1997 European Control Conference (ECC)
1-4 July 1997, Brussels, Belgium

AN LMI APPROACH TO SUB-OPTIMAL GUARANTEED COST CONTROL FOR UNCERTAIN TIME-DELAY SYSTEMS

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Keywords : Uncertain Time-Delay Systems, Norm Bounded Uncertainty, Quadratic Stabilization, Guaranteed Cost Control, Linear Matrix Inequalities.

Abstract

This paper presents results on the design of robust memoryless state feedback controller for uncertain time-delay systems with norm bounded uncertainty. It is proved that quadratic stabilization of uncertain time-delay systems is necessary and sufficient for the feasibility of an LMI problem. The robust state feedback controller can be constructed using the corresponding feasible solution of the LMI problem. A procedure is given to select a suitable state feedback controller that is also sub-optimal in the sense of minimizing a bound on an integral quadratic performance index.

1 Introduction

During the recent years, the problem of quadratic stabilization of uncertain linear systems with norm-bounded uncertainty has received considerable attention in the Control Community, e.g., see [1]-[3]. The stabilization approach in [3] is based on using a fixed quadratic Lyapunov function for uncertain systems which leads to a parameter dependent Riccati equation. Using any positive definite solution of this parameter dependent Riccati equation, a robust controller is constructed that quadratically stabilizes the linear uncertain system. In [4], the problem of designing a quadratic stabilizing controller which also satisfies an adequate level of performance for closed loop system is proposed. The performance is measured with the standard linear quadratic index. Then an upper bound for this index is defined. Finally, the corresponding robust controller that minimises this bound is found by a numerical search for the stabilizing solution of the parameter dependent Riccati equation.

The problem of designing robust controllers for uncertain time-delay systems has attracted a number of researchers' attention, e.g., see [5]-[7]. Again the Lyapunov approach is used to design a robust controller. However, the problem of designing a controller which would guarantee an adequate level of linear quadratic performance is still under question, especially when no assumption of matching conditions is considered. In [8], this problem is addressed for a particular class of uncertain time-delay systems.

In this paper, we propose linear matrix inequality (LMI) approach to the stabilization problem for uncertain time-delay systems. It should be noted that the idea of employing LMIs to stabilize linear time-delay systems was first proposed in [12]. The results of [12] are a special case of the results of this paper. Moreover, our approach also guarantees an upper bound on the linear quadratic performance measure. In section 2, we define the notion of quadratic stability for uncertain time-delay systems, then it is proved that quadratic stability can be stated as an LMI feasibility problem. The LMI formulation also helps us to find the least upper bound for a linear quadratic cost function by solving a convex minimization. Section 3 extends the results of section 2 to the synthesis case. Quadratic stabilization with considering a performance level for closed loop system is stated in an LMI minimization procedure. Section 4 contains one example that illustrates our approach.

2 Quadratic Stability of Uncertain Time-Delay Systems

We consider the uncertain time-delay system described by the state equation of the form:

$$\dot{x}(t) = (A+D\Delta(t)E_1)x(t)+(F+D\Delta(t)E_3)x(t-\tau), \quad (2.1)$$

where $x(t) \in R^n$ is the state and $\Delta(t) \in R^{p\times q}$ is a matrix of uncertain parameters satisfying the bound $\Delta'(t)\Delta(t) \leq I$. The rest of the matrices are real constant matrices with compatible dimensions and τ is a scalar number represent-

ISBN 978-3-9524269-0-6

ing the delay in the system. The initial condition is specified as $(x(0), x(s)) = (x_0, \phi(s))$ where $\phi(\cdot) \in L_2[-\tau, 0]$.

For system (2.1), we define the Lyapunov functional

$$V(x,t) = x'(t)Px(t) + \int_0^\tau x(t-s)'Nx(t-s)ds$$

where P and N are symmetric positive definite matrices. It is straightforward to show that

$$\dot{V}(x,t) = \begin{bmatrix} x(t)' & x(t-\tau)' \end{bmatrix} \times \begin{bmatrix} (A + D\Delta E_1)'P + P(A + D\Delta E_1) + N & P(F + D\Delta E_3) \\ (F + D\Delta E_3)'P & -N \end{bmatrix} \times \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}. \quad (2.2)$$

If $\dot{V} < 0$, then $x(t) \to 0$ as $t \to \infty$. This will guarantee the asymptotic stability of uncertain time-delay system (2.1).

Definition 2.1 The uncertain time-delay system (2.1) is quadratically stable if there exist $P > 0$ and $N > 0$, such that:

$$\begin{bmatrix} (A + D\Delta E_1)'P + P(A + D\Delta E_1) + N & P(F + D\Delta E_3) \\ (F + D\Delta E_3)'P & -N \end{bmatrix} < 0 \quad (2.3)$$

for all Δ satisfying $\Delta'\Delta \leq I$.

If an uncertain time-delay system is quadratically stable, it is straightforward to show that for any admissible uncertainty $\Delta(t)$, the resulting realization of uncertain time-delay system will be asymptotically stable.

The next theorem reduces the question of quadratic stability of the system (2.1) to the solvability of a quadratic matrix inequality.

Theorem 2.1 *The uncertain time-delay system (2.1) is quadratically stable, if and only if, there exist symmetric $P > 0, N > 0$ and $\epsilon > 0$ such that:*

$$\begin{bmatrix} \begin{pmatrix} A'P + PA + \epsilon PDD'P \\ +\frac{1}{\epsilon}E_1'E_1 + N \end{pmatrix} & PF + \frac{1}{\epsilon}E_1'E_3 \\ F'P + \frac{1}{\epsilon}E_3'E_1 & -N + \frac{1}{\epsilon}E_3'E_3 \end{bmatrix} < 0. \quad (2.4)$$

Proof The proof is given in the complete version of the paper. □

Now, suppose P and N are positive definite solutions of quadratic matrix inequality (2.4) for some value of ϵ. Hence $\bar{P} = \epsilon P$ and $\bar{N} = \epsilon N$ satisfy

$$\begin{bmatrix} \begin{pmatrix} A'\bar{P} + \bar{P}A + \bar{P}DD'\bar{P} \\ +E_1'E_1 + \bar{N} \end{pmatrix} & \bar{P}F + E_1'E_3 \\ F'\bar{P} + E_3'E_1 & -\bar{N} + E_3'E_3 \end{bmatrix} < 0. \quad (2.5)$$

Also, from this it follows that $\bar{N} - E_3'E_3 > 0$. Conversely, if condition (2.5) is satisfied, then it follows that (2.4) is satisfied with $\epsilon = 1$. Thus, in order to check the quadratic stability of uncertain time-delay system (2.1), we need to find positive definite matrices $\bar{P}$ and $\bar{N}$ such that (2.5) is satisfied. Using Schur complements, quadratic matrix inequality (2.5) will be equivalent to the inequality

$$\begin{bmatrix} A'\bar{P} + \bar{P}A + E_1'E_1 + \bar{N} & \bar{P}D & \bar{P}F + E_1'E_3 \\ D'\bar{P} & -I & 0 \\ F'\bar{P} + E_3'E_1 & 0 & -\bar{N} + E_3'E_3 \end{bmatrix} < 0,$$

which is an LMI in the matrix variables $\bar{P}$ and $\bar{N}$. Therefore the problem of determining the quadratic stability of the uncertain time-delay system (2.1) can be considered as an LMI feasibility problem. This problem is convex and can be effectively solved by corresponding LMI software, e.g. see [11]. If this problem is infeasible, it necessarily means that system (2.1) is not quadratically stable.

Considering the Strict Bounded Real Lemma, (Theorem 2.1 in [9]), it is possible to state quadratic stability condition (2.5) in terms of a Riccati equation or an H_∞ norm condition.

Theorem 2.2 *The following statements are equivalent:*
(I) The uncertain time-delay system (2.1) is quadratically stable.
(II) The LMI

$$\begin{bmatrix} A'P + PA + E_1'E_1 + N & PD & PF + E_1'E_3 \\ D'P & -I & 0 \\ F'P + E_3'E_1 & 0 & -N + E_3'E_3 \end{bmatrix} < 0,$$

has a positive definite solution P and N.
(III) There exists a matrix $N > E_3'E_3$ such that the Riccati equation

$$(A + F(N - E_3'E_3)^{-1}E_3'E_1)'P +$$
$$P(A + F(N - E_3'E_3)^{-1}E_3'E_1) +$$
$$P(DD' + F(N - E_3'E_3)^{-1}F')P +$$
$$E_1'(I + E_3(N - E_3'E_3)^{-1}E_3')E_1 + N = 0$$

has a positive definite stabilizing solution P.
(IV) There exists a matrix $N > E_3'E_3$ such that $A + F(N - E_3'E_3)^{-1}E_3'E_1$ is stable and

$$\left\| \begin{bmatrix} E_1 \\ N^{1/2} \\ (N - E_3'E_3)^{-1/2}E_3'E_1 \end{bmatrix} (sI - A - F(N - E_3'E_3)^{-1}E_3'E_1)^{-1} \begin{bmatrix} D & F(N - E_3'E_3)^{-1/2} \end{bmatrix} \right\|_\infty < 1.$$

Proof The proof is established by applying Schur complements to the matrix inequality (2.5) and then using the Strict Bounded Real Lemma. □

Observation 2.1 The three conditions

$$(i) \ A \text{ is a stability matrix},$$

$$(ii) \ \|E_1(sI - A)^{-1}D\|_\infty < 1,$$

(iii) $\|N^{1/2}(sI - A)^{-1}FN^{-1/2}\|_\infty < 1, \quad N > 0$

are necessary for quadratic stability of system (2.1). For the third condition we assume that $E_1 = E_3 = 0$

Observation 2.2 Consider the case that $E_1 = E_3 = 0$. In this case, (2.1) represents a linear time-delay system without uncertainty. By using the result of Theorem 2.2, this time-delay system is quadratically stable if and only if A is stable and

$$\| N^{1/2}(sI - A)^{-1}FN^{-1/2} \|_\infty < 1.$$

for some $N > 0$. Therefore quadratic stability of delay system can be considered as a scaled H_∞ analysis problem with scaling matrix $N > 0$.

Now assume that the system (2.1) is quadratically stable, then the next step is to consider the cost function

$$J = \int_0^\infty x'(t)R_1x(t)dt, \tag{2.6}$$

with $R_1 > 0$ as a performance measure for this system. We are interested in finding the least upper bound for this cost function.

Theorem 2.3 *Consider the system (2.1) and cost function (2.6) and suppose that there exist $P > 0$ and $N > 0$ such that*

$$\begin{bmatrix} \begin{pmatrix} (A+D\Delta E_1)'P + P(A+D\Delta E_1) \\ +N + R_1 \end{pmatrix} & P(F+D\Delta E_3) \\ (F+D\Delta E_3)'P & -N \end{bmatrix} < 0, \tag{2.7}$$

for all Δ such that $\Delta'\Delta \leq I$. Then the system (2.1) is quadratically stable and the cost function (2.6) satisfies the bound

$$J \leq x_0'Px_0 + \int_{-\tau}^0 x'(s)Nx(s)ds. \tag{2.8}$$

for all $\Delta(t)$ satisfying $\Delta(t)'\Delta(t) \leq I$.
Conversely, if the system (2.1) is quadratically stable, then given any $R_1 > 0$, there will exist $\hat{P} > 0$ and $\hat{N} > 0$ such that (2.7) is satisfied.

Proof The proof is given in the complete version of the paper. □

Now if we consider the result of Theorem 2.1, matrix inequality (2.7) is equivalent to the existence of positive definite matrices P and N, such that

$$\begin{bmatrix} \begin{pmatrix} A'P + PA + \\ \frac{1}{\epsilon}E_1'E_1 + N + R_1 \end{pmatrix} & PD & PF + \frac{1}{\epsilon}E_1'E_3 \\ D'P & -\frac{1}{\epsilon}I & 0 \\ F'P + \frac{1}{\epsilon}E_3'E_1 & 0 & -N + \frac{1}{\epsilon}E_3'E_3 \end{bmatrix} < 0, \tag{2.9}$$

for some value of $\epsilon > 0$. In this case, Theorem 2.3 suggests a corresponding upper bound for cost function (2.6). This bound is a linear and convex function in matrix variables P and N. Hence, the problem of finding the least upper bound for cost function J amongst all the possible choices of P, N and ϵ which satisfy (2.9) will be an LMI Eigenvalue Problem, see [12]. This problem is a convex optimization problem and is stated as follows:

$$\text{minimize } x_0'Px_0 + \int_{-\tau}^0 x'(s)Nx(s)ds$$

subject to (2.9) and $P > 0, \ N > 0, \ \epsilon > 0$.

The solution of this problem gives the global minimum for upper bound (2.8).

Remark 2.1 Suppose that the system (2.1) is quadratically stable and cost function (2.6) is defined with some weighting matrix R_1. Then the minimum of the upper bound (2.8) is achieved for some value of the parameters P, N and ϵ, e.g. P_1, N_1, ϵ_1. Using Schur complements, we can relate the Riccati inequality

$$\begin{aligned} &(A + F(\epsilon N - E_3'E_3)^{-1}E_3'E_1)'P + P(A + F(\epsilon N - E_3'E_3)^{-1}E_3'E_1) + \\ &\epsilon P(DD' + F(\epsilon N - E_3'E_3)^{-1}F')P + \\ &\frac{1}{\epsilon}E_1'(I + E_3(\epsilon N - E_3'E_3)^{-1}E_3')E_1 + N + R_1 < 0 \end{aligned} \tag{2.10}$$

to the LMI (2.9). Since we have assumed that the uncertain time-delay system (2.1) is quadratically stable, there will exist some $P_1 > 0$ which satisfy (2.9) and (2.10) for the corresponding values N_1 and ϵ_1. Now if we apply the Strict Bounded Real Lemma of [9], it follows that the Riccati equation corresponding to (2.10) will have a stabilizing solution $\tilde{P}_1 < P_1$. In this case, if N_1 and R_1 are positive definite, then $\tilde{P}_1$ is also positive definite. Hence, the minimum in LMI Eigenvalue Problem defined above occurs at P_1 which tends to the value of $\tilde{P}_1$.

3 Quadratic Stabilization of Uncertain Time-Delay Systems

In this section we consider the problem of designing state feedback quadratically stabilizing controllers for uncertain time-delay systems. The system under consideration is described by the state equation:

$$\begin{aligned} \dot{x}(t) = (A + D\Delta(t)E_1)x(t) + (F + D\Delta(t)E_3)x(t-\tau) + \\ (B + D\Delta(t)E_2)u(t). \end{aligned} \tag{3.1}$$

For this system, $u(t) = Kx(t)$ is a quadratically stabilizing controller, if the closed loop uncertain time-delay system

$$\begin{aligned} \dot{x}(t) = [A + BK + D\Delta(t)(E_1 + E_2K)]x(t) + \\ (F + D\Delta(t)E_3)x(t-\tau). \end{aligned} \tag{3.2}$$

is quadratically stable. Based on Definition 2.1, this requires that there exist $P > 0, \ N > 0$ such that

$$\begin{bmatrix} \begin{pmatrix} (A+BK)'P + P(A+BK) + \\ (D\Delta(E_1+E_2K))'P + \\ P(D\Delta(E_1+E_2K)) + \\ N \end{pmatrix} & P(F+D\Delta E_3) \\ (F+D\Delta E_3)'P & -N \end{bmatrix} < 0,$$

for all Δ satisfying $\Delta'\Delta \leq I$.

Theorem 2.2 implies that $Kx(t)$ is quadratically stabilizing if and only if there exist $P > 0$, and $N > 0$, such that quadratic matrix inequality

$$\begin{bmatrix} \begin{pmatrix} (A+BK)'P+ \\ P(A+BK)+ \\ PDD'P+N+ \\ (E_1+E_2K)'\times \\ (E_1+E_2K) \end{pmatrix} & \begin{pmatrix} PF+ \\ (E_1+E_2K)'E_3 \end{pmatrix} \\ \begin{pmatrix} F'P+ \\ E_3'(E_1+E_2K) \end{pmatrix} & -N+E_3'E_3 \end{bmatrix} < 0, \tag{3.3}$$

is satisfied. This problem is not jointly convex in matrix variables P, N and K. However, using the method of changing the variables as in [12], we can obtain an equivalent condition which is an LMI in its matrix variables.

First, we multiply every block of the above matrix inequality on the left and on the right by P^{-1}. Then, we define new matrix variables $Q = P^{-1}$, $Y = KP^{-1}$ and $M = P^{-1}NP^{-1}$. Hence we obtain the condition

$$\begin{bmatrix} \begin{pmatrix} QA'+AQ+ \\ BY+Y'B'+ \\ DD'+M+ \\ (E_1Q+E_2Y)'\times \\ (E_1Q+E_2Y) \end{pmatrix} & \begin{pmatrix} FQ+ \\ (E_1Q+E_2Y)'E_3Q \end{pmatrix} \\ \begin{pmatrix} QF'+ \\ QE_3'(E_1Q+E_2Y) \end{pmatrix} & -M+QE_3'E_3Q \end{bmatrix} < 0.$$

Next, by applying Schur complements and a standard matrix inversion formula, (see [13]), it follows that the above matrix inequality is equivalent to the inequality

$$\begin{bmatrix} \begin{pmatrix} QA'+AQ+ \\ BY+Y'B'+ \\ DD'+M \end{pmatrix} & FQ & (E_1Q+E_2Y)' \\ QF' & -M & QE_3' \\ (E_1Q+E_2Y) & E_3Q & -I \end{bmatrix} < 0. \tag{3.4}$$

Since this matrix inequality is an LMI in matrix variables $Q > 0$, $M > 0$ and Y, the quadratic stabilization of uncertain time-delay system (3.1) can be considered as an LMI feasibility problem. Whenever this problem is infeasible, the uncertain time-delay system (3.1) is not quadratically stabilizable by memoryless state feedback.

Observation 3.1 We have proved that the uncertain time-delay system (3.1) is quadratically stabilizable if and only if the matrix inequality (3.3) has a solution. We now consider some necessary conditions for the existence of such solution. Clearly it is required that the pair (A, B) is stabilizable. Now assume that $E_3 = 0$ and E_2 is full rank. Hence we have an uncertain time-delay system as follows:

$$\dot{x}(t) = (A + D\Delta(t)E_1)x(t) + Fx(t-\tau) + (B + D\Delta(t)E_2)u(t). \tag{3.5}$$

For this system, matrix inequality (3.3) is equivalent to the inequality

$$(A+BK)'P + P(A+BK) + PDD'P + PFN^{-1}F'P +$$
$$(E_1 + E_2K)'(E_1 + E_2K) + N < 0.$$

If we use the method of completing the square, we obtain the Riccati inequality

$$(A - B(E_2'E_2)^{-1}E_2'E_1)'P + P(A - B(E_2'E_2)^{-1}E_2'E_1)+$$
$$PDD'P + PFN^{-1}F'P - PB(E_2'E_2)^{-1}B'P+$$
$$E_1'\{I - E_2(E_2'E_2)^{-1}E_2'\}E_1 + N < 0, \tag{3.6}$$

which is required to satisfy for $N > 0$. Now assume that for some $N > 0$, this Riccati inequality has a positive definite solution P. Hence, the corresponding stabilizing controller will be

$$K = -(E_2'E_2)^{-1}[B'P + E_2'E_1].$$

Thus, (3.5) is quadratically stabilizable, if and only if, (3.6) has a positive definite solution P for some value of $N > 0$. This condition is also a necessary condition for quadratic stabilization of (3.1). We can also relax the full rank assumption on E_2. This leads to a more complicated expression in place of the inequality (3.6).

Now assume that (3.6) is satisfied for some value of $P > 0$ and $N > 0$. We note that $I - E_2(E_2'E_2)^{-1}E_2'$ is positive semi-definite. Therefore, using some results on the solution of algebraic Riccati equation and Riccati inequality, (see [10]), it is proved that the algebraic Riccati equation

$$(A - B(E_2'E_2)^{-1}E_2'E_1)'P + P(A - B(E_2'E_2)^{-1}E_2'E_1)+$$
$$PDD'P + PFN^{-1}F'P - PB(E_2'E_2)^{-1}B'P+$$
$$E_1'\{I - E_2(E_2'E_2)^{-1}E_2'\}E_1 + N = 0,$$

will have a positive definite stabilizing solution. However, we need to search over $N > 0$ to find if this algebraic Riccati equation has a suitable solution. In a particular case, if we assume that $N = \delta I$, this search is reduced to a search over real variable δ. This case is similar to the result of [8]. In our approach we do not require any assumption on the form of N. Moreover, we carry out the required numerical search by solving the corresponding LMI problem.

We now consider the problem of finding a suitable feedback controller K, which not only quadratically stabilizes the system (3.1), but also guarantees some level of performance for closed loop system. Similar to our results in section 2, associated with system (3.1), we consider the cost function

$$J = \int_0^\infty \{x'(t)R_1x(t) + u'(t)R_2u(t)\}dt \tag{3.7}$$

where R_1 and R_2 are positive definite matrices.

Theorem 3.1 *Consider the system (3.1) and cost function (3.7) and suppose that there exist matrices $Q > 0$, $M > 0$ and Y such that*

$$\begin{bmatrix} \begin{pmatrix} Q(A+D\Delta E_1)'+ \\ (A+D\Delta E_1)Q+ \\ (B+D\Delta E_2)Y+ \\ Y'(B+D\Delta E_2)'+ \\ M+QR_1Q+Y'R_2Y \end{pmatrix} & (F+D\Delta E_3)Q \\ Q(F+D\Delta E_3)' & -M \end{bmatrix} < 0, \tag{3.8}$$

for all Δ such that $\Delta'\Delta \le I$. Then the control law $u(t) = Kx(t)$, $K = YQ^{-1}$, quadratically stabilizes the uncertain time-delay system (3.1). Furthermore, the corresponding value of the cost (3.7) satisfies the bound

$$J \le x_0'Q^{-1}x_0 + \int_{-\tau}^{0} x'(s)Q^{-1}MQ^{-1}x(s)ds. \tag{3.9}$$

for all admissible $\Delta(t)$ satisfying $\Delta'(t)\Delta(t) \le I$.
Conversely, for any control law $u(t) = Kx(t)$ such that the resulting closed loop system (3.2) is quadratically stable and any given weighting matrices R_1 and R_2 in the cost function (3.7), there exist positive definite matrices $\hat{Q}$ and $\hat{M}$ that satisfy (3.8) with $\hat{Y} = K\hat{Q}$.

Proof The proof is given in the complete version of the paper. □

The matrix inequality (3.8) is equivalent to the LMI

$$\begin{bmatrix} QA'+AQ+BY+Y'B'+\epsilon DD'+M & FQ & (E_1Q+E_2Y)' & QR_1^{1/2} & Y'R_2^{1/2} \\ QF' & -M & QE_3' & 0 & 0 \\ (E_1Q+E_2Y) & E_3Q & -\epsilon I & 0 & 0 \\ R_1^{1/2}Q & 0 & 0 & -I & 0 \\ R_2^{1/2}Y & 0 & 0 & 0 & -I \end{bmatrix} < 0. \tag{3.10}$$

In contrast to our result in section 2, here, the corresponding upper bound for cost function (3.9) is apparently not a convex function in Q and M. Hence, finding the minimum of this upper bound can not be considered as an LMI Eigenvalue Problem. Instead, we propose another method to find a sub-optimal value for this bound.

First, we consider the LMI Eigenvalue Problem:

minimize $x_0'Q^{-1}x_0$

subject to (3.10) and $Q > 0, M > 0, \epsilon > 0, \text{trace}(M) = \gamma$.

Here we add an additional constraint for the trace of M. Then we vary this γ in a certain domain. For each value of γ, we find the solution of the above LMI Eigenvalue Problem. Using this solution, we calculate the upper bound (3.9). Therefore we consider this bound as a function of γ and we select its minimum as a sub-optimal value for the corresponding upper bound of the cost function (3.7). It should be noted that in some problems, this minimum indeed tends to the global minimum of this upper bound as we see in our example in the next section.

4 Illustrative Example

In this section, we present one example to illustrate our stabilization approach developed in sections 2-3.

Example Consider the uncertain time-delay system

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 0 & -2+\Delta(t) \end{bmatrix} x(t) +$$

$$\begin{bmatrix} 0 & 0 \\ 0.5+\Delta(t) & \Delta(t) \end{bmatrix} x(t-\tau) + \begin{bmatrix} -1 \\ 2+0.5\Delta(t) \end{bmatrix} u(t),$$

where $\tau = 1$, $x_1(t) = e^{t+1}$ and $x_2(t) = 0$ for $t \in [-1, 0]$. We wish to construct a suitable quadratically stabilizing controller for this system such that a corresponding upper bound for the cost

$$J = \int_0^{\infty} (x_1(t)^2 + x_2(t)^2 + u(t)^2)dt$$

is obtained. Thus, we apply our approach to find the sub-optimal state feedback controller. We have:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \; F = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \end{bmatrix}, \; B = \begin{bmatrix} -1 \\ 2 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; E_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \; E_2 = 0.5, \; E_3 = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

First we assume that $E_2 = E_3 = 0$. In this case our example will be the same as the example in [8]. Our approach in section 3 results in the following plot for upper bound (3.9) with respect to the trace(M) in Figure 4.1. The minimum occurs at trace(M)=0.26 and the corre-

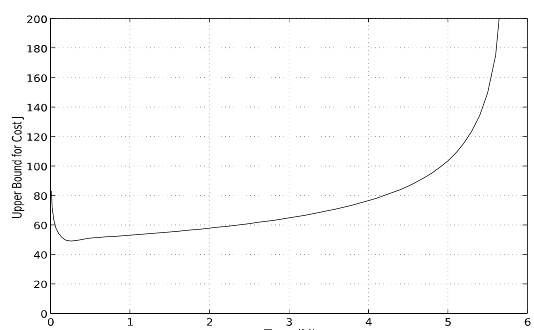

Figure 4.1: Upper Bound for Closed loop Cost Versus Trace(M)

sponding value for upper bound (3.9) is 49.196. Other variables are as follows:

$$Q = \begin{bmatrix} 0.244 & -0.480 \\ -0.480 & 2.716 \end{bmatrix}, M = \begin{bmatrix} 0.053 & -0.105 \\ -0.105 & 0.207 \end{bmatrix},$$

$$Y = \begin{bmatrix} 1.0000 & -2.0000 \end{bmatrix},$$

$$\epsilon = 1.364, \text{ and } K = \begin{bmatrix} 4.056 & -0.020 \end{bmatrix}.$$

It should be noted that the stabilizing feedback and the corresponding value of upper bound are similar to the result of [8]. The definition of quadratic stability for this class of uncertain time-delay systems ($E_3 = 0$) in [8] can be considered as a special case of our definition. Therefore, for this example we will obtain similar results. It is proved that these results are optimal in the sense of the definition given in [8].

Now we consider the case that E_2 and E_3 are not zero. Figure 4.2 shows the result for this case. The minimum

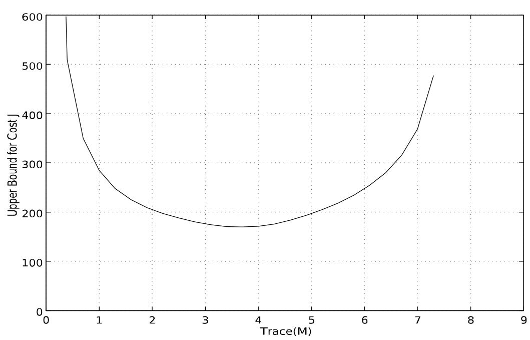

Figure 4.2: Upper Bound for Closed Loop Cost Versus Trace(M)

occurs for trace(M)=3.66 and the corresponding value of the upper bound is 169.605. Other variables are

$$Q = \begin{bmatrix} 0.199 & -0.622 \\ -0.622 & 2.543 \end{bmatrix} M = \begin{bmatrix} 0.149 & -0.719 \\ -0.719 & 3.511 \end{bmatrix}$$

$$Y = \begin{bmatrix} 1.110 & -2.812 \end{bmatrix},$$

$$\epsilon = 1.750, \text{ and } K = \begin{bmatrix} 9.027 & 1.101 \end{bmatrix}.$$

5 Conclusions

The main idea in this paper is the equality between the quadratic stabilization of linear uncertain time-delay systems and the feasibility of an LMI problem. We have also considered a linear quadratic performance measure for closed loop stable system. Using the LMI formulation, we select a suitable memoryless state feedback controller that guarantees a sub-optimal value for the upper bound of this performance measure.

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