Prime Numbers in Short Intervals

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Abstract

The Riemann zeta function, $\zeta(s)$, is central to number theory and our understanding of the distribution of the prime numbers. This thesis presents some of the known results in this area before exploring and expanding upon the work done by Dudek in [11] on primes in short intervals.

In Chapter 1, we first explore the history of prime numbers with a focus on primes in intervals. The relationship between primes, the Riemann zeta function and its zeros is then discussed. We then present the concepts of zero density estimates and zero free regions along with the Riemann–von Mangoldt explicit formula. These feature in the following chapters.

It was shown by Ingham [20] in 1937 that there exist primes between sufficiently large pairs of consecutive cubes. Chapter 2 explores how this result can be made explict. In §2.1, we present the proof given by Dudek that there is a prime between n^3 and $(n+1)^3$ for all $n \geq \exp(\exp(33.3))$. This makes Ingham's result explicit. Dudek's proof relies on does deriving an explicit version of the truncated Riemann–von Mangoldt explicit formula and making use of zero density estimates and zero free regions. The approach used is to apply known results to arrive at a statement of the form

$$\theta_{x,h} > h - f(x,h,k,A,c) - g(x,h,k) - E(x,h,k) \ ,$$

for given functions f, g, E. The goal is to show that $\theta_{x,h}$ is positive. First $E(x,h,k)$ is bounded and then an assumption of a $50-50$ split is used to reduce the problem to the following criterion, the satisfaction of which guarantees a prime in the interval $(x, x+3x^{2/3}] \subset (n^3, (n+1)^3]$. We wish to satisfy

$$f(x,h,k,A,c) < \frac{1}{2}(1-10^{-3})h$$

and

$$g(x,h,k) < \frac{1}{2}(1-10^{-3})h \ .$$

A solution can be found using *Mathematica*, yielding the explicit result.

In §2.2, we make some minor changes to Dudek's proof in order to obtain a slightly stronger result.

Chapter 3 examines how the approach used by Dudek can be generalised to give explicit results for primes between mth powers. After detailing this process in §3.1, we go on to make some substantial improvements. Dudek proves that there is a prime in the interval $(n^m, (n+1)^m]$ for all $n \geq 1$ when $m \geq 4.971 \times 10^9$. By removing the assumption of a $50-50$ split which was inherited from the cubes case, and making use of an explicit version of the prime number theorem, we are able to improve this to

$$\exists p \in (n^m, (n+1)^m], \forall n \geq 1, \forall m \geq 1438989 \ .$$

Chapter 4 contains detail on how *Mathematica* was used to obtain and verify the results in this thesis. It examines the functions that were written, how the functions were used, why they were useful and the limitations of their use.

Contents

Notation

Here we outline some notation.

Unless otherwise stated:

p is a prime,

$m, n \in \mathbb{N}$,

ε is a positive constant.

Big O notation:

One writes

$$f(x) = g(x) + O(h(x))$$

if there exists a positive real number M and a real number x_0 such that

$$\big|f(x) - g(x)\big| < M \times \big|h(x)\big|$$

for all $x > x_0$.

Explicit Big O notation:

If one wishes to be explicit then $O^*(\cdot)$ can be used. We write

$$f(x) = g(x) + O^*(h(x)))$$

to denote that

$$\big|f(x) - g(x)\big| < \big|h(x)\big|$$

for all $x > x_0$.

Asymptotic Equivalence:

We write

$$f(s) \sim g(s) \text{ (as } s \to s_0)$$

if

$$\lim_{s \to s_0} \frac{f(s)}{g(s)} = 1 \ .$$

Chapter 1

Introduction

1.1 What is a prime and where can I find one?

An integer p is called *prime* if $p > 1$ and the only positive divisors of p are 1 and p. Any positive integer $n > 1$ which is not prime is called *composite*. While primes are simple to define, their distribution is not well understood.

Primes are of interest not only because of their elusive nature but because they are the building blocks of multiplicative structure.

Theorem 1.1.1 (Fundamental Theorem of Arithmetic)**.** *Every integer greater than* 1 *can be expressed as a product of primes in a unique way (up to ordering).*

To build our picture of the distribution of primes we might consider the following questions. How many primes are there? How far apart can primes be? How densely do primes occur?

Theorem 1.1.2. *There are infinitely many primes [11].*

Proof. Suppose, for contradiction, that there are only finitely many prime numbers, $p_1, \ldots, p_k$. Then let

$$N = 1 + \prod_{i=1}^{k} p_i \ .$$

We see, by construction, that N cannot be divisible by any prime $p_1, \ldots, p_k$, so by by the Fundamental Theorem of Arithmetic, N must be divisible by some other prime that has not been listed, a contradiction. □

In 1737 Euler showed that the series

$$\sum_{n=1}^{\infty} \frac{1}{p_n}$$

diverges, where p_n denotes the nth prime number [19]. This is a stronger result than there being infinitely many primes, because Euler's celebrated identity, known as the Basel problem, shows that the sum of inverse squares converges to

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \ .$$

The Basel problem implies that for sufficiently large n, intervals of the form $[1, n]$ contain more primes than squares.

Interestingly, even though both the harmonic series and the sum of inverse primes diverge, the following result shows that the primes are much less dense than the natural numbers.

Theorem 1.1.3. *There exist arbitrarily large gaps between consecutive primes.*

Proof. Consider the sequence of consecutive numbers

$$(n+1)! + 2, (n+1)! + 3, \ldots, (n+1)! + (n+1) \ .$$

Each of these numbers is composite, since i is a factor of $(n+1)!+i$ for $2 \leq i \leq n+1$. Hence we have constructed a sequence of n consecutive composite numbers. □

1.2 The prime counting function and the prime number theorem

Of interest in the study of primes is the number of primes that do not exceed x, which can be denoted by the prime counting function

$$\pi(x) := \left|\{p \in \mathbb{N} \mid p \leq x\}\right|$$

Given that there are infinitely many primes, it is natural to ask what, if any, asymptotic behaviour $\pi(x)$ has. This question is answered by *the prime number theorem.* Conjectured by Gauss (1792) and Legendre (1798) [1], the prime number theorem is as follows.

Theorem 1.2.1 (Prime number theorem)**.** *The asymptotic distribution of the prime numbers among the positive integers can be described by*

$$\pi(x) \sim \frac{x}{\log x} .$$

Neither Gauss nor Legendre proved the prime number theorem. The first real attempt at a proof was given by Chebyshev, who in 1848 [6] proved that if the limit

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log x}} = \lim_{x\to\infty} \frac{\pi(x)\log x}{x}$$

exists it must be equal to 1. Later he proved the following.

Lemma 1.2.2 (Chebyshev)**.** *There exist constants A_1 and A_2, with $0.922 < A_1 < 1$ and $1 < A_2 < 1.105$, such that for all $x > 1$*

$$A_1 < \frac{\pi(x)\log x}{x} < A_2 .$$

He was however, unable to show that the limit actually existed. Riemann attempted to give a proof of the prime number theorem but was not successful. By building on Riemann's work, Hadamard [17] and independently de la Vallée Poussin [9] almost simultaneously proved the prime number theorem in 1896 [15].

The work of Chebyshev, Riemann, Hadamard and de la Vallée Poussin made use of the *Riemann zeta function.*

Definition 1.2.3 (The Riemann zeta function)**.** The Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $s = \sigma + it$ is a complex variable with $\text{Re}(s) = \sigma > 1$.

Though Riemann was not successful in his attempt to prove the prime number theorem, he was the first to consider treating s as a complex rather than a real variable [33] and to use the theory of analytic functions [19]. This enabled him to establish many properties of the zeta function and show that the prime number theorem was related to the locations of the zeros of the zeta function [15]. He also made the following conjecture which remains an open problem in number theory.

Conjecture 1.2.4 (The Riemann hypothesis)**.** *If* $\rho = \beta + i\gamma$ *is a zero of* $\zeta(s)$ *and* $0 < \beta < 1$*, then* $\beta = \frac{1}{2}$*.*

There has been a number of results that verify the Riemann hypothesis for all $\rho = \beta + i\gamma$ with $|\gamma| \leq H_0$ for some H_0. The largest value for H_0 was given by Platt [29] in 2017.

Theorem 1.2.5 (Platt, [29])**.** *The Riemann hypothesis is true for all* $s = \sigma + it$ *with* $|t| \leq H_0 = 3.061 \times 10^{10}$*.*

The Riemann hypothesis would be a very strong result about the distribution of the prime numbers. Even without its proof we can still make some progress. We instead bring our attention to questions about primes in intervals where we may have more success.

1.3 Primes in intervals

A classical, and as of yet unresolved, question about primes is Legendre's conjecture.

Conjecture 1.3.1 (Legendre's conjecture)**.** *There exists a prime, p, between any two consecutive squares. That is for all* $x \geq 1$*, there exists p such that*

$$p \in (x^2, (x+1)^2] \ .$$

There has been little progress on the problem of primes between squares. This is demonstrated by the fact that, even on assuming the Riemann hypothesis, no one has been successful in furnishing a proof.

The length is the interval $(x^2, (x+1)^2]$ is $2x+1$ which is small when compared to its starting point, x^2. We can classify an interval by its size relative to its starting point.

Definition 1.3.2 (Short Interval)**.** An interval of the form $(x, x+h(x))$ is a *short interval* if

$$\lim_{x\to\infty} \frac{h(x)}{x} = 0 \ .$$

Results about primes in short intervals can be used to prove results about primes in larger intervals and for this reason they are usually more difficult to prove.

One well-known result of primes in intervals is Bertrand's Postulate, which was proved by Chebyshev.

Theorem 1.3.3 (Bertrand's Postulate)**.** *For all* $x \geq 1$*, there exists* $p \in (x, 2x]$*.*

The constant 2 here is the best possible because the first interval $(1, 2]$ contains only a single prime on the boundary. If we relax the condition to $x \geq x_0 > 1$ then the constant 2 can be improved.

For example, consider the following 'Bertrand type' result, proved by Nagura, [27].

Theorem 1.3.4. *For all* $x > 25$*, there exists a prime, p, such that* $p \in (x, \frac{6}{5}x]$ *.*

More generally, we can find other results of the form

$$\exists p \in (x, (1+\delta)x) \text{ for } x \geq x_0 \ ,$$

which can be derived from the prime number theorem with error term [9], [15] (ch.5),

$$\Pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right) \ .$$

Some examples of such results come courtesy of Schoenfeld [34], Ramaré–Saouter [32] and Kadiri–Lumley [24]. In increasing order of δ, they are as follows.

Lemma 1.3.5 (Schoenfeld, [34])**.** *There is a prime in the interval* $(x, (1+\delta)x]$ *for all* $x \geq 2010760$ *and* $\delta = 1/16597$*.*

Next, we present one of the many results given by Ramaré–Saouter in Table 1 of their paper [32], rephrased to be consistent with the notation we are using here.

Lemma 1.3.6 (Ramareé–Saouter [32])**.** *There is a prime in the interval* $(x, (1+\delta)x]$ *for all* $x \geq 9.5 \times 10^{19}$ *where* $\delta = 1/81353846$*.*

Finally, we present one of the results of Kadiri - - Lumley [24], which has the smallest value for δ. This has also been rephrased to match our notation.

Lemma 1.3.7 (Kadiri–Lumley [24]). *There is a prime in the interval* $(x, (1+\delta)x]$ *for all* $x \geq 1.4 \times 10^{65}$ *where* $\delta = 1/2442159713$.

In 2015, Platt [28] computed $\pi(10^{24})$ which demonstrates that $\pi(x)$ can be calculated for x up to this size. This means it is sensible to consider these generalised Bertrand results of the form $\exists p \in (x, (1+\delta)x]$ for $x \geq x_0$, especially those of Ramaré and Saouter, because modern computational power is sufficient to examine the cases where $x < x_0$.

All of these 'Bertrand type' intervals are still too large to be short intervals. The interval $(x^2, (x+1)^2)$ from Legendre's conjecture is a short interval. While resolving Legendre's conjecture still seems well out of reach, there has been progress on the related question of the existence of primes between cubes and more generally between mth powers, all of which are short intervals.

In order to explore the progress that has been made on primes between cubes, we first present some results about short intervals of the form $(x, x+x^\theta)$ for $\theta \in (0,1)$.

It was shown by Hoheisel [18] in 1930 that for sufficiently large x and $\theta = 1 - \frac{1}{33000}$, there exists p, such that $p \in (x, x+x^\theta)$. That is, with finitely many exceptions, there is a prime in the interval $(x, x+x^{\frac{32999}{33000}})$.

By building on Hoheisel's ideas, Ingham [20] proved if there exists a $c > 0$ such that $\zeta(\frac{1}{2}+it) = O(t^c)$, then for $\theta = \frac{1+4c}{2+4c} + \varepsilon$, and x sufficiently large, there is a prime in the interval $(x, x+x^\theta)$.

Hardy and Littlewood proved this bound holds for $c = \frac{1}{6} + \varepsilon$ which gives $\theta = \frac{5}{8} + \varepsilon$. This is not the best value for c. It has been improved many times, most recently to $c = \frac{13}{84} + \varepsilon$ which was achieved by Bourgain [4] in 2017. This value of c gives $\theta = \frac{34}{55} \approx 0.618$.

The result with the smallest θ is due to Baker, Harman and Pintz [2].

Theorem 1.3.8 (Baker, Harman, Pintz, [2]). *There is a prime in the interval* $(x, x+x^{0.525})$ *for sufficiently large* x.

If we were to use Ingham's result to improve this value of $\theta = 0.525$, we would require

$$\theta = \frac{1+4c}{2+4c} < 0.525 \Rightarrow c < \frac{1}{38} \approx 0.026$$

which is much smaller than the current strongest result, $c = \frac{13}{84} \approx 0.155$.

The value of $\theta = 0.525$ is significant because the interval $(x, x + x^{0.525})$ is close to the size of the interval

$$(x, x + 2x^{0.5} + 1)$$

predicted by Legendre's conjecture.

1.3.1 Primes between cubes

Lemma 1.3.9. *For sufficiently large n, there exists $p \in (n^3, (n+1)^3)$.*

Proof. We will make use of Hardy and Littlewood's result above, that there is a prime in the interval $(x, x + x^{5/8+\varepsilon})$, for x sufficiently large. Although this is not the strongest result known, it is sufficient for this proof. By letting $x = n^3$ we observe that by choosing $0 < \varepsilon < \frac{1}{8}$ we get

$$(x, x + x^{5/8+\varepsilon}) = (n^3, n^3 + (n^3)^{\frac{5}{8}+\varepsilon}) \subset (n^3, n^3 + n^2) \subset (n^3, (n+1)^3)\ .$$

Therefore there must be a prime in the interval $(n^3, (n+1)^3)$ for sufficiently large n. $\square$

1.4 The von Mangoldt function and Chebyshev's ψ and θ functions

We now introduce the von Mangoldt function $\Lambda(n)$ and Chebyshev's functions $\psi(x)$ and $\theta(x)$. These will prove to be notationally useful.

Definition 1.4.1 (The von Mangoldt function)**.** Define the von Mangoldt function, $\Lambda : \mathbb{N} \to \mathbb{R}$, to be

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^m \text{ for some } m, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.4.2 (Chebyshev's ψ function [15])**.** Define Chebyshev's ψ function, $\psi : \mathbb{R} \to \mathbb{R}$, to be

$$\psi(x) := \sum_{n \le x} \Lambda(n) = \sum_{p^m \le x} \log p\ .$$

Definition 1.4.3 (Chebyshev's θ function [15])**.** Define Chebyshev's θ function, $\theta : \mathbb{R} \to \mathbb{R}$, to be

$$\theta(x) := \sum_{p \leq x} \log p \ .$$

1.5 The Riemann zeta function and its relationship with primes

The Riemann zeta function is initially defined on $\mathrm{Re}(s) = \sigma > 1$ because this is where the sum converges absolutely and uniformly. The Riemann zeta function also has an equivalent formula on the domain $\sigma > 1$, first observed by Euler. The following is know as Euler's identity

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} , \tag{1.1}$$

which is also convergent for $\sigma > 1$ [22]. The equivalence between these two expressions can be seen, as in [35], by observing that for $p^{-s} < 1$ we have

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{j=0}^{\infty} p^{-js} = \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right) .$$

Upon taking the product of these terms as in (1.1) we arrive at

$$\begin{aligned} \zeta(s) &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \\ &= \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} , \end{aligned}$$

where the final step can be concluded from Theorem 1.1.1, the Fundamental Theorem of Arithmetic [35].

The significance of the Riemann zeta function in the theory of prime numbers arises as a result of the way in which it combines two expressions, one of which contains primes explicitly while the other does not.

Of particular interest are the locations of the zeros of the Riemann zeta function, as it can be shown that this gives us information about the location of prime numbers.

By considering the product formula (1.1), which remains valid for $\sigma > 1$, we can see that there can be no zeros of the zeta function with $\sigma > 1$ as a convergent product can only be zero if one of its terms is zero. It was proved independently by Hadamard and de la Vallée Poussin in 1896 that there are no zeros of $\zeta(s)$ on the line $\sigma = 1$ [35]. Both of these mathematicians had the objective of proving the prime number theorem, Theorem 1.2.1.

1.5.1 Some properties of $\zeta(s)$

The Riemann zeta function can be extended analytically to the whole of the complex plane with the exception of a simple pole at $s = 1$. It satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s)\zeta(1-s) \, , \tag{1.2}$$

where $\Gamma : \mathbb{C} \setminus \{z \in \mathbb{Z} \mid z \leq 0\} \to \mathbb{C}$ denotes the Gamma function, defined as

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx \, .$$

The Gamma function, which is an extension of the factorial function satisfies $\Gamma(n) = (n-1)!$ for all positive integers n. For all complex numbers in its domain, it satisfies $\Gamma(s+1) = s\Gamma(s)$.

The functional equation (1.2) can be put into a more symmetrical form by using the properties of $\Gamma(s)$. We introduce the function $\xi : \mathbb{C} \to \mathbb{C}$, defined as

$$\xi(s) = \tfrac{1}{2} s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s) \, ,$$

which is entire. The ξ function has a simpler functional equation,

$$\xi(1-s) = \xi(s)$$

by which the symmetry of its zeros may be easily observed. This is helpful because its zeros are closely linked with those of $\zeta(s)$ [19].

Theorem 1.5.1. *(i) The zeros of $\xi(s)$ (if any exist) are all situated in the strip $0 \leq \sigma \leq 1$, and lie symmetrically about the lines $t = 0$ and $\sigma = \frac{1}{2}$.*
(ii) The zeros of $\zeta(s)$ are identical (in position and order of multiplicity) with those of $\xi(s)$, except that $\zeta(s)$ has a simple zero at each of the points $s = -2, -4, -6, \ldots$.

To get some insight into why this is the case, we can first express $\xi(s)$ as

$$\xi(s) = (s-1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s+1)\zeta(s) = h(s)\zeta(s),$$

by using the property that $\Gamma(n+1) = n\Gamma(n)$. Now we know by Euler's product formula that $\zeta(s)$ has no zeros in $\sigma > 1$ and the same is true of $h(s)$. Hence $\xi(s)$ has no zeros in $\sigma > 1$ and, because $\xi(s) = \xi(1-s)$ there are also no zeros in $\sigma < 0$. Therefore all zeros of $\xi(s)$ lie in $0 \leq \sigma \leq 1$.

Any zeros of $\xi(s)$ will be symmetrical about the real axis because $\xi(\sigma+it) = \overline{\xi(\overline{\sigma+it})}$, and so $\xi(\sigma+it)$ and $\xi(\sigma-it)$ are conjugates. There is also symmetry about the point $s = \frac{1}{2}$ since $\xi(s) = \xi(1-s)$. Together these imply that the zeros of $\xi(s)$ are symmetric about the line $\sigma = \frac{1}{2}$.

The zeros of $\zeta(s)$ can differ from those of $\xi(s)$ only where $h(s)$ has zeros or poles. The only zero of $h(s)$ is at $s = 1$ and this is not a zero of either $\zeta(s)$ or $\xi(s)$ because it is a pole of $\zeta(s)$ and $\xi(1) = \frac{1}{2}$. The poles of $h(s)$ are at $-2, -4, -6, \ldots$, exactly the places where $\zeta(s)$ is zero but $\xi(s)$ is not zero.

This means that there are no zeros, $\rho = \beta + i\gamma$, of the Riemann zeta function with $\beta > 1$ and that the zeros can be broken into two categories. The zeros at the negative even integers are called *trivial zeros* and the zeros that lie within the so called *critical strip* of $0 \leq \beta \leq 1$ are called *non-trivial zeros.*

Hadamard and de la Vallée Poussin proved the prime number theorem in 1896 by showing the equivalent statement that there are no zeros of $\zeta(s)$ on the line $\sigma = 1$.

One result that can be shown using the prime number theorem is the following,

Proposition 1.5.2. *Let p_n be the nth prime number. Then $p_n \sim n \log n$ as $n \to \infty$.*

Proof. Since $\frac{\pi(x)\log x}{x} \to 1$ as $x \to \infty$ and $\pi(p_n) = n$, we have, as $n \to \infty$,

$$\frac{n \log p_n}{p_n} \to 1 \tag{1.3}$$

$$\log n + \log\log p_n - \log p_n \to 0$$

$$\frac{\log n}{\log p_n} + \frac{\log\log p_n}{\log p_n} \to 1$$

$$\frac{\log n}{\log p_n} \to 1\ . \tag{1.4}$$

Combining (1.3) and (1.4) proves the proposition. □

In fact, Proposition 1.5.1 is equivalent to the prime number theorem.

1.6 Zero density estimates and zero-free regions

While the Riemann hypothesis remains unresolved, there has been some progress in narrowing down the locations of the non-trivial zeros of the Riemann zeta function. We already know that there are no zeros with $\mathrm{Re}(s) > 1$ and the prime number theorem improves this to $\mathrm{Re}(s) \geq 1$. Trying to expand on this idea we introduce zero-free regions and zero density estimates.

A *zero-free region* is a region of the critical strip in which there are no zeros of the Riemann zeta function. We have already seen that the half-plane $\sigma \geq 1$ is a zero-free region.

Hadamard [17] and de la Vallée Poussin [9] proved that all non-trivial zeros of the zeta function, $\rho = \beta + i\gamma$, with $\gamma \geq 3$ lie in the region

$$\sigma < 1 - \frac{1}{c_1 \log^9|t|} ,$$

for some $c_1 > 0$. This result was later improved by de la Vallée Poussin to

$$\beta < 1 - \frac{1}{c_2 \log|\gamma|} ,$$

again for $\gamma \geq 3$, where c_2 is a positive constant. This is referred to as the classical zero-free region.

This was the best upper bound for twenty years until Littlewood proved in 1922 that for all non-trivial zeros of the zeta function, $\rho = \beta + i\gamma$, with $\gamma \geq 9$ satisfy

$$\beta < 1 - \frac{\log\log|\gamma|}{c_3 \log|\gamma|}$$

for some positive constant c_3 [35].

The largest known zero-free region is due to Korobov [25] and Vinogradov [38], who showed independently that for sufficiently large $|t|$, the Riemann zeta function has no zeros in the region defined by

$$\sigma < 1 - \frac{1}{c_4(\log\log|t|)^{\frac{1}{3}}(\log|t|)^{\frac{2}{3}}} ,$$

for c_4 a positive constant.

Ford [16] established an explicit region of this type.

Lemma 1.6.1. *For $T \geq 3$, there are no zeros of $\zeta(s)$ in the region defined by $\sigma \geq 1 - \nu(T)$, where*

$$\nu(T) = \frac{1}{57.54 \log^{2/3} T (\log\log T)^{1/3}} .$$

The best explicit constant for the classical zero-free region is due to Mossinghoff and Trudgian [26] who showed that for $|\gamma| \geq 3$,

$$\beta \leq 1 - \frac{1}{5.573412 \log|\gamma|} . \tag{1.5}$$

In order to define what a *zero density estimate* is, we first define the functions $N(T)$ and $N(\sigma, T)$. First, $N(T)$ is the number of zeros, $\rho = \beta + i\gamma$, in the critical strip, such that $0 < \gamma < T$. Then, we have $N(\sigma, T)$ is the number of zeros, $\rho = \beta + i\gamma$, such that $0 < \gamma < T$ and $\beta > \sigma$. A *zero density estimate* is an upper bound on $N(\sigma, T)$ [22].

We first present some of what is known about $N(T)$, starting with the following result from Titchmarsh, [35].

Theorem 1.6.2. *As $T \to \infty$,*

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T) .$$

Therefore $N(T) \sim \frac{T}{2\pi} \log \frac{T}{2\pi}$, so that in particular, $N(T) = O(T \log T)$.

This asymptotic result is made explicit by using Corollary 1 of [36].

Theorem 1.6.3. *We have*

$$N(T) < \frac{T \log T}{2\pi}$$

for all $T > 15$.

For $N(\sigma, T)$, while there are many asymptotic results, there are very few explicit results. If the Riemann hypothesis was true, then we would have

$$N(\sigma, T) = 0 \text{ for all } \sigma > \frac{1}{2} ,$$

but even without the Riemann hypothesis, from what we know about $N(T)$ we can say

$$N(\sigma,T) \le N(T) < \frac{T\log T}{2\pi}$$

for $\frac{1}{2} < \sigma < 1$, for $T > 15$.

A stronger result than this, due to Bohr and Landau, is presented in Titchmarsh [35].

Theorem 1.6.4 (Bohr and Landau, [3])**.** *For any fixed σ greater than $\frac{1}{2}$,*

$$N(\sigma,T) = O(T)\ .$$

This theorem, in combination with the fact that $N(T) \sim \frac{T}{2\pi}\log\frac{T}{2\pi}$ means that all but an infinitesimal proportion of the zeros of $\zeta(s)$ lie in the strip $\frac{1}{2} - \delta < \sigma < \frac{1}{2} + \delta$, for any fixed δ [35]. A stronger result, presented in Titchmarsh [35], is due to Ingham.

Theorem 1.6.5 (Ingham [21])**.** *We have, for $\frac{1}{2} \le \sigma \le 1$,*

$$N(\sigma,T) = O(T^{3(1-\sigma)/(2-\sigma)}\log^5 T)\ .$$

Another result presented in Titchmarsh [35] is that

Theorem 1.6.6. *If $\zeta(\frac{1}{2}+it) = O(t^c\log^{c'} t)$, where $c' < \frac{3}{2}$, then*

$$N(\sigma,T) = O(T^{2(1+2c)(1-\sigma)}\log^5 T)$$

uniformly for $\frac{1}{2} \le \sigma \le 1$.

Ramaré achieved an explicit version of this by using Hardy and Littlewood's result that $\zeta(\frac{1}{2}+it) = O(t^{1/6+\varepsilon})$, which was made explicit by Trudgian [36].

Theorem 1.6.7 (Ramaré [31])**.** *If we let $T \ge 2000$ and $\sigma \ge 0.52$, then*

$$N(\sigma,T) \le 9.7(3T)^{8(1-\sigma)/3}\log^{5-2\sigma} T + 103\log^2 T\ .$$

This estimate is useful because it is explicit. Note that as $\frac{1}{6}+\varepsilon$ is not the smallest known c such that $\zeta(\frac{1}{2}+it) = O(t^c)$, stronger explicit results could be proved by using smaller values of c, though this would require some work.

Kadiri also proved explicit bounds for $N(\sigma,T)$ valid in the range $T \ge H_0$, where $H_0 = 3.061 \times 10^{10}$. This H_0 is the height to which the Riemann hypothesis has been verified — see Theorem 1.2.5.

Theorem 1.6.8 (Kadiri [23]). *Let $\sigma \geq 0.55$ and $T \geq H_0$. Let σ_0 and H be such that $0.5208 < \sigma_0 < 0.9723$, $\sigma_0 < \sigma$, and $10^3 \leq H \leq H_0$. There there exist b_1, b_2, b_3, positive constants depending on σ, σ_0, H, such that:*

$$N(\sigma, T) \leq b_1(T - H) + b_2 \log TH + b_3 \ .$$

This can be rewritten as $N(\sigma, T) \leq c_1 T + c_2 \log T + c_3$, for $T \geq H + 0$. Values for b_i and c_i are presented in Table 1 of Kadiri's paper [23].

For example, for $\sigma \geq \frac{17}{20}$ and $T \geq H_0$, we have

$$N(\sigma, T) \leq 0.5561T + 0.7586 \log T - 268658 \ .$$

This is better than Theorem 1.6.7 for only finitely many T.

1.7 Riemann-von Mangoldt explicit formula

The Riemann-von Mangoldt explicit formula provides a connection between the location of primes and zeros of the Riemann zeta function. It is as follows,

$$\psi(x) = \sum_{p^m \leq x} \log p = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2} \log(1 - x^{-2}) \ , \tag{1.6}$$

where the left hand sum is over prime powers and the right hand sum is over the non-trivial zeros, $\rho = \beta + i\gamma$, of $\zeta(s)$.

In some circumstances, it is more useful to consider a truncated version of the formula, in which the sum on the right hand side runs over non-trivial zeros below a certain point, rather than over all non-trivial zeros [39]. It is as follows,

$$\psi(x) = x - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + O\left(\frac{x \log^2 x}{T} \right) \ . \tag{1.7}$$

1.8 Some results on the Riemann hypothesis

We conclude this chapter by reviewing some results that assume the Riemann hypothesis. If the Riemann hypothesis is true, then results about $N(\sigma, T)$ which are used to give results about primes in intervals, can be replaced by $N(\sigma, T) = 0$ for $\sigma > \frac{1}{2}$. This is much stronger leads to correspondingly stronger results about primes.

Two results on the Riemann hypothesis given by Dudek [10], [11] are the following.

Theorem 1.8.1. *Suppose the Riemann hypothesis is true. Then there is a prime in the interval* $(x - \frac{4}{\pi}\sqrt{x}\log x, x]$ *for all* $x \geq 2$.

This explicit result of a theorem of Cramér [8] which states it is possible to find a positive constant c such that $\pi(x + c\sqrt{x}\log x) - \pi(x) > \sqrt{x}$ for $x \geq 2$. It is also close to proving Legendre's conjecture.

Theorem 1.8.2. *Suppose the Riemann hypothesis is true and let* $\varepsilon > 0$. *Then there is a prime in the interval* $(x - (1+\varepsilon)\sqrt{x}\log x, x]$ *for all sufficiently large values of* x.

Dudek's result, Theorem 1.9.2, gives the constant in the theorem of Cramér as $c = 1$. This has been improved to $c = \frac{21}{25}$ in [5].

Another result from [5] is

Theorem 1.8.3. *Assume the Riemann hypothesis. Then for* $x \geq 4$ *there is always a prime in the interval* $[x, x + \frac{22}{25}\sqrt{x}\log x]$.

This takes 5 pages to prove, which is a lot of work to reduce $\frac{4}{\pi}$ to $\frac{22}{25}$. This demonstrates the difficulty of these problems.

There are many other interesting properties and results on $\zeta(s)$ which could be included here. We have presented a selection of results that are useful for examining primes in intervals, which is the primary focus of this thesis. For more details on the zeta function and its properties, we refer the reader to the excellent text by Titchmarsh [35].

Chapter 2

Primes between consecutive cubes

In this chapter we first explore Dudek's proof that there is a prime between consecutive cubes, n^3 and $(n+1)^3$, for all $n \geq \exp(\exp(33.3))$ [11]. We then provide some minor improvements to the result.

2.1 Dudek's proof of primes between sufficiently large consecutive cubes

To motivate the investigation into the existence of primes between consecutive cubes we recall Legendre's conjecture, that there exists a prime between every pair of consecutive squares. This question, though it is very simple to state, is unresolved, even on assumption of the Riemann hypothesis.

The question of existence of primes between cubes is easier in the sense that the interval $(x^3, (x+1)^3)$ contains the interval $(y^2, (y+1)^2)$ where $x^3 = y^2$ and so primes between squares implies primes between cubes. In fact primes between mth powers implies primes between any powers larger than m.

We first note a result of Dudek [11], which will be used in this chapter to prove that there are primes between sufficiently large consecutive cubes.

Theorem 2.1.1. *Let $x > e^{60}$ be half an odd integer and suppose that $50 < T < x$.*

Then

$$\psi(x) = x - \sum_{|\gamma| \le T} \frac{x^\rho}{\rho} + O^*\left(\frac{2x\log^2 x}{T}\right). \tag{2.1}$$

This is an explicit version of the truncated Riemann-von Mangoldt explict formula. What makes this an explicit version is the constant '2' and the use of the $O^*(\cdot)$ in contrast to the $O(\cdot)$ used in the standard truncated Riemann-von-Mangoldt explicit formula (1.7).

Lemma 2.1.2. *There is a prime in the interval* $(x, x + 3x^{2/3})$ *for all* $x \le e^{60}$.

Dudek [11] states that this result comes from an application of Theorem 2 in Ramaré and Saouter's paper [32], which states that there is a prime in the interval $(x(1-\delta), x]$ for all $x > x_0$ where δ is given as a function of x_0 in Table 1 of their paper. Dudek claims that one only needs to check that the Ramaré–Saouter interval is contained in the short interval, to confirm that there is a prime in $(x, x + 3x^{2/3})$ for all $x \le e^{60}$, but this is not quite correct. A complete proof is as follows.

Proof. Theorem 2 of Ramaré and Saouter [32] can be used to show that there is a prime in the interval $(x, x + 3x^{2/3}]$ in the way that Dudek suggests, but only for $e^{46} \le x \le e^{60.7732}$.

As an example, taking the first row of Table 1 in [32] gives the following version of Theorem 2 in [32].

Lemma 2.1.3. *Let* y *be a real number such that* $y \ge e^{46}$. *Then the interval* $((1-\delta)y, y]$ *contains at least one prime where* $\delta = 1/81353847$ *[32]. Alternatively, for* $x \ge e^{46}(1-\delta) \in \mathbb{R}$, *there is a prime in the interval* $(x, x\frac{1}{1-\delta}]$ *where* $\delta = 1/81353847$.

It is easy to check that the interval $(x, x\frac{\Delta}{\Delta-1}]$ is contained in the short interval $(x, x + 3x^{2/3}]$ when $x \le e^{59.93}$, whence there is a prime in the interval $(x, x + 3x^{2/3}]$ for $e^{46} \le x \le e^{59.93}$.

By also considering the other rows of Table 1 of [32], it can be seen that there is a prime in the interval $(x, x+3x^{2/3}]$ for $e^{46} \le x \le e^{60.7732}$. This leaves a large gap from $x = 1$ to $x = e^{46}$ which Dudek does not address.

We now consider Theorem 3 of Ramaré and Saouter [32]. We have rearranged it in the same way as Lemma 2.1.3 to get

Lemma 2.1.4. *Let x be a real number larger than $10726905041(1-\delta)$ where $\delta = 1/28314000$. Then the interval $(x, x\frac{1}{1-\delta}]$ contains a prime.*

By testing when this interval in contained in the short interval $(x, x+3x^{2/3}]$, we can conclude there must be a prime in the interval $(x, x+3x^{2/3}]$ for $e^{23.096} < x < e^{54.7724}$.

Next recall Lemma 1.3.5, that there is a prime in the interval $(x, x(1-\delta)]$ for $x \geq 2010760$ where $\delta = 1/16596$.

Again, we can see that there is a prime in the interval $(x, x+3x^{2/3}]$ for $e^{14.514} < x < e^{32.4466}$.

Now, $e^{14.514} < 3 \times 10^6$ and $\pi(3 \times 10^6) = 216816$. This is a small enough number of primes that the remaining cases can easily be checked using a computer. By stitching all of these intervals together, it can now be seen that there is a prime in the interval $(x, x+3x^{2/3}]$ for all $x \leq e^{60}$. □

Having shown there is a prime in $(x, x+3x^{2/3}]$ for $x \leq e^{60}$ we may now assume that $x > e^{60}$. This allows us to use Dudek's explicit version of the truncated Riemann–von Mangoldt explicit formula (2.1.1), to find how large n must be so that we can guarantee a prime in the interval $(x, x+3x^{2/3}]$.

We now prove the following.

Theorem 2.1.5 (Dudek [11])**.** *There is a prime between n^3 and $(n+1)^3$ for all $n \geq \exp(\exp(33.3))$.*

Proof. Note that Cheng [7] claims to give this result for the range $n \geq \exp\exp(15)$ however there are errors in the proof. For a brief discussion of these issues see [11].

We first introduce the following notation which will be convenient for expressing the problem.

$$\theta_{x,h} = \theta(x+h) - \theta(x) = \sum_{x<p\leq x+h} \log \ .$$

This notation is helpful because $\theta_{x,h}$ is positive if and only if there is a prime in the interval $(x, x+h]$. To use this notation for primes between cubes we set $h = 3x^{2/3}$.

Note that we could use $h = 3x^{2/3} + 3x^{1/3} + 1$ but the impact on the result would be negligible given that we are considering $x > e^{60}$. It is also simpler to use $h = 3x^{2/3}$ in the ensuing analysis.

If we can show, for some x_0, that $\theta_{x,h} > 0$ for all $x > x_0$, then there is a prime in the interval $(x, x + 3x^{2/3}]$ for all $x > x_0$. By substituting $x = n^3$ we would get that there is a prime in the interval $(n^3, n^3 + 3n^2] \subset (n^3, (n+1)^3]$ for all $n > n_0 = {x_0}^{1/3}$.

Theorem 2.1.1 is applied to get that

$$\begin{aligned}\psi(x+h) - \psi(x) = \ & (x+h) - \sum_{|\gamma| \le T} \frac{(x+h)^\rho}{\rho} + O^*\left(\frac{2(x+h)\log^2(x+h)}{T}\right) \qquad (2.2)\\ & - x + \sum_{|\gamma| \le T} \frac{x^\rho}{\rho} - O^*\left(\frac{2x\log^2 x}{T}\right)\\ & \ge h - \left|\sum_{|\gamma| \le T} \frac{(x+h)^\rho - x^\rho}{\rho}\right| - \frac{4(x+h)\log^2(x+h)}{T}\ . \qquad (2.3)\end{aligned}$$

This is information about $\psi(x+h) - \psi(x)$ rather than $\theta(x+h) - \theta(x)$ but, as

$$\psi(x) = \sum_{p^m < x} \log p$$

and

$$\theta(x) = \sum_{p < x} \log p$$

we see that these functions are not too different. In fact

$$\psi(x) = \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + \dots$$

so

$$\psi(x) - \theta(x) \sim x^{1/2}\ .$$

An explicit version of this is the following combination of Proposition 3.1 of Dusart [12] and Corollary 2 of Platt and Trudgian [30].

Lemma 2.1.6. *If* $x \ge 121$, *then*

$$0.9999x^{1/2} < \psi(x) - \theta(x) < (1 + 7.5 \times 10^{-7})x^{1/2} + 3x^{1/3}\ .$$

We can apply this lemma to (2.3) to yield

$$\begin{aligned}
\theta_{x,h} &= \theta(x+h) - \theta(x) \\
&> \Big(\psi(x+h) - (1+7.5\times10^{-7})(x+h)^{1/2} + 3(x+h)^{1/3}\Big) - \Big(\psi(x) - 0.9999x^{1/2}\Big) \\
&= (\psi(x+h) - \psi(x)) + 0.9999x^{1/2} - (1+7.5\times10^{-7})(x+h)^{1/2} - 3(x+h)^{1/3} \\
&> h - \left|\sum_{|\gamma|<T} \frac{(x+h)^\rho - x^\rho}{\rho}\right| - \frac{4(x+h)\log^2(x+h)}{T} \\
&\quad - (1+7.5\times10^{-7})(x+h)^{1/2} - 3(x+h)^{1/3} + 0.9999x^{1/2}
\end{aligned}$$

The sum in this inequality needs to be estimated, so it is assigned a variable,

$$S = \left|\sum_{|\gamma|<T} \frac{(x+h)^\rho - x^\rho}{\rho}\right| .$$

It can be seen that

$$S = \left|\sum_{|\gamma|<T} \int_x^{x+h} t^{\rho-1}\, dt\right| \le \sum_{|\gamma|<T} \int_x^{x+h} t^{\beta-1}\, dt \le h \sum_{|\gamma|<T} x^{\beta-1} . \tag{2.4}$$

At this stage, we pause to introduce some useful identities and bounds. First, the identity

$$\sum_{|\gamma|<T} (x^{\beta-1} - x^{-1}) = \sum_{|\gamma|<T} \int_0^\beta x^{\sigma-1}\log x\, d\sigma = \int_0^1 \sum_{\beta>\sigma,\, |\gamma|<T} x^{\sigma-1}\log x\, d\sigma$$

can be reformulated to give,

$$\begin{aligned}
\sum_{|\gamma|<T} x^{\beta-1} &= \sum_{|\gamma|<T} x^{-1} + \int_0^1 \sum_{\beta>\sigma,\, |\gamma|<T} x^{\sigma-1}\log x\, d\sigma \\
&= 2x^{-1}N(T) + 2\int_0^1 N(\sigma,T)x^{\sigma-1}\log x\, d\sigma ,
\end{aligned}$$

recalling that $N(T)$ equals the number of zeros with $0 < \gamma < T$ and $N(\sigma, T)$ is the number of these zeros with $\beta > \sigma$.

This brings our estimation of S to

$$S \le h \times \left(2x^{-1}N(T) + 2\int_0^1 N(\sigma,T)x^{\sigma-1}\log x\, d\sigma\right) . \tag{2.5}$$

Recall Theorem 1.6.3: for all $T > 15$, we have

$$N(T) < \frac{T \log T}{2\pi} \ . \tag{2.6}$$

Now recall the zero density estimate Theorem 1.6.7: for $T \geq 2000$ and $\sigma \geq 0.52$, we have

$$N(\sigma, T) \leq 9.7(3T)^{8(1-\sigma)/3} \log^{5-2\sigma} T + 103 \log^2 T \ .$$

We notice when comparing the two terms in this bound that $103 \log^2 T$ is not significant when compared to $9.7(3T)^{8(1-\sigma)/3} \log^{5-2\sigma} T$.

We also observe that for $0 \leq \sigma \leq 5/8$ this bound is at best

$$N(\sigma, T) \leq 9.7 \times 3T \log^{15/4} T + 103 \log^2 T \ ,$$

which is worse than taking $N(\sigma, T) \leq N(T)$ and using (2.6). As σ approaches 1 however the bound approaches

$$9.7 \log^3 T + 103 \log^2 T$$

which is noticeably better than the bound for $N(T)$.

In order to examine the impact that improving the coefficient 9.7 in the above result might have, Dudek replaces it by A in the working that follows. This is examined in [10].

It is also necessary to use the following zero-free region, derived by Ford, [16]. Recall Lemma 1.6.1, that for $T \geq 3$ there are no zeros in the region $\sigma \geq 1 - \nu(T)$.

As in the case of the 9.7 being replaced by an A, Dudek replaces the value of 57.54 by c to make it easy to see what impact an improvement would have on the final result. Again, this is examined in [10].

Now that a stock of 'tools' has been established, work may continue on estimating the sum S, by applying the results above to (2.5). Break the integral into three parts. From the discussion above, for $0 \leq \sigma \leq 5/8$, we may as well bound $N(\sigma, T)$ by $N(T)$. For $5/8 \leq \sigma \leq 1 - \nu(T)$, Theorem 1.6.7 can be used. The final piece of the integral equals 0 as a result of Lemma 1.6.1. We have,

$$
\begin{aligned}
\sum_{|\gamma|<T} x^{\beta-1} &= 2x^{-1}N(T) + 2\int_0^1 N(\sigma,T)x^{\sigma-1}\log x \, d\sigma \\
&= 2x^{-1}N(T) + 2\int_0^{5/8} N(\sigma,T)x^{\sigma-1}\log x d\sigma + 2\int_{5/8}^{1-\nu(T)} N(\sigma,T)x^{\sigma-1}\log x \, d\sigma \\
&< 2x^{-1}N(T) + 2x^{-1}N(T)\log x\int_0^{5/8} x^{\sigma}\, d\sigma \\
&\quad + 2Ax^{-1}(3T)^{8/3}\log x\log^5 T\int_{5/8}^{1-\nu(T)}\left(\frac{x}{(3T)^{8/3}\log^2 T}\right)^{\sigma} d\sigma \\
&\quad + 103x^{-1}\log x\log^2 T\int_{5/8}^{1-\nu(T)} x^{\sigma}\, d\sigma \qquad (2.7)
\end{aligned}
$$

Dudek's omits the working on account of it being “routine but highly tedious”.

A parameter k is now introduced to give the relationship between T and x where $k \in (\frac{2}{3}, 1)$. Then, $T = T(k,x)$ is set to be the solution to

$$
\frac{x}{(3T)^{8/3}\log^2 T} = \exp(\log^k x) \qquad (2.8)
$$

which means that

$$
x = \exp(\log^k x)(3T)^{8/3}\log^2 T > T^{8/3} .
$$

By integrating (2.7), applying (2.8), using the bound (2.6) for $N(T)$ and considering that $\log T < \frac{3}{8}\log x$, the sum simplifies to

$$
\begin{aligned}
\sum_{|\gamma|<T} x^{\beta-1} &< \frac{e^{-\frac{3}{8}\log^k x}\log^{1/4} x}{3^{3/4}8^{1/4}\pi} + \frac{27A}{256}\log^{4-k} x(e^{-\nu(T)\log^k x} - e^{-(3/8)\log^k x}) \\
&\quad + \frac{927A}{32}\log^2 x(e^{-\nu(T)\log x} - x^{-3/8}) . \qquad (2.9)
\end{aligned}
$$

We estimate one of the exponential terms involving $\nu(T)$,

$$
\begin{aligned}
e^{-\nu(T)\log x} &= \exp\left(-\frac{\log x}{c\log^{2/3} T(\log\log T)^{1/3}}\right) \\
&< \exp\left(-\frac{4}{3^{2/3}c}\left(\frac{\log x}{\log\log x}\right)^{1/3}\right) . \qquad (2.10)
\end{aligned}
$$

Next, by expanding (2.9) and using (2.10), Dudek observes that

$$-\frac{27}{256}(\log x)^{4-k}e^{-(3/8)\log^k x}+\frac{927A}{32}\log^2 x(e^{-\nu(T)\log x}-x^{-3/8})<0$$

by looking at the dominant terms, verifying by dividing though by A and insisting that $c \le 57.54$, $k \in (\frac{2}{3}, 1)$ and $x > e^{60}$.

This allows us to conclude that

$$\sum_{|\gamma|<T} x^{\beta-1} < \frac{e^{\frac{3}{8}\log^k x}\log^{1/4} x}{3^{3/4}8^{1/4}\pi}+\frac{27A}{256}(\log x)^{4-k}e^{-\nu(T)\log^k x}\,.$$

We are omitting some terms by doing this, however the omitted terms are small compared to those that are retained.

With a little further work dealing with the remaining exponential term in the same way as the previous one, we obtain

$$\theta_{x,h} > h - f(x,h,k,A,c) - g(x,h,k) - E(x,h,k)$$

where

$$f(x,h,k,A,c)=\frac{27Ah}{256}(\log x)^{4-k}\exp\left(-\frac{4}{3^{2/3}c}\frac{\log^{k-2/3} x}{(\log\log x)^{1/3}}\right)$$

$$g(x,h,k)=12\left(\frac{3}{8}\right)^{3/4}\frac{(x+h)\log^{11/4}(x+h)}{x^{3/8}}\exp\left(\frac{3}{8}\log^k x\right)$$

$$\begin{aligned}E(x,h,k)=&\frac{h(\log x)^{1/4}\exp(-\frac{3}{8}\log^k x)}{6^{3/4}\pi}+(1+7.5\times 10^{-7})(x+h)^{1/2}\\&+3(x+h)^{1/3}+0.9999x^{1/2}\,.\end{aligned}$$

There is a slight error in Dudek's work that has been corrected here. The $E(x,h,k)$ that Dudek presents is the negative of the $E(x,h,k)$ given here which is the correct one. This error does not carry though to the final answer though, so it is of little importance.

The error term $E(x,h,k)$ is then bounded, noting that $x > e^{60}$, $h = 3x^{2/3}$ and the fact that $k = \frac{2}{3}$ will give the worst possible error term. Indeed,

$$\frac{E(x,h,k)}{h}=\frac{E(x,3x^{2/3},\frac{2}{3})}{3x^{2/3}}<10^{-3}\,. \tag{2.11}$$

In order to simplify the problem somewhat, without significantly changing the quality of the solution, one aims to satisfy (2.12) and (2.13) simultaneously. We do this by insisting that

$$f(x,h,k,A,c) < \frac{1}{2}(1-10^{-3})h \tag{2.12}$$

and

$$g(x,h,k) < \frac{1}{2}(1-10^{-3})h \ . \tag{2.13}$$

Satisfying both (2.12) and (2.13) will mean that

$$\begin{aligned}\theta_{x,h} &> h - f(x,h,k,A,c) - g(x,h,k) - E(x,h,k)\\ &> h - \tfrac{1}{2}(1-10^{-3})h - \tfrac{1}{2}(1-10^{-3})h - 10^{-3}h\\ &\geq 0 \ ,\end{aligned}$$

owing to (2.11), which ensures that there is a prime in the interval $[x, x+h)$.

This approach simplifies the problem by allocating half of the 'space' that is left over after the error term $E(x,h,k)$ is bounded to the $f(x,h,k,A,c)$ term and the other half to $g(x,h,k)$ term. Dudek claims that this $50-50$ split does not have a significant impact on the final solutions. He considers the better-than-possible scenario where, rather than the current $\frac{1}{2}(1-10^{-3})h$, the right hand side of each of (2.12) and (2.13) is equal to h.

For interest, removing the $50-50$ split gives an actual improvement which is almost as good as what can be obtained in the better-than-possible scenario above.

For (2.12), take the logarithm of both sides and set $x = e^y$ to get

$$\log\left(\frac{27A}{256}\right) + (4-k)\log y - \frac{4}{3^{2/3}c}\frac{y^{k-2/3}}{\log^{1/3} y} < \log\left(\frac{1}{2}(1-10^{-3})\right) \ . \tag{2.14}$$

We treat (2.13) similarly. First notice that

$$\frac{g(x,h,k)}{h} < \frac{2\log^{11/4} x}{x^{1/24}} \exp\left(\frac{3}{8}\log^k x\right) \ . \tag{2.15}$$

Applying the same method as before, setting $x = e^y$ and taking the logarithm of both sides, we get that we need

$$\frac{11}{4}\log y + \frac{3}{8}y^k - \frac{1}{24}y < \log\left(\frac{1}{4}(1-10^{-3})\right) \ . \tag{2.16}$$

A value of $k \in (\frac{2}{3}, 1)$ will be chosen to satisfy both (2.14) and (2.16) while keeping y as small as possible. Using *Mathematica*, and choosing $k = 0.9359$, both (2.14) and (2.16) are satisfied for $y > 8 \times 10^{14}$ or $n = x^{1/3} > \exp(\exp(33.217))$. In other words, there will be a prime in the interval $[n^3, (n+1)^3)$ for all $n > \exp(\exp(33.217))$.

□

2.2 Improvements for primes between cubes

There are a few places where improvements can be made.

First, rather than using the assumption of a $50-50$ split to simplify the problem, we introduce a parameter λ in place of the $\frac{1}{2}$ and 'split the space' that is left over after the error term into $\lambda(1-10^{-3})$ and $(1-\lambda)(1-10^{-3})$ instead. This means that we will now need to satisfy both

$$f(x,h,k,A,c) < \lambda(1-10^{-3})h$$

and

$$g(x,h,k) < (1-\lambda)(1-10^{-3})h$$

instead of (2.12) and (2.13) to guarantee a prime in the interval $[x, x+3x^{2/3})$.

The same method of setting $x = e^y$ and taking the logarithm on both sides means that (2.14) is now replaced by

$$\begin{aligned} f(x,h,k,A,c) &< \lambda(1-10^{-3})h \\ \frac{27A}{256}(\log e^y)^{4-k}\exp\left(-\frac{4}{3^{2/3}c}\frac{\log^{k-2/3} e^y}{(\log\log e^y)^{1/3}}\right) &< \lambda(1-10^{-3}) \\ \log\left(\frac{27A}{256}\right) + \log(y^{4-k}) + \left(-\frac{4}{3^{2/3}c}\frac{y^{k-2/3}}{(\log(y))^{1/3}}\right) &< \log(\lambda(1-10^{-3})) \\ \log\left(\frac{27A}{256}\right) + (4-k)\log(y) - \frac{4}{3^{2/3}c}\frac{y^{k-2/3}}{\log^{1/3} y} &< \log(\lambda(1-10^{-3}))\ , \end{aligned} \tag{2.17}$$

and similarly (2.16) is updated to

$$\frac{11}{4}\log y + \frac{3}{8}y^k - \frac{1}{24}y < \log\left(\frac{(1-\lambda)}{2}(1-10^{-3})\right)\ .$$

A second improvement can be made by reducing the value of the constant '2' in (2.15) slightly. We have,

$$
\begin{aligned}
\frac{g(x,h,k)}{h} &= 12\frac{3}{8}^{3/4}\frac{(x+h)\log^{11/4}(x+h)}{x^{3/8}h}\exp(\frac{3}{8}\log^k x) \\
&= \frac{3^{3/4}(1+3x^{-1/3})}{2^{1/4}} \times \frac{x\log^{11/4}(x+3x^{2/3})}{x^{25/24}}\exp(\frac{3}{8}\log^k x) \\
&< 1.91683 \times \frac{\log^{11/4}(x(1+3x^{-1/3}))}{x^{1/24}}\exp(\frac{3}{8}\log^k x) \\
&= 1.91683 \times \frac{(\log(x)+\log(1+3x^{-1/3}))^{11/4}}{x^{1/24}}\exp(\frac{3}{8}\log^k x) \\
&= 1.91683\frac{\log(x)^{11/4}}{x^{1/24}}\exp(\frac{3}{8}\log^k x) \times \left(1+\frac{\log(1+3x^{-1/3})}{\log(x)}\right)^{11/4} \\
&\leq 1.91684\frac{\log(x)^{11/4}}{x^{1/24}}\exp(\frac{3}{8}\log^k x) \ .
\end{aligned}
$$

This uses that $x > e^{60}$. We retain the 1.91684, which Dudek rounds to 2, leading to a very slight improvement in the results that will be obtained. More generally, with m not necessarily equal to 3, we will use the following bound in place of (2.15)

$$\frac{g(x,h,k)}{h} < \frac{5.78052}{m} \times \frac{\log(x)^{11/4}}{x^{(3/8-1/m)}}\exp(\frac{3}{8}\log^k x) \ .$$

This change carries though to (2.13), which including the changes that occurred when the parameter λ was introduced, is replaced by

$$\frac{11}{4}\log(y) - \left(\frac{3}{8}-\frac{1}{m}\right)y + \frac{3}{8}y^k < \log\left(\frac{(1-\lambda)m}{5.78052}(1-10^{-3})\right) \ . \tag{2.18}$$

We now need to satisfy (2.17) and (2.18) in order to guarantee a prime in the interval $[n^3,(n+10^3)$.

We note that the potential improvement to the result through a better bound on $E(x,h,k)$ is negligible. Even with the impossibly good $E(x,h,k) < 0$, the amount of improvement to final answer is tiny when compared to other improvements, being only $x^{1/3} = \exp(\exp(33.2142))$ as opposed to $\exp(\exp(33.217))$.

The "Manipulate" function in *Mathematica* is then used to find values of k and λ so that (2.17) and (2.18) are satisfied simultaneously. Setting $k = 0.935906887387$, $\lambda = 1 - 10^{-15}$ we can take $y = 7.7794 \times 10^{14}$ with (2.17) and (2.18) both satisfied. This value for y, since $x = e^y$ and $n = x^{1/3}$, corresponds to $n = \exp(\exp(33.1891))$.

Chapter 3

Extension to higher powers

All of the machinery that was developed in proving the primes between cubes result can also be used for higher powers. If we choose $m > 3$ and find $n_0(m)$ such that for all $n > n_0(m)$ there is a prime between n^m and $(n+1)^m$, we expect that $n_0(m)$ should decrease as m increases because the intervals grow. One might then ask, given that it has been shown that there is a prime in the interval $[n^3, (n+1)^3)$ for all n larger than $\exp(\exp(33.3))$, can we find a value of m such that there is a prime in the interval $[n^m, (n+1)^m)$ for all $n > 1$? If so, how large is m?

This question is also answered by Dudek in [11], but it is done in a way that shows how the work on cubes can be extended to higher powers without regard for the optimality of the result, to quote "we shall leave it to others to attempt to bring the value [of m] down" [10] (p41).

In this section we will explore the way in which we can extend the results of Chapter 2 to mth powers. We first detail Dudek's work on this topic and then refine the process and improve the result.

Note that §3.1 presents Dudek's work without implementing the changes and improvements made in Chapter 2 while §3.2 includes the modifications from Chapter 2 as well as making further improvements to the result.

3.1 Dudek's work on higher powers

To apply the method used for primes between cubes to primes between mth powers we must change the values of m and h that are used. In the cubes case, $h = 3x^{2/3}$ but for mth powers h is set to be $mx^{(m-1)/m}$. We could instead take h to be

$$(x^{1/m}+1)^m - (x^{1/m})^m = mx^{(m-1)/m} + \binom{m}{2} x^{(m-2)/m} + \dots ,$$

but, as Dudek notes, "the improvements [would be] negligible" [10] (p43).

The two inequalities that need to be satisfied now become (as they appear in [11])

$$\log\left(\frac{27A}{256}\right) + (4-k)\log y - \frac{4}{3^{2/3}c}\frac{y^{k-2/3}}{\log^{1/3} y} < \log\left(\frac{1}{2}(1-10^{-3})\right) \tag{3.1}$$

and

$$\frac{11}{4}\log y - \left(\frac{3}{8} - \frac{1}{m}\right) y + \frac{3}{8} y^k < \log\left(\frac{m}{12}(1-10^{-3})\right) . \tag{3.2}$$

Dudek [11] gives the following table,

m	k	$\log\log n_0$
4	0.9635	29.240
5	0.9741	27.820
6	0.9796	27.230
7	0.983	26.427
1000	0.9998	19.807

Table 3.1: Results for $\log\log n_0$.

from which the result for $m = 1000$ is used to prove the following theorem.

Theorem 3.1.1. *Let $m \geq 4.971 \times 10^9$. Then there is a prime between n^m and $(n+1)^m$ for all $n \geq 1$.*

Before this can be proved, the following lemma is required.

Lemma 3.1.2. *For all $m > m_0 = 1000$, there is a prime in the interval $[n^m, (n+1)^m)$ for all*

$$n \geq N_0 = \exp\left(\frac{1000\exp(19.807)}{m}\right) .$$

Proof. From Table 3.1 we know there is a prime in the interval $[n^{1000}, (n+1)^{1000})$ for all $n \geq \exp(\exp(19.807))$, so, by replacing n by $n^{m/1000}$ we get that for all $n^{m/1000} \geq \exp(\exp(19.807))$ there is a prime in the interval

$$[(n^{m/1000})^{1000}, (n^{m/1000}+1)^{1000}) \subseteq [n^m, (n+1)^m) \, .$$

Since

$$n^{m/1000} \geq \exp(\exp(19.807)) \, ,$$

then

$$n \geq \exp\left(\frac{1000\exp(19.807)}{m}\right) \, , \tag{3.3}$$

and the lemma follows. □

This proof made use of the result for $m = 1000$ from Table 3.1, but there is no particular reason to choose $m = 1000$. The result can be generalised as follows.

Lemma 3.1.3. *Choose $m_0 \geq 3$. If there is a prime in the interval $[n^{m_0}, (n+1)^{m_0})$ for all $n \geq \exp(\exp(L))$ for some $L(m_0)$, then the following holds.*

For all $m > m_0$, there is a prime in the interval

$$[n^m, (n+1)^m)$$

for all

$$n \geq N_0 = \exp\left(\frac{m_0 \exp(L(m_0))}{m}\right) \, .$$

We will call m_0 the *jumping off point*. We shall discuss in §3.2 how to make a sensible choice for the jumping off point.

Lemma 3.1.4. *There is a prime in the interval $[n^m, (n+1)^m)$ for all $n \geq 1$, where $m = 1000\exp(19.807) \approx 4 \times 10^{11}$.*

Proof. This can be seen by applying Lemma 3.1.2 and then applying Theorem 1.3.3, Bertrand's postulate. We have $m = 1000\exp(19.807) > 1000$ so by Lemma 3.1.2 there is a prime in the interval $[n^m, (n+1)^m)$ for all

$$n \geq \exp\left(\frac{1000\exp(19.807)}{1000\exp(19.807)}\right) = e \, .$$

This leaves the cases $n = 1$ and $n = 2$. Bertrand's postulate shows that there is a prime in each of the intervals $(1, 2]$ and $(2^m, 2 \times 2^m]$. It is clear that

$$(1, 2] \subset [1^m, 2^m)$$

and

$$(2^m, 2 \times 2^m] \subset [2^m, 3^m)$$

for all $m \geq 1000 \exp(19.807)$. This extends the result from $n \geq e$ to $n \geq 1$. $\square$

There is a better approach which will give the smaller m seen in Theorem 3.1.1. We now present Dudek's proof from [11].

Proof. The following corollary of Trudgian [37] is used.

Corollary 3.1.5. *For all $x \geq 2898242$ there exists a prime in the interval*

$$\left[x, x\left(1 + \frac{1}{111 \log^2 x}\right)\right] .$$

Setting $x = n^m$, we see that the above interval lies within $[n^m, n^m + mn^{m-1})$ by rearranging as follows

$$\begin{aligned} n^m\left(1 + \frac{1}{111 \log^2 n^m}\right) &\leq n^m + mn^{m-1} \\ \frac{n^m}{111 \log^2 n^m} &\leq mn^{m-1} \\ \frac{1}{111 \log^2 n^m} &\leq \frac{m}{n} \\ \frac{n}{m} &\leq 111 m^2 \log^2 n \\ \frac{n}{\log^2 n} &\leq 111 m^3 . \end{aligned} \tag{3.4}$$

To find the smallest m such that both (3.3) and (3.4) are satisfied we solve these inequalities simultaneously. By substituting (3.3) directly into (3.4) we get

$$\frac{\exp\left(\frac{1000\exp(19.807)}{m}\right)}{\log^2\left(\exp\left(\frac{1000\exp(19.807)}{m}\right)\right)} = 111m^3$$

$$\exp\left(\frac{1000\exp(19.807)}{m}\right) = 111m^3\left(\frac{1000\exp(19.807)}{m}\right)^2$$

$$\exp\left(\frac{1000\exp(19.807)}{m}\right) = 111m\left(1000\exp(19.807)\right)^2 .$$

Mathematica is then used to get the bound $m \geq 4.971 \times 10^9$ as required. □

3.2 Improvements for higher powers

While the assumption of a $50-50$ split has little impact on the result in the cubes case, this is not the case for higher powers. Notice that with the assumption of a $50-50$ split removed, (3.1) (now with the improvements to the constant 2 and the introduction of λ included) becomes

$$\log\left(\frac{27A}{256}\right) + (4-k)\log(y) - \frac{4}{3^{2/3}c}\frac{y^{k-2/3}}{\log^{1/3}y} < \log(\lambda(1-10^{-3})) , \tag{3.5}$$

which is independent of m. On the other hand, (3.2) (now also with the improvements to the constant 2 and the introduction of λ included) becomes

$$\frac{11}{4}\log(y) - \left(\frac{3}{8}-\frac{1}{m}\right)y + \frac{3}{8}y^k < \log\left(\frac{(1-\lambda)m}{5.78052}(1-10^{-3})\right) , \tag{3.6}$$

which does depend on the value of m.

As m increases, the left hand side of (3.6) decreases, while the right hand side increases in value. This means that as m increases, (3.6) becomes easier to satisfy.

On the other hand, (3.5) is independent of m which means that the assumption of a $50-50$ split originally used for primes between consecutive cubes may perturb the solution away from the optimal value. This is where the parameter λ, which was introduced to gain a mild improvement in the cubes case, becomes more important.

Table 3.2 shows the results achieved by Dudek in [11] (columns 2 and 3) and the improvements obtained by introducing λ in combination with the other changes and improvements from Chapter 2 (columns 4 to 6). The value of λ for all of the results in columns 4 to 6 is $\lambda = 1 - 10^{-15}$.

Recall that $y = \log x$ where $x = n^3$ and note that y is the variable in (3.5) and (3.6) that we wish to minimise.

m	k	$\log\log(n_0)$	k	$\log\log(n_0)$	y
3	0.9359	33.217	0.935906887387	33.1891	7.7794×10^{14}
4	0.9635	29.240	0.96395749	29.1098	1.7550×10^{13}
5	0.9741	27.820	0.974	27.7098	5.4100×10^{12}
6	0.9796	27.230	0.979508006	26.9143	2.9300×10^{12}
7	0.983	26.427	0.983	26.3864	2.0164×10^{12}
10	-	-	0.98878166654	25.4237	1.1000×10^{12}
100	-	-	0.9989	22.1121	4.0100×10^{11}
1000	0.9998	19.807	0.999853	19.7253	3.68616×10^{11}
10000	-	-	0.99997	17.4119	3.64652×10^{11}
100000	-	-	0.999999	15.1062	3.6353×10^{11}
1000000	-	-	0.9999997	12.8035	3.635×10^{11}

Table 3.2: Dudek's results and improvements.

We now consider the impact of choosing a jumping off point that is different from the $m_0 = 1000$ used by Dudek [11]. Rephrased slightly, Lemma 3.1.3 states, that for all $m > m_0$, there is a prime in the interval $[n^m, (n+1)^m)$ for all

$$n \geq N_0 = \exp\left(\frac{m_0 \times \exp(LLN_0(m_0))}{m}\right), \tag{3.7}$$

where $LLN_0(m_0)$ is the $\log\log(n_0)$ value that corresponds to m_0 in Table 3.2 above.

It can be seen that

$$\begin{aligned} m_0 \times \exp(LLN_0(m_0)) &= m_0 \times \exp(\log\log n_0) \\ &= m_0 \times \exp\left(\log \frac{y(m_0)}{m_0}\right) \\ &= y(m_0), \end{aligned}$$

that is, the numerator of the fraction in (3.7) is simply the y value corresponding to m_0 in Table 3.2 above.

We now consider what makes a good choice for the jumping off point m_0. We want to find the smallest possible m for which we can show there is a prime in the interval

$[n^m, (n+1)^m)$ for all $n \geq 1$. This will be achieved by an application of Lemma 3.1.3, which will give the result for all $n \geq N_0$ for some N_0. We shall then cover the gap between 1 and N_0 with some form of the prime number theorem.

Suppose that we have found the largest value that N_0 for which we can cover the gap, that is fix N_0. Our goal is to make m as small as we can, so since rearranging (3.7) gives

$$m = \frac{m_0 \times \exp(LLN_0(m_0))}{\log(N_0)} ,$$

and since N_0 is fixed, we should choose m_0 such that

$$m_0 \times \exp(LLN_0(m_0)) = y(m_0) = y$$

is as small as possible. By inspecting Table 3.2 we can see that y gets smaller as m_0 gets larger. However, very little improvement in the value of y is observed by increasing m_0 from 1000 to 10000. By setting $m_0 = 1000000$, $k = 0.9999997$ and $\lambda = 1 - 10^{-20}$, we get what appears to be a good value for $y(m_0)$ is $y = 3.635 \times 10^{11}$.

One approach that was used in order to try and reduce the value of m was to use a result similar to Corollary 3.1.5 but which guarantees a prime in smaller intervals.

One such result that we might try to use is the following result of Dusart [14].

Corollary 3.2.1. *For all $x \geq 468991632$, there exists a prime p such that*

$$x < p \leq x\left(1 + \frac{1/5000}{\log^2 x}\right) .$$

By setting $x = n^m$ this tells us that for all $n^m \geq 468991632$, there is a prime p such that

$$n^m < p \leq n^m\left(1 + \frac{1/5000}{\log^2 n^m}\right) .$$

Of interest is how small m can be such that we have

$$[n^m, n^m + mn^{m-1}) \subset [n^m, (n+1)^m)$$

for all $n < N_0 = \exp\left(\frac{y(m_0)}{m}\right)$. To find the smallest value of m we solve simultaneously

$$n = \exp\left(\frac{3.635 \times 10^{11}}{m}\right)$$

and

$$n^m\left(1+\frac{1/5000}{\log^2 n^m}\right)=n^m+mn^{m-1}\ .$$

By simplifying

$$\begin{aligned}n^m\left(1+\frac{1/5000}{\log^2 n^m}\right)&=n^m+mn^{m-1}\\ \frac{1}{m^2\log^2 n}&=5000mn^{-1}\\ \frac{n}{\log^2 n}&=5000m^3\ ,\end{aligned}$$

and substituting for n, and rearranging to get

$$\exp\left(\frac{3.635\times 10^{11}}{m}\right)=5000m(3.635\times 10^{11})^2\ .$$

Using *Mathematica* we get $m\geq 4.02244\times 10^9$, an improvement over Dudek's result of $m\geq 4.971\times 10^9$.

A better result can be achieved, however, by instead making use of a different result of Dusart [13].

Corollary 3.2.2. *Let* $R_0=5.69693$. *Then, for* $x\geq 3$,

$$\max\{\left|\theta(x)-x\right|,\left|\psi(x)-x)\right|\}<\sqrt{\frac{8}{\pi {R_0}^{1/2}}}x(\log x)^{1/4}e^{-\sqrt{(\log x)/R_0}}\ . \qquad (3.8)$$

This is an explicit version of the prime number theorem which we can use to formulate a statement about $\theta_{x,h}=\theta(x+h)-\theta(x)$. Note that R_0 - the zero-free region constant - has been improved by Mossinghoff and Trudgian [26], recall (1.5). This would lead to (3.8) with R_0 replaced by 5.573412, but only for $x\geq x_0=e^{300}$ [14]. It would be possible to lower x_0 with some effort, however the improvements would be very small. For this reason we simply use (3.8).

For convenience of notation we introduce the function

$$B(x)=\sqrt{\frac{8}{\pi {R_0}^{1/2}}}x(\log x)^{1/4}e^{-\sqrt{(\log x)/R_0}}\ .$$

It is not hard to see that

$$\theta_{x,h}>h-B(x+h)-B(x)\ .$$

If we can show $h - B(x+h) - B(x) > 0$ then we can guarantee that there will be a prime between x and $x+h$. We let $x = n^m$ and $h = mx^{m-1}$. We want to choose a value for m that is as small as possible such that, using Lemma 3.1.2 there will be a prime in the interval $[n^m, (n+1)^m)$ for all $n \geq N_0$ and for all $n < N_0$ we will have $mn^{m-1} - B(n^m + mn^{m-1}) - B(n^m) > 0$ in order to 'cover the gap'.

The smallest value for $y(m_0)$ that was obtained was $y(m_0) = 3.635 \times 10^{11}$ for $m_0 = 1000000$. This was the value used in these calculations. Note that this means we cannot achieve a value for m which is smaller than 1000000.

By using *Mathematica*, we test which values of m are such that

$$mn^{m-1} - B(n^m + mn^{m-1}) - B(n^m) > 0$$

for all $n < N_0 = \exp\left(\frac{y(m_0)}{m}\right)$. The smallest integer m for which this is the case is $m = 1438989$. This allows us to conclude that for all $n \geq 1$, there exists a prime p such that

$$p \in [n^{1438989}, (n+1)^{1438989})$$

This $m = 1438989$ is a substantial improvement over Dudek's result.

Chapter 4

Using *Mathematica*

Mathematica was the computational powerhouse used to check vast numbers of candidate combinations of y, k and λ, which form the basis of the results given in this thesis. The main ways in which *Mathematica* has been used are explored in this chapter along with the process of developing some of the more significant functions.

4.1 Using "Manipulate"

Mathematica's "Manipulate" function was used to investigate whether certain values of m, y, k and λ satisfied the two required inequalities (3.5) and (3.6). "Manipulate" takes an expression, its variables, and the range of values that each of variable may take. The variables in the expression can then be varied interactively and this is how "Manipulate" can be used to find a quasi-optimal solution through an iterative process.

For instance, to find a small value of y when $m = 3$, we input to the "Manipulate" function both inequalities (3.5) and (3.6). The variables we input are y, k and λ, for which we give the ranges $[10^{14}, 10^{15}]$, $[\frac{2}{3}, 1]$ and $[0, 1]$ respectively, with initial values of $y = 8.5 \times 10^{14}$, $k = 0.9$ and $\lambda = 0.9$. The"Manipulate" function immediately shows that our initial y, k and λ do not satisfy both inequalities. We now slowly increase the value of k to find the point $k = 0.93$ at which both inequalities are satisfied. We can further perturb the values of k and λ whilst always decreasing y through an iterative process. It was found that setting $k = 0.9359$ and $\lambda = 0.99$ will satisfy both of the inequalities with $y = 8.1 \times 10^{14}$.

This process is very imprecise and even more time consuming and tedious. Given that values of y, k and λ are needed for many different values of m, it is worth investing some time to write a function that automates this process.

4.2 Developing the 'GoodAnswer' function

In order to explore the impact of λ, a function was written which, with input m, returns a 'good' value of y along with the corresponding k and λ values. While the function does not necessarily return the smallest possible value for y, it is relatively time efficient and returns a reasonably good value by operating recursively and 'zooming in' on promising values of y, k and λ.

4.2.1 The 'bestValuesLambdaMod' function

The 'bestValuesLambdaMod' function is the first step towards being able to take a value of m and return a reasonably good value for y. The function takes parameters y_{start}, y_{stop}, y_{step}, k_{start}, k_{stop}, k_{step}, λ_{start}, λ_{stop} and λ_{step}, which are starting values, stopping values and step sizes for each of the three parameters y, k and λ respectively. It tests which combinations of y, k and λ satisfy both of the inequalities and returns an optimal triple $\{y, k, \lambda\}$ for which both inequalities are satisfied but contains the minimal such y.

To find this optimal triple, the 'bestValuesLambdaMod' function builds a table containing elements of the form $\{y, k, \lambda, \text{ineq1}, \text{ineq2}\}$. Each of 'ineq1' and 'ineq2' equals either 'True' or 'False' based on whether each inequality is satisfied (True) or not (False) by the particular values of y, k and λ in that array. The 'bestValuesLambdaMod' function then returns an array of the form $\{y, k, \lambda, \text{True}, \text{True}\}$ with the smallest y-value. The 'bestValuesLambdaMod' function can be viewed below.

Algorithm 4.2.1: bestValuesLambdaMod

Input : Value for m and start stop and step size of each of y, k λ
Output: A triple $\{y, k, \lambda\}$ such that each of the two inequalities are satisfied and such that $y = \min\{y_i \mid \exists\{y_i, k_i, \lambda_i\}$ satisfying both inequalites$\}$

initialization
$list_y$ = list of y values generated by the start, stop and step sizes for y
$list_k$ = list of y values generated by the start, stop and step sizes for k
$list_\lambda$ = list of y values generated by the start, stop and step sizes for λ
working triples = empty list

for $y \in list_y$ **:**
  for $k \in list_k$ **:**
    for $\lambda \in list_\lambda$ **:**
      if $\{y, k, \lambda\}$ *satisfies both inequalities* **:**
        add $\{y, k, \lambda\}$ to the working triples list
return the first element of the working triples list

In principle, this function is sufficient to find an excellent value for y. However, this would require testing a large range of y, k and λ values with very small step sizes, which would be very inefficient. There are approximately

$$N_y = \frac{y_{\text{stop}} - y_{\text{start}}}{y_{\text{step}}}$$

different values of y that are tested and similarly

$$N_k = \frac{k_{\text{stop}} - k_{\text{start}}}{k_{\text{step}}} \text{ and } N_\lambda = \frac{\lambda_{\text{stop}} - \lambda_{\text{start}}}{\lambda_{\text{step}}}$$

different values of k and λ tested respectively. This means that the total number of arrays $\{y, k, \lambda, \text{ineq1}, \text{ineq2}\}$ that need to be tested is $N_y \times N_k \times N_\lambda$ and, if it is assumed that the run time is linear in the number of arrays tested, this gives a good representation of the time complexity of this function.

In order to use this function to find a good value for y it was necessary to run the function with very small step sizes for k and λ, say 0.0001. As $k \in (2/3, 1)$ and $\lambda \in (0, 1)$ this would put $N_k = \frac{1}{3} \times 10^4$ and $N_\lambda = 10^4$. For the case of $m = 3$, a sensible search range for y might be from 7.5×10^{14} to 8.0×10^{14} with a step size of 10^{12}, giving $N_y = 51$, resulting in a total of 1.7×10^9 arrays that need to be checked.

The run time for the 'bestValuesLambdaMod' function over different numbers of arrays was measured, the results for which are displayed in Table 4.1.

Number of arrays to be tested	Time taken (seconds)
500	0.002268
5 000	0.028585
50 000	0.21099
500 000	2.13531
5 000 000	22.4414

Table 4.1: Running time vs. arrays to be tested.

When fitted with a linear trend line with the y-intercept set to 0, the result was the equation

$$y = 4 \times 10^{-6}x$$

with an R^2 value of 0.99997. This suggests that to run this function for the single value of $m = 3$ with the parameters specified above about would take approximately

$$4 \times 10^{-6} \times 1.7 \times 10^{9} = 3.8 \times 10^{3} \text{ seconds}$$

which is about 1 hour.

This is not practical for our purposes, given that we need to find good y values for many different values of m, so some time was invested to make a more efficient function. The general approach to this is to run the 'bestValuesLambdaMod' function over a wide range of y, k and λ values but with big step sizes. This makes the execution fast, even if the y-value obtained is not very optimal. This gives a 'general idea' of where good solutions can be found.

We will then 'zoom in' on this promising area and run the function again on a smaller interval and with smaller step sizes. By doing a more detailed search on smaller intervals, the number of arrays of y, k and λ that need to be checked is smaller, preventing the running time from being impractical. This 'zooming in' process will be repeated a number of times until a 'good' answer is reached.

One point that is worth noting is that the running time of 'bestValuesLambdaMod' is roughly independent of m, only depending on the number of values for each of y, k and λ that are being tested. This is convenient as the function will be used to investigate large values of m.

4.2.2 Detailed outline of the recursive process

The 'zooming in' mentioned in the previous section is achieved by writing a recursive function. The recursive process works by searching over a very coarse grid of values for y, k and λ, finding the best solution and then 'zooming in' on those values to refine the solution. This method can be used because the inequalities are continuous, meaning that as long as the initial grid is not 'too coarse' then a good solution can be found this way.

To illustrate how the process works, the following example is included for $m = 4$.

The 'bestValuesLambdaMod' function takes parameters as follows,

$$\text{bestValuesLambdaMod}[m, \{y_{\text{start}}, y_{\text{stop}}, y_{\text{step}}, k_{\text{start}}, k_{\text{stop}}, k_{\text{step}}, \lambda_{\text{start}}, \lambda_{\text{stop}}, \lambda_{\text{step}}\}] \ .$$

It returns an array $\{y, k, \lambda, \text{'Inequality 1'}, \text{'Inequality 2'}\}$. Here 'Inequality 1' and 'Inequality 2' are the values obtained by subtracting the right hand side from the left hand side of (3.5) and (3.6) respectively and hence should be positive for (3.5) and (3.6) to be satisfied. We abbreviate 'bestValuesLambdaMod' to 'bVLM'.

The iterations that are run are:

$$\begin{aligned}
&\text{bVLM}[4, \{10.0^{13}, 10.0^{16}, 10.0^{13}, 0.9, 0.99, 0.01, 0.5, 0.9, 0.1\}] \\
&= \{3.0 * 10^{13}, 0.96, 0.5, 0.450375, 4.9862 * 10^{11}\} \\[2ex]
&\text{bVLM}[4, \{2.0 * 10^{13}, 3.0 * 10^{13}, 10.0^{12}, 0.9, 0.99, 0.01, 0.5, 0.9, 0.1\}] \\
&= \{2.9 * 10^{13}, 0.96, 0.8, 0.112858, 4.77735 * 10^{11}\} \\[2ex]
&\text{bVLM}[4, \{1.6 * 10^{13}, 2.9 * 10^{13}, 10.0^{11}, 0.95, 0.97, 0.001, 0.5, 0.9, 0.1\}] \\
&= \{1.8 * 10^{13}, 0.964, 0.5, 0.0262827, 3.55145 * 10^{8}\} \\[2ex]
&\text{bVLM}[4, \{1.6 * 10^{13}, 1.8 * 10^{13}, 10.0^{11}, 0.959, 0.969, 0.0001, 0.5, 0.99, 0.01\}] \\
&= \{1.77 * 10^{13}, 0.9639, 0.97, 0.0052656, 5.75222 * 10^{9}\} \qquad (4.1)
\end{aligned}$$

The noteworthy parts of the process above are as follows; the value returned for y in any given step is set to be y_{stop} for the next iteration of the function; the 'search range' for k is centred around the value of k that was returned from the previous iteration; the value of k_{step} used in a given iteration will be the same as or smaller than the one preceding it.

This process had several steps so a number of preliminary functions were written to perform important roles in the final function.

4.2.3 The 'Calc' functions

There are three 'Calc Functions', 'CalcY', 'CalcK' and 'CalcL'. These determine, based on the y, k and λ values that attained the best result in the previous iteration of the algorithm, good choices for the starting value, stopping value and step size of each parameter for the next iteration. In other words, they automate the process of choosing where and how much to 'zoom in' for the next iteration.

'CalcY'

'CalcY' takes a list of y-values from previous iterations and returns y_{start}, y_{stop}, y_{step} to use for the next iteration. It has default start, stop and step values that it returns in the event that there are no previous y-values (as is the case when beginning the first iteration). Since we are looking for a small y, it is never necessary to consider values for y larger than those returned in previous iterations. Therefore the function specifies the stopping value for the next iteration to be the previous best y-value. The function sets the starting value and step size to be two orders of magnitude smaller than the stopping value.

The default start, stop and step size values for y are $10^{13}, 10^{16}$ and 10^{13} respectively. These are such that, in combination with the default start, stop and step size values for k and λ, there will be a combination of values of y, k, λ satisfying both (3.5) and (3.6) for $m = 3$. In other words it guarantees that there will be a solution. For $m > 3$, the inequalities are easier to satisfy and so any value of y that satisfies (3.5) and (3.6) for $m = 3$ will satisfy the inequalities for $m > 3$, therefore there will always be a solution.

'CalcY' also controls when the recursive process terminates by returning the string 'finish', instead of '$\{y_{\text{start}}, y_{\text{stop}}, y_{\text{step}}\}$', when a fixed number of iterations has been completed.

The following illustrates how 'CalcY' works.

Algorithm 4.2.2: CalcY

Input : List of previous y values
Output: Values for $y_{\text{start}}, y_{\text{stop}}, y_{\text{step}}$

initialization;
if *length of prevY is greater than 20* **:**
  return 'finish';
else:
  if *there are no previous y values* **:**
    return $\{10^{13}, 10^{16}, 10^{13}\}$;
  elif *there is 1 previous y value* **:**
    return $\{y \times 10^{-3}, y, y \times 10^{-3}\}$;
  else:
    y=most recent previous y value;
    return $\{y \times 10^{-2}, y, y \times 10^{-2}\}$;

'CalcK'

The function 'CalcK' plays the same role as the 'CalcY' function but for determining an appropriate starting value, stopping value and step size for k for the next iteration of the algorithm. Again, the starting value, stopping value and step size that this function returns are dependent on 'best values' of k from previous iterations of the algorithm. There are also default values for the case in which there are no previous k values.

As noted above, the previous k value gives the middle of the new search interval. This is because the next k values could be either larger or smaller. The step size decreases by one order of magnitude smaller each iteration, making each search is more refined than the one before it. Algorithm 4.2.3 shows how this function works.

Algorithm 4.2.3: CalcK

Input : List of previous k values
Output: Values for $k_{\text{start}}, k_{\text{stop}}, k_{\text{step}}$

initialization;
if *there are no previous k values* **:**
  return $\{0.93, 0.99.0.01\}$;
else:
  $k =$ most recent previous k value;
  $d = \min(\{\text{number of decimals in } k, 12\})$;
  return $\{k - 10^{-d}, k + 10^{-d}, 10^{-(d+1)}\}$;

'CalcL'

The output of the function 'CalcL' depends on the previous values of its parameter in slightly different ways than the other two 'Calc' functions. The potential for improvement in the y-value by increasing the number of decimals in λ varies depending of what λ is. If λ is quite close to 0.5, very little improvement is expected in y by allowing more decimal places for λ. However, when λ is much closer to 1, there is more potential for improvement.

When λ is less than 0.95 we restrict the number of decimal places. This keeps N_λ from being large unnecessarily, keeping running time small. Algorithm 4.2.4 shows how this function works.

Algorithm 4.2.4: CalcL

Input : List of previous λ values
Output: Values for λ_{start}, λ_{stop}, λ_{step}

initialization
λ = most recent previous λ value
if *there are no previous λ values* **:**
    return $\{0.5, 0.9, 0.1\}$
elif $\lambda = 0.5$ **:**
    return $\{0.5, 0.9, 0.1\}$
elif $\lambda < 0.95$ **:**
    return $\{\lambda, 0.99, 0.01\}$
else:
    x equals the minimum of the number of decimals in λ * and 4
    $\delta = 10^{-x}$
    return $\{\lambda, 1 - \delta, \delta\}$

4.2.4 The 'GoodAnswer' function

Now that we have the 'Calc' functions to automate the process of choosing sensible starting values, stopping values and step sizes for y, k and λ at each iteration, we introduce the 'GoodAnswer' function which automates the entire recursive process explored above. It executes the 'bestValuesLambdaMod' function a number of times while using the 'Calc' functions to determine where to search for a good answer based on the results of the previous iteration.

The results obtained by using this function can be seen in the Table 4.2.

m	k	λ	y
3	0.935871	0.9996	7.82208×10^{14}
4	0.963942	0.9999	1.75830×10^{13}
5	0.974	0.9	5.43000×10^{12}
6	0.9795	0.99	2.93436×10^{12}
7	0.983	0.95	2.01633×10^{12}
10	0.988753	0.99	1.10372×10^{12}
100	0.9989	0.98	4.04271×10^{11}
1000	0.999853	0.9998	3.68616×10^{11}
10000	0.99997	0.9924	3.64652×10^{11}
100000	0.999966	0.9999	3.64652×10^{11}
1000000	0.999966	0.9999	3.64652×10^{11}

Table 4.2: Results from 'GoodAnswer' function.

There are a few things to notice in Table 4.2. First, for m equals 10000, 100000 and 1000000, the function returns the same value of y. This is because the level of precision used in the function is not high enough, so rounding means that the same result is returned. Second, the value of λ is not increasing as m increases. The way the function was written means that if several values of λ work for the smallest y-value, then the smallest of them is returned. This is why λ is not increasing even though the theory suggests it should because (3.5) is independent of m and (3.6) gets easier to satisfy as m increases.

We also notice from Table 4.2 that the values for λ are all close to 1. By looking at the output of the 'bestValuesLambdaMod' function in (4.1), we see that the output values for 'Inequality 1' are very small while those for 'Inequality 2' are large. This indicates that 'Inequality 1'> 0 is the condition which is harder to satisfy, not 'Inequality 2'> 0. This means that (3.5) is harder to satisfy than (3.6), and so it makes sense that λ would be close to 1.

After testing some different values of λ close to 1, we conclude that searching only over two variables, y and k, and fixing $\lambda = 1 - 10^{-15}$ will give results of the same quality but the running time will be shorter, or our search over y and k may be finer without increasing the running time.

4.3 Rework of the 'GoodAnswer' function

Here we implement the changes mentioned above. Rather than checking over many values of λ, we now use $\lambda = 1 - 10^{-15}$ and search only over y and k. The 'CalcY' and 'CalcK' functions are replaced by functions 'CalcY4K' and 'CalcK4K' which do the same thing but give smaller step sizes and use higher precision, as seen in the Algorithms 4.3.1 and 4.3.2 respectively.

Algorithm 4.3.1: CalcY4K

Input : List of previous y values
Output: Values for y_{start}, y_{stop}, y_{step}

initialization
if *length of prevY is greater than 15* **:**
    return "finish"
else:
    if *there are no previous y values* **:**
        return $\{10^{13}, 10^{16}, 10^{13}\}$
    else:
        y=most recent previous y value
        return $\{y \times 10^{-3}, y, y \times 10^{-3}\}$

Algorithm 4.3.2: CalcK4K

Input : List of previous k values
Output: Values for k_{start}, k_{stop}, k_{step}

initialization
if *there are no previous k values* **:**
    return $\{0.93, 0.99.0.01\}$
else:
    k = most recent previous k value
    $d = \min(\{\text{number of decimals in } k, 12\})$
    return $\{k - 10^{-d}, k + 10^{-d}, 10^{-(d+1)}\}$

The results of these changes are presented in Table 4.3. Note that $\lambda = 1 - 10^{-15}$ for all rows.

m	y	k
3	7.823×10^{14}	$\frac{29245952517}{31250000000}$
4	1.759×10^{13}	$\frac{24098551}{25000000}$
5	5.41×10^{12}	$\frac{487}{500}$
6	2.945×10^{12}	$\frac{97946116427}{100000000000}$
7	2.030×10^{12}	$\frac{983}{1000}$
10	1.010×10^{12}	$\frac{19775674533}{20000000000}$
100	4.033×10^{11}	$\frac{4994567541}{5000000000}$
1000	3.682×10^{11}	$\frac{99986583}{100000000}$
10 000	3.678×10^{11}	$\frac{1249845407}{1250000000}$
100 000	3.678×10^{11}	$\frac{1249845407}{1250000000}$
1 000 000	3.678×10^{11}	$\frac{1249845407}{1250000000}$

Table 4.3: Results for 'GoodAnswer' function with fixed λ.

The values for y in the table are all rounded up to ensure that they do actually satisfy both of the inequalities for the specified k value. The values of k in the table are expressed exactly as fractions because the satisfaction of the inequalities is very sensitive to small changes in k. Presenting them in this way ensures that the information given is correct.

4.4 Precision

We notice that in Table 4.3 that the result for m equals each of 10000, 100000 and 1000000 are the same. This is not because y cannot be made any smaller by considering these larger values of m. Rather, it is because the function still does not use a high enough amount of precision to achieve a smaller y, even though it uses more that 'GoodAnswer'.

While the automated process does not use high enough precision to give different values for large m, calculations can be carried out 'by hand' using the process outlined in 4.1 to achieve some smaller values for y. In particular, this was done for $m = 100000$ and $m = 1000000$, which is how the results in the final two rows of Table 3.2 were reached.

Conclusions

Several improvements to results on primes in intervals have been achieved.

First, we have made a slight improvement to Dudek's result in [11] that there is a prime between n^3 and $(n+1)^3$ for $n \geq \exp(\exp(33.217))$ to $n \geq \exp(\exp(33.1981))$.

Second, a substantial improvement has been made for primes between mth powers. Dudek proves that there is a prime between n^m and $(n+1)^m$ for all $n \geq 1$ for all $m \geq 4.971 \times 10^9$. We have reduced the value of m to $m \geq 1\,438\,989$.

To improve the results here even further with the existing theory presented here, one could modify the algorithms used so that they are more precise. Though there are surely improvements to be made in this way, they are probably quite small.

Another possibility for achieving stronger results would be to incorporate Kadiri's [23] zero density estimate. This is a stronger result than Theorem 1.6.7 on certain intervals and could be used in combination with Ramaré's result, Theorem 1.6.7, and Ford's zero free region, Lemma 1.6.1, for the work in Chapter 2.

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