

Switching Reconstruction of Digraphs

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Abstract: Switching about a vertex in a digraph means to reverse the direction of every edge incident with that vertex. Bondy and Mercier introduced the problem of whether a digraph can be reconstructed up to isomorphism from the multiset of isomorphism types of digraphs obtained by switching about each vertex. Since the largest known nonreconstructible oriented graphs have eight vertices, it is natural to ask whether there are any larger nonreconstructible graphs. In this article, we continue the investigation of this question. We find that there are exactly 44 nonreconstructible oriented graphs whose underlying undirected graphs have maximum degree at most 2. We also determine the full set of switching-stable oriented graphs, which are those graphs for which all switchings return a digraph isomorphic to the original. © 2013 Wiley Periodicals, Inc. *J. Graph Theory* 00: 1–18, 2013

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1. INTRODUCTION

In combinatorics, a reconstruction problem asks whether a combinatorial object can be reconstructed from its “deck,” where the deck is a multiset of objects that are slight modifications of the original object. For example, in Kelly and Ulam’s reconstruction problem [5, 10] the deck of an n -vertex graph G is the multiset of n graphs that are each obtained by deleting one vertex of G . Kelly and Ulam conjectured that every graph on at least three vertices is reconstructible from its vertex-deleted deck. Another way of stating this is that nonisomorphic graphs on at least three vertices have different decks. Following their work, numerous other types of decks and their associated reconstruction problems have been considered. Among them is Stanley’s switching reconstruction problem [9], where the elements of the deck of a graph are obtained by choosing a vertex v and then replacing all edges incident with v with nonedges and vice versa. As with the previous reconstruction problem, it is conjectured that there are only finitely many graphs that cannot be reconstructed from their deck. For an overview of the various reconstruction variants we refer the reader to the existing surveys [1, 2, 8].

Recently, Bondy and Mercier [3, 7] introduced further switching reconstruction problems. In their context, the switching deck of a digraph contains all digraphs that are obtained by reversing the orientation of all edges incident with a specified vertex. In their article, Bondy and Mercier observe that a subgraph-counting result of Ellingham and Royle [4] applies to switching of connected digraphs. They provide an alternative proof that also applies to disconnected graphs and thereby show that for a digraph on n vertices the number of induced copies of any digraph on k vertices is reconstructible whenever n is not of the form $4j$ for $j \leq \lfloor k/2 \rfloor$.

Bondy and Mercier also list all decks of the oriented graphs on four vertices and, using this, derive many sets of switching nonreconstructible digraphs on four and eight vertices. Our computer search has shown that there are, in fact, 5,559 nonreconstructible oriented graphs on eight vertices. They ask whether there exist nonreconstructible oriented graphs on more than eight vertices, which, by their theorem, must have at least 12 vertices, since the number of vertices must be divisible by four. They also ask which oriented graphs have the property that all switchings are isomorphic. As examples of such digraphs they mention vertex transitive digraphs and switching-stable digraphs, i.e. digraphs for which every switching is isomorphic to the digraph itself.

A. Our Results

By an *oriented graph* we mean a digraph without loops, parallel edges, or 2-cycles. Ignoring the orientations of the edges of an oriented graph gives its *underlying undirected graph*. When we refer to the *degree* of a vertex of an oriented graph, we always mean the degree in the underlying undirected graph; otherwise we will write “in-degree” or “out-degree.”

In this article, we classify all switching-stable oriented graphs. We show that an oriented graph is switching-stable if and only if each of its components has one, two, or four vertices and each component on four vertices is isomorphic to the oriented cycle that has a unique directed path of length 3.

We also determine all nonreconstructible oriented graphs with maximum degree at most 2. There are exactly 44. They yield 29 sets $\{G, H\}$ of non-isomorphic oriented graphs of maximum degree at most 2 such that G and H have the same deck. To

determine all these oriented graphs, we prove with combinatorial arguments that there are none on more than 30 vertices and then with an efficient enumeration algorithm using the computer determine all nonreconstructible oriented graphs with at most 30 vertices.

B. Structure of the Paper

We first formally define the digraph switching reconstruction problem and the related notions (Section 2). We then classify all switching-stable oriented graphs (Section 3) and devise some properties of switching-stable sets (Section 4). To classify all nonreconstructible oriented graphs of maximum degree at most 2 we determine all paths (Section 5) and all cycles (Section 6) that are not reconstructible from their t -decks. An analysis of disconnected nonreconstructible oriented graphs (Section 7) then allows us to assemble the statements and determine all nonreconstructible oriented graphs of maximum degree at most 2 (Section 8).

2. PRELIMINARIES

In this article, we consider labeled graphs that are mostly directed. For any graph G , we denote by $\langle G \rangle$ the isomorphism class of G . The sign “=” means equality (never isomorphism). The automorphism group of G , denoted $\text{Aut}(G)$, acts on the vertex set of G , which we will always denote as V . We denote by S_n the symmetric group on V . For $\gamma, \delta \in S_n$, by v^γ we mean $\gamma(v)$, by $v^{\gamma\delta}$ we mean $(v^\gamma)^\delta$ and by G^γ we mean the graph on the same vertices as G for which $v^\gamma \rightarrow w^\gamma$ is an edge if and only if $v \rightarrow w$ is an edge of G .

For a digraph G and a vertex $v \in V$, G_v is the *switching of G at vertex v* , that is, the graph that equals G except that the direction of all edges incident with v is reversed. G_{vw} means $(G_v)_w$. Easy properties are: $G_{vv} = G$, $G_{vw} = G_{wv}$, and $(G_v)^\gamma = (G^\gamma)_{v^\gamma}$, for $v, w \in V$, $\gamma \in S_n$. For a multiset $W = \{w_1, \dots, w_k\}$ where $w_1, \dots, w_k \in V$, G_W means $G_{w_1 \dots w_k}$. Clearly, G_W depends only on the parity of the multiplicities of elements of W , and if W is a set (not multiset) we have $G_{V \setminus W} = G_W$. The property $(G_v)^\gamma = (G^\gamma)_{v^\gamma}$ generalizes to $(G_W)^\gamma = (G^\gamma)_{W^\gamma}$ for $W \subseteq V$, $\gamma \in S_n$, from which it follows that $G_X = G^\gamma$ and $G_Y = G^\delta$ together imply $G_{YX^\delta} = G^{\gamma\delta}$. Besides these easy properties, we also frequently need the following observations:

Lemma 1. *If $G_W = G^\delta$ then $(G_v)_{Wv^\delta} = (G_v)^\delta$.*

Proof. Using the fact that $(G_v)^\delta = (G^\delta)_{v^\delta}$, we get $(G_v)_{Wv^\delta} = (G_W)_{v^\delta} = (G^\delta)_{v^\delta} = (G_v)^\delta$. ■

Lemma 2. *If G is a connected oriented graph and $W, W' \subseteq V$ satisfy $G_W = G_{W'}$, then $W' = W$ or $W' = V \setminus W$.*

Proof. Supposing otherwise, there are adjacent vertices v, v' such that $v \in W \triangle W'$ and $v' \notin W \triangle W'$. Then G_W and $G_{W'}$ differ in the orientation of the edge between v and v' . ■

The *deck* of a digraph G is the multiset $\mathcal{C}(G) = \{\langle G_v \rangle \mid v \in V\}$. We say that the digraph G is *nonreconstructible* if there is a digraph H , not isomorphic to G , such

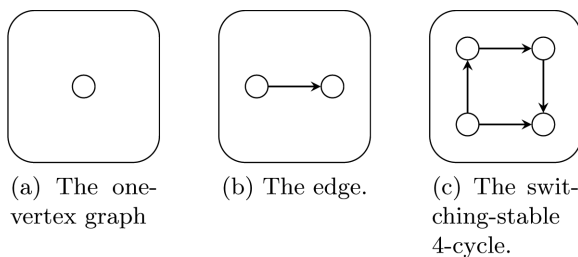


FIGURE 1. All connected switching-stable oriented graphs.

that $\mathcal{C}(H) = \mathcal{C}(G)$, and *reconstructible* otherwise. Since it will aid the proof of our main theorem, we generalize this concept as follows. For $t \in \mathbb{N}$, the t -deck of G , is the deck of G augmented by t copies of $\langle G \rangle$. By counting the number of copies of $\langle G \rangle$ that occur in the t -decks of two nonisomorphic digraphs G and H , we find that there is at most one t for which G and H have the same t -deck.

3. SWITCHING-STABLE DIGRAPHS

Following [3], we call a digraph D *switching-stable* if D_v is isomorphic to D for all vertices v . Examples of switching-stable digraphs are shown in Figure 1. The figure shows a graph on one vertex, a graph on two vertices with one edge and a graph that has four vertices v_1, v_2, v_3, v_4 and four edges $v_1v_2, v_2v_3, v_3v_4, v_1v_4$. In fact, as the following theorem shows, all switching-stable oriented graphs are obtained from these graphs by forming disjoint unions.

Theorem 1. *An oriented graph is switching-stable if and only if each of its components is either an isolated vertex, an isolated edge, or the oriented 4-cycle that has a unique directed path of length 3, i.e. all components are isomorphic to one of the graphs shown in Figure 1.*

Proof. Since a digraph is switching-stable if and only if all of its components are switching-stable, it is sufficient to consider connected oriented graphs. An easy search among the connected oriented graphs with at most four vertices shows that only the three stated examples are switching-stable. Thus, our task is to prove there are none that are larger.

Let D be a switching-stable oriented graph with $n > 4$ vertices and underlying undirected graph G . Let S_n be the symmetric group on the vertex set V of G . Define

$$\Gamma(D) = \{\gamma \in S_n \mid D_W = D^\gamma \text{ for some } W \subseteq V\}.$$

Since $\Gamma(D)$ is closed under composition ($D_X = D^\gamma$ and $D_Y = D^\delta$ imply $D_{X \cup Y} = D^{\gamma\delta}$), $\Gamma(D)$ is a group. Note that $\Gamma(D)$ contains the automorphism group $\text{Aut}(D)$.

If $D_X = D_Y$ for $X, Y \subseteq V$, then either $X = Y$ or $X = V \setminus Y$, by Lemma 2. Therefore, precisely 2^{n-1} different labeled oriented graphs arise by switching subsets of the vertices in D . Furthermore, switching preserves the underlying undirected graph G , so we have

$$\text{Aut}(D) \leq \Gamma(D) \leq \text{Aut}(G), \quad (1)$$

where $\text{Aut}(D)$ has index 2^{n-1} in $\Gamma(D)$.

The number 2^n cannot divide $n!$, while 2^{n-1} divides $n!$ only when n itself is a power of 2. Thus we can assume $n = 2^k$ for some k and that $\text{Aut}(G)$ contains a Sylow 2-subgroup of S_n , say Λ . Every Sylow 2-subgroup of S_n is an iterated wreath product of Z_2 with itself (i.e. the action on the leaves of a complete binary tree of the automorphism group of the tree). This means that G belongs to a particularly simple class of graphs, which we now construct.

Consider the sequence of permutation groups $S_1 = \Lambda_0, \Lambda_1, \dots, \Lambda_k = \Lambda$, where Λ_0 acts on a single point and, for $1 \leq i \leq k$, $\Lambda_i = \Lambda_{i-1} \wr Z_2$ is the standard permutation representation of the wreath product on 2^i points. That is, Λ_i is imprimitive with two blocks of size 2^{i-1} that can be exchanged by an element of order 2 and whose block-wise stabilizer is $\Lambda_{i-1} \times \Lambda_{i-1}$. This structure of $\Lambda_k \leq \text{Aut}(G)$ implies that G consists of two disjoint copies of some graph on which Λ_{k-1} acts, with these two copies either joined by no edges or by all possible edges.

Applying this logic recursively we find that $G = G_k$ where $G_0, G_1, \dots, G_k$ is some sequence of graphs constructed as follows.

- (a) Graph G_0 is the complete graph with one vertex, K_1 .
- (b) For $1 \leq i \leq k$, graph G_i is formed by taking two disjoint copies of G_{i-1} and adding either no more edges or all possible edges between the two copies. Note that G_i has 2^i vertices and $\Lambda_i \leq \text{Aut}(G_i)$.

Since we are assuming G to be connected, in the final step when G_k is made from two copies of G_{k-1} we must completely join the two copies. Thus G is a regular graph of degree $d \geq \frac{1}{2}n$ whose complement is disconnected.

By switching about neighbors of a vertex v of D , we can achieve that v has any out-degree in $\{0, 1, \dots, d\}$ we choose. Since D is switching-stable, this implies that each of the out-degrees $0, 1, \dots, d$ occurs in D . Let v_0 be a vertex of out-degree 0. Switching about v_0 changes the out-degree of v_0 to d , and decreases the out-degrees of its neighbors by 1. The only way this can fail to change the degree sequence of D is if the neighbors of v_0 have the distinct out-degrees $1, 2, \dots, d$. Now suppose there is a second vertex v'_0 of out-degree 0. Since $d \geq \frac{1}{2}n$, v_0 and v'_0 have at least one common neighbor, but then switching about v'_0 changes the set of out-degrees of the neighbors of v_0 , which we just proved is impossible. So v_0 is the only vertex of out-degree 0. Similarly, D has exactly one vertex v_d of out-degree d , and its neighbors have out-degrees $0, 1, \dots, d-1$.

Now, we divide the argument into two cases depending on whether or not v_0 and v_d lie in the same component of the complement $\bar{G}$. Recall that there are at least two such components.

First assume that v_0 and v_d lie in different components of $\bar{G}$. Then every vertex of D is a neighbor of either v_0 or v_d . Since $\text{Aut}(D)$ fixes v_0 and v_d , and their respective neighbors have unique out-degrees as we showed above, we have $|\text{Aut}(D)| = 1$. By (1), this implies $\Gamma(D) = \Lambda$. As explained above, Λ has a block-system consisting of two blocks W_0, W_1 of the same size, and all $\frac{1}{4}n^2$ edges between W_0 and W_1 are present. Therefore, for any vertex v , $D_v = D^{\gamma(v)}$ where $\gamma(v) \in \Lambda$ satisfies $\{W_0^{\gamma(v)}, W_1^{\gamma(v)}\} = \{W_0, W_1\}$. Moreover, if e is the number of edges of D directed from W_0 to W_1 , then $\frac{1}{4}n^2 - e$ is the number of edges directed the other way. Switching about a vertex must preserve the invariant $\{e, \frac{1}{4}n^2 - e\}$ consisting of the counts of edges directed from one side of the partition to the other. If $v \in W_0$ has x out-neighbors in W_1 , after switching about v there are $e + \frac{1}{2}n - 2x$ edges from W_0 to W_1 , so we must have $e + \frac{1}{2}n - 2x \in \{e, \frac{1}{4}n^2 - e\}$. Similarly, if w in W_1

has x out-neighbors in W_0 , after switching about w there are $e - \frac{1}{2}n + 2x$ edges from W_0 to W_1 , so we must have $e - \frac{1}{2}n + 2x \in \{e, \frac{1}{4}n^2 - e\}$. These constraints only allow x to have the three values $\frac{1}{4}n$, $\frac{1}{4}n + e - \frac{1}{8}n^2$, and $\frac{1}{4}n - e + \frac{1}{8}n^2$. However for the vertex v_0 , any value for x from 0 to $\frac{1}{2}n$ can be achieved by switching about its neighbors. Since $n \geq 8$ (recall that n is a power of 2), this is a contradiction.

Finally, assume that v_0 and v_d lie in the same component of $\overline{G}$, say the component induced by vertex set X . If $\overline{G}$ has exactly two components, then $\text{Aut}(G)$ preserves them as a blocks and the counting argument we used in the previous paragraph applies. If the number of components is greater than 2, it is still a power of 2 (by the construction of G) and $\text{Aut}(G)$ preserves the partition of V into those components. Switching about a vertex w not in X cannot create a vertex of out-degree 0 or d outside X , since all vertices not in X are adjacent to v_0 and to v_d . Therefore, the isomorphism that maps D to D_v preserves the set $V \setminus X$, and so the subgraph of D induced by $V \setminus X$ is switching-stable. However, it is connected and has an order that is not a power of 2, which contradicts what we proved above. ■

4. SWITCHING-STABLE SETS OF DIGRAPHS

A set M of digraphs is called *switching-stable* if every switching of a digraph in M is isomorphic to a digraph in M . Examples of switching-stable sets are sets that only contain switching-stable digraphs and sets that contain all orientations of an undirected graph. We now argue that if a small switching-stable set consists of orientations of a large connected graph, then the automorphism group of the graph must be large.

Lemma 3. *Let M be a set of orientations of a connected graph G on n vertices. If M is switching-stable then $2^{n-1} \leq |M| |\text{Aut}(G)|$.*

Proof. Suppose $m = |M|$. Since M is switching-stable, by the pigeonhole principle there must be digraphs $D, D' \in M$ (possibly equal) such that D_W is isomorphic to D' for at least $2^n/m$ sets $W \subseteq V$. Recalling that $D_W = D_{V \setminus W}$, it follows that for any fixed $v \in V$ there are $2^{n-1}/m$ sets W with $v \in W$ such that D_W is isomorphic to D' . Thus for each such W there is an automorphism δ_W of G such that $D_W = (D')^{\delta_W}$. If for two such sets W and W' we have $\delta_W = \delta_{W'}$, then $D_W = D_{W'}$, which cannot happen by Lemma 2. Thus all automorphisms δ_W are distinct, which proves the lemma. ■

Corollary 1. *Let M be a set of orientations of a connected graph G on n vertices of maximum degree at most 2. If M is switching-stable, then $2^{n-1} \leq 2n |M|$.*

Proof. This follows from the previous lemma since every connected n -vertex graph of maximum degree at most 2 has an automorphism group of size at most $2n$. ■

5. PATHS

In this section, we consider only oriented graphs with an underlying graph that is a path. We characterize all pairs of nonisomorphic oriented paths that have the same t -deck for

some $t \in \mathbb{N}$. The reconstructibility of paths on more than four vertices from their decks was shown by Mercier [7]. For our purposes, we need to extend this characterization to t -decks. Consequently, we cannot assume that the number of vertices of the paths are even and we require an alternative proof. It turns out that the pairs we want to characterize can only consist of oriented paths with three or four vertices.

Theorem 2. *Let t be a nonnegative integer. If P_1 and P_2 are two oriented paths on at least five vertices that have the same t -deck, then P_1 and P_2 are isomorphic.*

Proof. Let t be a nonnegative integer and let P be an oriented path on at least five vertices. We first argue that in the t -deck of P we can identify the two cards corresponding to switchings about the end vertices and that we can identify the two cards corresponding to switchings about the penultimate vertices. To show this, we first identify the set M of cards corresponding to switchings of P about one of the outer four vertices. We divide the cards into two groups according to the parity of the number of end edges pointing outwards. If there is a card with both end edges pointing outwards and a card with both end edges pointing inwards then M is the group of parity 0. Otherwise M is the group of parity 1.

By a similar argument, considering only cards in M and the parity of the number of the penultimate edges pointing outwards, we can distinguish the cards in M according to whether an end vertex or a neighbor of an end vertex was switched. This implies that P' , the induced subpath of P obtained by deleting the end vertices, is reconstructible.

Suppose P is a nonreconstructible oriented path on at least five vertices. By what we just observed, we can identify a card in the deck of P in which no leaf and no vertex adjacent to a leaf has been switched. Thus, we can reconstruct the number of end edges of P pointing outwards. Since we can also identify a card in which an end vertex has been switched, this implies that among the leafs of P there is one vertex with out-degree 0 and one with in-degree 0.

Since the oriented path P' obtained by deleting the end vertices has been determined, two possible reconstructions remain ($\rightarrow P' \rightarrow$ and $\leftarrow P' \leftarrow$). By considering the switchings about the penultimate vertices, we see that these only have the same deck if P' is the same as its reverse, in which case they are isomorphic. ■

The theorem says that for our purposes we only need to consider oriented paths on at most four vertices. Figure 2 depicts all families of nonisomorphic oriented paths that have the same t -deck for some $t \in \mathbb{N}$.

6. CYCLES

In this section, we consider only graphs whose underlying undirected graph is a cycle. We determine all pairs of nonisomorphic oriented graphs of this type that have the same t -deck for some $t \in \mathbb{N}$. The automorphism group of an undirected cycle on n vertices is the dihedral group D_n , which consists of n reflections and n rotations (the identity is a rotation).

Lemma 4. *Suppose that for $n \geq 30$, there are nonisomorphic oriented cycles G and H with the same t -deck for some t . Then there is a set $W \subseteq V$ of size 2 or 4 such that $G_W = G'$ for some rotation γ of order at least 5.*

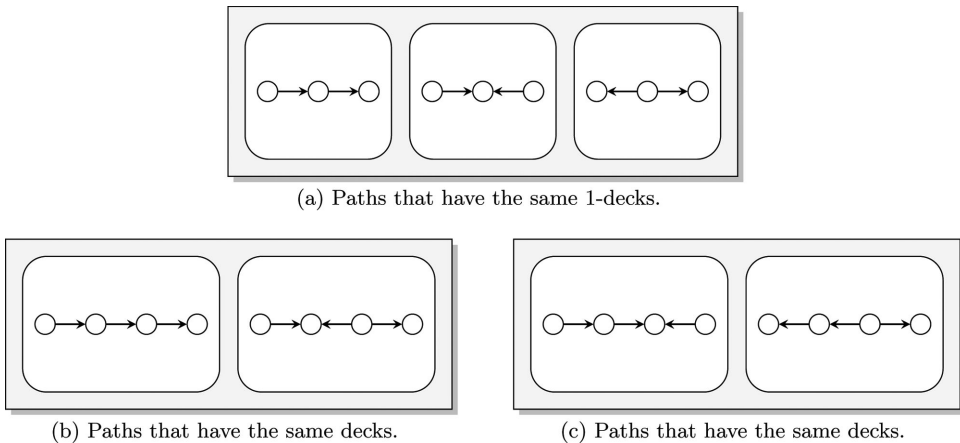


FIGURE 2. The three families of paths which have the same t -decks. The three paths in (a) have the same 1-decks, the other two pairs of graphs have the same decks.

Proof. For some labeling of G and H , the common t -deck of G and H can be indexed as $\{\langle J(i) \rangle : 1 \leq i \leq n+t\}$, so that $J(i) = H$ for $i = 1, \dots, t$, $J(i) = G$ for $i = n+1, \dots, n+t$, $\langle G_v \rangle = \langle J(v) \rangle$ for $i = 1, \dots, n$, and $\langle H_v \rangle = \langle J(v+t) \rangle$ for $i = 1, \dots, n$.

We will first find eight sets $W \subseteq V$ of size 2 such that $G_W = G^{\gamma(W)}$, where the permutations $\gamma(W)$ are nontrivial and distinct. We do this in two cases, depending on whether $t \leq 4$.

First suppose that $t \leq 4$. For each $v = t+1, \dots, n$, there is some w such that $G_{vw} \in \langle H \rangle$. We know that $w \neq v$ since H is not isomorphic to G . Then there are at least $n-t-2$ vertices x of G_{vw} other than v and w , such that for some $y \neq x$, $G_{vwxy} \in \langle G \rangle$. By Lemma 2, $G_{vwxy} = G^{\gamma(v,x)}$ for some nontrivial $\gamma(v,x) \in D_n$. We had $(n-t)(n-t-2)$ choices of (v,x) , but the permutations obtained might not be all different. By Lemma 2, $\gamma(v,x)$ in fact depends injectively on which elements of the multiset $\{v, w, x, y\}$ occur an odd number of times. Any such multiset with four distinct elements can be obtained from at most eight choices of (v,x) . The other possibility is that $y = v$ or $y = w$, so that exactly two elements, say a, b , of the multiset $\{v, w, x, y\}$ occur an odd number of times. Any such set $\{a, b\}$ can occur at most once for each value of v , that is, at most $n-t$ times. Finally, there are $2n-1$ nontrivial elements in D_n . If k is the number of choices (v,x) for which we obtain a 2-set, we must have $8(2n-k-1) + (n-t)k \geq (n-t)(n-t-2)$, which implies $k \geq 9$ when $t \leq 4$ and $n \geq 30$.

Next suppose that $t \geq 5$. For $v = 1, \dots, 5$, G_v is isomorphic to H , then for each such v , there are five values of w , corresponding to the vertices of H labeled $n+1, \dots, n+5$, for which G_{vw} is isomorphic to G . Discounting the possibility $v = w$, and noting that $G_{vw} = G_{wv}$, we find 10 sets $\{v, w\}$ such that $G_{vw} = G^{\delta(v,w)}$ for nontrivial distinct $\delta(v,w) \in D_n$.

The proof so far shows that we have a set $\mathcal{W}$ of eight distinct sets of size 2 such that for each $W \in \mathcal{W}$, $G_W = G^{\gamma(W)}$, where the permutations $\gamma(W) \in D_n$ are nontrivial and distinct.

If, for some $W \in \mathcal{W}$, $\gamma(W)$ is a rotation of order at least five, we are done, so assume that is not the case. We will find such a rotation by combining two sets U, W using the rule $G_{U^{\gamma(W)}W} = G^{\gamma(U)\gamma(W)}$. However, we need to avoid the case where $U^{\gamma(W)} = W$, since

we want the symmetric difference of $U^{\gamma(W)}$ and W to have size 2 or 4. This means that for each $W \in \mathcal{W}$, there is at most one $U \in \mathcal{W}$, namely $W^{-\gamma(W)}$, excluded for this reason. We will call the product $\gamma(U)\gamma(W)$ *acceptable* if $U \neq W^{-\gamma(W)}$.

First suppose that $\mathcal{W}$ contains four rotations. If there is one, γ_1 , of order 2 or 4, and two γ_2, γ_3 , of order 3, then $\gamma_2\gamma_1$ and $\gamma_3\gamma_1$ have order 6 or 12 and at least one of them is acceptable. The same argument holds if γ_1 has order 3 and γ_2, γ_3 have order 2 or 4.

If $\mathcal{W}$ does not contain four rotations, it has at least five reflections $\sigma_1, \dots, \sigma_5$. The product $\sigma_1\sigma_1$ is trivial and therefore not acceptable. Thus, the four products $\gamma_2 = \sigma_2\sigma_1, \dots, \gamma_5 = \sigma_5\sigma_1$ are acceptable and in particular nontrivial. If one of the nontrivial rotations $\gamma_2, \dots, \gamma_5$ has order at least five, we are done, so assume that is not the case. By the same argument as before there must be indices $i, j, k \in \{2, \dots, 5\}$ with $i \neq j$ such that $\gamma_i\gamma_k^{-1}$ and $\gamma_j\gamma_k^{-1}$ have order 6 or 12. Noting that $\gamma_i\gamma_k^{-1} = \sigma_i\sigma_1\sigma_1\sigma_k = \sigma_i\sigma_k$ and $\gamma_j\gamma_k^{-1} = \sigma_j\sigma_1\sigma_1\sigma_k = \sigma_j\sigma_k$ we conclude that one of the two products must be acceptable and has order at least 5.

One of these two possibilities (four rotations or five reflections) must happen if we have eight distinct nontrivial elements of D_n in $\mathcal{W}$. This shows that there is a set $W \subseteq V$ of size 2 or 4 such that $G_W = G^\gamma$ for some rotation γ of order at least 5. ■

Let δ be a rotation. We define $W(G, \delta)$ to be the set W that satisfies $G_W = G^\delta$ and $|W| < n/2$, if it exists. Note that the set W is unique if it exists. Let $\text{dist}(G, \delta)$ be the set of pairwise distances (lengths of shortest paths on the undirected cycle) between the elements of $W(G, \delta)$. Note that $|W(G, \delta)| = |W(G, \delta^{-1})|$ and $\text{dist}(G, \delta) = \text{dist}(G, \delta^{-1})$, since $W(G, \delta^{-1}) = W^{\delta^{-1}}$.

Lemma 5. *Let G be an oriented cycle and let δ be a nontrivial rotation such that $G_W = G^\delta$ for some $W \subseteq V$ with $n > 2|W| + 8$. Then $W(G_v, \delta)$ is well defined for each $v \in V$. Furthermore, for any oriented cycle H that has an extended deck that is also an extended deck of G , the set $W(H, \delta)$ is well defined and $|W(H, \delta)| = \max\{|W(G_v, \delta)| - 2 \mid v \in V\} = |W|$.*

Proof. The fact that $W(G_v, \delta)$ is well defined follows from Lemma 1. If H is an oriented cycle that has an extended deck that is also an extended deck of G , then there is a vertex v such that $G_v = H$ or there are two not necessarily distinct vertices v, w such that $G_{vw} = H$. Thus, by Lemma 1, $H_{W \vee v^\delta} = (G_v)_{W \vee v^\delta} = (G_v)^\delta = H^\delta$ or $H_{W \vee v^\delta w^\delta} = (G_{vw})_{W \vee v^\delta w^\delta} = (G_{vw})^\delta = H^\delta$. Since $|W| + 4 < n/2$, in the first case, we have $W' = W(H, \delta) = W \triangle \{v, v^\delta\}$, and in the second case we have $W' = W(H, \delta) = W \triangle \{v, v^\delta\} \triangle \{w, w^\delta\}$. In either case $W(H, \delta)$ is well defined.

For all v it holds that $|W'| - 2 \leq |W' \triangle \{v, v^\delta\}| \leq |W'| + 2$. Since $n > 2(|W| + 4) \geq 2|W'|$, there exists a u such that $W' \triangle \{u, u^\delta\}$ has size $|W'| + 2$. Thus $|W'| = \max\{|W(G_v, \delta)| - 2 \mid v \in V\}$. ■

Note that the fact that $|W(G_v, \delta)| = |W(G_v, \delta^{-1})|$ for all $v \in V$ implies that the value of $\max\{|W(G_v, \delta)| - 2 \mid v \in V\}$ depends only on the deck. The previous lemma thus says in other words that $|W|$ can be reconstructed, for any particular δ .

Lemma 6. *Let G be an oriented cycle. Suppose there exists a nontrivial rotation δ that is not of order 2, such that for $W = W(G, \delta)$ we have $|W| \geq 2$ and $n > 4|W| + 4$, then $\text{dist}(G, \delta)$ is reconstructible from every extended deck.*

Proof. Let r be the integer that is equal to the distance between a vertex and its image under δ . Since $|W| \geq 2$ and $n > 4|W| + 4$ together imply $n > 2|W| + 8$, Lemma 5 applies and $|W|$ can be reconstructed. First note that for any integer d the value of $|\{v \mid d \in \text{dist}(G_v, \delta)\}|$ depends only on the deck.

- For $d \notin \text{dist}(G, \delta)$ and $d \neq r$ we have $|\{v \mid d \in \text{dist}(G_v, \delta)\}| \leq 4|W|$.
- For $d \in \text{dist}(G, \delta)$ and $d \neq r$ we have $|\{v \mid d \in \text{dist}(G_v, \delta)\}| \geq n - 4$.

These inequalities remain true if the sets on the left hand side range over all cards in an extended deck of G . Thus if $4|W| < n - 4$, then we can decide whether d is in $\text{dist}(G, \delta)$, for $d \neq r$. To show that it is also possible to decide $r \in \text{dist}(G, \delta)$, we note that $r \in \text{dist}(G, \delta)$ if and only if there exist a vertex $v \in V$ such that $|\text{dist}(G_v, \delta)| = |W| - 2$. ■

Lemma 7. *Let G be an oriented cycle. Suppose there exists a nontrivial rotation δ that rotates by $r < n/2$ positions, such that for $W = W(G, \delta)$ we have $r \notin \text{dist}(G, \delta)$, $|W| \geq 2$, and $n > 4|W| + 4$. Then G is reconstructible from every extended deck.*

Proof. The inequalities $|W| \geq 2$ and $n > 4|W| + 4$ imply that $n > 2|W| + 8$. By Lemma 5, for any oriented cycle H that has an extended deck that is also an extended deck of G we have $|W(H, \delta)| = |W(G, \delta)|$. We show the existence of a vertex v' such that

1. $v', (v')^\delta \notin W$,
2. v' is not at distance r from any vertex in W , and
3. $(v')^\delta$ does not have distance r from any vertex in W .

We call such a vertex *good*. We count the number of vertices that violate one of the conditions: There are $|W|$ vertices in W and $|W|$ vertices in W^δ , thus there are at most $2|W|$ vertices that violate Condition 1. There are at most $|W|$ vertices that violate Condition 2 but do not violate Condition 1. There are at most $|W|$ vertices that violate Condition 3 but do not violate Condition 1. Thus in total we conclude that at most $2|W| + |W| + |W|$ vertices are not good. Since $n > 4|W|$, there is a good vertex.

Let v' be a good vertex. We show that for all $v'' \neq v'$ the set $W(G_{v'v''}, \delta) = W \triangle \{v', (v')^\delta\} \triangle \{v'', (v'')^\delta\}$ is not of size $|W|$. Since v' is good, the set $W \triangle \{v', (v')^\delta\} = W \cup \{v', (v')^\delta\}$ is of size $|W| + 2$. Furthermore, since v' is good and the distance r is not in $\text{dist}(G, \delta)$, the only two vertices in $W \cup \{v', (v')^\delta\}$ that are distance r apart are v' and $(v')^\delta$. Since $r \neq n/2$, for $v'' \neq v'$ the set $\{v', (v')^\delta\}$ is not equal to the set $\{v'', (v'')^\delta\}$. This shows that $W \triangle \{v', (v')^\delta\} \triangle \{v'', (v'')^\delta\}$ is not of size $|W|$.

Let H be an oriented cycle that has an extended deck that is also an extended deck of G . Consider the card $G_{v'}$ in this extended deck. Since $|W(H, \delta)| = |W|$, we conclude that $H \neq G_{v'}$ and that $H \neq (G_{v'})_{v''}$ for all $v'' \neq v'$. Thus $H = (G_{v'})_{v'} = G$. ■

Lemma 8. *Let G be an oriented cycle on $n > 28$ vertices. If there is a nontrivial rotation δ that is not of order 2, such that $|W(G, \delta)| = 4$ then G is reconstructible from every extended deck.*

Proof. Let r be the integer that is equal to the distance between a vertex and its image under δ . Define $D = \text{dist}(G, \delta)$. By the previous lemma, we may assume that $r \in D$. The conditions on $W = W(G, \delta)$ imply that $n > 2|W| + 8$ and $n > 4|W| + 4$. Thus, by

Lemmas 5 and 6, for any graph H that has an extended deck that is also an extended deck of G , we have $|W(H, \delta)| = 4$ and $\text{dist}(H, \delta) = D$.

We show the existence of a vertex $v' \in V$ such that

1. $v', (v')^\delta \notin W$,
2. v' is not at distance r from any vertex in W , and
3. for at least three distinct vertices $x_1, x_2, x_3 \in W$ the distance from v' to x_i is not contained in D .

We call such a vertex *good*. We count the number of vertices that violate one of the conditions: There are four vertices in W and four vertices in W^δ . However, since $r \in D$, one vertex lies in $W \cap W^\delta$, thus there are at most seven vertices that violate Condition 1. Since $r \in D$, there are at most three vertices that violate Condition 2 but do not violate Condition 1. We now bound the number of vertices for which at least two of the distances to vertices in W are contained in D . We double count the ordered pairs (x, u) with $x \in W, u \in V$, where u does not satisfy Condition 1 or 2 and the distance between x and u is in D . There are at most $|D| \cdot 2 \cdot 4 - 4 \cdot 3$ such pairs, since there are $4 \cdot 3$ pairs (x, u) with $x, u \in W$. Thus there are at most $(8|D| - 12)/2$ vertices not in W with two or less distances to vertices in W not contained in D . Thus in total we conclude that at most $7 + 3 + 4|D| - 6 \leq 28$ vertices are not good. Since $n > 28$, there is a good vertex.

Let v' be a good vertex. Since v' is good, the set $W \triangle \{v', (v')^\delta\} = W \cup \{v', (v')^\delta\}$ is of size $|W| + 2$. We now show that for all $v'' \neq v'$ the set $W'' = W \triangle \{v', (v')^\delta\} \triangle \{v'', (v'')^\delta\}$ has two vertices of a distance not contained in D , or the set $|W''|$ is not of size $|W|$.

Case 1: If $(v'')^\delta = v'$ then $v'' \notin W \triangle \{v', (v')^\delta\}$, since with the exception of $(v')^\delta$ no vertex in $W \cup \{v', (v')^\delta\}$ has distance r from v' and $r \neq n/2$. Thus $|W''| \neq |W|$.

Case 2: If $v'' \neq v'$ and $(v'')^\delta \neq v'$, then $v' \in W''$. Three of the vertices in W are at a distance to v' that is not contained in D . At least one of these vertices is contained in W'' .

Let H be a graph that has an extended deck that is also an extended deck of G . Consider the card $G_{v'}$ in this extended deck. Since $|W(H, \delta)| = 4$ and $D = \text{dist}(H, \delta)$, we conclude that $H \neq G_{v'}$ and that $H \neq (G_{v'})_{v''}$ for all $v'' \neq v'$. Thus $H = (G_{v'})_{v'} = G$. ■

Lemma 9. *Let $W = W(G, \delta)$ where δ is a nontrivial rotation of order at least five rotating by r positions. If $|W| = 2$, $n > 12$ and $r \in \text{dist}(G, \delta)$, then G is reconstructible from every extended deck.*

Proof. The assumptions $|W| = 2$ and $n > 12$ imply that $n > 2|W| + 8$.

For the sake of contradiction we assume G is not reconstructible. By the assumptions, W is of the form $W = \{w_1, w_2\}$ with $w_2 = w_1^\delta$. Define $w_0 = w_1^{\delta-1}$ and $w_3 = w_2^\delta$.

Since $W(G_v, \delta) = \{w_1, w_2\} \triangle \{v, v^\delta\}$ for any vertex v , it holds that $|W(G_v, \delta)| = 2$ if and only if $v \in \{w_0, w_2\}$. Since $G_{w_1 w_2} = G^\delta$, we have $(G_{w_0})^\delta = (G_{w_0})_{w_1 w_2 w_0 w_1} = G_{w_2}$. Thus, all cards G_v with $|W(G_v, \delta)| = 2$ are isomorphic. We can therefore identify the isomorphism type of the card G_{w_2} from the deck. Note that $W(G_{w_2}, \delta) = \{w_1, w_3\}$.

We first show that for every vertex $v' \notin \{w_1, w_2\}$ the set $W(G_{w_2 v'}, \delta) = \{w_1, w_3\} \triangle \{v', v'^\delta\}$ is either not of size 2, or it is of size two, but its two vertices are not at distance r : Indeed, whenever $v' \notin \{w_0, w_1, w_2, w_3\}$, then $\{w_1, w_3\} \triangle \{v', (v')^\delta\}$ is not of size two. If $v' \in \{w_0, w_3\}$ then $\{w_1, w_3\} \triangle \{v', (v')^\delta\}$ is of the form $\{u, u^{\delta^3}\}$ for some vertex $u \in V$ and thus, since δ is of order at least five, does not contain two vertices at distance r .

Since $(w_1)^\delta = w_2$, we know that $((G_{w_2})_{w_1})^\delta = G = (G_{w_2})_{w_2}$. Let H be a graph that has an extended deck that is also an extended deck of G . For all $v' \notin \{w_1, w_2\}$ we have

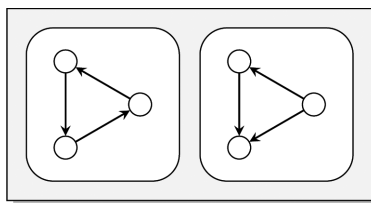


FIGURE 3. The 3-cycles that have the same 1-decks.

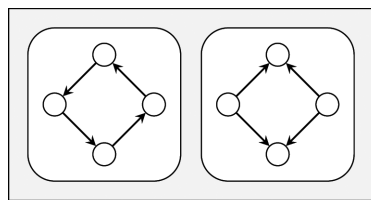


FIGURE 4. The 4-cycles that have the same decks.

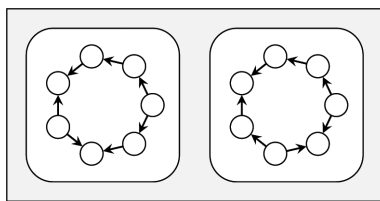


FIGURE 5. The 7-cycles that have the same 1-decks.

that $|W(G_{w_2v'}, \delta)| \neq 2$ or $r \notin W(G_{w_2v'}, \delta)$. Thus, for all $v' \notin \{w_1, w_2\}$, $H \neq G_{w_2v'}$. Since δ is not of order 3, $\text{dist}(G, \delta) \neq \text{dist}(G_{w_2}, \delta)$ and thus $H \neq G_{w_2}$. For all $v' \in \{w_1, w_2\}$ the graph $G_{w_2v'}$ is isomorphic to G . Therefore H is isomorphic to G . In any case, we obtain a contradiction. ■

Theorem 3. *Every oriented cycle on at least 30 vertices is reconstructible from each of its extended decks.*

Proof. Suppose G is an oriented cycle on $n \geq 30$ vertices that is not reconstructible from one of its extended decks. By Lemma 4 there is a set $W \subseteq V$ of size 2 or 4 such that $G_W = G^\gamma$ for some rotation δ of order at least five. Suppose δ operates by rotating G by r positions. If $r \notin \text{dist}(G, \delta)$ then G is reconstructible by Lemma 7. If $r \in \text{dist}(G, \delta)$, then G is reconstructible by Lemma 9 if $|W| = 2$ and by Lemma 8 if $|W| = 4$. ■

Figures 3–8 show families of nonisomorphic oriented cycles that have the same t -deck for some $t \in \mathbb{Z}$. In fact, Figure 6 shows two graphs that have the same (-1) -decks, where the (-1) -deck of graph D is obtained from the deck of D by deleting $\langle D \rangle$. All graphs have been computed using the graph generation package nauty [6] developed by the first author. Together with Theorem 3, this computation also demonstrates that there are no other examples.

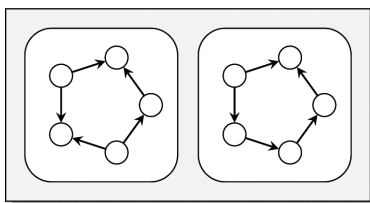
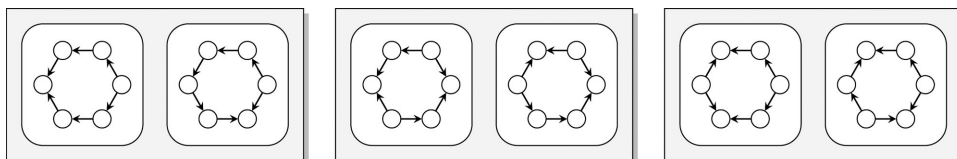
FIGURE 6. The 5-cycles that have the same (-1) -decks.

FIGURE 7. The three families of 6-cycles which have the same 2-decks.

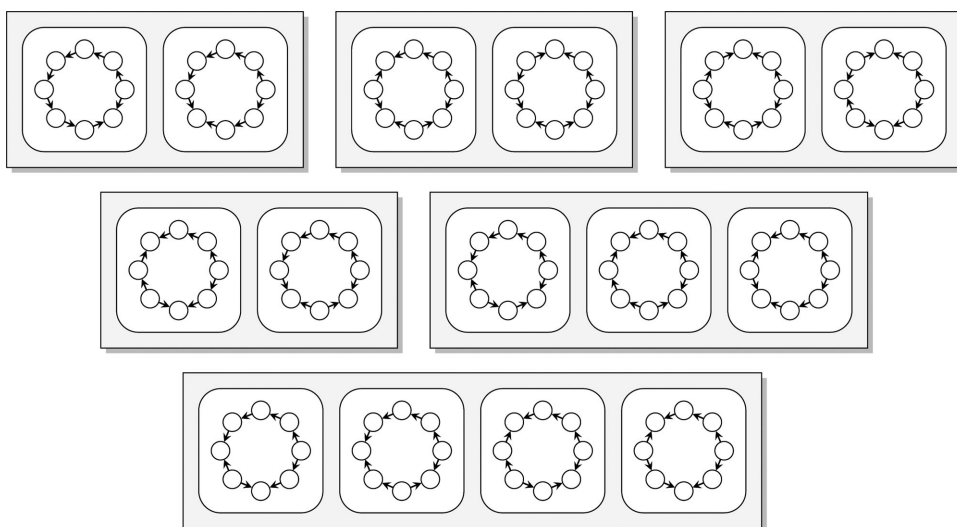


FIGURE 8. The six families of 8-cycles which have the same decks.

7. DISCONNECTED GRAPHS

In this section, we consider disconnected digraphs. For a digraph C and a digraph G let $n_C(G)$ be the number of components of G that are isomorphic to C . We call two components of a digraph *switching adjacent* if one can be obtained from the other by switching about a vertex.

Lemma 10. *If a digraph G contains at least two nonisomorphic switching adjacent components, then G is reconstructible.*

Proof. Suppose G contains the nonisomorphic switching adjacent components A and B . Let H be a digraph that has the same deck as G . Since the deck contains a card that has at least two A components, one of the components of H is A . By symmetry B is

also a component of H . Thus, the number $n_A(H)$ of A components of H is the maximum number of A components appearing in a card minus 1 and thus depends only on the deck. By symmetry the number $n_B(H)$ of B components in H also only depends on the deck. There is a card X with $n_A(H) + 1$ components isomorphic to A and $n_B(H) - 1$ components isomorphic to B . The graph H , and thus also the graph G , must be isomorphic to the graph obtained from the card X by replacing a component isomorphic to A with a component isomorphic to B . ■

Lemma 11. *If a digraph G contains at least three components of which at least two are not switching-stable, then G is reconstructible.*

Proof. Let H be a digraph that has the same deck as G . By the previous lemma we may assume that all nonisomorphic components of G are not switching adjacent, and similarly for H . Let A and B be components of G that are not switching-stable. Then there is card containing both of them, so at least one, say A , is a component of H . Consider a card X with the fewest components isomorphic to A . Since A is not switching-stable, this card corresponds to switching G about a vertex in A , and similarly for H . Since A is not switching adjacent to any component of G or H that is not isomorphic to A , both G and H are the graph obtained by replacing the unique component of X switching adjacent to A by a copy of A . That is, G and H are isomorphic. ■

For a digraph G , a *possible component* is a connected digraph that appears as some component in some digraph H that has the same deck as G . A *definite component* is a connected digraph that appears as a component in every digraph that has the same deck as G .

Lemma 12. *If a digraph G contains exactly two components of different sizes of which the smaller one is not switching-stable, then G is reconstructible.*

Proof. Suppose the components are A and B and A is of smaller size. Then A appears in more than half the cards and is thus definite. Since A is not switching-stable, there is a card CB with C nonisomorphic to A . Since A is definite and B cannot be a switching of A this implies that B is definite. ■

Lemma 13. *If G is a digraph that consists of two components with the same number of vertices then G is reconstructible or the possible components form a switching-stable set of size at most four.*

Proof. Suppose $G = A + B$. By Lemma 10 we may assume that A and B are not simultaneously nonisomorphic and switching adjacent.

We observe that every possible component appears at least $n/2$ times in the deck. Moreover every switching of a possible component is in the deck. We distinguish cases according to the number of possible components of G .

Four possible components: If G has exactly four nonisomorphic possible components, then they form a switching-stable set.

Three possible components: If G has exactly three nonisomorphic possible components then two of them must be adjacent. If there is a reconstruction that has two isomorphic components, then this component appears n times in the deck. The other two possible components appear $n/2$ times, so the three components form a switching-stable set.

Suppose now G does not have isomorphic components. Since G does not have adjacent components and does not have isomorphic components, it is not the case that all three possible components are adjacent. If one of the three possible components is adjacent to both other components, then G must be the graph that consists of the nonadjacent components. Otherwise suppose A and B are adjacent and C is not adjacent to either of them. Since A and B both appear at least $n/2$ times and C cannot switch to A or B , A must always switch to B and B must always switch to A . Thus if A appears in the deck then $G = B + C$. Otherwise $G = A + C$. In any case G is reconstructible.

Two possible components: If G has exactly two nonisomorphic possible components A and B , then G is isomorphic to $A + B$ if they are not adjacent. We can thus assume that $G \neq A + B$ by Lemma 10. Thus, $G = A + A$ or $G = B + B$. Either the set $\{A, B\}$ is switching-stable or the deck contains a card of the form $A + C$ or a card of the form $B + C$ with $C \notin \{A, B\}$, implying that one of the components A or B is definite.

One possible component: If G has exactly one nonisomorphic possible component then G is isomorphic to the disjoint union of this component with itself. ■

Theorem 4. *If G and H are disconnected nonisomorphic digraphs with the same deck, then*

1. *G and H each have exactly two components, and the possible components of G and H are of the same size and form a switching-stable set of size at most four, or*
2. *G and H each have exactly one component that is not switching-stable, and these two components have the same t -deck for some $t \in \mathbb{N}$.*

Proof. We first argue that if G and H both have exactly one component that is not switching-stable then the two nonswitching-stable components in G and H have the same t -deck for some $t \in \mathbb{N}$. The number of switching-stable components of each isomorphism type in every card of a deck is invariant over all cards and the same for the original graph. Let t be the number of vertices contained in switching-stable components of G . This implies that t is also the number of vertices contained in switching-stable components of H . The multiset of nonswitching-stable components in cards of G is equal to the t -deck of G . The analogous statements holds for H that shows that the two nonswitching-stable components in G and H have the same t -deck.

If G and H have more than three components then by Lemma 11 they both have exactly one component that is not switching-stable, showing that the second option holds.

If G and H have two components, then either by Lemma 12 the second option holds or by Lemma 13 the possible components of G form a switching-stable set of size at most four. By definition, all components of H are possible components of G . ■

We now describe all oriented graphs with maximum degree at most 2 that satisfy Property 1 from the previous theorem. Let G and H be a pair of such oriented graphs. Let c be the size of a largest possible component of G then by Corollary 1 we have $2^{c-1} \leq 4 \cdot c \cdot 2$, which implies $c \leq 6$.

Note that every oriented path on n vertices can be switched into every other oriented path on n vertices. Since there are more than four oriented paths on five vertices, an oriented path contained in a switching-stable set of size at most four has at most four vertices. Figure 9 shows the unions of oriented paths that have Property 1 from the previous theorem.

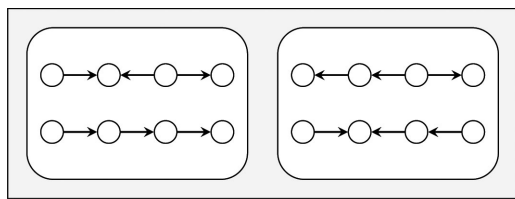


FIGURE 9. Disjoint unions of paths which have the same decks.

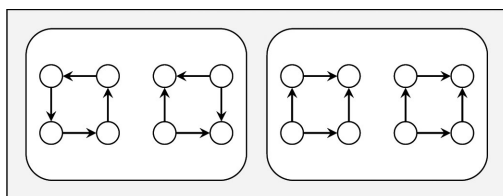


FIGURE 10. Disjoint unions of cycles which have the same decks.

Checking which pairs of cycles of length up to 6 are nonreconstructible can be done by hand or by a computer search. Figures 9 and 10 show all pairs of oriented graphs of maximum degree at most two that satisfy Property 1 from the previous theorem.

8. NONRECONSTRUCTIBLE GRAPHS OF MAXIMUM DEGREE AT MOST TWO

We assemble the results from the previous sections to characterize all nonreconstructible graphs with an underlying graph of maximum degree at most 2.

Lemma 14. *Let G and H be two digraphs. Let $G' \dot{\cup} S_G$ and $H' \dot{\cup} S_H$ be the decomposition of the graphs obtained by splitting off the parts S_G and S_H that contain all switching-stable connected components. Then G and H have the same deck if and only if S_G and S_H are isomorphic graphs on t vertices say, and G' and H' have the same t -deck.*

Proof. The lemma follows directly from the observation that the t -decks of G' and H' are obtained by removing all switching-stable components in each card in the deck of G and H , respectively. ■

Theorem 5. *The pairs $\{G, H\}$ of nonisomorphic oriented graphs with maximum degree at most 2 that have the same deck are exactly the following.*

1. *The oriented graphs obtained from paths in Figure 2 (A) by adding an isolated vertex.*
2. *Two oriented paths on four vertices that are in one of the two families in Figure 2 (B) and (C).*
3. *The oriented graphs obtained from the two 3-cycles shown in Figure 3 by adding an isolated vertices.*
4. *The two oriented 4-cycles in Figure 4.*

5. The oriented graphs obtained from the two 6-cycles that are in one of the three pairs in Figure 7 by adding either two isolated vertices or an isolated edge.
6. The oriented graphs obtained from the two 7-cycles shown in Figure 5 by adding an isolated vertex.
7. Two oriented 8-cycles that are in one of the six families of Figure 8.
8. The oriented forests shown in Figure 9.
9. The unions of oriented cycles shown in Figure 10.

In summary, on four vertices there are four families of size 2 and 1 family of size 3. On eight vertices there are 13 families of size 2, one family of size 3, and one family of size 4.

9. CONCLUDING REMARKS

As we mentioned earlier, all pairs of nonisomorphic oriented graphs with the same deck have an order that is a multiple of 4, and the largest known have eight vertices [3]. These include 20 families of two tournaments, four families of three tournaments, and two families of four tournaments with the same deck. One of the latter families is shown in Figure 11.

It is too expensive to test by exhaustive enumeration whether all 12-vertex oriented graphs are determined by their decks. However, we performed a partial search on that size, finding that all 154,108,311,168 tournaments, all 506,454,795 orientations of graphs

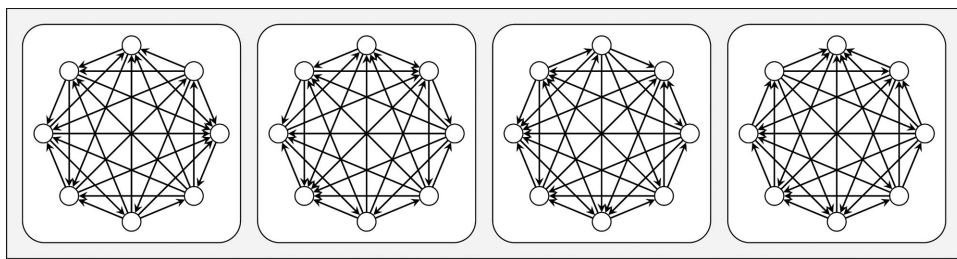


FIGURE 11. A family of four 8-vertex tournaments that have the same decks.

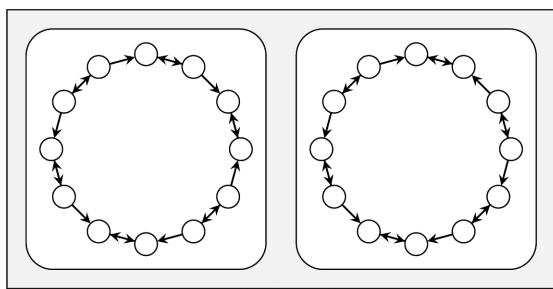


FIGURE 12. These 12-cycles contain 2-gons, depicted as two headed arrows. They have the same decks.

with maximum degree at most 3, and all 16,895,298,180 orientations of quartic graphs are determined by their decks.

In this article, we have restricted ourselves to oriented graphs, since a cycle of length 2 is not changed by a switching. Nevertheless, Figure 12 shows that new, structurally different reconstruction problems emerge when 2-cycles are allowed. The two graphs shown have the same deck. No similar pairs of cycles occur on 13–20 vertices.

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