Searching for Diophantine quintuples

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Abstract

We consider Diophantine quintuples $\{a, b, c, d, e\}$. These are sets of positive integers, the product of any two elements of which is one less than a perfect square. It is conjectured that there are no Diophantine quintuples; we improve on current estimates to show that there are at most $5.441 \cdot 10^{26}$ Diophantine quintuples.

1 Introduction

Define a Diophantine m-tuple as a set of m positive integers $\{a_1, \ldots, a_m\}$ with $a_1 < a_2 < \ldots < a_m$, such that $a_i a_j + 1$ is a perfect square for all $1 \le i < j \le m$. Throughout the rest of this article we frequently refer to m-tuples, and not to Diophantine m-tuples.

It is conjectured that there are no quintuples — see [2, 15]. Successive authors (see, e.g., Table 1 in [18]) have reduced the bound on the possible number of quintuples. The best such published bound is $2.3 \cdot 10^{29}$ by Trudgian [18]. The purpose of this paper is to improve on this in the following theorem.

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Theorem 1. *There are at most* $5.441 \cdot 10^{26}$ *Diophantine quintuples.*

In §2 we collect some ancillary results that aid the computational search for quintuples. In §3 we obtain bounds on the relative sizes of elements in a quintuple. We use this in §4 with results on linear forms of logarithms to obtain upper bounds on the second-largest element in a quintuple. In §5 we examine some number-theoretic sums, which enable us to bound the total number of quintuples. We present two new arguments in §6 that enable us to make a further saving, and ultimately to prove Theorem 1.

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2 Discards

It is known that every triple $\{a, b, c\}$ can be extended to a quadruple of a certain form. This is dubbed the 'regular' quadruple and is denoted as $\{a, b, c, d_+\}$. If a double or a triple cannot be extended to a non-regular quadruple, then it cannot be extended to a quintuple. We call such doubles or triples *discards*. The doubles $\{k, k+2\}$ [13] (see also [4]) are discards for $k \geq 1$. For an extensive list of discards, one may see [18, §2.1]. The following result allows us to recognise many discards.

Lemma 2.1. *Let* $\{a, b, c, d\}$ *be a Diophantine quadruple with* $a < b < c < d_+ < d$.

- *If* $b < 2\,a$ *then* $b > 21000$.
- *If* $2\,a \leq b \leq 12\,a$ *then* $b > 130000$.
- *If* $b > 12\,a$ *then* $b > 4001$.

Proof. The only difference between this lemma and Lemma 3.4 in [6] is the exclusion of the value $b = 4001$ in the last case. Indeed, a pair $\{a, 4001\}$ with $12a < 4001$ cannot be extended because the equation $4001a + 1 = r^2$ has unique integer solution $r < 4001$, namely $r = 4000$, which entails $a = 3999$. □

Lemma 2.2. ([7, Theorems 1.1, 1.2]) and ([6, Theorem 1.1]) *Let* $\{a, b, c, d, e\}$ *be a quintuple with* $a < b < c < d < e$ *and put* $g = \gcd(a, b)$. *Then* $b > 3ag$. *If moreover* $c > a + b + 2\sqrt{ab+1}$ *then* $b > \max\{24\,ag, 2\,a^{3/2}g^2\}$.

Lemma 2.3. ([6, Theorem 1.3]) *Let* $\{a, b, c, d, e\}$ *be a quintuple with* $a < b < c < d < e$ *and* $c = a + b + 2\sqrt{ab+1}$. *Then* $b < a^3$ *and* $\gcd(b, c) = 1$. *In particular, at least one of* a, b *is odd.*

Examination of the relative size of entries in a quintuple has the following outcome.

Lemma 2.4. *Any quintuple* $\{a, b, c, d, e\}$ *with* $a < b < c < d < e$ *must be of one of the types listed below:*

(A) $4a < b$ *and* $4ab + b + a < c < b^{3/2}$,

(B) $4a < b$ *and* $c = a + b + 2\sqrt{ab+1}$,

(C) $4a < b$ *and* $c > b^{3/2}$,

(D) $b < 4a$ *and* $c = a + b + 2\sqrt{ab+1}$.

Proof. According to [12, Lemma 4.2] or [5, Lemma 2.1], a Diophantine quintuple which is not of the kind described in the present lemma satisfies either $d > b^5$ or $c = 4r(r-a)(b-r) < b^3$, where $r = \sqrt{ab+1}$. The existence of quintuples of the former type is prohibited by Theorem 1.1 in [5], while the latter type is excluded in Subsection 2.2 from [18] with the help of Lemma 2.2 above. □

3 Exploiting the connection with Pellian equations

The entries in a quadruple are severely restricted in that they appear as coefficients of three generalized Pell equations that must have at least one common solution in positive integers. Each component of such a solution is obtained as a common term of two second-order linearly recurrent sequences, giving rise to relations of the type $z = v_m = w_n$ for some positive integers m and n. A key ingredient in the study of Diophantine sets is a relationship between the parameters m, n, and the values in the set in question.

Our next result is of this kind. It improves on several versions already in the literature — see, e.g., [5, 18, 20].

Proposition 3.1. *Let* $\{A, B, C, D\}$ *be a quadruple with* $A < B < C < D$ *for which* $v_{2m} = w_{2n}$ *has a solution with* $2n \geq m \geq n \geq 2$, $m \geq 3$. *Suppose that one has* $v_0 = w_0 = \varepsilon$, $v_1 = C + Sv_0$, $w_0 = C + Tw_0$, *where* $\varepsilon = \pm 1$, $S = \sqrt{AC+1}$ *and* $T = \sqrt{BC+1}$. *Assume further that* $A \geq A_0$, $B \geq B_0$, $C \geq C_0$, $B > \rho A$ *for some positive integers* A_0, B_0, C_0, *and a real number* $\rho > 1$. *Then*

$$m > \alpha B^{-1/2} C^{1/2},$$

where α is any real number satisfying both inequalities

$$\alpha^2 + \left(1 + \tfrac{1}{2}B_0^{-1}C_0^{-1}\right)\alpha \le 4, \tag{1}$$

$$3\alpha^2 + \left(4B_0(\lambda + \rho^{-1/2}) + 2(\lambda + \rho^{1/2})C_0^{-1}\right)\alpha \le 4B_0, \tag{2}$$

with $\lambda = (A_0+1)^{1/2}(\rho A_0 + 1)^{-1/2}$.

Moreover, if $C^\tau \ge \beta B$ for some positive real numbers β and τ then

$$m > \alpha\beta^{1/2}C^{(1-\tau)/2}.$$

Proof. We assume that $m \le \alpha B^{-1/2}C^{1/2}$ and aim at establishing a contradiction if α is too small. We use a method involving congruences, which was introduced in [11]. We start from the congruence (see, e.g., [9, Lemma 4])

$$\varepsilon A m^2 + Sm \equiv \varepsilon B n^2 + Tn \pmod{4C}. \tag{3}$$

Since

$$|Am^2 - Bn^2| < \max\{Am^2, Bn^2\} \le Bm^2 \le \alpha^2 C$$

and

$$\begin{aligned}|Sm - Tn| &< \max\{Sm, Tn\} \le Tm \le \alpha B^{-1/2}C^{1/2}\sqrt{BC+1}\\ &< \alpha B^{-1/2}C^{1/2}\left(B^{1/2}C^{1/2} + \tfrac{1}{2}B^{-1/2}C^{-1/2}\right) \le \alpha\left(1 + \tfrac{1}{2}B_0^{-1}C_0^{-1}\right)C,\end{aligned}$$

then, if α satisfies (1), the congruence (3) becomes the equality $Am^2 - Bn^2 = \varepsilon(Tn - Sm)$. Multiplication by $Tn + Sm$ followed by rearrangements results in the equality

$$(Bn^2 - Am^2)\left(C + \varepsilon(Tn + Sm)\right) = m^2 - n^2. \tag{4}$$

Note that $Bn^2 = Am^2$ entails $m^2 = n^2$, so that $A = B$: a contradiction. Hence, for $m = n$ one necessarily has $C = Tn + Sm$, while for $m > n$ one finds that $Bn^2 - Am^2$ divides the positive integer $m^2 - n^2$, so that $m^2 - n^2 \ge |Am^2 - Bn^2|$. This gives the following inequality

$$\frac{m^2}{n^2} \ge \frac{B+1}{A+1}.$$

Having in view the lower bounds for A and B, we obtain

$$\frac{m^2}{n^2} > \frac{\rho A + 1}{A+1} \ge \frac{\rho A_0 + 1}{A_0 + 1} = \frac{1}{\lambda^2}.$$

From (4), $m \le 2n$, and the definitions of S and T, we conclude that

$$\begin{aligned} C &\le Tn + Sm + m^2 - n^2 < \lambda m\sqrt{BC+1} + m\sqrt{AC+1} + \tfrac{3}{4}\,m^2 \\ &\le \tfrac{3}{4}\,\alpha^2 B^{-1}C + \alpha B^{-1/2}C^{1/2}\left(\lambda\sqrt{BC+1} + \sqrt{\rho^{-1}BC+1}\right) \\ &< \tfrac{3}{4}\,\alpha^2 B^{-1}C + \alpha C\left(\lambda\big(1+\tfrac{1}{2}B^{-1}C^{-1}\big) + \rho^{-1/2}\big(1+\tfrac{1}{2}\rho B^{-1}C^{-1}\big)\right) \\ &\le \tfrac{3}{4}\,\alpha^2 B_0^{-1}C + \alpha C\left(\lambda\big(1+\tfrac{1}{2}B_0^{-1}C_0^{-1}\big) + \rho^{-1/2}\big(1+\tfrac{1}{2}\rho B_0^{-1}C_0^{-1}\big)\right). \end{aligned}$$

The last expression is at most C if α satisfies the inequality (2), whence the first inequality in the conclusion of our proposition. The second one is readily obtained from what we have just proved and the hypothesis $C^\tau \ge \beta B$. □

Lemma 3.1. *If $\{a,b,c,d,e\}$ is a quintuple with $a<b<c<d<e$ then the following bounds for m hold:*
(A) $m > 3.3022d^{1/4}$, (B) $m > 1.5002d^{2/7}$, (C) $m > 2.0604d^{3/10}$,
(D) $m > 1.0080d^{1/3}$.

Proof. This is an application of the result just proved for $(A,B,C) = (a,b,d)$ in cases (A)–(C) and for $(A,B,C) = (a,c,d)$ in the remaining case. We use Proposition 3.1 with carefully chosen values for parameters in ranges suggested by Lemmas 2.1–2.3. To do so we first require the existence of a solution $v_{2m} = w_{2n}$ subject to hypotheses of Proposition 3.1 — this was shown in [14]. It is also known that one has $d > 4abc + a + b + c$ (see, for instance, the proof of Lemma 6 in [10]).

In case (A) Lemma 2.2 hints to consider separately values of a less than 144 since then one has $B = b > \max\{24a, 2a^{3/2}\} = 24a = 24A$. We conduct a short computer search to find potential triples. For example, after exploring the domain $1 \le a \le 143$, $4002 \le b \le 21000$ we know that there can be no triple with $a \le 143$, $b \le 4094$, and $4ab < c < b^{1.5}$ but, since $4095 + 1 = 64^2$, $139128 + 1 = 373^2$, $4095 \cdot 139128 + 1 = 23869^2$, and $4 \cdot 4095 + 4095 + 1 < 139128 < 4095^{1.5}$ we conclude that $B_0 = 4095$. Similarly, we find that $\frac{b}{a} \ge \frac{4095}{8} > 511$ in the same domain. For the unexplored region where $b \ge 21001$, the minimum value of the fraction b/a is obviously at least $21001/143 > 146$, so that we can safely consider $\rho = 146$. Clearly we must put $A_0 = 1$. From

$$C = d > 4abc + a + b + c > (4ab+1)(4ab+a+b) > (16a^2+4a)b^2$$

it follows that $\tau = 1/2$, $\beta = (16A_0^2 + 4A_0)^{1/2}$, $C_0 > 3.35 \cdot 10^8$ are admissible choices. Both inequalities (1) and (2) are satisfied by $\alpha = 1.56155$.

Still in case (A), when $a \geq 144$ one puts $A_0 = 144$, $B_0 = 4002$ (by Lemma 2.1), $\rho = 24$ (see Lemma 2.2), $\tau = 1/2$, $\beta = (16A_0^2 + 4A_0)^{1/2}$, whence $C_0 > 5.32 \cdot 10^{12}$ and $\alpha = 1.56155$.

Having in view Lemma 2.1, in case (B) we first examine the subcase $4a < b \leq 12a$. Then $B_0 = 130001$, which implies $A_0 = 10834$ and $\rho = 4$. From

$$c > b\big(1 + 12^{-1} + 2 \cdot 12^{-1/2}\big) = (1 + 12^{-1/2})^2 B$$

and $a^3 > b$ it follows that

$$C = d > (4ab + 1)(a + b + 2r) > 4(1 + 12^{-1/2})^2 ab^2 > 4(1 + 12^{-1/2})^2 B^{7/3},$$

so that $\tau = 3/7$, $\beta = (2 + 3^{-1/2})^{6/7}$, $C_0 = 5.68 \cdot 10^{12}$. For these choices it is readily obtained that $\alpha = 0.9999$ is permissible.

The other possibility in case (B) is to have $b > 12a$. Convenient values of parameters are $\rho = 12$, $A_0 = 16$ (from $a^3 > b > 4000$), $B_0 = 4002$, $\tau = 3/7$, $\beta = 2^{6/7}$, $C_0 = 1.01 \cdot 10^9$. for which the same value $\alpha = 0.9999$ works.

Case (C) is similar to case (A). Now, for $a \leq 143$ we see that we can take $A_0 = 1$, $B_0 = 4004$, $\rho = 28$. As

$$C > 4abc > 4ab^{5/2} > 4.05 \cdot 10^9 =: C_0,$$

we further get $\tau = 2/5$, $\beta = 4^{2/5}$, whence again $\alpha = 1.56155$. In the complementary subcase $a \geq 144$, admissible values are $A_0 = 144$, $B_0 = 4002$, $\rho = 24$, $\tau = 2/5$, $\beta = 576^{2/5}$, $C_0 = 5.83 \cdot 10^{11}$. Plugging these specializations into Proposition 3.1, we obtain the same value for α.

Finally, in case (D) we have $A = a < b/3$, $B = c = a + b + 2\sqrt{ab+1} > (1 + 3^{1/2})^2 A$, $B \leq a + b + 2\sqrt{3^{-1}b(b-1)+1} < (1 + 3^{-1/2})^2 b$, and

$$C = d > 4abc > b^2 c > (1 + 3^{-1/2})^{-4} B^3.$$

Therefore, $\rho = (1+3^{1/2})^2$, $\tau = 1/3$, and $\beta = (1+3^{-1/2})^{-4/3}$. From $130001 \leq b < 4a$, we have $A_0 = 32501$, whence $B_0 > 292504$, and $C_0 > 4.04 \cdot 10^{15}$. From (1) and (2) we obtain $\alpha = 1.3660$. □

For future reference, the values used in the previous proof are given in Table 1.

The values of α, and hence the bounds on m in Lemma 3.1, rely on the computational bounds in Lemma 2.1. While it is tempting to extend these computations, such an extension would have almost no effect on the values of α. Consider, for example, case (A): sending B_0, C_0 to infinity in (1) gives $\alpha^2 + \alpha \leq 4$. Therefore the optimal value of α is $1.5615528\ldots$,

Type	A_0	B_0	C_0	ρ	β	τ
(AI)	1	4095	$3.35 \cdot 10^8$	146	$20^{1/2}$	1/2
(AII)	144	4002	$5.32 \cdot 10^{12}$	24	$24 \cdot 577^{1/2}$	1/2
(BI)	10834	130001	$5.68 \cdot 10^{12}$	4	$(2+3^{-1/2})^{6/7}$	3/7
(BII)	16	4002	$1.01 \cdot 10^9$	12	$2^{6/7}$	3/7
(CI)	1	4004	$4.05 \cdot 10^9$	28	$4^{2/5}$	2/5
(CII)	144	4002	$5.83 \cdot 10^{11}$	24	$576^{2/5}$	2/5
(D)	32501	292504	$4.04 \cdot 10^{15}$	$(1+3^{1/2})^2$	$(1+3^{-1/2})^{-4/3}$	1/3

Table 1: Parameter values for various types of Diophantine quintuples.

whereas we have $\alpha = 1.56155$. Likewise, in case (D) the optimal value is $\frac{1}{2}(1+\sqrt{3}) = 1.366025\ldots$, whereas we have 1.3660. It seems that a new idea is needed to improve substantially on the lower bounds on m.

4 Employing linear forms in the logarithm

The lower bounds for the index m given in the previous section can be complemented by inequalities derived from upper bounds for linear forms of logarithms of algebraic numbers. To this end, we apply a result from [1] that turns out to be the most convenient in the present context.

Theorem 4.1 (Aleksentsev)**.** *Let Λ be a linear form in logarithms of n multiplicatively independent totally real algebraic numbers $\alpha_1, \ldots \alpha_n$, with rational coefficients $b_1, \ldots, b_n$. Let $h(\alpha_j)$ denote the absolute logarithmic height of α_j for $1 \le j \le n$. Let d be the degree of the number field $\mathcal{K} = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$, and let $A_j = \max(dh(\alpha_j), |\log \alpha_j|, 1)$. Finally, let*

$$E = \max\left(\max_{1\le i,j\le n}\left\{\frac{|b_i|}{A_j} + \frac{|b_j|}{A_i}\right\}, 3\right). \tag{5}$$

Then

$$\log|\Lambda| \ge -5.3n^{-n+1/2}(n+1)^{n+1}(n+8)^2(n+5)(31.44)^n d^2 (\log E) A_1 \cdots A_n \log(3nd).$$

We have used the first displayed equation on [1, p. 2] to define E in (5): this makes our application easier. We apply Theorem 4.1 for $d = 4, n = 3$ and to

$$\Lambda = j \log \alpha_1 - k \log \alpha_2 + \log \alpha_3,$$

with

$$\alpha_1 = S + \sqrt{AC}, \quad \alpha_2 = T + \sqrt{BC}, \quad \alpha_3 = \frac{\sqrt{B}(\sqrt{C} \pm \sqrt{A})}{\sqrt{A}\left(\sqrt{C} \pm \sqrt{B}\right)},$$

where the signs coincide. More precisely, we take $(A, B, C) = (a, b, d)$ in cases (A)–(C) and $(A, B, C) = (a, c, d)$ in case (D). Consequently, by [14] one has $4 \le k = 2n \le j = 2m$ and $j \le 2k$. Moreover, we shall assume that $j \ge 1000$.

For our purposes we do not need the exact values of A_j and E as defined in Theorem 4.1: decent estimates will suffice. To find these estimates we proceed as follows, keeping the notation and hypotheses of Proposition 3.1 and supposing additionally that $C \le C_1$ for a certain integer C_1.

We begin by noting that one has

$$\begin{aligned} 2\log\alpha_1 &< \log(4AC+4) \le \log\bigl(4\rho^{-1}(B-1)C+4\bigr) < \log(4\rho^{-1}BC) \\ &< \log(4\rho^{-1}\beta^{-1}C^{1+\tau}) \end{aligned}$$

provided that $\rho A \le B - 1$. This clearly follows from $\rho A < B$ when ρ is integer, as in cases (A)–(C). In case (D) we have $b \ge 3a+1$, so that (cf. the proof of Lemma 3.1)

$$B = c = a + b + 2\sqrt{ab+1} > 1 + (1+3^{1/2})^2 a = 1 + \rho A.$$

In each of the cases (A)–(D) we have $\beta\rho > 4$, whence

$$A_1 < g_1(\beta, \rho, \tau, C_1)\log C,$$

with

$$g_1(\beta, \rho, \tau, C_1) := 1 + \tau + \frac{\log 4 - \log(\beta\rho)}{\log C_1}.$$

We readily obtain the following lower bound on A_1

$$A_1 > \log(4AC) > g_2(A_0, C_1)\log C,$$

with

$$g_2(A_0, C_1) := 1 + \frac{\log 4 + \log A_0}{\log C_1}.$$

Similar relations hold for A_2, namely

$$2\log\alpha_2 < \log(4BC+4) < \log(4\beta^{-1}C^{1+\tau}+4),$$

which implies the upper bound

$$A_2 < g_3(\beta, \tau, e)\log C,$$

where

$$g_3(\beta, \tau, e) := 1 + \tau + \frac{\log 4 + \log(\beta^{-1} + e^{-1-\tau})}{\log e},$$

and $e = C_0$ in the cases (B), (CI), and (D) (when $\beta < 4$) and $e = C_1$ in the remaining cases (A) and (CII). An easily-derived lower bound for A_2 is

$$A_2 > g_4(B_0, C_1)\log C,$$

with

$$g_4(B_0, C_1) := 1 + \frac{\log 4 + \log B_0}{\log C_1}.$$

The inequalities

$$\frac{\sqrt{B}}{\sqrt{A}} \cdot \frac{\sqrt{C}+\sqrt{A}}{\sqrt{C}-\sqrt{B}} > \frac{\sqrt{B}}{\sqrt{A}} \cdot \frac{\sqrt{C}+\sqrt{A}}{\sqrt{C}+\sqrt{B}} > 1, \quad \frac{\sqrt{B}}{\sqrt{A}} \cdot \frac{\sqrt{C}-\sqrt{A}}{\sqrt{C}-\sqrt{B}} > 1$$

are obvious. The modulus of the fourth algebraic conjugate of α_3 is also greater than 1 precisely when $\sqrt{C}(\sqrt{B}-\sqrt{A}) > 2\sqrt{AB}$. This inequality holds whenever

$$\rho B_0^{1-\tau}(\rho^{1/2}-1)^{2\tau} > 2^{2\tau}. \tag{6}$$

It is easy to check that (6) is satisfied in each of the cases (A)–(D). One now obtains

$$A_3 = 4\mathrm{h}(\alpha_3) = \log\left(\frac{B^2(C-A)^2}{g}\right),$$

where g is the content of the polynomial $A^2(C-B)^2X^4 + 4A^2B(C-B)X^3 + 2AB(3AB - AC - BC - C^2)X^2 + 4AB^2(C-A)X + B^2(C-A)^2$. Since g is at most the smallest of the coefficients, which is $4A^2B(C-B)$, one has

$$\log\left(\frac{B(C-A)^2}{4A^2(C-B)}\right) \le A_3 \le \log\bigl(B^2(C-A)^2\bigr).$$

Note that $B(C-A) < \beta^{-1}C^{1+\tau}$ readily implies

$$A_3 < g_5(\beta, \tau, f)\log C,$$

with

$$g_5(\beta, \tau, f) := 2 + 2\tau - \frac{2\log\beta}{\log f}$$

and $f = C_1$ if $\beta > 1$ and $f = C_0$ if $\beta < 1$. A lower bound for A_3 is obtained with the help of the inequalities

$$A_3 \ge \log\left(\frac{B(C-A)^2}{4A^2(C-B)}\right) > \log\left(\frac{\beta\rho^2C^{1-\tau}(1-A_0C_1^{-1})^2}{4(1-\rho A_0C_1^{-1})}\right),$$

which entail

$$A_3 > g_6(\beta, \rho, \tau, A_0, C_1)\log C,$$

where

$$g_6(\beta,\rho,\tau,A_0,C_1) := 1-\tau+\frac{\log(\frac{1}{4}\beta\rho^2)+2\log(1-A_0C_1^{-1})-\log(1-4C_1^{-1})}{\log C_1}.$$

On noting that for all relevant values of parameters one has $g_2(A_0,C_1) < g_4(B_0,C_1)$ and using the inequality $g_2 > g_6$ (which follows, for $C_1 > 10^{12}$, from $16C_1^{2\tau}(1-\beta^{-1}) > \beta\rho^2$ if $\beta > 1$ and from $16(1+3^{-1/2})^{8/3}C_1^{2/3}(1-\rho A_0C_1^{-1}) > \rho^3$ in case (D)) as well as the above mentioned relation $j \geq \max\{k, 1000\}$, we find that we may take

$$E \leq \frac{2j}{g_6(\beta,\rho,\tau,A_0,C_1)\log C_0}.$$

Note that the right side of this inequality is bigger than 3, otherwise from this and $C_0 < 10^{72}$ (consequence of [18, Theorem 3]) it would follow

$$2j \leq 3g_6(\beta,\rho,\tau,A_0,C_1)\log C_0 < 6\log C_0 < 6\log 10^{72} < 995.$$

Hence, Theorem 4.1 yields the following corollary.

Corollary 4.2.

$$-\log\Lambda \leq 1.5013\cdot 10^{11}g_3g_5(2\log\alpha_1)(\log^2 C)\log\left(\frac{2j}{g_6\log C_0}\right).$$

Corollary 4.2 bounds Λ from below; we can bound Λ from above using Eq. (4.1) in [14], which states that

$$0 < \Lambda < \frac{8}{3}AC\alpha_1^{-2j}.$$

Comparison with Corollary 4.2 gives the main result of this section.

Proposition 4.3.

$$j < 1.50131\cdot 10^{11}g_3g_5(\log^2 C)\log\left(\frac{2j}{g_6\log C_0}\right).$$

Set $j = 2m$ in Proposition 4.3 and use Lemma 3.1 with the values given in Table 1 and $C_1 = 10^{72.188}$ in all cases, as per [5, Theorem 1.2]. We thus get a new upper bound on d that we take as C_1 in a new iteration of this procedure. Slightly better bounds result by taking much higher C_0 (just below the value for C_1 considered in the same iteration). This game makes sense as long as it decreases the exponent of 10 in the upper bound for d by at least one thousandth.

For example, in case (D) we start with $(C_0, C_1) = (4.04 \cdot 10^{15}, 10^{72.188})$, which shows that $d < 10^{51.514}$. Taking this as our new value for C_1 we find that $d < 10^{51.514}$ — that is, there is no noticeable change.

We now increase C_0 to $10^{51.414}$: thus we are assuming that $d \geq 10^{51.414}$ (if not, then we shall settle with $d < 10^{51.414}$). This shows that $d < 10^{51.416}$. Finally, though we may take this number as our new C_1 and iterate once more, we find no noticeable improvement. We therefore conclude that $d < 10^{51.416} < 2.603 \cdot 10^{51}$. We continue in this way, and record our computations in the following theorem.

Theorem 2. *If $\{a, b, c, d, e\}$ is a quintuple with $a < b < c < d < e$ then the following bounds for d hold:*

(A) $d < 10^{67.859} < 7.228 \cdot 10^{67}$, (B) $d < 10^{60.057} < 1.141 \cdot 10^{60}$,
(C) $d < 10^{56.528} < 3.373 \cdot 10^{56}$, (D) $d < 10^{51.416} < 2.603 \cdot 10^{51}$.

We close this section with a remark concerning the size of the smallest entry in a quintuple arising in case (A). Although it has no immediate bearing on the next section, further improvements on d should enable future researchers to enumerate all possible triples. Recording the maximal size of a should aid this goal.

Proposition 4.4. *The only quintuples that could arise from case* (A) *are those in which $a < 7.4 \cdot 10^7$.*

Proof. The triples in case (A) must satisfy $b^{3/2} > c > 4ab + b + a$, so that, in particular $a < b^{1/2}/4$. Some quick computations give that for $A_0 = 7.4 \cdot 10^7$ one obtains $d < 6.1 \cdot 10^{50}$. From $d > 4abc > 16a^2b^2 > (16a^2)^3$ it then follows $a < 7.29 \cdot 10^7$, a contradiction. □

5 Bounding the total number of quintuples

In this section we combine the methods of [5] and [18] in bounding certain arithmetical sums. We require the following lemma.

Lemma 5.1 (Lemma 13 in [18]). *For all $x \geq 1$ we have*

$$\sum_{n \leq x} \frac{2^{\omega(n)}}{n} \leq 3\pi^{-2} \log^2 x + 1.3948 \log x + 0.4107 + 3.253x^{-1/3},$$
$$\sum_{n \leq x} 2^{\omega(n)} \leq 6\pi^{-2} x \log x + 0.787x + 8.14x^{2/3} - 0.3762.$$

One can show, using Perron's formula and calculating residues, that

$$\sum_{n\le x} 2^{\omega(n)} \sim \frac{6}{\pi^2} x\log x + \frac{6}{\pi^4}\left(\pi^2(2\gamma-1) - 12\zeta'(2)\right)x,$$
$$\sum_{n\le x} \frac{2^{\omega(n)}}{n} \sim \frac{3}{\pi^2} x\log x + \frac{12}{\pi^4}\left(\pi^2\gamma - 6\zeta'(2)\right)x,$$

where

$$\frac{6}{\pi^4}\left(\pi^2(2\gamma-1) - 12\zeta'(2)\right) = 0.78687\ldots, \quad \frac{6}{\pi^4}\left(\pi^2(2\gamma-1) - 12\zeta'(2)\right) = 1.39479\ldots.$$

This shows that up to three decimal places, the bounds in Lemma 5.1 agree with the asymptotic expansions to the first two terms.

We also require bounds on $d(n)$, the number of divisors of n, and the related $d_H(n)$, which counts the number of divisors of n that do not exceed H. The function $d_H(n^2-1)$ arises naturally when considering the number of doubles $\{a, b\}$ satisfying certain restrictions.

Very recently, Dudek [8] considered partial sums of $d(n^2-1)$ and proved

$$\sum_{2\le n\le N} d(n^2-1) \sim \frac{6}{\pi^2} N\log^2 N. \tag{7}$$

This improves, asymptotically, on the bound with leading term $9\pi^{-2}N\log^2 N$ as given in [5]. We make Dudek's result explicit in the following lemma.

Lemma 5.2. *Let $d_H(n)$ denote the number of positive integers e such that $e|n$ and $e\le H$. Then, for any $N\ge 2$ and $H\ge 1$ we have*

$$\sum_{n=2}^{N} d_H(n^2-1) \le N\left(\frac{6}{\pi^2}\log^2 H + 2.369\log H + 6.175 + 12.071H^{-1/3}\right).$$

Let $g(d)$ denote the number of solutions to $x^2\equiv 1 \pmod d$ where $0\le x\le d-1$. Furthermore, let $Q(x, d)$ denote the number of positive $n\le x$ such that $n^2\equiv 1 \pmod d$. It follows that $Q(d, d) = g(d)$ and that $Q(x, d)\le g(d)(x/d+1)$. We therefore have

$$\begin{aligned}\sum_{2\le n\le N} d_H(n^2-1) &= 2\sum_{d\le H}\sum_{\substack{d<n\le N\\ n^2\equiv 1 \pmod d}} 1 = 2\sum_{d\le H}(Q(N,d)-Q(d,d))\\ &\le 2N\sum_{d\le H}\frac{g(d)}{d},\end{aligned}\tag{8}$$

To proceed we need to present two lemmas. Lemma 5.3 was proved by Berkane, Bordellès and Ramaré in [3]; Lemma 5.4 was proved by Ramaré in [17]. We quote the versions in [18] which correct two small misprints. In what follows we use the notation $f(x) = \vartheta(g(x))$ to mean $|f(x)| \le g(x)$ for all x under consideration.

Lemma 5.3 (Lemma 13 in [18]). *For all $t > 0$*

$$\sum_{n\le t} \frac{d(n)}{n} = \frac{1}{2}\log^2 t + 2\gamma \log t + \gamma^2 - 2\gamma_1 + \vartheta(1.16t^{-1/3}),$$

where γ is Euler's constant and γ_1 is the second Stieltjes constant, which satisfies $-0.07282 < \gamma_1 < -0.07281$.

Lemma 5.4 (Lemma 14 in [18]). *Let $\{g_n\}_{n\ge1}, \{h_n\}_{n\ge1}$ and $\{k_n\}_{n\ge1}$ be three sequences of complex numbers satisfying $g = h * k$: that is g is the Dirichlet convolution of h and k. Let $H(s) = \sum_{n\ge1} h_n n^{-s}$ and $H^*(s) = \sum_{n\ge1} |h_n| n^{-s}$, where $H^*(s)$ converges for $\Re(s) \ge -\frac{1}{3}$. If there are four constants A, B, C and D satisfying*

$$\sum_{n\le t} k_n = A\log^2 t + B\log t + C + \vartheta(Dt^{-1/3}), \quad (t > 0),$$

then

$$\sum_{n\le t} g_n = u\log^2 t + v\log t + w + \vartheta(Dt^{-1/3}H^*(-1/3)),$$

and

$$\sum_{n\le t} ng_n = Ut\log t + Vt + W + \vartheta(2.5Dt^{2/3}H^*(-1/3)),$$

where

$$u = AH(0), \quad v = 2AH'(0) + BH(0), \quad w = AH''(0) + BH'(0) + CH(0),$$
$$U = 2AH(0), \quad V = -2AH(0) + 2AH'(0) + BH(0),$$
$$W = A(H''(0) - 2H'(0) + 2H(0)) + B(H'(0) - H(0)) + CH(0).$$

Let

$$(9)\quad F(s) = \sum_{d=1}^{\infty} \frac{g(d)/d}{d^s}, \;\; H(s) = \frac{\left(1 + \frac{1}{2^{s+1}} + \frac{2}{4^{s+1}} + \frac{4}{8^{s+1}-4^{s+1}}\right)\left(\frac{1-2^{-(s+1)}}{1+2^{-(s+1)}}\right)}{\zeta(2(s+1))}.$$

Dudek shows, half-way down page 4 in [8], that

$$(10)\qquad F(s) = \zeta^2(s+1)H(s).$$

Since $\sum_{n=1}^{\infty} d(n)n^{-s} = \zeta^2(s)$, this suggests that we apply Lemma 5.4 with $g_n = g(n)/n, k_n = d(n)/n$ and with $h(n)$ the coefficients of the Dirichlet series $H(s)$ in (9). Since $g(d)$ is multiplicative we can determine its values at prime powers by Lemma 2.1 in [8]. This shows that

$$\text{(11)}\quad g(2)=1,\quad g(4)=g(p^{e_1})=2,\quad g(2^{e_2})=4,\quad (p\text{ odd}, e_1\geq 1, e_2\geq 3).$$

By (10) we may compare Euler products and use (11) to show that

$$h(1)=1,\quad h(p)=0,\quad h(p^2)=-1,\quad h(p^{e_1})=0,\quad (p\text{ odd}, e_1\geq 3),$$
$$h(2)=-1,\quad h(4)=h(8)=1,\quad h(16)=-2,\quad h(2^{e_2})=0,\quad (e_2\geq 5).$$

This shows that

$$H(s)=\prod_{p>2}\left(1-\frac{1}{p^{2(s+1)}}\right)\left(1-\frac{1}{2^{s+1}}+\frac{1}{2^{2(s+1)}}+\frac{1}{2^{3(s+1)}}-\frac{2}{2^{4(s+1)}}\right),$$
$$H^*(-\tfrac{1}{3})=\prod_{p}\left(1+\frac{1}{p^{4/3}}\right)\frac{\left(1+\frac{1}{2^{2/3}}+\frac{1}{4^{2/3}}+\frac{1}{8^{2/3}}+\frac{2}{16^{2/3}}\right)}{\left(1+\frac{1}{2^{4/3}}\right)}.$$

Since $\prod_p\left(1+p^{-4/3}\right)=\zeta(4/3)/\zeta(8/3)$ we conclude that

$$H^*(-\tfrac{1}{3})\leq 5.203.$$

We may therefore apply Lemmas 5.3 and 5.4. We find that

$$\text{(12)}\quad H(0)=\frac{6}{\pi^2},\quad 0.4822\leq H'(0)\leq 0.4823,\quad 4.4784\leq H''(0)\leq 4.4785.$$

Indeed, we have complicated but exact expressions for $H'(0)$ and $H''(0)$ — we have merely given the decimal approximation in (12). This shows that

$$\text{(13)}\qquad \sum_{d\leq N}\frac{g(d)}{d}\leq\frac{3}{\pi^2}\log^2 N+1.1842\log N+3.0871+6.0355N^{-1/3}.$$

Inserting (13) into (8) completes the proof of Lemma 5.2.

We now proceed to examine the number of quintuples that could arise from each of the triples (A)–(D).

5.1 Case (A)

This is the most damaging case in our considerations. We have $r<(d/16)^{1/4}$, whence, by Theorem 2 we have $r<4.611\cdot 10^{16}=R_A$. Using Lemma 5.2 we find that the number of doubles is at most

$$\frac{1}{2}\sum_{r=3}^{R_A}d_{R_A}(r^2-1)<2.288\cdot 10^{19}.$$

Since $b < (d/20)^{1/2} < 1.9011 \cdot 10^{33}$ we find that b could have as many as 23 distinct prime factors. As explained in [5, p. 216], this information allows one to conclude that there are at most $3 \cdot 4 \cdot 2^{24}$ possibilities to extend a Diophantine double $\{a, b\}$ with $b > 4a$ to a Diophantine quintuple. We therefore find that the number of quintuples is bounded by

$$3 \cdot 4 \cdot 2^{24} \cdot 2.288 \cdot 10^{19} \leq 4.605 \cdot 10^{27}. \tag{14}$$

Since the number of possible quintuples originating from case (A) is by far the largest, we devote §6 to reducing this number slightly.

5.2 Case (B)

Since $b > 4a$ we have $b > 2r$, whence $c > 4r + a$. Since $d > 4abc$ this shows that $d > 4(r^2 - 1)(4r + 2) > 16r^3$. From Theorem 2 we therefore have $r \leq 4.147 \cdot 10^{19} = R_B$. By Lemma 5.2 the number of doubles $\{a, b\}$ is at most

$$\frac{1}{2} \sum_{r=3}^{R_B} d_{R_B}(r^2 - 1) < 2.807 \cdot 10^{22}.$$

Since there are at most four ways of extending a quadruple to a quintuple we find that the total number of quintuples is bounded above by

$$1.123 \cdot 10^{23}. \tag{15}$$

5.3 Case (C)

We proceed as in case 2(iii) in [18]. We consider the cases $a > \eta$ and $a \leq \eta$ and optimise over η. In the former case, we have $d > 4abc > 4\eta b^{5/2}$ so that $b < (d/(4\eta))^{2/5} := N_{3a}$. Hence, by Lemma 3.3 in [12], the number of quintuples is at most

$$\frac{N_{3a}}{6} (\log N_{3a} + 2)^3 \cdot 8 \cdot 5 \cdot 4. \tag{16}$$

When $a \leq \eta$, we have $b < (d/(4a))^{2/5}$ so that $r^2 = ab+1 < a(d/(4a))^{2/5} + 1$. Thus

$$r < \sqrt{1 + \left(\frac{\eta^3 d^2}{16}\right)^{1/5}} = N_{3b}.$$

We apply Lemma 5.2 with $H = \eta$ and $N = N_{3b}$. Since $b < (d/4)^{2/5} < 2.35 \cdot 10^{22}$ we have $\omega(b) \leq 17$. Following the proof in [12] we deduce that the number of quintuples is at most

$$4 \cdot 2^{17} \cdot 5 \cdot 4 \cdot N_{3b} \left(\frac{6}{\pi^2} \log^2 \eta + 2.369 \log \eta + 6.175 + 12.071 \eta^{-1/3}\right). \tag{17}$$

We find that we can minimise the maximum of (16) and (17) at $\eta = 1.51 \cdot 10^{11}$. Hence the number of quintuples is at most

$$3.214 \cdot 10^{24}. \tag{18}$$

5.4 Case (D)

We have $b < (4d/9)^{1/3}$ so that, by Theorem 2, we have $b < 1.05 \cdot 10^{17} = R_D$. The number of doubles $\{a, b\}$ is therefore bounded by $2\sum_{b=4}^{R_D} 2^{\omega(b)}$. We use this and Lemma 5.1 to prove that the number of quintuples is at most

$$2.07 \cdot 10^{19}. \tag{19}$$

6 Improvements to case (A)

Here we investigate two methods. The first, in §6.1, reduces the bound on $\omega(b)$ from 23 to 22, thereby saving a factor of 2 in the estimate recorded in (14). The second, in §6.2, splits up the sum over b with $\omega(b)$ held constant. This saves a factor of about 4.23.

6.1 Removing one prime factor from b

Let $(p_n)_{n\in\mathbb{N}}$ denote the sequence of prime numbers, and consider those b satisfying

$$b_0 := \prod_{i=1}^{23} p_i \approx 2.67 \cdot 10^{32} \leq b < 1.9011 \cdot 10^{33}, \quad \omega(b) = 23. \tag{20}$$

We aim at enumerating all such b in (20). We shall show that none of these values of b can appear as the second-smallest element of a quintuple. This then shows that $\omega(b) \leq 22$, and leads immediately to a saving of a factor of 2 in (14).

Suppose $\{a, b, c, d, e\}$ is a quintuple. In case (A), Theorem 2 gives the bound $d < UD := 10^{67.859}$. When b is restricted as in (20) we find that 2 divides b, since, if not, the smallest b can be is $\prod_{i=1}^{23} p_i/2 \cdot p_{24} > 1.18 \cdot 10^{34}$. Continuing in this way we find that $2, 3, 5, 7, 11$ must all divide b.

From $4a(4a+1)b^2 < UD$ it then follows $a \leq 7$. Moreover, as the corresponding r is odd, ab is a multiple of 8, whence $b \equiv 0 \pmod 8$ for odd a and $b \equiv 0 \pmod 4$ for $a \equiv 2 \pmod 4$. Hence, each such b is obtained from $b_1 = b_1(a)$ by replacing v of its factors $p_6, \ldots, p_{23}$ by other v primes p_{k_1},

..., p_{k_v}, where $24 \leq k_1 < \cdots < k_v$, and then multiplying by some positive integer q such that the result is at most

$$UB = UB(a, UD) := UD^{1/2}(16a^2 + 4a)^{-1/2}.$$

Here $b_1(a) = 4b_0$ if a is odd, $b_1(a) = 2b_0$ if $a = 2, 6$, and $b_1(a) = b_0$ otherwise.

We now present a detailed exposition of the idea sketched above. All computations have been performed with GP scripts [16]. Clearly, the maximal v is determined from the condition

$$\frac{p_{24}p_{25}\cdots p_{23+v}}{p_{23}p_{22}\cdots p_{24-v}} < \frac{UB}{b_1}.$$

A short computer search gives $v = 3$ for $a = 2$ or 4; $v = 2$ for $a = 1$; $v = 0$ for the other values $a \leq 7$.

Next for each $u = 1, 2, \ldots, v$ we look for the largest index $K = K(u)$ satisfying

$$\frac{p_{24}p_{25}\cdots p_{23+u}p_K}{p_{23}p_{22}\cdots p_{24-u}} < \frac{UB}{b_1}$$

and the smallest J verifying

$$\frac{p_{24}p_{25}\cdots p_{23+u}}{p_{23}p_{22}\cdots p_{25-u}p_J} < \frac{UB}{b_1}.$$

After that we determined all integers $24 \leq k_1 < \cdots < k_u \leq K$ and $23 \geq j_1 > \cdots > j_u \geq J$ such that

$$\frac{p_{k_1}p_{k_2}\cdots p_{k_u}}{p_{j_1}p_{j_2}\cdots p_{j_u}} < \frac{UB}{b_1}.$$

Each such tuple $(k_1, \ldots, k_u, j_1, \ldots, j_u)$ gives rise to

$$\left\lfloor \frac{UB p_{j_1}p_{j_2}\cdots p_{j_u}}{b_1 p_{k_1}p_{k_2}\cdots p_{k_u}} \right\rfloor$$

candidates for the largest entry in a Diophantine couple $\{a, b\}$.

Since the bound $UD = 10^{67.859}$ found in case (A) entails $UB(a, UD) < 10^{33.9295}(16a^2 + 4a)^{-1/2}$, for $1 \leq a \leq 7$ one has

$$\frac{UB(a, UD)}{b_1(a)} \leq \frac{UB(4, UD)}{b_1(4)} < 2.$$

Therefore, the multiplier q mentioned above must be equal to 1.

For each value of b identified using the above method, we are able to show easily that there is no corresponding quadruple. This shows that $\omega(b) \leq 22$. In theory there is nothing stopping us from playing this trick again. However, when we search for $\omega(b) = 22$ we find that we could have over four thousand primes dividing b. This appears to be orders of magnitude harder than the $\omega(b) = 23$ case.

6.2 Bounding b in different ranges

We have $ab+1=r^2$. Note first that $d(r^2-1)$ is even (it is odd if and only if $r^2-1=s^2$ which implies that $(r+s)(r-s)=1$ — a contradiction). Since $d(r^2-1)$ counts the number of divisors of r^2-1 it follows that $\frac{1}{2}d(r^2-1)$ counts the number of *pairs* of divisors $\{a,b\}$ with $a<b$. Now each a corresponds with exactly one b (and hence one pair corresponds with exactly one value of a): therefore $\frac{1}{2}d(r^2-1)$ is actually counting the divisors a. Furthermore, note that

$$r^2-1=ab>a^2. \tag{21}$$

Therefore $\frac{1}{2}d(r^2-1)$ is actually counting all those a with $a<\sqrt{r^2-1}$. Whence for a fixed r we wish to count

$$\frac{1}{2}d_{\sqrt{r^2-1}}(r^2-1).$$

If $r\le R$ then summing over r shows that the number of pairs $\{a,b\}$ is at most

$$\frac{1}{2}\sum_{r=3}^{R} d_{\sqrt{r^2-1}}(r^2-1)\le\frac{1}{2}\sum_{r=3}^{R} d_r(r^2-1)<\frac{1}{2}\sum_{r=3}^{R} d_R(r^2-1). \tag{22}$$

Now, we can make a slight improvement on (22). Since, for case (A) quadruples we have $b>4a$, we can improve on (21) to show that $r^2-1=ab>4a^2$. Therefore, we amend (22) to show that the total number of pairs is at most

$$\frac{1}{2}\sum_{r=3}^{R} d_{R/2}(r^2-1).$$

One can go further than this. Let $N(\alpha,\beta)$ be the number of quintuples with $\alpha a<b\le\beta a$, for some $\beta>\alpha\ge 4$. It then follows that for integers m_i satisfying $4=m_0<m_1<\ldots<m_k$ the total number of quintuples is bounded above by

$$N(4,m_1)+N(m_1,m_2)+\cdots+N(m_{k-1},m_k)+N(m_k,\infty),$$

where $N(m_k,\infty)$ means all those pairs $\{a,b\}$ such that $b>m_k a$. With the exception of $N(m_k,\infty)$, each number is of the form $N(m_j,m_{j+1})$.

Take $m_j a<b\le m_{j+1}a$. Since $d>4ab(4ab+a+b)>16a^2b^2>16b^4/(m_{j+1})^2$ we have

$$b<\frac{d^{1/4}(m_{j+1})^{1/2}}{2}. \tag{23}$$

We also have

$$r^2 - 1 = ab > m_j a^2 \Rightarrow a < R/\sqrt{m_j}. \tag{24}$$

By taking m_j large we ensure that the bound on a in (24) is small. We now look at $\omega(b)$ for b satisfying (23). We want to choose m_{j+1} to be as large as possible such that we do not increase $\omega(b)$. For example, when $j = 0$ we are considering $4a < b \le m_1 a$. We find, using $d \le 7.228 \times 10^{67}$, that $\omega(b) \le 14$ provided that $m_1 \le 177$. Also, for m_2 we find that we can take $m_2 \le 499686$ and still ensure that $\omega(b) \le 15$. We continue in this way, contenting ourselves with estimates on m_j that are accurate to one decimal place. We find, using *Mathematica* [19], that we may take

$$(m_3, m_4, m_5, m_6, m_7, m_8) = (1.7{\cdot}10^9, 6.4{\cdot}10^{12}, 2.9{\cdot}10^{16}, 1.4{\cdot}10^{20}, 7.8{\cdot}10^{23}, 4.8{\cdot}10^{27}).$$

We know, from §6.1, that there are at most 22 distinct prime factors of b. Therefore we have that the number of quintuples is at most

$$3{\cdot}2\left(2^{15}\sum_{r=3}^{R} d_{R/2}(r^2-1) + 2^{16}\sum_{r=3}^{R} d_{R/\sqrt{177}}(r^2-1) + \cdots + 2^{23}\sum_{r=3}^{R} d_{R/\sqrt{4.8\cdot 10^{27}}}(r^2-1)\right).$$

We find that the above is no more than

$$5.4075 \cdot 10^{26}. \tag{25}$$

Using (15), (18), (19) and (25) we complete the proof of Theorem 1.

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