CORRECTIONS

p. 6-7 The statement "Therefore, h_α is a proper subspace ... when $\alpha < 0$ " is true provided H is unbounded.

p. 9-10 For the definition of the maximal multiplication operator see Kato, T., *Perturbation Theory for Linear Operators*, 2nd edition, Springer-Verlag Berlin 1984. Kato also proves (p. 299-302) that the Laplacian can be represented as multiplication by $|x|^2$.

p. 83 References described as "Canberra preprint" are Research Reports issued by the Mathematical Sciences Research Centre, The Australian National University.

COMMUTATORS AND GENERATORS

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I declare that this thesis is my own work and that all sources used in it have been acknowledged.

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Abstract

Since the late 1960's mathematicians working mainly in the area of quantum field theory have used commutator estimates to prove self-adjointness of operators on Hilbert space. Recently, these results were extended and codified into a theory in the papers [BaR], [Rob 1], and [Rob 2]. The following thesis is based largely upon the above reports.

Given a self-adjoint operator H on a Hilbert space h, spectral theory can be used to construct a decreasing set of subspaces $h_\alpha = \mathcal{D}((I+H^2)^{\alpha/2})$, $(\alpha \geq 0)$, and $h_\infty = \bigcap_{\alpha \geq 0} h_\alpha$. Let K be a symmetric operator with domain h_∞. Then, if [H, K] is bounded in a certain sense, K is essentially self-adjoint. If the higher order commutators $(\mathrm{ad}H)^n(K)$ are also bounded, the action of $V_t = \exp\{i\,t\,\overline{K}\}$ (or $T_t = \exp\{-\,t\,\overline{K}\}$ if K is positive) on the spaces h_α can be described. Commutator estimates are also useful in proving that two symmetric operators with domain h_∞ commute. There are applications to operators which are polynomials in x and i d/dx on $L^2(\mathbb{R})$. Some types of first and second-order ordinary, differential operators can also be treated.

Similar results occur in the more general setting of a Banach space B. Let σ_t be a one-parameter C_0-group of

isometries on B with generator H and define the spaces B_n, $(n \in N)$, and B_∞ by

$$B_n = \mathcal{D}(H^n) \quad , \quad B_\infty = \bigcap_{n \in N} B_n \, .$$

In addition, let K be a dissipative operator with domain B_∞. In this situation bounds on the commutator $[\sigma_t, K]$ are of use in showing that $\bar{K}$ generates a one-parameter semigroup of contractions τ_t on B. Again, the existence of higher order commutator bounds ensures that τ_t acts regularly on the spaces B_n.

We define the Lipshitz spaces $B_{n+\frac{1}{2}}$ $(n \in N)$ by

$$B_{n+\frac{1}{2}} = \left\{ a \in B_n \, : \, \sup_{|t| \leq 1} \left\{ \frac{\|(\sigma_t - I) H^n a\|}{|t|} \right\} < \infty \right\} .$$

Commutator estimates are of relevance in characterizing operators K such that $K B_\infty \subseteq B_{n+\frac{1}{2}}$.

CONTENTS

CH1. COMMUTATOR THEOREMS IN HILBERT SPACE.

§1. The Hilbert space setting.

In this chapter h will denote a fixed Hilbert space and H a fixed self-adjoint operator on h . Using H , we can construct a decreasing sequence of spaces $\{h_\alpha\}_{\alpha \in \mathbb{R}}$. For $\alpha \geq 0$ we define h_α by

(1.1.1)

$$h_\alpha = \mathcal{D}\left((1+H^2)^{\frac{\alpha}{2}}\right)$$

and make it into a Hilbert space by assigning it the inner product $<\,,\,>_\alpha$ defined by

$$(1.1.2) \quad <a,b>_\alpha = <(1+H^2)^{\frac{\alpha}{2}} a, (1+H^2)^{\frac{\alpha}{2}} b> \qquad \forall a, b \in h_\alpha$$

so we have

$$\| a \|_\alpha = \sqrt{<a,a>_\alpha} = \|(1+H^2)^{\frac{\alpha}{2}} a\| \quad \text{for } a \in h_\alpha .$$

For $\alpha < 0$, h_α is defined to be the completion of h under the inner product (1.1.2). Therefore, h_α is a proper subspace of h when $\alpha > 0$ and h is a proper subspace of h_α when $\alpha < 0$.

If we use the spectral theorem to represent H as multiplication by the real-valued function $\hbar(\cdot)$ on $L^2(M,dm)$, then $h_\alpha = L^2(M, (1+\hbar^2(m))^{\alpha/2}\, dm)$, $(\alpha \in \mathbb{R})$.

We also define the spaces $h_{-\infty}$ and h_∞ by

$$h_{-\infty} = \bigcup_{\alpha \in \mathbb{R}} h_\alpha \quad , \quad h_\infty = \bigcap_{\alpha \in \mathbb{R}} h_\alpha \, .$$

The space h_∞ is a Frechet [illegible] when its topology is determined by the family of norms $\{\|\cdot\|_\alpha \, ; \, \alpha \in \mathbb{R}\}$. We note that h_∞ is a core for H^K for any $k \in N$.

If $H \geq 0$, then $\mathcal{D}((1+H)^\alpha) = h_\alpha$ and we can introduce a second inner product $< , >'_\alpha$ on the spaces h_α;
$<a,b>'_\alpha = <(1+H)^\alpha a, (1+H)^\alpha b>$, $\| a \|'_\alpha = \|(1+H)^\alpha a\|$ $\forall a, b \in h_\alpha$.
Since

$$\| a \|_\alpha \leq \| a \|'_\alpha \leq 2^{\frac{\alpha}{2}} \| a \|_\alpha$$

the two norms are equivalent on h_α.

We can also define the operator $(i+H)^\alpha$ by spectral theory for any $\alpha \in \mathbb{R}$. This is possible since

$$\sigma(i+H) \cap \mathbb{R} = \emptyset .$$

~~We then have~~ $\cancel{\mathcal{D}((i+H)^{\alpha}) = h_\alpha}$ and

$$<(i+H)^{\alpha}a , (i+H)^{\alpha}b> = <a, b>_\alpha \quad \text{for } a,b \in h_\alpha .$$

We let $\mathfrak{L}(h_\alpha, h_\beta)$ be the space of bounded linear operators from h_α to h_β. Similarly, $\mathfrak{L}_0(h_\alpha, h_\beta)$ is the space of linear operators from h_∞ to h_β which are bounded as operators from h_α to h_β. We have the following set of equivalences for an operator K with $\mathcal{D}(K) = h_\infty$;

$$K \in \mathfrak{L}_0(h_\beta, h_\alpha)$$

$\Leftrightarrow$ $(1+H^2)^{-\alpha/2} K (1+H^2)^{-\beta/2}$ is bounded as an operator on h

$\Leftrightarrow$ $\exists\, c \geq 0$ such that

$$| <a, Kb> | \leq c \, \| a \|_\alpha \, \| b \|_\beta \qquad \forall a, b \in h_\infty .$$

Let $K : h_\infty \to h$ be a closable linear operator. Then, since h_∞ is a Frechet space, we can deduce using the closed graph theorem that $\exists\, c \geq 0$ and $p \geq 0$ such that

$$\| Ka \| \leq c \, \| a \|_p \qquad \forall\, a \in h_\infty$$

or, equivalently, we can deduce that the operator $K(1+H^2)^{-p/2}$ is bounded on h. In particular if $K: h_\infty \to h$ is symmetric such a p exists.

§2. Examples.

A. Let $h = L^2(\mathbb{R})$ and

$$H = i \overline{\frac{d}{dx} \Big|_{C_c^\infty(\mathbb{R})}}$$

Then H is self-adjoint. We can use the Fourier transform $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ to represent H as multiplication by x on $L^2(\mathbb{R})$;
i.e. for $\varphi \in \mathcal{D}(H)$

$$(\mathcal{F}(H\varphi))(x) = x(\mathcal{F}\varphi)(x) \qquad x \in \mathbb{R}$$

and $\mathcal{D}(H) = \mathcal{F}^{-1}\mathcal{D}(M_x)$ where M_x is the maximal multiplication operator by x on $L^2(\mathbb{R})$.

In this case

$$h_\alpha = \mathcal{D}((1+H^2)^{\alpha/2}) = \mathcal{F}^{-1}\mathcal{D}(M_{(1+x^2)^{\alpha/2}})$$

so,

$$h_\alpha = \{ \varphi \in L^2(\mathbb{R}) : (1+x^2)^{\alpha/2} \mathcal{F} \varphi \in L^2(\mathbb{R}) \} .$$

Similarly, choosing $h = L^2(\mathbb{R}^n)$ and

$$H = \overline{- \Delta|_{C_c^\infty(\mathbb{R}^n)}}$$

we can use the Fourier transform to represent H as multiplication by

$$|x|^2 = \sum_{i=1}^{n} x_i^2 \text{ on } L^2(\mathbb{R}^n) .$$

Therefore,

$$h_\alpha = \mathcal{F}^{-1} \mathcal{D}(M_{(1+|x|^4)^{\alpha/2}}) .$$

So, for $m \in N$, $h_{m/2}$ is the Sobolev space $H^m(\mathbb{R}^n)$ and $\|\cdot\|_{m/2}$ is equivalent to the Sobolev norm.

B. Fractional powers of a positive operator.

Pazy [Paz] represents fractional powers of certain operators A in the form

$$A^{-\alpha} = \frac{\sin \pi \alpha}{\pi} \int_0^\infty dt\, t^{-\alpha} (t + A)^{-1} \qquad 0 < \alpha < 1$$

and if $x \in \mathcal{D}(A)$ he gives the formula

$$A^{\alpha}x = \frac{\sin \pi \alpha}{\pi} \int_0^{\infty} dt\, t^{\alpha-1} A(t+A)^{-1} x .$$

These representations hold if $A \geq 1$. We also have the formula

$$(1.2.1) \qquad A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} dt\, t^{\alpha-1} \exp \{-tA\} \qquad \alpha > 0$$

where the integral converges in the uniform operator topology and $\exp\{-tA\}$ is the semigroup generated by $A \geq 1$.

Let

$$H = 1/2 \overline{\left(-\frac{d^2}{dx^2} + x^2 - 1\right)|_{\mathcal{S}(\mathbb{R})}}$$

where $\mathcal{S}(\mathbb{R})$ is the space of rapidly decreasing functions on $\mathbb{R}$.

Then $H \geq 0$. From [Dav] p.180-2 we know that

$$\left(\exp \{-tH\}f\right)(x) = \int_{y=-\infty}^{\infty} dy\, K_t(x,y) f(y) \qquad \forall f \in L^2(\mathbb{R})$$

where

$$K_t(x,y) = \left\{\pi(1-e^{-2t})\right\}^{-1/2} \exp\left\{\frac{4\,x\,y e^{-t} - (x^2+y^2)(1+e^{-2t})}{2(1-e^{-2t})}\right\}.$$

Therefore, from (1.2.1),

$$\left((1+H)^{-\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} dt \int_{-\infty}^{\infty} dy \; t^{\alpha-1} e^{-t} K_t(x,y) \; f(y) \quad \forall f \in \mathbf{L}^2(\mathbb{R}).$$

§3. Notation and structure of the results.

Section 4 below provides a summary of the commutator theorems on Hilbert spaces. It is first necessary to clear up some matters of notation and to comment on the structure of the results.

If K is an operator on h, the operator $(\mathrm{ad}H)^n(K)$ is defined recursively by $(\mathrm{ad}H)(K) = [H,K] = HK - KH$ and $(\mathrm{ad}H)^{n+1}(K) = [H,(\mathrm{ad}H)^n(K)]$. If however, K is unbounded with domain h_∞ this definition may not make sense. In this situation $(\mathrm{ad}H)(K)$ is to be interpreted as a quadratic form on $h_\infty \times h_\infty$. The commutator bounds which occur in §4 are quadratic form bounds. For example, the bound

$$| < a, (\mathrm{ad}H)(K)b>| \le c\| a \|_{1/2} \| b \|_{1/2} \qquad \forall a, b \in h_\infty$$

which is imposed in theorem 3 should be read as

$$| < Ha, Kb > - < a, KHb > | \le c\| a \|_{1/2} \| b \|_{1/2} \qquad \forall a, b \in h_\infty .$$

For a commutator bound of the form

$$(1.3.1) \qquad | < a,(\mathrm{ad}H)^n(K)b > | \le c\| a \|_\alpha \| b \|_\beta \qquad \forall a, b \in h_\infty$$

the index of the bound is defined to be $\alpha + \beta$. For fixed index $\alpha+ \beta$ we can use interpolation theory to decide what choice of α and β in (1.3.1) provides the strongest bound and what choice provides the weakest. The following result, which occurs as observation 2.4 in [Rob 2] , is particularly useful in this regard.

Theorem 1. Let s be a symmetric sesquilinear form over $h_\infty \times h_\infty$ and suppose

$$| s(a,b)| \le c\| a \|_\alpha \| b \|_\beta \qquad \forall a, b \in h_\infty$$

for some $\alpha, \beta \in \mathbb{R}$. Then

$$| s(a,b)| \le c \| a \|_{\alpha+\gamma} \| b \|_{\beta-\gamma} \qquad \forall a, b \in h_\infty$$

for all $\gamma \in [0, \beta - \alpha]$ if $\beta \ge \alpha$ or $\gamma \in [\beta - \alpha, 0]$ if $\beta \le \alpha$.

Therefore, the strongest commutator bounds of index $\alpha + \beta$ are of the form

$$| < a, (\mathrm{adH})^n(K)b >| \leq c\| a \| \, \| b \|_{\alpha+\beta} \qquad \forall a, b \in h_\infty$$

or

$$| < a, (\mathrm{adH})^n(K)b >| \leq c\| a \|_{\alpha+\beta} \| b \| \qquad \forall a, b \in h_\infty$$

and the weakest is of the form

$$| < a, (\mathrm{adH})^n(K)b >| \leq c\| a \|_{(\alpha+\beta)/2} \| b \|_{(\alpha+\beta)/2} \qquad \forall a, b \in h_\infty .$$

This implies, for example, that the self-adjointness of $\overline{K}$ in theorem 5 follows from theorem 3.

The bounds mentioned above can be expressed in another way. For example, (1.3.1) holds if and only if

$$(1.3.2) \qquad \| (z_1 + H)^{-\alpha}(\mathrm{adH})^n(K)(z_2 + H)^{-\beta} \| < \infty$$

where $z_1, z_2 \in \rho(H) \setminus \mathbb{R}$ and $(z_1 + H)^{-\alpha}$ is defined by spectral theory. (1.3.2) is to be interpreted in the quadratic form sense.

The results of §4 are concerned with deducing properties of an operator K which is symmetric and has domain h_∞. If K satisfies certain types of commutator bounds then we can show it is essentially self-adjoint on h_∞ and that the semigroups

$V_t = \exp\{i t \overline{K}\}$ and $T_t = \exp\{- t \overline{K}\}$ (if K is positive) are also semigroups on h_α for prescribed values of α. It is also sometimes possible to prove that two such operators K_1 and K_2 commute in the sense of unbounded operators.

If K is symmetric, then a bound of index one on the single commutator $[H,K]$ produces results. If K is positive, a bound of index two on the double commutator is also sufficient. The positivity of H allows the conclusions to be strengthened.

The following lemma ([Rob 2] p.24) indicates a relationship between single and double commutator bounds and will be used later.

Lemma 2. Let K be an operator from h_∞ into h and suppose that

$$| < a,Kb >| \leq k_0 \; \| a \|_{\alpha_0} \; \| b \|_{\beta_0} \qquad \forall a, b \in h_\infty$$

$$| < a,(\mathrm{ad}H)^2(K)b >| \leq k_2 \; \| a \|_{\alpha_2} \; \| b \|_{\beta_2}$$

Then

$$| < a,(\mathrm{ad}H)(K)b>| \leq k_1 \; \| a \|_{\alpha_1} \; \| b \|_{\beta_1} \qquad \forall a, b \in h_\infty$$

for some $k_1 \geq 0$ where $\alpha_1 = \alpha_0 \vee \alpha_2$, $\beta_1 = \beta_0 \vee \beta_2$.

§4. The theorems.

In this section we will state without proof the results of commutator theory on Hilbert space. Proofs can be found in [Rob 2] unless otherwise stated. Beneath each theorem we detail its history and variants.

Theorem 3. : Let $K: h_\infty \to h$ be symmetric and suppose that *for some $p \in \mathbb{N}$*

$$\| Ka \| \le c \| a \|_p \qquad \forall \alpha \in h_\infty$$

(1.4.1) $$| < a,(\mathrm{ad}H)(K)b > | \le k \| a \|_{1/2} \| b \|_{1/2} \qquad \forall a, b \in h_\infty .$$

Then $\overline{K}$ is essentially self-adjoint on any core for H^P and $\overline{K}\, h_{p+1} \subseteq h_{\frac{1}{2}}$.

Remarks:

1. Nelson [Nel] proved that K is essentially self-adjoint on h_∞ if it satisfies the additional estimate

$$| < a, Kb >| \le c_1 \| a \|_{1/2} \| b \|_{1/2} \qquad \forall a, b \in h_\infty .$$

2. Faris and Lavine [FaL] prove the following theorem which is

similar to, but weaker than, Theorem 3.

Theorem 3'. Let K be symmetric and $\mathcal{D}(K) = \mathcal{D}(H) = h_1$. Suppose $H \geq 0$ and that

(1.4.2) $\quad \pm i\,[H,K] \leq c\,H$ as quadratic forms for some $c < \infty$.

Then K is essentially self-adjoint.

(1.4.2) implies (1.4.1) by the following argument. If

$$|\langle a, [H,K]a\rangle| \leq c \langle a, Ha\rangle \qquad \forall\, a \in h_1$$

then, using the polarization identity, we have

$$|\langle a, [H,K]b\rangle| = \left|\sum_{\ell=0}^{3} \frac{i^{\ell}}{4} \langle a + i^{\ell} b, [H,K](a + i^{\ell} b)\rangle\right|$$

$$\leq \frac{1}{4} \sum_{\ell=0}^{3} |\langle a + i^{\ell} b, [H,K](a + i^{\ell} b)\rangle|$$

$$\leq \frac{1}{4} \sum_{\ell=0}^{3} c\, \| H^{\frac{1}{2}}(a + i^{\ell} b) \|^2$$

$$\leq c\{\| H^{\frac{1}{2}} a \|^2 + 2 \| H^{\frac{1}{2}} a \| \, \| H^{\frac{1}{2}} b \| + \| H^{\frac{1}{2}} b \|^2$$

$$\therefore \sup_{\substack{a,b \in h_\infty \\ \| H^{\frac{1}{2}} a \| \le 1 \\ \| H^{\frac{1}{2}} b \| \le 1}} | < a, [H,K]b > | \le c .$$

so, $| < a, [H,K]b > | \le c \, \| H^{1/2} a \| \, \| H^{1/2} b \|$

$$\le c \, \| a \|_{1/2} \, \| b \|_{1/2} \qquad \forall a, b \in h_\infty .$$

3. Both McBryan [McB] and Glimm and Jaffe [Gℓ J2] proved theorem 3 under the assumption that H is semi-bounded.

4. The original proof of theorem 3 was given in [DrS].

__Theorem 4.__ Suppose the conditions of theorem 3 hold with $H \ge 0$. Then for each $\alpha \in [0,1/2]$ the unitary group $V_t = \exp\{i t \bar{K}\}$ has the properties:

(i) $V_t h_\alpha = h_\alpha \qquad \forall t \in \mathbb{R}$.

(ii) $\| V_t a \|'_\alpha \le e^{|t| \frac{\alpha k}{2}} \| a \|'_\alpha \qquad \forall t \in \mathbb{R}, a \in h_\alpha$

(iii) $V|_{h_\alpha}$ is $\| \cdot \|'_\alpha$ - continuous (and therefore $\| \cdot \|_\alpha$ - continuous) .

Remark:

The only previous result of this kind was a theorem of Faris and Lavine ([FaL] theorem 2) which stated that under the conditions of theorem 3'

$$V_t h_{1/2} = h_{1/2} \qquad \forall t \in \mathbb{R} .$$

Theorem 5: Let $K: h_\infty \to h$ be symmetric and suppose that

$$\| Ka \| \leq C \| a \|_p \qquad \forall a \in h_\infty$$

$$| < a, (adH)(K)b > | \leq k \| a \| \, \| b \|_1 \qquad \forall a, b \in h_\infty .$$

Then $\overline{K}$ is essentially self-adjoint on any core for H^p.

Moreover, $\overline{K} h_{p+1} \subseteq h_1$ and the unitary group $V_t = \exp \{i t \overline{K}\}$ satisfies the following for each $\alpha \in [0,1]$:

(i) $\quad V_t h_\alpha = h_\alpha \qquad \forall t \in \mathbb{R}$

(ii) $\quad \| V_t a \|_\alpha \leq e^{|t| \alpha k} \| a \|_\alpha \qquad \forall t \in \mathbb{R}, a \in h_\alpha$

(iii) $\quad V|_{h_\alpha}$ is $\| \cdot \|_\alpha$ - continuous .

Remarks:

1. Glimm and Jaffe ([Gl J1] theorem 1.2, p.1570) prove theorem 5 in the special case that H is semi-bounded and $p = 1$.

2. The following result of McBryan ([McB] theorem 3) is similar to theorem 5.

Theorem 5' : Suppose $H \geq 0$ and let $K : h_\infty \to h$ be symmetric. Suppose also that

(1.4.3) $$|< a, Kb >| \leq c \| a \|_p \| b \|_q \qquad \forall a, b \in h_\infty$$
for some $p,q \in N$.

$$|< a,(adH)(K)b >| \leq k\| a \| \| b \|_1 \qquad \forall a, b \in h_\infty .$$

Then $\overline{K}$ is essentially self-adjoint on any core for H^{p+q}. Moreover, $\overline{K}\, h_{p+q+1} \subseteq h_1$ and if $V_t = \exp\{i\, t\, \overline{K}\}$ then

(i) $V_t\, h_1 = h_1 \qquad \forall\, t \in \mathbb{R}$

(ii) $V|_{h_1}$ is $\|\cdot\|_1$- continuous .

More generally, if (1.4.3) holds and there are constants

k_s , $s = 0, 1, \ldots, r-1$ such that

$$|< a, (\mathrm{ad}H)(K)b >| \leq k_s \| a \|_{-s} \| b \|_{1+s} \qquad \forall a, b \in h_\infty$$

then $\overline{K}$ is essentially self-adjoint on any core for H^{p+q} and $\overline{K}\, h_{p+q+s} \subseteq h_s$ for $s = 0,1,\ldots, r$. In addition, if $V_t = \exp\{i\, t\, \overline{K}\}$ then for $s = 0,1,\ldots, r$ we have :

(i) $\quad V_t\, h_s = h_s \qquad\qquad \forall\, t \in \mathbb{R}$

(ii) $\quad V\, |_{h_s}$ is $\|\cdot\|_s$ - continuous.

3. The self-adjointness part of theorem 5 was proved in [DrS].

Theorems 3 and 5 prove self-adjointness of $\overline{K}$ under bounds of index one. Self-adjointness of $\overline{K}$ follows from a more general type of commutator bound, called a mixed estimate of index one.

Theorem 6. Let $K: h_\infty \to h$ be symmetric and suppose that

$$\| Ka \| \leq c \| a \|_p$$

$$|< a, (\mathrm{ad}H)(K)b >| \leq \sum_{i=1}^{N} k_i \| a \|_{\beta_i} \| b \|_{1-\beta_i} \qquad \forall a, b \in h_\infty$$

where $\beta_i \in [0,1]\ \forall i$.

Then $\bar{K}$ is essentially self-adjoint on any core for H^p .

Remark:

Theorem 6 can be established by using the proof of theorem 2.1 in [Rob2] to show that $\pm i\bar{K}$ generate contraction semigroups. Alternatively, the proof of theorem 1 in [DrS] can be modified to deal with this situation.

As mentioned in the introduction, theorems 3 and 4 use the weakest commutator bound of index one, while theorem 5 uses the strongest such bound. For intermediate bounds we have the following:

Theorem 7. Let $K : h_\infty \to h$ be symmetric and suppose that

$$\| Ka \| \le c \| a \|_p$$

$$| \langle a,(\mathrm{ad}H)(K)b \rangle | \le k \| a \|_{\frac{1+\beta}{2}} \| b \|_{\frac{1-\beta}{2}} \qquad \forall a, b \in h_\infty$$

where $\beta \in [-1, 1]$. Then $\bar{K}$ is essentially self-adjoint on any core for H^p, $\bar{K}\, h_{p+1} \subseteq h_{\frac{1+|\beta|}{2}}$, and for each $\alpha \in [0, \frac{1+|\beta|}{2})$

the unitary group $V_t = \exp\{i\, t\, \bar{K}\}$ satisfies:

(i) $\quad V_t\, h_\alpha = h_\alpha \qquad\qquad \forall\, t \in \mathbb{R}$

(ii) $\quad V|_{h_\alpha}$ is $\|\cdot\|_\alpha$ - continuous.

Theorems 8 to 12 below show that when K is assumed to be semi-bounded, second order commutator bounds of index two become important.

Theorem 8: Let $K: h_\infty \to h$ be positive and suppose that

$$\| Ka \| \le c\| a \|_p \qquad\qquad \forall a \in h_\infty$$

(1.4.4) $\quad | < a, (adH)^2(K)b > | \le k \| a \|_1 \| b \|_1 \qquad \forall a, b \in h_\infty.$

Then $\bar{K}$ is essentially self-adjoint on any core for H^p and $\bar{K}\, h_{p+2} \subseteq h_1$.

Remarks:

(1) $\quad$(1.4.4) holds if $\pm(adH)^2(K) \le c_1 H^2$ for some $c_1 \ge 0$ as quadratic forms on $h_\infty \times h_\infty$.

(2) Theorem 8 was proved in [DrS] under the additional assumption that H is semibounded.

Theorem 9: If the conditions of theorem 8 hold and $H \geq 0$ then for each $\alpha \in [0, 1]$ the contraction semigroup $T_t = \exp\{-t\bar{K}\}$ has the following properties

(i) $T_t h_\alpha \subseteq h_\alpha \qquad \forall\, t \geq 0$

(ii) $\| T_t a \|'_\alpha \leq e^{t\frac{\alpha^2 k}{2}} \| a \|'_\alpha \qquad \forall\, t \geq 0,\ a \in h_\alpha$

(iii) $T|_{h_\alpha}$ is $\|\cdot\|'_\alpha$ - continuous.

Theorem 10: Let $K : h_\infty \to h$ be positive and suppose that

$$\| Ka \| \leq c \| a \|_p \qquad \forall\, a \in h_\infty$$

$$| < a, (\mathrm{ad}H)^2(K)b > | \leq k \| a \| \| b \|_2 \qquad \forall a, b \in h_\infty .$$

Then $\bar{K}$ is essentially self-adjoint on any core for H^p and $\bar{K}\, h_{p+2} \subseteq h_2$. Moreover, for each $\alpha \in [0, 2]$ the contraction semigroup $T_t = \exp\{-t\bar{K}\}$ satisfies:

(i) $T_t h_\alpha \subseteq h_\alpha$ $\qquad \forall\ t \geq 0$

(ii) $\| T_t a \|_\alpha \leq e^{t\frac{\alpha^2 k}{2}} \| a \|_\alpha$ $\qquad \forall\ t \geq 0,\ a \in h_\alpha$

(iii) $T|_{h_\alpha}$ is $\|\cdot\|_\alpha$ - continuous.

Remark:

Theorem 10 is proved in [DrS] under the additional hypothesis that there are constants k_m , $m = 3,4,\ldots,p$ such that

$$|< a, (\mathrm{ad}H)^m(K)b >| \leq k_m \| a \| \| b \|_m \qquad \forall a, b \in h_\infty$$

and that

$$|< a, (\mathrm{ad}H)(K)b >| \leq d \| a \| \| b \|_2 \qquad \forall a, b \in h_\infty .$$

As remarked by Driessler and Summers, the proof of theorem 4 in [DrS] can be used to establish the following theorem which is a partial analogue to theorem 6.

Theorem 11: Let $H \geq 0$ and suppose $K: h_\infty \to h$ is positive. Suppose also that

$$\| Ka \| \leq c \| a \|_p$$

$$| < a, (adH)^2(K)b > | \leq \sum_{i=1}^{N} k_i \| a \|_{\gamma_i} \| b \|_{2-\gamma_i} \qquad \forall a, b \in h_\infty$$

where $\gamma_i \in [0, 2] \; \forall_i$.

Then $\overline{K}$ is essentially self-adjoint on any core for H^p.

We can also use the proof of theorem 2.10 in [Rob 2] to obtain a similar result if H is not semi-bounded.

Theorem 12 : Let $K : h_\infty \to h$ be positive and suppose that

$$\| Ka \| \leq c \| a \|_p$$

$$| < a, (adH)^2(K)b > | \leq \sum_{i=1}^{N} k_i \| a \|_{\gamma_i} \| b \|_{2-\gamma_i} \qquad \forall a, b \in h_\infty$$

where $\gamma_i \in (0, 2) \; \forall_i$. Then $\overline{K}$ is essentially self-adjoint on any core for H^p.

Theorems 13 and 14 below show that if there are appropriate bounds on the higher order commutators of H and K then we can achieve better invariance properties for the action of the group $V_t = \exp \{i t \overline{K}\}$ or semigroup $T_t = \exp \{- t \overline{K}\}$ on the spaces h_α.

Theorem 13 : Let $K : h_\infty \to h$ be symmetric and suppose that *for some $m \in \mathbb{N}$*

$$\| Ka \| \le c \| a \|_p \qquad \forall a, b \in h_\infty$$

$$(1.4.5) \quad |< a, (\mathrm{adH})^q(K)b >| \le k_q \| a \| \| b \|_q \qquad q = 1,2,\dots,2m \ .$$

Then $\bar{K} h_{p+2m} \subseteq h_{2m}$ and for each $\alpha \in [0, 2m]$ the unitary group $V_t = \exp\{i t \bar{K}\}$ satisfies;

(i) $\quad V_t h_\alpha = h_\alpha \qquad \forall t \in \mathbb{R}$

(ii) $\quad \| V_t a \|_\alpha \le e^{|t| \frac{k\alpha}{2}} \| a \|_\alpha \qquad \forall t \in \mathbb{R}, a \in h_\alpha$

for some $k \ge 0$ *(k is independant of α).*

(iii) $\quad V|_{h_\alpha}$ is $\|\cdot\|_\alpha$ - continuous .

<u>**Theorem 14 :**</u> Let $K : h_\infty \to h$ be positive and suppose that

$$\| Ka \| \le c \| a \|_p \qquad \forall a, b \in h_\infty$$

$$(1.4.6) \quad |< a, (\mathrm{adH})^q(K)b >| \le k_q \| a \| \| b \|_q \qquad q = 2,3,\dots,2m \ .$$

Then $\bar{K}\, h_{p+2m} \subseteq h_{2m}$ and for each $\alpha \in [0, 2m]$ the semigroup $T_t = \exp\{-t\bar{K}\}$ satisfies

(i) $T_t h_\alpha \subseteq h_\alpha$ $\qquad \forall\, t \geq 0$

(ii) $\| T_t a \|_\alpha \leq e^{t\alpha^2 k} \| a \|_\alpha$ $\qquad \forall\, t \geq 0,\ a \in h_\alpha$

k is independant of α.

(iii) $T|_{h_\alpha}$ is $\|\cdot\|_\alpha$ - continuous .

Remark:

The statement that $\bar{K}\, h_{p+2m} \subseteq h_{2m}$ in theorems 13 and 14 requires proof. For $a \in h_\infty$, $b \in h_{p+2m}$ we have the identity

$$(1.4.7) \qquad \langle H^{2m} a, \bar{K} b\rangle = \langle a, \bar{K} H^{2m} b\rangle + \sum_{s=1}^{2m} \binom{2m}{s} \langle a, (\mathrm{ad}H)^s (\bar{K})\, H^{2m-s} b\rangle .$$

Therefore (1.4.5) immediately implies $\bar{K} b \in \mathcal{D}(H^{2m})$ in the situation of theorem 13. From (1.4.6) and lemma 2 we deduce that $|\langle a, (\mathrm{ad}H)(K)\, b\rangle| \leq k_1 \| a \| \, \| b \|_{2vp}$ $\forall a, b \in h_\infty$ and this extends to $b \in h_{p+2m}$ by continuity. Therefore, (1.4.7) implies $\bar{K}\, h_{p+2m} \subseteq \mathcal{D}(H^{2m})$ in theorem 14.

The remaining results in this section deal with situations in which two unbounded operators K_1 and K_2 commute.

Theorem 15: Let $K_1, K_2 : h_\infty \to h$ be symmetric and suppose they each satisfy one of the following two sets of conditions:

$$1. \quad \begin{cases} |\langle a, Kb \rangle| \leq k_0 \, \| a \|_{\frac{1}{2}} \, \| b \|_{\frac{1}{2}} \\ \\ |\langle a, (\mathrm{adH})(K)\, b \rangle| \leq k_1 \, \| a \| \, \| b \|_1 \end{cases} \qquad \forall a, b \in h_\infty$$

$$2. \quad \begin{cases} K \geq 0 \\ |\langle a, Kb \rangle| \leq k_0 \, \| a \|_1 \, \| b \|_1 \\ |\langle a, (\mathrm{adH})^2(K)\, b \rangle| \leq k_2 \, \| a \| \, \| b \|_2 \end{cases} \qquad \forall a, b \in h_\infty$$

Further assume that

$$\langle K_1 a, K_2 b \rangle = \langle K_2 a, K_1 b \rangle \quad \forall a, b \in h_\infty .$$

Then the self-adjoint closures of $\bar{K}_1$ and $\bar{K}_2$ commute.

Theorem 16: Let $K_1, K_2 : h_\infty \to h$ be symmetric and suppose that they each satisfy one of the following two sets of conditions:

$$1.\quad \begin{cases} \text{(i)} & |\langle a, Kb\rangle| \le k_0 \, \|a\|_m \, \|b\|_m & \forall a, b \in h_\infty \\ \text{(ii)} & |\langle a, (\mathrm{ad}H)^q(K)\, b\rangle| \le k_q \, \|a\| \, \|b\|_q & q = 1,2,\ldots,2m \end{cases}$$

$$2.\quad \begin{cases} \text{(i)} & K \ge 0 \\ \text{(ii)} & |\langle a, Kb\rangle| \le k_0 \, \|a\| \, \|b\|_{2m} & \forall a, b \in h_\infty \\ & & q = 2,3,\ldots,2m\,. \\ \text{(iii)} & |\langle a, (\mathrm{ad}H)^q(K)\, b\rangle| \le k_q \, \|a\| \, \|b\|_q \end{cases}$$

Further assume that

$$\langle K_1 a, K_2 b\rangle = \langle K_2 a, K_1 b\rangle \qquad \forall a, b \in h_\infty .$$

Then the self-adjoint closures $\overline{K}_1$ and $\overline{K}_2$ commute.

Proof: Since the proof is only a slight modification of the proof of theorem 4.5 in [Rob 2] it will only be sketched. There are three cases to consider.

<u>CASE I.</u> Both K_1 and K_2 satisfy 1.

From I(i) and I(ii) with $q = 1$ we deduce using corollary 4.3 of [Rob 2] that $h_{2m} \subseteq \mathcal{D}(\overline{K}_j)$ $j = 1,2$.

If $V_t^j = \exp\{i\,t\,\overline{K}_j\}$ $(j = 1,2)$ then $\forall\,\alpha \in [0, 2m]$ $V_t^j\, h_\alpha = h_\alpha$ by theorem 13. In particular $V_t^j\, h_{2m} = h_{2m}$ and so the calculation on p.51 of [Rob 2] is valid for $a,b \in h_{2m}$.

CASE II. Both K_1 and K_2 satisfy 2.

Condition 2(ii) implies $h_{2m} \subseteq \mathcal{D}(\overline{K}_j)$ $j = 1,2$. If $T_t^j = \exp\{-\,t\,\overline{K}_j\}$ then $\forall\,\alpha \in [0, 2m]$ $T_t^j\, h_\alpha \subseteq h_{2\alpha}$ by theorem 14. In particular $T_t^j\, h_{2m} \subseteq h_{2m}$ and so a similar calculation to the one mentioned in Case I can be performed for $a, b \in h_{2m}$.

CASE III. K_1 satisfies 1 and K_2 satisfies 2.

As above, we have $h_{2m} \subseteq \mathcal{D}(\overline{K}_j)$ $j = 1,2$ and

$$< \overline{K}_1\, a, \overline{K}_2\, b > \; = \; < \overline{K}_2\, a, \overline{K}_1\, b > \qquad \forall a, b \in h_{2m}.$$

Define K_α for $\alpha > 0$ by

$$K_\alpha = \frac{1}{\alpha} \int_0^\alpha dt\; U_t\, K_1\, U_{-t}$$

where $U_t = \exp\{i\,t\,H\}$. Then

$$| < a, (\mathrm{adH})^q(K_\alpha)\, b > | \le \frac{1}{\alpha}\int_0^\alpha dt\, | < U_{-t}\, a, (\mathrm{adH})^q(K_1)\, U_{-t} b > |$$

$$\le k_q \, \| a \| \, \| b \|_q \qquad \forall a, b \in h_\infty$$

$$q = 1,2,\ldots,2m\ .$$

So K_α satisfies 1 and $\mathcal{D}(\overline{K}_\alpha) \supseteq h_{2m}$.

Let $V_t^\alpha = \exp\{i\, t\, \overline{K}_\alpha\}$. Then by theorem 13

$$V_t^\alpha h_\beta = h_\beta \qquad \forall\, \beta \in [0, 2m]\ .$$

In particular $V_t^\alpha h_{2m} = h_{2m}$.

If $T_t = \exp\{-\, t\, \overline{K}_2\}$, then

$$T_t\, h_\alpha \subseteq h_\alpha \qquad \forall\, \alpha \in [0, 2m],\ t \ge 0$$

by theorem 14. The calculation on p.53 of [Rob 2] now shows

(1.4.8) $| < a, (V_s^\alpha\, T_t - T_t\, V_s^\alpha)\, b > |$

$$\le \int_0^s du \int_0^t dv\ \{ \| \overline{K}_2\, T_v\, V_{-u}^\alpha\, a \| \cdot \| (\overline{K}_\alpha - \overline{K}_1)\, T_{t-v}\, V_{s-u}^\alpha\, b \|$$

$$+ \| (\overline{K}_\alpha - \overline{K}_1)\, T_v\, V_{-u}^\alpha\, a \| \cdot \| \overline{K}_2\, T_{t-v}\, V_{s-u}^\alpha\, b \| \} \qquad \forall a, b \in h_{2m}\ .$$

We treat each term in the integrand separately.

$$\|\bar{K}_2 T_v V_{-u}^{\alpha} a\|^2 \le \langle V_{-u}^{\alpha} a, T_{\frac{v}{2}} \bar{K}_2 T_{\frac{v}{2}} V_{-u}^{\alpha} a \rangle \|\bar{K}_2 T_v\|$$

$$\le k_0 \| T_{\frac{v}{2}} V_{-u}^{\alpha} a\| \cdot \| T_{\frac{v}{2}} V_{-u}^{\alpha} a\|_{2m}\, c_1 v^{-1}$$

for some $c_1 \ge 0$ by condition 2(ii) and spectral theory. We have

$$\| T_{\frac{v}{2}} V_{-u}^{\alpha} a \| \le \|V_{-u}^{\alpha} a\| = \| a \|$$

and

$$\| T_{\frac{v}{2}} V_{-u}^{\alpha} a \|_{2m} \le e^{\frac{vk(2m)^2}{2}} \| V_{-u}^{\alpha} a \|_{2m}$$

$$\le e^{\frac{vk(2m)^2}{2}} e^{|u| \frac{k'2m}{2}} \| a \|_{2m}$$

for some $k, k' \ge 0$, where the first inequality comes from theorem 14 and the second from theorem 13.

Therefore, there is a $c \ge 0$ such that

$$(1.4.9)\qquad \|\bar{K}_2\, T_v\, V^{\alpha}_{-u}\, a\| \le c\, v^{-\frac{1}{2}}\, \|a\|^{\frac{1}{2}}\, \|a\|^{\frac{1}{2}}_{2m} \qquad \forall a \in h_{2m},\ |u| \le 1,\ 0 \le v \le 1 .$$

We can also show as on p.54 of [Rob 2] that there is a $c' \ge 0$ such that

$$(1.4.10)\qquad \|(\bar{K}_{\alpha} - \bar{K}_1)\, T_v\, V^{\alpha}_{-u}\, a\,| \le \alpha\, c'\, \|a\|_1 \qquad \forall a, \in h_{2m},\ |u| \le 1,\ 0 \le v \le 1 .$$

From (1.4.8), (1.4.9), and (1.4.10) we have

$$(1.4.11)\qquad |\langle a, (V^{\alpha}_s\, T_t - T_t\, V^{\alpha}_s)\, b\rangle| \le \alpha\, \gamma_1 \|a\|^{\frac{1}{2}} \|a\|^{\frac{1}{2}}_{2m} \|b\|_1 + \alpha\, \gamma_2 \|a\|_1\, \|b\|^{\frac{1}{2}} \|b\|^{\frac{1}{2}}_{2m}$$

for some $\gamma_1, \gamma_2 \ge 0$, $\forall\, a, b \in h_{2m}$, $|s| \le 1$, $0 \le t \le 1$.

For $a \in h_{\infty}$,

$$\|(K_\alpha - K_1)\, a\,\| \le \alpha^{-1} \int_0^\alpha dt\ \{\|(U_t - I)\, K_1 a\| + \|K_1 (U_{-t} - I) a\|\}$$

$$\le \alpha^{-1} \int_0^\alpha dt\ \{\|(U_t - I)\, K_1 a\| + \|(U_{-t} - I) a\|_{2m}$$

$$\to 0 \ \text{ as } \ \alpha \to 0\ .$$

Therefore, $V_t^\alpha \to V_t = \exp\{i\, t\, \overline{K}_1\}$ strongly as $\alpha \to 0$ by theorem 3.1.28 of [BrR] .

For $a, b \in h_{2m}$, (1.4.11) implies

$$< a, (V_s\, T_t - T_t\, V_s)\, b > = \lim_{\alpha \to 0} < a, (V_s^\alpha\, T_t - T_t\, V_s^\alpha)\, b > = 0$$

and so $\overline{K}_1$ and $\overline{K}_2$ commute.

<u>Theorem 17:</u> Let $H \ge 0$ and let $K_1, K_2 : h_\infty \to h$ be symmetric and satisfy

(1.4.12) $| < a, K_i\, b >| \le k_0\ \|\, a\, \|_{1/2}\ \|\, b\, \|_{1/2}$ $\qquad \forall a, b \in h_\infty$ (i=1,2)

(1.4.13) $| < a, (\mathrm{ad}H)(K_i)\, b >| \le k_1\ \|\, a\, \|_{1/2} \|\, b\, \|_{1/2}$

$\forall a, b \in h_\infty$ (i=1,2) .

Further assume that

$$< K_1\, a, K_2\, b > = < K_2\, a, K_1\, b > \qquad \forall a, b \in h_\infty .$$

Then $\bar{K}_1$ and $\bar{K}_2$ commute.

Remarks:

(1) Condition (1.4.13) is implied by the condition

$$| < a, (\mathrm{adH})^2(K_i)\, b > | \le k_1 \, \| a \|_{1/2} \, \| b \|_{1/2} \qquad \forall a, b \in h_\infty$$

and (1.4.12) because of lemma 2.

(2) In [G&J2] Glimm and Jaffe prove the following theorem which is weaker than theorem 17.

<u>**Theorem 17'**</u>. Let $H \ge 0$ and suppose $K_1, K_2 : \mathcal{D} \to h$ are symmetric where $\mathcal{D} \subseteq h_\infty$ is a core for H. Suppose also that

$$\|K_i\, a\| \le k_0 \, \| a \|_1$$

$$| < a, (\mathrm{adH})(K_i)\, b > | \le k_1 \, \| a \| \, \| b \|_1 \qquad \forall a, b \in \mathcal{D}, \quad i = 1,2$$

$$| < a, (\mathrm{adH})^2(K_i)\, b > | \le k_2 \, \| a \|_{1/2} \, \| b \|_{1/2} .$$

Further assume that

$$< K_1 a, K_2 b > = < K_2 a, K_1 b > \qquad \forall a, b \in \mathcal{D} .$$

Then $\overline{K}_1$ and $\overline{K}_2$ commute.

The following theorem comes from [DrS] .

Theorem 18: Let $H \geq 0$ and suppose $K_1 , K_2 : h_\infty \to h$ are symmetric with $K_2 \geq 0$. Assume

$$\| K_i a \| \leq k_0 \| a \|_2 \qquad \forall a \in h_\infty , i = 1,2$$

$$| < a, (\mathrm{ad}H)^q(K_1) b > | \leq k_q \| a \| \| b \|_q \qquad \forall a, b \in h_\infty \quad q = 1, 2$$

$$| < a, (\mathrm{ad}H)^2(K_2) b > | \leq c_1 \| a \|_1 \| b \|_1 \qquad \forall a, b \in h_\infty .$$

Assume also that

$$<K_1 a, K_2 b> = <K_2 a, K_1 b> \qquad \forall a, b \in \mathcal{D}$$

where $\mathcal{D} \subseteq h_\infty$ is a core for H^2 . Then $\overline{K}_1$ and $\overline{K}_2$ commute.

§5. Applications of commutator theorems.

§5.1. The following example was first mentioned in a commutator theory setting in [DrS].

Choose $h = L^2(\mathbb{R})$ and define maps A and $A^\dagger$ on $\mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$ by

$$(1.5.1) \qquad A = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right) \qquad , \qquad A^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right) .$$

Then $A, A^\dagger : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$.

Let $H = A^\dagger A$ with $\mathcal{D}(H) = \mathcal{S}(\mathbb{R})$. Then

$$H = \frac{1}{2}\left(- \frac{d^2}{dx^2} + x^2 - 1\right) .$$

We have the following result which is proved in [Dav] p180-1, lemma 7.12.

Lemma 19. If $\varphi_n \in \mathcal{S}(\mathbb{R})$ are defined for $n = 0, 1,2,...$ by

$$\varphi_n(x) = (n!)^{-\frac{1}{2}}\, 2^{\frac{n}{2}}\, i^{-n}\, \pi^{-\frac{3}{4}}\, e^{\frac{x^2}{2}} \int_{\mathbb{R}} du\, u^n \exp\{-u^2+2\,i\,xu\}$$

then

$$H\,\varphi_n = n\,\varphi_n .$$

The operator H is essentially self-adjoint on $\mathcal{S}(\mathbb{R})$ and is in fact essentially self-adjoint on $\mathcal{D} = \text{span}\,\{\varphi_n\}$, so $H \geq 0$.

[$\{\varphi_n\}$ are called the Hermite functions and form a complete orthonormal system for $\mathbf{L}^2(\mathbb{R})$].

Note that

$$A\,\varphi_0 = 0\,,\ A\,\varphi_n = n^{1/2}\varphi_{n-1}\ (n \geq 1)\,,\ \text{and}\ A^{\dagger}\varphi_n = (n+1)^{1/2}\varphi_{n+1}\ \forall_n\,.$$

The next result was given by Driessler and Summers in [DrS].

__Proposition 20:__ If P is any fourth order polynomial in x and i d/dx that forms a semi-bounded operator on $\mathbf{L}^2(\mathbb{R})$, then P is essentially self-adjoint on any core for H^2 . (e.g. $\mathcal{D}$ is such a core).

Proof: The proposition is proved by showing

(1.5.2) $\|P(H+4)^{-2}\| < \infty$ and $\|(\text{adH})^2(P)\,(H+4)^{-2}\| < \infty$

and then using theorem 10.

From (1.5.1) we can derive

$$(1.5.3) \quad x = \frac{1}{\sqrt{2}}(A+A^\dagger) \ , \ i\frac{d}{dx} = \frac{i}{\sqrt{2}}(A-A^\dagger) \quad \text{on } \mathcal{S}(\mathbb{R}) .$$

Therefore, if P is a fourth order polynomial in x and $i\, d/dx$ we can use (1.5.3) to make P a fourth order polynomial in A and $A^\dagger$ (we are always assuming that $\mathcal{D}(P) = \mathcal{S}(\mathbb{R})$).

We now show that P can be written as

$$(1.5.4) \quad P = \sum_{m=-4}^{4} P_m \quad \text{where} \quad [H, P_m] = mP_m$$

In order to do this we examine the properties of polynomials P in A and $A^\dagger$ of order 4 or less. The commutation identity

$$(1.5.5) \qquad [A, A^\dagger] = I$$

holds on $\mathcal{S}(\mathbb{R})$ and can be used to rearrange the terms of P so that each term has the form $(A^\dagger)^k A^\ell$ for some $k, \ell \geq 0$ with $k+\ell \leq 4$.

(e.g. $A A^\dagger A = (A^\dagger A + I)A = A^\dagger A^2 + A$).

Now we can show by calculation that

$$[H, (A^\dagger)^k A^\ell] = (k-\ell)(A^\dagger)^k A^\ell .$$

This gives the required decomposition (1.5.4).

If P is written in the form (1.5.4) then

$$[H, P] = \sum_{m=-4}^{4} mP_m \quad \text{and} \quad (\mathrm{ad}H)^2(P) = \sum_{m=-4}^{4} m^2 P_m \ .$$

Therefore, (1.5.2) follows if it can be shown that

(1.5.6) $\quad \|(A^\dagger)^k A^\ell (H+4)^{-2}\| < \infty \ . \ \forall\, k, \ell \geq 0$ such that $k+\ell \leq 4$.

We have:

$$(1.5.7) \quad (A^\dagger)^k A^\ell \varphi_n = \begin{cases} \prod_{i=1}^{k} \prod_{j=0}^{\ell-1} (n-\ell+i)^{\frac{1}{2}} (n-j)^{\frac{1}{2}} \varphi_{n+k-\ell} & \forall\, n \geq \ell \\ 0 & \forall\, n < \ell \end{cases}$$

$$(H+4)^{-2}\varphi_n = (n+4)^{-2}\varphi_n \qquad \forall\, n \geq 0 \ .$$

Therefore,

$$(1.5.8) \quad (A^{\dagger})^{k} A^{\ell} (H+4)^{-2} \varphi_n = \begin{cases} (n+4)^{-2} \prod_{i=1}^{k} \prod_{j=0}^{\ell-1} (n-\ell+i)^{\frac{1}{2}} (n-j)^{\frac{1}{2}} \varphi_{n+k-\ell} & \forall\, n \geq \ell \\ 0 & \forall\, n < \ell \end{cases}$$

Since the set $\{\varphi_n\}$ is complete and orthonormal and $k+\ell \leq 4$, (1.5.8) implies

$$\|(A^{\dagger})^{k} A^{\ell} (H+4)^{-2} \varphi\| \leq \| \varphi \| \qquad \forall\, \varphi \in \boldsymbol{S}(\mathbb{R}),\ k, \ell \geq 0,\ k+\ell \leq 4$$

and so (1.5.6) holds.

Using the other results of §4 we can say a little more about this situation.

If P is a fourth order polynomial in x and $i\,d/dx$ which is semi-bounded as an operator on $L^2(\mathbb{R})$ then from (1.5.4) we have

$$(\mathrm{ad}H)^{q}(P) = \sum_{m=-4}^{4} m^{q} P_m \,.$$

It follows that

$$\|(\mathrm{ad}H)^q(P)\,(H+4)^{-2}\| < \infty \qquad\qquad q = 1, 2, 3\ldots\ .$$

Therefore, theorem 14 can be used to show that

$$P h_{2+2m} \subseteq h_{2m} \;\; \forall\, m \in N \;,\;\; P h_\infty \subseteq h_\infty$$

and $\forall\, \alpha \geq 0$ the semigroup $T_t = \exp\{-\,t\,P\}$ satisfies

(i) $\quad T_t h_\alpha \subseteq h_\alpha \qquad\qquad \forall\, t \geq 0$

(ii) $\quad T|_{h_\alpha}$ is $\|\cdot\|_\alpha$ - continuous .

We also have the following result:

<u>**Theorem 21.**</u> Let P be a second-order polynomial in x and $i\,d/dx$ which acts as a symmetric operator on $L^2(\mathbb{R})$. Then P is essentially self-adjoint on any core for H . (Again $\mathcal{D}$ is such a core).

Proof: Any second-order polynomial P in x and $i\,d/dx$ can be made into a second order polynomial in A and $A^\dagger$ using (1.5.3). (1.5.5) now allows us to assume that each term in P is of the form $(A^\dagger)^k A^\ell$ where $k, \ell \geq 0$ and $k+\ell \leq 2$. Therefore, we can write P as

(1.5.9) $$P = \sum_{m=-2}^{2} P_m \quad \text{where } [H, P_m] = mP_m .$$

To prove the theorem we show

(1.5.10) $$\|P(H+2)^{-1}\| < \infty \text{ and } \|[H,P](H+2)^{-1}\| < \infty$$

and then use theorem 5. (1.5.10) will follow from (1.5.9) if we can show

$$\|(A^{\dagger})^{k} A^{\ell}(H+2)^{-1}\| < \infty \quad \text{for } k, \ell \geq 0 \;\; k+\ell \leq 2 .$$

We have

(1.5.11) $$(A^{\dagger})^{k} A^{\ell} (H+2)^{-1} \varphi_n = \begin{cases} (n+2)^{-1} \prod_{i=1}^{k} \prod_{j=0}^{\ell-1} (n-\ell+i)^{\frac{1}{2}} (n-j)^{\frac{1}{2}} \varphi_{n+k-\ell} & n \geq \ell \\ 0 & n < \ell \end{cases}$$

Therefore,

$$\|(A^{\dagger})^{k} A^{\ell}(H+2)^{-1} \varphi\| \leq \|\varphi\| \qquad \forall \varphi \in \mathcal{S}(\mathbb{R})$$

and the theorem is proved.

In the situation of theorem 21 we can apply theorem 13 to deduce properties of the unitary group $V_t = \exp\{i\, t\, \mathbb{P}\}$. For we can use (1.5.9) and (1.5.11) to show that

$$\|(\mathrm{ad}H)^q\,(\mathbb{P})\,(H+2)^{-1}\| < \infty \qquad\qquad q = 1,2,3,\ldots \quad .$$

Therefore, from theorem 13 we have

$$\mathbb{P}\, h_{1+2m} \subseteq h_{2m} \qquad \forall\, m \in N\,, \qquad\qquad \mathbb{P}\, h_\infty \subseteq h_\infty$$

and V_t satisfies the following $\forall\, \alpha \geq 0$:

(i) $\quad V_t\, h_\alpha = h_\alpha \qquad\qquad \forall\, t \in \mathbb{R}$

(ii) $\quad V|_{h_\alpha}$ is $\|\cdot\|_\alpha$ – continuous .

§5.2

Commutator theory can be used to handle certain types of first or second-order ordinary differential operators on $\mathbf{L}^2(\mathbb{R})$ i.e. operators of the form

(1.5.12) $\qquad Q_1\, H + Q_0$

(1.5.13) or $\;Q_2 H^2 + Q_1\, H + Q_0$

where $H = i\,d/dx$ and the Q_i are functions on $\mathbb{R}$.

The results depend upon the differentiability of the Q_i and the boundedness of D^jQ_i $i \in \{0,1,2\}$ $j \in N$.

In this section we choose $h = \mathbf{L}^2(\mathbb{R})$ and let H be the closure of

$$i \frac{d}{dx} \Big|_{C_c^\infty(\mathbb{R})} .$$

Then H is self-adjoint and $C_c^\infty(\mathbb{R})$ is a core for H^p $p = 1,2,3,\ldots$.

We first discuss first-order ordinary differential operators. Let K be of the form (1.5.12) with $\mathcal{D}(K) = C_c^\infty(\mathbb{R})$. Suppose for $i = 1,2$ that $Q_i, Q'_i,\ldots,Q_i^{(\ell)}$ exist and are bounded and that $Q_i^{(\ell+1)}$ exists. Then, because of the form of K and the boundedness of Q_1 and Q_0, there is a constant $c \geq 0$ such that

$$\| Kf \| \leq c \| f \|_1 \qquad \forall f \in C_c^\infty(\mathbb{R}) .$$

So, there is a $c_1 \geq 0$ such that for $p = 1, 2,\ldots, \ell$

$$\| Kf \| \leq c_1 \|(1+H^{2p})^{1/2p} f\| \qquad \forall f \in C_c^\infty(\mathbb{R})$$

or,

$$\| Kf \| \leq c_1 \, \|f\|_1^{H^p} \qquad \forall\, f \in C_c^{\infty}(\mathbb{R})$$

where the notation $\|\cdot\|^{H^p}$ has an obvious meaning.

We also have for $p = 1,2,\dots,\ell$

$$(1.5.14) \qquad (\mathrm{ad}H)^p(K) = \sum_{j=0}^{p} R_{j,p} H^j$$

where the coefficient functions $R_{j,p}$ are linear combinations of Q_i', Q_i'',...,$Q_i^{(p)}$. Since each of these is bounded, so is $R_{j,p}$ $j = 0,1,\dots,p$.

Therefore, there is a $k \geq 0$ such that

$$\|(\mathrm{ad}H^p)(K)\, f\| \leq k \,\| f \|_p \qquad \forall\, f \in C_c^{\infty}(\mathbb{R})$$

or

$$\|(\mathrm{ad}H^p)(K)\; f\| \leq k_1 \,\| \; f \|_1^{H^p} \qquad \forall\, f \in C_c^{\infty}(\mathbb{R}).$$

We can now apply theorem 5 with $p = 1,2,\dots,\ell$ to obtain:

Theorem 22 : Let $K = Q_1 H + Q_0$ be a symmetric operator on $L^2(\mathbb{R})$ with $\mathcal{D}(K) = C_c^\infty(\mathbb{R})$ and suppose $Q_i^{(m)}$ exist for $m = 0, 1, \ldots, \ell+1$ and are bounded for $m = 0, 1, \ldots, \ell$. Then K is essentially self-adjoint on $C_c^\infty(\mathbb{R})$. In addition $\bar{K} h_{p+1} \subseteq h_p$ for $p = 1, 2, \ldots, \ell$ and for $\alpha \in [0, \ell]$ the unitary group $V_t = \exp\{i t \bar{K}\}$ satisfies:

(i) $V_t h_\alpha = h_\alpha \quad \forall t \in \mathbb{R}$

(ii) $V|_{h_\alpha}$ is $\|\cdot\|_\alpha$ - continuous

In particular, if $Q_0, Q_1 \in C_c^\infty(\mathbb{R})$ then $\bar{K} h_{p+1} \subseteq h_p \ \forall p \in N$, $\bar{K} h_\infty \subseteq h_\infty$ and V_t satisfies (i) and (ii) above $\forall \alpha \geq 0$.

Note: A similar result can be achieved via theorem 13.

Two examples of symmetric first order ordinary differential operators on $L^2(\mathbb{R})$ are

$$K_1 = Q_1 H + HQ_1 + Q_0 = 2Q_1 H + iQ_1' + Q_0$$

where Q_0, Q_1 are real-valued functions, and

$$K_2 = \bar{Q}_1 HQ_1 + Q_0 = |Q_1|^2 H + i\bar{Q}_1 Q_1' + Q_0$$

where Q_1 is complex-valued and Q_0 is real-valued.

We now discuss second-order ordinary differential operators. Let K be of the form (1.5.13) with $\mathcal{D}(K) = C_c^\infty(\mathbb{R})$. Suppose K is semi-bounded as an operator on $\mathbf{L}^2(\mathbb{R})$ and that $Q_i, Q_i', \dots, Q_i^{(2\ell+1)}$ exist with $Q_i, Q_i', \dots, Q_i^{(2\ell)}$ bounded.

Then, because of the form of K and the boundedness of Q_i there is a $c \geq 0$ such that

$$\| Kf \| \leq c \| f \|_2 \qquad \forall f \in C_c^\infty(\mathbb{R}) .$$

(1.5.14) holds for $p = 1, 2, \dots, 2\ell$,where again each $R_{j,p}$ is a linear combination of Q_i' , Q_i'',...,$Q_i^{(p)}$ and is therefore bounded. This implies that there are constants $k_p \geq 0$ $p = 2,3,\dots,2\ell$ such that

$$\|(\mathrm{ad}H)^p(K) f\| \leq k_p \| f \|_p \qquad \forall f \in C_c^\infty(\mathbb{R}) .$$

Therefore, we may use theorem 14 and the fact that $C_c^\infty(\mathbb{R})$ is a core for H^2 to obtain :

<u>**Theorem 23 :**</u> Let $K = Q_2H^2 + Q_1H + Q_0$ be a semi-bounded operator on $\mathbf{L}^2(\mathbb{R})$ with $\mathcal{D}(K) = C_c^\infty(\mathbb{R})$. Suppose $Q_i^{(m)}$ exists for $m = 0, 1, 2, \dots, 2\ell + 1$ and $i = 0,1,2$ and that it is bounded for $m = 0, 1,2,..,2\ell$ and $i = 0,1,2$. Then $\overline{K}$ is essentially

self-adjoint on any core for H^2. In addition $\overline{K} h_{2+2m} \subseteq h_{2m}$ $m = 0,1,2,\ldots,\ell$ and $\forall\, \alpha \in [0,2\ell]$ the semigroup $T_t = \exp\{-t\,\overline{K}\}$ satisfies:

(i) $T_t h_\alpha \subseteq h_\alpha$ $\qquad \forall\, t \geq 0$

(ii) $T|_{h_\alpha}$ is $\|\cdot\|_\alpha$ - continuous.

In particular if $Q_0, Q_1, Q_2 \in C_c^\infty(\mathbb{R})$ then $\overline{K} h_{2+2m} \subseteq h_{2m}$ $\forall\, m \in N$, $\overline{K} h_\infty \subseteq h_\infty$ and T_t satisfies (i) and (ii) above $\forall\, \alpha \geq 0$.

Note: A similar result can be obtained via theorem 10 with H^ℓ in place of H.

Two examples of positive second-order ordinary differential operators on $\mathbf{L}^2(\mathbb{R})$ are

$$K_1 = H\,Q_1 H + Q_0 = Q_1 H^2 + iQ_1{}'\,H + Q_0$$

where Q_0, Q_1 are positive functions on $\mathbb{R}$, and

$$K_2 = \overline{Q}_1 H^2 Q_1 + Q_0 = |Q_1|^2 H^2 + 2i\overline{Q}_1\, Q_1{}'\,H - \overline{Q}_1 Q_1{}'' + Q_0$$

where Q_1 is a complex-valued function on $\mathbb{R}$ and Q_0 is positive.

§5.3.

As a final application of commutator theorems we present some results of Faris and Lavine [FaL] who used their theorems to analyse Schrodinger operators. Here Δ is the Laplacian on $\mathbb{R}^n$.

Theorem 24: Let V , W be real-valued measurable functions on $\mathbb{R}^n$ with $V \in L^2{}_{loc}(\mathbb{R}^n)$ and $V(x) \geq -cx^2 - d$.

Suppose also that

(i) There is a dense set $\mathcal{D} \subset \mathcal{D}(-\Delta) \cap \mathcal{D}(V) \cap \mathcal{D}(W)$ which is invariant under the operations x_j and

$$\frac{1}{i}\frac{\partial}{\partial x_j}$$

and such that $-\Delta + V + W + 2cx^2$ is essentially self-adjoint on $\mathcal{D}$.

(ii) For some $a < 1$, $-a\Delta + W$ is bounded below on $\mathcal{D}$.

Then $-\Delta + V + W$ is essentially self-adjoint on $\mathcal{D}$.

The following result is a corollary of theorem 24.

Theorem 25: Let V_1, V_2 be real-valued measureable functions on $\mathbb{R}^n$ satisfying :

(i) $V_2 \in L^p(\mathbb{R}^n)$ with $p \geq 2$ if $n \leq 3$, $p > 2$ if $n = 4$, and $p \geq n/2$ if $n \geq 5$.

(ii) $V_1 \geq -cx^2 - d$ for some c, d ; $V_1 \in L^2{}_{loc}(\mathbb{R}^n)$.

Then $-\Delta + V_1 + V_2$ is essentially self-adjoint on $C_0{}^\infty(\mathbb{R}^n)$.

CH 2. COMMUTATOR THEORY IN BANACH SPACE.

§1 . Introduction.

Commutator theorems were originally proved in Hilbert space since this is part of the framework within which quantum field theory operates. However, some of the results mentioned in CH.1 §4 have extensions to Banach space. Before providing these extensions it is necessary to set the scene.

In this section and the next B will denote a Banach space and σ_t will be a one-parameter C_0-group of isometries on B with generator H. We will define the spaces $B_n (n \in N)$ and B_∞ by

$$B_n = \mathcal{D}(H^n) \quad , \quad B_\infty = \bigcap_{n\geq 1} B_n \ .$$

We also define the Lipschitz spaces $B_{n+1/2}$ as follows

$$B_{n+\frac{1}{2}} = \{ a \in B_n \ : \ \sup_{|t|\leq 1} \frac{\|(\sigma_t - I)H^n a\|}{|t|} < \infty \} \ .$$

Each B_n is a Banach space when equipped with the norm $\|\cdot\|_n$ defined by

$$\| a \|_n = \sup_{m=0,1,\dots,n} \| H^m a \| \qquad \text{for } a \in B_n \ .$$

B_∞ is a Frechet space when it is given the topology determined by the set of norms $\{\|\cdot\|_n \ : n \in N\}$.

$B_{n+1/2}$ becomes a Banach space if it is given the norm $\|\cdot\|_{n+1/2}$ defined by

$$\| a \|_{n+\frac{1}{2}} = \| a \|_n \vee \left\{ \sup_{|t|\leq 1} \frac{\|(\sigma_t - I)H^n a\|}{|t|} \right\} .$$

We note that if K is a closable linear operator on B with $\mathcal{D}(K) = B_\infty$ then, by the closed graph theorem, there is $c \geq 0$ and $p \in N$ such that

(2.1.1) $\quad \| Ka \| \le c \| a \|_p \qquad\qquad \forall\, a \in B_\infty$.

When working on a Banach space B we need to redefine what we mean by a commutator bound. This is because we can no longer define $[H, K]$ as a quadratic form on $h_\infty \times h_\infty$. On B the n t h commutator bound will take the form

(2.1.2) $\quad \| (\mathrm{ad}\sigma_t)^n(K)\, a \| \le k|t|^n \| a \|_q \qquad\qquad \forall\, a \in B_\infty,\ |t| \le 1$

for some $q \in N$.

When B is a Hilbert space, (2.1.2) is equivalent to the bound (1.3.1) with $\alpha = 0$, $\beta = q$.

In order to indicate a relationship between the commutator theorems on Hilbert space and those on Banach space we note that on a Hilbert space K is symmetric if and only if $\pm\, i\, K$ is dissipative. We also note that by Stone's theorem the generators of unitary groups are of the form $i\overline{K}$ where $\overline{K}$ is self-adjoint.

§2. The theorems.

Proofs of the following theorems can be found in the two research reports [BaR] and [Rob 1].

<u>Theorem 26:</u> Let $K : B_\infty \to B$ be a linear operator satisfying

1. K is dissipative

2. $\|[\sigma_t, K]\, a\| \le k_1 \, |t| \, \|a\|_1 \qquad \forall\, a \in B_\infty, \; |t| \le 1 .$

Then $\bar{K}$ generates a C_0-semigroup of contractions τ on B. If $K B_\infty \subseteq B_1$ then

(i) $\tau_t B_1 \subseteq B_1 \qquad \forall\, t \ge 0$

(ii) $\| \tau_t a \|_1 \le e^{k_1 t} \| a \|_1 \qquad \forall\, t \ge 0 , a \in B_1$

(iii) $\tau|_{B_1}$ is $\|\cdot\|_1$ - continuous .

Theorem 27 : Let $K : B_\infty \to B$ be a linear operator satisfying 1. and 2. of theorem 26. Suppose also that K satisfies:

3. $\| Ka \| \le c \, \| a \|_{2n-1} \qquad \forall\, a \in B_\infty$

4. $\| (\mathrm{ad}\sigma_t)^m (K) a \| \le k_m |t|^m \| a \|_m \qquad \forall\, a \in B_\infty, \; |t| \le 1, \; m = 2,3,\dots,2n .$

Then, if τ_t is the semigroup of contractions generated by $\bar{K}$, the following hold for $m = 0,1,2,\dots,2n-1$:

(i) $\tau_t B_m \subseteq B_m \qquad \forall\, t \geq 0$

(ii) $\| \tau_t a \|_m \leq \|a\|_m \exp \left\{ \sum_{r=1}^{m} \binom{m}{r} k_r\, t \right\} \qquad \forall\, t \geq 0,\ a \in B_m$

(iii) $\tau|_{B_m}$ is $\|\cdot\|_m$ - continuous.

If, in addition, $K\, B_\infty \subseteq B_{2n}$ then (i), (ii), and (iii) above also hold for $m = 2n$.

Remarks:

(1) If σ_t is a general C_0-group and not a group of isometries then conditions 1 - 3 of theorems 26 and 27 and condition $4'$ below are enough to show that (i), (ii), and (iii) hold for $m = 0,1,2,\ldots,2n-1$.

$4'$. $\qquad \| (\mathrm{ad}\sigma_t)^m(K)\, a \| \leq k_m |t|^m \| a \|_{m \vee (2n-1)} \qquad \forall a \in B_\infty,\ |t| \leq 1,\ m=2,3,\ldots,2n$.

(2) The final statement of theorem 27 requires proof. Assume $K\, B_\infty \subseteq B_{2n}$ and for $m = 1,2,\ldots,2n$ let

$\omega_m = \sum_{r=1}^{m} \binom{m}{r} k_r$. From (ii) of theorem 27 we know that

$K + \omega_{2n-1}$ is $\|\cdot\|_{2n-1}$ - dissipative. Therefore, if $\varepsilon > 0$

$$\|[1+\varepsilon(K+\omega_{2n})]a\|_{2n-1} \geq [1+\varepsilon(\omega_{2n}-\omega_{2n-1})]\|a\|_{2n-1} \ \forall a \in B_\infty$$

$$\geq [1+\varepsilon(\omega_{2n}-\omega_{2n-1})]\|a\|_{2n-1}$$

(2.2.1)
$$-\varepsilon(\omega_{2n}-\omega_{2n-1})\|a\|_{2n}$$

We have the identity

$$[A, B^\ell] = -\sum_{r=1}^{\ell}\binom{\ell}{r}\left((\mathrm{ad}B)^r(A)\right)B^{\ell-r}.$$

Therefore, if $H_t = \dfrac{\sigma_t - I}{t}$, we have

$$H_t^{2n}[1+\varepsilon(K+\omega_{2n})]a = [1+\varepsilon(K+\omega_{2n})]H_t^{2n}a$$

$$+\ \varepsilon\sum_{r=1}^{2n}\binom{2n}{r}\frac{\left((\mathrm{ad}\sigma_t)^r(K)\right)H_t^{2n-r}a}{t^r}$$

$$\Rightarrow \|H_t^{2n}[1+\varepsilon(K+\omega_{2n})]a\| \geq \|[1+\varepsilon(K+\omega_{2n})]H_t^{2n}a\|$$

$$-\varepsilon\sum_{r=1}^{2n}\binom{2n}{r}k_r\|H_t^{2n-r}a\|_r$$

$$\geq (1+\varepsilon\omega_{2n})\|H_t^{2n}a\| - \varepsilon\sum_{r=1}^{2n}\binom{2n}{r}k_r\|H_t^{2n-r}a\|_r$$

where the dissipativity of K has been used. If we now take the limit as $t \to 0$ and use $KB_\infty \subseteq B_{2n}$ we get

$$\| H^{2n}[1+\varepsilon(K+\omega_{2n})]a \| \geq (1+\varepsilon\omega_{2n})\| H^{2n} a \| - \varepsilon \sum_{r=1}^{2n} \binom{2n}{r} k_r \| H^{2n-r} a \|_r$$

$$\geq (1+\varepsilon\omega_{2n})\| H^{2n} a \| - \varepsilon\omega_{2n} \| a \|_{2n} . \tag{2.2.2}$$

(2.2.1) and (2.2.2) show that $K+\omega_{2n}$ is $\|\cdot\|_{2n}$ - dissipative. We can also show that $[1+\varepsilon(K+\omega_{2n})]B_\infty$ is $\|\cdot\|_{2n}$ - dense in B_{2n} by using the proof of a similar statement in theorem 2.1 of [BaR]. The result follows from the Hille-Yosida theorem.

Theorem 28: Let $K : B_\infty \to B$ satisfy

1. K is dissipative

2. $\|[\sigma_t , K]a \| \leq k_1 |t| \| a \|_1 \qquad \forall t \in \mathbb{R}, a \in B_\infty$

3. $\|Ka\| \leq c \| a \|_1 \qquad \forall a \in B_\infty .$

If τ_t is the semigroup generated by K then

(i) $\quad \tau_t \, B_{\frac{1}{2}} \subseteq B_{\frac{1}{2}} \qquad\qquad \forall\, t \geq 0$

(ii) $\quad \|\tau_t a\|_{\frac{1}{2}} \leq e^{k_1 t} \|a\|_{\frac{1}{2}} \qquad\qquad \forall\, t \geq 0 \,,\, a \in B_{\frac{1}{2}} \,.$

If $n = 1$ in theorem 27, we can get more detailed information about the action of τ_t on B .

<u>**Theorem 29.**</u> Suppose $K : B_\infty \to B$ satisfies conditions 1 - 4 of theorems 26 and 27 with $n = 1$. Then τ_t also has the following properties:

(i) $\quad \tau_t \, B_{\frac{3}{2}} \subseteq B_{\frac{3}{2}} \qquad\qquad \forall\, t \geq 0$

(ii) $\quad \|\tau_t a\|_{\frac{3}{2}} \leq e^{(k_2 + 2k_1)t} \|a\|_{\frac{3}{2}} \qquad\qquad \forall\, t \geq 0,\, a \in B_{\frac{3}{2}} \,.$

Before closing this section we will briefly comment on the fact that certain commutator theorems in Hilbert space seem to have no Banach space analogues. For positive operators K on a Hilbert space h , theorems 8-12 show that second order commutator bounds of index two are sufficient to deduce generation properties of $\overline{K}$. However, there is no corresponding result for Banach spaces B , i.e. no particular type of operator

$K : B_\infty \to B$ is known such that the bound

$$\| (ad\sigma_t)^2(K)\, a \| \le k\, |t|^2\, \| a \|_2 \qquad \forall\, |t| \le 1\ ,\ a \in B_\infty$$

is sufficient to imply that $\overline{K}$ generates a C_0- semigroup on B.

§3.

In this section, B will denote a Banach space, $\sigma_t(t \ge 0)$ is a one-parameter C_0-semigroup of contractions on B with generator H, and K is a closable linear operator from B_∞ to B. We let p_0, k_0 be numbers such that

$$\| Ka \| \le k_0\, \| a \|_{p_0} \qquad \forall\, a \in B_\infty\ .$$

The Taylor series argument in the following lemma is due to Derek Robinson and is also used in the proof of lemma 2.11 of [Rob 2].

__Lemma 30 .__ Suppose there are constants $k_{n+1} \ge 0$ and $p_{n+1} \in N$ such that

$$\| (ad\sigma_t)^{n+1}(K)a \| \le k_{n+1}\, t^{n+1}\, \| a \|_{p_{n+1}} \qquad \forall\, a \in B_\infty\ ,\ t \ge 0\ .$$

Then for $m = 0, 1, 2, \ldots, n$ there are constants $C_m, D_m > 0$ such that

$$\| (\mathrm{ad}\sigma_t)^m (K)\, a \| \le t^m (C_m \| a \|_{p_0} + D_m \| a \|_{p_{n+1}}) \qquad \forall\, a \in B_\infty,\ t \ge 0 .$$

(i.e. the bound on the $(n+1)$st commutator implies a similar bound on the lower order commutators).

Proof: If $L : B_\infty \to B$ is a linear operator satisfying an inequality of the form

$$\| La \| \le c \| a \|_d \qquad \forall\, a \in B_\infty$$

then an operator $S_{s,t}(L)$ can be defined on B_∞ by

$$S_{s,t}(L)a = \big(\exp\{(s/t)(T_t - I)\}(L)\big)a \qquad a \in B_\infty$$

where $T_t(L) = \sigma_t L \sigma_{-t}$ and the above exponential is defined by a power series, i.e.

$$S_{s,t}(L)a = \sum_{m \ge 0} \left(\frac{s}{t}\right)^m \frac{(T_t - I)^m (L)a}{m!} .$$

The series is norm convergent because of the bound on L

and

$$
\begin{aligned}
\|S_{s,t}(L)a\| &= \left\|\left(\exp\{\tfrac{s}{t}(T_t - I)\}(L)\right)a\right\| \\
&= e^{-\frac{s}{t}}\left\|\left(\exp\{\tfrac{s}{t}T_t\}\right)(L)a\right\| \\
&\le e^{-\frac{s}{t}}\sum_{m\ge 0}\left(\frac{s}{t}\right)^m \frac{\|T_t^{\,m}(L)a\|}{m!} \\
&\le e^{-\frac{s}{t}}\sum_{m\ge 0}\left(\frac{s}{t}\right)^m \frac{c\,\|a\|_d}{m!} \\
&= c\,\|a\|_d
\end{aligned}
$$

since $\|T_t^m(L)a\| = \|\sigma_{mt}L\sigma_{-mt}a\| \le \|L\sigma_{-mt}a\|$
$\le c\|\sigma_{-mt}a\|_d \le c\|a\|_d$

We also have $(T_t - I)^m(L)a = (\mathrm{ad}\,\sigma_t)^m(L)\sigma_{-mt}a$, so

$$S_{s,t}(L)a = \sum_{m\ge 0}\left(\frac{s}{t}\right)^m \frac{(\mathrm{ad}\;\sigma_t)^m(L)\;\sigma_{-mt}a}{m!}\;.$$

Therefore,

$$\frac{d^k}{ds^k}\left(S_{s,t}(L)a\right) = \sum_{m\geq 0}\left(\frac{s}{t}\right)^m \frac{(\mathrm{ad}\,\sigma_t)^{m+k}(L)\,\sigma_{-(m+k)t}a}{t^k\,m!}$$

$$= \sum_{m\geq 0}\left(\frac{s}{t}\right)^m \frac{(\mathrm{ad}\,\sigma_t)^m\left((\mathrm{ad}\,\sigma_t)^k(L)\,\sigma_{-kt}\right)\sigma_{-mt}a}{t^k\,m!}$$

$$= S_{s,t}\left(\frac{(\mathrm{ad}\,\sigma_t)^k(L)\,\sigma_{-kt}}{t^k}\right)a\ .$$

Therefore, putting $L = K$ and using the Taylor series expansion :

$$f(x) = \sum_{m=0}^{n} \frac{f^m(0)\,x^m}{m!} + \int_0^x dr\,\frac{(x-r)^n}{n!}\,f^{(n+1)}(r)$$

gives

$$S_{s,t}(K)a = \sum_{m=0}^{n}\left(\frac{s}{t}\right)^m \frac{(\mathrm{ad}\,\sigma_t)^m(K)\,\sigma_{-mt}a}{m!}$$

$$+ \int_0^s dr\,\frac{(s-r)^n}{n!}\,S_{r,t}\left(\frac{(\mathrm{ad}\,\sigma_t)^{n+1}(K)\,\sigma_{-(n+1)t}}{t^{n+1}}\right)a\ .$$

Let $a_m(t) = \dfrac{(\mathrm{ad}\,\sigma_t)^m(K)\,\sigma_{-mt}a}{m!\,t^m}$

Then,

$$\sum_{m=1}^{n} s^m a_m(t) = S_{s,t}(K)a - Ka$$

$$- \int_0^s dr \, \frac{(s-r)^n}{n!} \, S_{r,t}\left(\frac{(\mathrm{ad}\,\sigma_t)^{n+1}(K)\,\sigma_{-(n+1)t}}{t^{n+1}}\right)a$$

Choose n values $s_1, s_2, \ldots, s_n$ such that the matrix

$$M = \begin{pmatrix} s_1 & s_1^2 & \cdots & s_1^n \\ s_2 & s_2^2 & \cdots & s_2^n \\ \vdots & \vdots & & \vdots \\ s_n & s_n^2 & \cdots & s_n^n \end{pmatrix}$$

is invertible. Since the Vandemonde determinant

$$\begin{vmatrix} 1 & s_1 & \cdots & s_1^n \\ 1 & s_2 & \cdots & s_2^n \\ \vdots & \vdots & & \vdots \\ 1 & s_{n+1} & \cdots & s_{n+1}^n \end{vmatrix} = \prod_{1 \le p < q \le n+1} (s_q - s_p)$$

det M $\neq 0$ whenever $s_1,\ldots,s_n$ are distinct and non zero. If we invert the system of equations

$$M \begin{pmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \end{pmatrix} = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

where

$$b_i(t) = S_{s_i,t}(K)a - Ka$$

$$- \int_0^{s_i} dr \, \frac{(s_i - r)^n}{n!} \, S_{r,t} \left(\frac{(\mathrm{ad}\sigma_t)^{n+1}(K)\sigma_{-(n+1)t}}{t^{n+1}} \right) a$$

we get the following expression for $a_m(t)$, m = 1,...,n

$$a_m(t) = \sum_{i=1}^{n} C_{mi}\Big\{S_{s_i,t}(K)a - Ka$$

$$- \int_0^{s_i} dr \, \frac{(s_i - r)^n}{n!} \, S_{r,t} \Big(\frac{(ad\sigma_t)^{n+1}(K)\sigma_{-(n+1)t}}{t^{n+1}}\Big)a \Big\}$$

where $M^{-1} = (C_{m,i})$. So

$$\Big\| \frac{(ad\sigma_t)^m (K)\sigma_{-m\,t}a}{t^m} \Big\| \le m! \sum_{i=1}^{n} |C_{m,i}| \Big\{ \|S_{s_i,t}(K)a\| + \|Ka\|$$

$$+ \int_0^{s_i} dr \, \frac{(s_i - r)^n}{n!} \, \Big\|S_{r,t}\Big(\frac{(ad\sigma_t)^{n+1}(K)\sigma_{-(n+1)t}}{t^{n+1}}\Big)a\Big\|$$

$$\le m! \sum_{i=1}^{n} |C_{m,i}| \Big\{ 2k_0 \, \| a \|_{p_0} + k_{n+1} \, \| a \|_{p_{n+1}} \int_0^{s_i} dr \, \frac{(s_i - r)^n}{n!} \Big\}$$

since $\|S_{r,t}(K)a\| \le k_0 \, \| a \|_{p_0}$ $\qquad \forall\, a \in B_\infty$

and $\Big\| \dfrac{(ad\sigma_t)^{n+1}(K)a}{t^{n+1}} \Big\| \le k_{n+1} \, \| a \|_{p_{n+1}}$ $\qquad \forall\, a \in B_\infty \, , t \ge 0$.

Therefore, there are constants C_m , D_m (m = 0, 1,...,n) such that

$$\left\| \frac{(\mathrm{ad}\sigma_t)^m(K)a}{t^m} \right\| \le C_m \| a \|_{p_0} + D_m \| a \|_{p_{n+1}} \qquad \forall\, a \in B_\infty ,\, t \ge 0 .$$

The following two results were proved by Derek Robinson:

<u>Lemma 31</u> : The following conditions are equivalent:

1. $a \in D(H^n)$

2. $\lim\limits_{t \to 0} \dfrac{(\sigma_t - I)^n}{t^n} a$ exists

3. there is a sequence $t_m > 0$ such that $t_m \to 0$ and

$$\lim_{m \to \infty} \frac{(\sigma_{t_m} - I)^n a}{t_m^n} \quad \text{exists .}$$

Moreover, if the conditions are satisfied then,

$$H^n a = \lim_{t \to 0} \frac{(\sigma_t - I)^n}{t^n} a = \lim_{m \to \infty} t_m^{-n} (\sigma_{t_m} - I)^n a \; .$$

Proof: $1 \Rightarrow 2$. This follows from the identity

$$\frac{(\sigma_t - I)^n}{t^n} a = t^{-n} \int_0^t ds_1 \int_0^t ds_2 \ldots \int_0^t ds_n \, \sigma_{s_1 + \ldots + s_n} H^n a$$

and the strong continuity of σ_t. This also shows that

$$H^n a = \lim_{t \to 0} \frac{(\sigma_t - I)^n}{t^n} a$$

for $a \in \mathcal{D}(H^n)$.

$2 \Rightarrow 3$. This is straightforward.

$3 \Rightarrow 1$. By definition $a \in \mathcal{D}(H^n)$ if, and only if, the multiple limit

$$\lim_{u_1 \to 0} \ldots \lim_{u_h \to 0} \left(\prod_{i=1}^{n} \frac{(\sigma_{u_i} - I)}{u_i} \right) a$$

exists. If the limit exists, it equals $H^n a$. The identity

$$(\sigma_t - I) \int_0^u ds\, \sigma_s\, a = (\sigma_u - I) \int_0^t ds\, \sigma_s\, a$$

implies

$$(2.3.1) \quad (\sigma_t - I)^n \int_0^{u_1} ds_1 \ldots \int_0^{u_n} ds_n\, \sigma_{s_1 + \ldots + s_n}\, a = \prod_{i=1}^{n} (\sigma_{u_i} - I) \int_0^t ds_1 \ldots \int_0^t ds_n\, \sigma_{s_1 + \ldots + s_n}\, a\ .$$

If,

$$b = \lim_{t_m \to 0} \frac{(\sigma_{t_m} - I)^n}{t_m^n}\, a$$

exists, then (2.3.1) gives

$$\Big(\prod_{i=0}^{n} u_i\Big)^{-1} \int_0^{u_1} ds_1 \ldots \int_0^{u_n} ds_n \, \sigma_{s_1+\ldots+s_n} b$$

$$= \lim_{t_m \to 0} t_m^{-n} \int_0^{t_m} ds_1 \ldots \int_0^{t_m} ds_n \, \sigma_{s_1+\cdots+s_n} \Big(\prod_{i=1}^{n} \frac{(\sigma_{u_i} - I)}{u_i} \Big) a$$

$$= \Big(\prod_{i=1}^{n} \frac{(\sigma_{u_i} - I)}{u_i} \Big) a$$

where the last equality follows from the strong continuity of σ. The left hand side of (2.3.2) converges to b in the multiple limit

$\lim_{u_1 \to 0} \ldots \lim_{u_n \to 0}$ by strong continuity. Therefore,

$a \in \mathcal{D}(H^n)$ and $b = H^n a$.

Lemma 32: The following are equivalent for each $a \in B$.

1. $\sup\limits_{0<t\leq 1} \left\{ \left\| \frac{(\sigma_t - I)^{n+1}}{t^{n+1}} a \right\| \right\} < \infty$

2. $\sup\limits_{0<t_i<1} \left\{ \left\| \prod\limits_{i=0}^{n} \frac{(\sigma_{t_i} - I)}{t_i} a \right\| \right\} < \infty$

3. $a \in B_{n+\frac{1}{2}}$.

Proof: 3⇒2. This follows from the identity

$$\prod_{i=0}^{n} (\sigma_{t_i} - I)a = \int_0^{t_1} ds_1 \dots \int_0^{t_n} ds_n \, \sigma_{s_1+\dots+s_n} (\sigma_{t_0} - I)H^n a$$

2. ⇒1. This is straightforward.

1. ⇒3. Let

$$C = \sup_{0<t\leq 1} \left\{ \left\| \frac{(\sigma_t - I)^{n+1}}{t^{n+1}} a \right\| \right\} .$$

We will use the identity

$$(2.3.3)\quad (\sigma_t - I)^n - \left[2(\sigma_{\frac{t}{2}} - I)\right]^n = \left[\left(\frac{(\sigma_{\frac{t}{2}} + I)}{2}\right)^n - I\right]2^n (\sigma_{\frac{t}{2}} - I)^n$$

$$= 2^{n-1}(\sigma_{\frac{t}{2}} - I)^{n+1} \sum_{m=0}^{n-1} \left(\frac{(\sigma_{\frac{t}{2}} + I)}{2}\right)^m .$$

If we fix $t : 0 < t \leq 1$ and put $t_j = \frac{t}{2^j}$ we get the following after dividing (2.3.3) by t_j^n:

$$\left[\frac{(\sigma_{t_j} - I)}{t_j}\right]^n - \left[\frac{(\sigma_{t_{j+1}} - I)}{t_{j+1}}\right]^n = \frac{t}{2^{j+2}} \left[\frac{(\sigma_{t_{j+1}} - I)}{t_{j+1}}\right]^{n+1} \sum_{m=0}^{n-1} \left[\frac{(\sigma_{t_{j+1}} + I)}{2}\right]^m .$$

Therefore, summing over j from k to $\ell - 1$ gives

$$(2.3.4)\quad \left[\frac{(\sigma_{t_k} - I)}{t_k}\right]^n - \left[\frac{\sigma_{t_\ell} - I}{t_\ell}\right]^n = \sum_{j=k}^{\ell-1} \left(\frac{t}{2^{j+2}}\right) \left[\frac{(\sigma_{t_{j+1}} - I)}{t_{j+1}}\right]^{n+1} \sum_{m=0}^{n-1} \left[\frac{(\sigma_{t_{j+1}} + I)}{2}\right]^m .$$

Since $\|\sigma_s\| \leq 1 \;\forall\, s \geq 0$ we have

$$\left\| \left[\frac{(\sigma_{t_k} - I)}{t_k}\right]^n a - \left[\frac{(\sigma_{t_\ell} - I)}{t_\ell}\right]^n a \right\| \leq Cn \sum_{j=k}^{\ell-1} \left(\frac{t}{2^{j+2}}\right) \qquad 0 < t \leq 1 .$$

Therefore, the limit

$$\lim_{k\to\infty} \left[\frac{(\sigma_{t_k} - I)}{t_k}\right]^n a$$

exists and so, by lemma 31, $a \in \mathcal{D}(H^n)$.

We now put $k = 0$ in (2.3.4) and multiply the result by

$$\frac{\sigma_t - I}{t}$$

to get:

$$\left[\frac{(\sigma_t - I)}{t}\right]\left[\frac{(\sigma_{t_\ell} - I)}{t_\ell}\right]^n = \left[\frac{(\sigma_t - I)}{t}\right]^{n+1} - \sum_{j=0}^{\ell-1} \frac{(\sigma_t - I)}{2^{j+2}} \left[\frac{(\sigma_{t_{j+1}} - I)}{t_{j+1}}\right]^{n+1}$$

$$\sum_{m=0}^{n-1} \left[\frac{(\sigma_{t_{j+1}} + I)}{2}\right]^m$$

Therefore,

$$(2.3.5)\quad \left\| \left[\frac{(\sigma_t - I)}{t}\right]\left[\frac{(\sigma_{t_\ell} - I)}{t_\ell}\right]^n a \right\| \le C\left(1+n\sum_{j=0}^{\ell-1} 2^{-j-1}\right).$$

Since $a \in \mathcal{D}(H^n)$, so is

$$\frac{(\sigma_t - I)a}{t}$$

and we have

$$\frac{(\sigma_t - I)}{t} H^n a = H^n \frac{(\sigma_t - I)a}{t}$$

$$= \lim_{\ell\to\infty}\left[\frac{\sigma_{t_\ell} - I}{t_\ell}\right]^n \frac{(\sigma_t - I)a}{t}.$$

So, from (2.3.5) we have

$$\left\| \frac{(\sigma_t - I)}{t} H^n a \right\| \le C\left(1+n\sum_{j=0}^{\infty} 2^{-j-1}\right) = C(1+n).\quad \forall t : 0 < t \le 1 .$$

Thus, 3 holds and the proof of the lemma is complete.

We now present the final result in this section. It gives commutator bounds that are equivalent to the statement $KB_\infty \subseteq B_{n+1/2}$.

Proposition 33 : The following are equivalent for each $n \in N$:

1. there exist constants $k_{n+1} > 0$ and $p_{n+1} \in N$ such that

$$\|(ad\sigma_t)^{n+1}(K)a\| \leq k_{n+1} t^{n+1} \|a\|_{p_{n+1}} \qquad \forall\, a \in B_\infty ,\ t \geq 0$$

2. there exist constants $k'_{n+1} > 0$ and $q_{n+1} \in N$ such that

$$\|\prod_{i=0}^{n} (ad\sigma_{t_i})(K)a\| \leq k'_{n+1} \prod_{i=0}^{n} t_i \|a\|_{q_{n+1}} \qquad \forall a \in B_\infty ,\ t_i \geq 0$$

3. $KB_\infty \subseteq B_n$ and there exist constants $k''_{n+1} > 0$ and $r_{n+1} \in N$ such that

$$\|(ad\sigma_t)((adH)^n(K))a\| \leq k''_{n+1} t \|a\|_{r_{n+1}} \qquad \forall\, a \in B_\infty ,\ t \geq 0$$

4. $KB_\infty \subseteq B_{n+1/2}$.

Proof: 1⇒4. If 1 holds then we can use lemma 30 to deduce the existence ot constants $C_m : m = 0, 1,...,n+1$ such that

$$\|(ad\sigma_t)^m(K)a\| \le C_m \|a\|_q t^m \quad \forall a \in B_\infty ,\ t \ge 0$$

where $q = \sup\{p_0 , p_{n+1}\}$. Therefore, it can be proved by induction that there are constants $D_m > 0$ $m = 0,1,...,n+1$ such that

$$\|(\sigma_t\text{-}I)^m Ka\| \le D_m t^m \|a\|_{q+m} \quad \forall a \in B_\infty ,\ t \ge 0 .$$

The first step follows since

$$\|(\sigma_t\text{-}I)Ka\| \le \|[\sigma_t, K]a\| + \|K(\sigma_t\text{-}I)a\|$$

$$\le C_1 t\|a\|_q + k_0\|(\sigma_t\text{-}I)a\|_{p_0}$$

$$\le (C_1 + k_0)t\|a\|_{q+1}$$

where the result

$$\|(\sigma_t\text{-}I)\ a\| = \|\int_0^t ds\sigma_s Ha\| \le t\|a\|_1$$

has been used.

In general,

$$(\sigma_t\text{-}I)^m \; Ka = (ad\sigma_t)^m(K)a - \sum_{r=0}^{m-1} (-1)^{m-r}\binom{m}{r}(\sigma_t\text{-}I)^r K(\sigma_t\text{-}I)^{m-r} a$$

The induction hypothesis implies

$$\|(\sigma_t\text{-}I)^r K(\sigma_t\text{-}I)^{m-r}a\| \le D_r\, t^r \,\|(\sigma_t\text{-}I)^{m-r}a\|_{q+r}$$

$$\le D_r\, t^r(t^{m-r}\|a\|_{q+r+(m-r)})$$

$$= D_r\, t^m\|a\|_{q+m}$$

for $0 \le r \le m\text{-}1$.

This, together with the bound on $\|(ad\sigma_t)^m(K)a\|$, gives the required bound on $\|(\sigma_t\text{-}I)^m Ka\|$ for $m = 0,1,\ldots,n+1$. Lemma 32 can now be used to deduce that $KB_\infty \subseteq B_{n+1/2}$.

$4 \Rightarrow 3$. If $KB_\infty \subseteq B_{n+1/2}$ then, by uniform boundedness, there are constants $\ell_m > 0$, $s_m \in N$ $(m=0,1,\ldots,n+1)$ such that

$$\|(\sigma_t\text{-}I)^m Ka\| \le \ell_m t^m \,\|a\|_{s_m} \quad \forall a \in B_\infty,\ t\ge 0 .$$

Therefore, from lemma 31

$$\|H^m K H^{n-m}(\sigma_t - I)a\| = \lim_{s\to 0} \left\| \frac{(\sigma_s - I)^m}{s^m} K H^{n-m}(\sigma_t - I)a \right\|$$

$$\leq \ell_m \|H^{n-m}(\sigma_t - I)a\|_{s_m}$$

$$\leq \ell_m \|(\sigma_t - I)a\|_{s_m + n - m}$$

$$\leq t\, \ell_m \; \|a\|_{s_m + n - m + 1} \tag{2.3.6}$$

for $m = 0, l, \ldots, n$.

We also have, from the definition of $B_{n+1/2}$

$$\|(\sigma_t - I)H^n K a\| = \mathcal{O}(t) \qquad \forall a \in B_\infty \tag{2.3.7}$$

and, for $m = 0, 1, \ldots, n-1$, we have

$$\|(\sigma_t - I)H^m K H^{n-m} a\| \leq t\, \|H^{m+1} K H^{n-m} a\| = \mathcal{O}(t) \quad \forall a \in B_\infty . \tag{2.3.8}$$

If we now expand $(\mathrm{ad}\sigma_t)((\mathrm{ad}H)^n(K))a$ and use (2.3.6), (2.3.7), and (2.3.8) we get

$$\|(\mathrm{ad}\sigma_t)((\mathrm{ad}H)^n(K))a\| = \mathcal{O}(t) \qquad\qquad \forall a \in B_\infty .$$

Therefore, 3. follows from uniform boundedness.

$3 \Rightarrow 2$. Let $a \in B_\infty$, and let $t_0, t_1, \ldots, t_n \geq 0$.

Define $F : \mathbb{R}^n \to B$ by

$$F(u_1, \ldots, u_n) = \left[\sigma_{t_0}, \sigma_{\sum_1^n u_i} K \sigma_{\sum_{j=1}^n t_j - \sum_{i=1}^n u_i} \right] a .$$

Then,

$$\frac{\partial^n F}{\partial_{u_1} \ldots \partial_{u_n}} = \left[\sigma_{t_0}, \sigma_{\sum_{i=1}^n u_i} \left((\mathrm{ad} H)^n (K)\right) \sigma_{\sum_1^n t_j - \sum_1^n u_i} \right] a$$

$$= \sigma_{\sum_1^n u_i} (\mathrm{ad}\sigma_{t_0}) \left((\mathrm{ad} H)^n (K)\right) \sigma_{\sum_1^n t_j - \sum_1^n u_i} a .$$

In addition we have

$$\int_0^{t_1} du_1 \ldots \int_0^{t_n} du_n \frac{\partial^n F}{\partial u_1 \ldots \partial u_n} = \prod_{i=0}^n (\mathrm{ad}\sigma_{t_i})(K) a .$$

Therefore, if 3. holds

$$\left\| \prod_{i=0}^n (\mathrm{ad}\sigma_{t_i})(K) a \right\| \leq k''_{n+1} \left(\prod_{i=0}^n t_i \right) \| a \|_{r_{n+1}}$$

since

$$\left\| \frac{\partial^n F}{\partial u_1 \cdots \partial u_n} \right\| \le \left\| (\mathrm{ad}\,\sigma_{t_0}) \left((\mathrm{ad}H)^n(K) \right) \sigma_{\sum_1^n t_j - \sum_1^n u_i}\, a \right\|$$

$$\le k''_{n+1} t_0 \left\| \sigma_{\sum_1^n t_j - \sum_1^n u_i}\, a \right\|_{r_{n+1}}$$

$$\le k''_{n+1} t_0 \| a \|_{r_{n+1}}$$

2.⇒1. Straightforward.

<u>Remark:</u> It can be assumed that $k_{n+1} = k'_{n+1} = k''_{n+1}$ and that $p_{n+1} = q_{n+1} = r_{n+1}$ in Proposition 33 by the following argument. The proofs of $3 \Rightarrow 2$ and $2 \Rightarrow 1$ show that it can be assumed $k_{n+1} \le k'_{n+1} \le k''_{n+1}$ and $p_{n+1} \le q_{n+1} \le r_{n+1}$. Therefore, it is sufficient to show that given k_{n+1} and p_{n+1} , the constants k''_{n+1} and r_{n+1} can be chosen so that $k''_{n+1} \le k_{n+1}$ and $r_{n+1} \le p_{n+1}$.

Let $t > 0$ and put $t_m = t \cdot 2^{-m}$. Then we have the identity

$$\left[\sigma_t, [\sigma_{t_m}, K] \right] = \sum_{j=1}^{2^m} \sigma_{(j-1)t_m} \left[\sigma_{t_m}, [\sigma_{t_m}, K] \right] \sigma_{(2^m - j)t_m} .$$

Putting $(\mathrm{ad}\sigma_{t_m})^{n-1}(K)$ in place of K gives

$$(\mathrm{ad}\sigma_t)\big((\mathrm{ad}\sigma_{t_m})^n(K)\big) = \sum_{j=1}^{2^m} \sigma_{(j-1)t_m}(\mathrm{ad}\sigma_{t_m})^{n+1}(K)\sigma_{(2^m-j)t_m} .$$

If

$$\|(\mathrm{ad}\sigma_t)^{n+1}(K)a\| \le k_{n+1}t^{n+1}\|a\|_{p_{n+1}} \qquad \forall t \ge 0,\ a \in B_\infty$$

then $KB_\infty \subseteq B_n$ from the proposition, and so

$$\|(\mathrm{ad}\sigma_t)\big((\mathrm{ad}H)^n(K)\big)a\| = \lim_{m\to\infty} \frac{1}{t_m^n}\|(\mathrm{ad}\sigma_t)\big((\mathrm{ad}\sigma_{t_m})^n(K)\big)a\|$$

$$\le \lim_{m\to\infty} \sum_{j=1}^{2^m} t_m^{-n}\,\|\sigma_{(j-1)t_m}\big((\mathrm{ad}\sigma_{t_m})^{n+1}(K)\big)\sigma_{(2^m-j)t_m}a\|$$

$$\le \lim_{m\to\infty} \sum_{j=1}^{2^m} t_m^{-n}\,\|\big((\mathrm{ad}\sigma_{t_m})^{n+1}(K)\big)\sigma_{(2^m-j)t_m}a\|$$

$$\leq \lim_{m\to\infty} \sum_{j=1}^{2^m} t_m^{-n} k_{n+1} t_m^{n+1} \|\sigma_{(2^m-j)t_m} a\|_{p_{n+1}}$$

$$\leq \lim_{m\to\infty} 2^m k_{n+1} t_m \| a \|_{p_{n+1}}$$

$$= k_{n+1} t \| a \|_{p_{n+1}} .$$

Therefore, equality of the constants may be assumed.

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