Performance Analysis and Coherent Guaranteed Cost Control for Uncertain Quantum Systems Using Small Gain and Popov Methods

Chengdi Xiang, Ian R. Petersen and Daoyi Dong

***Abstract*—This paper extends applications of the quantum small gain and Popov methods from existing results on robust stability to performance analysis results for a class of uncertain quantum systems. This class of systems involves a nominal linear quantum system and is subject to quadratic perturbations in the system Hamiltonian. Based on these two methods, coherent guaranteed cost controllers are designed for a given quantum system to achieve improved control performance. An illustrative example also shows that the quantum Popov approach can obtain less conservative results than the quantum small gain approach for the same uncertain quantum system.**

I. INTRODUCTION

Due to recent advances in quantum and nano-technology, there has been a considerable attention focusing on research in the area of quantum feedback control systems; e.g., [1]-[17]. In particular, robust control has been recognized as a critical issue in quantum control systems; e.g., [1], [2]. A majority of papers in the area of quantum feedback control only consider classical (non-quantum) controllers. In this case, analog and digital electronic devices may be involved and quantum measurements are required in the feedback loop. Due to the limitations imposed by quantum mechanics on the use of quantum measurement, recent research has considered the design of coherent quantum controllers to achieve improved performance. In this case, the controller itself is a quantum system; e.g., [1], [2] and [4]. In the linear case, the quantum system is often described by linear quantum stochastic differential equations (QSDEs) that require physical realizability conditions in terms of constraints on the system matrices to represent physically meaningful systems.

As opposed to using QSDEs, the papers [5], [6] have introduced a set of parameterizations (S, L, H) to represent a class of open quantum systems, where the matrix S, together with the vector L, describes the interface between the system and the field, and the operator H represents the energy of the system. The advantage of using a triple (S, L, H) is that this framework automatically represents a physically realizable quantum system. Therefore, in this paper, a coherent guaranteed cost controller is designed based on (S, L, H) and the physical realizability condition does not need to be considered.

The small gain theorem and the Popov approach are two of the most important methods for the analysis of robust stability in classical control. The paper [7] has applied the small gain method to obtain a robust stability result for uncertain quantum systems. The small gain method has also been extended to the robust stability analysis of quantum systems with different perturbations and applications (e.g., see [8], [9]). The paper [10] has introduced a quantum version of the Popov stability criterion in terms of a frequency domain condition [18], and has shown that this condition is less conservative than the stability result using the small gain theorem [7].

In this paper, we extend the quantum small gain method in [7] and the Popov type approach in [10] from robust stability analysis to robust performance analysis for a class of uncertain quantum systems with quadratic perturbation Hamiltonian. A coherent controller is designed using the small gain approach and the Popov approach for this class of systems, where a guaranteed bound on a cost function is derived in terms of linear matrix inequality (LMI) conditions. Although preliminary versions of the results in this paper have been presented in the conference papers [15] and [16], this paper presents complete proofs of the main results and also considers a different example from the one in [15]. Detailed descriptions can be found in the extended arXiv version of this paper [17].

This paper is organized as follows. In Section II, we define the general class of quantum systems under consideration. We then present a set of quadratic Hamiltonian perturbations in Section III. In Section IV, a small gain approach and a Popov type method are used to analyze the performance of the given system. Section V presents the results on coherent guaranteed cost controller design. An illustrative example is presented in Section VI and the conclusion is included in Section VII.

II. QUANTUM SYSTEMS

In this section, we describe the general class of quantum systems under consideration, which is defined by parameters (S, L, H). Here $H = H_1 + H_2$, H_1 is a known self-adjoint operator on the underlying Hilbert space referred to as the nominal Hamiltonian and H_2 is a self-adjoint operator on the underlying Hilbert space referred to as the perturbation Hamiltonian contained in a specified set of Hamiltonians $\mathcal{W}$; e.g., [5], [6]. The set $\mathcal{W}$ can correspond to a set of exosystems (see, [5]). The corresponding generator for this class of quantum systems is given by $\mathcal{G}(X) = -i[X, H] + \mathcal{L}(X)$, where $\mathcal{L}(X) = \frac{1}{2}L^\dagger[X, L] + \frac{1}{2}[L^\dagger, X]L$. Here, $[X, H] = XH - HX$ describes the commutator between two operators and the notation † refers to the adjoint transpose of a vector of operators. Based on a QSDE, the triple (S, L, H), together with the corresponding generators, defines the Heisenberg evolution $X(t)$ of an operator X [5]. The results presented in this paper will build on the following results from [10].

Lemma 1: [10] Consider an open quantum system defined by (S, L, H) and suppose there exist non-negative self-adjoint operators V and W on the underlying Hilbert space such that

C. Xiang, I. R. Petersen and D. Dong are with the School of Engineering and Information Technology, University of New South Wales, Canberra ACT 2600, Australia. {elyssaxiang, i.r.petersen, daoyidong}@gmail.com. This work was supported by the Australian Research Council (DP130101658, FL110100020).

$\mathcal{G}(V) + W \leq \lambda$, where λ is a real number. Then for any plant state, we have $\limsup_{T\to\infty} \frac{1}{T}\int_0^T \langle W(t)\rangle dt \leq \lambda$. Here $W(t)$ denotes the Heisenberg evolution of the operator W and $\langle\cdot\rangle$ denotes quantum expectation; e.g., see [5] and [10].

In this paper, the nominal system is considered to be a linear quantum system. We assume that H_1 is in the following form

$$H_1 = \frac{1}{2}\begin{bmatrix} a^\dagger & a^T \end{bmatrix} M \begin{bmatrix} a \\ a^\# \end{bmatrix} \tag{1}$$

where $M \in \mathbb{C}^{2n\times 2n}$ is a Hermitian matrix and has the following form with $M_1 = M_1^\dagger$, $M_2 = M_2^T$ and $M = \begin{bmatrix} M_1 & M_2 \\ M_2^\# & M_1^\# \end{bmatrix}$. Here a is a vector of annihilation operators on the underlying Hilbert space and $a^\#$ is the corresponding vector of creation operators. In the case of matrices, the notation † refers to the complex conjugate transpose of a matrix. In the case of vectors of operators, the notation $^\#$ refers to the vector of adjoint operators, and in the case of complex matrices, this notation refers to the complex conjugate matrix. The canonical commutation relations between annihilation and creation operators are described in the following way [2]

$$\begin{aligned} &\left[\begin{bmatrix} a \\ a^\# \end{bmatrix}, \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger \right] = \begin{bmatrix} a \\ a^\# \end{bmatrix}\begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger \\ &- \left(\begin{bmatrix} a \\ a^\# \end{bmatrix}^\# \begin{bmatrix} a \\ a^\# \end{bmatrix}^T \right)^T = J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}. \end{aligned}$$

The coupling vector L is assumed to be of the form

$$L = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix} = \tilde{N} \begin{bmatrix} a \\ a^\# \end{bmatrix} \tag{2}$$

where $N_1 \in \mathbb{C}^{m\times n}$ and $N_2 \in \mathbb{C}^{m\times n}$. We also write

$$\begin{bmatrix} L \\ L^\# \end{bmatrix} = N \begin{bmatrix} a \\ a^\# \end{bmatrix} = \begin{bmatrix} N_1 & N_2 \\ N_2^\# & N_1^\# \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix}.$$

When the nominal Hamiltonian H is a quadratic function of the creation and annihilation operators as shown in (1) and the coupling operator vector is a linear function of the creation and annihilation operators, the nominal system corresponds to a linear quantum system (see, [2]).

We consider self-adjoint "Lyapunov" operators V:

$$V = \begin{bmatrix} a^\dagger & a^T \end{bmatrix} P \begin{bmatrix} a \\ a^\# \end{bmatrix} \tag{3}$$

where $P \in \mathbb{C}^{2n\times 2n}$ is a positive definite Hermitian matrix of the form

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^\# & P_1^\# \end{bmatrix}. \tag{4}$$

We then consider a set of non-negative self-adjoint operators $\mathcal{P}$ defined as

$$\mathcal{P} = \left\{ \begin{array}{ll} V & \text{of the form (3) such that } P > 0 \text{ is a} \\ & \text{Hermitian matrix of the form (4)} \end{array} \right\}.$$

III. PERTURBATIONS OF THE HAMILTONIAN

This section defines the quadratic Hamiltonian perturbations (e.g., see [7]) for the quantum system under consideration. We first define two general sets of Hamiltonians in terms of a commutator decomposition, and then present two specific sets of quadratic Hamiltonian perturbations.

A. Commutator Decomposition

For the set of non-negative self-adjoint operators $\mathcal{P}$ and given real parameters $\gamma > 0, \delta \geq 0$, a particular set of perturbation Hamiltonians $\mathcal{W}_1$ is defined in terms of the commutator decomposition

$$[V, H_2] = [V, z^\dagger]w - w^\dagger[z, V] \tag{5}$$

for $V \in \mathcal{P}$, where w and z are given vectors of operators. $\mathcal{W}_1$ is then defined in terms of sector bound condition:

$$w^\dagger w \leq \frac{1}{\gamma^2} z^\dagger z + \delta. \tag{6}$$

We define $\mathcal{W}_1 = \left\{ \begin{array}{c} H_2 : \exists\ w, z \text{ such that (6) is satisfied} \\ \text{and (5) is satisfied } \forall\ V \in \mathcal{P} \end{array} \right\}$.

B. Alternative Commutator Decomposition

Given a set of non-negative operators $\mathcal{P}$, a self-adjoint operator H_1, a coupling operator L, real parameters $\beta \geq 0$ $\gamma > 0$, and a set of Popov scaling parameters $\Theta \subset [0, \infty)$, we define a set of perturbation Hamiltonians $\mathcal{W}_2$ in terms of the commutator decompositions [10]

$$\begin{aligned} [V - \theta H_1, H_2] &= [V - \theta H_1, z^\dagger]w - w^\dagger[z, V - \theta H_1], \\ \mathcal{L}(H_2) &\leq \mathcal{L}(z^\dagger)w + w^\dagger\mathcal{L}(z) + \beta[z, L]^\dagger[z, L] \end{aligned} \tag{7}$$

for $V \in \mathcal{P}$ and $\theta \in \Theta$, where w and z are given vectors of operators. Note that (5) and (7) correspond to a general quadratic perturbation of the Hamiltonian. This set $\mathcal{W}_2$ is then defined in terms of the sector bound condition

$$(w - \frac{1}{\gamma}z)^\dagger(w - \frac{1}{\gamma}z) \leq \frac{1}{\gamma^2} z^\dagger z. \tag{8}$$

We define $\mathcal{W}_2 = \left\{ \begin{array}{c} H_2 \geq 0 : \exists\ w, z \text{ such that (7) and (8)} \\ \text{are satisfied } \forall\ V \in \mathcal{P},\ \theta \in \Theta \end{array} \right\}$.

C. Quadratic Hamiltonian Perturbation

We consider a set of quadratic perturbation Hamiltonians that is in the form

$$H_2 = \frac{1}{2}\begin{bmatrix} \zeta^\dagger & \zeta^T \end{bmatrix} \Delta \begin{bmatrix} \zeta \\ \zeta^\# \end{bmatrix} \tag{9}$$

where $\zeta = E_1 a + E_2 a^\#$ and $\Delta \in \mathbb{C}^{2m\times 2m}$ is a Hermitian matrix of the form

$$\Delta = \begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_2^\# & \Delta_1^\# \end{bmatrix} \tag{10}$$

with $\Delta_1 = \Delta_1^\dagger$ and $\Delta_2 = \Delta_2^T$.

Since the nominal system is linear, we use the relationship:

$$z = \begin{bmatrix} \zeta \\ \zeta^\# \end{bmatrix} = \begin{bmatrix} E_1 & E_2 \\ E_2^\# & E_1^\# \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix} = E \begin{bmatrix} a \\ a^\# \end{bmatrix}. \tag{11}$$

Then we have $H_2 = \frac{1}{2}\begin{bmatrix} a^\dagger & a^T \end{bmatrix} E^\dagger \Delta E \begin{bmatrix} a \\ a^\# \end{bmatrix}$.

When the matrix Δ is subject to the norm bound

$$\|\Delta\| \leq \frac{2}{\gamma}, \tag{12}$$

where $\|.\|$ refers to the matrix induced norm, we define

$$\mathcal{W}_3 = \left\{ \begin{array}{c} H_2 \text{ of the form (9) and (10) such that} \\ \text{condition (12) is satisfied} \end{array} \right\}.$$

In [7], it has been proven that for any set of self-adjoint operators $\mathcal{P}$,

$$\mathcal{W}_3 \subset \mathcal{W}_1. \tag{13}$$

When the matrix Δ is subject to the bounds

$$0 \leq \Delta \leq \frac{4}{\gamma} I, \tag{14}$$

we define $\mathcal{W}_4 = \left\{ \begin{array}{c} H_2 \text{ of the form (9) and (10) such} \\ \text{that condition (14) is satisfied} \end{array} \right\}$. In [10], it has been proven that if $[z, L]$ is a constant vector, then for any set of self-adjoint operators $\mathcal{P}$,

$$\mathcal{W}_4 \subset \mathcal{W}_2. \tag{15}$$

IV. PERFORMANCE ANALYSIS

In this section, we provide several results on performance analysis for quantum systems subject to a quadratic perturbation Hamiltonian. Also, the associated cost function is defined in the following way:

$$\mathcal{J} = \limsup_{T\to\infty} \frac{1}{T} \int_0^T \langle \left[\begin{array}{cc} a^\dagger & a^T \end{array} \right] R \left[\begin{array}{c} a \\ a^\# \end{array} \right] \rangle dt \tag{16}$$

where $R > 0$. We denote

$$W = \left[\begin{array}{cc} a^\dagger & a^T \end{array} \right] R \left[\begin{array}{c} a \\ a^\# \end{array} \right]. \tag{17}$$

Now we present two theorems (Theorem 1 and Theorem 2) which can be used to analyze the performance of the given quantum systems using a quantum small gain method and a Popov type approach, respectively.

A. Performance Analysis Using the Small Gain Approach

Theorem 1: Consider an uncertain quantum system (S, L, H), where $H = H_1 + H_2$, H_1 is in the form of (1), L is of the form (2) and $H_2 \in \mathcal{W}_3$. If $F = -iJM - \frac{1}{2}JN^\dagger JN$ is Hurwitz, and

$$\left[\begin{array}{cc} F^\dagger P + PF + \frac{E^\dagger E}{\gamma^2\tau^2} + R & 2PJE^\dagger \\ 2EJP & -I/\tau^2 \end{array} \right] < 0 \tag{18}$$

has a solution $P > 0$ in the form of (4) and $\tau > 0$, then

$$\mathcal{J} = \limsup_{T\to\infty} \frac{1}{T} \int_0^T \langle W(t) \rangle dt \leq \tilde{\lambda} + \frac{\delta}{\tau^2} \tag{19}$$

where $W(t)$ is in the form of (17) and

$$\tilde{\lambda} = \mathrm{Tr}(PJN^\dagger \left[\begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right] NJ). \tag{20}$$

In order to prove this theorem, we need the following lemma.

Lemma 2: Consider an open quantum system (S, L, H) where $H = H_1 + H_2$ and $H_2 \in \mathcal{W}_1$, and the set of non-negative self-adjoint operators $\mathcal{P}$. If there exist a $V \in \mathcal{P}$ and real constants $\tilde{\lambda} \geq 0$, $\tau > 0$ such that

$$-i[V, H_1] + \mathcal{L}(V) + \tau^2 [V, z^\dagger][z, V] + \frac{1}{\gamma^2\tau^2} z^\dagger z + W \leq \tilde{\lambda}, \tag{21}$$

then

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \langle W(t) \rangle dt \leq \tilde{\lambda} + \frac{\delta}{\tau^2}, \quad \forall t \geq 0.$$

Proof: Since $V \in \mathcal{P}$ and $H_2 \in \mathcal{W}_1$, we have

$$\mathcal{G}(V) = -i[V, H_1] + \mathcal{L}(V) - i[V, z^\dagger]w + iw^\dagger [z, V], \tag{22}$$

$$\begin{aligned} 0 &\leq (\tau[V, z^\dagger] - \frac{i}{\tau} w^\dagger)(\tau[V, z^\dagger] - \frac{i}{\tau} w^\dagger)^\dagger \\ &= \tau^2 [V, z^\dagger][z, V] + i[V, z^\dagger]w - iw^\dagger [z, V] + \frac{w^\dagger w}{\tau^2}. \end{aligned} \tag{23}$$

Substituting (22) into (23) and using the sector bound condition (6), the following inequality is obtained:

$$\mathcal{G}(V) \leq -i[V, H_1] + \mathcal{L}(V) + \tau^2 [V, z^\dagger][z, V] + \frac{1}{\gamma^2\tau^2} z^\dagger z + \frac{\delta}{\tau^2}.$$

Hence, $\mathcal{G}(V) + W \leq \tilde{\lambda} + \frac{\delta}{\tau^2}$. Consequently, the conclusion in the lemma follows from Lemma 1. □

Proof of Theorem 1: Using the Schur complement [19], the inequality (18) is equivalent to

$$F^\dagger P + PF + 4\tau^2 PJE^\dagger EJP + \frac{E^\dagger E}{\gamma^2\tau^2} + R < 0. \tag{24}$$

If the Riccati inequality (24) has a solution $P > 0$ of the form (4) and $\tau > 0$, according to Lemma 4.2 of [7], we have

$$\begin{aligned} &-i[V, H_1] + \mathcal{L}(V) + \tau^2 [V, z^\dagger][z, V] + \frac{1}{\gamma^2\tau^2} z^\dagger z + W = \\ &\left[\begin{array}{c} a \\ a^\# \end{array} \right]^\dagger \left(\begin{array}{c} F^\dagger P + PF + 4\tau^2 PJE^\dagger EJP \\ + \frac{E^\dagger E}{\gamma^2\tau^2} + R \end{array} \right) \left[\begin{array}{c} a \\ a^\# \end{array} \right] \\ &+ \mathrm{Tr}(PJN^\dagger \left[\begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right] NJ). \end{aligned}$$

Therefore, it follows from (18) that condition (21) is satisfied with $\tilde{\lambda} = \mathrm{Tr}(PJN^\dagger \left[\begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right] NJ) \geq 0$.

Then, according to the relationship (13) and Lemma 2, we have results (19) and (20). □

B. Performance Analysis Using the Popov Approach

Theorem 2: Consider an uncertain quantum system (S, L, H), where $H = H_1 + H_2$, H_1 is in the form of (1), L is of the form (2) and $H_2 \in \mathcal{W}_4$. If $F = -iJM - \frac{1}{2}JN^\dagger JN$ is Hurwitz, and

$$\left[\begin{array}{cc} PF + F^\dagger P + R & -2iPJE^\dagger + E^\dagger + \theta F^\dagger E^\dagger \\ 2iEJP + E + \theta EF & -\gamma I \end{array} \right] < 0 \tag{25}$$

has a solution $P > 0$ in the form of (4) for some $\theta \geq 0$, then

$$\mathcal{J} = \limsup_{T\to\infty} \frac{1}{T} \int_0^T \langle W(t) \rangle dt \leq \lambda \tag{26}$$

where $W(t)$ is in the form of (17), $\Sigma = \left[\begin{array}{cc} 0 & I \\ I & 0 \end{array} \right]$ and

$$\lambda = \mathrm{Tr}(PJN^\dagger \left[\begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right] NJ) + \frac{4\theta}{\gamma} \tilde{N}^\# \Sigma JE^\dagger EJ\Sigma \tilde{N}^T. \tag{27}$$

Proof: Using the Schur complement, (25) is equivalent to

$$\begin{aligned} &PF + F^\dagger P + \frac{1}{\gamma}(-2iPJE^\dagger + E^\dagger + \theta F^\dagger E^\dagger) \\ &\times (2iEJP + E + \theta EF) + R < 0. \end{aligned} \tag{28}$$

According to Lemma 4.2 of [7], Lemma 5 and Lemma 6 of [10], we have

$$
\begin{aligned}
&-i[V,H_1]+\mathcal{L}(V)+\frac{1}{\gamma}(i[z,V-\theta H_1]+\theta\mathcal{L}(z)+z)^\dagger \\
&\times(i[z,V-\theta H_1]+\theta\mathcal{L}(z)+z)+\frac{4\theta}{\gamma}[z,L]^\dagger[z,L]+W \\
&=\left[\begin{array}{c} a \\ a^\# \end{array}\right]^\dagger \left(\begin{array}{l} PF+F^\dagger P \\ +\frac{1}{\gamma}(-2iPJE^\dagger+E^\dagger+\theta F^\dagger E^\dagger) \\ \times(2iEJP+E+\theta EF)+R \end{array}\right)\left[\begin{array}{c} a \\ a^\# \end{array}\right] \\
&\quad+\mathrm{Tr}(PJN^\dagger\left[\begin{array}{cc} I & 0 \\ 0 & 0 \end{array}\right]NJ)+\frac{4\theta}{\gamma}\tilde{N}^\#\Sigma JE^\dagger EJ\Sigma\tilde{N}^T.
\end{aligned} \tag{29}
$$

Results (26) and (27) are obtained by using (29), the relationship (15) and Theorem 1 in [10]. □

V. GUARANTEED COST CONTROLLER DESIGN

In this section, we design a coherent guaranteed cost controller for the uncertain quantum system subject to a quadratic perturbation Hamiltonian to make the control system not only stable but also to achieve an adequate level of performance. The coherent controller is realized by adding a controller Hamiltonian H_3. H_3 is assumed to be in the form

$$
H_3=\frac{1}{2}\left[\begin{array}{cc} a^\dagger & a^T \end{array}\right]K\left[\begin{array}{c} a \\ a^\# \end{array}\right] \tag{30}
$$

where $K\in\mathbb{C}^{2n\times 2n}$ is a Hermitian matrix of the form $K=\left[\begin{array}{cc} K_1 & K_2 \\ K_2^\# & K_1^\# \end{array}\right]$ and $K_1=K_1^\dagger$, $K_2=K_2^T$. Associated with this system is the cost function $\mathcal{J}$

$$
\mathcal{J}=\limsup_{T\to\infty}\frac{1}{T}\int_0^\infty\langle\left[\begin{array}{c} a \\ a^\# \end{array}\right]^\dagger(R+\rho K^2)\left[\begin{array}{c} a \\ a^\# \end{array}\right]\rangle dt \tag{31}
$$

where $\rho\in(0,\infty)$ is a weighting factor. We let

$$
W=\left[\begin{array}{c} a \\ a^\# \end{array}\right]^\dagger(R+\rho K^2)\left[\begin{array}{c} a \\ a^\# \end{array}\right]. \tag{32}
$$

The following theorems (Theorem 3 and Theorem 4) present our main results on coherent guaranteed cost controller design for the given quantum system using a quantum small gain method and a Popov type approach, respectively.

A. Controller Design Using the Small Gain Approach

Theorem 3: Consider an uncertain quantum system (S,L,H), where $H=H_1+H_2+H_3$, H_1 is in the form of (1), L is of the form (2), $H_2\in\mathcal{W}_3$ and the controller Hamiltonian H_3 is in the form of (30). With $Q=P^{-1}$, $Y=KQ$ and $F=-iJM-\frac{1}{2}JN^\dagger JN$, if there exist a matrix $Q=q*I$ (q is a constant scalar and I is the identity matrix), a Hermitian matrix Y and a constant $\tau>0$, such that

$$
\left[\begin{array}{cccc} A+4\tau^2JE^\dagger EJ & Y & qR^{\frac{1}{2}} & qE^\dagger \\ Y & -I/\rho & 0 & 0 \\ qR^{\frac{1}{2}} & 0 & -I & 0 \\ qE & 0 & 0 & -\gamma^2\tau^2 I \end{array}\right]<0 \tag{33}
$$

where $A=qF^\dagger+Fq+iYJ-iJY$, then the associated cost function satisfies the bound

$$
\mathcal{J}=\limsup_{T\to\infty}\frac{1}{T}\int_0^T\langle W(t)\rangle dt\le\tilde{\lambda}+\frac{\delta}{\tau^2} \tag{34}
$$

where $W(t)$ is in the form of (32) and

$$
\tilde{\lambda}=\mathrm{Tr}(PJN^\dagger\left[\begin{array}{cc} I & 0 \\ 0 & 0 \end{array}\right]NJ). \tag{35}
$$

Proof: Suppose the conditions of the theorem are satisfied. Using the Schur complement, (33) is equivalent to

$$
\left[\begin{array}{ccc} A+4\tau^2JE^\dagger EJ+\rho YY & qR^{\frac{1}{2}} & qE^\dagger \\ qR^{\frac{1}{2}} & -I & 0 \\ qE & 0 & -\gamma^2\tau^2 I \end{array}\right]<0. \tag{36}
$$

Applying the Schur complement twice, (36) is equivalent to

$$
\begin{aligned}
&qF^\dagger+Fq+iYJ-iJY+4\tau^2JE^\dagger EJ \\
&+\rho YY+q^2(\frac{E^\dagger E}{\gamma^2\tau^2}+R)<0.
\end{aligned} \tag{37}
$$

Substituting $Y=Kq=qK^\dagger$ into (37), we obtain

$$
\begin{aligned}
&q(F-iJK)^\dagger+(F-iJK)q+4\tau^2JE^\dagger EJ \\
&+q^2(\frac{E^\dagger E}{\gamma^2\tau^2}+R+\rho K^2)<0.
\end{aligned} \tag{38}
$$

Since $P=Q^{-1}$, premultiplying and postmultiplying this inequality by the matrix P, we have

$$
\begin{aligned}
&(F-iJK)^\dagger P+P(F-iJK)+4\tau^2PJE^\dagger EJP \\
&+\frac{E^\dagger E}{\gamma^2\tau^2}+R+\rho K^2<0.
\end{aligned} \tag{39}
$$

It follows straightforwardly from (39) that $F-iJK$ is Hurwitz. We also know that

$$
\begin{aligned}
&-i[V,H_1+H_3]+\mathcal{L}(V)+\tau^2[V,z^\dagger][z,V]+\frac{1}{\gamma^2\tau^2}z^\dagger z+W \\
&=\left[\begin{array}{c} a \\ a^\# \end{array}\right]^\dagger\left(\begin{array}{l} (F-iJK)^\dagger P+P(F-iJK) \\ +4\tau^2PJE^\dagger EJP+E^\dagger E/(\gamma^2\tau^2) \\ +R+\rho K^2 \end{array}\right)\left[\begin{array}{c} a \\ a^\# \end{array}\right] \\
&\quad+\mathrm{Tr}(PJN^\dagger\left[\begin{array}{cc} I & 0 \\ 0 & 0 \end{array}\right]NJ).
\end{aligned} \tag{40}
$$

According to the relationship (13) and Lemma 2, we have results (34) and (35). □

Remark 1: In order to design a coherent controller which minimizes the cost bound (34) in Theorem 3, we need to formulate an inequality

$$
\mathrm{Tr}(PJN^\dagger\left[\begin{array}{cc} I & 0 \\ 0 & 0 \end{array}\right]NJ)+\frac{\delta}{\tau^2}\le\xi. \tag{41}
$$

We know that $P=Q^{-1}=q^{-1}I$ and apply the Schur complement twice. It is clear that (41) is equivalent to

$$
\left[\begin{array}{ccc} -\xi & \delta^{\frac{1}{2}} & C^{\frac{1}{2}} \\ \delta^{\frac{1}{2}} & -\tau^2 & 0 \\ C^{\frac{1}{2}} & 0 & -q \end{array}\right]\le 0. \tag{42}
$$

where $C=\mathrm{Tr}(JN^\dagger\left[\begin{array}{cc} I & 0 \\ 0 & 0 \end{array}\right]NJ)$. Hence, Theorem 3 can be formulated as the minimization of ξ subject to (42) and (33), which is a standard LMI problem.

B. Controller Design Using the Popov Approach

Theorem 4: Consider an uncertain quantum system (S, L, H), where $H = H_1 + H_2 + H_3$, H_1 is in the form of (1), L is of the form (2), $H_2 \in \mathcal{W}_4$, the controller Hamiltonian H_3 is in the form of (30). With $Q = P^{-1}$, $Y = KQ$ and $F = -iJM - \frac{1}{2}JN^\dagger JN$, if there exist a matrix $Q = q * I$ (q is a constant scalar and I is the identity matrix), a Hermitian matrix Y and a constant $\theta > 0$, such that

$$\begin{bmatrix} A & B^\dagger & Y & qR^{\frac{1}{2}} \\ B & -\gamma I & 0 & 0 \\ Y & 0 & -I/\rho & 0 \\ qR^{\frac{1}{2}} & 0 & 0 & -I \end{bmatrix} < 0 \tag{43}$$

where $A = Fq + qF^\dagger - iJY + iYJ$ and $B = 2iEJ + Eq + \theta EFq - i\theta EJY$, then the associated cost function satisfies the bound

$$\mathcal{J} = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \langle W(t)\rangle dt \le \lambda \tag{44}$$

where $W(t)$ is in the form of (32) and

$$\lambda = \mathrm{Tr}(PJN^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} NJ) + \frac{4\theta}{\gamma}\tilde{N}^\# \Sigma JE^\dagger EJ\Sigma\tilde{N}^T. \tag{45}$$

Proof: Suppose the conditions of the theorem are satisfied. Using the Schur complement twice, (43) is equivalent to

$$\begin{aligned} &Fq + qF^\dagger - iJY + iYJ + q^2R + \rho YY + \\ &\frac{1}{\gamma}(-2iJE^\dagger + qE^\dagger + \theta qF^\dagger E^\dagger + i\theta YJE^\dagger) \\ &\times (2iEJ + Eq + \theta EFq - i\theta EJY) < 0. \end{aligned} \tag{46}$$

By substituting $Y = Kq = qK^\dagger$ into (46) and premultiplying and postmultiplying this inequality by the matrix P, we have

$$\begin{aligned} &P(F - iJK) + (F - iJK)^\dagger P \\ &+ \frac{1}{\gamma}(-2iPJE^\dagger + E^\dagger + \theta(F - iJK)^\dagger E^\dagger) \\ &\times (2iEJP + E + \theta E(F - iJK)) + R + \rho K^2 < 0. \end{aligned} \tag{47}$$

It follows straightforwardly from (47) that $F - iJK$ is Hurwitz. We also know that

$$\begin{aligned} &-i[V, H_1 + H_3] + \mathcal{L}(V) + \frac{4\theta}{\gamma}[z, L]^\dagger [z, L] + W \\ &+ \frac{1}{\gamma}(i[z, V - \theta(H_1 + H_3)] + \theta\mathcal{L}(z) + z)^\dagger \\ &\times (i[z, V - \theta(H_1 + H_3)] + \theta\mathcal{L}(z) + z) \\ &= \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger \tilde{M} \begin{bmatrix} a \\ a^\# \end{bmatrix} \\ &\quad + \mathrm{Tr}(PJN^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} NJ) + \frac{4\theta}{\gamma}\tilde{N}^\# \Sigma JE^\dagger EJ\Sigma\tilde{N}^T \end{aligned}$$

where $\tilde{M} = P(F - iJK) + (F - iJK)^\dagger P + R + \rho K^2 + \frac{1}{\gamma}(-2iPJE^\dagger + E^\dagger + \theta(F - iJK)^\dagger E^\dagger) \times (2iEJP + E + \theta E(F - iJK))$.

It follows from the relationship (15) and Theorem 1 in [10] that results (44) and (45) are satisfied. □

Remark 2: For each fixed value of θ, the problem is an LMI problem. Then, we can iterate on $\theta \in [0, \infty)$ and choose the value which minimizes the cost bound (45) in Theorem 4.

VI. ILLUSTRATIVE EXAMPLE

In order to illustrate our methods and compare their performance, we consider the same quantum system in [16]. The system corresponds to a degenerate parametric amplifier and its (S, L, H) description has the following form

$$H = \frac{1}{2}i((a^\dagger)^2 - a^2), S = I, L = \sqrt{\kappa}a.$$

We let the perturbation Hamiltonian be

$$H_2 = \frac{1}{2}\begin{bmatrix} a^\dagger & a^T \end{bmatrix}\begin{bmatrix} 1 & 0.5i \\ -0.5i & 1 \end{bmatrix}\begin{bmatrix} a \\ a^\# \end{bmatrix}$$

and the nominal Hamiltonian be

$$H_1 = \frac{1}{2}\begin{bmatrix} a^\dagger & a^T \end{bmatrix}\begin{bmatrix} -1 & 0.5i \\ -0.5i & -1 \end{bmatrix}\begin{bmatrix} a \\ a^\# \end{bmatrix}$$

so that $H_1 + H_2 = H$. The corresponding parameters considered in Theorems 1-4 are as follows:

$$M = \begin{bmatrix} -1 & 0.5i \\ -0.5i & -1 \end{bmatrix}, N = \begin{bmatrix} \sqrt{\kappa} & 0 \\ 0 & \sqrt{\kappa} \end{bmatrix}, E = I$$

$$F = \begin{bmatrix} -\frac{\kappa}{2} + i & 0.5 \\ 0.5 & -\frac{\kappa}{2} - i \end{bmatrix}, \Delta = \begin{bmatrix} 1 & 0.5i \\ -0.5i & 1 \end{bmatrix}.$$

To illustrate Theorem 1 and Theorem 3, we consider $H_2 \in \mathcal{W}_3$. Hence, $\gamma = 1$ is chosen to satisfy (12). The performance using the small gain approach for the uncertain quantum system is shown in Fig. 1(a). In Fig. 1(a), the dashed line represents the cost bound for the linear quantum system considered in Theorem 1 as a function of the parameter κ. The solid line shows the system performance with the coherent controller designed in Theorem 3. Compared to the performance without a controller, the coherent controller can guarantee that the system is stable for a larger range of the damping parameter κ and gives the system improved performance.

Now we illustrate one approach to realizing the desired controller. For instance, when $\kappa = 4.5$, by using the controller design method in Theorem 3, we have the desired controller Hamiltonian as

$$H_3 = \frac{1}{2}\begin{bmatrix} a^\dagger & a^T \end{bmatrix}\begin{bmatrix} 0 & -0.5i \\ 0.5i & 0 \end{bmatrix}\begin{bmatrix} a \\ a^\# \end{bmatrix}. \tag{48}$$

This controller Hamiltonian can be realized by connecting the degenerate parametric amplifier with a static squeezer as shown in Fig. 2. This static squeezer is a static Bogoliubov component which corresponds to the Bogoliubov transformation [20], [21]. Also, we have the following definition:

Definition 1: (see [20], [21]) A static Bogoliubov component is a component that implements the Bogoliubov transformation: $\begin{bmatrix} dy(t) \\ dy^\#(t) \end{bmatrix} = \mathbf{B}\begin{bmatrix} du(t) \\ du^\#(t) \end{bmatrix}$, where $\mathbf{B} = \begin{bmatrix} B_1 & B_2 \\ B_2^\# & B_1^\# \end{bmatrix}$, $\mathbf{B}^\dagger J\mathbf{B} = J$.

To realize H_3 in (48), we let the matrix $\mathbf{B} = \begin{bmatrix} \frac{5}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{5}{4} \end{bmatrix}$ which satisfies the Bogoliubov condition $\mathbf{B}^\dagger J\mathbf{B} = J$, and $\tilde{\kappa} = \frac{1}{3}$. Then, it is straightforward to verify [17] that the overall Hamiltonian of the closed loop system is $H = H_1 + H_2 +$

H_3. That is the scheme in Fig. 2 which realizes the designed coherent controller.

Note that the interconnection shown in Fig. 2 corresponds to an algebraic loop which is not defined if $\mathbf{B} = I$. However, we will not choose $\mathbf{B}$ equal to this value. Also, in practice, a static squeezer is in fact an approximation to a dynamic squeezer and so the algebraic loop would not occur in practice. In addition, for each value of $\mathbf{B} \neq I$ satisfying Definition 1, it is straightforward to show that the QSDEs components to the interconnection shown in Fig. 2 result from an (S, L, H) description and thus correspond to a valid quantum system.

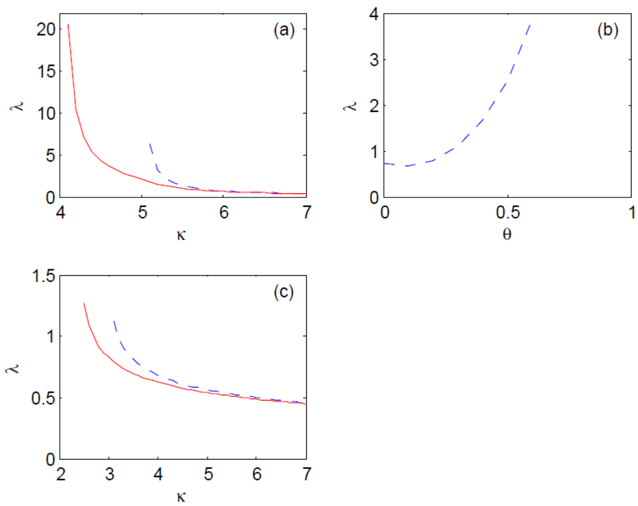

Fig. 1. (a) Guaranteed cost bounds for an uncertain quantum system with a controller (solid line) and without a controller (dashed line) using the small gain approach. (b) Performance versus θ. (c) Guaranteed cost bounds for the uncertain quantum system with a controller (solid line) and without a controller (dashed line) using the Popov approach.

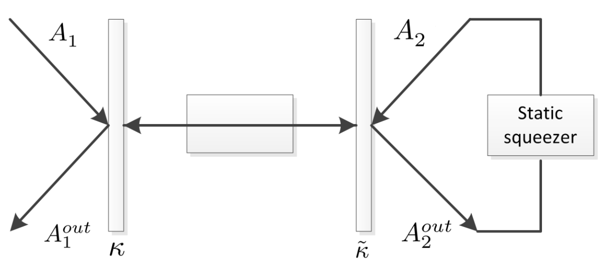

Fig. 2. Degenerate parametric amplifier coupled to a static squeezer.

To illustrate Theorem 2 and Theorem 4, we consider $H_2 \in \mathcal{W}_4$. Hence, $\gamma = 2$ is chosen to satisfy (14). The results using the Popov approach are shown in Fig. 1(b) and Fig. 1(c). Fig. 1(b) demonstrates how to choose the value of θ. We consider the same example as above with $\kappa = 3.8$ and iterate on $\theta \in [0, 1]$. Fig. 1(b) shows the cost bound for this quantum system obtained in Theorem 4 as a function of the parameter θ. It is clear that the minimal cost bound is achieved when $\theta = 0.1$. Therefore, we choose $\theta = 0.1$ for $\kappa = 3.8$ and use a similar method to choose θ for other values of κ.

In Fig. 1(c), the dashed line shows the performance for the given system considered in Theorem 2 and the solid line describes the cost bound for the linear quantum system with the coherent controller considered in Theorem 4. As can be seen in Fig. 1(c), the system with a controller has better performance than the case without a controller.

Also, we can observe that the method in Theorem 3 can only make the quantum system stable for $\kappa > 4$ in the example. Therefore, compared with the results in Fig. 1(a), the Popov method obtains a lower cost bound and a larger range of robust stability as shown in Fig. 1(c). This is as expected, since the Popov approach allows for a more general class of Lyapunov functions than the small gain approach.

VII. CONCLUSION

In this paper, the small gain method and the Popov approach, respectively, are used to analyze the performance of an uncertain linear quantum system subject to a quadratic perturbation in the system Hamiltonian. Then, we add a coherent controller to make the given system not only stable but also to achieve improved performance. An example is presented to demonstrate the system performance.

REFERENCES

[1] M. R. James, H. I. Nurdin, and I. R. Petersen, "H^∞ control of linear quantum stochastic systems," *IEEE Transactions on Automatic Control*, vol. 53, no. 8, pp. 1787-1803, 2008.

[2] I. R. Petersen, "Quantum linear systems theory," *in Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems*, Budapest, Hungary, 2010.

[3] D. Dong and I. R. Petersen, "Sliding mode control of two-level quantum systems," *Automatica*, vol. 48, pp. 725-735, 2012.

[4] M. Yanagisawa and H. Kimura, "Transfer function approach to quantum control-part I: Dynamics of quantum feedback systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 12, pp. 2107-2120, 2003.

[5] M. James and J. Gough, "Quantum dissipative systems and feedback control design by interconnection," *IEEE Transactions on Automatic Control*, vol. 55, no. 8, pp. 1806-1820, 2010.

[6] J. Gough and M. R. James, "The series product and its application to quantum feedforward and feedback networks," *IEEE Transactions on Automatic Control*, vol. 54, no. 11, pp. 2530-2544, 2009.

[7] I. R. Petersen, V. Ugrinovskii, and M. R. James, "Robust stability of uncertain linear quantum systems," *Philosophical Transactions of the Royal Society A*, vol. 370, no. 1979, pp. 5354-5363, 2012.

[8] I. R. Petersen, V. Ugrinovskii, and M. R. James, "Robust stability of quantum systems with a nonlinear coupling operator," *in Proceedings of the 51st IEEE Conference on Decision and Control,* Maui, 2012.

[9] I. R. Petersen, "Quantum robust stability of a small Josephson junction in a resonant cavity," *in 2012 IEEE Multi-conference on Systems and Control,* Dubrovnik, Croatia, 2012.

[10] M. R. James, I. R. Petersen, and V. Ugrinovskii, "A Popov stability condition for uncertain linear quantum systems," *in Proceedings of the 2013 American Control Conference*, Washington, DC, USA, 2013.

[11] J. E. Gough, M. R. James, and H. I. Nurdin, "Squeezing components in linear quantum feedback networks," *Physical Review A*, vol. 81, p. 023804, 2010.

[12] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control*, Cambridge, U.K.: Cambridge University Press, 2010.

[13] D. Dong and I. R. Petersen, "Quantum control theory and applications: a survey," *IET Control Theory & Applications*, vol. 4, pp. 2651-2671, 2010.

[14] H. I. Nurdin, M. R. James, and I. R. Petersen, "Coherent quantum LQG control," *Automatica*, vol. 45, no. 8, pp. 1837-1846, 2009.

[15] C. Xiang, I. R. Petersen and D. Dong, "Performance analysis and coherent guaranteed cost control for uncertain quantum systems," *in the Proceedings of the 2014 European Control Conference,* Strasbourg, France, 2014.

[16] C. Xiang, I. R. Petersen and D. Dong, "A Popov approach to performance analysis and coherent guaranteed cost control for uncertain quantum systems," *in the Proceedings of the 2014 Australian Control Conference,* Canberra, Australia, 2014.

[17] C. Xiang, I. R. Petersen and D. Dong, "Performance analysis and coherent guaranteed cost control for uncertain quantum systems using small gain and Popov methods," arXiv: 1508.06377v2.

[18] H. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ, USA: Prentice-Hall, 2002.

[19] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*. Philadelphia, PA: SIAM, 1994.

[20] J. E. Gough, M. R. James, and H. I. Nurdin, "Squeezing components in linear quantum feedback networks," *Physical Review A*, vol. 81, p. 023804, 2010.

[21] S. L. Vuglar and I. R. Petersen, "Singular perturbation approximations for general linear quantum systems," *in Proceedings of the 2012 Australian Control Conference*, Sydney, Australia, 2012.