implicitly in a precise statement of the semantics for Quine's system.

Secondly, various restrictions that appear to be needed to prevent nonsignificance are not stated by Quine. For example the possibility that some operators should apply to 0-adic predicates is admitted, indeed required, by Quine: yet it seems to be nonsignificant to apply either kind of permutation to a 0-adic predicate, and the definition of cropping does not work for a 0-adic predicate (even though in ordinary terms $(\exists x)p$ is significant.) This kind of defect is not an especially major one, but explanation of the cases in question is required.

Thirdly, Curry in [18] remarks (of an earlier version of Quine's system, but still applicable to the present system) that there are other regular combinators besides those definable in Quine's system. In particular an elementary cancellator $Canc_{\pi_k \pi_{k+1}}$ with the property

$$(g_{\pi_k})(Canc_{\pi_k \pi_{k+1}}(Invnun_{\pi_{k+1} \pi_k}\, g_{\pi_k}) = g_{\pi_k})$$

is not definable. (A formal proof of this fact could be had by using the result that any two translations, into quantifier theory, of a wff of Quine's system are logically equivalent, whereas this no longer holds if $Canc_{\pi_k \pi_{k+1}}$ is admitted into the system). This means that in some sense (a rather technical one) Quine's system is incomplete, even though it does what it was designed to do, namely to provide an exactly adequate means for the representation of (the closed wffs of) first-order quantification theory.

Fourthly, as one would expect from its correspondence with first-order quantification theory, there is no actual logic of "predicate-functors" in Quine's "predicate-functor logic". There is no provision for variable "predicate-functors", and functors on predicate-functors, i.e. modifier modifiers, are explicitly eschewed as belonging to dizzy realms. There is also no provision for j-ary modifiers where $j > 0$, and such modifiers have a definite role in the symbolization of prepositional phrases in English. Hence Quine's system cannot even begin to deal with the basic concerns of the logic of predicate modifiers.

1.4 Kinds of inclusiveness of modifiers

One of the main difficulties of the theory of predicate modifiers is to account properly for the difference between

(A) Jones killed Smith with a knife
$\therefore$ Jones killed Smith

and

(B) Jones killed Smith in a dream
$\therefore$ Jones killed Smith .

In [48], Montague says that because of (B), where the conclusion may be false even though the premise is true, (A) is not a logically valid argument. That is, Montague argues that because the schema

(S) $$f_{\pi_2^2\iota}\, x_\iota\, g_{\pi_2}\, y_\iota\, z_\iota$$
$$\therefore \quad g_{\pi_2}\, y_\iota\, z_\iota$$

does not always yield valid arguments, and because (A) is an instance of (S), then (A) is not a logically valid argument.

Let us compare this situation with

(C) Jones is taller than Smith
$\therefore$ Smith is not taller than Jones

and

(D) Jones is liked by Smith
$\therefore$ Smith is not liked by Jones .

Argument (D) shows that the schema

(T) $$g_{\pi_2} y_\iota z_\iota$$

$$\therefore \quad \sim g_{\pi_2} z_\iota y_\iota$$

does not always yield valid arguments: but does it follow from this that (C) is not a logically valid argument? If we define

$$Asym_{o\pi_2} =_{df} \lambda f_{\pi_2} (y_\iota)(z_\iota)(f_{\pi_2} y_\iota z_\iota \supset (\sim f_{\pi_2} z_\iota y_\iota))$$

then the schema

(T') $$g_{\pi_2} y_\iota z_\iota$$

$$Asym_{o\pi_2} g_{\pi_2}$$

$$\therefore \quad \sim g_{\pi_2} z_\iota y_\iota$$

is valid, and since the relation of being taller than is asymmetrical we are justified in adding this fact as an extra premise to (C), and thus validating it according to (T'). Now, regardless of whether this extra premise is counted as an empirical premise or as a meaning postulate in the Carnapian sense or as having some other status again, there is a good sense in which (C) <u>is</u> a valid argument since it becomes an instance of the valid schema (T') once this extra premise is added.

This leads us to look for properties of modifiers analogous to the familiar and useful properties of dyadic relations, in order that arguments like (A) can be made valid by the addition of extra premises using these properties of modifiers. To this end, we begin by making the (temporary) definitions

$$Imp_o =_{df} (f_{\pi_k^2} g_{\pi_k} y_\iota^{((k))} \supset g_{\pi_k} y_\iota^{((k))})$$

$$Qimp_o =_{df} (y_\iota^{((k))}) Imp_o$$

and $Imp_o^j =_{df} (f_{\pi_k^2 \iota j}\, x_\iota^{((j))}\, g_{\pi_k}\, y_\iota^{((k))} \supset g_{\pi_k}\, y_\iota^{((k))})$

$$Qimp_o^j =_{df} (y_\iota^{((k))})\, Imp_o^j \quad ,$$

and then using these we define a property of modifiers corresponding to inclusiveness, of [45] and [57], or the "subproperty" property of [48]:

$$Incl_{o(\pi_k^2)} =_{df} \lambda f_{\pi_k^2}(g_{\pi_k})\, Qimp_o$$

for 0-ary modifiers and

$$Incl_{o(\pi_k^2 \iota j)} =_{df} \lambda f_{\pi_k^2 \iota j}\, (x_\iota^{((j))})(g_{\pi_k})\, Qimp_o^j$$

for j-ary modifiers, $j \geqslant 1$. Then the schema

(S_1')

$$\begin{array}{l} f_{\pi_2^2 \iota}\, x_\iota\, g_{\pi_2}\, y_\iota\, z_\iota \\ Incl_{o(\pi_2^2 \iota)}\, f_{\pi_2^2 \iota} \\ \therefore \quad g_{\pi_2}\, y_\iota\, z_\iota \end{array}$$

is obviously valid, and we might hope to use it to validate (A), after the addition of a suitable extra premise. However, the extra premise would require us to assert that "with x" is an inclusive modifier, for any individual term x, and this is made dubious by examples like

(E) I accepted his account with reservations

∴ I accepted his account

and

(E') The theorem that the continuum has no well-ordering was proved with the Axiom of Determinateness

∴ The theorem that the continuum has no well-ordering was proved .

Thus we are led to define the following weaker notion of inclusiveness for j-ary modifiers:

$$Incl^{n}_{o(\pi_k^2\iota j)\iota} =_{df} \lambda x_\iota^{(n)} \lambda f_{\pi_k^2 \iota j} (x_\iota^{((n-1))})(x_\iota^{(n+1)}) \ldots (x_\iota^{(j)})(g_{\pi_k}) \, Qimp_o^{j} \quad ,$$

where $1 \leq n \leq j$. This property says, roughly, that the j-ary modifier $f_{\pi_k^2 \iota j}$ is inclusive at the nth place for the individual $x_\iota^{(n)}$. Then the schema

(S$'_2$)

$$f_{\pi_2^2\iota} x_\iota \, g_{\pi_2} y_\iota z_\iota$$

$$Incl^{1}_{o(\pi_2^2\iota)\iota} \, x_\iota \, f_{\pi_2^2\iota}$$

$$\therefore \quad g_{\pi_2} y_\iota z_\iota$$

is valid, and the additional premise now required in argument (A) is that "with a knife" is inclusive, and this is weaker than our previous premise that "with x" is inclusive for any x.

In each of (S), (S$'_1$) and (S$'_2$) we have assumed that "a knife" is an individual, appropriately symbolized by x_ι. More generally, and probably more correctly, we should symbolize the premise of argument (A) by

$$(\exists x_\iota)(h_{\pi_1} x_\iota \; \& \; f_{\pi_2^2\iota} x_\iota \, g_{\pi_2} y_\iota z_\iota) \quad ,$$

where h_{π_1} symbolizes the predicate of being a knife. We can now define a notion of inclusiveness for j-ary modifiers appropriate to this case, and of which our two previous notions are special cases:

$$Incl^{n}_{o(\pi_k^2\iota j)\pi_1} =_{df} \lambda h_{\pi_1} \lambda f_{\pi_k^2\iota j} (x_\iota^{((j))})(g_{\pi_k})(h_{\pi_1} x_\iota^{(n)} \supset Qimp_o^{j}) \; .$$

This property appears in the schema

(S$'_3$)

$$(\exists x_\iota)(h_{\pi_1} x_\iota \; \& \; f_{\pi_2^2\iota} x_\iota \, g_{\pi_2} y_\iota z_\iota)$$

$$Incl^{1}_{o(\pi_2^2\iota)\pi_1} \, h_{\pi_1} \, f_{\pi_2^2\iota}$$

$$\therefore \quad g_{\pi_2} y_\iota z_\iota \quad ,$$

and the added premise required to validate (A) is now the premise that "with x" is inclusive for any knife x.

Our last notion of inclusiveness is more general than the previous two notions, since we have

$$Incl_{o(\pi_k^2\iota j)} = Incl^{n}_{o(\pi_k^2\iota j)\pi_1}\,\lambda x_\iota(x_\iota = x_\iota)$$

(for any $n, 1 \leqslant n \leqslant k$), and

$$Incl^{n}_{o(\pi_k^2\iota j)\iota} = \lambda x_\iota(Incl^{n}_{o(\pi_k^2\iota j)\pi_1}\,\lambda y_\iota(y_\iota = x_\iota))$$

(for any $n, 1 \leqslant n \leqslant k$).

To gain yet another notion of inclusiveness, we may argue that "with a knife" is not inclusive for all predicates, but only for things appropriately done with a knife, such as killing, injuring, chopping, slicing and frightening, and that the examples

(F) The captor helped the prisoner along with a knife

$\therefore$ The captor helped the prisoner along

(with the premise taken in the ironic context of the prisoner having the knife at his back),

(F') Jack failed to open to door with a knife

$\therefore$ Jack failed to open the door

(due to N. Griffin), and

(F") Sam Orr ate his peas fastidiously with a knife

$\therefore$ Sam Orr ate his peas fastidiously

provide cases of predicates for which "with a knife" is not inclusive. To follow out this line of argument we can define

$$Incl^{n}_{o(\pi_k^2\iota j)\pi_1(o\pi_k)} =_{df} \lambda m_{o\pi_k}\lambda h_{\pi_1}\lambda f_{\pi_k^2\iota j}(x_\iota^{("j")})(g_{\pi_k})(m_{o\pi_k}g_{\pi_k} \supset . h_{\pi_1}x_\iota^{(n)} \supset Qimp_o^{j})$$

in which $\lambda m_{o\pi_k}$ abstracts with respect to properties of k-adic predicates. Our new argument schema is

$$(S_4') \qquad (\exists x_\iota)(h_{\pi_1}x_\iota \,\&\, f_{\pi_2^2\iota}x_\iota g_{\pi_2}y_\iota z_\iota)$$

$$m_{o\pi_2}g_{\pi_2} \,\&\, Incl^{1}_{o(\pi_2^2\iota)\pi_1(o\pi_2)}\,m_{o\pi_2}h_{\pi_1}f_{\pi_2^2\iota}$$

$$\therefore \quad g_{\pi_2}y_\iota z_\iota$$

and the added premise required to validate (A) is that "with a knife" is inclusive for a certain kind of predicate, and that "killing" is a predicate of this kind.

It may well be, as suggested by C.L. Hamblin, that "with" is simply ambiguous in English just as much as "bank" or "fire" is ambiguous. If so then it would be pointless to expect to be able to symbolize it with the same modifier $f_{\pi_2^2\iota}$ in all cases: however, even if it is that different senses of "with" just have the different kinds of properties that have arisen in our discussion, this is sufficient to show that we need to be able to define and codify various different properties of modifiers, of wider or narrower applicability.

Finally we define the most narrow notion of inclusiveness of all:

$$Incl_{o(\pi_k^2\iota j)\iota j\pi_k\iota^k} =_{df} \lambda y_\iota^{((k))} \lambda g_{\pi_k} \lambda x_\iota^{((j))} \lambda f_{\pi_k^2\iota j}\, Imp_o^{j} \quad .$$

A schema using this notion is

(S$_5'$)
$$f_{\pi_2^2\iota}\, x_\iota\, g_{\pi_2}\, y_\iota\, z_\iota$$
$$Incl_{o(\pi_2^2\iota)\iota\pi_2\iota^2}\, y_\iota\, z_\iota\, g_{\pi_2}\, x_\iota\, f_{\pi_2^2\iota}$$
$$\therefore \quad g_{\pi_2}\, y_\iota\, z_\iota$$

and it is no surprise that this schema is valid, since by undoing the various abbreviations in the second premise, and applying Henkin's rules of inference II and III, we find that the second premise and

$$f_{\pi_2^2\iota}\, x_\iota\, g_{\pi_2}\, y_\iota\, z_\iota \supset g_{\pi_2}\, y_\iota\, z_\iota$$

are interderivable (and, given Henkin's axiom 10, they are hence identical). So schema (S$_5'$) validates argument (A) with the extra premise: that Jones killed Smith with a knife materially implies that Jones killed Smith. This premise is indeed a property of the modifier "with" but it is scarcely a property of the general, classificatory, kind that we set out to find. The point is that we have no purely formal syntactical criteria to delimit the kind of modifier properties that we are looking for, and that it remains for us to gain experience in the classification and use of modifiers before we can judge which definitions of modifier properties are to be retained and generally applied.

1.5 Survey of inclusiveness properties

We now seek to survey all the ways of defining a modifier property that could reasonably be called an inclusiveness property for a modifier of type $\pi_k^2 \iota^j$.

(An attenuated version of the survey will apply to modifiers of type π_k^2: we do not carry this as a separate case). Our survey generalizes on the various definitions that we made, partly from philosophico-genetic considerations and partly from formal considerations, in the preceding section.

Firstly we describe the construction of **Lattice One**. This lattice has 2^{j+k+1} elements, with $Incl_{o(\pi_k^2\iota^j)}$ as its greatest element and $Incl_{o(\pi_k^2\iota^j)\iota^j\pi_k\iota^k}$ as its least element. (Actually the lattice has the structure of the finite Boolean algebra with $j+k+1$ atoms, but there is no clear intuitive construal for the complementation operator in the algebra, so we refer to it just as a lattice and not as a Boolean algebra.) The immediate predecessors of $Incl_{o(\pi_k^2\iota^j)}$ (i.e. the properties covered by $Incl_{o(\pi_k^2\iota^j)}$) are the following $j+k+1$ properties:

(i) the j properties $Incl^n_{o(\pi_k^2\iota^j)\iota}$ for $1 \leq n \leq j$;

(ii) the property $Incl^p_{o(\pi_k^2\iota^j)\pi_k}$ defined by

$$Incl^p_{o(\pi_k^2\iota^j)\pi_k} =_{df} \lambda g_{\pi_k} \lambda f_{\pi_k^2\iota^j} (x_\iota^{(j)})\, Qimp_o^j \quad ;$$

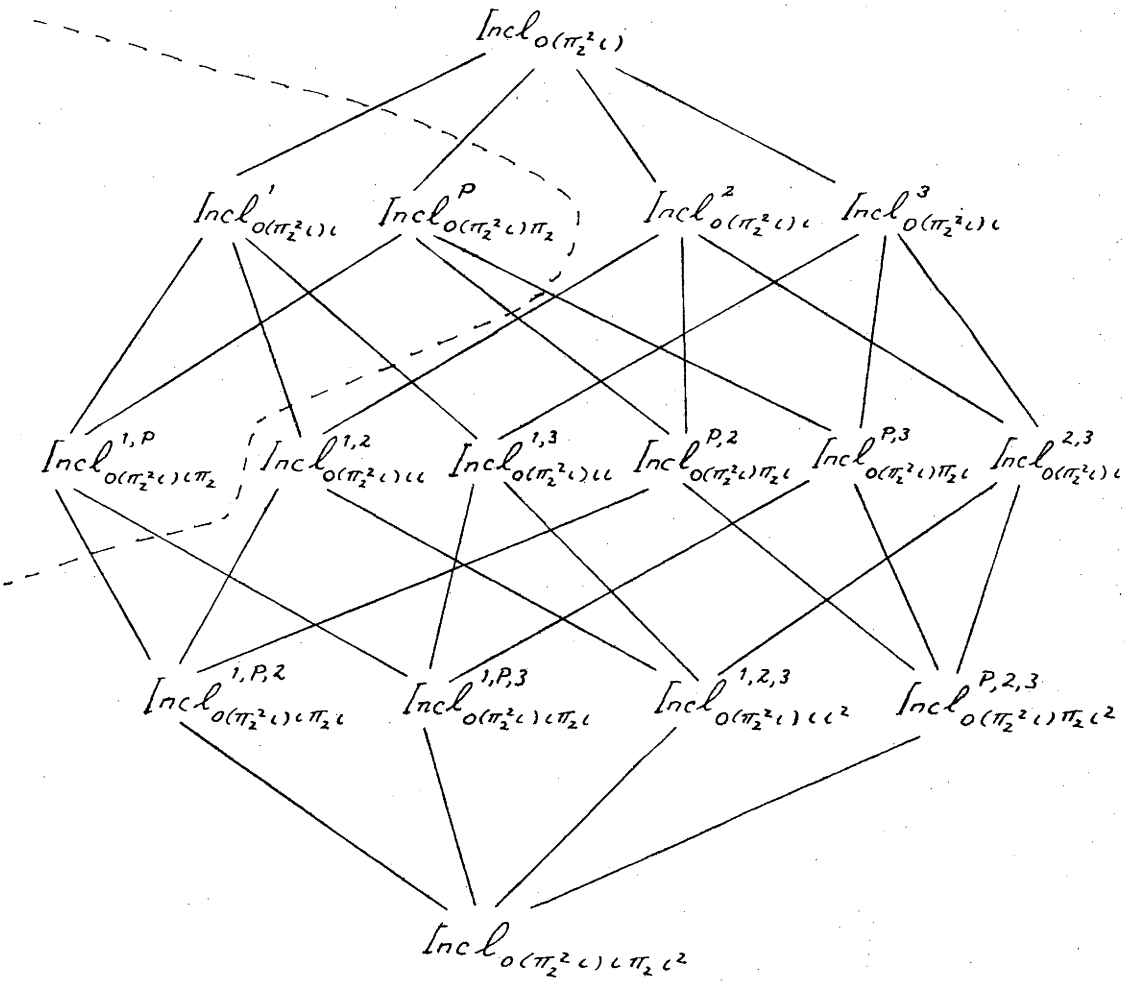
(iii) the k properties $Incl^{j+m}_{o(\pi_k^2\iota^j)\iota}$ for $1 \leq m \leq k$,

defined by

$$Incl^{j+m}_{o(\pi_k^2\iota^j)\iota} =_{df} \lambda y_\iota^{(m)} \lambda f_{\pi_k^2\iota^j} (x_\iota^{((j))})(g_{\pi_k})(y_\iota^{((m-1))})(y_\iota^{(m+1)}) \ldots (y_\iota^{(k)})\, Imp_o^j \ .$$

In short, each predecessor property is gained by deleting one of the universal quantifications from the definition of $Incl_{o(\pi_k^2\iota^j)}$, replacing it by an abstraction operator at the front of the definiens, adding an appropriate type symbol to the type symbol of $Incl$, and adding an appropriate index to $Incl$. We use the index P if we are forming a property of g_{π_k} , and we order the indices $1,\ldots,j, P, j+1,\ldots,j+k$: the index P is unnecessary for properties in **Lattice One**, since the situation is sufficiently indicated by the occurrence of π_k in the type symbol, but it will be needed for properties in **Lattice Three** and we introduce it now for uniformity. We find predecessors as we proceed down **Lattice One** by the same general procedure of removing universal quantifications, and adding abstraction operators, type symbols and indices: for example the index $1, 3, P, 5$, for a property of a modifier of type $\pi_2^2\iota^3$, shows that $\lambda y_\iota^{(2)} \lambda g_{\pi_2} \lambda x_\iota^{(1)} \lambda x_\iota^{(3)}$ appears in the definiens, and the corresponding type symbol is $o(\pi_2^2\iota^3)\iota^2\pi_2\iota$.

We illustrate by setting out all of **Lattice One** for a modifier of type $\pi_2^2\iota$:



(We drop the indices from the least element, since they are no longer necessary: we could write the index *1,P,2,3* if we wished). From the simple kind of considerations of the previous section, the properties within the dotted line are those most likely to be used from **Lattice One.**

Our next lattice, **Lattice Two**, houses properties like those introduced in (S'_3) and (S'_4) in §1.4. It has the same number of elements and the same greatest element as **Lattice One**, but its least element is $Incl_{o(\pi_k^2\iota^j)\pi_1^j(o\pi_k)\pi_1^k}$ defined by

$$Incl_{o(\pi_k^2\iota^j)\pi_1^j(o\pi_k)\pi_1^k} =_{df} \lambda n_{\pi_1}^{((k))} \lambda m_{o\pi_k} \lambda h_{\pi_1}^{((j))} \lambda f_{\pi_k^2\iota^j} (y_\iota^{((k))})(g_{\pi_k}) (x_\iota^{((j))}) (n_{\pi_1}^{(1)} y_\iota^{(1)} \supset. \ldots n_{\pi_1}^{(k)} y_\iota^{(k)} \supset. m_{o\pi_k} g_{\pi_k} \supset. h_{\pi_1}^{(1)} x_\iota^{(1)} \supset. \ldots h_{\pi_1}^{(j)} x_\iota^{(j)} \supset. Imp_o^j) .$$

In general each definition of a property in **Two** can be gained from the definition of a corresponding property in **One**: in the definiendum each separate type symbol after $o(\pi_k^2\iota^j)$ is prefaced by o, i.e. α is changed to $o\alpha$ while the indexing system remains the same; in the definiens each λv_α, other than $\lambda f_{\pi_k^2\iota^j}$, is changed to $\lambda w_{o\alpha}$, for some distinct $w_{o\alpha}$ in each case, and the quantification (v_α) and the hypothesis $w_{o\alpha} v_\alpha$ are inserted in appropriate places (effectively restricting the quantification (v_α) to $w_{o\alpha}$).

When schemata are formed using modifier properties from **Lattice Two**, either the additional properties $w_{o\alpha}$ appear in the premises (as h_{π_1} does in (S'_3)), or they are introduced together with the modifier property (as $m_{o\pi_2}$ is in (S'_4)).

From each entry in **Two** we can define the corresponding property in **One** by generalizing on the identity

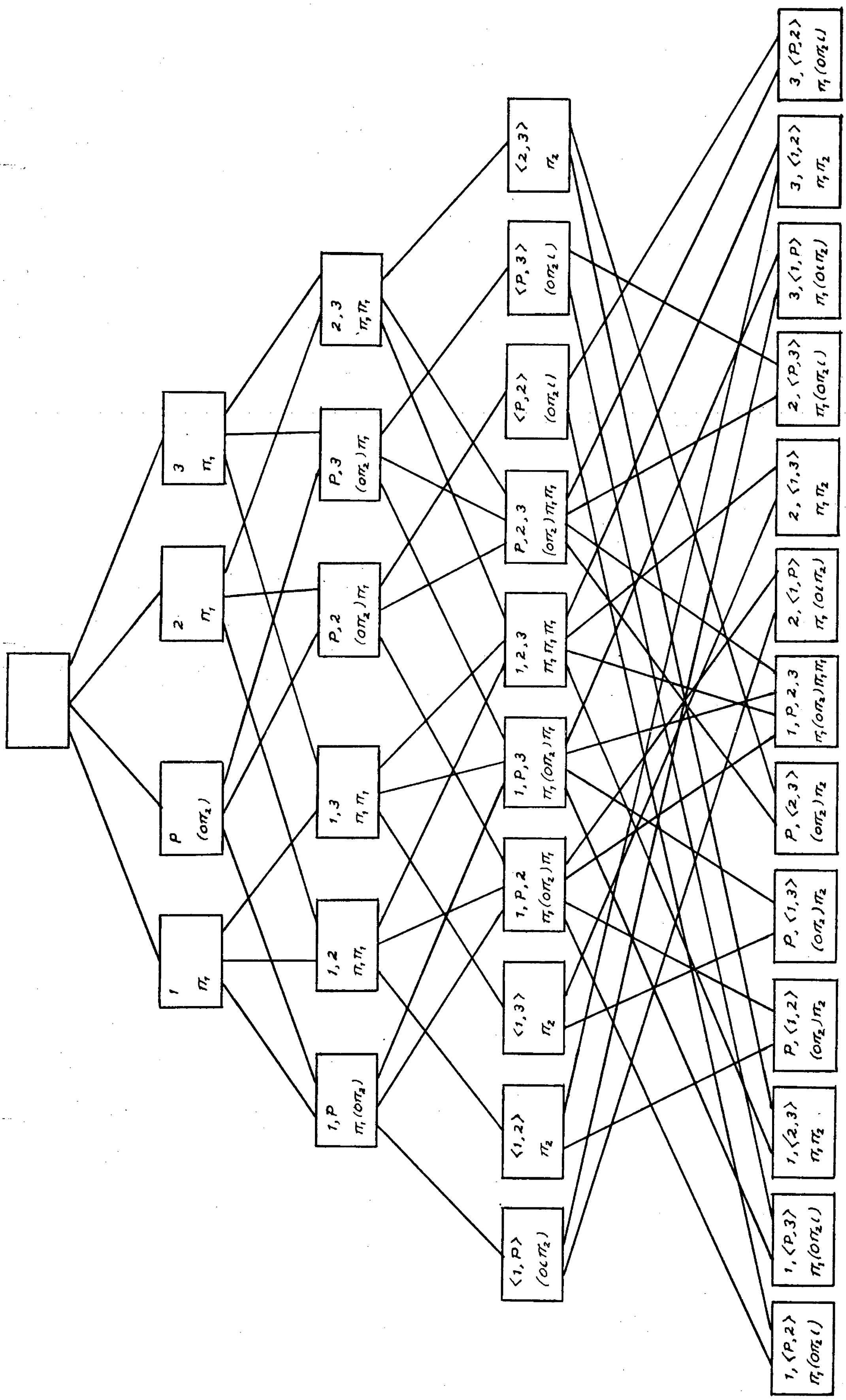
$$Incl^n_{o(\pi_k^2\iota^j)\iota} = \lambda x_\iota (Incl^n_{o(\pi_k^2\iota^j)\pi_1} \lambda y_\iota (y_\iota = x_\iota))$$

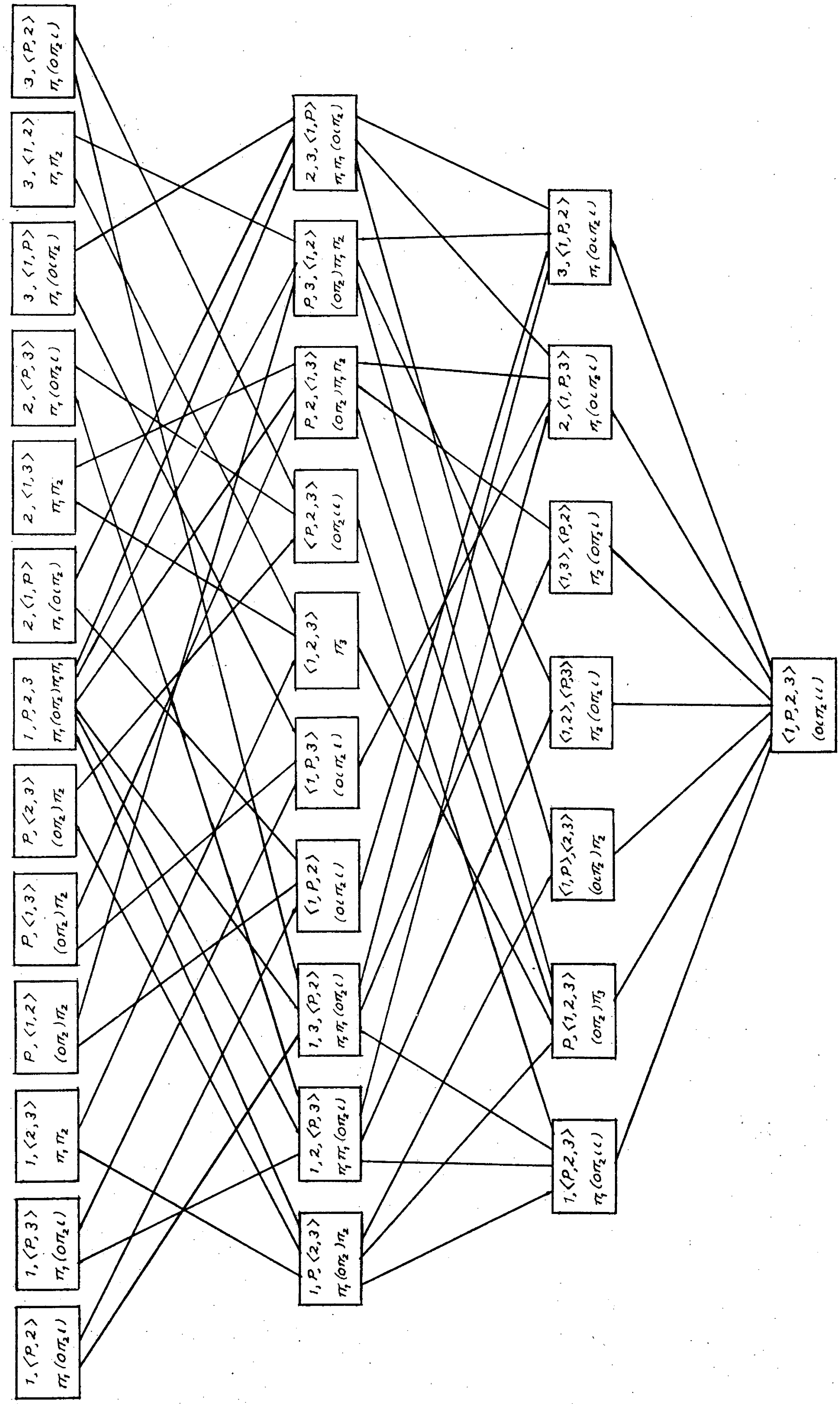
that we noticed in the previous section. And within **Two** we can proceed from any property to any property that covers it by using tautological properties like $N_{\pi_1^2}(K_{\pi_1^3} h_{\pi_1} (N_{\pi_1^2} h_{\pi_1}))$ or $\lambda x_\iota (x_\iota = x_\iota)$. Thus from the least element $Incl_{o(\pi_k^2\iota^j)\pi_1^j(o\pi_k)\pi_1^k}$ of **Two** we can generate all the properties in **One** and **Two**: the sheer syntactical length of the definiens of this property helps to make this assertion plausible.

Lattice Three is something else. We now generalize on the kind of definitions used to define the properties in **Two** by admitting relations (of any adicity up to $j+k+1$), as well as properties, to hold for the x_ι's, y_ι's and g_{π_k} . A typical example is the property

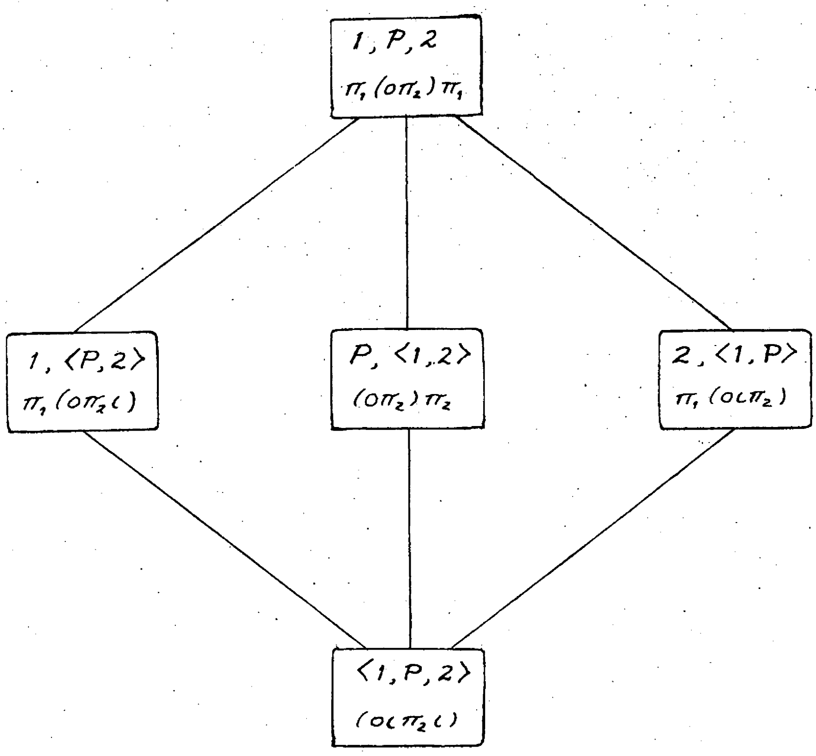
$$Incl^{P,\langle 1,2\rangle}_{o(\pi_2{}^2\iota)(o\pi_2)(\pi_2)} =_{df} \lambda h_{\pi_2}\lambda m_{o\pi_2}(x_\iota^{\text{“}(1)\text{”}})(g_{\pi_2})(y_\iota^{\text{“}(2)\text{”}})(h_{\pi_2}x_\iota^{\text{“}(1)\text{”}}y_\iota^{\text{“}(1)\text{”}} \supset . \, m_{o\pi_2}g_{\pi_2} \supset . \, Imp_0{}')$$

where some dyadic relation h_{π_2} is required to hold between $x_\iota^{\text{“}(1)\text{”}}$ and $y_\iota^{\text{“}(1)\text{”}}$. In the indexing system a finite tuple of our previous indices shows that a relation is required to hold of the individuals, or the property, in the places named by these indices. To illustrate **Lattice Three** we set out firstly its upper portion then its lower portion for modifiers of type $\pi_2{}^2\iota$ (we give just the index and the portion of the type symbol after $o(\pi_2{}^2\iota)$ for each $Incl$ property in the lattice):


1
π1
P
(oπ2)
2
π1
3
π1
1, P
π1(oπ2)
1, 2
π1π1
1, 3
π1π1
P, 2
(oπ2)π1
P, 3
(oπ2)π1
2, 3
π1π1
⟨1, P⟩
(oιπ2)
⟨1, 2⟩
π2
⟨1, 3⟩
π2
1, P, 2
π1(oπ2)π1
1, P, 3
π1(oπ2)π1
1, 2, 3
π1π1π1
P, 2, 3
(oπ2)π1π1
⟨P, 2⟩
(oπ2ι)
⟨P, 3⟩
(oπ2ι)
⟨2, 3⟩
π2
1, ⟨P, 2⟩
π1(oπ2ι)
1, ⟨P, 3⟩
π1(oπ2ι)
1, ⟨2, 3⟩
π1π2
P, ⟨1, 2⟩
(oπ2)π2
P, ⟨1, 3⟩
(oπ2)π2
P, ⟨2, 3⟩
(oπ2)π2
1, P, 2, 3
π1(oπ2)π1π1
2, ⟨1, P⟩
π1(oιπ2)
2, ⟨1, 3⟩
π1π2
2, ⟨P, 3⟩
π1(oπ2ι)
3, ⟨1, P⟩
π1(oιπ2)
3, ⟨1, 2⟩
π1π2
3, ⟨P, 2⟩
π1(oπ2ι)


$1, \langle P,2\rangle$ $\pi_1(o\pi_2\iota)$
$1, \langle P,3\rangle$ $\pi_1(o\pi_2\iota)$
$1, \langle 2,3\rangle$ $\pi_1\pi_2$
$P, \langle 1,2\rangle$ $(o\pi_2)\pi_2$
$P, \langle 1,3\rangle$ $(o\pi_2)\pi_2$
$P, \langle 2,3\rangle$ $(o\pi_2)\pi_2$
$1, P, 2, 3$ $\pi_1(o\pi_2)\pi_1\pi_1$
$2, \langle 1,P\rangle$ $\pi_1(o\iota\pi_2)$
$2, \langle 1,3\rangle$ $\pi_1\pi_2$
$2, \langle P,3\rangle$ $\pi_1(o\pi_2\iota)$
$3, \langle 1,P\rangle$ $\pi_1(o\iota\pi_2)$
$3, \langle 1,2\rangle$ $\pi_1\pi_2$
$3, \langle P,2\rangle$ $\pi_1(o\pi_2\iota)$
$1, P, \langle 2,3\rangle$ $\pi_1(o\pi_2)\pi_2$
$1, 2, \langle P,3\rangle$ $\pi_1\pi_1(o\pi_2\iota)$
$1, 3, \langle P,2\rangle$ $\pi_1\pi_1(o\pi_2\iota)$
$\langle 1,P,2\rangle$ $(o\iota\pi_2\iota)$
$\langle 1,P,3\rangle$ $(o\iota\pi_2\iota)$
$\langle 1,2,3\rangle$ π_3
$\langle P,2,3\rangle$ $(o\pi_2\iota\iota)$
$P, 2, \langle 1,3\rangle$ $(o\pi_2)\pi_1\pi_2$
$P, 3, \langle 1,2\rangle$ $(o\pi_2)\pi_1\pi_2$
$2, 3, \langle 1,P\rangle$ $\pi_1\pi_1(o\iota\pi_2)$
$1, \langle P,2,3\rangle$ $\pi_1(o\pi_2\iota\iota)$
$P, \langle 1,2,3\rangle$ $(o\pi_2)\pi_3$
$\langle 1,P\rangle, \langle 2,3\rangle$ $(o\iota\pi_2)\pi_2$
$\langle 1,2\rangle, \langle P,3\rangle$ $\pi_2(o\pi_2\iota)$
$\langle 1,3\rangle, \langle P,2\rangle$ $\pi_2(o\pi_2\iota)$
$2, \langle 1,P,3\rangle$ $\pi_1(o\iota\pi_2\iota)$
$3, \langle 1,P,2\rangle$ $\pi_1(o\iota\pi_2\iota)$
$\langle 1,P,2,3\rangle$ $(o\iota\pi_2\iota\iota)$

It can be seen that **Lattice Three** will in general contain **Lattice Two** as a proper sublattice. **Three** is not a Boolean algebra in general, since e.g. the lattice given for properties of modifiers of type $\pi_2{}^2\iota$ has 52 elements, and 52 is not of the form 2^n. In fact, **Three** is in general not distributive: the lattice given contains



as a sublattice, and hence fails Dedekind's test for distributivity of a lattice.

The lattices serve the usual taxonomic properties of lattices: if an inclusiveness property of modifiers is required to validate an inference, and if a particular inclusiveness property will do so, we can investigate properties above it, in the appropriate lattice, in order to see if some more general inclusiveness property can be asserted of the modifier in question. Conversely if a given property is too strong we can investigate properties below it in a lattice, in order to find a weaker property which can be asserted with more justification and which will still validate the inference in question. The lattices also show which inclusiveness properties are independent of each other, and of two independent properties it may be necessary to use both when dealing with a number of different inferences using a particular modifier.

"Within" the interstices between the vertices of the reticulations and decussations of the networks forming lattices **One, Two** and **Three**, there are further lattices (actually they are atomless Boolean algebras this time). For example consider the interstice

$$\begin{tikzcd} & \mathit{Incl}_{o(\pi_2^2\iota)} \\ \mathit{Incl}'_{o(\pi_2^2\iota)\pi_1} \arrow[ur, dash] & \end{tikzcd}$$

in **Lattice Two** for $\pi_2^2\iota$. For any given wff B_{π_1} , $\mathit{Incl}'_{o(\pi_2^2\iota)\pi_1} B_{\pi_1}$ is a wff of type $o(\pi_2^2\iota)$, and hence is a property of modifiers of type $\pi_2^2\iota$. If B_{π_1} is equivalent to $\lambda x_\iota(x_\iota = x_\iota)$, then $\mathit{Incl}'_{o(\pi_2^2\iota)\pi_1} B_{\pi_1}$ is the property $\mathit{Incl}_{o(\pi_2^2\iota)}$, already in **Two**, but otherwise we gain a new property of modifiers when we fix a place in $\mathit{Incl}'_{o(\pi_2^2\iota)\pi_1}$ to a particular wff B_{π_1} . If B_{π_1} is equivalent to $\lambda x_\iota(\sim(x_\iota = x_\iota))$, we have the least element of this interstitial lattice - "least" because it is least difficult for a modifier $f_{\pi_2^2\iota}$ to have this property, since all such modifiers have it. There are no atoms in the interstitial algebra, for much the same reason that the Lindenbaum algebra of propositional calculus is atomless: for any proposed atom $\mathit{Incl}'_{o(\pi_2^2\iota)\pi_1} A_{\pi_1}$, where A_{π_1} is not equivalent to $\lambda x_\iota(\sim(x_\iota = x_\iota))$, there is a lesser element

$Incl^{1}_{O(\pi_2^2\iota)\pi_1}(K_{\pi_1^3} A_{\pi_1} v_{\pi_1})$, where v_{π_1} is a variable not in A_{π_1} .

1.6 Kinds of commutativity of modifiers

By comparing

(G) Jones killed Smith with a knife in the park
∴ Jones killed Smith in the park with a knife

with (example due to P. O'Carroll)

(H) The monkey stood on the blue box on the red box
∴ The monkey stood on the red box on the blue box,

we see that in some cases the order of modification of two modifiers is irrelevant, in which case we will say in general that the modifiers are commutative (with each other), but that in other cases the order of modification is relevant, and arguments like (H) are invalid. In argument (G) the inference works in either direction, i.e. the premise and conclusion are equivalent, but we can have cases where modifiers commute in one direction only. As a simple artificial example, let us define the verum modifier by

$$V_{\pi_k^2} =_{df} \lambda f_{\pi_k} \lambda x_\iota^{((k))}(x_\iota^{(1)} = x_\iota^{(1)}) \quad :$$

then the argument schema

$$N_{\pi_k^2}(V_{\pi_k^2} f_{\pi_k}) y_\iota^{((k))}$$
$$\therefore \quad V_{\pi_k^2}(N_{\pi_k^2} f_{\pi_k}) y_\iota^{((k))}$$

is (trivially) valid, while

$$V_{\pi_k^2}(N_{\pi_k^2} f_{\pi_k}) y_\iota^{((k))}$$
$$\therefore \quad N_{\pi_k^2}(V_{\pi_k^2} f_{\pi_k}) y_\iota^{((k))}$$

is (equally trivially) invalid.

These considerations lead us to lay down the temporary definitions

$$CCom_0 =_{df} (f_{\pi_k^2}(g_{\pi_k^2} h_{\pi_k})y_\iota^{((k))} \supset g_{\pi_k^2}(f_{\pi_k^2} h_{\pi_k})y_\iota^{((k))})$$
$$CCom_0^m =_{df} (f_{\pi_k^2\iota^m} x_\iota^{((m))}(g_{\pi_k^2} h_{\pi_k})y_\iota^{((k))} \supset g_{\pi_k^2}(f_{\pi_k^2\iota^m} x_\iota^{((m))} h_{\pi_k})y_\iota^{((k))})$$
$$CCom_0^{m,j} =_{df} (f_{\pi_k^2\iota^m} x_\iota^{((m))}(g_{\pi_k^2\iota^j} z_\iota^{((j))} h_{\pi_k})y_\iota^{((k))} \supset g_{\pi_k^2\iota^j} z_\iota^{((j))}(f_{\pi_k^2\iota^m} x_\iota^{((m))} h_{\pi_k})y_\iota^{((k))}) \quad ,$$

and similar definitions for $ECom_0$, $ECom_0^m$, $ECom_0^{m,j}$ in which the main operator $\supset$ is replaced by $\equiv$. Then various strong forms of commutativity for modifiers can be defined, along the lines of

$$EComm_{0(\pi_k^2\iota^m)(\pi_k^2\iota^j)} =_{df} \lambda g_{\pi_k^2\iota^j} \lambda f_{\pi_k^2\iota^m}(z_\iota^{((j))})(x_\iota^{((m))})(y_\iota^{((k))})(h_{\pi_k}) ECom_0^{m,j} \quad .$$

These strong forms can then be weakened by analogy with the way in which the strong forms of inclusiveness were weakened in §1.4 and §1.5. Similar, but more ramified, lattices will result, and the interstitial algebras will be like the direct product of two of our previous interstitial algebras. No special problems are known concerning the details of these structures, and we omit any further discussion of them.

We will say that a modifier is universally commutative if it commutes (in some sense) with all modifiers: three strong kinds of universal commutativity for 0-ary modifiers are

$$Univ\,L\,CComm_{0(\pi_k^2)} =_{df} \lambda f_{\pi_k^2}(g_{\pi_k^2}) CComm_{0(\pi_k^2)^2}\, g_{\pi_k^2} f_{\pi_k^2}$$

$$Univ\,R\,CComm_{0(\pi_k^2)} =_{df} \lambda g_{\pi_k^2}(f_{\pi_k^2}) CComm_{0(\pi_k^2)^2}\, g_{\pi_k^2} f_{\pi_k^2}$$

and

$$Univ\,EComm_{0(\pi_k^2)} =_{df} \lambda f_{\pi_k^2}(g_{\pi_k^2}) EComm_{0(\pi_k^2)^2}\, g_{\pi_k^2} f_{\pi_k^2} \quad .$$

The falsum modifier $F_{\pi_k^2}$ $(= \lambda f_{\pi_k}(N_{\pi_k^2}(V_{\pi_k^2} f_{\pi_k})))$ is universally left C Commutative, the verum modifier is universally right C Commutative, and the identity modifier is universally E Commutative.

1.7 Predicates associated with modifiers

Let us begin another Montague-style argument by comparing

(I) This is a grey mouse
$\therefore$ This is grey and this is a mouse

with

(J) This is a big mouse
$\therefore$ This is big and this is a mouse :

we see that (I) is not logically valid, since it has the same logical form as (J), and (J) is not valid. So, as with arguments (A) and (C), we need to find a property of the modifier "grey" that is not enjoyed by the modifier "big", even though both are inclusive modifiers (at least when applied to the predicate "mouse").

In [45], we called the property in question detachability, while Montague calls it the intersection-property. Here we begin with a property weaker than these: we firstly write

$$Sep_o =_{df} (f_{\pi_k^2}\, g_{\pi_k}\, y_\iota^{((k))} \supset .\; g_{\pi_k}\, y_\iota^{((k))} \;\&\; h_{\pi_k}\, y_\iota^{((k))})$$

$$Qsep_o =_{df} (y_\iota^{((k))})\, Sep_o$$

$$Sep_o^j =_{df} (f_{\pi_k^2 \iota^j}\, x_\iota^{((j))}\, g_{\pi_k}\, y_\iota^{((k))} \supset .\; g_{\pi_k}\, y_\iota^{((k))} \;\&\; h_{\pi_{k+j}}\, y_\iota^{((k))}\, x_\iota^{((j))})$$

$$Qsep_o^j =_{df} (y_\iota^{((k))})\, Sep_o^j$$

as temporary (and formally evil) definitions, and then get down to business with

$$Separate_{o\pi_k^2} =_{df} \lambda f_{\pi_k^2} (\exists h_{\pi_k})(g_{\pi_k})\, Qsep_o$$

and

$$Separate_{o(\pi_k^2 \iota^j)} =_{df} \lambda f_{\pi_k^2 \iota^j} (\exists h_{\pi_{k+j}})(x_\iota^{((j))})(g_{\pi_k})\, Qsep_o^j$$

as definitions of <u>separability</u> for 0-ary and j-ary modifiers. As with inclusiveness and commutativity, these strong properties of modifiers may well be too strong when we wish to assert them of particular modifiers. For instance, "grey" is not separable (at least not intuitively so) in the usage "the existing legislation leaves a grey area here", and grey hair and grey horses are not always grey. Most colour words seem to have such extended or idiomatic uses in which they are not separable - we have blue movies and blue laws, purple patches, red herrings and red hot mommas, white papers, black bans, green soldiers, pink politicians, the yellow press, and so on.

Thus we are led to weaker separability properties such as

$$Separate_{o(\pi_k{}^2)\pi_k} =_{df} \lambda g_{\pi_k} \lambda f_{\pi_k{}^2} (\exists h_{\pi_k}) Qsep_0$$

and

$$Separate_{o(\pi_k{}^2)(o\pi_k)} =_{df} \lambda m_{o\pi_k} \lambda f_{\pi_k{}^2} (\exists h_{\pi_k})(g_{\pi_k})(m_{o\pi_k} g_{\pi_k} \supset Qsep_0) .$$

The formal pattern of the definitions of these separability properties follows that for the inclusiveness properties: we will get three similar lattices for separability of j-ary modifiers, cut-down versions of these for 0-ary modifiers, and "interstitial" atomless Boolean algebras. At each corresponding point in the respective lattices, the separability property is strictly stronger than the inclusiveness property (except at the least elements of the interstitial algebras, where the two properties are trivially equivalent).

Each modifier has a <u>separator</u>, as per the definitions

$$Separator_{\pi_k(\pi_k{}^2)} =_{df} \lambda f_{\pi_k{}^2} (\iota h_{\pi_k})(g_{\pi_k}) Qsep_0$$

$$Separator_{\pi_{k+j}(\pi_k{}^2\iota^j)} =_{df} \lambda f_{\pi_k{}^2\iota^j} (\iota h_{\pi_{k+j}})(x_\iota^{((j))})(g_{\pi_k}) Qsep_0^j$$

$$Separator_{\pi_k(\pi_k{}^2)(o\pi_k)} =_{df} \lambda m_{o\pi_k} \lambda f_{\pi_k{}^2} (\iota h_{\pi_k})(g_{\pi_k})(m_{o\pi_k} g_{\pi_k} \supset Qsep_0)$$

and so on. If a modifier is not separable, then (in any model) its separator is an arbitrary (but fixed) predicate of the appropriate type: if a modifier is separable then its separator is any h_{π_k} or $h_{\pi_{k+j}}$ satisfying the existential quantification in the definition of separability. (In general there will be a family of such properties, closed under the formation of supersets.) Formulae such as

$$\mathit{Separate}_{o(\pi_k^2)} f_{\pi_k^2} \;\&\; f_{\pi_k^2} g_{\pi_k} y_\iota^{((k))} \supset.$$
$$\mathit{Separator}_{\pi_k(\pi_k^2)} f_{\pi_k^2} y_\iota^{((k))} \;\&\; g_{\pi_k} y_\iota^{((k))}$$

are valid, and provide weak type-theoretic versions of the "only if" half of the Quinean "reduction" $(FG)x \equiv . Fx \,\&\, Gx$. (Of course Quine's formula is ill-formed by our criteria, since the first F and the second F should be different.)

Now to distinguish between our sample arguments (I) and (J), each of which are instances of the schema

$$f_{\pi_1^2} g_{\pi_1} x_\iota$$
$$\therefore \quad g_{\pi_1} x_\iota \;\&\; h_{\pi_1} x_\iota$$

we see that (I) can be validated by the schema

$$f_{\pi_1^2} g_{\pi_1} x_\iota$$
$$m_{o\pi_1} g_{\pi_1} \;\&\; \mathit{Separate}_{o(\pi_1^2)(o\pi_1)} m_{o\pi_1} f_{\pi_1^2}$$
$$\therefore \quad g_{\pi_1} x_\iota \;\&\; \mathit{Separator}_{\pi_1(\pi_1^2)(o\pi_1)} m_{o\pi_1} f_{\pi_1^2} x_\iota \quad ,$$

where the additional premise requires us to assert that "grey" is separable for a (large) class of predicates which includes "mouse" (but excludes "hair" and "horse") and where the "grey" in the conclusion of (I) is taken as any appropriate separator for the modifier "grey". The argument (J) cannot be validated by the same schema, since the extra premise would require us to assert that "big" is separable for the same class of predicates for which "grey" is separable, and this is false. However, we can validate (J) by the schema

$$f_{\pi_1{}^2}\, g_{\pi_1}\, x_\iota$$

$$Separate_{o(\pi_1{}^2)\pi_1}\, g_{\pi_1}\, f_{\pi_1{}^2}$$

$$\therefore \quad g_{\pi_1} x_\iota \;\&\; Separator_{\pi_1(\pi_1{}^2)\pi_1}\, g_{\pi_1}\, f_{\pi_1{}^2}\, x_\iota$$

where now the "big" in the conclusion is any separator for the modifier "big" and the predicate "mouse": in English, such a separator is the predicate "big for a mouse" or "big as mice go".

Depending on one's attitude to the mess that is the English language, one will react with pleasure or horror to the following fact, made fairly explicit in the preceding paragraph: so far as the type-theoretic formalism is concerned, nothing can be both a modifier and a predicate. The words "grey" in (I) and "big" in (J) require different symbols in the respective premises and conclusion: in the general schema for the two arguments they are symbolized by $f_{\pi_1{}^2}$ in the premise and h_{π_1} in the conclusion . In the conclusion of the schema that validates (I), h_{π_1} becomes $Separator_{\pi_1(\pi_1{}^2)(o\pi_1)}\, m_{o\pi_1}\, f_{\pi_1{}^2}$, and this is still different from $f_{\pi_1{}^2}$ even though it bears a certain relation to it. And when we say that (J) can be validated, we trade on the fact that since the two "big"'s are in any event required to be different, we can take the second "big" as an appropriate separator, such as "big for a mouse", whence the argument becomes intuitively valid.

1.8 Modifiers associated with predicates

As a kind of converse to separable modifiers, we have affixable predicates. To illustrate, consider the arguments

(K) This is red and this is a ball
$\therefore$ This is a red ball

and

(L) This is big and this is an elephant
$\therefore$ This is a big elephant

Initially at least, (L) would not be counted as valid, and so the general schema common to (K) and (L) is not valid. But we do want to count (K) as valid, and so we want to find an additional premise whereby it can be validated. We define temporarily

$$Con_o =_{df} (g_{\pi_k} y_\iota^{((k))} \& h_{\pi_k} y_\iota^{((k))} \supset f_{\pi_k^2} h_{\pi_k} y_\iota^{((k))})$$

and

$$Qcon_o =_{df} (y_\iota^{((k))}) Con_o ,$$

and then define affixability by

$$Affix_{o\pi_k} =_{df} \lambda g_{\pi_k} (\exists f_{\pi_k^2})(h_{\pi_k}) Qcon_o$$

or

$$Affix_{o\pi_k(o\pi_k)} =_{df} \lambda m_{o\pi_k} \lambda g_{\pi_k} (\exists f_{\pi_k^2})(h_{\pi_k})(m_{o\pi_k} h_{\pi_k} \supset Qcon_o)$$

and so on. Corresponding to separators we have <u>affixers</u>, of which some sample definitions are

$$Affixer_{\pi_k^2 \pi_k} =_{df} \lambda g_{\pi_k} (\iota f_{\pi_k^2})(h_{\pi_k}) Qcon_o$$

and

$$Affixer_{\pi_k^2 \pi_k (o\pi_k)} =_{df} \lambda m_{o\pi_k} \lambda g_{\pi_k} (\iota f_{\pi_k^2})(h_{\pi_k})(m_{o\pi_k} h_{\pi_k} \supset Qcon_o) .$$

Again the choice-function aspect of ι in type theory is being exploited: like separators, affixers are not in general unique, even for affixable predicates. We do not need to consider j-ary modifiers as possible affixers: if, say $(\exists x_\iota)(\exists f_{\pi_k^2 \iota}) A_o$ holds, then so does $(\exists f_{\pi_k^2}) A_o$ since we can put $f_{\pi_k^2} = (\iota f_{\pi_k^2 \iota})(\exists x_\iota) A_o (\iota x_\iota)(\exists f_{\pi_k^2 \iota}) A_o$.

Using the notions of affixability and affixers, the following schemata are valid:

$$g_{\pi_k} y_\iota^{((k))} \ \& \ h_{\pi_k} y_\iota^{((k))}$$
$$Affix_{o\pi_k} g_{\pi_k}$$
$$\therefore \quad Affixer_{\pi_k^2 \pi_k} g_{\pi_k} h_{\pi_k} y_\iota^{((k))}$$

and

$$g_{\pi_k} y_\iota^{((k))} \ \& \ h_{\pi_k} y_\iota^{((k))}$$
$$m_{o\pi_k} h_{\pi_k} \ \& \ Affix_{o\pi_k (o\pi_k)} m_{o\pi_k} g_{\pi_k}$$
$$\therefore \quad Affixer_{\pi_k^2 \pi_k (o\pi_k)} m_{o\pi_k} g_{\pi_k} h_{\pi_k} y_\iota^{((k))} \quad ,$$

and the argument (K) can be validated by the first of these if we can assert that "red" is affixable <u>tout court</u>, and by the second schema if we can assert that "red" is affixable to each predicate in some class of which "ball" is a member.

In many cases, very weak affixability properties will have to be used. In the arguments

(M) Gunsynd was a champion and a miler
$\therefore$ Gunsynd was a champion miler

and

(N) Gunsynd was a champion and a stallion
$\therefore$ Gunsynd was a champion stallion

we have to know specific facts about the horse Gunsynd, not just about champions, milers and stallions, in order to know that the conclusion of (M) is true while that of (N) is false (simply because Gunsynd has not yet (1973) had racing progeny). Thus we can use an affixability property like

$$Affix^{m}_{o\pi_k (o\pi_k)\iota} =_{df} \lambda y_\iota^{(m)} \lambda m_{o\pi_k} \lambda g_{\pi_k} (\exists f_{\pi_k^2})(h_{\pi_k})(y_\iota^{((m-1))})$$
$$(y_\iota^{(m+1)}) \ldots (y_\iota^{(k)}) \, Con_o$$

and a corresponding affixer operator in the schema

$$g_{\pi_1} x_\iota \;\&\; h_{\pi_1} x_\iota$$
$$m_{o\pi_1} h_{\pi_1} \;\&\; Affix^{1}_{o\pi_1 (o\pi_1)\iota}\, x_\iota\, m_{o\pi_1}\, g_{\pi_1}$$
$$\therefore \quad Affixer^{1}_{\pi_1{}^2\pi_1 (o\pi_1)\iota}\, x_\iota\, m_{o\pi_1}\, g_{\pi_1}\, h_{\pi_1}\, x_\iota$$

to validate (M), where the extra premise requires the affixability of "champion" for Gunsynd and a smallish class of predicates ({miler, racehorse, grey horse, middle distance horse,...}) to which "miler" belongs, but "stallion" does not. This extra premise uses an affixability property well down the lattice of such properties for type π_1 , but no substantially stronger premise seems warranted in this case.

1.9 Separability and affixability combined

As well as being intuitively valid themselves, arguments (I) and (K) also have valid converses: the fact that the converses of (J) and (L) are also valid shows that there is no entailment between the validity of (I) and (K) and the validity of their converses. Thus we can investigate, as a separate question, the conditions for validity of the whole schema

$$\left\{ \left(\begin{array}{l} f_{\pi_k{}^2}\, g_{\pi_k}\, y_\iota^{((k))} \\ \therefore\; g_{\pi_k}\, y_\iota^{((k))} \;\&\; h_{\pi_k}\, y_\iota^{((k))} \end{array} \right) \text{ and } \left(\begin{array}{l} g_{\pi_k}\, y_\iota^{((k))} \;\&\; h_{\pi_k}\, y_\iota^{((k))} \\ \therefore\; f_{\pi_k{}^2}\, g_{\pi_k}\, y_\iota^{((k))} \end{array} \right) \right\}$$

and it generalization to j-ary modifiers. As temporary definitions, Equ_o, $Qequ_o$, Equ_o^j and $Qequ_o^j$ are the same as $Sep_o, \ldots, Qsep_o^j$ except that the main operator of the definiens is in each case $\equiv$ rather than $\supset$. Then we define detachability properties such as

$$Detach_{o(\pi_k^2)} =_{df} \lambda f_{\pi_k^2} (\exists h_{\pi_k})(g_{\pi_k}) Qequ_o$$

$$Detach_{o(\pi_k^2(j))} =_{df} \lambda f_{\pi_k^2(j)} (\exists h_{\pi_{k+j}})(x_\iota^{((j))})(g_{\pi_k}) Qequ_o^j$$

$$Detach_{o(\pi_k^2)\pi_k} =_{df} \lambda g_{\pi_k} \lambda f_{\pi_k^2} (\exists h_{\pi_k}) Qequ_o$$

and

$$Detach_{o(\pi_k^2)(o\pi_k)} =_{df} \lambda m_{o\pi_k} \lambda f_{\pi_k^2} (\exists h_{\pi_k})(g_{\pi_k})(m_{o\pi_k} g_{\pi_k} \supset Qequ_o) \quad ,$$

and it is obvious that each of these is strictly stronger than the corresponding separability property. We gain definitions of various kinds of detachers by replacing $(\exists h_{\pi_k})$ (or $(\exists h_{\pi_{k+j}})$) by $(\imath h_{\pi_k})$ (or $(\imath h_{\pi_{k+j}})$) in the definitions of the detachability properties: for non-detachable modifiers, the detachers are arbitrary predicates as for non-separable modifiers, but for detachable modifiers the $\imath$ now functions as a definite description operator, since a detachable modifier has a unique detacher. The detacher of a detachable modifier is the smallest predicate among its separators.

In the other direction, we define <u>attachability</u> of predicates by definitions like

$$Attach_{o\pi_k} =_{df} \lambda h_{\pi_k} (\exists f_{\pi_k^2})(g_{\pi_k}) Qequ_o$$

and

$$Attach_{o\pi_k(o\pi_k)} =_{df} \lambda m_{o\pi_k} \lambda h_{\pi_k} (\exists f_{\pi_k^2})(g_{\pi_k})(m_{o\pi_k} g_{\pi_k} \supset Qequ_o)$$

and so on. The question arises as to whether <u>every</u> predicate h_{π_k} is attachable (and hence also affixable) since it appears that for any h_{π_k} we can always find a suitable $f_{\pi_k^2}$ simply by defining

$$f_{\pi_k^2} = \lambda g_{\pi_k} \lambda y_\iota^{((k))} (g_{\pi_k} y_\iota^{((k))} \;\&\; h_{\pi_k} y_\iota^{((k))}) \quad .$$

The answer to this question depends upon the model theory of our type theory: if we are considering only the intended models, then indeed all predicates are attachable for the reasons given; but if we are considering also general models (in Henkin's terminology) then it is possible that the function corresponding to $f_{\pi_k^2}$, as defined, does not belong to the domain $D_{\pi_k^2}$. Hence $(h_{\pi_k})(\exists f_{\pi_k^2})(g_{\pi_k})Qequ_o$ is valid in the standard sense but may not be valid in the general sense (we have not actually produced a general countermodel to it). We want to leave open the question of whether it is appropriate to include general models in the semantics of modifier theory: we would not do so merely to achieve axiomatizability, but sad facts like the denumerability of the set of English expressions might provide pressure to do so. We continue, therefore, to treat attachability and affixability as genuine properties of modifiers.

We define <u>attachers</u> in the familiar way:

$$Attacher_{\pi_k^2\pi_k} =_{df} \lambda h_{\pi_k}(\imath f_{\pi_k^2})(g_{\pi_k})Qequ_o$$

$$Attacher_{\pi_k^2\pi_k(o\pi_k)} =_{df} \lambda m_{o\pi_k}\lambda h_{\pi_k}(\imath f_{\pi_k^2})(g_{\pi_k})(m_{o\pi_k}g_{\pi_k} \supset Qequ_o)$$

et cetera. For attachable predicates, the attacher is a unique modifier, given in effect by the definition of $f_{\pi_k^2}$ in the preceding paragraph. Attachers and detachers are related simply by the following valid formulae:

$$Attach_{o\pi_k}g_{\pi_k} \supset Detacher_{\pi_k(\pi_k^2)}(Attacher_{\pi_k^2\pi_k}g_{\pi_k}) = g_{\pi_k}$$

(and hence

$$Attach_{o\pi_k}g_{\pi_k} \supset Detach_{o(\pi_k^2)}(Attacher_{\pi_k^2\pi_k}g_{\pi_k}) \qquad)$$

and

$$Detach_{o(\pi_k^2)}f_{\pi_k^2} \supset Attacher_{\pi_k^2\pi_k}(Detacher_{\pi_k(\pi_k^2)}f_{\pi_k^2}) = f_{\pi_k^2}$$

(and hence

$$Detach_{o(\pi_k^2)}f_{\pi_k^2} \supset Attach_{o\pi_k}(Detacher_{\pi_k(\pi_k^2)}f_{\pi_k^2}) \qquad).$$

The following schemata are valid:

$$f_{\pi_k^2}\, g_{\pi_k}\, y_\iota^{((k))}$$

$$Detach_{o(\pi_k^2)}\, f_{\pi_k^2}$$

$$\therefore \quad g_{\pi_k}\, y_\iota^{((k))} \;\&\; Detacher_{\pi_k(\pi_k^2)}\, f_{\pi_k^2}\, y_\iota^{((k))}$$

and

$$g_{\pi_k}\, y_\iota^{((k))} \;\&\; h_{\pi_k}\, y_\iota^{((k))}$$

$$Attach_{o\pi_k}\, h_{\pi_k}$$

$$\therefore \quad Attacher_{\pi_k^2\pi_k}\, h_{\pi_k}\, g_{\pi_k}\, y_\iota^{((k))} \quad ,$$

and hence by previous identities so are

$$Attacher_{\pi_k^2\pi_k}\, h_{\pi_k}\, g_{\pi_k}\, y_\iota^{((k))}$$

$$Attach_{o\pi_k}\, h_{\pi_k}$$

$$\therefore \quad g_{\pi_k}\, y_\iota^{((k))} \;\&\; h_{\pi_k}\, y_\iota^{((k))}$$

and

$$g_{\pi_k}\, y_\iota^{((k))} \;\&\; Detacher_{\pi_k(\pi_k^2)}\, f_{\pi_k^2}\, y_\iota^{((k))}$$

$$Detach_{o(\pi_k^2)}\, f_{\pi_k^2}$$

$$\therefore \quad f_{\pi_k^2}\, g_{\pi_k}\, y_\iota^{((k))} \quad .$$

Hence the schema with which we began the section can be validated entirely in terms of detachability and detachers if it can be expressed in the form

$$\left(\begin{array}{l} \qquad f_{\pi_k^2}\, g_{\pi_k}\, y_\iota^{((k))} \\ \therefore \quad g_{\pi_k}\, y_\iota^{((k))} \;\&\; Detacher_{\pi_k(\pi_k^2)}\, f_{\pi_k^2}\, y_\iota^{((k))} \end{array}\right) \quad \text{and} \quad \left(\text{conversely}\right)$$

where $Detach_{o(\pi_k^2)}\, f_{\pi_k^2}$, and in terms wholly of attachability and attachers if it can be expressed in the form

$$\left(\begin{array}{l} g_{\pi_k} y_\iota^{((k))} \,\&\, h_{\pi_k} y_\iota^{((k))} \\ \therefore\ \mathit{Attacher}_{\pi_k^2 \pi_k}\, h_{\pi_k}\, g_{\pi_k}\, y_\iota^{((k))} \end{array} \right) \quad \text{and} \quad \left(\text{conversely} \right)$$

where $\mathit{Attach}_{o\pi_k}\, h_{\pi_k}$.

1.10 Relations between modifiers and individuals

We begin by considering three arguments:

(O) This is a Shakespeare play
∴ This was written by Shakespeare and is a play

(P) This is a Robert Carrier recipe
∴ This was written by Robert Carrier and is a recipe

(Q) This is a Molotov cocktail
∴ This is named after Molotov and is a cocktail

Here (Q) shows that when individuals are used in "modification" position they do not always operate as inclusive modifiers, since Molotov cocktails although related to Molotov in some way, are not cocktails. A similar example is yielded by the pedestrian crossing arrangement called a "Barnes dance". The arguments (O) and (P) show that even when we have an inclusive kind of modification, the relation involved is not always the same: Robert Carrier recipes are collected, tested and published, but not necessarily written, by Robert Carrier; Diesel engines were invented by R. Diesel; Melba toast is named after Melba (because of her predilection for it); Menzies governments were led by Menzies; Herbrand-Gödel-Kleene recursive functions were first defined, less or more precisely, by Herbrand, Gödel and Kleene, and so on. And of course we do not always have the same relation connected with the same individual:

Menzies wit is not led, but rather produced, by Menzies; and the Menzies era was not exactly led, but rather dominated politically, by Menzies. Again, if more than one individual is involved then different relations may be connected with each of them: a Gilbert and Sullivan opera is one whose words were written by Gilbert and whose music was composed by Sullivan; and the Löwenheim-Skolem theorem is the result of a proof by Löwenheim significantly extended by Skolem.

Our various examples lead us to the following general schema:

$$f_{\pi_k 2 \iota j}\, x_\iota^{((j))}\, g_{\pi_k}\, y_\iota^{((k))}$$

$$\therefore\ h^{(1)}_{\pi_{k+1}\pi_k}\, g_{\pi_k}\, x_\iota^{(1)}\, y_\iota^{((k))} \;\&\ldots\&\; h^{(j)}_{\pi_{k+1}\pi_k}\, g_{\pi_k}\, x_\iota^{(j)}\, y_\iota^{((k))} \;\&\; g_{\pi_k}\, y_\iota^{((k))} \quad .$$

In this schema the individuals $x_\iota^{(1)}, \ldots, x_\iota^{(j)}$ are those appearing in "modifier" position: when j such individuals appear there is an implicit j-ary modifier operating, and this is made explicit in the type-theoretic notation of the schema by the modifier $f_{\pi_k 2 \iota j}$. In the conclusion the relation $h^{(n)}_{\pi_{k+1}\pi_k}$, for $1 \leqslant n \leqslant j$, is the relation connected with predicate g_{π_k} and individual $x_\iota^{(n)}$. For instance if g_{π_1} is the predicate "opera", $x_\iota^{(1)}$ is Gilbert and $x_\iota^{(2)}$ is Sullivan, then $h^{(1)}_{\pi_2\pi_1} g_{\pi_1}$ is the relation of writing the words of, and $h^{(2)}_{\pi_2\pi_1} g_{\pi_1}$ is the relation of composing the music of.

A more general situation would be to have a single relation $h_{\pi_{k+j}\pi_k}$ appearing in the schema

$$f_{\pi_k 2 \iota j}\, x_\iota^{((j))}\, g_{\pi_k}\, y_\iota^{((k))}$$

$$\therefore\ h_{\pi_{k+j}\pi_k}\, g_{\pi_k}\, x_\iota^{((j))}\, y_\iota^{((k))} \;\&\; g_{\pi_k}\, y_\iota^{((k))} \quad .$$

Our j different relations $h^{(n)}_{\pi_{k+1}\pi_k}$ could be extracted from $h_{\pi_{k+j}\pi_k}$ by the definitions

$$h^{(n)}_{\pi_{k+1}\pi_k} =_{df} \lambda g_{\pi_k} \lambda x_\iota^{(n)} \lambda y_\iota^{((k))} (\exists x_\iota^{((n-1))})(\exists x_\iota^{(n+1)}) \ldots (\exists x_\iota^{(j)}) h_{\pi_{k+j}\pi_k}\, g_{\pi_k}\, x_\iota^{((j))}\, y_\iota^{((k))}$$

if there were one separate and identifiable relation connected with each individual. The case where there are j separate relations is probably more common in English, but we will consider the more general case, where there is a single relation $h_{\pi_{k+j}\pi_k}$, in what follows.

To express the relation $h_{\pi_{k+j}\pi_k}$ as a function of $x_\iota^{(1)}, \cdots, x_\iota^{(j)}$ and g_{π_k}, we want firstly to express the $f_{\pi_k{}^2\iota^j}$ as such a function. This is done by introducing a primitive, undefined, function

$$Modind_{(\pi_k{}^2\iota^j)\iota^j\pi_k}$$

which associates a j-ary modifier with j individuals and a k-adic predicate: we then define

$$Associator_{(\pi_{k+j}\pi_k)\iota^j\pi_k} =_{df} \lambda g_{\pi_k} \lambda x_\iota^{((j))} Separator_{\pi_{k+j}(\pi_k{}^2\iota^j)} (Modind_{(\pi_k{}^2\iota^j)\iota^j\pi_k}\, g_{\pi_k}\, x_\iota^{((j))}) \quad .$$

We may also define $DAssociator_{(\pi_{k+j}\pi_k)\iota^j\pi_k}$, with Detacher in place of Separator in the definiens, if we are considering as well the converses of our schemata involving individuals in modifier position.

Given the definition of the Associator, the following is a valid schema:

$$Modind_{(\pi_k{}^2\iota^j)\iota^j\pi_k}\, g_{\pi_k}\, x_\iota^{((j))}\, x_\iota^{((j))}\, g_{\pi_k}\, y_\iota^{((k))}$$

$$Separate_{o(\pi_k{}^2\iota^j)} (Modind_{(\pi_k{}^2\iota^j)\iota^j\pi_k}\, g_{\pi_k}\, x_\iota^{((j))})$$

$$\therefore\ Associator_{(\pi_{k+j}\pi_k)\iota^j\pi_k}\, g_{\pi_k}\, x_\iota^{((j))}\, g_{\pi_k}\, x_\iota^{((j))}\, y_\iota^{((k))}\ \&\ g_{\pi_k}\, y_\iota^{((k))} \quad .$$

In this schema the primitive function *Modind* expresses the modifier $f_{\pi_k{}^2\iota^j}$ as a function of the individuals

$x_\iota^{(1)}, \ldots, x_\iota^{(j)}$ and the predicate g_{π_k} : then we add the premise that this modifier is separable, and then by the defined function Associator we express the relation $h_{\pi_{k+j}\pi_k}$ in the conclusion as a function of $x_\iota^{(1)}, \ldots, x_\iota^{(j)}$ and g_{π_k} .

Conversely, we might suppose that the Associator relation is primitive and undefined. This is more like what happens in practice, where we have the (empirical) data that the relation "... writes..." is associated with Shakespeare and plays, "... invents..." with Diesel and engines, etc. In this case the *Modind* modifier may be defined as the Affixer, or the Attacher, of the Associator of the individuals and predicate: we omit the details of this construction.

1.11 Function modifiers

Again we begin with some sample arguments:

(A_1) Tom is Joe's natural father
$\therefore$ Tom is Joe's father

and

(B_1) Tom is Joe's step-father
$\therefore$ Tom is Joe's father

Here we are supposing that "father" is being used as a function (from individuals to individuals) and that "natural" and "step-" are being used as function modifiers. Thus formally the schema common to (A_1) and (B_1) is:

$$x_\iota = f_{(\iota^2)^2}\, g_{\iota^2}\, y_\iota$$

$$\therefore \quad x_\iota = g_{\iota^2}\, y_\iota$$

where g_{ι^2} is a function and $f_{(\iota^2)^2}$ is a function modifier. (It is amusing to note that the "lowest" type of numbers in Church's [8] are of type ι', which is $(\iota^2)^2$ in our notation: thus the "simplest" of Church's numbers are function modifiers!)

We can define properties of function modifiers quite analogous to some of those for predicate modifiers - as argument (B_1) shows, there is some point in doing so, at least for inclusiveness, since not all function modifiers are inclusive. Thus we may define

$$Incl_{o((\iota^2)^2)} =_{df} \lambda f_{(\iota^2)^2} (g_{\iota^2})(y_\iota)(f_{(\iota^2)^2}\, g_{\iota^2}\, y_\iota = g_{\iota^2}\, y_\iota)$$

$$Incl_{o((\iota^2)^2)(o(\iota^2))} =_{df} \lambda n_{o(\iota^2)} \lambda f_{(\iota^2)^2} (g_{\iota^2})(y_\iota)(n_{o(\iota^2)}\, g_{\iota^2} \supset . f_{(\iota^2)^2}\, g_{\iota^2}\, y_\iota = g_{\iota^2}\, y_\iota)$$

and so on. The analogue to detachability is gained by replacing intersection of properties by composition of functions, and we call such function modifiers <u>composable</u>. A sample definition is

$$Compose_{o((\iota^2)^2)} =_{df} \lambda f_{(\iota^2)^2} (\exists h_{\iota^2})(g_{\iota^2})(f_{(\iota^2)^2}\, g_{\iota^2}\, y_\iota = h_{\iota^2}(g_{\iota^2}\, y_\iota)) .$$

The analogy between composable function modifiers and detachable predicate modifiers is not very strong, since e.g. not all composable function modifiers are inclusive.

In the well-known way, we can translate functions into their associated relations, using the function defined in

$$Relfun_{\pi_2(\iota^2)} =_{df} \lambda g_{\iota^2} (\imath h_{\pi_2})(x_\iota)(y_\iota)(h_{\pi_2}\, x_\iota\, y_\iota \equiv . g_{\iota^2}\, x_\iota = y_\iota)$$

and its generalization to k-adic functions ($(k+1)$-adic relations).

$$Relfun_{\pi_{k+1}(\iota^{k+1})} =_{df} \lambda g_{\iota^{k+1}} (\imath h_{\pi_{k+1}})(x_\iota^{((k+1))})(h_{\pi_{k+1}}\, x_\iota^{((k+1))} \equiv . g_{\iota^{k+1}}\, x_\iota^{((k))} = x_\iota^{(k+1)}) .$$

To translate function modifiers into predicate modifiers the natural question to ask is whether the diagram

$$\begin{tikzcd}
\iota^k \arrow[r, "Relfun_{\pi_k(\iota^k)}"] \arrow[d, "f_{(\iota^k)^2}"'] & \pi_k \arrow[d, dashed, "g_{\pi_k^2}"] \\
\iota^k \arrow[r, "Relfun_{\pi_k(\iota^k)}"] & \pi_k
\end{tikzcd}$$

is uniquely solvable (for $g_{\pi_k^2}$). (More precisely, we are asking this question of the semantical interpretation of the diagram, with sets D_{ι^k} and D_{π_k} rather than type symbols ι^k and π_k, and functions $V_\varphi(f_{(\iota^k)^2})$ etc.) In general the diagram will not be uniquely solvable, since the function $g_{\pi_k^2}$ has to be defined on all of the domain of type π_k, but the range of $Relfun_{\pi_k(\iota^k)}$ is in general a proper subset of this domain (i.e. $Relfun$ is not a surjection). Hence the values of $g_{\pi_k^2}$ can be arbitrary for arguments not in the range of $Relfun_{\pi_k(\iota^k)}$, yet the diagram will still commute. However $g_{\pi_k^2}$ is uniquely determined "wherever it matters", and we can use

$$Predmfunm_{(\pi_k^2)((\iota^k)^2)} =_{df} \lambda f_{(\iota^k)^2} (\imath g_{\pi_k^2})(h_{\iota^k})(g_{\pi_k^2}(Relfun_{\pi_k(\iota^k)} h_{\iota^k}) = Relfun_{\pi_k(\iota^k)}(f_{(\iota^k)^2} h_{\iota^k}))$$

as the definition of a function associating a predicate modifier with a function modifier. The next natural question to ask is whether inclusiveness of $Predmfunm_{(\pi_k^2)((\iota^k)^2)} f_{(\iota^k)^2}$ transfers to inclusiveness of $f_{(\iota^k)^2}$, that is, whether the following wff is valid:

$$Incl_{o(\pi_k^2)}(Predmfunm_{(\pi_k^2)((\iota^k)^2)} f_{(\iota^k)^2}) \supset Incl_{o((\iota^k)^2)} f_{(\iota^k)^2}$$

(clearly the converse implication fails, for the arbitrary values of $Predmfunm_{(\pi_k^2)((\iota^k)^2)} f_{(\iota^k)^2}$ at least). Indeed this wff is valid, and in consequence in some cases we can deal with function modifiers and their inclusiveness properties via their associated predicate modifiers and their inclusiveness properties.

In the case of inclusiveness, we have given the same general label to properties of both predicate modifiers and function modifiers: we did this because of the fairly clear analogy between the properties, the analogy which was put into formal shape in the preceding paragraph. A more general question concerns the corresponding properties (if any) of function modifiers for properties like commutativity, separability and detachability of predicate modifiers, and conversely the corresponding property (if any) of predicate modifiers for composability of function modifiers. We do not investigate this general question, which we suspect may best be phrased and dealt with in combinatory logic.

1.12 Iterated modifiers and nonsignificance

If we compare

(C_1) I will do it now now
$\therefore$ I will do it now

with

(D_1) I will do it in a day's time in a day's time
$\therefore$ I will do it in a day's time

and

(E_1) Jones killed Smith with a knife with a knife
$\therefore$ Jones killed Smith with a knife

we see that some modifiers, like "now" in (C_1), are <u>collapsible</u>, in the sense that further modification past the first is redundant; some, as in (D_1), are not collapsible since further modifications make a definite difference to what is said; and that in cases like the premise of (E_1) it appears that repeated modification has led to nonsignificance. In [12] Clark proposes, in effect, that all modifiers (or at least all 0-ary modifiers) should be treated as collapsible: temporal modifiers as in (D_1) provide immediate refutations of this proposal, as do examples like

(F_1) The monkey stood on a box on a box
to get his bananas,

for which the monkey evokes a different conditioned response from an observer than if he had merely stood on a box, and

(G_1) Jones killed Smith in a dream in a dream,

in which the deed is another level of reality removed from his merely killing him in a dream. In fact it seems that rather few modifiers are collapsible: apart from those like "now", the identity, verum and falsum modifiers, and some constructed from closure operators in mathematics, we are hard put to it to find examples. There are rhetorical flights, such as

(H_1) Greg Chappell drove wonderfully, wonderfully

or

(I_1) Nureyev danced beautifully, beautifully

where we may as well treat the modifier as collapsible, but otherwise there is a decided shortage of naturally-occurring collapsible modifiers.

As formal definitions, we define the iteration operator on 0-ary modifiers by

$$Iterate_{(\pi_k{}^2)^2} =_{df} \lambda f_{\pi_k^2} \lambda h_{\pi_k} \lambda y_\iota^{((k))} (f_{\pi_k^2} (f_{\pi_k^2} h_{\pi_k}) y_\iota^{((k))}) \quad ,$$

thence a modifier is collapsible if it is identical to its first iteration, as per

$$Collapse_{o(\pi_k{}^2)} =_{df} \lambda f_{\pi_k^2} (Iterate_{(\pi_k{}^2)^2} f_{\pi_k^2} = f_{\pi_k^2}) \quad .$$

We could of course separate the identity in the definiens into two separate implications, and gain two notions each weaker than full-scale collapsibility; examples from classical tense logics can be used to show that this may sometimes be a fruitful procedure. Taking our cue from the notion of a multi-stage topology (see e.g. Nöbeling

[51]), or for that matter from the modal systems $S4^n$ (with semantics as in [56]), we can also define weaker collapsibility properties by defining a generalized iteration operator with the recursion

$$^{0}Iterate_{(\pi_k{}^2)^2} = \lambda f_{\pi_k{}^2} f_{\pi_k{}^2}$$

$$^{n+1}Iterate_{(\pi_k{}^2)^2} = \lambda f_{\pi_k{}^2} \lambda h_{\pi_k} \lambda y_\iota^{((k))} (f_{\pi_k{}^2} ({}^{n}Iterate_{(\pi_k{}^2)^2} f_{\pi_k{}^2}) h_{\pi_k} y_\iota^{((k))})$$

and thence the definition

$$^{n}Collapse_{o(\pi_k{}^2)} =_{df} \lambda f_{\pi_k{}^2} ({}^{n+1}Iterate_{(\pi_k{}^2)^2} f_{\pi_k{}^2} = {}^{n}Iterate_{(\pi_k{}^2)^2} f_{\pi_k{}^2}) \quad .$$

Such a property might arise in the analysis of a modifier such as "intentionally", which even though not collapsible <u>tout court</u>, may be argued to be ^{k}collapsible for some smallish $k > 1$.

For modifiers other than 0-ary, we have strong collapsibility properties as defined by

$$Collapse_{o(\pi_k{}^2\iota j)} =_{df} \lambda f_{\pi_k{}^2\iota j} (x_\iota^{((j))}) (Iterate_{(\pi_k{}^2)^2} (f_{\pi_k{}^2\iota j} x_\iota^{((j))}) = f_{\pi_k{}^2\iota j} x_\iota^{((j))}) \quad ,$$

and weaker properties where collapsibility is not asserted for all individuals in the modifier but only for a particular individual or for individuals with a particular property. The pattern of all this is familiar from our treatment of the various lattices of properties already considered.

The fact that, as in (E_1), iterated modification can lead to nonsignificance is just a particular case of the fact that modification in general can lead to non-significance. Simple examples such as

(J_1) Jack slept quickly

or

(K_1) Mary sat on a prime number

(with "on" treated as a 1-ary modifier) are sufficient to establish this point. Indeed this point was one of

Ryle's main philosophical concerns; in [74] p.15 Ryle remarks on the question "Why cannot a traveller reach London gradually?" and on its answer-sketch that "gradually" won't "go with" verbs like "reach". Naturally Goddard and Routley in [24] have not failed to notice this point, even though they do not give it detailed treatment.

The converse point, namely that modification may prevent nonsignificance, does not seem to have had as much currency. An example to illustrate this point is that

(L_1) The forecast is for weather

seems to be nonsignificant, whereas

(M_1) The forecast is for fine weather

is certainly significant, the difference being the presence of the modifier "fine".

In order to deal properly with these points we need to combine modifier theory with significance theory. The method proposed by Lewis with a certain amount of distaste in [44], whereby nonsignificant results of modification are supposed to have a "null intension", is not adequate since it appears that it would make all furious sleeps into fat numbers, and also because it appears that it could not handle the "converse" point of our last paragraph. We need to take nonsignificance seriously as a propositional value, in the spirit of Goddard and Routley, and the basic syntactical mechanism of Church's formulation of type theory is not inconsistent with our doing so, since the syntax nowhere requires that the type o shall have a domain $\mathcal{D}_o$ consisting of just the two classical truth-values. Thus our type-theoretic framework, with semantics other than those given by Henkin, will not necessarily be too limited for a project of combining modifier theory with significance theory, but the project is beyond the scope of this monograph.

1.13 Modifiers and propositional operators

Traditional grammar, apparently, proposed that what we would call non-inclusive modifiers should be treated as propositional operators, along the lines of

(N_1) He is a possible saint

which is to be treated as

(O_1) It is possible that he is a saint,

according to the theory. Prima facie, examples like

(P_1) He is nearly finished,

which cannot be treated as

(Q_1) It is nearly the case that he is finished,

show that the theory in question is far too narrow and does not give a viable account of modifiers in general or non-inclusive modifiers in particular. But the basic lesson of our attempts to formalize English within type theory is that we must use different symbols whenever a word has a different syntactical role, and so there is no reason why the "nearly" in (P_1) should be retained when we form a propositional operator, and there is no reason why the nonsensical (Q_1) should result: it may well be that some other propositional operator could be used to match the modifier "nearly" as it appears in (P_1).

Following the pattern of many of our earlier definitions, let us try out the definition

$$\mathit{Propmod}_{(oo)(\pi_k^2)} =_{df} \lambda f_{\pi_k^2} (\lambda h_{oo})(g_{\pi_k})(y_\iota^{((k))})(f_{\pi_k^2}\, g_{\pi_k}\, y_\iota^{((k))} \equiv h_{oo}(g_{\pi_k}\, y_\iota^{((k))}))\,.$$

If this definition worked, then $f_{\pi_k^2}\, g_{\pi_k}\, y_\iota^{((k))}$ and $\mathit{Propmod}_{(oo)(\pi_k^2)} f_{\pi_k^2} (g_{\pi_k}\, y_\iota^{((k))})$ would be logically equivalent, and so every predicate modifier could in effect be put into propositional operator position, provided the right *Propmod* were chosen in each case. However the subjunctive mood in the last sentence is chosen deliberately: the situation is counterfactual and the definition does not work as intended since the condition on h_{oo} will in most cases not be satisfied.

For example, if $D_\iota = \{a, b\} = V_\varphi\, g_{\pi_1}$ and $V_\varphi\, f_{\pi_1^2}$ maps $\{a, b\}$ to $\{a\}$, then the condition is not satisfied since, in this model, h_{oo} is required both to map T to T and to map T to F as the quantifier $(y_\iota^{(1)})$ ranges over $\{a, b\}$. This problem will not be fixed merely by intensionalizing the whole set-up (in the way to be suggested in §2, or any other way that I know of), since it will then just arise again in a slightly more complicated form; nor will it be fixed by restricting our attention to non-inclusive modifiers, since we can easily give an example of the problem using such a modifier.

The problem can be fixed, in a rather trivial fashion, by laying down

$$Propmod_{(oo)(\pi_k^2)\pi_k \iota^k} =_{df} \lambda y_\iota^{((k))} \lambda g_{\pi_k} \lambda f_{\pi_k^2} (\imath h_{oo})(f_{\pi_k^2}\, g_{\pi_k}\, y_\iota^{((k))} \equiv h_{oo}(g_{\pi_k}\, y_\iota^{((k))})).$$

Now the propositional operator h_{oo}, formed by the *Propmod* function, is a function not only of the modifier $f_{\pi_k^2}$ but of the predicate g_{π_k} and the individuals $y_\iota^{(1)}, \ldots, y_\iota^{(k)}$ as well. (It can be shown that the same problem will arise if either the predicate or the individuals are not included as the function's arguments.) This fix is so trivial that it can hardly be used to support the theory that we began with.

The argument against the $Propmod_{(oo)(\pi_k^2)}$ function is a cardinality argument: indeed, the cardinality aspects of the situation are just the opposite to the theory in question. We can decently define a function which matches a predicate modifier with every propositional operator, by inverting the definition of $Propmod_{(oo)(\pi_k^2)}$ thus:

$$Modprop_{\pi_k^2(oo)} =_{df} \lambda h_{oo} (\imath f_{\pi_k^2})(g_{\pi_k})(y_\iota^{((k))})(f_{\pi_k^2}\, g_{\pi_k}\, y_\iota^{((k))} \equiv h_{oo}(g_{\pi_k}\, y_\iota^{((k))})),$$

and, for dyadic propositional operators,

$$Modprop_{\pi_k^3(ooo)} =_{df} \lambda h_{ooo} (f_{\pi_k^3})(g_{\pi_k})(m_{\pi_k})(y_\iota^{((k))}) (f_{\pi_k^3} g_{\pi_k} m_{\pi_k} y_\iota^{((k))} \equiv h_{ooo} (g_{\pi_k} y_\iota^{((k))})(m_{\pi_k} y_\iota^{((k))})) .$$

There is nothing much new in this, since e.g. if h_{oo} is N_{oo} then $Modprop_{\pi_k^2(oo)} h_{oo} = N_{\pi_k^2}$; and if h_{ooo} is conjunction then $Modprop_{\pi_k^3(ooo)} h_{ooo}$ is $K_{\pi_k^3}$.

Thus we find no way in which the theory of predicate modifiers can be "reduced" to a theory of propositional operators, and we suggest that the converse "reduction", while not of any special interest, is possible (in fact e.g. Quine's predicate-functor logic contains this converse reduction as a proper part).

1.14 Prepositions and relations

A point that arises often in symbolizing English is that a preposition, symbolized as a 1-ary modifier, turns a monadic predicate into a dyadic relation: the dyadic relation may otherwise have been symbolized directly with a variable of type π_2. For example

(A$_2$) Mr. Whitlam walks to The Lodge

may be symbolized either by

$$g_{\pi_2} x_\iota y_\iota$$

or

$$f_{\pi_1^2\iota} y_\iota g_{\pi_1} x_\iota ,$$

with $f_{\pi_1^2\iota}$ for the preposition "to". Obviously we can define a function forming a dyadic relation from a preposition and a monadic predicate:

$$Dmprep_{\pi_2\pi_1(\pi_1^2\iota)} =_{df} \lambda f_{\pi_1^2\iota} \lambda g_{\pi_1} \lambda x_\iota \lambda y_\iota (f_{\pi_1^2\iota} y_\iota g_{\pi_1} x_\iota) ,$$

and this can be generalized to

$$Dmprep_{\pi_{k+j}\pi_k(\pi_k^2\iota^j)} =_{df} \lambda f_{\pi_k^2\iota^j} \lambda g_{\pi_k} \lambda y_\iota^{((k))} \lambda x_\iota^{((j))} (f_{\pi_k^2\iota^j} x_\iota^{((j))} g_{\pi_k} y_\iota^{((k))}) .$$

Construed differently, for any preposition symbolized as a 1-ary modifier there is a heteradic modifier: for $f_{\pi_1{}^2\iota}$ there is the heteradic modifier $Dmprep_{\pi_2\pi_1(\pi_1{}^2\iota)}f_{\pi_1{}^2\iota}$ of type $\pi_2\pi_1$. In general a j-ary modifier $f_{\pi_k{}^2\iota j}$ yields a heteradic modifier

$Dmprep_{\pi_{k+j}\pi_k(\pi_k{}^2\iota j)}f_{\pi_k{}^2\iota j}$ of type $\pi_{k+j}\pi_k$. These heteradic modifiers raise the adicity of the predicate they modify: the only other modifier that we have encountered with this property is Quine's $Invnun_{\pi_{k+1}\pi_k}$, the padding modifier.

It is worth noticing that sometimes in English the same word can be used as a monadic predicate and a dyadic relation (grammatically speaking, the same verb may be both intransitive and transitive), but that the dyadic relation differs from any $Dmprep$ of the monadic predicate. For example, "walks" can be both monadic and dyadic (in American at least), but its dyadic usage is quite different from the $Dmprep$ "walks to". Similarly the dyadic "fights" is different from the $Dmprep$ "fights for", although it is not much different, if at all, from the $Dmprep$ "fights with". Moreover in some of these cases the monadic predicate is more aptly retrieved from the $Dmprep$ than from the dyadic relation: for instance if h_{π_2} is the dyadic "walks", h_{π_1} is the monadic "walks" and $f_{\pi_1{}^2\iota}$ is "to", then we certainly do not have

$$h_{\pi_1} = S^2_{\pi_1\pi_2}h_{\pi_2}$$

since one can walk without walking something (a dog, etc.) but apart from cases of completely random walks we do have

$$h_{\pi_1} = S^2_{\pi_1\pi_2}(Dmprep_{\pi_2\pi_1(\pi_1{}^2\iota)}f_{\pi_1{}^2\iota}\,h_{\pi_1})$$

that is, x walks iff x walks to somewhere (roughly speaking).

It has often been remarked, e.g. by McCulloch [46] p.396, that logic has not yet supplied a decent classification and calculus of triadic relations. A fertile way of classifying such relations may well be to consider them as being formed by two applications of a $Dmprep$ operator from two prepositions, and then to use properties of the prepositions qua 1-ary modifiers to yield properties of the triadic relation. For instance the triadic relation

$$x \text{ fights (with) } y \text{ in } z$$

may be analyzed as

$$Dmprep_{\pi_3\pi_2(\pi_2{}^2\iota)}\, in_{\pi_2{}^2\iota}\,(Dmprep_{\pi_2\pi_1(\pi_1{}^2\iota)}\, with_{\pi_1{}^2\iota}\, fights_{\pi_1})x_\iota\, y_\iota\, z_\iota$$

whereupon e.g. its 2-3 "symmetry" can be treated in terms of the commutativity, in an extended sense, of $in_{\pi_2{}^2\iota}$ and $with_{\pi_1{}^2\iota}$, and so on.

1.15 The Fundamental Problem of modifier theory.

In the absence of a Fundamental Theorem, we have a Fundamental Problem in the theory of predicate modifiers. This problem, to which I gave a suggestion of the partial solution on p.641 of [57], does not seem to have been treated by other authors; in particular Montague in [48] nowhere adverts to the problem, even though it arises within the fragment of English that he treats.

In its simplest terms, the Problem is this: in English, the same modifier can modify predicates of different adicity, and the problem is to find what connections there are between a modifier's operations in the different cases. For example, consider

(B_2) John strikes rapidly

and

(C_2) John strikes Harry rapidly ;

in (B_2) a monadic predicate is modified, and in (C_2) a dyadic relation is modified by the same modifier "rapidly". In type theory, (B_2) and (C_2) are symbolized

$$f_{\pi_1^2}\, g_{\pi_1}\, x_\iota$$

and

$$f_{\pi_2^2}\, g_{\pi_2}\, x_\iota\, y_\iota$$

respectively: it is clear that there is no logical relation (save that of indifference) between the two, since both "strikes" and "rapidly" are symbolized by different variables in the two formulae. If we form the monadic "strikes" from the dyadic "strikes" of (C_2) by existential quantification, thus gaining

$$f_{\pi_1^2}\,(S^2_{\pi_1\pi_2}\, g_{\pi_2})\, x_\iota$$

as our symbolization of (B_2), we still have "rapidly" symbolized by different variables in each case, and there is still no logical relation as matters stand.

In order to state a relation between modifiers acting on predicates of different adicities, we introduce the following functions:

$$L^u_{\pi_k^2(\pi_{k+1}^2)} =_{df} \lambda f_{\pi_{k+1}^2}\,(\imath f_{\pi_k^2})(g_{\pi_{k+1}})(y_\iota^{((k))})\,(S^u_{\pi_k\pi_{k+1}}\,(f_{\pi_{k+1}^2}\, g_{\pi_{k+1}})\, y_\iota^{((k))} \supset f_{\pi_k^2}\,(S^u_{\pi_k\pi_{k+1}}\, g_{\pi_{k+1}})\, y_\iota^{((k))}) \;,$$

$$M^u_{\pi_k^2(\pi_{k+1}^2)} =_{df} \lambda f_{\pi_{k+1}^2}\,(\imath f_{\pi_k^2})(g_{\pi_{k+1}})(y_\iota^{((k))})\,(f_{\pi_k^2}\,(S^u_{\pi_k\pi_{k+1}}\, g_{\pi_{k+1}})\, y_\iota^{((k))} \supset S^u_{\pi_k\pi_{k+1}}\,(f_{\pi_{k+1}^2}\, g_{\pi_{k+1}})\, y_\iota^{((k))})$$

and

$$LM^u_{\pi_k^2(\pi_{k+1}^2)} =_{df} \lambda f_{\pi_{k+1}^2}\,(\imath f_{\pi_k^2})\,(f_{\pi_k^2} = L^u_{\pi_k^2(\pi_{k+1}^2)}\, f_{\pi_{k+1}^2} \;\&\; f_{\pi_k^2} = M^u_{\pi_k^2(\pi_{k+1}^2)}\, f_{\pi_{k+1}^2}) \;;$$

together with

$$LM^{u,v}_{\pi_k^2(\pi_{k+1}^2)} =_{df} \lambda f_{\pi_{k+1}^2}\,(\imath f_{\pi_k^2})\,(f_{\pi_k^2} = LM^u_{\pi_k^2(\pi_{k+1}^2)}\, f_{\pi_{k+1}^2} \;\&\; f_{\pi_k^2} = LM^v_{\pi_k^2(\pi_{k+1}^2)}\, f_{\pi_{k+1}^2})$$

etc.

These functions can be visualized by means of the diagram

$$\begin{array}{ccccc} \Pi_{k+1} & \xrightarrow{\quad f_{\Pi^2_{k+1}} \quad} & \Pi_{k+1} & & \Pi^2_{k+1} \\ \Big\downarrow S^u_{\Pi_k\,\Pi_{k+1}} & & \Big\downarrow S^u_{\Pi_k\,\Pi_{k+1}} & & \Big\downarrow X^u_{\Pi^2_k(\Pi^2_{k+1})} \\ \Pi_k & \xrightarrow{\quad X^u_{\Pi^2_k(\Pi^2_{k+1})}\, f_{\Pi^2_{k+1}} \quad} & \Pi_k & & \Pi^2_k \end{array}$$

If X is L, the result of passing through the top right of the square materially implies (or, set-theoretically , is a subset of) the result of passing through the bottom left of the square. If X is M, the converse implication holds; if X is LM then both implications hold, and then the diagram (or rather its set-theoretic interpretation) commutes in more-or-less the usual algebraic sense.
We can ask whether all the diagrams, for $1 \leqslant u \leqslant k+1$, are uniquely simultaneously solvable for X; or, what is the same question, whether the ι operator in the definition of $LM^{1,2,\ldots,k+1}_{\Pi^2_k(\Pi^2_{k+1})}$ is a definite description rather than a choice operator. And prior to this question, we can ask whether any pair of diagrams are always simultaneously solvable, that is whether the condition on $f_{\Pi^2_k}$ in the definition of $LM^{u,v}_{\Pi^2_k(\Pi^2_{k+1})}$ is always satisfiable. To answer the latter question in the negative, consider a model where $D_\iota = \{a,b\}$, $V_\varphi g_{\Pi_2} = \{\langle a,a\rangle\}$ (in effect: strictly speaking, $V_\varphi g_{\Pi_2} = \{\langle\{\langle T,a\rangle,\langle F,b\rangle\},a\rangle,\langle\{\langle F,a\rangle,\langle F,b\rangle\},b\rangle\}$) and $V_\varphi f_{\Pi^2_2}$ is the function which maps $\{\langle a,a\rangle\}$ to $\{\langle a,b\rangle\}$ and everything else in D_{Π_2} to $\{\}$. There is now no function $V_\varphi(X^u_{\Pi^2_1(\Pi^2_2)} f_{\Pi^2_2})$ for which the diagrams for both $u=1$ and $u=2$ commute: if $u=1$ the commuting diagram requires that

$V_\varphi\,(X^1_{\pi_1^2(\pi_2^2)}\, f_{\pi_2^2})(\{a\}) = \{a\}$ and if $u=2$ it requires that $V_\varphi\,(X^2_{\pi_1^2(\pi_2^2)}\, f_{\pi_2^2})(\{a\}) = \{b\}$.

In summary: in the definitions of $L^u_{\pi_k^2(\pi_{k+1}^2)}$ and $M^u_{\pi_k^2(\pi_{k+1}^2)}$ the $\imath$ operator is acting as a choice operator; in the definition of $LM^u_{\pi_k^2(\pi_{k+1}^2)}$ the $\imath$ is a definite description operator since the commutation of the diagram uniquely determines the LM function; and in the definition of $LM^{u,v}_{\pi_k^2(\pi_{k+1}^2)}$ the $\imath$ chooses an arbitrary element of $\mathcal{D}_{\pi_k^2}$ since in general the condition following it is not satisfied.

The question also arises as to whether we should consider a diagram

$$\begin{tikzcd}
\pi_{k+1} \arrow[rr, "Y_{\pi_{k+1}^2(\pi_k^2)}\, f_{\pi_k^2}"] & & \pi_{k+1} & & \pi_{k+1}^2 \\
\pi_k \arrow[u, "H_{\pi_{k+1},\pi_k}"] \arrow[rr, "f_{\pi_k^2}"] & & \pi_k \arrow[u, "H_{\pi_{k+1}\pi_k}"'] & & \pi_k^2 \arrow[u, "Y_{\pi_{k+1}^2(\pi_k^2)}"']
\end{tikzcd}$$

where H is some heteradic modifier which raises the adicity of the predicate it modifies, and where Y is some function from modifiers of type π_k^2 to modifiers of type π_{k+1}^2 .
I can find no natural choice of $H_{\pi_{k+1}\pi_k}$, except for more-or-less trivial dualizations of our previous diagram, for which we should consider solving the diagram for the function Y. In particular, neither Quine's padding modifier nor any of the modifiers $Dmprep_{\pi_{k+1},\pi_k(\pi_k^2)}\, f_{\pi_k^2}$ of §1.14 seem to give rise to examples paralleling the relationship between (B_2) and (C_2). However this does not rule out further consideration of the "dual" diagram as part of the solution to our Problem.

Returning to our examples (B_2) and (C_2), and using some of the functions between modifiers of different types, we have the valid schemata

$$f_{\pi_2^2}\, g_{\pi_2}\, x_\iota\, y_\iota$$
$$\therefore\quad L^2_{\pi_1^2(\pi_2^2)}\, f_{\pi_2^2}\,(S^2_{\pi_1\pi_2}\, g_{\pi_2})\, x_\iota$$

and thence

$$f_{\pi_2^2}\, g_{\pi_2}\, x_\iota\, y_\iota$$
$$\therefore\quad LM^2_{\pi_1^2(\pi_2^2)}\, f_{\pi_2^2}\,(S^2_{\pi_1\pi_2}\, g_{\pi_2})\, x_\iota \quad .$$

Thus the logical relation (of implication) between (C_2) and (B_2) can be captured if we form the monadic predicate "strikes" and the modifier "rapidly" in (B_2) from the dyadic "strikes" and the modifier "rapidly" in (C_2) via the functions $S^2_{\pi_1\pi_2}$ and $L^2_{\pi_1^2(\pi_2^2)}$ (or $LM^2_{\pi_1^2(\pi_2^2)}$) respectively.

This facile "solution" to our example of the general Problem raisesmore questions than it answers. Firstly there is the general question of knowing when and how to use the functions L, M or LM in translating from English to type theory: this question is similar to those that have already arisen concerning the use of functions such as *Detacher* , *Attacher* and *Dmprep* , and it is a fact of life (albeit a sad fact) about English that we have no hard and fast rules for translation into something more precise (on which see my <u>The Myth of Canonical Notation: an exercise in exorcise</u> (not forthcoming)). Secondly there are questions arising from the formal fact that the functions $LM^{u,v}_{\pi_k^2(\pi_{k+1}^2)}$ are not always well-defined in the usual sense. Consider

(D_2) Harry is struck rapidly :

taking our cue from (B_2) we could expect to symbolize this as

$$L^1_{\pi_1^2(\pi_2^2)} f_{\pi_2^2} (S^1_{\pi_1\pi_2} g_{\pi_2}) y_\iota$$

(or with $LM^1_{\pi_1^2(\pi_2^2)}$ in place of $L^1_{\pi_1^2(\pi_2^2)}$), whence the inference from (C_2) to (D_2) is indeed validated. However it may well be that, say, $LM^1_{\pi_1^2(\pi_2^2)} f_{\pi_2^2}$ differs from $LM^2_{\pi_1^2(\pi_2^2)} f_{\pi_2^2}$, and hence that the modifier "rapidly" in (D_2) is different, not only from that in (C_2) as it must be on syntactic grounds, but also from the modifier "rapidly" in (B_2) even though both are modifying monadic predicates. For example if $V_\varphi g_{\pi_2} = \{\langle a,b\rangle, \langle b,a\rangle\}$ and if $V_\varphi f_{\pi_2^2}(V_\varphi g_{\pi_2}) = \{\langle a,b\rangle\}$ then $V_\varphi (LM^1_{\pi_1^2(\pi_2^2)} f_{\pi_2^2})(\{a,b\}) = \{b\}$ but $V_\varphi (LM^2_{\pi_1^2(\pi_2^2)} f_{\pi_2^2})(\{a,b\}) = \{a\}$. Whether this is simply another fact about the mess of English, or whether it indicates a defect in our treatment of the Problem, is an open question.

A third question arises from the fact that even where we naturally have modifiers operating on predicates of different adicities, our functions L, M and LM do not give the sort of results that may have been expected. For example we have the constant modifiers $N_{\pi_k^2}$ for any $k \geqslant 1$: we can show, say, that $LM^2_{\pi_1^2(\pi_2^2)} N_{\pi_2^2}$ does not equal $N_{\pi_1^2}$, in general, by considering a model where $D_\iota = \{a,b\}$ and $V_\varphi g_{\pi_2} = \{\langle a,b\rangle, \langle b,a\rangle\}$. Then $V_\varphi N_{\pi_2^2}(V_\varphi g_{\pi_2}) = \{\langle a,a\rangle, \langle b,b\rangle\}$ and $V_\varphi (LM^2_{\pi_1^2(\pi_2^2)} N_{\pi_2^2})(D_\iota) = D_\iota$ whereas $V_\varphi N_{\pi_1^2}(D_\iota) = \{\}$. As a consequence we have difficulties in treating the relation between, say

(E_2) John did not strike Harry

and

(F_2) John did not strike .

If we symbolize the "not" in (F_2) by $N_{\pi_1^2}$, then the argument from (E_2) to (F_2) is not valid, as we would expect. But if we symbolize that "not" in (F_2) by $LM^2_{\pi_1^2(\pi_2^1)} N_{\pi_2^2}$, following the pattern of (B_2), then the argument from (E_2) to (F_2) becomes valid: this is possible since $LM^2_{\pi_1^2(\pi_2^1)} N_{\pi_2^2}$ is not $N_{\pi_1^2}$. It was this fact that led me, in effect, to reject the use of the functions L and LM in [57], and to consider only the function M as expressing a relation between modifiers acting on predicates of different adicities. I now propose that the question should be resolved by finding general conditions for the correct application of the functions L, M and LM (or others like them, gained from the dual diagram or anywhere else), and that particular examples such as that given should not count as ruling out the use of L or LM in all cases. (A first guess, to the effect that LM should only be used with detachable modifiers, is shown not to work by the example of the previous paragraph.)

Thus the Fundamental Problem is still with us, even for the simple case of the example given by (B_2) and (C_2). Further aspects of the Problem concern modifiers like "alone": in considering

(G_2) John drives alone

and

(H_2) John drives Mary alone

we see that in (G_2) "alone" means approximately "by himself", while in (H_2) "alone" is ambiguous between "only" and "by themselves". Taking just the latter case, which is closest to its meaning in (G_2), it would still be inappropriate to form the "alone" in (G_2) from that in (H_2) by means of an L, M or LM

function since (H_2) does not imply (G_2). However, there is still <u>some</u> relation between the two uses of "alone", and a further part of the general Problem is to explain the relation in cases like this.

Another aspect of the Problem is raised by Montague's remark in [48] that "and conversely" may be an example of a modifier that modifies dyadic but not monadic predicates. We might question whether

(I_2) John strikes, and conversely

means that John strikes and is struck, or whether it is nonsignificant. Another example is "in any order", which seems to modify any predicates except monadic ones:

(J_2) John, Joe and Jill are cousins, in any order

seems significant,

(K_2) Lasker, Capablanca and Alekhine were successive chess champions, in any order

is significant but false, while

(L_2) Lasker was a chess champion, in any order

seems nonsignificant. It is another aspect of the Fundamental Problem to given an explanation of the properties of the modifiers concerned that lead to examples of this kind.

A more complicated aspect of the Problem is illustrated by the operation of "from... to" in the examples

(M_2) The train went from Sydney to Melbourne

and

(N_2) The seas went from moderate to rough.

In each case "from...to" is a 2-ary modifier, and indicates a transition over some range, but in (M_2) this range is a range of points specified by the individual points Sydney and Melbourne (and the fact that trains run on railway tracks), while in (N_2) the range is a range of properties specified by the properties "moderate" and "rough" (and the fact that it is seas, not sandpaper or vin ordinaire, that we are talking about). The problem here is simply whether it is possible to make the last sentence more precise in terms of modifier theory, in particular whether we can find some functional relationship between the different cases of "from...to", or whether we are forced to treat the two cases of "from...to" as quite distinct logically.

1.16 An application

In this last section of §1 we discuss some of the basic terminology of the theory of recursive functions in terms of the logic of predicate modifiers. In all cases we try to use the terminology of Rogers [59], and we assume that "set", "function" etc., refer to the natural numbers only.

Firstly we will suppose that "recursive" is predicated only of functions, and not of sets, real numbers, functionals, ordinals etc. (see index of [59]). Now consider the various ways in which "recursive" may be modified, beginning with the modifier "primitive". Since the primitive recursive functions are a (proper) subset of the recursive functions, it follows that "primitive" is an inclusive modifier when modifying "recursive" (but perhaps not in general, since primitive art may not be art, etc.) If the type ι is the type of the natural numbers (i.e. $D_\iota = \omega$), then we can use

$r_{o(\iota^2)}$ for "recursive"

and

$p_{(o(\iota^2))^2}$ for "primitive".

Then, with a slight and obvious extension of our previous definitions (put (ι^2) for ι throughout), we have

$$Incl^{p}_{o((o(\iota^2))^2)(o(\iota^2))}\; r_{o(\iota^2)}\; p_{(o(\iota^2))^2}$$

which is our formal way of saying that $p_{(o(\iota^2))^2}$ is an inclusive modifier when applied to $r_{o(\iota^2)}$.

Given that "primitive" is inclusive we can go on to ask whether it is separable, and if so, what is its separator. Clearly when applied to $r_{o(\iota^2)}$ it is separable, since the primitive recursive functions are a fixed subset of the recursive functions (supposing that only monadic functions are considered): the question then is "what is the separator of $p_{(o(\iota^2))^2}$?", or metaphorically "what is a solution (or a least solution) for X in

f is primitive recursive
$\therefore$ f is X and f is recursive ?"

The least solution for X will be the detacher of $p_{(o(\iota^2))^2}$ as well as a separator, and the solution is simply "primitive recursive". When the primitive recursive functions are defined as the closure under certain operations of a certain basic set of functions, the words "primitive" and "recursive" nowhere appear separately i.e. the primitive recursive functions are not defined as some particular kind of recursive functions. So "primitive recursive" is here not precisely the result of modifying "recursive" with "primitive", but rather a property with a specific definition of its own.

The modifier "general" is one which falls into Clark's Case 1B., [12] p.329, at least when it modifies "recursive". That is, it is simply an identity modifier for this predicate (but not, of course, generally, since not all elections are general elections and not all models are general models). Clark, incidentally, does not have sufficient expressive power in his notation to formalize our rider "for this predicate": his example "extended" is indeed an identity modifier for "surface", but not for all predicates, since not all holidays are extended holidays and not all extended consistent systems are consistent systems.

Next we consider the (apparent) modifier "partial", as in

(O_2) f is a partial function

and

(P_2) f is a partial recursive function .

In (O_2), "partial" modifies "function", and in (P_2) it appears to modify either "recursive function" or "recursive" (in the latter case it would be a modifier modifier). But (despite the partial explanation in [59] p.65) (P_2) is actually something of a solecism: if f is a partial recursive function then f is a partial function (which may or may not be total) and f is recursive <u>as such functions go</u>, i.e. we may have to widen our original notion of recursiveness (depending on how we characterized it) so that non-total functions as well as total functions can be recursive. The type-theoretic notation as it stands is not well adapted for the discussion of partial functions, since a type $\iota\iota$ represents a function defined on all of $\mathcal{D}_\iota$, and there is no immediate way to form a type $\iota\mu$ where $\mathcal{D}_\mu$ is meant to be a subset of $\mathcal{D}_\iota$ (e.g. the type $\iota(o\iota)$ does not do the required job). Also, we cannot assume all partial functions to be arbitrarily completed to total functions, since it is impossible to complete some partial functions (this is the real point of introducing them in the first place). To deal effectively with partial functions we need to build Kleene's 3-valued logic into the bottom of our system, and this would take us beyond our present concerns. The upshot of all this is that insofar as "partial" is a modifier, it does not modify "recursive" but "function", and partial functions cannot be adequately dealt with in our present notation.

We turn now to the 1-ary modifier "in" (or perhaps more accurately "relatively ... in", although the "relatively" is usually left tacit) as it appears in the theory of relative recursiveness. For a start, this is not in general an inclusive modifier, since

(Q_2) f is recursive in g

does not entail

(R_2) f is recursive .

(In (Q_2) we write g for a (characteristic) function rather than a set, just to avoid having to predicate recursiveness of a set in what follows.) If, however, g is recursive then (Q_2) implies (R_2), so if we write

$$in_{(o(\iota^2))^2(\iota^2)} \qquad \text{for "in"}$$

then we have the following inclusiveness property from **Lattice Two** slightly extended:

$$Incl^{1}_{o((o(\iota^2))^2(\iota^2))(o(\iota^2))(o(o(\iota^2)))}\, m_{(o(o(\iota^2)))}\, r_{(o(\iota^2))}\, in_{(o(\iota^2))^2(\iota^2)} \quad ,$$

where $m_{(o(o(\iota^2)))}$ is simply $\lambda n_{o(\iota^2)}(n_{o(\iota^2)} = r_{o(\iota^2)})$.

De-mystifying this string of symbols results in the proposition that "in" is an inclusive modifier when applied to "recursive" provided that the individual (function) involved in the modification is itself recursive. If we had predicates, say $r'_{o(\iota^2)}$, $r''_{o(\iota^2)}$ etc. for other degrees of unsolvability, then we could lay down similar inclusiveness properties for "in" at each degree. And we could also consider "in" as a j-ary modifier for $j > 1$, since a function can be recursive in more than one function (we can plug any number of oracles into a Turing machine). Finally, we could express the fact that every function is recursive in itself by saying that the relation

$$Dmprep_{(o(\iota^2)^2)(o(\iota^2))((o(\iota^2))^2(\iota^2))}\, in_{(o(\iota^2))^2(\iota^2)}\, r_{o(\iota^2)}$$

is reflexive.

These examples suffice to show some of the uses and limitations of type-theoretic modifier theory in explaining and classifying pieces of a technical language. It would be possible to incorporate some such modifier theory in an axiomatic account of recursive function theory.

There are at least two ways in which our type-theoretical basis could be intensionalized. According to the first way, which we do not favour, the syntax is kept intact and the semantics is intensionalized. Thus the domain D_o , instead of being simply $\{T,F\}$ would now be $\{T,F\}^K$ where K is some set, of possible worlds or whatever; and the remainder of the semantics would be intensionalized in a similar fashion. In the light of the development to be presented, the main trouble with this "first way" is that it buries the possible worlds in the semantics, and the only reference to them is covert reference via modal operators like $\Box_{oo}$, which could be added to the syntax and evaluated semantically in the familiar way. According to the second way, which (as it goes without saying) we favour, the possible worlds appear explicitly in the syntax by way of a further type symbol. Thus we introduce a new basic type symbol κ , and the revised formation rules for type symbols are simply

1′. o, ι and κ are type symbols

2′. If α and β are type symbols, so is $(\alpha\beta)$.

The formation rules for wffs are precisely as before: since κ is a type symbol the stock of variables is automatically extended, so that in particular there are extra variables $f_\kappa, g_\kappa, h_\kappa, m_\kappa, n_\kappa, \ldots$. No new logical constants are introduced (save of course that there are constants like $\Pi_{(o(o\kappa))}$ involving the new type symbol κ), but to match things like accessibility relations or neighbourhoods for possible worlds we will countenance the introduction of extralogical constants governed by extralogical postulates.

The resulting set-up can be looked at in at least two different ways, viz. as a theory or metatheory of the formal semantics of modal logics, or more directly as an object language (with its own extensional semantics) in which substantial portions of intensional language can be formalized. The semantics appropriate for our extended syntax is certainly extensional, since the only further semantical prescription required is that D_k shall be a non-empty set: some may object that it is impossible for a syntax in which intensional language can be symbolized to have an extensional semantics, but we will let the development that follows provide its own answer to such an objection. Some also may object, perhaps with Benacerraf [3] in mind, that no particular set-theoretic explications of such peculiarly philosophical entities as propositions and properties and individual concepts can be correct, since there are several different explications provided by different systems of intensional logic. Our answer to this objection would run thus: just as it was a discovery, by Zermelo in the first instance, that formal surrogates for the natural numbers and operations on them are available within general set theory, so it has been a discovery, by the pioneers in the semantics of modal logic, that formal surrogates for propositions and so on are also available within general set theory (or, in particular, within our version of type theory). And, while we do not have such clear-cut adequacy conditions as those provided by the Dedekind-Peano axioms, we can test these formal surrogates against the usual kinds of jobs for which the philosophical entities are required, and we find that they come out of this test rather well. Finally some may also object that if intensional logic is actually so (superficially) complicated and ramified as we make it out to be, then it must be either wrong or useless. We leave such objectors to their own inadequate devices.

2.1 Propositions and propositional operators

The domain $\mathcal{D}_\kappa$ is to be construed as a set of possible worlds: in certain cases we may be inclined to take possible worlds as time-slots or as indices or reference points in the sense of Scott [64] or Montague [50], and in an enriched system we would be able to distinguish among different aspects or different notions of possible worlds. Given, anyway, that $\mathcal{D}_\kappa$ is our set of possible worlds, then $\mathcal{D}_{o\kappa}$ $(= \mathcal{D}_o{}^{\mathcal{D}_\kappa})$ is the set of subsets of worlds, or equivalently, of sequences of truth-values indexed by worlds. The elements of $\mathcal{D}_{o\kappa}$ thus amount to the by-now-usual representation of <u>propositions</u>: we have already anticipated, and indicated our answer to, objections about this kind of a reconstruction of a philosophical notion. It follows that variables $x_{o\kappa}, y_{o\kappa}, \ldots$ are propositional variables, and in general that wffs $A_{o\kappa}$ of type $o\kappa$ denote propositions: the system still has variables of type o , but these are just truth-value variables, and we need no longer suspend belief and assume that truth-values are propositions. Notice that we do not call the elements of $\mathcal{D}_{o\kappa}$ "propositional concepts", as Scott [64] p.154 proposes: despite Scott's reference to Carnap, the latter explicitly eschews the term "propositional concepts" in [7] p.21, and our terminology is in consonance with Church [10] p.11, whereby propositions are concepts of truth-values.

The constants N_{oo} and A_{ooo} are still with us, but as their type shows they are operators solely on truth-values, and <u>all</u> propositional operators have to be defined (or perhaps introduced as primitive). We can define propositional negation and inclusive disjunction thus

$$N_{(o\kappa)^2} =_{df} \lambda x_{o\kappa} \lambda w_\kappa (N_{oo} (x_{o\kappa} w_\kappa))$$

$$A_{(o\kappa)^3} =_{df} \lambda x_{o\kappa} \lambda y_{o\kappa} \lambda w_\kappa (A_{ooo} (x_{o\kappa} w_\kappa)(y_{o\kappa} w_\kappa)) \; :$$

in general, we will call a propositional operator, that is defined world-by-world in terms of a corresponding truth-value operator, a conventional propositional operator. So as well as the conventional negation and inclusive disjunction just defined, we will have conventional conjunction, material implication and so on, with definitions following the same pattern. We may write

$$(\sim x_{OK}) \quad \text{for} \quad (N_{(OK)^2}\, x_{OK})$$
$$(x_{OK} \vee y_{OK}) \quad \text{for} \quad (A_{(OK)^3}\, x_{OK}\, y_{OK})$$

etc., allowing the ordinary operators to be used in a (systematically) ambiguous fashion. This is a concession to readability, not a chink in our self- (and notationally-) imposed determination to be completely explicit about our typal indices.

As well as conventional propositional operators, we can define other propositional operators with the resources presently available. Most notably the availability of variables for, and quantification over, possible worlds allows us to define necessity of the $S5$ type (the universal necessity of [64] p.157) by

$$\Box_{(OK)^2} =_{df} \lambda x_{OK}\, \lambda w_K\, (v_K)(x_{OK}\, v_K) \quad .$$

In this definition the vacuity of the abstraction λw_K corresponds to known properties of $S5$ necessity. The definition typifies one aspect of our approach to intensional logic within type theory: what would normally be a part of the recursive semantical stipulations in the metatheory is written directly into a formula of our type-theoretic symbolism. But, recalling our introductory remarks on the topic, in our view this does not prevent the type-theoretic symbolism from being used directly as an object language. Perhaps surprisingly, we will find the operator $\Box_{(OK)^2}$ of no special utility: in most cases it will be easier simply to universally quantify over $\mathcal{D}_\kappa$ than to organize an expression into shape suitable for application of $\Box_{(o\kappa)2}$.

Other styles of necessitation operators can be defined by introducing various constants governed, perhaps, by extralogical postulates. We permanently appropriate the variable

$$g_\kappa$$

as a constant for the real world (in a tense-logical context we will use n_κ for the present world), and using this we can define $S0.5$ necessity à la Cresswell [15] by

$$\Box^{g_\kappa}_{(o\kappa)2} =_{df} (\imath f_{(o\kappa)2})(x_{o\kappa})(f_{(o\kappa)2}\, x_{o\kappa}\, g_\kappa \equiv (v_\kappa)(x_{o\kappa}\, v_\kappa)) \quad .$$

In this definition we trade on the choice-operator aspect of $\imath$ to gain the same effect as Cresswell's requirement that $V(\Box\alpha, w)$ shall be an arbitrary truth-value where w is not the real world.

The constant g_κ is not governed by postulates, but as might be expected it does figure in the following definition of <u>truth</u> for propositions:

$$Tru_{o(o\kappa)} =_{df} \lambda x_{o\kappa}(x_{o\kappa}\, g_\kappa) \quad ,$$

i.e. a proposition is true iff it is true at the real world. (We use 'Tru' rather than '$True$' to avoid any suggestion that we are giving a truth-definition for the whole type-theoretical language: the latter is embodied in Andrews's $[True]_{o2}$ in [1] p.83, with the definition and the transfinite type 2 trivally extended to take type κ into account.) This definition raises a philosophical objection to the effect that propositions are paradigmatically "truth-bearers", and should not have truth or falsity derivative upon such things as truth-values and the real world. Our answers to this objection are (a) that truth-values and the real world, rather than propositions per se, appear in the semantics of intensional systems, so why not make them appear explicitly in the language? and (b) our (or rather, the current and usual) analysis simply refutes the philosophical doctrine that propositions are the "bearers of truth"; they are no more (and no less) so than the function Weight from objects to numbers is a "bearer of numbers".

For further kinds of necessitation, other than $S5$ and $S0.5$, we introduce constants corresponding to the items in the model theory normally used to evaluate these operators. If we introduce a constant

$$S^{(1)}_{o\kappa^2}$$

then

$$\Box^{S^{(1)}}_{(o\kappa)^2} =_{df} \lambda x_{o\kappa} \lambda w_\kappa (v_\kappa)(S^{(1)}_{o\kappa^2} w_\kappa v_\kappa \supset x_{o\kappa} v_\kappa)$$

defines *K-ish* necessity à la Kripke, and added postulates for such a constant will yield operators within the ambit of the semi-normal systems. For instance for $S4$ necessity, with a constant $S^{(2)}_{o\kappa^2}$, we would lay down the postulates

$S^{(2)}(1) \qquad (w_\kappa)(S^{(2)}_{o\kappa^2} w_\kappa w_\kappa)$

$S^{(2)}(2) \qquad (w_\kappa)(v_\kappa)(u_\kappa)(S^{(2)}_{o\kappa^2} w_\kappa v_\kappa \;\&\; S^{(2)}_{o\kappa^2} v_\kappa u_\kappa \supset S^{(2)}_{o\kappa^2} w_\kappa u_\kappa)$

for reflexivity and transitivity respectively. A constant

$$Q^{(1)}_{o\kappa}$$

(actually a proposition asserting that things are queer) may be used in definitions such as

$$\Box^{S^{(2)}Q^{(1)}}_{(o\kappa)^2} =_{df} \lambda x_{o\kappa} \lambda w_\kappa (\sim Q^{(1)}_{o\kappa} w_\kappa \;\&\; (v_\kappa)(S^{(2)}_{o\kappa^2} w_\kappa v_\kappa \supset x_{o\kappa} v_\kappa)) \quad ,$$

and if the postulate

$Q^{(1)}(1) \qquad \sim Q^{(1)}_{o\kappa} g_\kappa$

is added, then $\Box^{S^{(2)}Q^{(1)}}_{(o\kappa)^2}$ is necessity of the $S3$ type.

"Operational" semantics (or at least one form of such semantics) will be represented by introducing constants like

$$O^{(1)}_{\kappa^3}$$

figuring in the definition

$$\Box^{O^{(1)}}_{(o\kappa)\iota} =_{df} \lambda x_{o\kappa} \lambda w_\kappa (v_\kappa)(x_{o\kappa}(O^{(1)}_{\kappa^3} w_\kappa v_\kappa)) \quad .$$

Again, postulates on $O^{(1)}_{\kappa^3}$, e.g. the existence of inverses of the operation, will yield various kinds of necessitation for the propositional operator defined in terms of it.

More generally, we can consider the "neighbourhood" semantics of Scott, Montague and Howard. (Warning: in general there is no condition that the neighbourhoods shall be open sets, and so further topological terms will not be apposite since the neighbourhoods to be discussed are not genuine neighbourhoods. E.g. there are no accumulation points in our topology, Horatio.) For the neighbourhood semantics we need operators associating a set of neighbourhoods with each world: formally we need constants like

$$Nbd^{(1)}_{o(o\kappa)\kappa} \quad .$$

which appear in definitions like

$$\Box^{Nbd^{(1)}}_{(o\kappa)\iota} =_{df} \lambda x_{o\kappa} \lambda w_\kappa (Nbd^{(1)}_{o(o\kappa)\kappa} w_\kappa x_{o\kappa}) \quad .$$

This definition looks even more trivial, from one point of view, in our notation than in ordinary semantic metatheory (as e.g. Scott [64], p.160, Montague [50] Def. III (6)): to say that the set of all worlds where a proposition is true is one of the neighbourhoods of a particular world is merely to say that the world and the proposition (which in our notation is indistinguishable from the set of worlds where it is true) are related by the neighbourhood relation (which itself is no different from a neighbourhood operator, in our notation). The ultimate step in triviality, from this point of view, comes when we consider the " "f" -version" of the Scott-Montague semantics, as set out in Hansson and Gärdenfors [26] pp.7-8. Their operator f maps subsets of worlds to subsets of worlds (i.e. $f \in \mathcal{P}(K)^{\mathcal{P}(K)}$), and thus we introduce constants like

$$F^{(1)}_{(OK)(OK)}$$

which figure in definitions like

$$\Box^{F^{(1)}}_{(OK)^2} =_{df} \lambda x_{OK} \lambda w_K \, (F^{(1)}_{(OK)(OK)} \, x_{OK} \, w_K) \quad .$$

It is apparent that $\Box^{F^{(1)}}_{(OK)^2} = F^{(1)}_{(OK)(OK)}$, and so in our notation we gain nothing at all from the "f"-version of the neighbourhood semantics: it does not even "somehow read differently in terms of sets" in our notation.

For each of our different kinds of necessitation operators, the definiens in their definition has the form $\lambda x_{OK} \lambda w_K A_O$. For a more general notation we could use A_O to distinguish the operator, i.e. we could write

$$\Box^{A_O}_{(OK)^2} =_{df} \lambda x_{OK} \lambda w_K A_O \quad .$$

Although obviously this could become rather unwieldy, some such notation would be needed where more than one operator depends on the same constant or constants: for example in the notation used to date we could not distinguish between $\Box^{S^{(1)}}_{(OK)^2}$ and

$$\lambda x_{OK} \lambda w_K (v_K)(S^{(1)}_{OK^2} \, v_K \, w_K \supset x_{OK} \, v_K) \quad ,$$

since both depend only on $S^{(1)}_{OK^2}$.

2.2 Dyadic propositional operators, entailment

As examples of dyadic propositional operators, other than the conventional disjunction, conjunction etc. and those like strict implication that are definable via these and monadic necessitation operators, we consider dyadic deontic operators as treated in von Wright [71] Ch. 1 §§ 7-12 and entailment as treated by Routley and Meyer [60] etc.

If I have read von Wright's terminology correctly, the following provide translations for his six different dyadic obligation operators O_1 through O_6 :

$$\Box^{/(1)}_{(o\kappa)^3} =_{df} \lambda x_{o\kappa}\lambda y_{o\kappa}\lambda w_\kappa (v_\kappa)(y_{o\kappa} v_\kappa \supset (u_\kappa)(S^{(3)}_{o\kappa^2} v_\kappa u_\kappa \supset x_{o\kappa} u_\kappa))$$

$$\Box^{/(2)}_{(o\kappa)^3} =_{df} \lambda x_{o\kappa}\lambda y_{o\kappa}\lambda w_\kappa (\exists v_\kappa)(y_{o\kappa} v_\kappa \,\&\, (u_\kappa)(S^{(3)}_{o\kappa^2} v_\kappa u_\kappa \supset x_{o\kappa} u_\kappa))$$

$$\Box^{/(3)}_{(o\kappa)^3} =_{df} \lambda x_{o\kappa}\lambda y_{o\kappa}\lambda w_\kappa (u_\kappa)(\sim(x_{o\kappa} u_\kappa) \supset (\exists v_\kappa)(\sim(S^{(3)}_{o\kappa^2} v_\kappa u_\kappa) \,\&\, y_{o\kappa} v_\kappa))$$

$$\Box^{/(4)}_{(o\kappa)^3} =_{df} \lambda x_{o\kappa}\lambda y_{o\kappa}\lambda w_\kappa (\exists u_\kappa)(\sim(x_{o\kappa} u_\kappa) \,\&\, (v_\kappa)(y_{o\kappa} v_\kappa \supset \sim(S^{(3)}_{o\kappa^2} v_\kappa u_\kappa)))$$

$$\Box^{/(5)}_{(o\kappa)^3} =_{df} \lambda x_{o\kappa}\lambda y_{o\kappa}\lambda w_\kappa (v_\kappa)(y_{o\kappa} v_\kappa \supset (\exists u_\kappa)(\sim(S^{(3)}_{o\kappa^2} v_\kappa u_\kappa) \,\&\, \sim(x_{o\kappa} u_\kappa)))$$

$$\Box^{/(6)}_{(o\kappa)^3} =_{df} \lambda x_{o\kappa}\lambda y_{o\kappa}\lambda w_\kappa (\exists v_\kappa)(y_{o\kappa} v_\kappa \,\&\, (\exists u_\kappa)(\sim(S^{(3)}_{o\kappa^2} v_\kappa u_\kappa) \,\&\, \sim(x_{o\kappa} u_\kappa))) .$$

In each case, $\Box^{/(i)}_{(o\kappa)^3} x_{o\kappa} y_{o\kappa}$ is meant to correspond with von Wright's $O_i(p/q)$, with $x_{o\kappa}$ for p and $y_{o\kappa}$ for q. Although we have given type $(o\kappa)^3$ to the operators, this has been achieved by use of a "vacuous" λw_κ in each definition: von Wright's operators, as he defines them, are not iterable and are in effect of type $o(o\kappa)^2$ rather than $(o\kappa)^3$. Each of the operators depends upon a dyadic relation for which we have used $S^{(3)}_{o\kappa^2}$: von Wright discusses a range of possible conditions on $S^{(3)}_{o\kappa^2}$, ranging from the very weak

$$(\exists v_\kappa)(\exists u_\kappa)\, S^{(3)}_{o\kappa^2} v_\kappa u_\kappa$$

through stronger conditions like

$$(v_\kappa)(\exists u_\kappa)\, S^{(3)}_{o\kappa^2} v_\kappa u_\kappa$$

to the anarchistic

$$(v_\kappa)(u_\kappa)\, S^{(3)}_{o\kappa^2} v_\kappa u_\kappa$$

(which, given the construal of $S^{(3)}_{o\kappa^2}$, says that every world is permitted in every world). He does not discuss conditions such as transitivity, since these are basically concerned with iterations of the modal operators and von Wright's original operators are not iterable.

Another treatment of dyadic deontic operators is given in Rescher [58] Ch. XVI, However, Rescher's prescriptions depend upon the use of an entailment operator, and thus are not adequate in the absence of any form of semantics for the latter. This is one among many reasons for setting out semantics for entailment in our notation, and we do this by copying with translation from Routley and Meyer.

The conventional conjunction and disjunction operators for propositions are used by Routley and Meyer, but in systems containing negation this operator is not conventional. It depends upon an operator $*$, for which we will introduce the constant

$$Star_{\kappa^2} ,$$

and is defined by

$$N^{*}_{(o\kappa)^2} =_{df} \lambda x_{o\kappa} \lambda w_{\kappa} \sim (x_{o\kappa} (Star_{\kappa^2} w_{\kappa})) .$$

We might notice that an incidental feature of our type-theoretical treatment is that <u>all</u> propositional operators have to be defined, and hence we can cater easily for non-conventional operators such as $N^{*}_{(o\kappa)^2}$ as well as the conventional $N_{(o\kappa)^2}$. We might also notice the sense in which the Routley-Meyer semantics admits or requires the use of "impossible worlds": the impossibility is relative to $N^{*}_{(o\kappa)^2}$, in the sense that for some world w_{κ}, both $x_{o\kappa} w_{\kappa}$ and $N^{*}_{(o\kappa)^2} x_{o\kappa} w_{\kappa}$ may hold for some proposition $x_{o\kappa}$. The impossibility does not hold for $N_{(o\kappa)^2}$, since we still cannot have both $x_{o\kappa} w_{\kappa}$ and $N_{(o\kappa)^2} x_{o\kappa} w_{\kappa}$ for any w_{κ} and $x_{o\kappa}$: thus the semantics for entailment does not require the kind of deracination of our ideas that might be suggested by heuristic remarks to the effect that it uses impossible worlds as well as possible worlds.

The entailment operator, which we write as $E^{R^{(1)}}_{(OK)^3}$, depends upon a triadic relation $R^{(1)}_{OK^3}$ according to the definition

$$E^{R^{(1)}}_{(OK)^3} =_{df} \lambda x_{OK} \lambda y_{OK} \lambda w_K (v_K)(u_K)(R^{(1)}_{OK^3} w_K v_K u_K \,\&\, x_{OK} v_K \supset y_{OK} u_K).$$

We may write

$$(x_{OK} \rightarrow y_{OK}) \quad for \quad (E^{R^{(1)}}_{(OK)^3} x_{OK} y_{OK}) \quad ,$$

provided we remember not only that $\rightarrow$ has lost any typal indices, but that the dependence on $R^{(1)}_{OK^3}$ is not shown in such shorthand notation.

The operator $Star_{K^2}$ and the relation $R^{(1)}_{OK^3}$ have to satisfy conditions appropriate for the strength of the entailment required. For example for the basic system $\mathcal{B}$ we can copy the following definitions and postulates from [63]: there is a one-place predicate O_{OK}, the definitions are

$d1.\quad o_K =_{df} (\imath w_K)(O_{OK} w_K)$

(o_K plays much the same role as g_K in modal systems)

and

$d2.\quad \leqslant_{OK^2} =_{df} \lambda w_K \lambda v_K (\exists x_K)(O_{OK} x_K \,\&\, R^{(1)}_{OK^3} x_K w_K v_K)$,

and the postulates are

$p1.\quad (x_K)(\leqslant_{OK^2} x_K x_K)$

$p2.\quad (x_K)(y_K)(w_K)(v_K)(\leqslant_{OK^2} x_K y_K \,\&\, R^{(1)}_{OK^3} y_K w_K v_K \supset R^{(1)}_{OK^3} x_K w_K v_K)$

$p3.\quad (x_K)(Star_{K^2}(Star_{K^2} x_K) = x_K)$

and

$p4.\quad (x_K)(y_K)(\leqslant_{OK^2} x_K y_K \supset \leqslant_{OK^2} (Star_{K^2} y_K)(Star_{K^2} x_K))$.

As well as these postulates, there is a modelling condition, which we may write as

$c1.\quad (x_{OK})(w_K)(v_K)(\leqslant_{OK^2} w_K v_K \,\&\, x_{OK} w_K \supset x_{OK} v_K)$.

The various conditions on $R^{(1)}_{o\kappa^3}$ and $Star_{\kappa^2}$ appear in a different form in our presentation than they do in [63], where they are stated in the semantical metalanguage rather than in the same language as the propositional connectives under analysis. Although we have not done so (nor do we have any general theorem that proves that it can be done), we should be able to show that these conditions have the same effect as they do in a more orthodox presentation: viz. that desirable basic principles of entailment such as $p \to p$, $\sim\sim p \to p$ and $p \to q \to . \sim q \to \sim p$ result from the definitions and conditions as laid down.

This provides a sample of the way in which a genuine entailment between propositions can be introduced into our system, and, incidentally, helps us to sustain a defence of type $o\kappa$ as appropriate for propositions. Obviously enough, our notation could encompass various kinds of generalizations of the relational semantics for entailment. For example we could introduce neighbourhood functions like

$$Nbd^{(1)}_{o(o\kappa)(o\kappa)\kappa}$$

and define

$$E^{Nbd^{(1)}}_{(o\kappa)^3} =_{df} \lambda x_{o\kappa}\, \lambda y_{o\kappa}\, \lambda w_{\kappa}\, (Nbd^{(1)}_{o(o\kappa)(o\kappa)\kappa}\, w_{\kappa}\, x_{o\kappa}\, y_{o\kappa}) \quad :$$

so far as the shape of the definition is concerned this has an equal generality, and an equal triviality coefficient to that of the neighbourhood semantics for one-place operators. However, the triviality does not extend to the formulation of postulates for $Nbd^{(1)}_{o(o\kappa)(o\kappa)\kappa}$ to match desirable principles of entailment, and we have in no way obviated the need for the kind of detailed work involved in completeness proofs.

The purpose of §§ 2.1 and 2.2 has been to specify the kind of entity that we take propositions to be, and to give examples of the way in which our extended type-theoretic system can be used to define various kinds of propositional operators; obviously we have not attempted to survey all the operators that have, for good reasons or ill, been defined in the literature on intensional logics. In §2.3 we sketch means whereby propositional operators defined by algebraic methods can also be incorporated into our notation.

2.3 Algebraic semantics

In the usual passage from a set algebra (e.g. a field of sets) to an abstract algebra (e.g. a Boolean algebra) we "lose" our original elements since the operators in the set algebra take subsets of these original elements as their arguments, and it is the subsets of the original elements that are represented as individuals in the abstract algebra. This "loss" is precisely contrary to the spirit of our extended type-theoretic language, since in it our intention is to refer explicitly to worlds whenever such reference is needed, and the worlds are lost in the passage to algebraic semantics for intensional logics. Nevertheless we now show how we can deal with algebraic semantics, even if the result is somewhat inelegant.

Firstly, since the nature of the individuals in an abstract algebra is irrelevant, we choose some subset of our existing individuals, by means of a constant $\mathcal{B}^{(1)}_{o\iota}$

to serve as the carrier of the algebra in question. We define the restricted quantifier

$$\Pi^{B}_{o(o\iota)} =_{df} \lambda f_{o\iota}(x_\iota)(\mathcal{B}^{(1)}_{o\iota}x_\iota \supset f_{o\iota}x_\iota)$$

with abbreviations $({}^{B}x_\iota)A_o$ for $\Pi^{B}_{o(o\iota)}(\lambda x_\iota A_o)$ and $(\exists^{B}x_\iota)A_o$ for $\sim({}^{B}x_\iota)(\sim A_o)$, to range over the carrier.

For non-degeneracy, we add a postulate

$$(\exists^{B}x_\iota)(\exists^{B}y_\iota)(x_\iota \neq y_\iota) \quad ,$$

but otherwise $\mathcal{B}^{(1)}$ is an arbitrary subset of individuals. Next we allocate constants

$$\underline{j}^{(1)}_{\iota 3} \ , \ \underline{m}^{(1)}_{\iota 3} \ , \ \underline{c}^{(1)}_{\iota 2}$$

to serve as the join, meet and complementation operators in the algebra. Although we do not, strictly speaking, have type-theoretic notation for it, we will let $\mathcal{B} = \langle \mathcal{B}^{(1)}, \underline{j}^{(1)}, \underline{m}^{(1)}, \underline{c}^{(1)} \rangle$ be an algebra; then we require sufficient postulates in order for $\mathcal{B}$ to be an algebra of the appropriate kind, e.g. a Boolean algebra or something weaker. For example, a (type-theoretic translation of a) set of postulates characterizing $\mathcal{B}$ as a Boolean algebra is

$$({}^{\mathcal{B}}x_\iota)({}^{\mathcal{B}}y_\iota)(\underline{j}^{(1)}_{\iota 3}\, x_\iota\, y_\iota = \underline{j}^{(1)}_{\iota 3}\, y_\iota\, x_\iota)$$

$$({}^{\mathcal{B}}x_\iota)({}^{\mathcal{B}}y_\iota)(\underline{m}^{(1)}_{\iota 3}\, x_\iota\, y_\iota = \underline{m}^{(1)}_{\iota 3}\, y_\iota\, x_\iota)$$

$$({}^{\mathcal{B}}x_\iota)({}^{\mathcal{B}}y_\iota)({}^{\mathcal{B}}z_\iota)(\underline{j}^{(1)}_{\iota 3}\, x_\iota (\underline{j}^{(1)}_{\iota 3}\, y_\iota\, z_\iota) = \underline{j}^{(1)}_{\iota 3} (\underline{j}^{(1)}_{\iota 3}\, x_\iota\, y_\iota)\, z_\iota)$$

$$({}^{\mathcal{B}}x_\iota)({}^{\mathcal{B}}y_\iota)({}^{\mathcal{B}}z_\iota)(\underline{m}^{(1)}_{\iota 3}\, x_\iota (\underline{m}^{(1)}_{\iota 3}\, y_\iota\, z_\iota) = \underline{m}^{(1)}_{\iota 3} (\underline{m}^{(1)}_{\iota 3}\, x_\iota\, y_\iota)\, z_\iota)$$

$$({}^{\mathcal{B}}x_\iota)({}^{\mathcal{B}}y_\iota)(\underline{j}^{(1)}_{\iota 3} (\underline{m}^{(1)}_{\iota 3}\, x_\iota\, y_\iota)\, y_\iota = y_\iota)$$

$$({}^{\mathcal{B}}x_\iota)({}^{\mathcal{B}}y_\iota)(\underline{m}^{(1)}_{\iota 3} (\underline{j}^{(1)}_{\iota 3}\, x_\iota\, y_\iota)\, y_\iota = y_\iota)$$

$$({}^{\mathcal{B}}x_\iota)({}^{\mathcal{B}}y_\iota)({}^{\mathcal{B}}z_\iota)(\underline{j}^{(1)}_{\iota 3}\, x_\iota (\underline{m}^{(1)}_{\iota 3}\, y_\iota\, z_\iota) = \underline{m}^{(1)}_{\iota 3} (\underline{j}^{(1)}_{\iota 3}\, x_\iota\, y_\iota)(\underline{j}^{(1)}_{\iota 3}\, x_\iota\, z_\iota))$$

$$({}^{\mathcal{B}}x_\iota)({}^{\mathcal{B}}y_\iota)(\underline{j}^{(1)}_{\iota 3} (\underline{m}^{(1)}_{\iota 3}\, x_\iota (\underline{c}^{(1)}_{\iota 2}\, x_\iota))\, y_\iota = y_\iota)$$

$$({}^{\mathcal{B}}x_\iota)({}^{\mathcal{B}}y_\iota)(\underline{m}^{(1)}_{\iota 3} (\underline{j}^{(1)}_{\iota 3}\, x_\iota (\underline{c}^{(1)}_{\iota 2}\, x_\iota))\, y_\iota = y_\iota) \quad .$$

Then $\mathcal{B}$ will be extended, to $\mathcal{B}^e$ say, by the addition of one or more further constants like $\underline{n}^{(1)}_{\iota 2}$ to match the intensional operators under consideration. Postulates like

$$({}^{\mathcal{B}}x_\iota)(\underline{m}^{(1)}_{\iota 3} (\underline{n}^{(1)}_{\iota 2}\, x_\iota)\, x_\iota = \underline{n}^{(1)}_{\iota 2}\, x_\iota)$$

(which in ordinary notation says that $\underset{\sim}{N}x \leqslant x$) will then be added to make $\mathcal{B}^e$ into a dual extension algebra, a closure algebra or whatever is required. This finishes the sketch of the definition of the algebra <u>per</u> <u>se</u>. The next step is to set up a homomorphism into the algebra, and the question is: where is the homomorphism <u>from</u>?

In the usual case, the homomorphism is from a formula algebra: so, in keeping with our attitude that the type-theoretic language may be used as an object language, we take the homomorphism from the formula algebra on wffs of type $o\kappa$. Such a homomorphism will be formalized by a function $f_{\iota(o\kappa)}$ with postulates

$$(x_{o\kappa})(B^{(1)}_{o\iota}(f_{\iota(o\kappa)}x_{o\kappa}))$$

$$(x_{o\kappa})(y_{o\kappa})(f_{\iota(o\kappa)}(x_{o\kappa} \vee y_{o\kappa}) = \underset{-}{\dot{j}}^{(1)}_{\iota^3}(f_{\iota(o\kappa)}x_{o\kappa})(f_{\iota(o\kappa)}y_{o\kappa}))$$

etc., and, supposing there is just one necessitation operator $\Box^{x}_{(o\kappa)^2}$ under consideration,

$$(x_{o\kappa})(f_{\iota(o\kappa)}(\Box^{x}_{(o\kappa)^2}x_{o\kappa}) = \underset{-}{n}^{(1)}_{\iota^2}(f_{\iota(o\kappa)}x_{o\kappa})) \quad .$$

By conjoining all the various postulates for an algebra and a homomorphism, we will be able to define a wff

$$Homo^{(1)}_{o(o\iota)(\iota^3)(\iota^3)(\iota^2)(\iota^2)(\iota(o\kappa))}f_{\iota(o\kappa)}\underset{-}{n}^{(1)}_{\iota^2}\underset{-}{c}^{(1)}_{\iota^2}\underset{-}{m}^{(1)}_{\iota^3}\underset{-}{\dot{j}}^{(1)}_{\iota^3}B^{(1)}_{o\iota}$$

to say that $f_{\iota(o\kappa)}$ is a homomorphism from the formula algebra on wffs of type $o\kappa$ to an algebra $\langle B^{(1)}, \underset{-}{\dot{j}}^{(1)}, \underset{-}{m}^{(1)}, \underset{-}{c}^{(1)}, \underset{-}{n}^{(1)}\rangle$ of the appropriate kind. Thence we will be able to define a wff

$$ATru^{(1)}_{o(o\iota)(\iota^3)(\iota^3)(\iota^2)(\iota^2)(\iota(o\kappa))(o\kappa)}x_{o\kappa}f_{\iota(o\kappa)}\underset{-}{n}^{(1)}_{\iota^2}\underset{-}{c}^{(1)}_{\iota^2}\underset{-}{m}^{(1)}_{\iota^3}\underset{-}{\dot{j}}^{(1)}_{\iota^3}B^{(1)}_{o\iota}$$

to say that the proposition $x_{o\kappa}$ is true in the algebra under the homomorphism. For example if the algebra is a Boolean algebra then the vital conjunct in the definiens of $ATru^{(1)}$ will be

$$(\exists^{B}x_{\iota})(f_{\iota(o\kappa)}x_{o\kappa} = \underset{-}{\dot{j}}^{(1)}_{\iota^3}x_{\iota}(\underset{-}{c}^{(1)}_{\iota^2}x_{\iota})) \quad ;$$

or else a filter $F^{(1)}_{o\iota}$ in the algebra will be characterized by the requirements

$$(x_\iota)(F^{(1)}_{o\iota}x_\iota \supset B^{(1)}_{o\iota}x_\iota)$$

$$(x_\iota)(y_\iota)(F^{(1)}_{o\iota}x_\iota \,\&\, F^{(1)}_{o\iota}y_\iota \supset F^{(1)}_{o\iota}(\underline{m}^{(1)}_{\iota^3}x_\iota y_\iota))$$

$$(x_\iota)({}^{B}y_\iota)(F^{(1)}_{o\iota}x_\iota \supset F^{(1)}_{o\iota}(\underline{j}^{(1)}_{\iota^3}x_\iota y_\iota)) \quad ,$$

and then the vital conjunct will be just $F^{(1)}_{o\iota}(f_{\iota(o\kappa)}x_{o\kappa})$.

Finally we will have a wff $AValid^{(1)}_{o(o\kappa)}x_{o\kappa}$

gained by universally quantifying $ATru^{(1)}$ over homomorphisms and algebras, to say that the $x_{o\kappa}$ is valid in the algebraic semantical system. We keep an index $^{(1)}$ on the wffs $ATru$ and $AValid$ since we may well want to use different definitions of truth and validity, and to compare the results thereof, within the one piece of algebraic semantics: for instance we might want to discuss the intersection of the wffs valid in two different kinds of algebras.

It will be seen that this account of algebraic semantics does not depend in any way on the type $o\kappa$ assigned to the propositions that are being assessed as true or valid, and no account is taken of the fact that a proposition can be true in quite a different sense, viz. according to $Tru_{o(o\kappa)}$ of §2.1. The first of these points has the consequence that the account can be generalized by replacing $o\kappa$ by any type α for which there is a formula algebra. This type α could be, say, π_k whence the formula algebra would be constructed with predicate modifier constants $N_{\pi_k^2}$ etc.; or it could be some primitive type, say φ, introduced as the type of purely syntactical entities. The second of the points made above is a partial consequence of the loss of worlds in algebraic semantics.

The sketch presented in this part follows a well-trodden path, and we have done no hard work towards its establishment. We have simply indicated the way in which algebras and homomorphisms can be defined, and how our propositions (or anything else) can be hooked up with the algebras and thence characterized as true or valid.

2.4 Individuals and individual concepts

The domain $\mathcal{D}_\iota$ of our semantics is a domain of individuals, with no variation from world to world. Each x_ι, or A_ι in general, denotes a member of this domain, and quantifications (x_ι) (or more precisely the operator $\Pi_{o(o\iota)}$) quantify "over" this domain. These individuals, we take it, are much the same as R. Thomason's substances, for which see Thomason and Stalnaker [68] and Thomason [69] and [70]. By analogy with the passage from truth-values to propositions, we might expect $\mathcal{D}_{\iota\kappa}$ to form a domain of intensional individuals, or individual concepts. That is, an individual concept is an indexed sequence of individuals in the same way that a proposition is an indexed sequence of truth-values: this usage of the term is very similar to that of Carnap [7] p.181, save that as usual one has to say "but Carnap's notion is relative to a language".

We now spell out a number of (what we take to be) good things that follow from our construal of $\mathcal{D}_\iota$ as a domain housing individuals and of $\mathcal{D}_{\iota\kappa}$ as a domain housing individual concepts, entities that until the 1960's and despite [7] would have been regarded as somewhat shadowy Doppelgänger of genuine individuals. The first of these good things is that we have variables over both individuals and individual concepts, just because we have variables of type ι and variables of type $\iota\kappa$. (Thus in a sense we have proper names for individual concepts, and some philosophers may object to this result). Having variables of the two different types is an advance over most systems of quantified intensional logic; such systems are usually tied to a one-sorted first-order quantificational basis, and divided into "necessary identity" systems with the individual variables in effect having type ι, and "contingent identity" systems with the individual variables having type $\iota\kappa$. This bifurcation appears, for example, in the presentation in Hughes and Cresswell [35] Ch. 11.

The second good thing is that no individual concept can be identical to an individual, since $x_{\iota\kappa} = y_\iota$ is ill-formed. What is well-formed is $x_{\iota\kappa} w_\kappa = y_\iota$, that is, roughly speaking, individual concepts take on substances as their values in each world. This simple syntactical fact has a number of philosophical consequences: for instance it obviates the necessity to make the kind of restrictions found in the last paragraph on p.362 of [68], and there is no question of an individual constant's being "more like 'Socrates' or 'Miss America'."

The third good thing is that we have perfectly good identity conditions for individual concepts viz. good old set-theoretic identity of functions. We can of course define contingent identity of individual concepts, by

$$Coincr_{o(\iota\kappa)^2} =_{df} \lambda x_{\iota\kappa} \lambda y_{\iota\kappa} (x_{\iota\kappa} g_\kappa = y_{\iota\kappa} g_\kappa)$$

(where $Coincr$ stands for "Coincidence at the real world" - later we will define a more general relation of coincidence). Obviously this contingent identity of individual concepts does not imply the full-scale identity of individual concepts that is already present in the system as $Q_{o(\iota\kappa)^2}$.

The fourth good thing is that we can define quite nicely Kripke's notions of "rigid designator" and "strongly rigid designator", as in [37] esp. pp.269 ff. We have firstly

$$Rigiddes_{o(\iota\kappa)} =_{df} \lambda x_{\iota\kappa} (\exists y_\iota)(w_\kappa)(x_{\iota\kappa} w_\kappa = y_\iota) ,$$

that is, a rigid designator is a designator (our type $\iota\kappa$) that takes on the same substance in each world. To define "strongly rigid designator" we need a constant

$$Dom_{o\iota\kappa}$$

such that for each world w_κ, $Dom_{o\iota\kappa} w_\kappa$ gives the substances <u>existing</u> in w_κ. Thence we have

$$Strongrigiddes_{o(\iota\kappa)} =_{df} \lambda x_{\iota\kappa} (\exists y_\iota)(w_\kappa)(Dom_{o\iota\kappa} w_\kappa y_\iota \;\&\; x_{\iota\kappa} w_\kappa = y_\iota) \;,$$

from which it follows that

$$Strongrigiddes_{o(\iota\kappa)} x_{\iota\kappa} \equiv . \; Rigiddes_{o(\iota\kappa)} x_{\iota\kappa} \;\&\; \Box_{(o\kappa)\kappa} (\lambda w_\kappa (Dom_{o\iota\kappa} w_\kappa (x_{\iota\kappa} w_\kappa))) g_\kappa \;,$$

that is, as Kripke says, a rigid designator which denotes a necessary existent is a strongly rigid designator.

Kripke says, on [37] p.270, "Those who have argued that to make sense of the notion of rigid designator, we must antecedently make sense of 'criteria of transworld identity' have precisely reversed the cart and the horse; it is because we can refer rigidly to Nixon, and stipulate that we are speaking of what might have happened to <u>him</u> (under certain circumstances), that 'transworld identifications' are unproblematic in such cases." From the point of view of our definition of $Rigiddes_{o(\iota\kappa)}$ the only identity that appears in the definiens is $Q_{o\iota\iota}$, that is identity which in no way depends on the intensionalizing apparatus introduced with the type κ . Thus this identity, whether or not it is called a 'transworld identity' (we take Kripke's quote marks to be those of disassociation), is indeed unproblematic. It may seem that there is some problem for us in accommodating names (or at least some of them) as rigid designators, since we might expect to symbolize a name like 'Nixon' as n_ι if we were proposing to use it as a rigid designator, and yet $Rigiddes_{o(\iota\kappa)}$ has type $o(\iota\kappa)$ and so cannot be predicated of n_ι , of type ι . The solution to this problem is simply to symbolize a name like 'Nixon' as $\lambda w_\kappa n_\iota$: this has type $\iota\kappa$, is always a rigid designator, but can be compared to other wffs of type $\iota\kappa$ which are not rigid designators.

2.5 Properties and descriptions

Despite our roster of good things about individual concepts in §2.4, the question still arises as to how adequately our individual concepts match the intuitive sense(s) of the notion: this question arises because individual concepts usually result from descriptive phrases, and we have not yet analyzed descriptive phrases (other than the particular case $\lambda w_\kappa n_\iota$) that gave rise to wffs of type $\iota\kappa$. For this analysis we need to consider how (intensional) properties are to be symbolized, since properties can be expected to appear in individual concept-forming descriptions.

Scott's advice in [64] p.156 is (in effect) to take all non-logical dyadic relations as having type $(o\kappa)(\iota\kappa)^2$, and so we could expect that in general he would advise taking an n-adic relation (-in-intension) as having type $(o\kappa)(\iota\kappa)^n$. Our advice is more ramified, but we hope the explicit typal indices will reduce its tortu(r)ous character: there are several different notions of properties, all can be fairly clearly exemplified, none is logically or philosophically primitive, and English grammar (e.g. the difference between verbs and predicative phrases formed from common nouns) gives us no guide as to the semantical kind of property involved. We advise that the appropriate notion be used in the appropriate place, and hope to give some guide to what we regard as "appropriate" in what follows.

For a start, let us take extensional properties of individuals, of type π_1 , and relativize to worlds to get type $\pi_1\kappa$ i.e. $o\iota\kappa$. We suggest that a property like "... is a Prime Minister of Australia (at world so-and-so)" is appropriately symbolized by a variable (temporarily a constant) such as $f_{\pi_1\kappa}$. For each world there is a subset of individuals who are Prime Ministers of Australia - given a world, it is then true or false of each individual whether he or she is a Prime Minister of Australia, and this latter truth or falsity does not vary (further) from world to world. From $f_{\pi_1\kappa}$ we can form the individual concept "the Prime Minister of Australia" by the definition

$$p_{\iota\kappa} =_{df} \lambda w_\kappa (\imath x_\iota)(f_{\pi_1\kappa} w_\kappa x_\iota) \qquad :$$

we take it that the form of this definition, in which the definiens does <u>not</u> begin with $(\imath x_{\iota\kappa})$, provides an illustration of how individual concepts may arise without starting out with proper names or variables for them.

Now let us suppose that Mr. Whitlam is an individual, and denote him by w_ι. We cannot say "The Prime Minister of Australia is Mr. Whitlam" in the form $p_{\iota\kappa} = w_\iota$, but instead have to take account of the unhappy fact of politics that the "is" has some kind of temporal element (at least: it also has an actualizing element, pinning the statement down to the real world and not, say, the fanciful world of the Leader of the Opposition). Most crudely, we can use n_κ for the present world, whence

$$p_{\iota\kappa} n_\kappa = w_\iota$$

is both well-formed and true. By λ-conversion, this is equivalent to $(\imath x_\iota) f_{\pi_1\kappa} n_\kappa x_\iota = w_\iota$, and then from this and the uniqueness of Prime Ministers of Australia, it follows that $f_{\pi_1\kappa} n_\kappa w_\iota$ holds. Slightly less crudely, we can abstract from the temporal (etc.) element of the "is", and form the proposition

$$\lambda x_\kappa (p_{\iota\kappa} x_\kappa = w_\iota)$$

as our symbolization of "The Prime Minister of Australia is Mr. Whitlam".

On the other hand, and partly contrary to Kripke's notion of names as rigid designators, we might suppose that "Mr. Whitlam" denotes an individual concept, since, say, in the political world of Canberra in 1930 the name would have denoted Mr. H.F.E. Whitlam and not Mr. E.G. Whitlam. If we use $w_{\iota\kappa}$ for this individual concept, and t_κ for the world of 1930, then we can (try to) say "The Prime Minister of Australia is Mr. Whitlam" in the form

$$p_{\iota\kappa} = w_{\iota\kappa} \qquad ,$$

but this is false since if functions are equal they have equal values for all arguments, and even though

$$p_{\iota\kappa} n_\kappa = w_{\iota\kappa} n_\kappa$$

is true,

$$p_{\iota\kappa} t_\kappa = w_{\iota\kappa} t_\kappa$$

is false since Mr. H.F.E. Whitlam was not Prime Minister in 1930.

So far, it seems that our example has shown that we can fruitfully and naturally use wffs of type $\pi_1 \kappa$ for properties. But there are also several other possibilities: for instance we might consider the type $(o\kappa)\iota$, mapping individuals to propositions rather than truth-values as does π_1 . Since set-theoretically we have

$$(A^B)^C \sim (A^C)^B \quad \text{(as well as } (A^B)^C \sim A^{B\times C} \text{)},$$

there is no set-theoretical difference between the type $o\iota\kappa$ and the type $o\kappa\iota$: but in certain cases the latter type will arise quite naturally and may be used when it does so. In particular it arises in quantificational contexts, as we will see in §2.7.

The next type we might consider is $o(\iota\kappa)$, or $o(\iota\kappa)^n$ in general. For a start, this type certainly arises formally, since $Rigiddes_{o(\iota\kappa)}$ and $Strongrigiddes_{o(\iota\kappa)}$ both have type $o(\iota\kappa)$. Also identity of individual concepts has this type (as is shown, e.g. if we unabbreviate $(x_{\iota\kappa} = y_{\iota\kappa})$ one stage to $(Q_{o(\iota\kappa)^2}\, x_{\iota\kappa}\, y_{\iota\kappa})$), and so does contingent identity of individual concepts as given by $Coincr_{o(\iota\kappa)^2}$ of §2.4. Less formally, we might argue that the property of being head of the Australian Government is a property of the Prime Minister of Australia qua individual concept, so that

$$h_{o(\iota\kappa)}$$

would be an appropriate symbol for this property, and

$$h_{o(\iota\kappa)}\, p_{\iota\kappa}$$

would be true. It is no argument against the use of type $o(\iota\kappa)$ that in some cases (not all cases, by simple cardinality considerations) a property of type $o(\iota\kappa)$ may be defined in terms of one of type π,κ. For example, if $h_{\pi,\kappa}$ were used instead of $h_{o(\iota\kappa)}$, for the property of being head of the Australian Government, by parity with our previous use of $f_{\pi,\kappa}$ for the property of being a Prime Minister of Australia, then we could define $h_{o(\iota\kappa)}$ by

$$h_{o(\iota\kappa)} =_{df} \lambda x_{\iota\kappa}\,(x_{\iota\kappa} = \lambda w_\kappa\,(\imath x_\iota)\, h_{\pi,\kappa}\, w_\kappa\, x_\iota) \quad .$$

Then $h_{o(\iota\kappa)}\, p_{\iota\kappa}$ follows from $(w_\kappa)(x_\iota)(f_{\pi,\kappa}\, w_\kappa\, x_\iota \supset h_{\pi,\kappa}\, w_\kappa\, x_\iota)$,

which would, I think, have been our original grounds for taking the property of being head to be a property of the individual concept of Prime Minister rather than of individuals. But all of this is just to track down a relationship between different possible formalizations, and is not to show that one is superior or logically prior to the other.

Scott's recommended type $(o\kappa)(\iota\kappa)^n$ also occurs formally. Coincidence of individual concepts, defined by

$$Coinc_{(o\kappa)(\iota\kappa)^2} =_{df} \lambda x_{\iota\kappa} \lambda y_{\iota\kappa} \lambda w_\kappa (x_{\iota\kappa} w_\kappa = y_{\iota\kappa} w_\kappa)$$

has such a type. From this general form of coincidence we can derive $Coincr_{o(\iota\kappa)^2}$, contingent identity of individual concepts, by the equality

$$Coincr_{o(\iota\kappa)^2} = \lambda x_{\iota\kappa} \lambda y_{\iota\kappa} (Coinc_{(o\kappa)(\iota\kappa)^2} x_{\iota\kappa} y_{\iota\kappa} g_\kappa) \quad .$$

(The distinction between $Coincr_{o(\iota\kappa)^2}$ and $Coinc_{(o\kappa)(\iota\kappa)^2}$ (<u>or</u> identity $Q_{o(\iota\kappa)^2}$) does not quite match Scott's distinction, [64] p.157, between = and ≡ since for Scott both = and ≡ have type $(o\kappa)(\iota\kappa)^2$.) We can also find informal contexts where type $(o\kappa)(\iota\kappa)$ might occur: for instance when we argued that the property of being head of the Australian Government was a property of the individual concept of Prime Minister of Australia, we did not take into account possible worlds where the individual concept of being Prime Minister still exists, defined say as head of Cabinet, but where some other official is head of the Government. If we admit such a possibility, then the property of being head of the Government would have type $(o\kappa)(\iota\kappa)$.

According to the semantics of Lewis [44], common nouns in effect take on properties of type π,κ while verb phrases take on properties of type $(o\kappa)(\iota\kappa)$. As our discussion has indicated, we find no special reason for this particular distinction: the different ways in which types of properties can arise seems not to be dependent upon accidents of English syntax. For instance we see no reason why the same considerations as apply to assigning type π,κ to "is a Prime Minister of Australia" should not apply equally to "is red" or "is a pig" or "grunts" (here we have been a bit cavalier about the role of "is" and "a" in Lewis's semantics). It is for this reason that we advised that English grammar gives no guide as to the (semantical) kind of property that should be chosen in a given context.

Finally, in this section we remark on the possibility of "mixed" properties for relations of adicity of two or more. Thus types like $o\iota(\iota\kappa)\iota$ and $(o\kappa)(\iota\kappa)^2\iota^3$ are possible in our formalism but not in a more restricted formalism geared to an ordinary first-order basis. Such properties may well be used in practice: e.g. in "Mr. Whitlam rebuked the least popular member of his Cabinet", the type assigned to the relation could be $o\iota(\iota\kappa)$, since, given the ingenuity and perseverance applied to achieving unpopularity, there would be dangers in assuming that the rebuke was directed to any particular individual.

2.6 Some philosophical applications

In this part we apply our formalized distinctions, between individuals and individual concepts and between their respective properties, to some specific philosophical problems. The first of these problems derives from a thesis of Geach's, as in [23] §§ 91, 109, that sentences of the form " x is the same F as y" are not to be analyzed as 'There is a z such that z is F and both x and y are identical to z': using terminology of N. Griffin's (and, later, borrowing examples from him) we say that the analysandum contains an assertion of relative identity, while the analysans contains assertions of absolute identity. Geach's thesis, or at least part of it, is that relative identities cannot be analyzed out so that only absolute identities remain. A further part of Geach's thesis is that relative identity is genuinely relative, i.e. that it is possible that for some x, y, F and G, x is the same F as y but x is not the same G as y.

In support of the latter claim we might encounter examples such as

(1) The Chancellor of the ANU is the same man as the Chairman of the Council for the Arts,

but

(2) The Chancellor of the ANU is not the same official as the Chairman of the Council for the Arts,

wherein "man" and "official" seem to provide a suitable F and G in the schematic form of Geach's thesis. However, if we introduce the following dictionary, we can sustain the rejected analysis of relative identities in terms of absolute identities in this case:

$c_{o\iota\kappa}$	the property of being a Chancellor of the ANU
$a_{o\iota\kappa}$	the property of being a Chairman of the Council for the Arts
$m_{o\iota\kappa}$	the property of being a man
$f_{o(\iota\kappa)\kappa}$	the property, of individual concepts, of being an official.

Then (1) can be symbolized as

$$(3)\quad (\exists x_\iota)(m_{o\iota\kappa} g_\kappa x_\iota \,\&\, (\imath y_\iota)(c_{o\iota\kappa} g_\kappa y_\iota) = x_\iota \,\&\, (\imath y_\iota)(a_{o\iota\kappa} g_\kappa y_\iota) = x_\iota)\ ,$$

a slightly complicated form of the rejected analysans, while (2) may be symbolized as

$$(4)\quad f_{o(\iota\kappa)\kappa} g_\kappa (\lambda w_\kappa (\imath y_\iota)(c_{o\iota\kappa} w_\kappa y_\iota)) \,\&\, f_{o(\iota\kappa)\kappa} g_\kappa (\lambda w_\kappa (\imath y_\iota)(a_{o\iota\kappa} w_\kappa y_\iota)) \,\&$$
$$\sim(\lambda w_\kappa (\imath y_\iota)(c_{o\iota\kappa} w_\kappa y_\iota) = \lambda w_\kappa (\imath y_\iota)(a_{o\iota\kappa} w_\kappa y_\iota))\ .$$

No dreadful consequences ensue from (3) and (4), contrary to what would happen if we used the simple (-minded) first-order dictionary

Cx	x is a Chancellor of the ANU
Ax	x is a Chairman of the Council for the Arts

Mx x is a man

Fx x is an official

and symbolizations

(5) $(\exists x)(Mx \mathbin{\&} (\imath y)Cy = x \mathbin{\&} (\imath y)Ay = x)$

(6) $F(\imath y)Cy \mathbin{\&} F(\imath y)Ay \mathbin{\&} (\imath y)Cy \neq (\imath y)Ay$:

outright contradiction ensues from (5) and (6).

The basic point of our symbolization of (1) and (2) as (3) and (4), and of the dictionary involved, is that being an official, $f_{o(\iota\kappa)\kappa}$, is not a property of individuals but rather a property of individual concepts. We then took (1), in effect, to be saying that the individual concepts of The Chancellor and The Chairman coincide at the real world, and that their common value is a man. By contrast we took (2) as saying that both individual concepts are not identical; from this it does not follow that they may not be "contingently identical" i.e. coincide at the real world. As an objection to this analysis, it may be argued that it is incorrect to allocate different types to "man" and "official", since propositions like

(7) The man in fancy dress is an official

and

(8) Not all officials are men

seem to be significant and to require that "man" and "official" have the same type. However, adding

$d_{o\iota\kappa}$ the property of being in fancy dress

to our dictionary, we can symbolize (7) and (8) as

(9) $(\exists y_{\iota\kappa})(f_{o(\iota\kappa)\kappa}\, g_\kappa\, y_{\iota\kappa} \mathbin{\&} y_{\iota\kappa}\, g_\kappa = (\imath x_\iota)(m_{o\iota\kappa}\, g_\kappa\, x_\iota \mathbin{\&} d_{o\iota\kappa}\, g_\kappa\, x_\iota))$

and

(10) $\sim(y_{\iota\kappa})(w_{\kappa})(f_{o(\iota\kappa)\kappa}\, w_{\kappa}\, y_{\iota\kappa} \supset m_{o\iota\kappa}\, w_{\kappa}\, (y_{\iota\kappa}\, w_{\kappa}))$.

Thus (7) and (8) do not require that we have an additional property of type $o\iota\kappa$, for being an official, in addition to $f_{o(\iota\kappa)\kappa}$; our analysis of (7) and (8) is somewhat more complex than a first-order quantificational analysis, but this is to be expected.

We turn now to Geach's example in [23] § 91; Geach argues that

(11) Heraclitus bathed in some river yesterday, and bathed in the same river today

and

(12) Heraclitus bathed in some water yesterday, and bathed in the same water today

cannot be given the rejected analysis, since the additional premise

(13) Whatever is a river is water

would then validate the argument from (11) to (12), and this argument is not valid. Our analysis of these propositions is motivated by the awkwardness of (13): it is not implausible that "river" and "water" should be of different types, since (13) does not readily paraphrase as "All rivers are water". If we take some (reasonably small) lumps of water as individuals - not too gross a move if one considers the meteorologists' treatment of (smallish) bubbles of air as individuals - then certain subsets of lumps of water, and perhaps other things e.g. rocks and sand, will be rivers, at any given time. Also, we need to treat "bathing in a river" as an ellipsis for "bathing in some water of a river", again not too gross a move if we consider the differences between bathing in some water and bathing in some river. So we can set up our dictionary

h_ι	Heraclitus
$w_{o\iota\kappa}$	the property of being water
$r_{o(o\iota\kappa)\kappa}$	the property, of time-dependent (or world-dependent) subsets of individuals, of being a river
y_κ	(the world of) yesterday
t_κ	(the world of) today
$b_{o\iota\iota\kappa}$	the relation (in intension) of bathing in

and them symbolize (11), (12) and (13) as

(14) $$(\exists v_{o\iota\kappa})(r_{o(o\iota\kappa)\kappa}\, y_\kappa\, v_{o\iota\kappa} \;\&\; r_{o(o\iota\kappa)\kappa}\, t_\kappa\, v_{o\iota\kappa} \;\&\; (\exists x_\iota)(v_{o\iota\kappa}\, y_\kappa\, x_\iota \;\&\; w_{o\iota\kappa}\, y_\kappa\, x_\iota \;\&\; b_{o\iota\iota\kappa}\, y_\kappa\, h_\iota\, x_\iota) \;\&\; (\exists x_\iota)(v_{o\iota\kappa}\, t_\kappa\, x_\iota \;\&\; w_{o\iota\kappa}\, t_\kappa\, x_\iota \;\&\; b_{o\iota\iota\kappa}\, t_\kappa\, h_\iota\, x_\iota)),$$

(15) $$(\exists x_\iota)(w_{o\iota\kappa}\, y_\kappa\, x_\iota \;\&\; b_{o\iota\iota\kappa}\, y_\kappa\, h_\iota\, x_\iota \;\&\; w_{o\iota\kappa}\, t_\kappa\, x_\iota \;\&\; b_{o\iota\iota\kappa}\, t_\kappa\, h_\iota\, x_\iota)$$

and

(16) $$(v_{o\iota\kappa})(z_\kappa)(r_{o(o\iota\kappa)\kappa}\, z_\kappa\, v_{o\iota\kappa} \supset (x_\iota)(v_{o\iota\kappa}\, z_\kappa\, x_\iota \supset w_{o\iota\kappa}\, z_\kappa\, x_\iota))$$

respectively. Even though the rejected analysis is used in (14) and (15), it is clear that (14) and (16) do not entail (15), and that only a fallacious move of the form $(\exists x)Fx \;\&\; (\exists x)Gx \;\therefore\; (\exists x)(Fx \;\&\; Gx)$ would allow a derivation of (15) from (14) and (16). A set-theoretic model with just two individual lumps of water, as well as Heraclitus, can easily be constructed to show formally that (14) and (16) do not entail (15). Moreover, there is still no entailment if a simpler analysis is used in which $w_{o\iota\kappa}$ and $r_{o(o\iota\kappa)\kappa}$ are replaced by $w_{o\iota}$ and $r_{o(o\iota\kappa)}$, on the assumption that waterhood and riverhood are time- (or world-) invariant properties.

Finally, in our response to the notion of relative identity, we deal with Griffin's example in which Cleopatra's Needle wears out and is replaced by another block of stone. Then we have

(17) The new block of stone is not the same block of stone as the old block of stone

but nevertheless

(18) The new block of stone is the same landmark (viz. Cleopatra's Needle) as the old block of stone.

Our analysis of this situation uses the dictionary

a_ι	the old block
b_ι	the new block
$s_{o\iota\kappa}$	the property of being a block of stone
$c_{o\iota\kappa}$	the property of being a block of stone, of a certain shape etc., in the place of Cleopatra's Needle
$\ell_{o(\iota\kappa)\kappa}$	the property, of individual concepts, of being a landmark
t_κ	then (when the old block was there)
n_κ	now

and the symbolizations, again with the analysis rejected by Geach,

(19) $s_{o\iota\kappa}\, t_\kappa\, a_\iota \;\&\; s_{o\iota\kappa}\, n_\kappa\, b_\iota \;\&\; a_\iota \neq b_\iota$

(20) $(\exists y_{\iota\kappa})(\ell_{o(\iota\kappa)\kappa}\, n_\kappa\, y_{\iota\kappa} \;\&\; y_{\iota\kappa} t_\kappa = a_\iota \;\&\; y_{\iota\kappa} n_\kappa = b_\iota \;\&\; y_{\iota\kappa} = \lambda w_\kappa (\imath x_\iota)\, c_{o\iota\kappa}\, w_\kappa\, x_\iota \;)$,

in which in (20) the last conjunct in the scope of $(\exists y_{\iota\kappa})$ is added to capture the parenthetical portion of (18). In these various examples we do not claim to have exhibited a general method for dealing with all cases of "relative identity", but we do claim to have shown that a formal distinction between individuals and individual concepts, and properties thereof, can be used to deal adequately with a typical range of cases.

The formal distinction can also be used to resolve such puzzles as the following argument, the invalidity of which seems to provide a counterexample to the indiscernibility of identicals:

(21) The President of the U.S.A. is elected every four years

(22) R.M. Nixon is the President of the U.S.A.

$\therefore$

(23) R.M. Nixon is elected every four years .

The appropriate symbols are

$p_{o\iota\kappa}$	the property of being President of the U.S.A.
td_{ι}	R.M. Nixon
n_{κ}	now
$f_{o(\iota\kappa)\kappa}$	the property, of individual concepts, of being elected every four years

whence the premises of the argument become

(24) $f_{o(\iota\kappa)\kappa}\, n_{\kappa}\, (\lambda w_{\kappa} (\imath x_{\iota}) p_{o\iota\kappa}\, w_{\kappa}\, x_{\iota})$

(25) $td_{\iota} = (\imath x_{\iota}) p_{o\iota\kappa}\, n_{\kappa}\, x_{\iota}$

and the conclusion is not even well-formed. Of course we could make it well-formed by using the rigid designator $\lambda w_{\kappa}\, td_{\iota}$, and writing

$\therefore$

(26) $f_{o(\iota\kappa)\kappa}\, n_{\kappa}\, (\lambda w_{\kappa}\, td_{\iota})$.

But even so, there is no valid inference from (24) and (25) to (26): there *is* a valid inference from (25) and (26) to

(27) $f_{o(\iota\kappa)\kappa}\, n_\kappa\, (\lambda w_\kappa (\imath x_\iota) p_{o\iota\kappa}\, n_\kappa\, x_\iota)$

or from (25) and (27) to (26), but clearly (24) and (27) make quite different assertions.

In order to argue for this particular analysis of the problem, we have to give reasons for saying that the property of being elected every four years, as it occurs in (21), is a property of individual concepts rather than individuals. Such reasons are not difficult to find: although (21) might depend upon the American Constitution, it does not depend upon empirical facts to the extent that (22) does; and (21) does not permit addition of a relative clause to yield

(28) The President of the U.S.A., whoever he may be from time to time, is elected every four years,

whereas something like

(29) The President of the U.S.A. is the most powerful man on earth, and will continue to be so for the next 20 years

does permit such a relative clause, and does more clearly predicate something of (a number of different) individuals rather than individual concepts. At least we can say that the difference between (21) and (29) shows that some constructions in English are ambiguous as to whether they assign properties to individuals or individual concepts, and if there is such an ambiguity we are entitled to resolve it in the way we chose in formulae (24) - (27).

2.7 Quantifiers and existence

The quantifiers $\Pi_{o(o\iota)}$ and $\Pi_{o(o(\iota\kappa))}$, "over" individuals and individual concepts respectively, are already part of the type-theoretical basis of our system and require no further explanation beyond their standard semantics. We have used them freely up to date, and have found no need to use any different kinds of quantification for individuals and individual concepts. However it is clear that other kinds of quantifiers have been used in quantified intensional systems; indeed the difference between systems often amounts precisely to different rules for evaluating quantified formula. In particular we need to find proposition-valued quantifiers in order for type-theoretic versions of wffs like $\Box(x)Fx$ to be well-formed, and for us to be able to deal with some of the classical problems of quantified modal logic such as the validity or otherwise of the Barcan formulae.

In most treatments of quantified modal logic, the domain of existing individuals varies from world to world: the constant $Dom_{o\iota\kappa}$ introduced in §2.4 matches this feature, and in effect provides a property of existence. This constant figures in our first definition of a proposition-forming quantifier thus:

$$\Lambda_{(o\kappa)(o\iota)} =_{df} \lambda x_{o\iota} \lambda w_\kappa (y_\iota)(Dom_{o\iota\kappa} w_\kappa y_\iota \supset x_{o\iota} y_\iota) \quad .$$

The abbreviation

$$(\Lambda x_\iota)A_o \qquad for \qquad \Lambda_{(o\kappa)(o\iota)}(\lambda x_\iota A_o)$$

gives us a normal-looking variable-binding quantification (Λx_ι) , and

$$(Vx_\iota)A_o \qquad for \qquad (\sim(\Lambda x_\iota)(\sim A_o))$$

gives us a corresponding existential quantification.

The quantifier $\Lambda_{(o\kappa)(o\iota)}$ is something of a hybrid insofar as it forms propositions from (extensional) subsets of individuals, of type $o\iota$; as such, it would not be found in any guise in an ordinary quantified modal logic. Two quantifiers that are not hybrid in this way are

$$\Lambda_{(o\kappa)(o\iota\kappa)} =_{df} \lambda f_{o\iota\kappa} \lambda w_\kappa (y_\iota)(Dom_{o\iota\kappa} w_\kappa y_\iota \supset f_{o\iota\kappa} w_\kappa y_\iota)$$

and

$$\Lambda_{(o\kappa)(o\kappa\iota)} =_{df} \lambda f_{o\kappa\iota} \lambda w_\kappa (y_\iota)(Dom_{o\iota\kappa} w_\kappa y_\iota \supset f_{o\kappa\iota} y_\iota w_\kappa) \ .$$

Using the latter, we have the abbreviation

$$(\Lambda^p x_\iota) A_{o\kappa} \quad \text{for} \quad \Lambda_{(o\kappa)(o\kappa\iota)} (\lambda x_\iota A_{o\kappa})$$

with V^p the usual dual of Λ^p. Insofar as we used $\Lambda_{(o\kappa)(o\kappa\iota)}$ rather than $\Lambda_{(o\kappa)(o\iota\kappa)}$, we are gaining some benefit from our policy of not being dogmatic about what we count as properties: in quantificational contexts we do tend to think firstly of the individual then of a proposition formed from the individual, and this is reflected formally in the type symbol $o\kappa\iota$ as compared to $o\iota\kappa$.

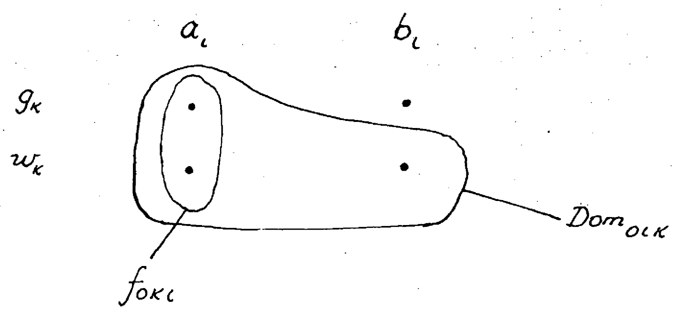
Our quantifier Λ^p is somewhat akin to Scott's $\forall\!\!\!/$, but we take over neither the infelicitous notation nor the suggestion that the quantifier itself incorporates "an indefinite index" ([64] p.149). The various quantifiers Λ_α depend upon the <u>function</u> $Dom_{o\iota\kappa}$, but, just because they are proposition-forming, they do <u>not</u> depend upon worlds w_κ: there is not a separate quantifier for each world, as the remarks in [64] might suggest.

The simple syntactical fact that $(\Lambda^p x_\iota) \Box_{(o\kappa)^2} (f_{o\kappa\iota} x_\iota)$ is well-formed shows that, contrary to a number of philosophical assertions, there is no reason to suppose that quantification into intensional contexts requires quantification over individual concepts: the quantification $(\Lambda^p x_\iota)$ is proposition-forming, but x_ι is not an individual concept. The Barcan formula may be written as

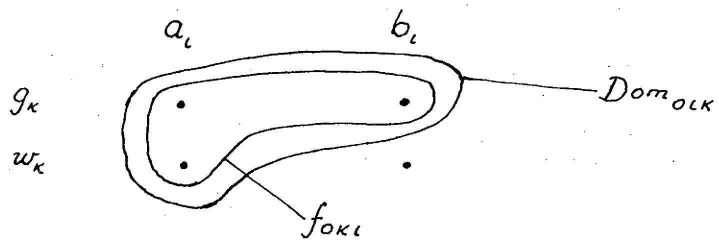
$$((\Lambda^p x_\iota) \Box_{(o\kappa)^2} (f_{o\kappa\iota} x_\iota) \supset \Box_{(o\kappa)^2} (\Lambda^p x_\iota)(f_{o\kappa\iota} x_\iota))$$

(with $\supset$ as a propositional operator, so that the formula has type $o\kappa$), or we can tack g_κ on the end to make it have type o. Similarly for the converse of the Barcan formula. The following is a picture of our version of the well-known

Kripke countermodel to the Barcan formula:



and similarly the following is a countermodel to the converse:



(wherein $f_{o\kappa\iota}$ is in effect the property of existence).

Just as much as quantification into intensional contexts does not necessitate quantification over individual concepts, so quantification over individual concepts need not be proposition-forming: we have already noted that the system contains $\Pi_{o(o(\iota\kappa))}$, a quantifier over individual concepts which is not proposition-forming. But we can also define quantifiers such as

$$\Xi_{(o\kappa)(o\kappa(\iota\kappa))} =_{df} \lambda f_{o\kappa(\iota\kappa)} \lambda w_\kappa (y_{\iota\kappa})(f_{o\kappa(\iota\kappa)} y_{\iota\kappa} w_\kappa)$$

and the "artificial"

$$\Xi_{(o\kappa)(o(\iota\kappa))} =_{df} \lambda f_{o(\iota\kappa)} \lambda w_\kappa (y_{\iota\kappa})(f_{o(\iota\kappa)} y_{\iota\kappa})$$

which are proposition-forming. We have the attendant abbreviations

$(Ax_{\iota\kappa})A_o$ for $\Xi_{(o\kappa)(o(\iota\kappa))}(\lambda x_{\iota\kappa} A_o)$

and

$(A^P x_{\iota\kappa})A_{o\kappa}$ for $\Xi_{(o\kappa)(o\kappa(\iota\kappa))}(\lambda x_{\iota\kappa} A_{o\kappa})$,

and E and E^P for the duals of A and A^P .

Because the definition of $\Xi_{(o\kappa)(o\kappa(\iota\kappa))}$ has an unrestricted quantification $(y_{\iota\kappa})$, over <u>all</u> individual concepts, it follows that the Barcan formula formed with A^p, viz.

$$((A^p x_{\iota\kappa})\, \Box_{(o\kappa)^2}\, (f_{o\kappa(\iota\kappa)} x_{\iota\kappa}) \supset \Box_{(o\kappa)^2}\, (A^p x_{\iota\kappa})(f_{o\kappa(\iota\kappa)} x_{\iota\kappa}))\, g_\kappa \quad ,$$

and its converse, are both valid. This provides a formal reason for considering a constant $Dom_{o(\iota\kappa)\kappa}$

to pick out a domain $Dom_{o(\iota\kappa)\kappa}\, w_\kappa$ of individual <u>concepts</u> for each world w_κ. In §2.8 we will find that some of Hintikka's ideas provide philosophical motivation for introducing such a constant; we can also provide some examples of a more homely nature. Consider, for example, the individual concept of the Dictator of Australia: despite some quondam Hawke-inspired fears, no such individual exists, even though the individual concept does, in the present world. But in a world where Australia did not exist, or a world with radically different political organization making it impossible even to conceive of dictators, it may well be argued that the individual concept does not exist. Thus we have motivation for introducing the quantifiers

$$\Upsilon_{(o\kappa)(o\kappa(\iota\kappa))} =_{df} \lambda f_{o\kappa(\iota\kappa)}\, \lambda w_\kappa\, (y_{\iota\kappa})\, (Dom_{o(\iota\kappa)\kappa}\, w_\kappa\, y_{\iota\kappa} \supset f_{o\kappa(\iota\kappa)}\, y_{\iota\kappa}\, w_\kappa)$$

and

$$\Upsilon_{(o\kappa)(o(\iota\kappa))} =_{df} \lambda f_{o(\iota\kappa)}\, \lambda w_\kappa\, (y_{\iota\kappa})\, (Dom_{o(\iota\kappa)\kappa}\, w_\kappa\, y_{\iota\kappa} \supset f_{o(\iota\kappa)}\, y_{\iota\kappa}) \quad ,$$

and thence the abbreviations

$$(\Lambda x_{\iota\kappa})A_o \quad \text{for} \quad \Upsilon_{(o\kappa)(o(\iota\kappa))}\, (\lambda x_{\iota\kappa} A_o)$$

and

$$(\Lambda^p x_{\iota\kappa})A_{o\kappa} \quad \text{for} \quad \Upsilon_{(o\kappa)(o\kappa(\iota\kappa))}\, (\lambda x_{\iota\kappa} A_{o\kappa}) \quad .$$

All of our various quantifiers have dyadic, and more generally polyadic, generalizations. Some of these generalizations may be defined in terms of the existing quantifiers, but some may not. For example the extensional dyadic quantifier $M_{o(o\iota)(o\iota)}$, where

$M_{o(o\iota)(o\iota)} f_{o\iota} g_{o\iota}$ is construed as saying that most of the x_ι's such that $f_{o\iota} x_\iota$ are also such that $g_{o\iota} x_\iota$, cannot be defined in terms of existing quantifiers but must be introduced as a primitive constant with a special evaluation rule in the semantics.

Several of our quantifiers may themselves reasonably be intensionalized. For example corresponding to $\Lambda_{(o\kappa)(o\kappa\iota)}$ we might have the intensionalized quantifier

$$\Lambda_{(o\kappa)(o\kappa\iota)\kappa} =_{df} \lambda x_\kappa \lambda f_{o\kappa\iota} \lambda w_\kappa (y_\iota)(Dom_{o\iota\kappa} x_\kappa y_\iota \supset f_{o\kappa\iota} y_\iota w_\kappa) \ .$$

In this quantifier, the individuals counted as existing for the purposes of quantification are those that exist in a world that may be different from the world where the proposition is being evaluated. A proposition in which such a quantifier could be used is "Bill remembers everyone of 1930", for which a symbolization is

$$\Lambda_{(o\kappa)(o\kappa\iota)\kappa} t_\kappa (\lambda x_\iota \lambda w_\kappa (rem_{o\iota\iota\kappa} w_\kappa b_\iota x_\iota)),$$

assuming D_ι to house only people and using an obvious dictionary. Such an intensionalized quantifier does indeed "incorporate an indefinite index", and an abbreviation such as

$$(\Lambda^{p}_{w_\kappa} x_\iota) A_{o\kappa} \quad \text{for} \quad \Lambda_{(o\kappa)(o\kappa\iota)\kappa} w_\kappa (\lambda x_\iota A_{o\kappa})$$

would be appropriate, if messy.

Given this plethora of different quantifiers, what is our advice concerning their use? As Scott notes, [64] p.161, there are good philosophical reasons for treating <u>many</u> modal operators in the one system: for specific examples see "applications" in [56], and Lakoff [38] §§ VII, VIII. Analogously, I suggest that there are good reasons for treating systems with <u>many</u> different quantifiers, ranging over individuals or individual concepts, forming truth-values or propositions, possibly depending on a property of existence, possibly having more than one argument, and possibly being themselves intensionalized. We have shown that such quantifiers are (mostly but not always) definable, and have philosophical motivation: it remains to work out the details of their properties and interconnections. In particular it will then be seen that there is often no precise basis for philosophical speculations about quantifying "into", "over", "through", "across" (or some other misleading-preposition-wise) intensional contexts. Throughout the rest of our work we will of course assiduously ignore the advice just ladled out: odd forms of quantification will be treated as charitably as a leprous gorilla at a wake, and nice easy things like $\Pi_{o(o\iota)}$ and $\Pi_{o(o(\iota\kappa))}$ will invariably be used.

2.8 Hintikka on individuating functions

In a series of papers, e.g. [30], [31], [33], and [34], Hintikka has expounded a type of semantics for quantified modal logic different from any so far formulated in our notation. Motivated specifically by Quine's objections to quantified modal logic, in some ways it is a cross between necessary identity and contingent identity systems. The exposition and formulation (and criticism) that follows is in some ways tentative, since unfortunately Hintikka does not formulate his semantics in a set-theoretically precise manner and so some of his prescriptions are open to misunderstanding (cf. e.g. Kaplan [36] fn. 27 p.137).

Hintikka begins by assuming, contrary say to Kripke, that there _is_ a problem about "cross-world identification of individuals": he says that merely to assume that the same individuals can crop up in different worlds is just to wish the problem away, not to solve it. His solution to the problem is to postulate the existence of what we might call a canonical set of individual concepts, relative to each world; this may be symbolized by a constant

$$Indiv_{o(\iota\kappa)\kappa}\ ,$$

where for each w_κ, the set $Indiv_{o(\iota\kappa)\kappa}w_\kappa$ is the set of individuating functions, in Hintikka's terminology, for w_κ. Since $Indiv_{o(\iota\kappa)\kappa}$ has the same type as $Dom_{o(\iota\kappa)\kappa}$, the question immediately arises as to the differences, if any, between these two constants.

There are two differences between $Indiv_{o(\iota\kappa)\kappa}$ and $Dom_{o(\iota\kappa)\kappa}$, one fairly small formal difference and one major difference in the way they are used in evaluating identity of individuals and quantified formulae. On the formal difference, Hintikka says "... we must often require that, given $f_1, f_2 \in \mathbf{F}$, if $f_1(\mu)=f_2(\mu)$ then $f_1(\lambda)=f_2(\lambda)$ for all alternatives λ to μ. In other words, an individual cannot 'split' when we move from a world to its alternatives." ([31] p.100). This requirement is given in a context where basically only one modal operator ("a believes that...") is being considered, but in general it seems that "all alternatives" means "all worlds accessible by any chain of alternativeness relations (not necessarily all the same relation)." Thus to formalize Hintikka's requirement of non-splitting we need to formalize the generalized ancestral relation involved in our construal of "all alternatives" - it is not enough simply to form the relative products of the ordinary ancestrals of all the relations involved.

To do this we firstly define the relative product, among alternativeness relations, by

$$Stroke_{(o\kappa^2)^3} =_{df} \lambda f_{o\kappa^2} \lambda g_{o\kappa^2} \lambda w_\kappa \lambda v_\kappa (\exists u_\kappa)(f_{o\kappa^2} w_\kappa u_\kappa \,\&\, g_{o\kappa^2} u_\kappa v_\kappa)\ ,$$

and we assume that a class $Altrel_{o(o\kappa^2)}$ of alternativeness relations is given (usually by enumeration , e.g.

$$Altrel_{o(o\kappa^2)} = \lambda f_{o\kappa^2}(f_{o\kappa^2} = S^{(1)}_{o\kappa^2} \vee f_{o\kappa^2} = S^{(2)}_{o\kappa^2} \vee f_{o\kappa^2} = S^{(3)}_{o\kappa^2})$$

puts $S^{(1)}$, $S^{(2)}$ and $S^{(3)}$ in $Altrel$). Then we close $Altrel$ under $Stroke$, i.e. we form $ClosAltrel$ by the definitions

$$SupClosAltrel_{o(o(o\kappa^2))} =_{df} \lambda h_{o(o\kappa^2)}((f_{o\kappa^2})(Altrel_{o(o\kappa^2)} f_{o\kappa^2} \supset h_{o(o\kappa^2)} f_{o\kappa^2}) \,\&$$
$$(f_{o\kappa^2})(g_{o\kappa^2})(h_{o(o\kappa^2)} f_{o\kappa^2} \,\&\, h_{o(o\kappa^2)} g_{o\kappa^2} \supset h_{o(o\kappa^2)}(Stroke_{(o\kappa^2)^3} f_{o\kappa^2} g_{o\kappa^2})))$$

and

$$ClosAltrel_{o(o\kappa^2)} =_{df} (\imath m_{o(o\kappa^2)})(SupClosAltrel_{o(o(o\kappa^2))} m_{o(o\kappa^2)} \,\&\, (n_{o(o\kappa^2)})$$
$$(SupClosAltrel_{o(o(o\kappa^2))} n_{o(o\kappa^2)} \supset (f_{o\kappa^2})(m_{o(o\kappa^2)} f_{o\kappa^2} \supset n_{o(o\kappa^2)} f_{o\kappa^2}))) .$$

Hintikka's condition may now be written

$$(w_\kappa)(f_{\iota\kappa})(g_{\iota\kappa})(Indiv_{o(\iota\kappa)\kappa} w_\kappa f_{\iota\kappa} \,\&\, Indiv_{o(\iota\kappa)\kappa} w_\kappa g_\kappa \supset . f_{\iota\kappa} w_\kappa = g_{\iota\kappa} w_\kappa \supset$$
$$(v_\kappa)(h_{o\kappa^2})(ClosAltrel_{o(o\kappa^2)} h_{o\kappa^2} \,\&\, h_{o\kappa^2} w_\kappa v_\kappa \supset f_{\iota\kappa} v_\kappa = g_{\iota\kappa} v_\kappa)),$$

wherein, roughly, $Indiv_{o(\iota\kappa)\kappa}$ is Hintikka's **F**, $f_{\iota\kappa}$ is f_1, $g_{\iota\kappa}$ is f_2, w_κ is μ, v_κ is λ and the quantification $(h_{o\kappa^2})$, restricted to $ClosAltrel_{o(o\kappa^2)}$, formalizes his "all alternatives."

So much for the formal difference between $Indiv_{o(\iota\kappa)\kappa}$ and an arbitrary $Dom_{o(\iota\kappa)\kappa}$. Now, for the way in which $Indiv_{o(\iota\kappa)\kappa}$ is used, we may set down the following definition of cross-world identity:

$$CWIdent_{o\kappa\iota\kappa\iota} =_{df} \lambda w_\kappa \lambda x_\iota \lambda v_\kappa \lambda y_\iota \lambda u_\kappa (\exists f_{\iota\kappa})(Indiv_{o(\iota\kappa)\kappa} w_\kappa f_{\iota\kappa} \,\&\, f_{\iota\kappa} v_\kappa = x_\iota \,\&\, f_{\iota\kappa} u_\kappa = y_\iota) .$$

This says that, from world w_κ, individual x_ι "in" v_κ is cross-world identical to individual y_ι "in" u_κ iff there is an individuating function in the set of such functions for w_κ, such that x_ι is the value of the function at v_κ and y_ι is the value of the function at u_κ. We see that, in our terms, it would be misleading to treat $CWIdent$ as <u>identity</u> between x_ι and y_ι, since these already have a perfectly good identity of their own viz.

that expressed by $x_\iota = y_\iota$, and this identity is in no way a function of worlds w_κ , v_κ and u_κ. But Hintikka (mostly in approximation-quotes, but see [31] p.136 ℓ.6) treats the values of individuating functions as "manifestations" or "roles" of things as they appear in each world, and requires that these "manifestations" or "roles" be linked by an individuating function before they can be said to be <u>of</u> the <u>same</u> individual.

It is unclear to me precisely what Hintikka <u>does</u> semantically with cross-world identity, if anything. If one could write a formula with, say, "(<u>a</u>" in model set μ and "<u>b</u>)" in model set λ and "=" between the model sets, then presumably the formula "(<u>a</u> = <u>b</u>)" would be true iff <u>a</u> and <u>b</u> were cross-world identical. But this never happens of course: so far as I can see, and some of Scott's remarks in [64] e.g. p.166 give support to the view, Hintikka evaluates singular terms in general as individual concepts, identity as our coincidence ($Coinc_{(o\kappa)(\iota\kappa)^2}$), and his quantifiers are restricted to $Indiv_{o(\iota\kappa)\kappa} w_\kappa$ at each world w_κ. In one respect this is an oversimplification, due to the way in which Hintikka's rules ($C.U_1$), ($C.E_1$) and ($C.U_m$), ($C.E_m$), on pp.124 and 127 of [34], are dependent upon syntactical features (the "modal profile") of the scope of the quantification. Åqvist [75] p.53 remarks pertinently on this aspect of Hintikka's theory, and says that Hintikka has actually got infinitely many quantifiers around. In general, it is not clear to me whether the definition

$$^{H}\Pi_{(o\kappa)(o\kappa(\iota\kappa))} =_{df} \lambda f_{o\kappa(\iota\kappa)} \lambda w_\kappa (x_{\iota\kappa})(Indiv_{o(\iota\kappa)\kappa} w_\kappa x_{\iota\kappa} \supset f_{o\kappa(\iota\kappa)} x_{\iota\kappa} w_\kappa)$$

and the abbreviation

$$(^{H}U x_{\iota\kappa}) A_{o\kappa} \quad \text{for} \quad ^{H}\Pi_{(o\kappa)(o\kappa(\iota\kappa))}(\lambda x_{\iota\kappa} A_{o\kappa}) \quad ,$$

with ^{H}E the dual of ^{H}U , captures the semantical content of Hintikka's rules or not; the conditions built into

$Indiv_{o(\iota\kappa)\kappa}$ guarantee the truth of Hintikka's extra premises in all cases, so far as I can see, and so our quantifier ${}^{H}\Pi_{(o\kappa)(o\kappa(\iota\kappa))}$ is in some sense the upper bound of Hintikka's infinitely many quantifiers.

In [29] and [32] Hintikka has made the point that in dealing with operators such as "a perceives that...", there may be two different kinds of individuation (which he calls physical and perceptual individuation); Scott suggests that there may be any number of different kinds of individuation. Hintikka discusses two pairs of quantifiers based on the two different methods of individuation, and gives intuitive readings for formulae involving the different quantifiers. In our notation, if we are right about ${}^{H}U$ and ${}^{H}E$, we can express the difference simply by introducing a further constant $PIndiv_{o(\iota\kappa)\kappa}$ (obeying the same kind of non-splitting condition as $Indiv_{o(\iota\kappa)\kappa}$, with the appropriate relations put into a new class $PAltrel_{o(o\kappa^2)}$) and then defining a quantifier ${}^{P}\Pi_{(o\kappa)(o\kappa(\iota\kappa))}$ based on this new constant in the way that ${}^{H}\Pi_{(o\kappa)(o\kappa(\iota\kappa))}$ is based on $Indiv_{o(\iota\kappa)\kappa}$.

The philosophical and formal possibility of different systems of individuation raises anew the spectre of relative identity à la Geach and Griffin, as dealt with in § 2.6. Certainly we will have two different notions of cross-world identity, and x_ι in v_κ may be identical to y_ι in u_κ according to one of these notions and not according to the other. But, if we are right, cross-world identity does not figure in Hintikka's evaluation of identity, and coincidence of individual concepts is not relative to the set $Indiv_{o(\iota\kappa)\kappa}$, so we will not have a kind of relative coincidence to yield a relative evaluation of identity. However, for all this, it is possible that some of Hintikka's methods might yield semantics for the notion of relative identity.

2.9 Stage theory and the analysis of individuals

Scott, in [64] pp. 166 ff., and Hintikka, as we have already noted in §2.8, both propose to analyze individuals in terms of their "states" or "stages" or "manifestations" or "roles" or "snapshots" in each world. That is, they take individuals to have type $\iota\kappa$, in our terminology, and Scott goes on to take individual concepts to have type $\iota\kappa\kappa$ (whereas for Hintikka individual concepts have the same type, but are not subject to the same constraints, as individuals). They both seem to assume tacitly that the domains of "states" are disjoint from world to world, that is that $x_{\iota\kappa} w_\kappa$ and $y_{\iota\kappa} v_\kappa$ cannot be identical unless $w_\kappa = v_\kappa$; in one sense this is just as much a metaphysical assumption as the assumption that there can be substances of type ι whose identity from world to world is unproblematical. Moreover the analysis of individuals in terms of stages seems to impose certain conditions on the set D_κ of possible worlds; for instance if we want to deal with some kind of continuing individuals then D_κ will have to be continuously ordered in some way, otherwise our individuals will blink out of existence during the jumps or gaps in D_κ. This cannot happen with individuals of type ι, but perhaps the objection is only a metaphysical one. Again, stage theory (as we will henceforth call the approach in question) does not have a natural limiting case when there is only one possible world; in ordinary extensional quantification theory we do not usually suppose that we have a domain of stages of individuals (although we can so construe the semantics if we wish).

Apart from such caveats about the ascription of type $\iota\kappa$ to individuals, we might also urge that, if individuals are to be assigned any type other than ι, then the type $\iota\kappa$ is not <u>sufficient</u> to provide an informative analysis of individuals. Along the lines of mereology or the calculus of individuals we might want to analyze individuals into their parts, perhaps atomic parts but not necessarily so, and we might want to have a relationship between an individual and some set of its parts. We might also want to give formal expression to the idea that an individual has to have a certain structure among its parts; for example that a human individual ceases to be an individual if he is, literally, decomposed, or that a ship ceases to be so if it is merely a pile of planks and a scattered collection of fittings. (P. Herbst has remarked in discussion that this latter idea both has a long philosophical history and is a metaphysical dead-end: we have no special commitment to the idea, but suggest that hitherto it has lacked the kind of formal basis that we hope to provide.) Again, we might want to be in a position to deal with questions concerning Geach's cat Tibbles, and the individual Tib consisting of Tibbles minus her tail: S. Voss has asked questions like "When Tibbles meows, how many individuals meow?", and we would like to see formally how the straight answer "One", favoured by Voss, could be justified.

We assume that we have available all the resources of the calculus of individuals, so that relations $<_{o\iota\iota}$ and $\ll_{o\iota\iota}$, for part and proper part respectively, the operator $+_{\iota^3}$ for individual sum, and the operator $Fu_{\iota(o\iota)}$ for the fusion of a class to an individual, in particular are available and require no further explanation. Stagewise extensions of these notions may be defined thus:

$$<_{o(\iota\kappa)^2} =_{df} \lambda x_{\iota\kappa} \lambda y_{\iota\kappa} (w_\kappa)(<_{o\iota\iota}(x_{\iota\kappa} w_\kappa)(y_{\iota\kappa} w_\kappa)),$$

and similarly for $\ll_{o(\iota\kappa)^2}$,

$$+_{(\iota\kappa)^3} =_{df} \lambda x_{\iota\kappa} \lambda y_{\iota\kappa} \lambda w_\kappa (+_{\iota^3}(x_{\iota\kappa} w_\kappa)(y_{\iota\kappa} w_\kappa))$$

and

$$Fu_{\iota\kappa(o\iota\kappa)} =_{df} \lambda f_{o\iota\kappa} \lambda w_\kappa (Fu_{\iota(o\iota)}(f_{o\iota\kappa} w_\kappa)) \quad .$$

If we take the members of D_ι seriously as stages then a a further kind of fusion may be defined, collecting the stages together into a single individual. Indeed there are two separate notions here: according to one we collect together all stages, and according to the other only existing stages are collected. These operators will be called wormifiers, to distinguish them from the kind of fusion operators already available, and their definitions are

$$Worm_{\iota(\iota\kappa)} =_{df} \lambda y_{\iota\kappa}(Fu_{\iota(o\iota)}\,\lambda x_\iota(\exists w_\kappa)(x_\iota = y_{\iota\kappa}w_\kappa))$$

$$EWorm_{\iota(\iota\kappa)} =_{df} \lambda y_{\iota\kappa}(Fu_{\iota(o\iota)}\,\lambda x_\iota(\exists w_\kappa)(x_\iota = y_{\iota\kappa}w_\kappa \;\&\; Dom_{o\iota\kappa}w_\kappa x_\iota)).$$

The trouble with such wormifiers is that even though the entities of type ι in the values of type $\iota\kappa$ have to be thought of as stages (the fusion of {Barton, Deakin,..., McMahon, Whitlam} scarcely provides us with an individual Prime Minister of Australia), the entities in the value of the wormifiers are still of type ι but are certainly not stages. Precisely how the ontological points involved here are to be reflected in the assignment of types is an open problem at this stage.

In order to illustrate the way in which the notion of structure might be formalized and applied, we consider the problem of Theseus as propounded by Hobbes and N. Griffin. According to this problem we have a ship, call it Theseus I, which is replaced plank-by-plank (ignore the sails etc.) resulting in a new ship, which we call Theseus II. Meanwhile the old planks have been stored away one-by-one, and then resurrected into a further ship, which we may call Theseus III. The problem is basically of the "What are we to say?" variety: are we to say that Theseus I is identical with both Theseus II and Theseus III, even though Theseus II is not identical with Theseus III, thereby denying that identity has the Euclidean property? If Theseus I is not identical with one of Theseus II or Theseus III, then which one? and what then is its relationship to the other one? And so on.

To give a description of this situation, we begin with the following definition of the field of a relation of type $o\iota\iota\kappa$:

$$Field_{(o\iota\kappa)(o\iota\iota\kappa)} =_{df} \lambda f_{o\iota\iota\kappa} \lambda w_\kappa \lambda x_\iota (\exists y_\iota)(f_{o\iota\iota\kappa} w_\kappa x_\iota y_\iota \vee f_{o\iota\iota\kappa} w_\kappa y_\iota x_\iota) .$$

We then introduce the relation

$$Plankarray_{o\iota\iota\kappa}$$

and the property

$$Plank_{o\iota\kappa}$$

connected by the fact that every individual in the array of planks is a plank, i.e.

$$(x_\iota)(w_\kappa)(Field_{(o\iota\kappa)(o\iota\iota\kappa)} Plankarray_{o\iota\iota\kappa} w_\kappa x_\iota \supset Plank_{o\iota\kappa} w_\kappa x_\iota) .$$

If bs_κ is the world of some time before the start of the shenanigans, then we assume that the array of planks is such that Theseus I is the fusion of the field of this relation, i.e.

$$ThI_\iota = Fu_{(\iota\kappa)(o\iota\kappa)}(Field_{(o\iota\kappa)(o\iota\iota\kappa)} Plankarray_{o\iota\iota\kappa}) bs_\kappa .$$

And if as_κ is some world after the shenanigans are over, then Theseus II is the fusion of the field of $Plankarray$ as well:

$$ThII_\iota = Fu_{(\iota\kappa)(o\iota\kappa)}(Field_{(o\iota\kappa)(o\iota\iota\kappa)} Plankarray_{o\iota\iota\kappa}) as_\kappa$$

Meanwhile, what has been happening to $Plankarray$? <u>At least</u> $Plankarray_{o\iota\iota\kappa} as_\kappa$ is isomorphic to $Plankarray_{o\iota\iota\kappa} bs_\kappa$, as per the definitions

$$Field_{(o\iota)(o\iota^2)} =_{df} \lambda f_{o\iota^2} \lambda x_\iota (\exists y_\iota)(f_{o\iota^2} x_\iota y_\iota \vee f_{o\iota^2} y_\iota x_\iota)$$

$$Bij_{o(\iota^2)(o\iota)^2} =_{df} \lambda f_{o\iota} \lambda g_{o\iota} \lambda h_{\iota^2}((x_\iota)(f_{o\iota} x_\iota \supset g_{o\iota}(h_{\iota^2} x_\iota)) \,\&\, (y_\iota)(g_{o\iota} y_\iota \supset (\exists x_\iota)(y_\iota = h_{\iota^2} x_\iota)) \,\&\, (x_\iota)(y_\iota)(f_{o\iota} x_\iota \,\&\, f_{o\iota} y_\iota \,\&\, h_{\iota^2} x_\iota = h_{\iota^2} y_\iota \supset x_\iota = y_\iota))$$

$$Iso_{o(\iota^2)(o\iota^2)^2} =_{df} \lambda m_{o\iota^2} \lambda n_{o\iota^2} \lambda h_{\iota^2}(Bij_{o(\iota^2)(o\iota)^2}(Field_{o\iota(o\iota^2)} m_{o\iota^2})(Field_{o\iota(o\iota^2)} n_{o\iota^2}) h_{\iota^2} \,\&\, (x_\iota)(y_\iota)(m_{o\iota^2} x_\iota y_\iota \supset n_{o\iota^2}(h_{\iota^2} x_\iota)(h_{\iota^2} y_\iota)))$$

$Isomorphic_{o(o\iota^2)^2} =_{df} \lambda m_{o\iota^2} \lambda n_{o\iota^2} (\exists h_{\iota^2}) Iso_{o(\iota^2)(o\iota^2)^2} m_{o\iota^2} n_{o\iota^2} h_{\iota^2}$.

Moreover, if we have suitable notions of betweenness and adjacency for the worlds involved, then $Plankarray_{o\iota\iota\kappa} x_\kappa$ and $Plankarray_{o\iota\iota\kappa} y_\kappa$ are isomorphic for any worlds x_κ and y_κ between bs_κ and as_κ , and if x_κ and y_κ are adjacent worlds then the fields of $Plankarray_{o\iota\iota\kappa} x_\kappa$ and $Plankarray_{o\iota\iota\kappa} y_\kappa$ are almost equal in the sense defined by

$Almost{=}_{o(o\iota)^2} =_{df} \lambda f_{o\iota} \lambda g_{o\iota} (\exists x_\iota)(\lambda y_\iota (f_{o\iota} y_\iota \& y_\iota \neq x_\iota) = \lambda y_\iota (g_{o\iota} y_\iota \& y_\iota \neq x_\iota))$.

These set-theoretic conditions on $Plankarray_{o\iota\iota\kappa}$ seem to tie down pretty accurately the relation between $Th I_\iota$ and $Th II_\iota$. The relation between $Th I_\iota$ and $Th III_\iota$, where $Th III_\iota = Fu_{(\iota\kappa)(o\iota\kappa)} (Field_{(o\iota\kappa)(o\iota\iota\kappa)} Plankarry'_{o\iota\iota\kappa}) as_\kappa$, depends on the relation between $Plankarray'_{o\iota\iota\kappa}$ and $Plankarray_{o\iota\iota\kappa}$. This relation is specified by the facts that for all x_κ between bs_κ and as_κ ,

$Field_{o\iota(o\iota\iota)} (Plankarray'_{o\iota\iota\kappa} x_\kappa) = Field_{o\iota(o\iota\iota)} (Plankarray_{o\iota\iota\kappa} bs_\kappa)$,

that is that $Plankarray'_{o\iota\iota\kappa}$ always holds the planks that were in Theseus I, no matter whether some are stacked up somewhere, and

$Plankarray'_{o\iota\iota\kappa} as_\kappa = Plankarray_{o\iota\iota\kappa} bs_\kappa$

that is at as_κ the planks are put back into the same structure they had at bs_κ . It follows from this that $Th III_\iota = Th I_\iota$: but this is a relation solely between individuals, and if the structure is taken into account then an individual-with-structure will have type $o\iota(o\iota\iota\kappa)$. During the changeover the structure $Plankarray'_{o\iota\iota\kappa}$ was different, and in the sense of individual-with-structure

Theseus III did not equal Theseus I during this period.

For the Geach-Voss problem about Tibbles and Tib, we need to postulate some canonical set $Vossind_{o(o\iota\kappa)\kappa}$ of structures, such that at each world w_κ the domain of existing individuals may be defined by

$$Dom_{o\iota\kappa} =_{df} \lambda w_\kappa \lambda x_\iota (\exists f_{o\iota\iota\kappa})(Vossind_{o(o\iota\iota\kappa)\kappa}\, w_\kappa f_{o\iota\iota\kappa}\ \& \ x_\iota = Fu_{\iota(o\iota)}(Field_{o\iota(o\iota\iota)}(f_{o\iota\iota\kappa} w_\kappa))) \quad .$$

Criteria for membership of $Vossind_{o(o\iota\iota\kappa)\kappa}\, w_\kappa$ may be given in terms of the kind of entities allowed in its field, and perhaps in terms of some formal relational properties akin to connectivity etc. In any event, an existence-quantifier based on $Dom_{o\iota\kappa}$ as defined will allow the natural answer "One" to Voss's question, and we have the beginnings of a formalism which will handle general problems of this kind.

This fairly prolix account is an example of what we mean by the "analysis of individuals", and is perhaps also an example of what Scott calls the fleshing-out of the concept of an individual (in stage theory). An example or two does not provide a whole theory: the latter awaits further development.

2.10 Functions, rules and rogators

In mathematics, functions are treated extensionally, as a class of ordered pairs obeying a suitable restriction. An apparent exception is category theory, where we have also to specify the domain and codomain (or at least the latter) of a function, since e.g. the identity function $I : A \to A$ differs from the insertion function $1_A : A \to B \quad (A \subsetneq B)$ even though they are the same set of ordered pairs. (They differ, e.g., in the well-definedness of their composition with other functions.) But this exception is only apparent, since extensionality is restored by treating a mapping as a triple $\langle F, A, B \rangle$ where F is the set of ordered pairs and A and B are the domain and codomain, not necessarily recoverable from F.

Occasionally the extensional treatment of functions leads to counterintuitive results: e.g. in recursive function theory the function f defined, for all x, by

$$f(x) = 1 \quad \text{if there is a sequence } 012\ldots9 \text{ in the decimal expansion of } \pi$$
$$= 2 \quad \text{otherwise},$$

is recursive, since it is either the constant 1 or the constant 2 function. Even though we cannot produce a machine that computes it, there is (non-constructively) such a machine, and that is all that matters. This is a clear indication that we are not treating this function according to any kind of <u>rule</u> given in its definition: such a treatment would require us actually to search through the decimal expansion of π, and this is not involved either in computing the constant 1 function or in computing the constant 2 function. Rogers [59] §1.3 suggests that algorithms for functions, as distinct from extensional functions themselves, serve as <u>names</u> for functions: but later (§1.8) uses P_x for a set of instructions and φ_x for the (partial) functions computed by these instructions. The fact that a <u>set of instructions</u> can be treated quite differently from a <u>name</u> shows that some further analysis of the extensional/intensional distinction in this area is called for.

On the philosophical side a number of points, about an extensional/intensional distinction for functions, are made in Sloman's [65]. Sloman distinguishes between functions, as classes or ordered pairs, and rogators, which are intensional entities such that extensional equivalence of rogators does not imply identity of rogators. He also (in Section B of [65]) makes some valuable distinctions between his function/rogator distinction and other distinctions e.g. some made by Frege, and by Russell involving "referentially opaque" functions like "the number of the day on which George IV first thought about x", for which a different day will result for argument $x =$ Scott than for argument $x =$ the author of Waverly, even though Scott = the author of Waverly.

Although in §12 of [65] Sloman argues against what will be essentially our approach (he reports that it was suggested to him by Montague and derives from work by Tarski), we now propose to investigate how well our notation might handle a distinction between extensional functions and (some formal surrogate for) intensional rogators. If possible, we want such a distinction to apply both to mathematical entities and to the more philosophical cases such as Sloman's "the town in which x was born."

Supposing that all our individuals have type ι (e.g. that numbers have this type and not, say Church's $(\iota\iota)(\iota\iota)$), then a function of one argument will have type $\iota\iota$. We ask, then, whether type $\iota\iota\kappa$ might serve for rogators, by analogy with our intensionalization of various other kinds of entity. Entities of type $\iota\iota\kappa$ are set-theoretically equivalent to entities of type $(\iota\kappa)\iota$, i.e. functions from individuals to individual concepts, but not to entities of type $\iota(\iota\kappa)$, i.e. functions from individual concepts to individuals. This distinction seems to match the difference between an example in Sloman's §7 and one in his §13: in the former we consider the function "the town in which x was born" and analyze the particular case "the town in which Aristotle's first pupil was born" with the dictionary

a_ι	Athens
s_ι	Stagyra
$f_{o\iota\kappa}$	the property of being Aristotle's first pupil
$b_{o\iota\iota\kappa}$	the relation of being born in.

(We assume everyone to be born in just one town.) We can define the rogator

$$t_{\iota\iota\kappa} = \lambda w_\kappa \lambda x_\iota (\imath y_\iota)(b_{o\iota\iota\kappa} w_\kappa x_\iota y_\iota) \quad ,$$

and can then explicate Sloman's points viz. that if Aristotle's first pupil was actually born in Athens then the value of the rogator is Athens, but if that individual had been born elsewhere (say Stagyra) then the value of the rogator would be different. This is just to say that

$$t_{\iota\iota\kappa} g_\kappa (\imath x_\iota)(f_{o\iota\kappa} g_\kappa x_\iota) = a_\iota$$

but that in alternative circumstances w_κ we have

$$(\imath x_\iota)(f_{o\iota\kappa} g_\kappa x_\iota) = (\imath x_\iota)(f_{o\iota\kappa} w_\kappa x_\iota) \qquad \text{but}$$

$$t_{\iota\iota\kappa} w_\kappa (\imath x_\iota)(f_{o\iota\kappa} g_\kappa x_\iota) = s_\iota \quad .$$

Of course, we might not have had $(\imath x_\iota)(f_{o\iota\kappa} g_\kappa x_\iota) = (\imath x_\iota)(f_{o\iota\kappa} w_\kappa x_\iota)$, but this does not bear on the "rogational" character of the example.

For Sloman's example in §13 we have to analyze "the day on which our chairman first thought about x", explaining the fact that for x = Bertrand Russell the day may well be different from that for x=the author of <u>The Principles of Mathematics</u>, even though Bertrand Russell is the author of <u>The Principles of Mathematics</u>. This seems simply to need the function $p_{\iota(\iota\kappa)}$, where $p_{\iota(\iota\kappa)} x_{\iota\kappa}$ is "the day (of type ι) on which our chairman first thought about individual concept $x_{\iota\kappa}$ (of type $\iota\kappa$)." Putting

b_ι	Bertrand Russell

and

$a_{o\iota\kappa}$	the property of being an author of <u>The Principles of Mathematics</u>,

we simply have that $p_{\iota(\iota\kappa)}(\lambda w_\kappa b_\iota)$ may differ from $p_{\iota(\iota\kappa)}(\lambda w_\kappa (\neg x_\iota)(a_{o\iota\kappa} w_\kappa x_\iota))$ even though $Coincr_{o(\iota\kappa)^2}(\lambda w_\kappa b_\iota)(\lambda w_\kappa (\neg x_\iota)(a_{o\iota\kappa} w_\kappa x_\iota))$ holds.

So much for making a formal distinction between functions (type $\iota\iota$), rogators (type $\iota\iota\kappa$) and another kind of intensional function (type $\iota(\iota\kappa)$). Can we also use our notation to shed light on the mathematical distinction between an extensional function and an algorithm for the function? In [65] §3 Sloman suggests that a "machine" for a function might be an explanation of rogators rather than functions, so we might expect a similar kind of notation to work in both cases.

It is fairly clear that algorithms or machines per se do not have type $\iota\iota\kappa$: we do not have to feed a possible world into an algorithm before it starts giving us a value for each argument. However, if we treat κ not as the type of possible worlds but as the type of possible transducing or encoding systems, from numbers to physical representations thereof (and back again), and if we think of a machine merely as operating on physical representations of numbers, then the computational function of a machine can indeed be treated as having type $\iota\iota\kappa$. For, the machine by itself merely does complicated things with voltages or currents or whatever, and one has to have a transducing system for these before the machine can be said actually to be doing a computation. As a trivial example, suppose there are two machines which respectively calculate the identity and the squaring function under some standard and intended transducing system. Now change the transducing system so that every number is represented by whatever normally represents the number one, as input to each machine, and each machine's output for this input is converted back to the number one. Then for this mad transducing system, both machines compute the constant one function; they coincide at that mad system. But they are not identical machines, since they compute decent and different functions under the standard system.

This gives us one kind of mathematical model for the type $\iota\iota\kappa$, but there is at least one other way of doing so. According to the second way, we construe D_κ itself as a class of otherwise unspecified and unanalyzed algorithms, and we introduce a constant $App_{\iota\iota\kappa}$

as an application function, whereby $App_{\iota\iota\kappa}\, w_\kappa$ is the function that results from the operation of algorithm w_κ . In broad terms, the function $App_{\iota\iota\kappa}$ explains how the algorithms in D_κ work, e.g. it describes the general operation of a set of instructions making up a Turing Machine or real computer, or it describes the method of calculating a function from a set of equations, or it describes what $\rightarrow$ and $\rightarrow\cdot$ etc. mean in a Markov algorithm. In a particular case this function may be more exactly specified, depending on the nature of the entities in D_κ (for Turing machines the application function is in a sense the universal machine). Under this conception of things, $App_{\iota\iota\kappa}$ is one big rogator, yielding functions $App_{\iota\iota\kappa}\, w_\kappa$ for each algorithm in D_κ: we may well have $App_{\iota\iota\kappa}\, w_\kappa = App_{\iota\iota\kappa}\, v_\kappa$ without having $w_\kappa = v_\kappa$, and this is now our formal statement of the fact that functions may be the same even though algorithms for them differ qua algorithms.

Further development of this second conception of things would require us to introduce a (fixed) Gödel numbering $Gn_{\kappa\iota}$

so that for each index x_ι , $Gn_{\kappa\iota}\, x_\iota$ is the algorithm or set of instructions with index x_ι (Rogers's P_x) and then $App_{\iota\iota\kappa}(Gn_{\kappa\iota}x_\iota)$ is the function with index x_ι (Rogers's φ_x). Development past this point would require a decent treatment of partial functions, as in §1.16, and we leave the matter there.

2.11. Second-order properties and relations

Various useful properties of properties, and relations between properties, can be defined in our type-theoretic formalism. For example we may define an actualizer as an "existence-entailing" property:

$$Actualizer_{o(\pi,\kappa)} =_{df} \lambda f_{\pi,\kappa}(x_\iota)(w_\kappa)(f_{\pi,\kappa}\, w_\kappa\, x_\iota \supset Dom_{o\iota\kappa}\, w_\kappa\, x_\iota) ,$$

or we can put

$$Actualizer_{o\kappa(\pi,\kappa)} =_{df} \lambda f_{\pi,\kappa}\, \lambda w_\kappa (x_\iota)(f_{\pi,\kappa}\, w_\kappa\, x_\iota \supset Dom_{o\iota\kappa}\, w_\kappa\, x_\iota)$$

if we want a proposition-forming property of properties. In general, a relation might be an actualizer in some places and not others, e.g. "x believes that y exists" might be taken to be an actualizer for x but not for y. Thus we might define

$$Actualizer^{i}_{o(\pi_n\kappa)} =_{df} \lambda f_{\pi_n\kappa}(w_\kappa)(x_\iota^{((n))})(f_{\pi_n\kappa}\, w_\kappa\, x_\iota^{((n))} \supset Dom_{o\iota\kappa}\, w_\kappa\, x_\iota^{(i)})$$

for $1 \leqslant i \leqslant n$. Again, some relations may be conditional actualizers in some places: for example "x is married to y" is a relation such that if x exists then so does y, (and conversely), but the relation is not an actualizer in either place. The most general expression for such a kind of actualization is embodied in the definition

$$Actualizer^{i_1,\dots,i_r;j}_{o(\pi_n\kappa)} =_{df} \lambda f_{\pi_n\kappa}(w_\kappa)(x_\iota^{((n))})(f_{\pi_n\kappa}\, w_\kappa\, x_\iota^{((n))} \supset. Dom_{o\iota\kappa}\, w_\kappa\, x_\iota^{(i_1)} \supset. \dots \supset. Dom_{o\iota\kappa}\, w_\kappa\, x_\iota^{(i_r)} \supset Dom_{o\iota\kappa}\, w_\kappa\, x_\iota^{(j)}) ,$$

whereby $f_{\pi_n\kappa}$ is an actualizer in the j^{th} place conditional upon the existence of the arguments in the places $i_1, \dots, i_r$ (where $1 \leqslant j \leqslant n$, $1 \leqslant r \leqslant n$, $1 \leqslant i_s \leqslant n$ for $1 \leqslant s \leqslant r$).

We can define a range of properties of properties along the lines of

$$Fictionalizer_{o(\pi_1\kappa)} =_{df} \lambda f_{\pi_1\kappa}(w_\kappa)(x_\iota)(f_{\pi_1\kappa} w_\kappa x_\iota \supset \sim Dom_{o\iota\kappa} w_\kappa x_\iota) \ ,$$

and we can have mixed actualization/fictionalization whereby existence in some places implies non-existence in other places, and so on. We can also generalize on our conditional actualizers thus:

$$Actualizer^{i_1,\ldots,i_r;j}_{o(\pi_n\kappa)(\pi_1\kappa)^r} =_{df} \lambda g^{((r))}_{\pi_1\kappa} \lambda f_{\pi_n\kappa}(w_\kappa)(x^{((n))}_\iota)(f_{\pi_n\kappa} w_\kappa x^{((n))}_\iota \supset . \ g^{(1)}_{\pi_1\kappa} w_\kappa x^{(i_1)}_\iota \supset . \ldots \supset . g^{(r)}_{\pi_1\kappa} w_\kappa x^{(i_r)}_\iota \supset Dom_{o\iota\kappa} w_\kappa x^{(j)}_\iota) \ ,$$

and proceed from here to generalized mixed actualizer/ fictionalizers.

Some philosophical motivation for the notion of actualizers is given in Prior [54] p.161; this is discussed further by Cocchiarella e.g. in [13] and [14], where he uses the term "e-attributes". These papers yield the suggestion that we could define

$$Dom_{o\iota\kappa} =_{df} \lambda w_\kappa \lambda x_\iota (\exists f_{\pi_1\kappa})(Actualizer_{o(\pi_1\kappa)} f_{\pi_1\kappa} \ \& \ f_{\pi_1\kappa} w_\kappa x_\iota)$$

if we took $Actualizer_{o(\pi_1\kappa)}$ as primitive.

Akin to rigid designators, we have "rigid" properties, defined by

$$Rigidprop_{o(\pi_1\kappa)} =_{df} \lambda f_{\pi_1\kappa}(\exists g_{o\iota})(w_\kappa)(f_{\pi_1\kappa} w_\kappa = g_{o\iota})$$

with a range of obvious generalizations. I am not sure whether Kripke in [37] argues that some species names, e.g. "tiger" denote such rigid properties. (Obviously any class of entities denoted by rigid designators forms a rigid property.)

Among relations between properties, perhaps the most philosophically interesting are those in the coincidence cluster. We begin with

$$Coinc_{(o\kappa)(\pi_1\kappa)^2} =_{df} \lambda f_{\pi_1\kappa} \lambda g_{\pi_1\kappa} \lambda w_\kappa (f_{\pi_1\kappa} w_\kappa = g_{\pi_1\kappa} w_\kappa)$$

and its simple generalization

$$Coinc_{(o\kappa)(\pi_n\kappa)^2} =_{df} \lambda f_{\pi_n\kappa} \lambda g_{\pi_n\kappa} \lambda w_\kappa (f_{\pi_n\kappa} w_\kappa = g_{\pi_n\kappa} w_\kappa) \quad :$$

at each world, two properties coincide iff their extensions at that world are identical. Since the extensions will in general include non-existent individuals at each world, a weaker relation of coincidence is defined by

$$DomCoinc_{(o\kappa)(\pi_n\kappa)^2} =_{df} \lambda f_{\pi_n\kappa}\, \lambda g_{\pi_n\kappa}\, \lambda w_\kappa\, (x_\iota^{((n))})(Dom_{o\iota\kappa}\, w_\kappa\, x_\iota^{(1)} \supset . \ldots \supset . \; Dom_{o\iota\kappa}\, w_\kappa\, x_\iota^{(n)} \supset . \; f_{\pi_n\kappa}\, w_\kappa\, x_\iota^{((n))} \equiv g_{\pi_n\kappa}\, w_\kappa\, x_\iota^{((n))}) ,$$

whereby the extensions are required to be identical only for existent individuals.

The two different kinds of coincidence yield two different kinds of contingent identity for properties, viz.

$$Coincr_{o(\pi_n\kappa)^2} =_{df} \lambda f_{\pi_n\kappa}\, \lambda g_{\pi_n\kappa}\, (Coinc_{(o\kappa)(\pi_n\kappa)^2}\, f_{\pi_n\kappa}\, g_{\pi_n\kappa}\, g_\kappa)$$

and

$$DomCoincr_{o(\pi_n\kappa)^2} =_{df} \lambda f_{\pi_n\kappa}\, \lambda g_{\pi_n\kappa}\, (DomCoinc_{(o\kappa)(\pi_n\kappa)^2}\, f_{\pi_n\kappa}\, g_{\pi_n\kappa}\, g_\kappa) .$$

The possibility of defining a reasonable notion of contingent identity of properties, parallel to that of contingent identity of individual concepts, provides a simple formalistic vindication of Deutscher's position in [21] p.73 ff., as against M.C. Bradley [4] and J.J.C. Smart [66] p.90. It is not clear to me whether $Coincr$ or $DomCoincr$ best fits Deutscher's ideas about such a contingent identity, but this does not count against the basic correctness of his point. Perhaps we could quibble over the term "contingent identity" in this context, with its suggestion that a <u>kind</u> of identity is involved: the only identity, for any of the entities in our analysis, is straight-out set-theoretic Leibnizian $Q_{o\alpha^2}$, but there are other

<u>equivalence relations</u> that may be usefully defined since they appear in philosophical discourse. The two relations $Coincr$ and $DomCoincr$ are such equivalence relations, and they answer closely to intuitive ideas of contingent identity, even though they are not identity relations "properly so-called".

2.12 Propositional quantifiers

The introduction of proposition-forming propositional quantifiers may be carried out along the same lines as the introduction of proposition-forming individual quantifiers in §2.7. The main difference is that the domain of propositions declared to exist at each world may have to satisfy various closure conditions, etc., that do not and cannot apply to domains of individuals.

Just as for individuals, the system retains the straight truth-value forming quantifiers $\Pi_{o(oo)}$ and $\Pi_{o(o(o\kappa))}$ as a matter of course: but to treat intensional systems with propositional quantifiers, such as those of Bull [5], Fine [22] and Cresswell [16], we introduce a constant

$$Dom_{o(o\kappa)\kappa}$$

(indeed, a neighbourhood function!) such that at each world w_κ, $Dom_{o(o\kappa)\kappa} w_\kappa$ is the domain of propositions at w_κ. Following Fine, we require at least that for each w_κ, the domain be non-empty, i.e.

$$(w_\kappa)(\exists x_{o\kappa})(Dom_{o(o\kappa)\kappa} w_\kappa x_{o\kappa}) \quad .$$

The weakest further condition that Fine proposes is that each $Dom_{o(o\kappa)\kappa} w_\kappa$ should be Boolean, i.e. that they should be closed under (conventional) negation and disjunction. This condition is expressed by

$$(w_\kappa)(x_{o\kappa})(Dom_{o(o\kappa)\kappa} w_\kappa x_{o\kappa} \supset Dom_{o(o\kappa)\kappa} w_\kappa (N_{(o\kappa)^2} x_{o\kappa}))$$

and

$$(w_\kappa)(x_{o\kappa})(y_{o\kappa})(Dom_{o(o\kappa)\kappa} w_\kappa x_{o\kappa} \supset . \ Dom_{o(o\kappa)\kappa} w_\kappa y_{o\kappa} \supset Dom_{o(o\kappa)\kappa} w_\kappa (A_{(o\kappa)^3} x_{o\kappa} y_{o\kappa})) \quad .$$

If we expected universal instantiation to hold for propositions formed from other than conventional propositional operators, then further closure conditions would be added. For example if the operators of an entailment system were involved, then we would require closure under $A_{(o\kappa)^3}$, $K_{(o\kappa)^3}$, $N^*_{(o\kappa)^2}$ and $E_{(o\kappa)^3}$; and if some modal operator $\Box^x_{(o\kappa)^2}$ were involved then we would require closure under it. The strongest possible condition on $Dom_{o(o\kappa)\kappa} w_\kappa$ is that it should select the full power set of the set of all worlds; this condition automatically satisfies all closure conditions, and is expressed simply by

$$(w_\kappa)(x_{o\kappa})(Dom_{o(o\kappa)\kappa} w_\kappa x_{o\kappa})$$

(given that only standard models appear in our underlying model theory).

Althoug h I know of no author who has found an urge to do so, it would be possible to have the domains of propositions satisfy different conditions depending on the kind of world involved. For instance closure under $\Box^x_{(o\kappa)^2}$ may not be required at queer worlds w_κ i.e. those where $Q_{o\kappa} w_\kappa$ holds; and for a quantified first-degree entailment system we might not expect closure under $E_{(o\kappa)^3}$ at worlds other than the real world.

Given a suitably restrained $Dom_{o(o\kappa)\kappa}$, we define the proposition-forming propositional quantifier $\Psi_{(o\kappa)(o(o\kappa))}$:

$$\Psi_{(o\kappa)(o(o\kappa))} =_{df} \lambda m_{o(o\kappa)} \lambda w_\kappa (x_{o\kappa})(Dom_{o(o\kappa)\kappa} w_\kappa x_{o\kappa} \supset m_{o(o\kappa)} x_{o\kappa}).$$

This quantifier is hybrid in the same way as $\Lambda_{(o\kappa)(o\iota)}$: a non-hybrid quantifier akin to $\Lambda_{(o\kappa)(o\kappa\iota)}$ is given by

$$\Psi_{(o\kappa)(o\kappa(o\kappa))} =_{df} \lambda m_{o\kappa(o\kappa)} \lambda w_\kappa (x_{o\kappa})(Dom_{o(o\kappa)\kappa} w_\kappa x_{o\kappa} \supset m_{o\kappa(o\kappa)} x_{o\kappa} w_\kappa).$$

Notice that in fact $\Psi_{(o\kappa)(o\kappa(o\kappa))}$ is a property of monadic propositional operators: $\Psi_{(o\kappa)(o\kappa(o\kappa))} \Box_{(o\kappa)^2}$ and $\Psi_{(o\kappa)(o\kappa(o\kappa))} N_{(o\kappa)^2}$ express forms of Rationalism and Vetoism respectively, albeit inconsistent ones given the conditions on $Dom_{o(o\kappa)\kappa}$. We may use the abbreviation

$$(\Lambda^p x_{o\kappa}) A_{o\kappa} \quad \text{for} \quad \Psi_{(o\kappa)(o\kappa(o\kappa))}(\lambda x_{o\kappa} A_{o\kappa}) \quad ,$$

whence Barcan-type formulae like

$$((\Lambda^{p}x_{o\kappa})(\Box_{(o\kappa)^2}A_{o\kappa}) \supset \Box_{(o\kappa)^2}(\Lambda^{p}x_{o\kappa})A_{o\kappa})g_{\kappa}$$

become well-formed. Similar conditions to those for individual domains $\mathcal{D}om_{o\iota\kappa}$ determine the validity or otherwise of this formula and its converse: in particular the strong "full-power-set" condition previously given is a sufficient, but not necessary, condition for the validity of both the Barcan formula and its converse.

The "second-order" aspect of Cresswell's [16] lies in its quantification over propositional operators as well as propositions. Along familiar lines, we may remark that we already have quantifiers such as $\Pi_{o(o((o\kappa)^2))}$, for quantification over monadic propositional operators, but that these quantifiers are not proposition-forming. Cresswell's condition of hereditariness ([16] 5.10 p.310), to some extent the analogue of the conditions on the domains $\mathcal{D}_{o\pi_1}$, $\mathcal{D}_{o\pi_2}$, ... in the frame of a Henkin general model, provides a minimum condition that a set of domains

$$\mathcal{D}om_{o(o\kappa)},\ \mathcal{D}om_{o((o\kappa)^2)},\ \mathcal{D}om_{o((o\kappa)^3)},\ \ldots$$

must satisfy in order to be used with quantifiers over propositional operators: such a condition would have to be built into any specification of domains of operators "existing" at each world if we proposed to define proposition-forming quantifiers over propositional operators.

2.13 A Bull-style tense logic

As a sample of a rather comprehensive propositional system we discuss some notions gained by what we might call an "ontologization" of the semantics of Bull's tense logic as set out in [6]. By our neologism we mean that we ignore the restrictions placed on Bull's system by its syntax, which are doubtless necessary for the completeness proofs to go through but which are not to the point here. We assume throughout this part that the set D_κ is a set of times - no other way of indexing possible worlds will be permitted, although a yet more comprehensive system would allow a wider class of worlds as well.

In Bull's semantics, the propositional variables $p, q, r, \ldots$ just take on propositions, of type $o\kappa$. There are also clock-propositional variables $a, b, c, \ldots$ and these take on clock-propositions, defined by

$$Clockprop_{o(o\kappa)} =_{df} \lambda x_{o\kappa} (\exists w_\kappa)(v_\kappa)(x_{o\kappa} v_\kappa \equiv . w_\kappa = v_\kappa) \quad .$$

Thus a clock-proposition is a proposition true at precisely one (instant of) time. We can recover that time from a clock-proposition by applying the function $Timeclock$, defined by

$$Timeclock_{\kappa(o\kappa)} =_{df} \lambda x_{o\kappa} (\imath w_\kappa)(x_{o\kappa} w_\kappa) \quad .$$

Bull's condition (iii) (p.284 of [6]) puts a syntactical requirement on his system, to the effect that there is a clock-propositional variable for every clock-proposition: we do not formalize this condition.

Bull's operators N, K, L, G, and H are respectively conventional negation and conjunction, universal necessity (= omnitemporality in this context), semi-normal necessity evaluated with a relation $S_{o\kappa^2}$ and semi-normal necessity evaluated with the converse of $S_{o\kappa^2}$. There is an operator Π for universal quantification over clock-propositions: we match this with the quantifier

$$^c\Phi_{(o\kappa)(o\kappa(o\kappa))} =_{df} \lambda m_{o\kappa(o\kappa)} \lambda w_\kappa (x_{o\kappa})(Clockprop_{o(o\kappa)} x_{o\kappa} \supset m_{o\kappa(o\kappa)} x_{o\kappa} w_\kappa),$$

and an abbreviation

$$({}^c\Pi x_{o\kappa}) A_{o\kappa} \quad \text{for} \quad {}^c\Phi_{(o\kappa)(o\kappa(o\kappa))}(\lambda x_{o\kappa} A_{o\kappa}) \quad .$$

Here $Clockprop_{o(o\kappa)}$ is acting as a domain, akin to

$Dom_{o(o\kappa)\kappa}\, w_\kappa$ in §2.12; however, unless there happen to be just two instants, $Clockprop_{o(o\kappa)}$ is not closed under negation, and unless there is only one instant it is not closed under conjunction or disjunction. So the quantifier ${}^{c}\Phi_{(o\kappa)(o\kappa(o\kappa))}$ differs from our previous propositional quantifiers at least to the extent that its domain in not Boolean: in the syntax of Bull's system, of course only clock-propositional variables may be used in instantiating Π, so the non-Boolean character of the domain has a syntactical correlate.

From this point, Bull's system is then changed so that reference to future-tense operators is omitted, and history-propositional variables u, v, w,... are introduced. To match Bull's definition of a path ([6] p.291) we define a history-proposition thus:

$$SubHistprop_{o(o\kappa)} =_{df} \lambda x_{o\kappa}((\exists w_\kappa)(x_{o\kappa} w_\kappa) \& (w_\kappa)(v_\kappa)(x_{o\kappa} w_\kappa \& x_{o\kappa} v_\kappa \supset . w_\kappa = v_\kappa \vee S_{o\kappa\kappa} w_\kappa v_\kappa \vee S_{o\kappa\kappa} v_\kappa w_\kappa))$$

$$Histprop_{o(o\kappa)} =_{df} \lambda x_{o\kappa}(SubHistprop_{o(o\kappa)} x_{o\kappa} \& (y_{o\kappa})((SubHistprop_{o(o\kappa)} y_{o\kappa} \& (w_\kappa)(x_{o\kappa} w_\kappa \supset y_{o\kappa} w_\kappa)) \supset x_{o\kappa} = y_{o\kappa})),$$

i.e. a Subhistory-proposition in a non-empty linearly ordered set of times, and a history-proposition is any $\subseteq$-maximal proposition among Subhistory-propositions. Thus of any two history-propositions, if one is a subset of the other then they are identical, and this is what Bull's postulate $CLCuvLCvu$

says of history-propositions. His postulate

$$CKTauTbuAUabUba$$

requires that any world in a history-proposition should be related to itself: this is not true of our notion and to gain compatability we should either delete $w_\kappa = v_\kappa$ from the definiens of $SubHistprop_{o(o\kappa)}$, or, what I regard as preferable, weaken Bull's postulate to

$$CKTauTbuALEabAUabUba.$$

Conversely, Bull's postulate

$$C\,Uab\,\Sigma u\,K\,Tau\,Tbu$$

could be strengthened to

$$K\,\Sigma u\,Tau\,C\,Uab\,\Sigma u\,K\,Tau\,Tbu$$

under our definition of history-propositions.

For quantification over history-propositions, Bull gives a substitutional evaluation and says (in effect) that it is weaker than the quantifier defined by

$$^{H}\Phi_{(o\kappa)(o\kappa(o\kappa))} =_{df} \lambda m_{o\kappa(o\kappa)} \lambda w_{\kappa} (x_{o\kappa})(Histprop_{o(o\kappa)}\, x_{o\kappa} \supset m_{o\kappa(o\kappa)}\, x_{o\kappa}\, w_{\kappa})$$

with abbreviation

$$(^{H}\Pi x_{o\kappa})A_{o\kappa} \qquad \text{for} \qquad {}^{H}\Phi_{(o\kappa)(o\kappa(o\kappa))}(\lambda x_{o\kappa} A_{o\kappa}) \qquad ,$$

and $^{H}\Sigma$ for the dual of $^{H}\Pi$. (Similarly to the domain of clock-propositions, the domain of history-propositions is only Boolean in exceptional cases.) Our $^{H}\Phi_{(o\kappa)(o\kappa(o\kappa))}$ is an example of our ontologization from Bull's system: regardless of its axiomatizability it is still a reasonable notion for quantification over (our version of) history-propositions.

Given history-propositions, Bull then defines G (and its dual F) as usual, and adds a further operator $\mathcal{F}$, for "it will be the case that...", defined (modulo our last two paragraphs) by

$$\mathcal{F}_{(o\kappa)^2} =_{df} \lambda x_{o\kappa} \lambda w_{\kappa} (^{H}\Pi y_{o\kappa})(y_{o\kappa} w_{\kappa} \supset (\exists v_{\kappa})(S_{o\kappa^2}\, w_{\kappa} v_{\kappa}\ \&\ y_{o\kappa} v_{\kappa}\ \&\ x_{o\kappa} v_{\kappa})) \quad :$$

thus $\mathcal{F}_{(o\kappa)^2}\, x_{o\kappa}\, w_{\kappa}$ holds iff $x_{o\kappa}$ is true at some time in the future in every history-proposition which includes w_{κ}. Bull's postulate

$$E\,Ta\,\mathcal{F}p\,\Pi u\,C\,Tau\,\Sigma b\,K\,Tbu\,K\,Uab\,Tbp$$

is almost literally a syntactical version of this condition.

Our notation suggests some fairly simple-minded additions to the structure that we have ontologized out of Bull's system. For instance the name "clock-proposition" is not quite appropriate since we might expect that the same assertion "It is n-o'clock" could be true in several different alternative futures, but this possibility is ruled out by the definition of $Clockprop_{o(o\kappa)}$. To formalize the notion that I have in mind, we need something like a metric tense logic, or a discrete ordering of the set of instants. Suppose $T_{o\kappa^2}$ is the "tomorrow" relation in such a discrete ordering: then for each w_κ we define

$$^{0}MClock_{o(o\kappa)\kappa} =_{df} \lambda w_\kappa \lambda x_{o\kappa}(v_\kappa)(x_{o\kappa} v_\kappa \equiv. w_\kappa = v_\kappa)$$

$$^{1}MClock_{o(o\kappa)\kappa} =_{df} \lambda w_\kappa \lambda x_{o\kappa}(v_\kappa)(x_{o\kappa} v_\kappa \equiv T_{o\kappa^2} w_\kappa v_\kappa)$$

$$^{-1}MClock_{o(o\kappa)\kappa} =_{df} \lambda w_\kappa \lambda x_{o\kappa}(v_\kappa)(x_{o\kappa} v_\kappa \equiv T_{o\kappa^2} v_\kappa w_\kappa)$$

$$^{2}MClock_{o(o\kappa)\kappa} =_{df} \lambda w_\kappa \lambda x_{o\kappa}(v_\kappa)(x_{o\kappa} v_\kappa \equiv (\exists u_\kappa)(T_{o\kappa^2} w_\kappa u_\kappa \,\&\, T_{o\kappa^2} u_\kappa v_\kappa)),$$

etc. Strangelovish thoughts like "Whatever anyone does, the bomb goes off at time t from now" could then be symbolized, a bit indirectly, as

$$(x_{o\kappa})(w_\kappa)(\,^{t}MClock_{o(o\kappa)\kappa}\, n_\kappa\, x_{o\kappa} \supset. x_{o\kappa} w_\kappa \supset b_{o\kappa} w_\kappa) \quad ,$$

where $b_{o\kappa}$ is the proposition that the bomb goes off.

2.14. Relations between domains

Inter alia, we have now introduced the following constants for domains at each world:

	$Dom_{o\iota\kappa}$	for individuals
	$Dom_{o(\iota\kappa)\kappa}$	for individual concepts
and	$Dom_{o(o\kappa)\kappa}$	for propositions.

As they stand, the various domains bear no relation to each other (apart from relations from world to world for domains of the same kind of entity, e.g. the inclusion condition

$$(w_\kappa)(v_\kappa)(S^{(1)}_{o\kappa\kappa} w_\kappa v_\kappa \supset (x_\iota)(Dom_{o\iota\kappa} w_\kappa x_\iota \supset Dom_{o\iota\kappa} v_\kappa x_\iota))$$

for the converse of the Barcan formula). But we could expect certain relations to hold between these domains: for instance we should be able to at least express Fine's Priorish suggestion, in [22] p.344, that "one may argue that a proposition exists in a given world iff the individuals which the proposition is about exist in that world", even if we do not agree with it. And we could further expect that relations between the various domains should be reflected in principles connecting the various quantifiers whose definitions depend upon those domains.

Let us begin by looking at some possible relations between $Dom_{o\iota\kappa}$ and $Dom_{o(\iota\kappa)\kappa}$. A condition which comes readily to mind is that if the value of any individual concept exists in a certain world, then the individual concept itself exists in that world: in symbols

$$(w_\kappa)(y_{\iota\kappa})(Dom_{o\iota\kappa} w_\kappa (y_{\iota\kappa} w_\kappa) \supset Dom_{o(\iota\kappa)\kappa} w_\kappa y_{\iota\kappa}) \quad .$$

A possible counterexample to this condition is given by letting

	w_κ	be the world of 1940
and	$y_{\iota\kappa}$	be the inventor of the tunnel diode.

Then $y_{\iota\kappa} w_\kappa$ indeed existed at 1940, qua individual, but we could argue that $y_{\iota\kappa}$ did not exist at 1940, qua individual concept, since tunnel diodes could not have been described in 1940, and so the concept of their inventor did not exist. If we accept this as a counterexample to the given condition, then a weaker condition not open to the counterexample is

$(w_\kappa)(y_{\iota\kappa})(Dom_{o\iota\kappa}w_\kappa(y_{\iota\kappa}w_\kappa) \supset . Rigiddes_{o(\iota\kappa)}y_{\iota\kappa} \supset Dom_{o(\iota\kappa)\kappa}w_\kappa y_{\iota\kappa})$,
whereby any rigid designator whose value exists in a certain world is itself required to exist in that world. A special case of this condition, which seems to be quite irrefutable, is

$$(w_\kappa)(x_\iota)(Dom_{o\iota\kappa}w_\kappa x_\iota \supset Dom_{o(\iota\kappa)\kappa}w_\kappa(\lambda v_\kappa x_\iota)) \quad ,$$

in which the rigid designator $\lambda v_\kappa x_\iota$ is required to exist in any world in which x_ι exists.

When we introduced the domain $Dom_{o(\iota\kappa)\kappa}$ in §2.7, we noted that existence of an individual concept in a given world does not in general imply the existence of its value in that world. This applies to rigid designators in particular: "the prime number between 7 and 9" is a rigid designator, it exists in many worlds including the present world, but its value does not exist in this world.

In order to express Fine's suggestion about the existence of propositions, we have to find some way of expressing some minimal idea of aboutness, since as it stands a proposition like $x_{o\kappa}$ bears no trace of what it is about. But we can, presumably, say that

if $x_{o\kappa} = f_{o\kappa\iota}y_\iota$ or $x_{o\kappa} = \lambda w_\kappa(f_{\pi_1\kappa}w_\kappa y_\iota)$

then $x_{o\kappa}$ is about y_ι ,

if $x_{o\kappa} = f_{o\kappa\iota^n}y_\iota^{((n))}$ or $x_{o\kappa} = \lambda w_\kappa(f_{\pi_n\kappa}w_\kappa y_\iota^{((n))})$

then $x_{o\kappa}$ is about $y_\iota^{(1)}, \ldots, y_\iota^{(n)}$,

if $x_{o\kappa} = f_{o\kappa(\iota\kappa)}y_{\iota\kappa}$

then $x_{o\kappa}$ is about $y_{\iota\kappa}$,

and if $x_{o\kappa} = f_{o\kappa(\iota\kappa)^n}y_{\iota\kappa}^{((n))}$

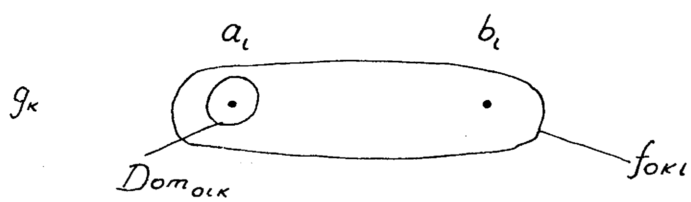
then $x_{o\kappa}$ is about $y_{\iota\kappa}^{(1)}, \ldots, y_{\iota\kappa}^{(n)}$:

these provide some minimal sufficient conditions on aboutness, but they are certainly not necessary conditions. In terms of these conditions, some versions of Fine's suggestion are

$$(w_\kappa)(x_{o\kappa})(f_{o\kappa\iota})(y_\iota)(x_{o\kappa} = f_{o\kappa\iota}\, y_\iota \supset .\ Dom_{o\iota\kappa}\, w_\kappa\, y_\iota \equiv Dom_{o(o\kappa)\kappa}\, w_\kappa\, x_{o\kappa}),$$

$$(w_\kappa)(x_{o\kappa})(f_{o\kappa\iota^n})(y_\iota^{((n))})(x_{o\kappa} = f_{o\kappa\iota^n}\, y_\iota^{((n))} \supset .\ Dom_{o\iota\kappa}\, w_\kappa\, y_\iota^{(1)} \,\&\, \dots \,\&\, Dom_{o\iota\kappa}\, w_\kappa\, y_\iota^{(n)} \equiv Dom_{o(o\kappa)\kappa}\, w_\kappa\, x_{o\kappa})$$

etc. However, as you have doubtless anticipated, these restrictions on $Dom_{o(o\kappa)\kappa}$ are inconsistent: if we have just



then $f_{o\kappa\iota}\, a_\iota = f_{o\kappa\iota}\, b_\iota$ ($= x_{o\kappa}$, say). Then $Dom_{o\iota\kappa}\, g_\kappa\, a_\iota$ requires that we have $Dom_{o(o\kappa)\kappa}\, g_\kappa\, x_{o\kappa}$, whereas $\sim Dom_{o\iota\kappa}\, g_\kappa\, b_\iota$ requires $\sim Dom_{o(o\kappa)\kappa}\, g_\kappa\, x_{o\kappa}$. Hence the "if and only if" in Fine's suggestion is too strong: either the "if" half or the "only if" half could be consistently accepted, other things being equal. Of these, the "if" itself is far more reasonable: it requires that propositions about existents themselves exist, whereas the "only if" half requires that propositions about non-existents do not exist. So we replace the $\equiv$ in our previous formulae by $\supset$, and shorten them a bit to get

$$(w_\kappa)(f_{o\kappa\iota})(y_\iota)(Dom_{o\iota\kappa}\, w_\kappa\, y_\iota \supset Dom_{o(o\kappa)\kappa}\, w_\kappa\, (f_{o\kappa\iota}\, y_\iota)),$$

$$(w_\kappa)(f_{o\kappa\iota^n})(y_\iota^{((n))})(Dom_{o\iota\kappa}\, w_\kappa\, y_\iota^{(1)} \supset . \dots \supset .\ Dom_{o\iota\kappa}\, w_\kappa\, y_\iota^{(n)} \supset Dom_{o(o\kappa)\kappa}\, w_\kappa\, (f_{o\kappa\iota^n}\, y_\iota^{((n))}))$$

etc. These conditions do not conflict with the Boolean closure conditions on each $Dom_{o(o\kappa)\kappa}\, w_\kappa$: the propositions required to exist by the conditions just given can be taken as the generators of a free algebra whose operators are conventional negation, disjunction and possibly other propositional operators, and this free algebra might suffice for $Dom_{o(o\kappa)\kappa} w_\kappa$. If we had accepted the "only if" half of Fine's suggestion, there would have been a conflict between the Boolean closure conditions and the conditions requiring that propositions about non-existents should not exist.

A further condition on the existence of propositions would be that various logically definable propositions should exist. For instance the clock-propositions of §2.13, and in general world-propositions defined in the same way but with D_κ construed more widely, should be required to exist (in the domain of propositions of each world). Again we might require that all assertions of existence, like

$$\lambda w_\kappa \, (Dom_{o\iota\kappa} w_\kappa x_\iota)$$

for <u>any</u> x_ι, should themselves exist. Boolean closure will then require that all assertions of non-existence also exist, and this may not find universal acceptance.

We are not able to satisfy the expectation of the last sentence of the introductory paragraph of this part: I can find no principles directly expressible in terms of the defined quantifiers of §2.7 whose satisfaction requires some of the given relations between domains to hold. This, I think, is because the possible worlds and domains are buried inside these quantifiers, and more explicit reference to them is needed to express the kind of relations between domains that we have considered. But there may well be more formulae of the Barcan type waiting to be discovered.

2.15 Predicate modifiers in intensional logics

Since in general the type of a predicate modifier involves both predicates and individuals, and since there are several intensional versions of predicates as well as an intensional version of individuals, it follows that there are many different possible types for predicate modifiers in intensional logics. Thus in type $\pi_k^{2}\iota^{j}$ we may put $(\pi_k\kappa)$, $(o\kappa\iota^{k})$ or $((o\kappa)(\iota\kappa)^{k})$ for π_k, and $(\iota\kappa)$ for any or all of the occurrences of ι, depending on the type of the predicate to be modified and the kind of individuals involved in the modification. It would also be possible to have predicate modifiers of a heterotypical type: for example we might define

$$Intch_{(o\iota\kappa)(o\kappa\iota)} =_{df} \lambda f_{o\kappa\iota}\lambda w_k \lambda x_\iota (f_{o\kappa\iota}\, x_\iota\, w_k) \quad ,$$

a heterotypical predicate modifier which simply "interchanges the type symbols κ and ι". So far as I can see, English provides us with no intuitive examples of such modifiers, but they may be useful in some technical applications: e.g. if only the quantifier $\Lambda_{(o\kappa)(o\iota\kappa)}$ were defined, then we could put

$$(\Lambda^{p} x_\iota) A_{o\kappa} \quad \text{for } \Lambda_{(o\kappa)(o\iota\kappa)}(Intch_{(o\iota\kappa)(o\kappa\iota)}(\lambda x_\iota A_{o\kappa}))$$

and this would have the same effect as our previous explanation of the abbreviation $(\Lambda^{p} x_\iota) A_{o\kappa}$.

As we would expect, extensionality principles like

$$(m_{(\pi,\kappa)^2})(f_{\pi,\kappa})(g_{\pi,\kappa})(f_{\pi,\kappa} = g_{\pi,\kappa} \supset m_{(\pi,\kappa)^2} f_{\pi,\kappa} = m_{(\pi,\kappa)^2} g_{\pi,\kappa})$$

hold for intensional predicate modifiers: they are a direct consequence of the definition of identity within type theory. But, again as we would expect, principles like

$$(m_{(\pi,\kappa)^2})(f_{\pi,\kappa})(g_{\pi,\kappa})(Coincr_{o(\pi,\kappa)^2} f_{\pi,\kappa}\, g_{\pi,\kappa} \supset Coincr_{o(\pi,\kappa)^2}(m_{(\pi,\kappa)^2} f_{\pi,\kappa})(m_{(\pi,\kappa)^2} g_{\pi,\kappa}))$$

fail in general, and this is in accord with examples like those of Clark [12] p.332 and Lewis [44] p.179, and also similar examples known for quite some time to I. Hinckfuss; e.g. even if it happened to be the case that all smokers were drivers, it would not follow that all fast smokers were fast drivers. I remain unconvinced about many such examples, since they often succumb to the treatment suggested in Malinas and Rennie [45] whereby e.g. "having a lung" is symbolized with a relation for "having" rather than as an unanalyzed predicate. But in any event an intensionalized theory of predicate modifiers will handle such examples if all else fails.

The various properties of predicate modifiers, such as inclusiveness etc., will now come in (at least) two grades. For example, let us write

$$Qimpp_o =_{df} (y_\iota)(m_{(\pi,\kappa)^2}\, g_{\pi,\kappa}\, w_\kappa\, y_\iota \supset g_{\pi,\kappa}\, w_\kappa\, y_\iota) \quad ,$$

whence we can define a necessary inclusiveness property

$$Incln_{o((\pi,\kappa)^2)} =_{df} \lambda m_{(\pi,\kappa)^2}\, (g_{\pi,\kappa})(w_\kappa)\, Qimpp_o$$

and a contingent inclusiveness property

$$Inclr_{o((\pi,\kappa)^2)} =_{df} \lambda m_{(\pi,\kappa)^2}\, (g_{\pi,\kappa})(\lambda w_\kappa\, Qimpp_o\, g_\kappa) \quad .$$

There may be yet further significant ramifications: e.g. necessary detachability ramifies according to whether the detacher is or is not a rigid property. Formally, let us write

$$Qequu_o =_{df} (y_\iota)(m_{(\pi,\kappa)^2}\, g_{\pi,\kappa}\, w_\kappa\, y_\iota \equiv .\, g_{\pi,\kappa}\, w_\kappa\, y_\iota \;\&\; h_{\pi,\kappa}\, w_\kappa\, y_\iota) \quad :$$

then we have a contingent detachability property

$$Detachr_{o((\pi,\kappa)^2)} =_{df} \lambda m_{(\pi,\kappa)^2}\, (\exists h_{\pi,\kappa})(g_{\pi,\kappa})(\lambda w_\kappa\, Qequu_o\, g_\kappa) \, ,$$

a necessary detachability property

$$Detachn_{o((\pi,\kappa)^2)} =_{df} \lambda m_{(\pi,\kappa)^2}\, (\exists h_{\pi,\kappa})(g_{\pi,\kappa})(w_\kappa)\, Qequu_o \quad ,$$

and a rigid necessary detachability property

$$Detachnrig_{o((\pi,\kappa)^2)} =_{df} \lambda m_{(\pi,\kappa)^2}\, (\exists h_{\pi,\kappa})(Rigidprop_{o(\pi,\kappa)}\, h_{\pi,\kappa} \;\& \\ (g_{\pi,\kappa})(w_\kappa)\, Qequu_o) \quad .$$

I doubt whether English is capable of supplying illustrative cases, but there is a clear formal difference between the two necessary detachability properties.

Next we discuss a topic in the logic of modifiers which has not arisen, so far as I know, in previous formal treatments of modifiers (it is not discussed e.g. in Lewis [44] or Montague [48]). The question is whether modifiers themselves might be intensionalized, i.e. whether they might have types like $(\pi,\kappa)^2\kappa$. Once posed, it is easy to answer the question in the affirmative: generally it seems reasonable to suppose that the operation of modifiers can vary from world to world, and specifically we can find examples from tensed discourse where such a type is required. Consider for example the modifier "contemporary" (as an adjective): this modifier varies from world to world (time-slot to time-slot) as becomes clear if we set out to explain the ambiguity of the sentence

(1) In the year 2000, no one will read contemporary books.

For this we use the dictionary

n_κ	the present world
t_κ	the world of the year 2000
$b_{\pi,\kappa}$	the property of being a book
$p_{\pi,\kappa}$	the property of being a person
$r_{\pi_2\kappa}$	the relation of reading
$c_{(\pi,\kappa)^2\kappa}$	the modifier "contemporary",

whence the ambiguity in (1) is that between

(2) $(y_\iota)(p_{\pi,\kappa}\, t_\kappa\, y_\iota \supset \sim(\exists x_\iota)(r_{\pi_2\kappa}\, t_\kappa\, y_\iota\, x_\iota \;\&\; c_{(\pi,\kappa)^2\kappa}\, n_\kappa\, b_{\pi,\kappa}\, t_\kappa\, x_\iota))$

and

(3) $(y_\iota)(p_{\pi,\kappa} t_\kappa y_\iota \supset \sim(\exists x_\iota)(r_{\pi,\kappa} t_\kappa y_\iota x_\iota \& c_{(\pi,\kappa)^2\kappa} t_\kappa b_{\pi,\kappa} t_\kappa x_\iota))$,

i.e. between "contemporary" meaning "contemporary as of now" ($c_{(\pi,\kappa)^2\kappa} n_\kappa$) or "contemporary as of then" ($c_{(\pi,\kappa)^2\kappa} t_\kappa$).

The admission of modifiers that are themselves intensionalized has as a consequence a further multiplication of the kinds of types that properties may have. Thus if we abstract the property of being a contemporary book from our previous example by the definition

$$cb_{\pi,\kappa\kappa} =_{df} \lambda w_\kappa \lambda v_\kappa \lambda x_\iota (c_{(\pi,\kappa)^2\kappa} w_\kappa b_{\pi,\kappa} v_\kappa x_\iota) \quad ,$$

we see that such a property has type $\pi,\kappa\kappa$, the first κ for the reference point of "contemporary" and the second κ for the normal reference point for a property. Once we accept that properties can have types like $\pi,\kappa\kappa$ (and that such types are not e.g. the types of <u>senses</u> of properties as Montague [49] would suggest) then we have the beginnings of an infinite sequence of types of properties and their modifiers. For, straight modifiers of properties of type $\pi,\kappa\kappa$ will have types like $(\pi,\kappa\kappa)^2$, but these modifiers may themselves be intensionalized to have type $(\pi,\kappa\kappa)^2\kappa$: thence from the result of their operation properties with type $\pi,\kappa\kappa\kappa$ may be abstracted, and so it goes on. Of course, we will soon run out of English examples, but our single example "contemporary book" is sufficient to show that more than one κ may appear at the end of the type of a property.

We might also consider whether there are any properties of modifiers especially applicable to intensionalized modifiers. One such property is the property of being a rigid modifier, defined by

$$Rigidmod_{o((\pi,\kappa)^2\kappa)} =_{df} \lambda f_{(\pi,\kappa)^2\kappa} (\exists g_{(\pi,\kappa)^2})(h_{\pi,\kappa})(w_\kappa)(f_{(\pi,\kappa)^2\kappa} w_\kappa h_{\pi,\kappa} = g_{(\pi,\kappa)^2} h_{\pi,\kappa}) ,$$

which generalizes to

$$Rigidmod_{o((\pi,\kappa)^2\kappa^{j+1})} =_{df} \lambda f_{(\pi,\kappa)^2\kappa^{j+1}} (\exists g_{(\pi,\kappa)^2\kappa^j})(h_{\pi,\kappa})(w_\kappa)(v_\kappa^{((j))})$$
$$(f_{(\pi,\kappa)^2\kappa^{j+1}} w_\kappa v_\kappa^{((j))} h_{\pi,\kappa} = g_{(\pi,\kappa)^2\kappa^j} v_\kappa^{((j))} h_{\pi,\kappa}) .$$

A modifier of type $(\pi,\kappa)^2\kappa^{j+1}$ can also be fully rigid, defined recursively with

$$Fullrigidmod_{o((\pi,\kappa)^2\kappa)} = Rigidmod_{o((\pi,\kappa)^2\kappa)}$$

as the basis clause and

$$Fullrigidmod_{o((\pi,\kappa)^2\kappa^{j+1})} =_{df} \lambda f_{(\pi,\kappa)^2\kappa^{j+1}} (\exists g_{(\pi,\kappa)^2\kappa^j})(h_{\pi,\kappa})(w_\kappa)(v_\kappa^{((j))})$$
$$(Fullrigidmod_{o((\pi,\kappa)^2\kappa^j)}\, g_{(\pi,\kappa)^2\kappa^j} \;\&\; f_{(\pi,\kappa)^2\kappa^{j+1}} w_\kappa v_\kappa^{((j))} h_{\pi,\kappa} = g_{(\pi,\kappa)^2\kappa^j} v_\kappa^{((j))} h_{\pi,\kappa}) .$$

Apart from the points already made, we envisage no special matters of concern arising in the amalgamation of predicate modifiers with intensional logics, and we feel that sufficient types for properties, properties of modifiers and so on have been exhibited to allow any particular problems to be categorized and dealt with in the framework available. In §2.16 we treat one such particular problem, that of modifiers related to the formation of positive and superlative forms of adjectives from their comparative forms.

2.16 Good, better, best, bested

In this part we set out to find modifiers that will form the positive and superlative forms of an adjective from its comparative form, treated as a dyadic relation. A first draft of this part was cooked by two events: (a) the realization that "Most *F's* are *G's* " cannot be symbolized with a monadic quantifier for "most" and (b) the appearance of Wallace's [73]. Cheered by the fact that our system can cope with these events, we press on, hoping that this part is not now so readily cookable.

Beginning with pure extensions, we define the heteradic modifier

$$Greatest_{\pi_1\pi_2} =_{df} \lambda g_{\pi_2} \lambda x_\iota (y_\iota)(x_\iota \neq y_\iota \supset g_{\pi_2} x_\iota y_\iota) \quad .$$

Thus if g_{π_2} is the relation of being a better boxer, and if m_ι is Muhammed Ali, then the latter's favorite assertion is symbolized by

$$Greatest_{\pi_1\pi_2}\, g_{\pi_2}\, m_\iota \quad .$$

The truth , or possibility, of such an assertion depends not only on properties of m_ι but on properties of g_{π_2} : unless g_{π_2} provides a decent kind of ordering then $Greatest_{\pi_1\pi_2}\, g_{\pi_2}$ will be the null class. Taking our guide from lattice theory, we can also define

$$Maximal_{\pi_1\pi_2} =_{df} \lambda g_{\pi_2} \lambda x_\iota (y_\iota)(g_{\pi_2} y_\iota x_\iota \supset y_\iota = x_\iota) \quad .$$

We do not spell out the difference between maximal elements and greatest elements, or the conditions under which $Maximal_{\pi_1\pi_2}\, g_{\pi_2}$ may be non-null even though $Greatest_{\pi_1\pi_2}\, g_{\pi_2}$ is null: these matters are well-known.

In order to form greatest elements from among some subset of the field of g_{π_2} , we define the (0-ary 2-place heteradic) modifier

$$Greatest_{\pi_1\pi_2\pi_1} =_{df} \lambda f_{\pi_1} \lambda g_{\pi_2} \lambda x_\iota (y_\iota)(f_{\pi_1} y_\iota \;\&\; x_\iota \neq y_\iota \supset g_{\pi_2} x_\iota y_\iota) \ .$$

This modifier does not require that $f_{\pi_1} x_\iota$ should hold, and so we can use it in, say, the definition of a complete lattice whereby for all f_{π_1} , $Greatest_{\pi_1\pi_2\pi_1} f_{\pi_1} g_{\pi_2}$ is required to be non-null. However in applications in English we will expect the greatest element of a set to belong to that set (all our sets are closed, as it were), and we define

$$EGreatest_{\pi_1\pi_2\pi_1} =_{df} \lambda f_{\pi_1} \lambda g_{\pi_2} \lambda x_\iota (Greatest_{\pi_1\pi_2\pi_1} f_{\pi_1} g_{\pi_2} x_\iota \;\&\; f_{\pi_1} x_\iota) \ .$$

Similar definitions can be given for $Maximal_{\pi_1\pi_2\pi_1}$ and $EMaximal_{\pi_1\pi_2\pi_1}$.

Our purely extensional modifiers can be extended simply enough to modifiers of properties by definitions like

$$EGreatest_{(\pi_1\kappa)(\pi_2\kappa)} =_{df} \lambda f_{\pi_2\kappa}\lambda w_\kappa (EGreatest_{\pi_1\pi_2}(f_{\pi_2\kappa} w_\kappa)) \quad :$$

nothing more complicated than such a world-by-world extension seems needed. But even so, we have still not got sufficient mechanism to deal with the relation "better" and the superlative "best", since "better" is notoriously not detachable and must be treated as a modifier and not via any detacher thereof. So if we treat "better" as a modifier of type $(\pi_2\kappa)(\pi_1\kappa)$, mapping e.g. the property of being a man to the relation of being a better man than, then "best" will be a modifier of type $(\pi_1\kappa)^2$, and we will need a "heteradic" modifier modifier of type $(\pi_1\kappa)^2((\pi_2\kappa)(\pi_1\kappa))$ to state the relationship between "better" and "best".

Thus our more general superlative-former is

$$Super_{(\pi_1\kappa)^2((\pi_2\kappa)(\pi_1\kappa))} =_{df} \lambda f_{(\pi_2\kappa)(\pi_1\kappa)}(\imath g_{(\pi_1\kappa)^2})(h_{\pi_1\kappa})(w_\kappa)(x_\iota)(g_{(\pi_1\kappa)^2}h_{\pi_1\kappa}w_\kappa x_\iota \equiv (y_\iota)(x_\iota \neq y_\iota \;\&\; h_{\pi_1\kappa}w_\kappa y_\iota \supset f_{(\pi_2\kappa)(\pi_1\kappa)}h_{\pi_1\kappa}w_\kappa x_\iota y_\iota)) \;.$$

If we define a necessary detacher for modifiers of type $(\pi_2\kappa)(\pi_1\kappa)$ by

$$Detachern_{(\pi_2\kappa)((\pi_2\kappa)(\pi_1\kappa))} =_{df} \lambda f_{(\pi_2\kappa)(\pi_1\kappa)}(\imath g_{\pi_2\kappa})(h_{\pi_1\kappa})(w_\kappa)(x_\iota)(y_\iota)(f_{(\pi_2\kappa)(\pi_1\kappa)}h_{\pi_1\kappa}w_\kappa x_\iota y_\iota \equiv . \; g_{\pi_2\kappa}w_\kappa x_\iota y_\iota \;\&\; h_{\pi_1\kappa}w_\kappa x_\iota \;\&\; h_{\pi_1\kappa}w_\kappa y_\iota)$$

then the diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
(\pi_2\kappa)(\pi_1\kappa) \arrow[r, "Super_{(\pi_1\kappa)^2((\pi_2\kappa)(\pi_1\kappa))}"] \arrow[d, "Detachern_{(\pi_2\kappa)((\pi_2\kappa)(\pi_1\kappa))}"'] & (\pi_1\kappa)^2 \arrow[d, "Detachern_{(\pi_1\kappa)((\pi_1\kappa)^2)}"] \\
\pi_2\kappa \arrow[r, "EGreatest_{(\pi_1\kappa)(\pi_2\kappa)}"] & \pi_1\kappa
\end{tikzcd}$$

expresses the relationship that we expect to hold for detachable modifiers; but in general the superlative-former cannot be defined in terms of modifiers like $EGreatest_{(\pi_1\kappa)(\pi_2\kappa)}$.

The argument

(1) Jack is a best man
(in which we do not have the idiomatic use of "best man")
(2) Bill is a man different from Jack
$\therefore$
(3) Jack is a better man than Bill
can now be dealt with by the dictionary

j_ι	Jack
b_ι	Bill
$m_{\pi_1\kappa}$	the property of being a man
$b_{(\pi_2\kappa)(\pi_1\kappa)}$	the modifier "better" ,

and the symbolization

(4) $Super_{(\pi_1\kappa)^2((\pi_2\kappa)(\pi_1\kappa))}\, b_{(\pi_2\kappa)(\pi_1\kappa)}\, m_{\pi_1\kappa}\, g_\kappa\, j_\iota$

(5) $m_{\pi_1\kappa}\, g_\kappa\, b_\iota \;\;\&\;\; b_\iota \neq j_\iota$

$\therefore$

(6) $b_{(\pi_2\kappa)(\pi_1\kappa)}\, m_{\pi_1\kappa}\, g_\kappa\, j_\iota\, b_\iota$;

this validates the argument without assuming that "better" is detachable, ie. that it is one fixed ordering relation among individuals. A treatment of the argument along the lines of Wallace's [73] p.773 would require this false assumption.

So much for superlatives and their complications. What now of the positive forms - "large" from "larger", "good" from "better", etc.? Firstly we set out a simple suggestion, which Wallace ascribes to C.H. Langford and rejects. For this, we need the dyadic quantifier $M_{o(o\iota)(o\iota)}$ introduced in §2.7, and we can use it in definitions like

$$U_{\pi_1\pi_2} =_{df} \lambda g_{\pi_2} \lambda x_\iota \, (My_\iota)(x_\iota \neq y_\iota \, , \, g_{\pi_2} x_\iota y_\iota) \quad ,$$

with $(Mx_\iota)(A_o, B_o)$ abbreviating $M_{o(o\iota)(o\iota)}(\lambda x_\iota A_o)(\lambda x_\iota B_o)$.

Then $U_{\pi_1\pi_2} g_{\pi_2} x_\iota$ holds iff x_ι has g_{π_2} to most other individuals y_ι , and this seems to be a reasonable start to a solution of the problem. We can go on to define

$$EU_{\pi_1\pi_2\pi_1} =_{df} \lambda f_{\pi_1} \lambda g_{\pi_2} \lambda x_\iota \, (f_{\pi_1} x_\iota \, \& \, (My_\iota)(x_\iota \neq y_\iota \, \& \, f_{\pi_1} y_\iota \, , \, g_{\pi_2} x_\iota y_\iota))$$

by analogy with $EGreatest_{\pi_1\pi_2\pi_1}$ and this makes $EU_{\pi_1\pi_2\pi_1} f_{\pi_1} g_{\pi_2} x_\iota$ hold iff x_ι is an f_{π_1} and has g_{π_2} to most other individuals that have f_{π_1} . Following along these lines we reach a positive-forming function, analogous to our superlative-forming function $Super$,

$$Positive_{(\pi_1\kappa)^2((\pi_2\kappa)(\pi_1\kappa))} =_{df} \lambda f_{(\pi_2\kappa)(\pi_1\kappa)} (\iota g_{(\pi_1\kappa)^2})(h_{\pi_1\kappa})(w_\kappa)(x_\iota)(g_{(\pi_1\kappa)^2} h_{\pi_1\kappa} w_\kappa x_\iota$$
$$\equiv (My_\iota)(x_\iota \neq y_\iota \, \& \, h_{\pi_1\kappa} w_\kappa y_\iota \, , \, f_{(\pi_2\kappa)(\pi_1\kappa)} h_{\pi_1\kappa} w_\kappa x_\iota y_\iota)) \, ,$$

simply by replacing "all" by "most" throughout.

We should now test $Positive$ against Wallace's objection, [73] p.777, "If I manufacture a distinctive line of lawn mower that includes a special diminutive version for midgets and children, it may very well be true that most Wallace lawn mowers are large Wallace lawn mowers." One reply to this objection is that it is not using "large" as a positive of "larger than": larger than for Wallace lawn mowers is being assessed by the usual criteria, but "large" is just a label for the non-diminutive models. If, say, there is a ratio of 3:1 between the non-diminutive and the diminutive models, then there are in fact no large Wallace lawn mowers according to the use of "large" as the positive of "larger than" and not in Wallace's sense.

I do not know how to assess this reply to Wallace's objection, and even if successful I do not know how to assess the standing of our *Positive* vis-à-vis Wang's paradox. However, if the defined form of *Positive* has to be abandoned, then the fact that "larger than" etc. are modifiers rather than predicates or relations in general would force some changes in Wallace's approach. In particular, I doubt whether Wallace's Mod function would be able to form "x is a good man" in a satisfactory way, and in general the α in $Mod(x,\alpha,\beta)$ should have the type of a modifier ($(\pi_2\kappa)(\pi_1\kappa)$) not that of a relation-in-intension ($\pi_2\kappa$).

2.17 Attribute theory and set theory

Within Church's formulation of the theory of types, we can define a form of the membership relation viz.

$$Epsilon_{o\alpha(o\alpha)} =_{df} \lambda x_{o\alpha} \lambda y_{\alpha} (x_{o\alpha} y_{\alpha}) \quad ,$$

and we may write $(y_\alpha \in x_{o\alpha})$ for $(Epsilon_{o\alpha(o\alpha)} x_{o\alpha} y_\alpha)$ (or for $(x_{o\alpha} y_\alpha)$ for that matter). Obviously not all of set theory can be reproduced using the relation thus defined, because of the limitations made explicit by the type symbols α and $o\alpha$ in the expression $y_\alpha \in x_{o\alpha}$ (and also for simple model-theoretic reasons). But this is not a serious bar to contrasting a portion of attribute theory with set theory according to the theory of types, since sufficient contrast should appear with "well-typed" attributes and a relation of having an attribute restricted in the same way as the membership relation just defined.

We take the intuitions of Lemmon's [43] as our starting point: we find that we cannot capture all of Lemmon's intuitions, nor can we interpret the axioms that do not involve non-well-typed attributes, let alone provide semantics for his whole system. We suggest that this is a pointer more to the need for a firm semantic basis for any axiomatization of attribute theory than to obvious inadequacy in our system.

To set out Lemmon's intuitions, we quote from his p.98:

" (i) the morning star is the evening star, but not necessarily so;

(ii) both the morning star and the evening star have the attribute of being the morning star, and belong to the extension of that attribute;

(iii) the morning star, but not the evening star, has the attribute of being necessarily the morning star, and the former, but not the latter, belongs to the extension of that attribute;

(iv) the unit class containing the morning star is not identical with the class of things that are necessarily the morning star, because the evening star belongs to the former but not to the (intensionally conceived) latter class;

(v) the class containing just the morning star and the evening star contains as a matter of fact exactly one thing."

For analysis, let us set up the dictionary

$m_{o\iota\kappa}$ the property of being the morning star, i.e. of being the last "star" clearly visible in the west at dawn,...

$e_{o\iota\kappa}$ the property of being the evening star.

Then we may define, in familiar fashion

$$ms_{\iota\kappa} = \lambda w_{\kappa}(\imath x_{\iota})\, m_{o\iota\kappa} w_{\kappa} x_{\iota}$$

and

$$es_{\iota\kappa} = \lambda w_{\kappa}(\imath x_{\iota})\, e_{o\iota\kappa} w_{\kappa} x_{\iota}$$

respectively for (the individual concepts of) the morning star and the evening star. Lemmon's (i) is then dealt with by the simple assertion

$$Coincr_{o(\iota\kappa)^2}\, ms_{\iota\kappa}\, es_{\iota\kappa} \;\&\; ms_{\iota\kappa} \neq es_{\iota\kappa} \quad .$$

For Lemmon's (ii), we need either to define an attribute from the property $m_{o\iota\kappa}$ in such a way that $ms_{\iota\kappa}$ and $es_{\iota\kappa}$ can have the attribute, or else to define a relation of having an attribute so that $m_{o\iota\kappa}$ is taken itself as an attribute that $ms_{\iota\kappa}$ and $es_{\iota\kappa}$ can (significantly) have. We consider the two possibilities in turn.

First we define

$$attm_{o(\iota\kappa)\kappa} = \lambda w_{\kappa} \lambda x_{\iota\kappa} (m_{o\iota\kappa} w_{\kappa} (x_{\iota\kappa} w_{\kappa})) \quad ,$$

where we intend $attm_{o(\iota\kappa)\kappa}$ to be the attribute of being the morning star (so that attributes are, in effect, properties of individual concepts). Then we can have both

$$attm_{o(\iota\kappa)\kappa}\, g_{\kappa}\, ms_{\iota\kappa}$$

and

$$attm_{o(\iota\kappa)\kappa}\, g_{\kappa}\, es_{\iota\kappa}$$

without having, in particular, $(w_{\kappa})(attm_{o(\iota\kappa)\kappa} w_{\kappa} es_{\iota\kappa})$. Moreover, the two formulae will do equally well for the last clause of (ii), since if a one-place attribute has type $o(\iota\kappa)\kappa$ we may form the extension of such an attribute by means of the function

$$Ext_{o(\iota\kappa)(o(\iota\kappa)\kappa)} =_{df} \lambda x_{o(\iota\kappa)\kappa} (x_{o(\iota\kappa)\kappa}\, g_{\kappa}) \quad .$$

Thus the extension of an attribute is a subset of individual concepts, and both $ms_{\iota\kappa}$ and $es_{\iota\kappa}$ belong to the extension of $attm_{o(\iota\kappa)\kappa}$. In this first approach, the relation of having an attribute is the same as the relation of belonging to the extension of the attribute: we could make this explicit by the definition

$$Have_{o(\iota\kappa)(o(\iota\kappa)\kappa)} =_{df} \lambda x_{o(\iota\kappa)\kappa} \lambda y_{\iota\kappa} (x_{o(\iota\kappa)\kappa}\, g_\kappa\, y_{\iota\kappa}) ,$$

whence the two formulae stating Lemmon's (ii) become long-windedly

$$Have_{o(\iota\kappa)(o(\iota\kappa)\kappa)}\, attm_{o(\iota\kappa)\kappa}\, ms_{\iota\kappa}$$

and

$$Have_{o(\iota\kappa)(o(\iota\kappa)\kappa)}\, attm_{o(\iota\kappa)\kappa}\, es_{\iota\kappa} .$$

On the second approach to attributes, and the having thereof, we change the type of the "having" relation. We now treat properties, of type $o\iota\kappa$, as attributes; more generally any wff of type $o\alpha\kappa$ may be an attribute of entities of type $\alpha\kappa$, so that $o\iota\kappa$ is the type of individual attributes. Entities of type $\alpha\kappa$ now have attributes according to the definition

$$^2Have_{o(\alpha\kappa)(o\alpha\kappa)} =_{df} \lambda x_{o\alpha\kappa} \lambda y_{\alpha\kappa} (x_{o\alpha\kappa}\, g_\kappa\, (y_{\alpha\kappa}\, g_\kappa))$$

and we write

$$(y_{\alpha\kappa}\ h\ x_{o\alpha\kappa}) \quad \text{for} \quad (^2Have_{o(\alpha\kappa)(o\alpha\kappa)}\, x_{o\alpha\kappa}\, y_{\alpha\kappa}) .$$

We also define

$$Ext_{o\alpha(o\alpha\kappa)} =_{df} \lambda x_{o\alpha\kappa} (x_{o\alpha\kappa}\, g_\kappa)$$

of which our previous $Ext_{o(\iota\kappa)(o(\iota\kappa)\kappa)}$ is a special case. Now Lemmon's (ii) may be stated thus:

$$(ms_{\iota\kappa}\ h\ m_{o\iota\kappa}) \,\&\, (es_{\iota\kappa}\ h\ m_{o\iota\kappa}) \,\&$$

$$(ms_{\iota\kappa}\, g_\kappa \in Ext_{o\iota(o\iota\kappa)}\, m_{o\iota\kappa}) \,\&\, (es_{\iota\kappa}\, g_\kappa \in Ext_{o\iota(o\iota\kappa)}\, m_{o\iota\kappa}) .$$

Keeping both approaches as options, we express (iii) according to each. On the first approach its expression is

$$Have_{o(\iota\kappa)(o(\iota\kappa)\kappa)}\, nattm_{o(\iota\kappa)\kappa}\, ms_{\iota\kappa} \,\&\, \sim(Have_{o(\iota\kappa)(o(\iota\kappa)\kappa)}\, nattm_{o(\iota\kappa)\kappa}\, es_{\iota\kappa})$$

where $\quad nattm_{o(\iota\kappa)\kappa} = \lambda w_\kappa \lambda x_{\iota\kappa} (v_\kappa)(m_{o\iota\kappa}\, v_\kappa\, (x_{\iota\kappa} v_\kappa))$

and where again the part about belonging to the extension of the attribute is built into the relation *Have* .

On the second approach we define

$$nm_{o\iota\kappa} =_{df} \lambda w_\kappa \lambda x_\iota (v_\kappa)(m_{o\iota\kappa} v_\kappa x_\iota)$$

as our necessary attribute of type $o\iota\kappa$, and then we have the expression for (iii):

$$(ms_{\iota\kappa}\ h\ nm_{o\iota\kappa}) \& \sim(es_{\iota\kappa}\ h\ nm_{o\iota\kappa}) \&$$

$$(ms_{\iota\kappa} g_\kappa \in Ext_{o\iota(o\iota\kappa)} nm_{o\iota\kappa}) \& \sim(es_{\iota\kappa} g_\kappa \in Ext_{o\iota(o\iota\kappa)} nm_{o\iota\kappa}),$$

and this is inconsistent since $ms_{\iota\kappa} g_\kappa = es_{\iota\kappa} g_\kappa$. Hence the second approach has to be rejected: our extensions, as it were, can't be too extensional since if they are outright contradiction results from Lemmon's intuitions.

For Lemmon's (iv), we define the unit class of the morning star by $ucm_{o\iota} = \lambda x_\iota (x_\iota = ms_{\iota\kappa} g_\kappa)$,

whence both $(ms_{\iota\kappa} g_\kappa \in ucm_{o\iota})$ and $(es_{\iota\kappa} g_\kappa \in ucm_{o\iota})$ hold, given Lemmon's (v), which amounts simply to $ms_{\iota\kappa} g_\kappa = es_{\iota\kappa} g_\kappa$. However, the "because" clause of (iv) causes problems: we cannot define the class of things which are necessarily the morning star as $\lambda x_\iota (v_\kappa)(x_\iota = ms_{\iota\kappa} g_\kappa)$, since clearly this is equal to $ucm_{o\iota}$. We can define the different class $\lambda x_\iota (v_\kappa)(x_\iota = ms_{\iota\kappa} v_\kappa)$ but I take it that this is the null class since there is no reason why (the substance) Venus should be the last "star" seen in the morning in all possible worlds. In fact Lemmon does not say that the class in question is <u>not</u> the null class, but it is reasonable to suppose that he wanted the morning star to belong to it. So we might suppose that the class in question is a class of individual concepts (this would be one construal of the parenthetical, but unclear, "intensionally conceived"), and that the definition $\lambda x_{\iota\kappa} (v_\kappa)(x_{\iota\kappa} = ms_{\iota\kappa})$ (or $\lambda x_{\iota\kappa}(x_{\iota\kappa} = ms_{\iota\kappa})$, or $\lambda x_{\iota\kappa} (v_\kappa)(x_{\iota\kappa} v_\kappa = ms_{\iota\kappa} v_\kappa)$) would serve: call this class $nucm_{o(\iota\kappa)}$. Then we have

$$(ms_{\iota\kappa} \in nucm_{o(\iota\kappa)}) \& \sim(es_{\iota\kappa} \in nucm_{o(\iota\kappa)}) ,$$

matching the "because" clause. But of course, there is no question that $nucm_{o(\iota\kappa)}$ should equal $ucm_{o\iota}$, since they have different types, so it is not clear that we have yet captured Lemmon's intentions.

Indeed it becomes clear that we have not done so when we consider the axioms of Lemmon's system. We are unable to give any account of the various appearances of the necessity operator $\Box$ in axioms A4-A6 on p.101 of [43]: Lemmon's introductory discussion has not, for us, provided a rationale for these appearances. Thus, as we said before, we cannot provide semantics for Lemmon's system: but, dear Brutus, we suggest that this time the fault lies not in ourselves.

2.18. Concluding remarks

Having set an all-time high so far as the definition/theorem ratio goes, some concluding remarks to justify this situation might be in order. We claim to have shown, what we set out to show, that Church's formulation of the simple theory of types provides a comprehensive and workable framework in which to deal with the logic of predicate modifiers and various aspects of intensional logics. In the former case, no syntactical or semantical extensions of Church's system are required: in a sense, there is no special logic of modifiers, except insofar as certain restricted fragments may be isolated and axiomatized or whatever. In the latter case the syntax and semantics are extended minimally, by κ and $\mathcal{D}_\kappa$: the result is a system of strong expressive power, rock-solid semantics, and with a wide range of philosophical applications. Even in those cases where we have reached no firm conclusions, we have at least been able to formulate the problems involved in some kind of accurate fashion.

We leave as an exercise the application of our system to venerable problems in modal logic, of the kind posed by Quine and others. The work is easy and the rewards, except for those who suppose that philosophical problems cannot be solved, are well worth the trouble.

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$$\mathcal{J}_2(\xi) = {}^{\wedge}\xi$$

$$\mathcal{J}_3(\xi) = {}^{\vee}\xi$$

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