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Some new results on algebraic Riccati equations arising in linear quadratic differential games and the stabilization of uncertain linear systems *

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Abstract: This paper presents some new results on algebraic Riccati equations arising in linear quadratic differential games. The first result is a uniqueness result for solutions of the Riccati equations under consideration. The second result is concerned with Riccati equations which arise in differential games in which the weighting on the minimizing control is allowed to approach zero. It is shown that if a certain minimum phase condition is satisfied then the corresponding solution to the Riccati equation will also approach zero.

Keywords: Riccati equations, Uniqueness results, Differential games, Cheap control.

1. Introduction

This paper is concerned with algebraic Riccati equations of the form

$$A'P + PA - PMP + Q = 0 \tag{1.1}$$

where $A \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{n \times n}$ and $M \in \mathbb{R}^{n \times n}$. It is assumed that the matrices M and Q are symmetric. However, no sign definiteness assumption will be placed on the matrix M. Riccati equations of this form arise in the stabilization of uncertain systems (see [1-3]), disturbance attenuation problems (see [4]) and in linear quadratic differential games (see [5]).

The main result of this paper is given in Section 3. This result is concerned with the algebraic Riccati equation which arises in linear quadratic differential games with cheap control; see [6]. The main result of [6] shows that if a certain strict minimum phase condition is satisfied then the value of the differential game will approach zero as the control weighting approaches zero. However, an important gap emerges in the results presented in [6]. Indeed, that paper shows that the strict minimum phase condition is a sufficient but not necessary condition for the limiting value of the game to be zero. In this paper, we close this gap by showing that a (non-strict) minimum phase condition is both necessary and sufficient for the limiting value of the game to be zero. This result is established by looking at the limiting value of the solution to the algebraic Riccati equation associated with the differential game. As mentioned in [6], this result has applications not only in the area of differential games but also in the stabilization of uncertain systems using high gain observer based feedback.

In order to establish the main result of this paper, we use some preliminary results concerning the existence and uniqueness of solutions to Riccati equations of the form (1.1). These results may be of independent interest and hence are stated separately in Section 2.

2. Existence and uniqueness results

Before stating our uniqueness result for Riccati equations of the form (1.1), we first define a notion of strong solutions to (1.1); see also [7].

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Definition 2.1. A symmetric matrix [1] P is said to be a *strong solution* to Riccati equation (1.1) if it satisfies (1.1) and furthermore, all eigenvalues λ of the matrix $A-MP$ satisfy $\mathrm{Re}(\lambda)\leqslant 0$.

Notation. Given symmetric matrices M and N, the notation $M\geqslant N$ refers to the fact that the matrix $M-N$ is positive-semidefinite. Similarly, the notation $M>N$ refers to the fact that the matrix $M-N$ is positive-definite.

Theorem 2.1. *Suppose that the Riccati equation*

$$A'P+PA-PMP+Q_1=0 \tag{2.1}$$

has a symmetric solution P_1. Then given any symmetric matrix $Q_2<Q_1$, the Riccati equation

$$A'P+PA-PMP+Q_2=0 \tag{2.2}$$

has at most one strong solution.

In order to prove this theorem, we will first establish the following lemma.

Lemma 2.1. *Suppose that the symmetric matrix Q is positive-definite. Then the Riccati equation*

$$A'P+PA-PMP+Q=0 \tag{2.3}$$

has at most one strong solution.

Proof. We first observe that all symmetric solutions to (2.3) must be nonsingular. Indeed, if P is a symmetric solution to (2.3) such that $Px=0$ for some $x\neq 0$, then

$$x'A'Px+x'PAx-x'PMPx+x'Qx=x'Qx=0.$$

This contradicts the fact that Q is positive-definite.

We now suppose P_1 and P_2 are two strong solutions to (2.3). Hence P_1 and P_2 are both nonsingular. Thus, we can define the two symmetric matrices $S_1=P^{-1}$ and $S_2=P_2^{-1}$. Both of these matrices will satisfy the Riccati equation

$$-AS-SA'-SQS+M=0. \tag{2.4}$$

Furthermore, it follows from Riccati equation (2.4) that

$$S_1(-A'-QS_1)=AS_1-M$$

and hence

$$S_1(-A'-QS_1)S_1^{-1}=A-MP_1.$$

That is, the matrix $-A'-QS_1$ is similar to the matrix $A-MP_1$. Therefore, S_1 is a strong solution to Riccati equation (2.4). Similarly, it also follows that S_2 is a strong solution (2.4).

We now observe that the matrix Q is positive-definite and hence the pair $(-A', Q^{1/2})$ is controllable. Therefore, using Theorem 3 of [8], it follows that there is at most one strong solution to (2.4). Hence $S_1=S_2$ and therefore $P_1=P_2$. □

Proof of Theorem 2.1. Suppose that P_1 is a symmetric solution to (2.1) and that P_2 and P_3 are strong solutions to (2.2). We first define the matrices $F=A-MP_1$, $U_2=P_1-P_2$, $U_3=P_1-P_3$ and $R=Q_1-Q_2>0$.

Claim: The matrices U_2 and U_3 are both strong solutions to the Riccati equation

$$F'U+UF+UMU+R=0. \tag{2.5}$$

We first establish this claim for the matrix U_2. Indeed

$$\begin{aligned}F'U_2&=(A-MP_1)'(P_1-P_2)\\&=A'P_1-A'P_2-P_1MP_1+P_1MP_2.\end{aligned}$$

Hence, using (2.1) and (2.2), it follows that

$$\begin{aligned}F'U_2+U_2F&=A'P_1+P_1A-A'P_2-P_2A\\&\quad-2P_1MP_1+P_1MP_2+P_2MP_1\\&=P_1MP_1-Q_1-P_2MP_2+Q_2\\&\quad-2P_1MP_1+P_1MP_2+P_2MP_1\\&=-R-U_2MU_2.\end{aligned}$$

Thus, U_2 is a symmetric solution to (2.5). Furthermore,

$$\begin{aligned}F+MU_2&=A-MP_1+M(P_1-P_2)\\&=A-MP_2.\end{aligned}$$

However, all eigenvalues λ of $A-MP_2$ satisfy $\mathrm{Re}(\lambda)\leqslant 0$ and hence U_2 must be a strong solution to (2.5).

Using an identical analysis to the above, it follows that U_3 is also a strong solution to (2.5). This completes the proof of the claim.

In order to complete the proof of the theorem, we now invoke Lemma 2.1. Using this lemma, it follows that Riccati equation (2.5) has at most one

[1] Throughout this paper, we will be concerned only with real matrices.

strong solution. Therefore $U_2 = U_3$ and hence $P_2 = P_3$. □

Theorem 2.2. *Let $M_1 \geqslant 0$ be a given symmetric matrix and suppose that the Riccati equation*

$$A'P + PA - PM_1P + Q = 0 \tag{2.6}$$

has a symmetric solution P_1. Furthermore, let $M_2 \geqslant 0$ be a given symmetric matrix such that $M_2 \leqslant M_1$ and the pair (A, M_2) is stabilizable. Then the Riccati equation

$$A'P + PA - PM_2P + Q = 0 \tag{2.7}$$

has a strong solution $P_2^+ \geqslant P_1$.

The following proof of this theorem is based on a personal communication from Dr David Clements at the University of New South Wales.

Proof. Let the matrix R be defined as follows:

$$\begin{aligned} R &\triangleq A'P_1 + P_1A - P_1M_2P_1 + Q \\ &= A'P_1 + P_1A - P_1M_1P_1 \\ &\quad + Q + P_1M_1P_1 - P_1M_2P_1 \\ &= P_1(M_1 - M_2)P_1 \\ &\geqslant 0. \end{aligned}$$

Furthermore, let $\tilde{A} \triangleq A - M_2P_1$. Given that the pair (A, M_2) is stabilizable, it follows that the pair $(\tilde{A}, M_2)$ will also be stabilizable. We now use Theorem 2.2 of [9] to conclude that the Riccati equation

$$\tilde{A}'P + P\tilde{A} - PM_2P + R = 0$$

has a maximal solution $P^+ \geqslant 0$. Letting $P_2 = P_1 + P^+ \geqslant P_1$, it follows that

$$\begin{aligned} &A'P_2 + P_2A - P_2M_2P_2 + Q \\ &\quad = A'P_1 + P_1A - P_1M_2P_1 + Q \\ &\qquad + \tilde{A}'P^+ + P^+\tilde{A} - P^+M_2P^+ \\ &\quad = \tilde{A}'P^+ + P^+\tilde{A} - P^+M_2P^+ + R \\ &\quad = 0. \end{aligned}$$

Thus, P_2 is a symmetric solution to (2.7). Furthermore, using Theorem 2.1 of [9], it now follows that (2.7) has a strong solution $P_2^+ \geqslant P_2 \geqslant P_1$. □

3. The main result

In this section, we consider the limiting behaviour of the solution to the Riccati equation

$$\begin{aligned} &A'P + PA - q^2PBB'P \\ &\quad + PDD'P + C'C + Q/q^2 = 0 \end{aligned} \tag{3.1}$$

as $q \to \infty$. In this Riccati equation $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{r\times n}$, and $Q \in \mathbb{R}^{n\times n}$. The matrix Q is assumed to be positive-definite and symmetric. It is shown in [5] that the solution of this Riccati equation is associated with the value of a certain differential game; see also [6]. Also associated with (3.1) is the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Dv(t), \\ y(t) &= Cx(t). \end{aligned} \tag{Σ}$$

In order to prove the main result of this section, we will assume that the system (Σ) satisfies the following standard assumptions:

A1. $[A, B]$ is stabilizable.
A2. $[C, A]$ is detectable.
A3. B is full rank.
A4. C is full rank.

It will also be assumed that the system (Σ) satisfies the following (non-strict) minimum phase condition.

A5. The transfer function

$$G(s) = C(sI - A)^{-1}B$$

is right invertible and minimum phase; i.e.,

$$\operatorname{rank}\begin{bmatrix} A - sI & B \\ C & 0 \end{bmatrix} = n + r, \quad \operatorname{Re}(s) > 0.$$

It is the weakening of this minimum phase condition from the strict minimum phase condition assumed in [6] which constitutes the most significant feature of our main result.

Note. In the statement of Assumption 5 in [6] there is a misleading typographical error. In fact the condition in [6] should read

$$\operatorname{rank}\begin{bmatrix} A - sI & B \\ C & 0 \end{bmatrix} = n + q, \quad \operatorname{Re}(s) \geqslant 0.$$

This is in contrast to the (non-strict) minimum phase condition given above.

Before stating our main result, we must first introduce a definition.

Definition 3.1. A positive-definite symmetric matrix P_1 satisfying (3.1) is said to be the *minimal positive-definite symmetric solution* to (3.1) if given any other positive-definite symmetric solution P_2 to (3.1), then $P_1 \leqslant P_2$.

We are now in a position to state the main result of this paper.

Theorem 3.1. *Suppose that the system* (Σ) *satisfies assumptions* A1–A5 *and that Riccati equation* (3.1) *has a positive-definite symmetric solution for all* $q \geqslant \bar{q}$. *Then for each* $q \geqslant \bar{q}$, (3.1) *has a minimal positive-definite solution* P_q^-. *Furthermore* $P_q^- \to 0$ *as* $q \to \infty$.

Remarks. As mentioned in the introduction, this result on the algebraic Riccati equation can be applied to the limiting behaviour of a linear quadratic differential game with cheap control. Indeed, consider the differential game with cost functional

$$J_q(u, v) = \int_0^\infty \left\{ y(t)'y(t) + \frac{1}{q^2}u(t)'u(t) - v(t)'v(t) \right\} \mathrm{d}t$$

and suppose the system under consideration is described by the state equations (Σ). In this differential game $u(t) \in \mathbb{R}^m$ is the minimizing control and $v(t) \in \mathbb{R}^r$ is the maximizing control. The value of this differential game will be denoted J_q^*.

Differential games of this form were studied in [6] by considering the Riccati equation (3.1). Indeed, as detailed on page 187 of [6], it follows that for any initial condition x_0, the value of the game satisfies the bound

$$J_q^* \leqslant x_0' P_q^- x_0$$

where P_q^- is the minimum positive-definite solution of (3.1). Hence, using the above theorem, it follows that if assumptions A1–A5 are satisfied then $J_q^* \to 0$ as $q \to \infty$.

From the above, it can be seen that for systems satisfying the regularity assumptions A1–A4, then assumption A5 is a sufficient condition for the limiting value of the game to be zero. Furthermore, it was shown on page 187 of [6] that our assumption A5 is a necessary condition for the limiting value of the game to be zero. Thus, we have a necessary and sufficient condition for the value of the game to be zero. This closes the gap mentioned in [6].

Before establishing Theorem 3.1, we must first introduce some additional notation and establish some preliminary lemmas.

Notation. Let $\mathscr{S} \subset \mathbb{R}^n$ be the maximal A, B-invariant subspace contained in ker C; e.g., see [10]. Furthermore, let $\mathscr{F}(\mathscr{S})$ denote the set of matrices F such that $\mathscr{S}$ is $(A + BF)$-invariant. Given any matrix $F \in \mathbb{R}^{m \times n}$, we denote $A + BF$ by A_F.

It follows from [10] that given any $F \in \mathscr{F}(\mathscr{S})$, there exists a state space transformation on (Σ) such that we can write

$$A_F = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}, \tag{3.2}$$

$$\mathscr{S} = \operatorname{span}\begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{3.3}$$

Furthermore, since $\mathscr{S} \subset \ker C$, it follows that C will be of the form $C = [0 \ \ C_2]$.

Lemma 3.1. *Suppose that the system* (Σ) *satisfies assumptions* A1–A5. *Then, there exists a matrix* $F_0 \in \mathscr{F}(\mathscr{S})$ *with the following property: Suppose* A_{F_0} *has been transformed into the form* (3.2). *Then, all eigenvalues* λ *of the block* A_1 *satisfy* $\mathrm{Re}(\lambda) \leqslant 0$.

Proof. The proof is identical to the proof of Lemma 3.1 in [6]. □

Lemma 3.2. *Suppose that the system* (Σ) *satisfies assumptions* A1–A5 *and that Riccati equation* (3.1) *has a positive-definite symmetric solution for all* $q \geqslant \bar{q}$. *Furthermore, let* F_0 *be defined as in Lemma* 3.1. *Then for each* $q \geqslant \bar{q}$, *the Riccati equation* (3.1) *has a minimum positive-definite solution* P_q^-. *Moreover, there exists a family of feedback matrices* $\{F_q \in \mathbb{R}^{m \times n}\colon q \in [\bar{q}, \infty)\}$ *such that the following conditions hold:*

(i) *The Riccati equation*

$$A_{F_q}'P + PA_{F_q} + PDD'P + C'C + F_q'F_q/q^2 + Q/q^2 = 0 \tag{3.4}$$

has a positive-definite symmetric solution P_q^+ for all $q \geqslant \tilde{q}$.

(ii) *There exists a positive-definite symmetric matrix $\tilde{P}$ such that $P_q^+ \leqslant \tilde{P}$ for all $q \geqslant \tilde{q}$.*

(iii) *There exists a positive-semidefinite matrix*

$$P^\infty = \begin{bmatrix} 0 & 0 \\ 0 & P_2^\infty \end{bmatrix} \quad (\textit{conforming with}(3.2))$$

and a strictly increasing sequence of integers $\{k_i\}_{i=1}^\infty$ such that $k_1 \geqslant \tilde{q}$ and $P_{k_i}^+ \to P^\infty$ as $i \to \infty$.

Proof. Suppose that Riccati equation (3.1) has a positive-definite symmetric solution P for $q \geqslant \bar{q}$. Hence, the symmetric matrix $S = P^{-1}$ will be a solution to the Riccati equation

$$AS + SA' - q^2BB' + DD' + SC'CS + SQS/q^2 = 0.$$

However, if we define Q^* to be the positive-definite matrix $Q^* = C'C + Q/q^2$, it follows that the pair $(A, (Q^*)^{1/2})$ will be controllable. Hence, using Theorem 2.1 of [9], it follows that the above Riccati equation will have a maximal symmetric solution $S^+ > 0$. Furthermore, the inverse of this matrix $P_q^- \triangleq (S^+)^{-1}$ will be the minimal positive-definite solution to (3.1).

We now consider the case of $q = \bar{q}$ and let $\tilde{P} = P_{\bar{q}}^-$. Using the definition of A_{F_0}, it follows that

$$A'_{F_0}\tilde{P} + \tilde{P}A_{F_0} - F_0'B'\tilde{P} - \tilde{P}BF_0 - \bar{q}^2\tilde{P}BB'\tilde{P} + \tilde{P}DD'\tilde{P} + C'C + Q/\bar{q}^2 = 0.$$

Let $x \in \mathbb{R}^n$ be given such that $B'\tilde{P}x = 0$, $x \neq 0$. Therefore

$$x'\left[A'_{F_0}\tilde{P} + \tilde{P}A_{F_0} + \tilde{P}DD'\tilde{P} + C'C\right]x = -x'Qx/q^2 < 0.$$

Hence, Finsler's Theorem (see [11]) implies that there exists a constant $\sigma > 0$ such that the matrix

$$-\hat{Q} = A'_{F_0}\tilde{P} + \tilde{P}A_{F_0} - \sigma\tilde{P}BB'\tilde{P} + \tilde{P}DD'\tilde{P} + C'C$$

is negative-definite. That is,

$$A'_{F_0}\tilde{P} + \tilde{P}A_{F_0} - \sigma\tilde{P}BB'\tilde{P} + \tilde{P}DD'\tilde{P} + C'C + \hat{Q} = 0. \tag{3.5}$$

Hence, $\tilde{S} = \tilde{P}^{-1}$ satisfies the Riccati equation

$$A_{F_0}\tilde{S} + \tilde{S}A'_{F_0} - \sigma BB' + DD' + \tilde{S}(C'C + \hat{Q})\tilde{S} = 0. \tag{3.6}$$

We now consider the Riccati equation

$$A_{F_0}S + SA'_{F_0} - \sigma BB' + DD' + S(C'C + \varepsilon^2\hat{Q})S = 0. \tag{3.7}$$

It follows from Theorem 2.2 that given any $\varepsilon \in (0, 1]$, this Riccati equation has a strong solution S_ε such that $S_\varepsilon > 0$. Furthermore, S_ε is monotone increasing as $\varepsilon \to 0$.

Given any S_ε defined as above, let $P_\varepsilon = S_\varepsilon^{-1}$. It follows that given any $\varepsilon \in (0, 1]$, the positive-definite symmetric matrix P_ε satisfies the Riccati equation

$$A'_{F_0}P_\varepsilon + P_\varepsilon A_{F_0} - \sigma P_\varepsilon BB'P_\varepsilon + P_\varepsilon DD'P_\varepsilon + C'C + \varepsilon^2\hat{Q} = 0. \tag{3.8}$$

We also observe from (3.7) that

$$S_\varepsilon\left(-A'_{F_0} - \left[C'C + \varepsilon^2\hat{Q}\right]S_\varepsilon\right) = A_{F_0}S_\varepsilon - \sigma BB' + DD'$$

and hence

$$S_\varepsilon\left(-A'_{F_0} - \left[C'C + \varepsilon^2\hat{Q}\right]S_\varepsilon\right)S_\varepsilon^{-1} = A_{F_0} - [\sigma BB' - DD']P_\varepsilon.$$

That is, the matrix

$$A_{F_0} - [\sigma BB' - DD']P_\varepsilon$$

is similar to the matrix

$$-A'_{F_0} - \left[C'C + \varepsilon^2\hat{Q}\right]S_\varepsilon.$$

Therefore, using the fact that S_ε is a strong solution to (3.7), it follows that P_ε is a strong solution to (3.8).

We now recall the definition of P_ε and note that P_ε is monotone decreasing as $\varepsilon \to 0$. Furthermore, $0 < P_\varepsilon \leqslant \tilde{P}$ for all $\varepsilon \in (0, 1]$. Thus, there exists a limiting matrix $\bar{P} \geqslant 0$ such that $P_\varepsilon \to \bar{P}$ as $\varepsilon \to 0$. Moreover, using the fact that for any $\varepsilon \in (0, 1]$, all eigenvalues λ of

$$A_{F_0} - [\sigma BB' - DD']P_\varepsilon$$

satisfy $\mathrm{Re}(\lambda) \leqslant 0$, it follows that all eigenvalues of

$$A_{F_0} - [\sigma BB' - DD']\bar{P}$$

satisfy $\mathrm{Re}(\lambda) \leqslant 0$. Hence, taking the limit as $\varepsilon \to 0$ in (3.8), it follows that $\bar{P}$ is a strong solution to the Riccati equation

$$A'_{F_0}P + PA_{F_0} - \sigma PBB'P + PDD'P + C'C = 0. \tag{3.9}$$

We are now in a position to construct the required family of feedback matrices $\{F_q \in \mathbb{R}^{m\times n}\colon q \in [\tilde{q}, \infty)\}$. Indeed, let F_0 be defined as in Lemma 3.1 and let $F_\infty = F_0 - \sigma B'\bar{P}/2$. Furthermore, choose the constant $\delta > 0$ such that

$$Q \leqslant \delta^2\hat{Q}/2 \quad \text{and} \quad F'_\infty F_\infty \leqslant \delta^2\hat{Q}/4.$$

Given any $q \geqslant \delta$, the corresponding matrix F_q is constructed as follows: Let $\varepsilon = \delta/q \leqslant 1$ and let P_ε be the corresponding solution to (3.8). Then

$$F_q \triangleq F_0 - \sigma B'P_\varepsilon/2.$$

It follows from this definition that $F_q \to F_\infty$ as $q \to \infty$. Hence, we can define the constant $\tilde{q} \geqslant \delta$ such that $F'_qF_q \leqslant \delta^2\hat{Q}/2$ for all $q \geqslant \tilde{q}$. Thus, the family of feedback matrices $\{F_q \in \mathbb{R}^{m\times n}\colon q \in [\tilde{q}, \infty)\}$ is such that $F'_qF_q + Q \leqslant \delta^2\hat{Q}$ for all $q \geqslant \tilde{q}$. Furthermore, it follows from (3.8) that given any $q \geqslant \tilde{q}$, if we let $\varepsilon = \delta/q$ then the matrix P_ε satisfies the Riccati equation

$$A'_{F_q}P + PA_{F_q} + PDD'P + C'C + \delta^2\hat{Q}/q^2 = 0.$$

Therefore $S_\varepsilon = P_\varepsilon^{-1}$ satisfies the Riccati equation

$$A_{F_q}S + SA'_{F_q} + DD' + S\{C'C + \delta^2\hat{Q}/q^2\}S = 0.$$

However,

$$\delta^2\hat{Q}/q^2 \geqslant (F'_qF_q + Q)/q^2 > 0$$

and hence using Theorem 2.2, it follows that the Riccati equation

$$A_{F_q}S + SA'_{F_q} + DD' + S\{C'C + (F'_qF_q + Q)/q^2\}S = 0$$

has a positive-definite symmetric solution $S^+ \geqslant S_\varepsilon > 0$. Thus, the matrix $P_q^+ = (S^+)^{-1} \leqslant S_\varepsilon^{-1} = P_\varepsilon$ satisfies Riccati equation (3.4). This completes the proof of part (i) of the lemma.

To establish part (ii) of the lemma, we observe that given any $q \geqslant \tilde{q}$ then $P_q^+ \leqslant P_{\delta/q} \leqslant \tilde{P}$ as required.

To establish part (iii) of the lemma, we first observe that given any $q \geqslant \tilde{q}$ then the matrix P_q^+ is contained in the compact set [2] $\mathscr{P} = \{P \in \mathbb{R}^{n\times n}\colon P \text{ is symmetric and } 0 \leqslant P \leqslant \tilde{P}\}$. Letting $\tilde{k}$ be any integer greater than $\tilde{q}$, we now define the infinite sequence of matrices $\{P_k^+\}_{k=\tilde{k}}^\infty \subset \mathscr{P}$. Given that the set $\mathscr{P}$ is compact, it follows that this sequence has a convergent subsequence $\{P_{k_i}^+\}_{i=1}^\infty$ such that $P_{k_i}^+ \to P^\infty \in \mathscr{P}$ as $i \to \infty$. Thus, in order to complete the proof of part (iii), it remains only to show that the matrix P^∞ is of the required form.

Towards this end, we first recall that given any i, if $\varepsilon = \delta/k_i$ then $0 \leqslant P_{k_i}^+ \leqslant P_\varepsilon$. Furthermore, $P_\varepsilon \to \bar{P}$ as $\varepsilon \to 0$ and hence

$$0 \leqslant P^\infty \leqslant \bar{P}. \tag{3.10}$$

Thus, we will now investigate the structure of the matrix $\bar{P}$.

Recall equation (3.6) and let $\tilde{Q} = \tilde{S}\hat{Q}\tilde{S}$. It follows that

$$A_{F_0}\tilde{S} + \tilde{S}A'_{F_0} - \sigma BB' + DD' + \tilde{S}C'C\tilde{S} + \tilde{Q} = 0. \tag{3.11}$$

We now partition the matrices B, D, $\tilde{Q}$ and $\tilde{S}$ to conform with (3.2) as follows:

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},$$

$$\tilde{Q} = \begin{bmatrix} \tilde{Q}_1 & \tilde{Q}_3 \\ \tilde{Q}'_3 & \tilde{Q}_2 \end{bmatrix}, \qquad \tilde{S} = \begin{bmatrix} \tilde{S}_1 & \tilde{S}_3 \\ \tilde{S}'_3 & \tilde{S}_2 \end{bmatrix}.$$

Substituting into (3.11) and taking the (2, 2) block, we conclude that $\tilde{S}_2 > 0$ satisfies the Riccati equation

$$A_2\tilde{S}_2 + \tilde{S}_2A'_2 - \sigma B_2B'_2 + D_2D_2 + \tilde{S}_2C'_2C_2\tilde{S}_2 + \tilde{Q}_2 = 0.$$

Letting $\tilde{P}_2 = \tilde{S}_2^{-1}$ and $\tilde{Q}_2 = \tilde{P}_2\tilde{Q}_2\tilde{P}_2$, it follows that $\tilde{S}_2$ satisfies the Riccati equation

$$-A_2\tilde{S}_2 - \tilde{S}_2A'_2 + [\sigma B_2B'_2 - D_2D'_2] - \tilde{S}_2(C'_2C_2 + \tilde{Q}_2)\tilde{S}_2 = 0.$$

It now follows from Theorem 2.2 that given any $\varepsilon \in (0, 1]$, the Riccati equation

$$-A_2S - SA'_2 + [\sigma B_2B'_2 - D_2D'_2] - S(C'_2C_2 + \varepsilon\tilde{Q}_2)S = 0 \tag{3.12}$$

[2] For example, using the trace norm topology on the set of positive-semidefinite matrices.

has a strong solution $\hat{S}_\varepsilon > \tilde{S} > 0$. Furthermore, $\hat{S}_\varepsilon$ is monotone increasing as $\varepsilon \to 0$.

Given any $\varepsilon > 0$, we define $\hat{P}_\varepsilon = \hat{S}_\varepsilon^{-1}$. Using (3.12) it is straightforward to verify that the matrix

$$A_2 - [\sigma B_2 B_2' - D_2 D_2'] \hat{P}_\varepsilon$$

is similar to

$$-A_2' - \left(C_2' C_2 + \varepsilon^2 \overline{Q}_2\right) \hat{S}_\varepsilon.$$

Hence, using the fact that $\hat{S}_\varepsilon$ is a strong solution to (3.12), it follows that $\hat{P}_\varepsilon$ is a strong solution to the Riccati equation

$$\begin{aligned} &A_2' \hat{P}_\varepsilon + \hat{P}_\varepsilon A_2 - \sigma \hat{P}_\varepsilon B_2 B_2' \hat{P}_\varepsilon \\ &\quad + \hat{P}_\varepsilon D_2 D_2' \hat{P}_\varepsilon + C_2' C_2 + \varepsilon^2 \overline{Q}_2 = 0. \end{aligned} \tag{3.13}$$

Moreover, $\hat{P}_\varepsilon$ is monotone decreasing as $\varepsilon \to 0$ and thus there exists a limiting matrix $\overline{P}_2$ such that $\hat{P}_\varepsilon \to \overline{P}_2$. Given that $\hat{P}_\varepsilon$ is a strong solution to (3.13) for all $\varepsilon \in (0, 1]$, it now follows that $\overline{P}_2$ will be a strong solution to the Riccati equation

$$\begin{aligned} &A_2' \overline{P}_2 + \overline{P}_2 A_2 - \sigma \overline{P}_2 B_2 B_2' \overline{P}_2 \\ &\quad + \overline{P}_2 D_2 D_2' \overline{P}_2 + C_2' C_2 = 0. \end{aligned} \tag{3.14}$$

Using this equation, it is straightforward to verify that the matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & \overline{P}_2 \end{bmatrix}$$

satisfies (3.9). Furthermore

$$A_{F_0} - [\sigma BB' - DD'] \begin{bmatrix} 0 & 0 \\ 0 & \overline{P}_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_3 - (\sigma B_1 B_2' - D_1 D_2') \overline{P}_2 \\ 0 & A_2 - (\sigma B_2 B_2' - D_2 D_2') \overline{P}_2 \end{bmatrix}.$$

Hence, using Lemma 3.1 and the fact that $\tilde{P}_2$ is a strong solution to (3.14), it follows that all eigenvalues of the matrix

$$A_{F_0} = [\sigma BB' - DD'] \begin{bmatrix} 0 & 0 \\ 0 & \overline{P}_2 \end{bmatrix}$$

will satisfy $\operatorname{Re}(\lambda) \leqslant 0$. Thus

$$\begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}_2 \end{bmatrix}$$

is in fact a strong solution to (3.9).

We now apply Theorem 2.1. Indeed, comparing Riccati equations (3.5 and (3.9), it follows from Theorem 2.1 that Riccati equation (3.9) will have at most one strong solution. However, we have previously established that $\overline{P}$ is a strong solution to (3.9). Thus

$$\overline{P} = \begin{bmatrix} 0 & 0 \\ 0 & \overline{P}_2 \end{bmatrix}.$$

Returning to (3.10), it now follows that P^∞ has the required form. □

Lemma 3.3. *Let $\{F_q\colon q \in [\tilde{q}, \infty)\}$ be defined as in Lemma 2.1 and let $q \geqslant \max\{\tilde{q}, \tilde{q}\}$ be given. Furthermore, let P_q^- be the corresponding minimal positive-definite symmetric solution to (3.1) and let P_q^+ be the corresponding solution to (3.4). Then*

$$P_q^- \leqslant P_q^+.$$

Proof. The proof is identical to the proof of Lemma 3.3 in [6]. □

Lemma 3.4. *Let $\{F_q \in \mathbb{R}^{m \times n}\colon q \in [\tilde{q}, \infty)\}$ be defined as in Lemma 3.2 and given any $q \geqslant \max\{\tilde{q}, \tilde{q}\}$, let P_q^- be defined as in Lemma 3.2. Then there exists a limiting matrix $\hat{P}$ such that $P_q^- \to \hat{P}$ as $q \to \infty$ and $\mathscr{S} \subset \ker \hat{P}$.*

Proof. If P_q^- is the minimal positive-definite solution to (3.1), it follows that $S_q = (P_q^-)^{-1}$ is the maximal solution to the Riccati equation

$$\begin{aligned} &AS + SA' - q^2 BB' + DD' \\ &\quad + S\left(C'C + Q/q^2\right) S = 0. \end{aligned}$$

Hence, using Theorem 2 of [8], it follows that S_q is monotone increasing as $q \to \infty$. Therefore P_q^- is monotone decreasing as $q \to \infty$. It now follows that there exists a matrix $\hat{P} \geqslant 0$ such that $P_q^- \to \hat{P}$ as $q \to \infty$.

In order to show that $\mathscr{S} \subset \ker \hat{P}$, we recall from Lemma 3.3 that for all

$$q \geqslant \max\{\tilde{q}, \tilde{q}\}, \quad 0 \leqslant P_q^- \leqslant P_q^+.$$

Furthermore, using part (iii) of Lemma 3.2, it follows that $0 \leqslant P_{k_i}^- \leqslant P_{k_i}^+$ for all $i = 1, 2, \ldots$. Taking the limit as $i \to \infty$, we conclude that $0 \leqslant \hat{P} \leqslant P^\infty$. Moreover, it follows from the form of P^∞ that $\mathscr{S} \subset \ker P^\infty$ and hence $\mathscr{S} \subset \ker \hat{P}$. □

Notation (see p. 447 of [12]). The *minimal controllable output injection subspace* $\mathscr{T} \subset \mathbb{R}^n$ is defined as follows:

$$\mathscr{T}_0 = \operatorname{Im} B;$$

$$\mathscr{T}_{i+1} = \operatorname{Im} B + A(\ker C \cap \mathscr{T}_i);$$

$$\mathscr{T} = \lim_{i \to \infty} \mathscr{T}_i.$$

Lemma 3.5 (see [6] for proof). $\mathscr{T} \subset \ker \hat{P}$.

Proof of Theorem 3.1. Let $q \geqslant \tilde{q}$ be given, it follows from Lemma 3.2 that Riccati equation (3.1) has a minimal positive-definite solution P_q^-. In order to show that $P_q^- \to 0$ as $q \to \infty$, it suffices to show the matrix $\hat{P}$ defined in Lemma 3.4 is zero. Indeed, it follows from Lemma 3.4 that $\mathscr{S} \subset \ker \hat{P}$. Furthermore, using Lemma 3.5, it follows that $\mathscr{S} + \mathscr{T} \subset \ker \hat{P}$. We now use the fact that the transfer function $G(s)$ is right invertible. It then follows from Theorem 3.24 of [13] that $\mathscr{S} + \mathscr{T} = \mathbb{R}^n$. Thus $\ker \hat{P} = \mathbb{R}^n$ and hence $\hat{P} = 0$. □

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