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On globally non-trivial almost-commutative manifolds

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Within the framework of Connes' noncommutative geometry, we define and study globally non-trivial (or topologically non-trivial) almost-commutative manifolds. In particular, we focus on those almost-commutative manifolds that lead to a description of a (classical) gauge theory on the underlying base manifold. Such an almost-commutative manifold is described in terms of a "principal module," which we build from a principal fibre bundle and a finite spectral triple. We also define the purely algebraic notion of "gauge modules," and show that this yields a proper subclass of the principal modules. We describe how a principal module leads to the description of a gauge theory, and we provide two basic yet illustrative examples. © 2014 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4898769>]

I. INTRODUCTION

The framework of Connes' noncommutative geometry¹ provides a generalisation of ordinary Riemannian spin manifolds to noncommutative manifolds. Within this framework, the special case of a (globally trivial) almost-commutative manifold has been shown to describe a (classical) gauge theory over a Riemannian spin manifold, which ultimately led to a description of the full Standard Model of high energy physics, including the Higgs mechanism and neutrino mixing.²

The gauge theories mentioned above are, by construction, topologically trivial (in the sense that the corresponding principal bundles are globally trivial bundles). The aim of this paper is to adapt the framework in order to allow for globally non-trivial gauge theories as well. Such a generalisation has previously been obtained for the special case of Yang-Mills theory.³

Let us briefly recall how a description of a gauge theory is obtained from an almost-commutative manifold in the globally trivial case (for a more detailed introduction, we refer to Ref. 4). We start with a smooth compact 4-dimensional Riemannian spin manifold M , which can be described in terms of a (real, even) spectral triple $(C^\infty(M), L^2(S), \not{D}, \gamma_5, J_M)$, where \not{D} is the Dirac operator on the spinor bundle $S \rightarrow M$, γ_5 is the grading operator and J_M is charge conjugation.⁵ If we take a real even finite spectral triple $(A_F, \mathcal{H}_F, D_F, \gamma_F, J_F)$, one can consider the product triple

$$M \times F := (C^\infty(M, A_F), L^2(S) \otimes \mathcal{H}_F, \not{D} \otimes \mathbb{I} + \gamma_5 \otimes D_F, \gamma_5 \otimes \gamma_F, J_M \otimes J_F). \quad (1)$$

For a real spectral triple $T = (\mathcal{A}, \mathcal{H}, D, J)$, we define its gauge group as

$$\mathcal{G}(T) := \{uJuJ^* \mid u \in \mathcal{U}(\mathcal{A})\} \simeq \mathcal{U}(\mathcal{A})/\mathcal{U}(\mathcal{A}_J), \quad (2)$$

where \mathcal{A}_J is the central subalgebra of \mathcal{A} consisting of all elements $a \in \mathcal{A}$ for which $aJ = Ja^*$. Now suppose we have a real even finite spectral triple $F = (A_F, \mathcal{H}_F, D_F, \gamma_F, J_F)$ with gauge

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group $G_F = \mathcal{G}(F)$. Then the product triple $M \times F$ defined above has gauge group $\mathcal{G}(M \times F) \simeq C^\infty(M, G_F)$ (at least when M is simply connected), which coincides with the “classical” notion of the gauge group of the (globally trivial) principal G_F -bundle $P = M \times G_F$. (The isomorphism $\mathcal{G}(M \times F) \simeq C^\infty(M, G_F)$, stated in Proposition 4.3 of Ref. 3 and Sec. 2.4.3 of Ref. 4, is only valid under some additional conditions, and simply connectedness of M is always sufficient. We shall prove this in general for the globally non-trivial case in Theorem 4.12.)

One can show that the inner fluctuations of the operator $\not{D} \otimes \mathbb{I} + \gamma_5 \otimes D_F$ yield gauge fields (i.e., connection forms on the principal bundle P) as well as scalar fields (which are interpreted as Higgs fields in the noncommutative Standard Model). Finally, the spectral action principle⁶ yields a (gauge-invariant) Lagrangian from the data of the triple $M \times F$.

This paper is organised as follows. We start in Sec. II by gathering some preliminary material. Sections II A–II C contain a brief introduction into (principal) fibre bundles and modules. In Sec. II D, we describe a sufficient condition for when sections of a quotient group bundle can be lifted. Finally, we recall the basics of spectral triples and unbounded Kasparov modules in Sec. II E. The reader who is familiar with these topics may wish to skip these preliminaries on a first reading.

In Sec. III, we describe the generalisation of the product triples $M \times F$ to (in general globally non-trivial) almost-commutative manifolds. We show that these almost-commutative manifolds are naturally given by the internal Kasparov product of an *internal space* I (replacing the finite spectral triple F) with the underlying manifold M .

While every globally trivial almost-commutative manifold describes a gauge theory, this no longer holds for arbitrary globally non-trivial almost-commutative manifolds. In Sec. IV, we therefore focus our attention on those internal spaces that will allow us to obtain a gauge theory. After briefly recalling the classification of finite spectral triples, we define the notion of a *principal module*, which is an internal space built from a finite spectral triple F and a principal G_F -bundle P over M . We show that the algebraic definition of the gauge group of a principal module (defined similarly to Eq. (2)) coincides precisely with the usual definition of the gauge group of P (i.e., the vertical automorphisms of P), provided that the underlying manifold M is simply connected.

One of the main ideas in the development of noncommutative geometry has been the translation of geometric data into (operator-)algebraic data. Whereas principal modules are constructed from geometric objects (namely principal fibre bundles), we devote Sec. V to the purely *algebraic* notion of what we call a *gauge module*. We prove that these gauge modules form a proper subclass of the principal modules, which are characterised by a lift of P to a principal $\mathcal{U}(A_F)$ -bundle (where A_F is the algebra of the finite spectral triple F).

By equipping a principal module with a connection and a “mass matrix,” we construct the corresponding *principal* almost-commutative manifold in Sec. VI. The remainder of this section is used to establish the main goal of this paper, namely, we describe in detail how this principal almost-commutative manifold describes a gauge theory on M . In Sec. VII, we provide two basic but illustrative examples of such gauge theories, namely Yang-Mills theory and electrodynamics. The Yang-Mills example in particular shows that not every principal module is a gauge module. However, we also show that the Yang-Mills example *is* a gauge module when the underlying manifold is simply connected and 4-dimensional. Hence on such manifolds we have no example of a principal module which is not a gauge module. We finish with an Outlook on possible future work.

Notation. All C^* -algebras and Hilbert modules will be denoted with capital letters (e.g., A, B, E, \dots), their smooth sub-algebras or pre- C^* -algebras (i.e., densely defined $*$ -sub-algebras that are closed under the holomorphic functional calculus) and Hilbert pre-modules will be denoted with curly letters (e.g., $\mathcal{A}, \mathcal{B}, \mathcal{E}, \dots$). The main exception to these conventions is the notation \mathcal{H} , which always denotes a complex Hilbert space. By M we denote a smooth connected compact Riemannian spin manifold. Bundles over M will be denoted with “typewriter font,” where we use B for algebra bundles, E for vector bundles, P for principal fibre bundles, G for group bundles, and S for the spinor bundle. Continuous (resp., smooth) sections of a bundle $E \rightarrow M$ will be denoted by $\Gamma(E)$ (resp., $\Gamma^\infty(E)$).

II. PRELIMINARIES

A. Fibre bundles

The definitions concerning fibre bundles in this paper may differ from the definitions in some other literature, including Ref. 3, so that we find it necessary to include a list of the definitions we use. All manifolds are assumed to be *smooth* and all maps between them are also assumed to be smooth.

Let $E \rightarrow M$ be a smooth fibre bundle (see, e.g., Ref. 7). A *local trivialisation* of E is denoted by (U, h_U) , where U is an open neighbourhood in M and $h_U : \pi^{-1}(U) \rightarrow U \times F$ is a diffeomorphism such that $\text{pr}_1 \circ h_U = \pi$. For two local trivialisations (U, h_U) and (V, h_V) for which $U \cap V \neq \emptyset$, we denote the corresponding *transition function* by $g_{UV} := h_V \circ h_U^{-1} \in C^\infty(U \cap V, \text{Diff}(F))$.

Definition 2.1. Let \mathcal{C} be some subcategory of the category of smooth manifolds, with objects $\text{Ob}_{\mathcal{C}}$ and morphisms $\text{Mor}_{\mathcal{C}}(A, B)$ for all objects $A, B \in \text{Ob}_{\mathcal{C}}$. Let M be a smooth manifold. A fibre bundle $\pi : E \rightarrow M$ with fibre F is called a \mathcal{C} -*bundle* if $F \in \text{Ob}_{\mathcal{C}}$ and if on each local trivialisation (U, h_U) the map $h_U|_{\pi^{-1}(x)} : \pi^{-1}(x) \rightarrow x \times F$ is an isomorphism in $\text{Mor}_{\mathcal{C}}(\pi^{-1}(x), F)$.

Let $\pi_1 : E_1 \rightarrow M$ and $\pi_2 : E_2 \rightarrow M$ be fibre bundles. A *bundle morphism* $\phi : E_1 \rightarrow E_2$ is a smooth map such that $\pi_2 \circ \phi = \pi_1$. If E_1 and E_2 are \mathcal{C} -bundles, then ϕ is called a \mathcal{C} -*bundle morphism* if $\phi|_{\pi_1^{-1}(x)} : \pi_1^{-1}(x) \rightarrow \pi_2^{-1}(x)$ is an element of $\text{Mor}_{\mathcal{C}}(\pi_1^{-1}(x), \pi_2^{-1}(x))$ for each $x \in M$.

Let $\pi : E \rightarrow M$ be a \mathcal{C} -bundle with fibre F . A fibre subbundle $\pi' : E' \rightarrow M$ with fibre F' is a \mathcal{C} -subbundle if $F' \in \text{Ob}_{\mathcal{C}}$ and there exist local trivialisations $\{(U, h_U)\}$ for E such that $h_U(E'|_U) \simeq U \times \iota(F')$, where ι is an injective morphism in $\text{Mor}_{\mathcal{C}}(F', F)$.

If \mathcal{C} is the category of *finite-dimensional* vector spaces, *finite-dimensional* $(*)$ -algebras, or Lie groups, then \mathcal{C} -bundles are referred to as vector bundles, $(*)$ -algebra bundles, or group bundles (respectively).

Remark 2.2. Note that according to Definition 2.1, a $(*)$ -algebra bundle is always locally trivial, in contrast with the definition of $(*)$ -algebra bundle in Ref. 3 (where the bundle is only assumed to be locally trivial as a *vector* bundle). The weaker notion given in Ref. 3 will here be referred to as *weak $(*)$ -algebra bundle*, following terminology of Ref. 8.

The space of smooth sections $\Gamma^\infty(E)$ of a vector bundle E is a finitely generated projective $C^\infty(M)$ -module, with pointwise addition and multiplication by $C^\infty(M)$. If $\phi : E_1 \rightarrow E_2$ is a vector bundle morphism, then

$$\phi_* : \Gamma^\infty(E_1) \rightarrow \Gamma^\infty(E_2), \quad (\phi_* s)(x) = \phi(s(x))$$

is a $C^\infty(M)$ -module morphism. By the Serre-Swan theorem,⁹ the assignment $E \mapsto \Gamma^\infty(E)$ on objects and the assignment $\phi \mapsto \phi_*$ on morphisms determines an equivalence between the category of smooth vector bundles over M and the category of finitely generated projective modules over $C^\infty(M)$.

Similarly, for a group bundle G , the sections $\Gamma^\infty(G)$ form a group with fibre-wise multiplication and inverse.

Example 2.3 (Unitary group bundle). If B is a unital $*$ -algebra bundle, we define the *unitary group bundle* of B as

$$\mathcal{U}(B) := \{b \in B \mid bb^* = b^*b = 1\}.$$

Then $\mathcal{U}(B)$ is a fibre subbundle of B , which forms a group bundle with group multiplication of $\mathcal{U}(B)_x = \mathcal{U}(B_x)$ inherited from the algebra multiplication of B_x , and group inverse given by the involution $*$. The sections of the unitary group bundle are equal to the unitary sections of the algebra bundle: $\Gamma^\infty(\mathcal{U}(B)) = \mathcal{U}(\Gamma^\infty(B))$.

Example 2.4 (Endomorphism bundle). Let $\pi_E : E \rightarrow M$ be a (hermitian) vector bundle with fibre V and local trivialisations (U, h_U^E) . Then the bundle of endomorphisms $\text{End}(E)$ is a unital

($*$ -)algebra bundle over M with fibre $\text{End}(V)$, and its local trivialisations $(U, h_U^{\text{End}(E)})$ are induced from (U, h_U^E) .

Theorem 2.5 (Ref. 3, Theorem 3.8). *Let M be a compact manifold. There is an equivalence between the category of (unital) weak ($*$ -)algebra bundles over M and the category of (unital) (involutive) $C^\infty(M)$ -module algebras that are finitely generated projective as $C^\infty(M)$ -modules.*

We again emphasise the difference between algebra bundles and weak algebra bundles as mentioned in Remark 2.2. We are grateful to Eli Hawkins who pointed out to us that a weak algebra bundle is locally trivial if and only if there exists a connection ∇ satisfying the Leibniz rule

$$\nabla(ab) = (\nabla a)b + a(\nabla b).$$

In the continuous case, however, it remains unclear what algebraic conditions one needs to impose on a $C(M)$ -module algebra $B = \Gamma(B)$, where B is a (continuous) weak algebra bundle, to ensure that the weak algebra bundle B is in fact locally trivial.

B. Principal fibre bundles and (classical) gauge theories

In this section, we briefly recall the definition of a principal fibre bundle, and some basic results. We refer to Chapter I of Ref. 7 and Ref. 10 for more details.

Definition 2.6. A principal fibre bundle P over M with structure group G (or a principal G -bundle for short) consists of a fibre bundle $P \xrightarrow{\pi} M$ equipped with a smooth right action of G that acts freely and transitively on the fibres, such that for a local trivialisation (U, h_U) of P , the map h_U intertwines the right action of G on $P|_U$ with the natural right action of G on $U \times G$.

One can construct a principal G -bundle P as soon as one knows its (G -valued) transition functions.

Theorem 2.7 (Reconstruction theorem, Chapter I, Proposition 5.2. of Ref. 7). *Let M be a compact manifold, G a Lie-group, and $\{U_i\}_{i \in I}$ an open covering of M . Suppose that for each $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, there is a smooth map $g_{ij}: U_i \cap U_j \rightarrow G$ such that $g_{ij}(x)g_{jk}(x)g_{ki}(x) = e$ for all $x \in U_i \cap U_j \cap U_k$. Then there exists a unique principal G -bundle P over M with the $\{U_i\}$ as trivialising neighbourhoods and the g_{ij} as transition functions.*

Definition 2.8. Let $\{(U_i, h_i)\}$ be a set of local trivialisations of P such that $\cup_i U_i = M$. A connection ω on P is a set of local \mathfrak{g} -valued 1-forms $\omega_i \in \Omega^1(U_i, \mathfrak{g})$ such that

$$\omega_j = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1} \omega_i g_{ij} \quad (3)$$

for i, j such that $U_i \cap U_j \neq \emptyset$.

Definition 2.9. Given an action ρ of G on a smooth manifold F , we define the associated bundle $P \times_\rho F$ (or $P \times_G F$) as the quotient of the product manifold $P \times F$ with respect to the equivalence relation given by $(pg, f) \sim (p, \rho(g)f)$. If $F \in \text{Ob}_{\mathcal{C}}$ and $\rho(g) \in \text{Mor}_{\mathcal{C}}(F, F)$ for all $g \in G$, then $P \times_\rho F$ is a \mathcal{C} -bundle.

Example 2.10. The adjoint bundle $\text{Ad } P$ is defined as the associated bundle $P \times_{\text{Ad}} G$ with respect to the adjoint action $\text{Ad}(g)h := ghg^{-1}$, ($g, h \in G$). The adjoint bundle is a group bundle with fibres isomorphic to G , and its sections $\Gamma^\infty(\text{Ad } P)$ then form a group with fibre-wise multiplication.

Definition 2.11. A gauge transformation of a principle G -bundle P is a principal bundle automorphism of P over $\text{id}: M \rightarrow M$, that is, a smooth invertible map $\phi: P \rightarrow P$ such that $\pi(\phi(p)) = \pi(p)$ and $\phi(pg) = \phi(p)g$ for all $p \in P$ and $g \in G$. The set of all such ϕ is called the gauge group $\mathcal{G}(P)$ of P , where the group multiplication is given by composition.

Theorem 2.12 (see, e.g., Chap. 3 of Ref. 10). *The gauge group $\mathcal{G}(P)$ is isomorphic to the group $\Gamma^\infty(\text{Ad}P)$.*

Definition 2.13. Let M be a manifold and G a Lie group. A *classical G -gauge theory* over M is a principal fibre bundle P with structure group G . Connections ω on P are also called *gauge potentials*.

More precisely, the bundle P forms the *setting* for a classical gauge theory. The particle fields can be described as sections of associated bundles of P . The description of the gauge theory is completed by specifying an *action functional*, which depends on the connection and on the particle fields, and which is invariant under the action of the gauge group.

1. Structure group

Let E be a vector bundle with fibre V . A set of transition functions $\{(U_i, g_{ij})\}$ on E is called a G -atlas if each transition function takes values in $G \subset GL(V)$. If E admits a G -atlas, then we say that E has *structure group* G . Given two G -atlases $\{(U_i, g_{ij})\}$ and $\{(U_i, g'_{ij})\}$ (where, after taking a common refinement, we may assume without loss of generality that both atlases are given on the same open covering $\{U_i\}$), we say that they are equivalent if there are functions $g_i \in C^\infty(U_i, G)$ such that (for all i, j)

$$g'_{ij}(x) = g_i(x)^{-1} g_{ij}(x) g_j(x), \quad \text{for all } x \in U_i \cap U_j.$$

Given a G -atlas $\{(U_i, g_{ij})\}$ on E , Theorem 2.7 constructs a unique principal G -bundle P , which only depends (up to isomorphism) on the equivalence class of the G -atlas. Conversely, a set of transition functions on P uniquely determines an equivalence class of G -atlases on the associated bundle $P \times_G V$.

Example 2.14. Let $E \rightarrow M$ be a complex vector bundle with fibre \mathbb{C}^N over a compact manifold M . Then all $U(N)$ -atlases on E are equivalent. Hence there is a unique (up to isomorphism) principal $U(N)$ -bundle P such that $E \simeq P \times_{U(N)} \mathbb{C}^N$.

Definition 2.15 (Lifting of structure group). Let $\phi: H \rightarrow G$ be a surjective group homomorphism. A principal G -bundle $P \rightarrow M$ is said to lift to a principal H -bundle $Q \rightarrow M$ along ϕ if there is a bundle morphism $\tau: Q \rightarrow P$ such that $\tau(qh) = \tau(q)\phi(h)$ for all $q \in Q, h \in H$. Equivalently, Q is a lift of P if

$$Q \times_\phi G \simeq P$$

as principal G -bundles.

If $\tau: Q \rightarrow P$ is such a lift and $\rho: G \rightarrow GL(V)$ is a finite-dimensional representation, then $Q \times_{\rho \circ \phi} V$ is isomorphic to $P \times_\rho V$. We stress that a lift need not always exist, and if it exists, it need not be unique.

C. Conjugate modules and vector bundles

In the construction of gauge modules in Sec. V we will make explicit use of the notion of a conjugate module. For completeness, we recall the definition of conjugate modules and vector bundles here. Since most of the modules are endowed with a hermitian structure, we recall the definition of a hermitian module first.

Definition 2.16. Let \mathcal{A} be a $*$ -algebra and let \mathcal{E} be a right \mathcal{A} -module. A (right) *hermitian structure* $(\cdot, \cdot)_{\mathcal{A}}: \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{A}$ on \mathcal{E} is a sesqui-linear map (anti-linear in the first variable) satisfying

$$(e_1, e_2 a)_{\mathcal{A}} = (e_1, e_2)_{\mathcal{A}} a; \quad (e_2, e_1)_{\mathcal{A}} = (e_1, e_2)_{\mathcal{A}}^*; \quad (e, e)_{\mathcal{A}} \geq 0; \quad (e, e)_{\mathcal{A}} = 0 \iff e = 0,$$

for all $a \in \mathcal{A}, e_1, e_2, e \in \mathcal{E}$. We also write (\cdot, \cdot) instead of $(\cdot, \cdot)_{\mathcal{A}}$ when no confusion can arise. A module endowed with a hermitian structure is also called a *hermitian module*. A left hermitian structure ${}_{\mathcal{A}}(\cdot, \cdot)$ is defined similarly.

A hermitian structure is called *non-degenerate* if the map

$$\mathcal{E} \rightarrow \mathcal{E}^* := \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A}), \quad e_0 \mapsto (e \mapsto (e_0, e))$$

is an anti-linear isomorphism. Note that the assumption that the hermitian structure is positive-definite already implies that the map $\mathcal{E} \rightarrow \mathcal{E}^*$ is injective. Non-degeneracy therefore requires surjectivity of this map.

A finitely generated projective right \mathcal{A} -module \mathcal{E} is of the form $p\mathcal{A}^N$, for some $N \in \mathbb{N}$ and some projection $p \in M_N(\mathcal{A})$. The restriction of the standard hermitian structure on \mathcal{A}^N then gives a non-degenerate hermitian structure on \mathcal{E} . If $\mathcal{A} = C^\infty(M)$ (so that $\mathcal{E} = \Gamma^\infty(\mathbb{E})$ for some vector bundle $\mathbb{E} \rightarrow M$ by the Serre-Swan theorem⁹), then the hermitian structure is non-degenerate if and only if it induces an inner product on each fibre of \mathbb{E} .

Definition 2.17. Let \mathcal{E} be an $\mathcal{A} - \mathcal{B}$ -bimodule with a (right) \mathcal{B} -valued hermitian structure $(\cdot, \cdot)_{\mathcal{B}}$. Its *conjugate module* $\bar{\mathcal{E}}$ is equal to \mathcal{E} itself as an additive group. It can naturally be endowed with a $\mathcal{B} - \mathcal{A}$ -bimodule structure and a (left) \mathcal{B} -valued hermitian structure ${}_{\mathcal{B}}(\cdot, \cdot)$ by setting

$$b\bar{e} := \overline{eb^*}, \quad \bar{e}a := \overline{a^*e}, \quad {}_{\mathcal{B}}(\bar{e}_1, \bar{e}_2) := (e_1, e_2)_{\mathcal{B}},$$

for all $a \in \mathcal{A}, b \in \mathcal{B}, e, e_1, e_2 \in \mathcal{E}$.

If $\mathcal{E} = \Gamma^\infty(\mathbb{E})$ is the $C^\infty(M)$ -module of sections of some (hermitian) vector bundle \mathbb{E} , then the conjugate module $\bar{\mathcal{E}}$ is equal to the $C^\infty(M)$ -module of sections of the conjugate vector bundle $\bar{\mathbb{E}}$ which is defined as:

Definition 2.18. Let $\mathbb{E} \rightarrow M$ be a complex vector bundle. Take $\bar{\mathbb{E}}$ to be equal to \mathbb{E} as fibres bundles over M , and write \bar{e} for the element in $\bar{\mathbb{E}}$ that corresponds to $e \in \mathbb{E}$ under this identification. The bundle $\bar{\mathbb{E}}$ is turned into a vector bundle over M by defining the vector space structure in $\bar{\mathbb{E}}_x$ by

$$(\bar{e}_1, \bar{e}_2) \mapsto \overline{e_1 + e_2}, \quad \lambda \cdot \bar{e} = \overline{\lambda e},$$

for all $\lambda \in \mathbb{C}, e, e_1, e_2 \in \mathbb{E}_x$. The vector bundle $\bar{\mathbb{E}} \rightarrow M$ is called the *conjugate vector bundle* of \mathbb{E} .

The identification $\mathbb{E} \ni e \mapsto \bar{e} \in \bar{\mathbb{E}}$ in the above definition is an anti-linear isomorphism of vector bundles.

A local trivialisation (U, h) of \mathbb{E} induces a local trivialisation of $\bar{\mathbb{E}}$ given by the map

$$\bar{h} : \pi_{\bar{\mathbb{E}}}^{-1}(U) \ni \bar{e} \mapsto \overline{he} \in U \times \bar{V},$$

where $\overline{(x, v)} := (x, \bar{v}) \in U \times \bar{V}$. If g_{ij} is a transition function between two local trivialisations (U_i, h_i) and (U_j, h_j) of \mathbb{E} , then the transition function \bar{g}_{ij} between the corresponding local trivialisations (U_i, \bar{h}_i) and (U_j, \bar{h}_j) is equal to

$$\bar{h}_i \circ \bar{h}_j^{-1}(x, \bar{v}) = \bar{h}_i \left(\overline{h_j^{-1}(x, v)} \right) = \overline{h_i h_j^{-1}(x, v)} = (x, \overline{g_{ij}(x)v}) = (x, \bar{v} \cdot g_{ij}(x)^*). \quad (4)$$

From here on, we consider $\mathcal{A} := C^\infty(M)$. Suppose that \mathcal{E} is a hermitian right \mathcal{A} -module with hermitian structure $(\cdot, \cdot)_{\mathcal{A}}$.

Definition 2.19. A *connection* ∇ on \mathcal{E} is a map $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(M)$ satisfying the rule

$$\nabla(ea) = \nabla(e)a + e \otimes da,$$

for all $e \in \mathcal{E}$ and $a \in \mathcal{A}$. The connection is called *hermitian* if

$$(\nabla e_1, e_2)_{\Omega^1(M)} + (e_1, \nabla e_2)_{\Omega^1(M)} = d(e_1, e_2)_{\mathcal{A}},$$

for all $e_1, e_2 \in \mathcal{E}$, where the map $(\cdot, \cdot)_{\Omega^1(M)} : \mathcal{E} \times (\mathcal{E} \otimes_{\mathcal{A}} \Omega^1(M)) \rightarrow \Omega^1(M)$ is defined as $(e_1, e_2 \otimes \alpha)_{\Omega^1(M)} := (e_1, e_2)_{\mathcal{A}} \alpha$. We then define $(\cdot, \cdot)_{\Omega^1(M)} : (\mathcal{E} \otimes_{\mathcal{A}} \Omega^1(M)) \times \mathcal{E} \rightarrow \Omega^1(M)$ as $(e_1 \otimes \alpha, e_2)_{\Omega^1(M)} := ((e_2, e_1 \otimes \alpha)_{\Omega^1(M)})^*$.

The conjugate connection $\overline{\nabla} : \overline{\mathcal{E}} \rightarrow \Omega^1(M) \otimes_{\mathcal{A}} \overline{\mathcal{E}}$ is given by

$$\overline{\nabla} \overline{e} = \overline{\nabla} e, \quad (e \in \mathcal{E}),$$

where $\overline{e \otimes \omega} = \omega^* \otimes \overline{e}$ for all $e \otimes \omega \in \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(M)$. Here $*$: $\Omega^1(M) \rightarrow \Omega^1(M)$ is defined as $(fdg)^* = f^*(dg^*)$. It then follows that $\overline{\nabla}$ is also hermitian for the map ${}_{\Omega^1(M)}(\cdot, \cdot) : (\Omega^1(M) \otimes_{\mathcal{A}} \overline{\mathcal{E}}) \times \overline{\mathcal{E}} \rightarrow \Omega^1(M)$ defined as ${}_{\Omega^1(M)}(\alpha \otimes \overline{e}_1, \overline{e}_2) := (e_1 \otimes \alpha^*, e_2)_{\Omega^1(M)} = \alpha(e_1, e_2)_{\mathcal{A}}$.

For a commutative algebra $\mathcal{A} = C^\infty(M)$ the notion of left and right modules are equivalent. If \mathcal{E} is a left \mathcal{A} -module with (left) \mathcal{A} -valued hermitian structure ${}_{\mathcal{A}}(\cdot, \cdot)$, then $(e_1, e_2)_{\mathcal{A}} := {}_{\mathcal{A}}(e_2, e_1)$ defines a right \mathcal{A} -valued hermitian structure on \mathcal{E} when it is seen as a right \mathcal{A} -module. If $\mathcal{A} = C^\infty(M)$, we will freely use this identification.

D. Covering maps

We observe that, for a surjective group bundle morphism $\phi : \mathbb{H} \rightarrow \mathbb{G}$, the induced map $\phi_* : \Gamma^\infty(\mathbb{H}) \rightarrow \Gamma^\infty(\mathbb{G})$ need *not* always be surjective, as the following example shows.

Example 2.20. Take $M = SO(3)$ and consider the *globally trivial* group bundles $\mathbb{H} = M \times U(2)$ and $\mathbb{G} = M \times PSU(2)$, with the obvious group bundle morphism $\phi : \mathbb{H} \rightarrow \mathbb{G}$ given by the quotient $U(2) \rightarrow PSU(2)$. Since \mathbb{H} and \mathbb{G} are globally trivial, we can make the identifications $\Gamma^\infty(\mathbb{H}) \simeq C^\infty(SO(3), U(2))$ and $\Gamma^\infty(\mathbb{G}) \simeq C^\infty(SO(3), PSU(2))$. Consider the map $f : SO(3) \rightarrow PSU(2)$ given by the identification of $PSU(2)$ with $SO(3)$, i.e., $f = \text{id}$ on $SO(3)$. If there exists a lift $\tilde{f} : SO(3) \rightarrow U(2)$ such that $f = \phi \circ \tilde{f}$, then \tilde{f} is nothing but a global section of the $U(1)$ -principal bundle $\pi : U(2) \rightarrow SO(3)$. However, as this bundle is not globally trivial (the fundamental group of $U(2)$ is \mathbb{Z} , whereas the fundamental group of $SO(3) \times U(1)$ is $\mathbb{Z}_2 \times \mathbb{Z}$), such a section does not exist. Hence the map f , seen as a section in $\Gamma^\infty(\mathbb{G})$, is not contained in the image of ϕ_* .

In this subsection, we aim to find sufficient conditions for the surjectivity of ϕ_* . In other words, we would like to have sufficient conditions to ensure that for any section $s : M \rightarrow \mathbb{G}$ there exists a lift $\tilde{s} : M \rightarrow \mathbb{H}$ such that $\phi_*(\tilde{s}) = s$. Though the existence of lifts for covering maps has been well-studied, we will typically be dealing with more general fibrations $\phi : \mathbb{H} \rightarrow \mathbb{G}$, for which the problem of existence of lifts is more complicated. We avoid this problem by reducing it to the case of covering maps, as follows.

Lemma 2.21. Let $p : E \rightarrow B$ be a fibration, and consider some map $f : M \rightarrow B$. Suppose there exists a submanifold $C \subset E$ such that $p|_C : C \rightarrow B$ is a covering space, satisfying $f_*(\pi_1(M, m)) \subset p_*(\pi_1(C, c))$, where $m \in M$ and $c \in C$ are such that $f(m) = p(c)$. Then there exists a lift $\tilde{f} : M \rightarrow E$ satisfying $p \circ \tilde{f} = f$ and $\tilde{f}(m) = c$.

Proof. Consider the diagram

$$\begin{array}{ccc} & C & \hookrightarrow E \\ & \uparrow \tilde{f}' & \downarrow p|_C \\ M & \xrightarrow{f} & B \xlongequal{\quad} B \\ & & \downarrow p \end{array}$$

The assumption $f_*(\pi_1(M, m)) \subset p_*(\pi_1(C, c))$ implies (see, e.g., Proposition 1.33 of Ref. 11) that there exists a lift $\tilde{f}' : M \rightarrow C$ satisfying $\tilde{f}'(m) = c$, and then we can simply define $\tilde{f} : M \rightarrow E$ as the composition $M \xrightarrow{\tilde{f}'} C \hookrightarrow E$. □

We now translate the above lemma into the setting of group bundles, where we will need it later.

Corollary 2.22. Let M be a simply connected manifold, and let \mathbb{G}, \mathbb{H} be group bundles over M . If \mathbb{G} is covered by a subbundle \mathbb{U} of \mathbb{H} via a group bundle morphism $\phi : \mathbb{H} \rightarrow \mathbb{G}$, then the map $\phi_* : \Gamma^\infty(\mathbb{H}) \rightarrow \Gamma^\infty(\mathbb{G})$, given by $s \mapsto \phi \circ s$, is surjective.

Proof. By assumption, $\phi|_U : U \rightarrow G$ is a covering space. Since $\pi_1(M, m)$ is trivial (by definition of simply-connectedness) it follows from Lemma 2.21 that each section $s : M \rightarrow G$ can be lifted to a section $\tilde{s} : M \rightarrow U \subset H$ such that $\phi_*(\tilde{s}) = s$. \square

E. Spectral triples and Kasparov modules

Spectral triples were introduced in Ref. 1 as a noncommutative analogue of a spin manifold.

Definition 2.23. A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ is given by an involutive unital algebra \mathcal{A} represented (faithfully) as bounded operators on a Hilbert space \mathcal{H} and a self-adjoint (generally unbounded) operator D with compact resolvent (or equivalently, $(1 + D^2)^{-1/2}$ is a compact operator) such that $a \cdot \text{Dom } D \subset \text{Dom } D$ and the commutator $[D, a]$ is bounded for each $a \in \mathcal{A}$.

A spectral triple is called *even* if there exists a \mathbb{Z}_2 -grading γ on \mathcal{H} that commutes with any $a \in \mathcal{A}$ and anti-commutes with D .

A spectral triple is called *real* if there exists an anti-unitary isomorphism $J : \mathcal{H} \rightarrow \mathcal{H}$ satisfying

$$\begin{aligned} J^2 &= \varepsilon, & JD &= \varepsilon' DJ, & J\gamma &= \varepsilon'' \gamma J \text{ (if } \gamma \text{ exists),} \\ [a, JbJ^*] &= 0, & [[D, a], JbJ^*] &= 0, & \forall a, b &\in \mathcal{A}. \end{aligned}$$

The signs $\varepsilon, \varepsilon',$ and ε'' determine the *KO-dimension* n modulo 8 of the real spectral triple, according to the following table:

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

We will refer to the conditions $[a, JbJ^*] = 0$ and $[[D, a], JbJ^*] = 0$ as the zeroth- and first-order condition, respectively.

Given an algebra \mathcal{A} , we define the *opposite algebra* as the vector space $\mathcal{A}^{\text{op}} := \{a^{\text{op}} \mid a \in \mathcal{A}\}$ with the *opposite product* $a^{\text{op}}b^{\text{op}} = (ba)^{\text{op}}$. For a real spectral triple, we therefore have a linear representation of \mathcal{A}^{op} on \mathcal{H} given by $a^{\text{op}} \mapsto Ja^*J^*$.

The notion of spectral triple can be seen as an unbounded version of a Fredholm module. The generalisation of Fredholm modules from Hilbert spaces to Hilbert modules was performed by Kasparov,¹² where for any two graded C^* -algebras A and B the set $KK(A, B)$ was defined as the set of equivalence classes of certain Kasparov $A - B$ -modules. In addition, there exists a Kasparov product $KK(A, B) \times KK(B, C) \rightarrow KK(A, C)$. More details can be found in, e.g., Ref. 13. Kasparov modules were subsequently generalised to the unbounded picture by Baaj and Julg.¹⁴ In this paper, we will only focus on the unbounded picture, which we briefly recall below.

Definition 2.24 (Ref. 14). Given \mathbb{Z}_2 -graded C^* -algebras A and B , an *unbounded Kasparov $A - B$ -module* $({}_{\phi(A)}E_B, D)$ is given by

- a \mathbb{Z}_2 -graded, countably generated, right Hilbert B -module E_B ;
- a \mathbb{Z}_2 -graded $*$ -homomorphism $\phi : A \rightarrow \text{End}_B(E)$;
- a self-adjoint, regular, odd operator $D : \text{Dom } D \subset E \rightarrow E$ such that, for all a in a dense sub-algebra \mathcal{A} of A , $\phi(a) \cdot \text{Dom } D \subset \text{Dom } D$ and $[D, \phi(a)]_{\pm}$ is (or extends to) a bounded endomorphism, and $\phi(a)(1 + D^2)^{-\frac{1}{2}}$ is a compact endomorphism (i.e., it lies in $\text{End}_B^0(E)$).

The set of all unbounded Kasparov $A - B$ -modules is denoted by $\Psi(A, B)$. We will often simply write ${}_A E_B$ instead of ${}_{\phi(A)} E_B$.

A right Hilbert \mathbb{C} -module is just a Hilbert space. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ may then be seen as an unbounded Kasparov $A - \mathbb{C}$ -module $({}_A \mathcal{H}_{\mathbb{C}}, D)$, where the C^* -closure A of \mathcal{A} is trivially graded.

There is a natural map from the unbounded picture to the bounded one. This map is defined by replacing the operator D in $(\phi_{(A)}E_B, D)$ by $b(D) = D(1 + D^2)^{-\frac{1}{2}}$, where $b : \mathbb{R} \rightarrow \mathbb{R}$ denotes the function $b(x) = x(1 + x^2)^{-\frac{1}{2}}$.

Theorem 2.25 (Theorems 17.10.7 and 17.11.4 of Ref. 13). *If $(\phi_{(A)}E_B, D) \in \Psi(A, B)$, then $(\phi_{(A)}E_B, b(D)) \in KK(A, B)$. Moreover, if A is separable and B is σ -unital, then this map $\Psi(A, B) \rightarrow KK(A, B)$ is surjective.*

The Kasparov product has an unbounded analogue. To be precise, we say that an unbounded Kasparov $A - C$ -module $(\phi_{(A)}E_C, D)$ represents the Kasparov product of two unbounded Kasparov modules $(\phi_{1(A)}E_{1B}, D_1)$ and $(\phi_{2(B)}E_{2C}, D_2)$ if $[(E, b(D))] \in KK(A, C)$ is the Kasparov product of $[(E_1, b(D_1))] \in KK(A, B)$ and $[(E_2, b(D_2))] \in KK(B, C)$, where the square brackets indicate that we take the equivalence class of the Kasparov-module.

We will show in Sec. III that the construction of an almost-commutative manifold as the product of an internal space I with the underlying manifold M corresponds to an unbounded Kasparov product on the level of KK -classes. Although this follows from the (more general) framework of Mesland,¹⁵ we will prove it directly using the following result.

Theorem 2.26 (Ref. 16). *Let $(\phi_{1(A)}E_B^1, D_1)$ and $(\phi_{2(B)}E_C^2, D_2)$ be unbounded Kasparov modules. Write $E := E^1 \hat{\otimes}_B E^2$, where $\hat{\otimes}$ denotes the graded tensor product. Suppose that $(\phi_{1(A) \otimes \text{id}}E_C, D)$ is an unbounded Kasparov module such that:*

(i) *for all e_1 in a dense subspace of $\phi_{1(A)}E^1$, the commutators*

$$\left[\begin{pmatrix} D & 0 \\ 0 & D_2 \end{pmatrix}, \begin{pmatrix} 0 & T_{e_1} \\ T_{e_1}^* & 0 \end{pmatrix} \right]$$

are bounded on $\text{Dom}(D \oplus D_2) \subset E \oplus E^2$, where $T_{e_1} : E^2 \rightarrow E$ is given by $T_{e_1}(e_2) = e_1 \otimes e_2$;

(ii) $\text{Dom}(D) \subset \text{Dom}(D_1 \hat{\otimes} 1)$;

(iii) $((D_1 \hat{\otimes} 1)e | De) + (De | (D_1 \hat{\otimes} 1)e) \geq K(e | e)$ for some $K \in \mathbb{R}$, for all $e \in \text{Dom}(D)$.

Then $(\phi_{1(A) \otimes \text{id}}E_C, D)$ represents the Kasparov product of $(\phi_{1(A)}E_B^1, D_1)$ and $(\phi_{2(B)}E_C^2, D_2)$.

III. ALMOST-COMMUTATIVE MANIFOLDS

Almost-commutative manifolds $M \times F$ of the form Eq. (1) were first studied in Ref. 17 and Refs. 18–21. They were later used in Refs. 2 and 22 to geometrically describe Yang-Mills theories and the Standard Model of elementary particles. The name almost-commutative manifolds was coined in Ref. 23, their classification started in Refs. 24 and 25.

Let M be a smooth compact even-dimensional Riemannian spin manifold. We assume (throughout this section) that M has dimension 4. The manifold M can be completely characterised⁵ by the real even spectral triple

$$(C^\infty(M), L^2(S), \not{D}, \gamma_5, J_M),$$

which is often referred to as the *canonical* spectral triple for M . Here S is a spinor bundle over M , $\not{D} = -ic \circ \nabla^S$ is the corresponding Dirac operator (where ∇^S is the lift of the Levi-Civita connection on M , and c denotes Clifford multiplication with the conventions $c(v)c(w) + c(w)c(v) = 2g(v, w)$ and $c(v)^* = c(v)$ for any $v, w \in \Gamma^\infty(T^*M)$), γ_5 is the grading of the spinor bundle, and J_M is the charge conjugation operator. Given a real even *finite* spectral triple $(A_F, \mathcal{H}_F, D_F, \gamma_F, J_F)$ (for which $\dim \mathcal{H}_F < \infty$), we can construct the product triple

$$M \times F := (C^\infty(M, A_F), L^2(S) \otimes \mathcal{H}_F, \not{D} \otimes \mathbb{1} + \gamma_5 \otimes D_F, \gamma_5 \otimes \gamma_F, J_M \otimes J_F).$$

Defining the (globally trivial) algebra bundle $B = M \times A_F$ and the (globally trivial) vector bundle $E = M \times \mathcal{H}_F$, we can rewrite $C^\infty(M, A_F) \simeq \Gamma^\infty(B)$ and $L^2(S) \otimes \mathcal{H}_F \simeq L^2(S \otimes E)$. The purpose of this section is to generalise the construction of $M \times F$ to *globally non-trivial* bundles over M . At the same time, we will put this generalised construction in the context of the Kasparov product between

unbounded Kasparov modules. The globally non-trivial case was first considered in Ref. 3 for the case of algebra bundles with fibre $M_N(\mathbb{C})$, and has also been studied more generally in Ref. 8.

A. The internal space

Definition 3.1. A (smooth) internal space I^∞ over a compact manifold M is given by the data

$$I^\infty := (\Gamma^\infty(\mathcal{B}), \Gamma^\infty(\mathcal{E}), D_I),$$

where \mathcal{E} is a hermitian vector bundle over M , \mathcal{B} is a unital $*$ -algebra subbundle of $\text{End}(\mathcal{E})$, and D_I is a hermitian element of $\Gamma^\infty(\text{End}(\mathcal{E})) \simeq \text{End}_{C^\infty(M)}(\Gamma^\infty(\mathcal{E}))$.

An internal space is called *even* if there is a grading γ_I , i.e., an endomorphism $\gamma_I \in \Gamma^\infty(\text{End}(\mathcal{E}))$ such that

$$\gamma_I^* = \gamma_I, \quad \gamma_I^2 = 1, \quad \gamma_I D_I = -D_I \gamma_I, \quad \gamma_I a = a \gamma_I \quad \forall a \in \Gamma^\infty(\mathcal{B}).$$

An even internal space is called *real* if there is a real structure J_I , i.e., an anti-unitary endomorphism J_I on \mathcal{E} such that

$$\begin{aligned} J_I^2 &= \varepsilon, & J_I D_I &= \varepsilon' D_I J_I, & J_I \gamma_I &= \varepsilon'' \gamma_I J_I, \\ [a, J b^* J^*] &= 0, & [[D_I, a], J b^* J^*] &= 0, & \forall a, b \in \Gamma^\infty(\mathcal{B}), \end{aligned}$$

where the signs determine the KO-dimension of the internal space according to the same table as in Definition 2.23.

Remark 3.2. The endomorphism D_I will be interpreted as a mass matrix describing the masses of the elementary particles. We would like to point out a few things about this mass matrix:

1. On a local trivialisation (say, around a point $x \in M$) we can view the endomorphism D_I as a matrix-valued function $D_I(x)$, but the precise form of this matrix $D_I(x)$ depends on the choice of local trivialisation. However, since the transition functions are unitary, two different choices of local trivialisations yield two unitarily equivalent mass matrices, and hence the eigenvalues of the matrix $D_I(x)$ (i.e., the masses of the particles) are independent of the choice of local trivialisation.
2. These eigenvalues of $D_I(x)$ are (by default) allowed to vary as a function of $x \in M$. In the standard (globally trivial) approach one can also make the (ad hoc) decision to promote the mass parameters to functions (although this is usually not done). However, this would be unnatural from the perspective that a (globally trivial) almost-commutative manifold is the (external) Kasparov product of a Riemannian spin manifold with a finite spectral triple. Instead, varying mass parameters are more naturally described by replacing the finite spectral triple by an internal space (which works equally well in the globally trivial case) and replacing the external by the internal Kasparov product. As such, the promotion of the mass parameters to functions becomes a natural attribute of our framework.
3. One could ask whether it is always possible to choose these mass parameters to be globally constant (as in the usual approach). We expect that this might not always be possible in the general globally non-trivial case, but it is unclear what the precise topological obstructions would be.

We shall write $\mathcal{A} = C^\infty(M)$, $\mathcal{B} = \Gamma^\infty(\mathcal{B})$, and $\mathcal{E} = \Gamma^\infty(\mathcal{E})$. Their respective C^* -closures are denoted by $A = C(M)$, $B = \Gamma(\mathcal{B})$, and $E = \Gamma(\mathcal{E})$.

Proposition 3.3. An even internal space $I^\infty = (\Gamma^\infty(\mathcal{B}), \Gamma^\infty(\mathcal{E}), D_I)$ yields an unbounded Kasparov $B - A$ -module $I = ({}_B \Gamma(\mathcal{E})_A, D_I)$.

Proof. The algebras A and B are trivially graded C^* -algebras, and $E = \Gamma(\mathcal{E})$ is a \mathbb{Z}_2 -graded, finitely generated projective, right Hilbert A -module, with a left action of B that commutes with the (right) action of A . The properties of γ_I guarantee that all conditions with respect to the grading are satisfied. For instance, the condition $(E^{(m)}, E^{(n)}) \subset A^{(m+n)}$, where $m, n \in \mathbb{Z}_2$, is satisfied, since the

condition $\gamma_I^* = \gamma_I$ implies that $\langle s, t \rangle = 0$ as soon as one of the arguments is odd and the other is even. The operator D_I is a bounded, self-adjoint, odd operator by definition (and hence it is automatically regular). The boundedness of D_I implies that $[D_I, b]$ is also bounded for all $b \in \mathcal{B}$.

For a compact manifold M the compact endomorphisms of the $C(M)$ -module $\Gamma(E)$ are exactly the sections of the endomorphism bundle $\text{End } E$, i.e., $\text{End}_{C(M)}^0(\Gamma(E)) = \Gamma(\text{End}(E))$ (since $\Gamma(\text{End}(E))$ is already unital, the compact endomorphisms of $\Gamma(E)$ are actually all the bounded endomorphisms, see, e.g., Proposition 3.9 of Ref. 26). Thus, $b(1 + D_I^2)^{-\frac{1}{2}}$ is compact for all $b \in \mathcal{B}$, because both $(1 + D_I^2)^{-\frac{1}{2}}$ and b are compact. Hence $({}_B\Gamma(E)_A, D_I)$ has all the properties mentioned in Definition 2.24. \square

B. The product space

Definition 3.4. Let $I^\infty := (\Gamma^\infty(B), \Gamma^\infty(E), D_I, \gamma_I, J_I)$ be a real even internal space over M , with M a compact 4-dimensional Riemannian spin manifold. Let ∇^I be a hermitian connection on E . We define a real even *almost-commutative manifold* to be

$$I^\infty \times_{\nabla} M := (\Gamma^\infty(B), L^2(E \otimes S), \not{D}_E + D_I \otimes \gamma_5, \gamma_I \otimes \gamma_5, J_I \otimes J_M),$$

where $L^2(E \otimes S) \simeq \Gamma(E) \otimes_{C(M)} L^2(S)$ are the L^2 -sections of the twisted spinor bundle $E \otimes S$, and \not{D}_E is the twisted Dirac operator

$$\not{D}_E := \mathbb{I} \otimes_{\nabla} \not{D} := \mathbb{I} \otimes \not{D} - i(\mathbb{I} \otimes c) \circ (\nabla^I \otimes \mathbb{I}).$$

Note that by definition the underlying manifold of an almost-commutative manifold is always assumed to be of dimension 4.

We note that our definition of almost-commutative manifolds fits within the slightly more general definition of almost-commutative spectral triples given in Definition 2.3 of Ref. 8.

The order of I^∞ and M in the notation $I^\infty \times_{\nabla} M$ is reversed in comparison with the order of F and M in $M \times F$. The reason is that the order $I^\infty \times_{\nabla} M$ is more natural from a KK -theoretical viewpoint, whereas the notation $M \times F$ for the globally trivial case is quite standard in the literature. In the remainder of this section, we show in detail that an almost-commutative manifold $I^\infty \times_{\nabla} M$ determines an unbounded Kasparov $B - \mathbb{C}$ -module (i.e., a spectral triple over \mathcal{B}) whose KK -class represents the Kasparov product between the KK -classes of the internal space I^∞ and the canonical spectral triple for M .

Proposition 3.5. Let $I^\infty = (\Gamma^\infty(B), \Gamma^\infty(E), D_I, \gamma_I, J_I)$ be a real even internal space over a compact Riemannian spin manifold M of even KO -dimension k . Let ∇^I be a hermitian connection on E that commutes with the grading γ_I , satisfies $\nabla_\mu^I J_I = J_I \nabla_\mu^I$, and is such that the induced connection $[\nabla^I, \cdot]$ on $\text{End } E$ restricts to a connection on B . Then the real even almost-commutative manifold $I^\infty \times_{\nabla} M$ is a real even spectral triple of KO -dimension $4 + k \pmod{8}$.

Proof. Let us write $D := \not{D}_E + D_I \otimes \gamma_5$. We need to show that $[D, a]$ is bounded for all $a \in \Gamma^\infty(B)$. Since D_I is bounded itself, we need only check this for the twisted Dirac operator \not{D}_E , and we find

$$[\not{D}_E, a] = -ic([\nabla^I, a]),$$

where, with some abuse of notation, we write $c(T\otimes\alpha) = T\otimes c(\alpha)$ for $T \in \Gamma^\infty(\text{End } E)$ and $\alpha \in \Omega^1(M)$. Hence for smooth a the commutator $[\not{D}_E, a]$ indeed acts as a bounded operator on $L^2(E \otimes S)$. Furthermore we need to show that D has compact resolvent, and (as M is compact) for this it is sufficient to show that D^2 (and hence D) is elliptic. The Lichnerowicz-Weitzenböck formula shows that the square of the twisted Dirac operator \not{D}_E is a generalised Laplacian, and hence is elliptic. The bounded (zeroth-order) perturbation $\not{D}_E \rightarrow \not{D}_E + D_I \otimes \gamma_5$ does not affect this ellipticity. Hence $I^\infty \times_{\nabla} M$ is indeed a spectral triple.

Given the grading operators γ_I and γ_5 , it is straightforward to check that $D(\gamma_I \otimes \gamma_5) = -(\gamma_I \otimes \gamma_5)D$, provided that $[\nabla^I, \gamma_I] = 0$.

Given the real structures J_I and J_M , the operator $J_I \otimes J_M$ is anti-unitary and satisfies

$$\begin{aligned} (J_I \otimes J_M)^2 &= -\varepsilon, & D(J_I \otimes J_M) &= (J_I \otimes J_M)D, \\ (J_I \otimes J_M)(\gamma_I \otimes \gamma_5) &= \varepsilon''(\gamma_I \otimes \gamma_5)(J_I \otimes J_M), \end{aligned} \tag{5}$$

where the signs $\varepsilon, \varepsilon''$ are determined by the KO-dimension k of J_I . The first equality in Eq. (5) is immediate from $J_M^2 = -1$ and $J_I^2 = \varepsilon$. Using the relations

$$J_M \not{D} = \not{D} J_M, \quad \gamma^\mu J_M = -J_M \gamma^\mu, \quad \gamma_5 J_M = J_M \gamma_5, \quad J_I D_I = D_I J_I, \quad \nabla_\mu^I J_I = J_I \nabla_\mu^I,$$

the second equality in Eq. (5) is checked by a local calculation (writing $(\mathbb{I} \otimes c) \circ (\nabla^I \otimes \mathbb{I}) = \nabla_\mu^I \otimes \gamma^\mu$):

$$\begin{aligned} D(J_I \otimes J_M)(s \otimes \psi) &= (J_I s) \otimes (\not{D} J_M \psi) - i(\nabla_\mu^I J_I s) \otimes (\gamma^\mu J_M \psi) + (D_I J_I s) \otimes (\gamma_5 J_M \psi) \\ &= (J_I s) \otimes (J_M \not{D} \psi) + i(J_I \nabla_\mu^I s) \otimes (J_M \gamma^\mu \psi) + (J_I D_I s) \otimes (J_M \gamma_5 \psi) \\ &= (J_I s) \otimes (J_M \not{D} \psi) - (J_I \nabla_\mu^I s) \otimes (J_M i \gamma^\mu \psi) + (J_I D_I s) \otimes (J_M \gamma_5 \psi) \\ &= (J_I \otimes J_M)D(s \otimes \psi). \end{aligned}$$

The third equality in Eq. (5) immediately follows from $[J_M, \gamma_5] = 0$ and $J_I \gamma_I = \varepsilon'' \gamma_I J_I$. From the values of $-\varepsilon$ and ε'' it is immediate that the KO-dimension of $I^\infty \times_{\nabla} M$ should be $4 + k \pmod{8}$ (see the table in Definition 2.23).

The zeroth-order condition on $I^\infty \times_{\nabla} M$ is immediate from the zeroth-order condition on I^∞ . Moreover,

$$[[\not{D}_E, a], JbJ^*] = -i[c([\nabla^I, a)], JbJ^*] = -ic([\nabla^I, a], JbJ^*) = 0,$$

because, by assumption, $[\nabla^I, a] \in \Gamma^\infty(\mathcal{B}) \otimes_{C^\infty(M)} \Omega^1(M)$, which commutes with JbJ^* . Together with the first-order condition on D_I , this implies that D satisfies the first-order condition. \square

For a real spectral triple $T = (\mathcal{A}, \mathcal{H}, D, J)$, the gauge group is defined in Definition 2.5 of Ref. 4 as

$$\mathcal{G}(T) := \{uJ u J^* \mid u \in \mathcal{U}(\mathcal{A})\} \simeq \mathcal{U}(\mathcal{A})/\mathcal{U}(\mathcal{A}_J), \tag{6}$$

where the central subalgebra \mathcal{A}_J is defined as $\mathcal{A}_J := \{a \in \mathcal{A} \mid aJ = Ja^*\}$. For the above almost-commutative manifold, we therefore obtain the gauge group

$$\mathcal{G}(I^\infty \times_{\nabla} M) = \mathcal{U}(\mathcal{B})/\mathcal{U}(\mathcal{B}_J),$$

for the real structure $J = J_I \otimes J_M$. However, since $\mathcal{B}_J \simeq \mathcal{B}_{J_I}$, we find that the gauge group of the almost-commutative manifold is completely determined by the internal space, and we write

$$\mathcal{G}(I^\infty \times_{\nabla} M) \simeq \mathcal{G}(I^\infty) := \{uJ_I u J_I^* \mid u \in \mathcal{U}(\mathcal{B})\}. \tag{7}$$

C. The Kasparov product

We now show that the product $I^\infty \times_{\nabla} M$ is an unbounded representative for the Kasparov product of the KK -classes of I^∞ and the canonical spectral triple for M . We first prove this for the cases where $D_I = 0$, and then show that the presence of D_I is irrelevant at the level of KK -classes.

Let I^∞ be an internal space over M , where $D_I = 0$, and consider the unbounded Kasparov module $I := ({}_B E_A, 0)$, where $E = \Gamma(\mathcal{E})$. We know from Proposition 3.5 that $I^\infty \times_{\nabla} M = (\mathcal{B}, L^2(\mathcal{E} \otimes \mathcal{S}), D)$ is a spectral triple, which thus yields an unbounded Kasparov module $I \times_{\nabla} M = ({}_B L^2(\mathcal{E} \otimes \mathcal{S})_{\mathbb{C}}, D) \in \Psi(\mathcal{B}, \mathbb{C})$ (Definition 2.24).

Proposition 3.6. The unbounded Kasparov module $I \times_{\nabla} M$ represents the Kasparov product of (the classes of) $I \in \Psi(\mathcal{B}, A)$ and $({}_A L^2(\mathcal{S})_{\mathbb{C}}, \not{D}) \in \Psi(A, \mathbb{C})$.

Proof. It suffices to check the conditions of Theorem 2.26. Since $D_I = 0$, Conditions (ii) and (iii) are trivial, and we only need to check Condition (i). For all e in a dense subspace of $BE = E$, we need to check boundedness of

$$\begin{aligned} DT_e - T_e \mathcal{D} & \quad \text{on } \text{Dom}(\mathcal{D}) \subset L^2(\mathbb{S}), \\ \mathcal{D}T_e^* - T_e^* D & \quad \text{on } \text{Dom}(D) \subset E \otimes_A L^2(\mathbb{S}) \simeq L^2(E \otimes \mathbb{S}), \end{aligned}$$

where $D = \mathcal{D}_E = -i(\mathbb{I} \otimes c) \circ (\mathbb{I} \otimes \nabla^S + \nabla^I \otimes \mathbb{I})$. For $\psi \in \text{Dom}(\mathcal{D})$, we obtain

$$(DT_e - T_e \mathcal{D})\psi = -i(\mathbb{I} \otimes c) \circ (\mathbb{I} \otimes \nabla^S + \nabla^I \otimes \mathbb{I})e \otimes \psi - e \otimes \mathcal{D}\psi = -ic(\nabla^I e) \otimes \psi,$$

which is indeed bounded for all e in the dense subspace \mathcal{E} . Next, for $f \otimes \psi \in \Gamma^\infty(B \otimes \mathbb{S}) \subset \text{Dom}(D)$ we obtain

$$(\mathcal{D}T_e^* - T_e^* D)(f \otimes \psi) = \mathcal{D}(e|f)\psi - (e|f)\mathcal{D}\psi + i(e|c(\nabla^I f))\psi = -ic(\nabla^I e|f)\psi,$$

where we have used the compatibility of the connection ∇^I with the hermitian form $(\cdot|\cdot)_A$, and so $\mathcal{D}T_e^* - T_e^* D$ is a zeroth-order differential operator for smooth e . \square

To prove a similar result for the case where $D_I \neq 0$, we use the following two lemmas.

Lemma 3.7. *If ${}_{\phi(B)}E_A$ is finitely generated projective as a right A -module, then for any self-adjoint, odd endomorphism $F \in \text{End}_A(E)$, the unbounded Kasparov $B - A$ -modules $({}_{\phi(B)}E_A, F)$ and $({}_{\phi(B)}E_A, 0)$ represent the same class in $KK(B, A)$.*

Proof. Since E is a finitely generated projective A -module, all bounded endomorphisms are in fact compact, i.e., $\text{End}_A(E) = \text{End}_A^0(E)$. The equivalence of the compact operators 0 and $b(F) = F(1 + F^2)^{-\frac{1}{2}}$ is then simply obtained via the operator homotopy $t \rightarrow tb(F)$, for $t \in [0, 1]$. Hence the modules $({}_{\phi(B)}E_A, b(F))$ and $({}_{\phi(B)}E_A, 0)$ are equivalent bounded Kasparov $B - A$ -modules. \square

Lemma 3.8 (see also Corollary 17 of Ref. 16). *Let $({}_{\phi(B)}E_A, D) \in \Psi(B, A)$ and let $T \in \text{End}_A(E)$ be self-adjoint and odd. Then*

1. $({}_{\phi(B)}E_A, D + T)$ is also an unbounded Kasparov module in $\Psi(B, A)$, and
2. $({}_{\phi(B)}E_A, D + T)$ and $({}_{\phi(B)}E_A, D)$ represent the same class in $KK(B, A)$.

Proof.

1. Since T is bounded and self-adjoint, it follows from the Kato-Rellich theorem for Hilbert modules (see Theorem 4.5 of Ref. 27) that the sum $D + T$ remains self-adjoint and regular. The only non-trivial thing to prove is that $D + T$ has compact resolvent, i.e., $\phi(b)(1 + (D + T)^2)^{-1/2} \in \text{End}_A^0(E)$ for all $b \in \mathcal{B} \subset B$. This is equivalent to showing that $\phi(b)(\pm i + D + T)^{-1}$ is compact. The operator $(\pm i + D + T)^{-1}$ maps E into $\text{Dom}(D + T) = \text{Dom } D$, so that $(\pm i + D)(\pm i + D + T)^{-1}$ is a well-defined bounded operator on E . From

$$\phi(b)(\pm i + D + T)^{-1} = \phi(b)(\pm i + D)^{-1}(\pm i + D)(\pm i + D + T)^{-1},$$

we then see that $\phi(b)(\pm i + D + T)^{-1}$ is compact.

2. The idea is to prove that $({}_{\phi(B)}E_A, D + T) \in \Psi(B, A)$ represents the Kasparov product $[({}_{\phi(B)}E_A, D)] \otimes_A [({}_{\phi(B)}E_A, 0)]$. It is enough to show that all the conditions in Theorem 2.26 are satisfied. First of all,

$$\begin{aligned} A \ni a & \mapsto (D + T)T_e(a) = ((D + T)e)a, \\ (f \otimes a) & \mapsto T_e^*(D + T)(f \otimes a) = ((D + T)e, f)_A a, \end{aligned}$$

are both clearly bounded on A and $\text{Dom}(D + T) = \text{Dom } D$, respectively, for all $e \in \text{Dom } D$. In particular, this holds for all $e \in \phi(\mathcal{B}) \text{Dom } D$, which is a dense subset of $\phi(B)E$. This proves that Condition (i) in Theorem 2.26 is satisfied.

Since $\text{Dom } D = \text{Dom}(D + T)$, Condition (ii) is also satisfied. For the final condition, a small calculation shows that

$$\begin{aligned} & ((D + T)e, De) + (De, (D + T)e) \\ &= ((D + T)e, (D + T)e) - ((D + T)e, Te) + (De, (D + T)e) \\ &= ((D + T)e, (D + T)e) - (Te, Te) + (De, De) \geq -\|T\|^2(e, e), \end{aligned}$$

for all $e \in \text{Dom } D$, since $((D + T)e, (D + T)e)$ and (De, De) are positive. \square

Corollary 3.9. The unbounded Kasparov module $I \times_{\nabla} M = ({}_B E \otimes_A L^2(\mathcal{S})_{\mathbb{C}}, \mathbb{I} \otimes_{\nabla} \mathcal{D} + D_I \otimes \gamma_5)$ represents the Kasparov product of $I = ({}_B E_A, D_I)$ with $({}_A L^2(\mathcal{S})_{\mathbb{C}}, \mathcal{D})$.

Proof. By Lemma 3.7 we know that $({}_B E_A, D_I)$ and $({}_B E_A, 0)$ represent the same Kasparov class. From Proposition 3.6, it then follows that the cycle $({}_B E \otimes_A L^2(\mathcal{S})_{\mathbb{C}}, \mathcal{D}_E)$ also represents the Kasparov product of $({}_B E_A, D_I)$ with $({}_A L^2(\mathcal{S})_{\mathbb{C}}, \mathcal{D})$. According to Lemma 3.8, the cycle $({}_B E \otimes_A L^2(\mathcal{S})_{\mathbb{C}}, \mathcal{D}_E + D_I \otimes \gamma_5)$ represents the same Kasparov class as $({}_B E \otimes_A L^2(\mathcal{S})_{\mathbb{C}}, \mathcal{D}_E)$, so it also represents this Kasparov product. \square

Remark 3.10. 1. The construction of $I \times_{\nabla} M$ via Kasparov products fits naturally in the framework of Mesland’s category of spectral triples,¹⁵ where the internal space I^{∞} with the connection ∇ can be seen as (a representative of) a morphism from the canonical triple for M to the almost-commutative manifold $I^{\infty} \times_{\nabla} M$.

2. As is clear from the above discussion, the presence of the operator D_I (or $D_I \otimes \gamma_5$) is completely irrelevant on the level of KK -classes. In this sense the KK -equivalence is too strong for our purposes, because in the models under consideration the presence of the operator D_I certainly does matter. We will describe in Sec. VI how this operator plays the role of a “mass matrix” for the elementary fermions of the gauge theory, and gives rise to the Higgs field in the noncommutative Standard Model (see also Sec. VII B for a concrete example of D_I as a mass matrix).

IV. PRINCIPAL MODULES

We would like to describe a classical gauge theory on a manifold M by considering an almost-commutative manifold $I^{\infty} \times_{\nabla} M$. For this purpose we now restrict our attention to a special case of internal spaces, which we call principal modules.

In Sec. IV A we first recall (part of) the classification of finite-dimensional real spectral triples that has been done by Krajewski²⁴ and by Paschke and Sitarz.²⁵ In Sec. IV B, we then define the notion of principal modules, and we show that, when the base manifold (which is of arbitrary dimension) is simply connected, the gauge group of a principal module (as defined for internal spaces in Eq. (7)) is isomorphic to the classical notion of the gauge group of a principal fibre bundle (as defined in Definition 2.11).

A. Real finite spectral triples

Finite-dimensional real spectral triples have been classified for the case of KO-dimension 0.^{24,25} With similar arguments, this can be generalised to arbitrary KO-dimension.²⁸ In the following theorem we give the result for complex algebras, while also setting the matrix $D_F = 0$. Below $c. c.$ denotes complex conjugation of the coefficients with respect to the standard basis of $\mathbb{C}^{m_{ij}}$.

Theorem 4.1. *Let $F := (A_F, \mathcal{H}_F, 0, J_F)$ be a real finite spectral triple over a complex $*$ -algebra A_F . Up to unitary equivalence, this triple is of the form*

$$A_F = \bigoplus_{i=1}^l M_{N_i}(\mathbb{C}), \quad \mathcal{H}_F = \bigoplus_{i,j=1}^l \mathcal{H}_{ij}, \quad \mathcal{H}_{ij} := \bigoplus_{i,j=1}^l M_{N_i, N_j}(\mathbb{C}) \otimes \mathbb{C}^{m_{ij}},$$

such that $m_{ij} = m_{ji}$, and the inner product on each copy of $M_{N_i, N_j}(\mathbb{C})$ is given by $\langle t_1, t_2 \rangle = \text{Tr}(t_1^* t_2)$. If $J_F^2 = \varepsilon$, then J_F acts on $\mathcal{H}_{ij} \oplus \mathcal{H}_{ji}$, ($i < j$), as

$$\begin{pmatrix} 0 & \varepsilon(\cdot)^* \\ (\cdot)^* & 0 \end{pmatrix} \otimes (\text{Id}_{m_{ij}} \circ \text{c. c.}).$$

If $J_F^2 = 1$, the real structure J_F acts on $\mathcal{H}_{ii} \simeq M_{N_i}(\mathbb{C}) \otimes \mathbb{C}^{m_{ii}}$ as

$$(\cdot)^* \otimes (\text{Id}_{m_{ii}} \circ \text{c. c.}).$$

If $J_F^2 = -1$, then m_{ii} is even and J_F acts on $(M_{N_i}(\mathbb{C}) \oplus M_{N_i}(\mathbb{C})) \otimes \mathbb{C}^{\frac{m_{ii}}{2}}$ as

$$\begin{pmatrix} 0 & -(\cdot)^* \\ (\cdot)^* & 0 \end{pmatrix} \otimes (\text{Id}_{\frac{m_{ii}}{2}} \circ \text{c. c.}).$$

The different copies of $M_{N_i, N_j}(\mathbb{C})$ (with respect to the above decomposition) in \mathcal{H}_{ij} are denoted by \mathcal{H}_{ij}^α , where $1 \leq \alpha \leq m_{ij}$.

Remark 4.2. For finite-dimensional complex vector spaces V and W , consider the linear isomorphism

$$L : V \otimes \overline{W} \rightarrow \text{Hom}(W, V), \quad v \otimes \overline{w} \mapsto (w' \mapsto v\langle w, w' \rangle), \quad v \in V, \quad w, w' \in W,$$

where \overline{W} denotes the conjugate vector space. Write $V_i = \mathbb{C}^{N_i}$, endowed with the standard inner product. Then the finite-dimensional Hilbert space \mathcal{H}_{ij} can also be put in the form

$$\mathcal{H}_F = \bigoplus_{(i,j) \in K} V_i \otimes \overline{V}_j,$$

endowed with its standard inner product. Here K is a *multiset* consisting of pairs in $I \times I$ such that the multiplicity of (i, j) is equal to (j, i) and such that the projection $K \rightarrow I$ on either of the factors is surjective (this last condition is equivalent to the faithfulness of the action of A_F on \mathcal{H}_F). The algebra $A_F \otimes A_F^{\text{op}}$ acts on a summand $V_i \otimes \overline{V}_j$ as

$$(a, b^{\text{op}})(v \otimes \overline{w}) = a_i v \otimes \overline{b_j^* w},$$

and the corresponding real structure on $V_i \otimes \overline{V}_j \rightarrow V_j \otimes \overline{V}_i$ is simply given by

$$J_F(v \otimes \overline{w}) = \pm w \otimes \overline{v},$$

where the signs are determined by the KO -dimension of F . We will use this form of the real finite spectral triple in Sec. V.

From now on we assume that every real finite spectral triple (with $D_F = 0$) is of the form as mentioned in Theorem 4.1. Later on, the algebra $(A_F)_{J_F}$ will also be of interest, so we conclude this subsection by determining its precise form.

Recall that, in general, for any real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$, the complex central subalgebra \mathcal{A}_J is defined as $\mathcal{A}_J = \{a \in \mathcal{A} \mid aJ = Ja^*\}$.

Proposition 4.3. *With notation as above, we have*

$$(A_F)_{J_F} = \left\{ a = \bigoplus_{i \in I} \lambda_i \text{id}_{N_i} \in A_F \mid \lambda_i \in \mathbb{C}; \lambda_i = \lambda_j \text{ if } \mathcal{H}_{ij} \neq \{0\} \right\}.$$

Proof. We can assume that J is in standard form. Write $A_F = \bigoplus_i M_{N_i}(\mathbb{C})$ and consider an element $a = \bigoplus_{i \in I} a_i \in A_F$. If $t \in \mathcal{H}_{ij}^\alpha$ ($1 \leq \alpha \leq m_{ij}$), then

$$a(J_F t) = \pm a_j t^* \quad \text{and} \quad J_F(a^* t) = \pm t^* a_i.$$

Choose $t^* = e_{kl}$, where $1 \leq k \leq N_j$ and $1 \leq l \leq N_i$. Then

$$(a_j e_{kl})_{\gamma\beta} = (a_j)_{\gamma k} \delta_{\beta l}, \quad \text{and} \quad (e_{kl} a_i)_{\gamma\beta} = \delta_{\gamma k} (a_i)_{l\beta}.$$

Therefore, $a J_F = J_F a^*$ if and only if

$$(a_j)_{\gamma k} \delta_{\beta l} = (a_i)_{l\beta} \delta_{\gamma k},$$

for all $1 \leq k, \gamma \leq N_j$ and $1 \leq l, \beta \leq N_i$. It follows that a_i, a_j are diagonal and $(a_j)_{kk} = (a_i)_{ll}$ for all $1 \leq k \leq N_j$ and $1 \leq l \leq N_i$. Hence, $a \in (A_F)_{J_F}$ if and only if each $a_i = \lambda_i \text{id}_{N_i}$ and $\lambda_i = \lambda_j$ if $\mathcal{H}_{ij} \neq \{0\}$. \square

The following definition is inspired by the proof of Proposition 4.3.

Definition 4.4. Let $A_F = \bigoplus_{i \in I} M_{N_i}(\mathbb{C})$ act on $\mathcal{H}_F = \bigoplus_{i,j \in I} \mathcal{H}_{ij}$ as above. We define an equivalence relation on I as follows. For $i \neq j \in I$, we set $i \sim j$ if there exists a sequence $i = i_0, \dots, i_k = j$ such that $\mathcal{H}_{i_m, i_{m+1}} \neq \{0\}$ for all $0 \leq m < k$. If $i \sim j$, we say that i is connected to j .

Proposition 4.3 in particular shows that $\mathbb{C} \subset (A_F)_{J_F} \subset Z(A_F)$.

Corollary 4.5. We have the isomorphism $(A_F)_{J_F} \simeq \bigoplus_{[i] \in I/\sim} \mathbb{C}$. In particular, the two extreme cases are:

- $(A_F)_{J_F} = Z(A_F)$ if and only if $\mathcal{H}_{ij} = 0$ for all $i \neq j$ (i.e., $I/\sim \simeq I$).
- $(A_F)_{J_F} = \mathbb{C}$ if and only if i is connected to j for all $i, j \in I$ (i.e., $I/\sim \simeq \{1\}$).

B. Principal modules

We now want to find spectral triples for gauge theories that are globally non-trivial. Recall from Definition 2.13 that a general gauge theory with structure group G_F on a manifold M is given by a principal G_F -bundle P over M (along with a prescribed action functional or Lagrangian).

If $(A_F, \mathcal{H}_F, D_F, J_F)$ is a finite-dimensional real spectral triple, then the corresponding gauge group G_F is given by (see also Eq. (6))

$$G_F := \{u J_F u J_F^* \mid u \in \mathcal{U}(A_F)\} \simeq \mathcal{U}(A_F) / \mathcal{U}((A_F)_{J_F}).$$

Such finite spectral triples can be used to describe globally trivial gauge theories over M (see the Introduction). Any finite spectral triple F automatically yields an internal space

$$I_F^\infty = (\Gamma^\infty(M \times A_F), \Gamma^\infty(M \times \mathcal{H}_F), D_F, J_F),$$

where now D_F and J_F are seen as constant bundle endomorphisms acting on the fibre \mathcal{H}_F . We now want to generalise this construction in order to describe globally non-trivial gauge theories. Of course, fibre-wise we want to obtain the finite-dimensional situation that has been explained in Sec. IV A.

The most straightforward way to obtain (examples of) globally non-trivial gauge theories over M would then be as follows (see also Lemma 2.5 of Ref. 8 and Ref. 3). Take any real finite spectral triple $F := (A_F, \mathcal{H}_F, D_F, J_F)$ with gauge group G_F , and let M be a smooth compact 4-dimensional Riemannian spin manifold. Take any principal G_F -bundle $P \rightarrow M$. We construct the globally non-trivial triple of the form

$$P \times_{G_F} F := (\Gamma^\infty(P \times_{G_F} A_F), \Gamma^\infty(P \times_{G_F} \mathcal{H}_F), D_P, 1 \times J_F).$$

Here D_P is an endomorphism acting on the vector bundle $P \times_{G_F} \mathcal{H}_F$ satisfying certain compatibility requirements (which we will specify later in Definition 6.1).

Remark 4.6. Note that (in contrast to Ref. 8) we do not require D_p to be of the form $1 \times D_F$, where D_F is a G_F -invariant operator on \mathcal{H}_F , as such an assumption is too strong for our purposes. In particular, in specific examples (such as the noncommutative Standard Model) that requirement would prevent the appearance of a scalar (Higgs-like) field through inner fluctuations.

For the remainder of this section we ignore the endomorphism D_p , since it is not relevant for the definition of the gauge group, and we define the following:

Definition 4.7. Let $F := (A_F, \mathcal{H}_F, 0, J_F)$ be a real finite spectral triple of the same form as in Theorem 4.1. Write G_F for the corresponding gauge group. Let M be a smooth compact Riemannian spin manifold and let $P \rightarrow M$ be any principal G_F -bundle. A triplet of the form

$$P \times_{G_F} F := (\Gamma^\infty(P \times_{G_F} A_F), \Gamma^\infty(P \times_{G_F} \mathcal{H}_F), 1 \times J_F),$$

is called a *principal G_F -module* over M (or $C^\infty(M)$) with fibre F . For brevity, we introduce the notation $B := P \times_{G_F} A_F$, $E := P \times_{G_F} \mathcal{H}_F$, $\mathcal{B} := \Gamma^\infty(B)$, $\mathcal{E} := \Gamma^\infty(E)$, and $J := 1 \times J_F$.

Remark 4.8. The principal fibre bundle P is an explicit ingredient in the definition of a principal module. From P we constructed the associated vector bundle $E = P \times_{G_F} \mathcal{H}_F$, and (as discussed in Sec. II B 1) P equips E with a unique equivalence class of G_F -atlases. Whenever we consider transition functions of E , we therefore assume that they form a G_F -atlas in the equivalence class obtained from P . Given a G_F -atlas, the vector bundle E inherits a hermitian structure from the inner product on \mathcal{H}_F , which is well-defined because the action of G_F on \mathcal{H}_F is unitary. For two equivalent G_F -atlases, the corresponding hermitian structures are isometric.

We stress that, given only the vector bundle E (with structure group G_F), we cannot reconstruct the principal G_F -bundle P . In order to reconstruct P , we also need to know the corresponding equivalence class of G_F -atlases.

Proposition 4.9. A principal module $P \times_{G_F} F$ is a real internal space $(\Gamma^\infty(P \times_{G_F} A_F), \Gamma^\infty(P \times_{G_F} \mathcal{H}_F), 0, 1 \times J_F)$ over M .

Proof. The action of G_F on A_F is given by conjugation when A_F is considered as a $*$ -subalgebra of $\text{End}(\mathcal{H}_F)$. Consequently, the fibre-wise action of the $*$ -algebra bundle $B = P \times_{G_F} A_F$ on E is well defined, and hence B is a unital $*$ -algebra subbundle of $\text{End}(E)$. The operator $D_I = 0$ is trivially a hermitian endomorphism. Since the operator J_F commutes with G_F , it induces a real structure J_x on each fibre of E . The operator $J = 1 \times J_F$ denotes the anti-linear operator on E that is induced by these real structures J_x on the fibres. □

Remark 4.10. Because $(uJ_F u J_F^*)a(J_F u^* J_F^* u^*) = uau^*$ for all $a \in A_F$, $u \in \mathcal{U}(A_F)$, we see that the given action of an element $uJ_F u J_F^* \in G_F$ on A_F coincides with the usual conjugation of the element $u \in \mathcal{U}(A_F)$. Since $(A_F)_{J_F} \subset Z(A_F)$, the map $\tau : G_F \ni uJ_F u J_F^* \mapsto \text{Ad}(uJ_F u J_F^*) = \text{Ad } u \in \text{Inn}(A_F)$ does not depend on the choice of u . Thus, the surjective map $\tau : G_F \rightarrow \mathcal{U}(A_F)/\mathcal{U}(Z(A_F)) \simeq \text{Inn}(A_F)$ is induced by the usual map $\mathcal{U}(A_F) \rightarrow \text{Inn}(A_F)$ (recall that G_F is the quotient $\mathcal{U}(A_F)/\mathcal{U}((A_F)_{J_F})$).

1. The gauge group

Consider a principal module $P \times_{G_F} F = (B, \mathcal{E}, J)$ over M . Using the classification of A_F and \mathcal{H}_F , as given in Sec. IV A, we can decompose the bundles $B = P \times_{G_F} A_F$ and $E = P \times_{G_F} \mathcal{H}_F$ in a similar way:

$$\begin{aligned}
 B &= \bigoplus_{i \in I} B_i, & B_i &= P \times_{G_F} M_{N_i}(\mathbb{C}), \\
 E &= \bigoplus_{i, j \in I} E_{ij}, & E_{ij} &= P \times_{G_F} \mathcal{H}_{ij}.
 \end{aligned}$$

Each vector bundle E_{ij} carries the obvious action by $B \otimes B^{\text{op}}$. Note, however, that even though $\mathcal{H}_{ij} = \mathbb{C}^{N_i} \otimes \mathbb{C}^{N_j} \otimes \mathbb{C}^{m_{ij}}$, E_{ij} is not necessarily of the form $E_i \otimes E_j \otimes \mathbb{C}^{m_{ij}}$ for some vector bundles E_i and E_j (see Sec. VII A for an example).

Note that, for the case $i = j$, the bundle E_{ii} is necessarily isomorphic to (a number of copies of) B_i . Indeed, the G_F -valued transition functions act on the fibres of E_{ii} , which are isomorphic to (copies of) $M_{N_i}(\mathbb{C})$, by conjugation with an element $u \in U(N_i)$, and are therefore inner automorphisms of the algebra $M_{N_i}(\mathbb{C})$. By Remark 4.10, these transition functions are equal to those for the $*$ -algebra bundle B_i .

Denote by $[i]$ the equivalence class of all $j \in I$ that are connected to i (see Definition IV.4). Write

$$B_{[i]} = \bigoplus_{s \in [i]} B_s,$$

and write $b_{[i]}$ for the projection of an element b onto $B_{[i]}$. As $(\mathcal{B}_{[i]})_J = C^\infty(M)$, we obtain (see also Corollary 4.5)

$$\mathcal{B}_J = \bigoplus_{[i] \in I/\sim} C^\infty(M).$$

The gauge group of the principal module $P \times_{G_F} F = (\mathcal{B}, \mathcal{E}, J)$ is defined as (see Eq. (7))

$$\mathcal{G}(P \times_{G_F} F) := \{uJuJ^* \mid u \in \mathcal{U}(\mathcal{B})\} \simeq \mathcal{U}(\mathcal{B})/\mathcal{U}(\mathcal{B}_J).$$

At the same time, a principal G_F -bundle $P \rightarrow M$ is equipped with the gauge group $\mathcal{G}(P) = \Gamma^\infty(\text{Ad } P)$ (see Sec. II B). We now aim at showing that for a principal module $P \times_{G_F} F$, the gauge groups $\mathcal{G}(P \times_{G_F} F)$ and $\mathcal{G}(P)$ coincide, provided that M is simply connected.

Consider the group bundle map

$$\phi : \mathcal{U}(\mathcal{B}) \simeq P \times_{G_F} \mathcal{U}(A_F) \rightarrow P \times_{G_F} \mathcal{U}(\mathcal{H}_F), \quad u_x \mapsto u_x J_x u_x J_x^*.$$

The image $\phi(\mathcal{U}(\mathcal{B}))$ is a group subbundle of $P \times_{G_F} \mathcal{U}(\mathcal{H}_F)$, with fibres isomorphic to G_F . In fact, this subbundle is isomorphic to the group bundle $\text{Ad } P$. The induced map ϕ_* on the sections $\mathcal{U}(\mathcal{B}) \simeq \mathcal{U}(\Gamma^\infty(\mathcal{B})) \rightarrow \mathcal{U}(\Gamma^\infty(\text{End}(\mathcal{E})))$ is precisely the map $u \mapsto uJuJ^*$, $u \in \mathcal{U}(\mathcal{B})$. Thus, ϕ_* maps $\mathcal{U}(\mathcal{B})$ into $\Gamma^\infty(\text{Ad } P)$. However, as discussed in Sec. II D, this map ϕ_* need not always be surjective. We will proceed by showing that in our case, under the assumption that M is simply connected, we do have surjectivity.

Proposition 4.11. *Let $P \times_{G_F} F$ be a principal module over M . There exists a group subbundle $U \subset \mathcal{U}(\mathcal{B})$ such that the restriction $\phi : U \rightarrow \text{Ad } P$ is a covering map.*

Proof. Consider the subbundle $E_{[i]} := B_{[i]} \cdot E$ (i.e., the subbundle on which $B_{[i]}$ acts non-trivially). Define the group subbundle

$$U := \{u \in \mathcal{U}(\mathcal{B}) \mid \det_{[i]} u_{[i]} = 1 \text{ for all } [i]\},$$

where $\det_{[i]} u_{[i]}$ denotes the fibrewise determinant of $u_{[i]}$ seen as an element of the bundle $\text{End } E_{[i]}$. Denote the rank of $E_{[i]}$ by $N_{[i]}$. Since any element $u \in \mathcal{U}(\mathcal{B})$ can be written as $u = vw$, where $v \in U$ and $w \in \mathcal{U}(B_J)$ (just take $w_{[i]} = (\det_{[i]} u_{[i]})^{\frac{1}{N_{[i]}}} \text{id}_{N_{[i]}}$ and $v = uw^{-1}$), the image $\phi(U)$ is equal to the image $\phi(\mathcal{U}(\mathcal{B})) = \text{Ad } P$.

Let us calculate the kernel $\phi_x : U_x \rightarrow (\text{Ad } P)_x$. Choose $u \in U_x \cap \ker \phi_x$. Since $u \in \ker \phi_x$, each $u_{[i]}$ is diagonal. Because $\det_{[i]} u_{[i]} = 1$, we obtain that $u_{[i]} = \lambda_{[i]} \text{id}_{N_{[i]}}$, where $\lambda_{[i]}$ is an $N_{[i]}$ -th root of unity. Since there are only finitely many equivalence classes $[i]$, the group $U_x \cap \ker \phi_x$ is finite.

The condition for a map to be a covering map is of a local nature, so we can assume that all bundles are globally trivial. In that case, it follows from the fact that $U_x \cap \ker \phi_x$ is finite, that $U \rightarrow \text{Ad } P$ is a covering map. \square

Combining Proposition 4.11 with Corollary 2.22 and Theorem 2.12 immediately yields the desired result:

Theorem 4.12. *Let $P \times_{G_F} F$ be a principal module over M . If M is simply connected, then*

$$\mathcal{G}(P \times_{G_F} F) \simeq \Gamma^\infty(\text{Ad } P) \simeq \mathcal{G}(P).$$

Remark 4.13. It follows from the above that for each element g of the gauge group $\mathcal{G}(P \times_{G_F} F)$, there exists a unitary section $u \in \mathcal{B}$ with (fibre-wise) determinant equal to 1, such that $g = uJuJ^*$. In this sense, the gauge group is *unimodular* by default. This only holds for complex algebras \mathcal{B} . For real algebras (including the one describing the noncommutative Standard Model^{2,22}), one needs to impose unimodularity by hand (see also Ref. 29 and references therein).

V. GAUGE MODULES

In Sec. IV B, we introduced the notion of principal modules, which have an entirely *geometric* nature. In this section, we introduce so-called *gauge modules*, which are of a purely *algebraic* nature. We show that each gauge module is in fact also a principal module, but unfortunately not all principal modules can be obtained from gauge modules.

Inspired by the standard form of finite spectral triples as obtained in Theorem 4.1 and Remark 4.2, we introduce the following definition, which might be considered an extension of Krajewski diagrams to the globally non-trivial case.

Definition 5.1. Let $\mathcal{A} := C^\infty(M)$. Suppose we are given a finite set of non-degenerate hermitian finitely generated projective \mathcal{A} -modules \mathcal{E}_i (for $i \in I = \{1, \dots, l\}$), and define the module algebras $\mathcal{B}_i := \text{End}_{\mathcal{A}}(\mathcal{E}_i)$. Take a *multiset* K consisting of pairs in $I \times I$ such that the multiplicity of (i, j) is equal to the multiplicity of (j, i) , and such that the projection $K \rightarrow I$ on either of the factors is surjective. Denote the multiplicity of the pair (i, j) by m_{ij} and write (i_α, j_α) ($1 \leq \alpha \leq m_{ij}$) to distinguish the pairs in K that occur more than once (see also Theorem 4.1 for this notation).

A *gauge module* $(\mathcal{B}, \mathcal{E}, J)$ is of the form

$$\mathcal{B} := \bigoplus_{i \in I} \mathcal{B}_i, \quad \mathcal{E} := \bigoplus_{(i,j) \in K} \mathcal{E}_i \otimes_{\mathcal{A}} \overline{\mathcal{E}}_j, \quad J : \mathcal{E}_i \otimes_{\mathcal{A}} \overline{\mathcal{E}}_j \rightarrow \mathcal{E}_j \otimes_{\mathcal{A}} \overline{\mathcal{E}}_i,$$

where J is of the same standard form as the finite operator J_F in Theorem 4.1 (and which depends on the value of $J^2 = \varepsilon = \pm 1$, e.g., $J_{ij}(e_{i_\alpha} \otimes \overline{e}_{j_\alpha}) = \varepsilon e_{j_\alpha} \otimes \overline{e}_{i_\alpha}$, for $e_{i_\alpha} \otimes \overline{e}_{j_\alpha} \in \mathcal{E}_{i_\alpha} \otimes \overline{\mathcal{E}}_{j_\alpha}$ if $j < i$).

The assumption that the projection $K \rightarrow I$ is surjective ensures that the action of \mathcal{B} on \mathcal{E} is faithful. From the Serre-Swan theorem, we know that each module \mathcal{E}_i is given by the smooth sections of a vector bundle $E_i \rightarrow M$. Because the hermitian structure on \mathcal{E}_i is non-degenerate, this yields a hermitian structure on E_i . By Theorem 2.5, the module algebra \mathcal{B}_i is given by the smooth sections of a unital weak $*$ -algebra bundle $B_i \rightarrow M$. Since $\mathcal{B}_i = \text{End}_{\mathcal{A}}(\mathcal{E}_i)$ we obtain $B_i = \text{End}(E_i)$. The local triviality of B_i then follows from the local triviality of E_i , which means that B_i is in fact a unital $*$ -algebra bundle.

As mentioned in Remark 4.8, given a principal module $P \times_{G_F} F = (\mathcal{B}, \mathcal{E}, J)$ (but not P itself), it is not possible to reconstruct P , unless we are given the equivalence class of G -atlases on the vector bundle $E = P \times_{G_F} \mathcal{H}_F$. However, we will show below that for gauge modules it is possible to uniquely reconstruct the corresponding principal G_F -bundle. The main distinctive feature of gauge modules is that the vector bundle E decomposes as a direct sum of tensor products of hermitian vector bundles E_i . To each E_i there uniquely (up to isomorphism) corresponds a principal $U(N_i)$ -bundle. From these principal $U(N_i)$ -bundles, we can subsequently construct the corresponding principal G_F -bundle P .

Proposition 5.2. *Let $(\mathcal{B}, \mathcal{E}, J)$ be a gauge module. Then:*

1. *There exist a real finite spectral triple $F = (A_F, \mathcal{H}_F, 0, J_F)$ and a principal $\mathcal{U}(A_F)$ -bundle Q such that $(\mathcal{B}, \mathcal{E}, J) = Q \times_{\mathcal{U}(A_F)} F$.*
2. *There exists a principal G_F -bundle P such that $(\mathcal{B}, \mathcal{E}, J) = P \times_{G_F} F$.*

Proof. 1. The gauge module $(\mathcal{B}, \mathcal{E}, J)$ is constructed from a given set of hermitian vector bundles E_i of rank N_i and the index (multi)sets I and K . By assumption $B_i = \text{End}(E_i)$, and so B_i has typical fibre $M_{N_i}(\mathbb{C})$. We define

$$A_F := \bigoplus_{i \in I} M_{N_i}(\mathbb{C}), \quad \mathcal{H}_F := \bigoplus_{(i,j) \in K} \mathbb{C}^{N_i} \otimes \overline{\mathbb{C}^{N_j}}.$$

For each E_i , there is a principal $U(N_i)$ -bundle Q_i (which is unique up to isomorphism) such that $E_i \simeq Q_i \times_{U(N_i)} \mathbb{C}^{N_i}$ (see Example 2.14). Let (U, u_{UV}^i) be a $U(N_i)$ -atlas on E_i corresponding to local trivialisations of Q_i . The transition functions u_{UV}^j of $\overline{E_j}$ are given by the right action of $(u_{UV}^j)^*$ on $\overline{\mathbb{C}^{N_j}}$ (see Eq. (4)), which is implemented as $(v_i \otimes \overline{w_j})(u_{UV}^j)^* = J u_{UV}^j J^* (v_i \otimes \overline{w_j})$. Hence we obtain transition functions for E of the form

$$g_{UV} = \bigoplus_{(i,j) \in K} u_{UV}^i \otimes (u_{UV}^j)^{\text{op}*} = \bigoplus_{(i,j) \in K} u_{UV}^i J u_{UV}^j J^*.$$

Writing $u_{UV} = \bigoplus_{i \in I} u_{UV}^i \in C^\infty(U \cap V, \mathcal{U}(A_F))$, we see that $g_{UV} = u_{UV} J u_{UV} J^* \in C^\infty(U \cap V, G_F)$. Since the u_{UV}^i are transition functions of $\pi_i : Q_i \rightarrow M$, we see that the u_{UV} are the transition functions of the principal $\mathcal{U}(A_F)$ -bundle

$$Q := Q_1 \times_M \cdots \times_M Q_l := \{(q_1, \dots, q_l) \in Q_1 \times \cdots \times Q_l \mid \pi_1(q_1) = \cdots = \pi_l(q_l)\}.$$

Since the action of u_{UV} on \mathcal{H}_F is given by $g_{UV} = u_{UV} J u_{UV} J^*$, we see that $E \simeq Q \times_{\mathcal{U}(A_F)} \mathcal{H}_F$ as hermitian vector bundles. As conjugation by u_{UV} coincides with conjugation by g_{UV} on the algebra A_F , we also have $B \simeq Q \times_{\mathcal{U}(A_F)} A_F$. It is straightforward to check that J is invariant under conjugation by a transition function g_{UV} , and hence it is simply of the form $J = 1 \times J_F$. Since J is an anti-unitary operator satisfying $J^2 = \varepsilon$ and the order-zero condition, it follows that J_F is a real structure on \mathcal{H}_F .

2. Given the principal $\mathcal{U}(A_F)$ -bundle Q from the first part of this lemma, we simply construct a principal G_F -bundle as

$$P := Q \times_{\mathcal{U}(A_F)} G_F,$$

where $u \in \mathcal{U}(A_F)$ acts on G_F as left multiplication by the element $u J_F u J_F^*$. The transition functions of P are given by $g_{UV} = u_{UV} J u_{UV} J^* \in C^\infty(U \cap V, G_F)$. It then straightforwardly follows that

$$P \times_{G_F} \mathcal{H}_F \simeq (Q \times_{\mathcal{U}(A_F)} G_F) \times_{G_F} \mathcal{H}_F \simeq Q \times_{\mathcal{U}(A_F)} \mathcal{H}_F \simeq E,$$

and similarly we obtain $P \times_{G_F} A_F \simeq B$. □

The above proposition shows that each gauge module is in fact a principal module $P \times_{G_F} F$ (where we can uniquely reconstruct F and P), where P can be lifted to a principal $\mathcal{U}(A_F)$ -bundle Q (which is unique up to isomorphism). We now show the converse, namely that a principal module $P \times_{G_F} F$ with a lift $\tau : Q \rightarrow P$ uniquely corresponds to a gauge module.

Proposition 5.3. Let $P \times_{G_F} F = (\mathcal{B}, \mathcal{E}, J)$ be a principal module, and suppose we have a principal $\mathcal{U}(A_F)$ -bundle Q that lifts P . Then Q naturally induces a gauge module structure on $(\mathcal{B}, \mathcal{E}, J)$.

Proof. As we have seen in Sec. IV A, the real finite spectral triple $F = (A_F, \mathcal{H}_F, 0, J_F)$ has a decomposition of the form

$$A_F = \bigoplus_{i \in I} M_{N_i}(\mathbb{C}), \quad \mathcal{H}_F = \bigoplus_{(i,j) \in K} \mathbb{C}^{N_i} \otimes \overline{\mathbb{C}^{N_j}}.$$

Thus we have $\mathcal{U}(A_F) = \times_{i \in I} U(N_i)$, and the principal $\mathcal{U}(A_F)$ -bundle Q then decomposes as $Q_1 \times_M \cdots \times_M Q_l$, where each Q_i is a principal $U(N_i)$ -bundle given by $Q_i := Q \times_{\mathcal{U}(A_F)} U(N_i)$. We then construct

$$B_i := Q \times_{\mathcal{U}(A_F)} M_{N_i}(\mathbb{C}) \simeq Q_i \times_{U(N_i)} M_{N_i}(\mathbb{C}), \quad E_i := Q \times_{\mathcal{U}(A_F)} \mathbb{C}^{N_i} \simeq Q_i \times_{U(N_i)} \mathbb{C}^{N_i},$$

where $\mathcal{U}(A_F) = \times_{i \in I} U(N_i)$ acts on \mathbb{C}^{N_i} as left multiplication by the factor $U(N_i)$, and on $M_{N_i}(\mathbb{C})$ as conjugation by $U(N_i)$. The bundle E_i naturally inherits a hermitian structure from the standard inner product on \mathbb{C}^{N_i} . Because Q lifts P , the bundles B and E corresponding to the principal module $P \times_{G_F} F$ are in fact of the form

$$B := Q \times_{\mathcal{U}(A_F)} A_F = \bigoplus_{i \in I} B_i, \quad E := Q \times_{\mathcal{U}(A_F)} \mathcal{H}_F = \bigoplus_{(i,j) \in K} E_i \otimes \overline{E_j}.$$

Furthermore, as the transition functions of B_i are given by conjugation by the transition functions of E_i , and as its fibre equals $M_{N_i}(\mathbb{C}) = \text{End}(\mathbb{C}^{N_i})$, it follows that $B_i = \text{End}(E_i)$ and B_i acts as such on E . Hence we have shown that the principal module $P \times_{G_F} F$ is equal to the gauge module given by the modules $\mathcal{E}_i := \Gamma^\infty(E_i)$ and the real structure $J = 1 \times J_F$. \square

The previous two propositions then lead us to the main result of this section:

Theorem 5.4. *A gauge module is characterised uniquely (up to isomorphism) by a principal module $P \times_{G_F} F$ for which there exists a principal $\mathcal{U}(A_F)$ -bundle Q that lifts P .*

Proof. Given a gauge module, we have shown in Proposition 5.2 that we can uniquely construct a real finite spectral triple $F = (A_F, \mathcal{H}_F, 0, J_F)$, a principal G_F -bundle P , and a principal $\mathcal{U}(A_F)$ -bundle Q that lifts P . Conversely, given such F , P , and Q , Proposition 5.3 shows that $P \times_{G_F} F$ is in fact given by a gauge module. These constructions are inverse to each other. \square

Remark 5.5. 1. If there exists a principal $\mathcal{U}(A_F)$ -bundle Q that lifts P , then Q is unique up to isomorphism, because each principal $U(N_i)$ -bundle Q_i is unique up to isomorphism (cf. Example 2.14).

2. Every *globally trivial* principal module, constructed from a finite spectral triple F and the principal bundle $P = M \times_{G_F}$, is in fact a gauge module, with the lift $Q = M \times \mathcal{U}(A_F)$.

3. An example of a principal module that is (in general) *not* a gauge module (except when for instance the underlying manifold is simply connected and 4-dimensional) is described in Sec. VII A.

VI. GAUGE THEORY

In this section, we show how principal modules describe gauge theories on 4-dimensional compact spin manifolds. First we will introduce a “mass matrix.” Viewing the (now massive) principal module as an internal space and endowing it with a (suitable) connection, we can then use it to construct an almost-commutative manifold. Subsequently, we determine the inner fluctuations and provide an explicit formula for the spectral action of this almost-commutative manifold. We end this section by stating our main result, namely that such an almost-commutative manifold indeed describes a gauge theory in the sense of Definition 2.13.

A. Principal almost-commutative manifolds

Definition 6.1. Consider a principal module $P \times_{G_F} F = (\mathcal{B}, \mathcal{E}, J_I)$ (from here on we include a subscript I in order to differentiate between the different operators occurring). In order to be able to describe massive gauge theories, we now introduce a “mass matrix”

$$D_I \in \Gamma^\infty(\text{End}(E)) \simeq \text{End}_{\mathcal{A}}(\mathcal{E}),$$

satisfying

$$D_I = D_I^*, \quad D_I J_I = \varepsilon' J_I D_I, \quad [[D_I, a], J_I b J_I^*] = 0 \quad \forall a, b \in \mathcal{B},$$

where the sign ε' (along with the signs $\varepsilon, \varepsilon''$ obtained through the finite spectral triple F) is determined by the KO-dimension according to the same table as in Definition 2.23. We then call $I_P^\infty := (\mathcal{B}, \mathcal{E}, D_I, J_I)$ a massive principal module over M . We say I_P^∞ is even if there exists a grading operator γ_I on \mathcal{E} such that $D_I \gamma_I = -\gamma_I D_I, \gamma_I J_I = \varepsilon'' J_I \gamma_I$ and $a \gamma_I = \gamma_I a$ for all $a \in \mathcal{B}$.

It is an immediate consequence of the definition that a massive principal module over M is a real internal space over M . If $(\mathcal{B}, \mathcal{E}, J_I)$ is in fact a gauge module, we shall call $(\mathcal{B}, \mathcal{E}, D_I, J_I)$ a massive gauge module.

Let $P \times_{G_F} F$ be a principal module. Denote by \mathfrak{g}_F the Lie algebra of the structure group G_F . Take a connection on P , i.e., for each local trivialisation (U_i, h_i) of P we have a (local) \mathfrak{g}_F -valued 1-form $\omega_i \in \Omega^1(U_i, \mathfrak{g}_F)$ such that

$$\omega_j = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1} \omega_i g_{ij}$$

for all i, j such that $U_i \cap U_j \neq \emptyset$ (see Definition 2.8). These connection one-forms yield a connection $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(M)$ by defining locally (i.e., on local trivialisations (U_i, h_i) of E that are induced by those of P) the expression

$$\nabla|_{U_i} := h_i^{-1} \circ (d + \omega_i) \circ h_i,$$

where d is the exterior derivative acting on the components of the local trivialisation. The transformation property of ω_i ensures that ∇ is globally well-defined. Connections on \mathcal{E} of this form are also referred to as G_F -compatible connections, or simply G_F -connections.

Consider the associated vector bundle $\text{ad } P := P \times_{\text{ad}} \mathfrak{g}_F$, where ad is the adjoint action of G_F on \mathfrak{g}_F . Since \mathfrak{g}_F is (isomorphic to) the image of $\mathfrak{u}(A_F)$ in $\mathfrak{u}(\mathcal{H}_F)$ under the map $t \mapsto t + J_F t J_F^*$, the bundle $\text{ad } P$ is (isomorphic to) the image of $\mathfrak{u}(B)$ in $\mathfrak{u}(E)$ under the map $\tau : t \mapsto t + J_I t J_I^*$. The kernel of this map is equal to the set of all elements $t \in \mathfrak{u}(B)$ satisfying $t = -J_I t J_I^* = J_I t^* J_I^*$, or equivalently,

$$\ker \tau = \{t \in \mathfrak{u}(B) \mid t J_I = J_I t^*\} = \mathfrak{u}(B_J).$$

Hence we see that $\text{ad } P$ is isomorphic to $\mathfrak{u}(B)/\mathfrak{u}(B_J)$. In particular, $\mathfrak{g}_F = \mathfrak{u}(A_F)/\mathfrak{u}((A_F)_{J_F})$.

Lemma 6.2. The induced map $\tau : \mathfrak{u}(B) \rightarrow \Gamma^\infty(\text{ad } P)$ is surjective, and

$$\Gamma^\infty(\text{ad } P) \simeq \mathfrak{u}(B)/\mathfrak{u}(B_J).$$

Moreover, $\text{ad } P$ is isomorphic to the subbundle

$$\mathfrak{u} = \{t \in \mathfrak{u}(B) \mid \text{Tr}_{[i]} t_{[i]} = 0 \text{ for all } [i]\}$$

of $\mathfrak{u}(B)$, where $\text{Tr}_{[i]} t_{[i]}$ denotes the fibrewise trace of $t_{[i]}$ seen as an element of the bundle $\text{End } E_{[i]}$, and $\mathfrak{u}(B) = \ker \tau \oplus \mathfrak{u}$, with $\ker \tau = \mathfrak{u}(B_J)$.

Proof. Though the first two statements follow immediately from the exactness of the Serre-Swan equivalence functor Γ^∞ , we prove them directly by showing that $\text{ad } P$ is isomorphic to the subbundle \mathfrak{u} (compare also Proposition 4.11). Indeed, every $t \in \mathfrak{u}(B)$ can be written as $s + q$, where $s \in \mathfrak{u}$ and $q \in \mathfrak{u}(B_J)$ (just take $q_{[i]} = \frac{1}{N_{[i]}} \text{Tr}_{[i]}(t_{[i]}) \cdot \text{id}_{[i]}$ and $s = t - q$). Hence $\tau|_{\mathfrak{u}}$ is surjective.

Suppose now that $t \in \ker \tau|_{\mathfrak{u}}$. Because $t \in \ker \tau$, we obtain $t_{[i]} = \lambda_{[i]} \text{id}_{N_{[i]}}$, where $\lambda_{[i]} \in i\mathbb{R}$ (see Proposition 4.3). Since $t \in \mathfrak{u}$, each of the $t_{[i]}$ is traceless. Hence each of the $\lambda_{[i]}$ is zero, and consequently, the kernel of $\tau|_{\mathfrak{u}}$ is trivial. \square

Lemma 6.3. Let $P \times_{G_F} F = (\mathcal{B}, \mathcal{E}, J_I, \gamma_I)$ be an even principal module. Any G_F -compatible connection ∇ on \mathcal{E} commutes with the real structure J_I (in the sense that $\nabla_\mu J_I = J_I \nabla_\mu$) and the grading γ_I .

Proof. It is sufficient to show that J_F and γ_F commute with elements in \mathfrak{g}_F . Any element in \mathfrak{g}_F is of the form $t + J_F t J_F^*$, with $t \in \mathfrak{u}(A_F)$. In particular, J_F commutes with these elements. Since

γ_F commutes with elements in A_F , and (anti-)commutes with J_F , the grading γ_F commutes with elements in \mathfrak{g}_F , too. □

If the principal module is obtained from a gauge module $(\mathcal{B}, \mathcal{E}, J_I)$, we can construct such a G_F -connection explicitly as follows. Consider the decomposition $\mathcal{E} = \bigoplus_{(i,j)} \mathcal{E}_i \otimes_A \overline{\mathcal{E}}_j$, and choose a hermitian connection ∇^i on each \mathcal{E}_i . We define

$$\nabla := \bigoplus_{(i,j)} \left(\nabla^i \otimes \mathbb{I} + \mathbb{I} \otimes \overline{\nabla^j} \right),$$

where the conjugate connection $\overline{\nabla^j}$ is defined in Sec. II C. In order to see that ∇ corresponds to a connection on the principal bundle P , we first need to check that its local connection one-forms take values in the Lie algebra \mathfrak{g}_F . If (U, h^i_U) are local trivialisations of \mathcal{E}_i , we can write $\nabla^i|_U = (h^i_U)^{-1} \circ (d + \omega^i_U) \circ h^i_U$ for some local connection one-forms $\omega^i_U \in \Omega^1(U, \mathfrak{u}(N_i))$. The connection ∇ then locally has the connection 1-form

$$\omega_U := \bigoplus_{(i,j)} \left(\omega^i_U \otimes \mathbb{I} + \mathbb{I} \otimes (\omega^j_U)^{\text{op}} \right) \in \Omega^1(U, A_F \otimes A_F^{\text{op}}).$$

This ensures that $[\nabla, \cdot]$ yields a connection on $\mathcal{B} \otimes \mathcal{B}^{\text{op}}$, which preserves \mathcal{B} and \mathcal{B}^{op} . Writing $t_U = \bigoplus_{i \in I} \omega^i_U$, we can write $\omega_U = t_U + J_F t_U J_F^* \in \Omega^1(U, \mathfrak{g}_F)$. To verify that ω_U defines a connection on the principal G_F -bundle P we need to show that ω_U transforms correctly under the G_F -valued transition functions.

So, consider two neighbourhoods U and V such that $U \cap V \neq \emptyset$, and let $u = \times u_i \in C^\infty(U \cap V, \mathcal{U}(A_F))$ be a transition function for the principal $\mathcal{U}(A_F)$ -bundle Q . The corresponding transition function for the principal G_F -bundle P is $g := u J_F u J_F^*$. Since the ω^i_U are connection forms on E_i , t_U transforms as

$$t_V = \bigoplus_{i \in I} \omega^i_V = \bigoplus_{i \in I} (u_i^* \omega^i_U u_i + u_i^* du_i) = u^* t_U u + u^* du.$$

We then see that

$$\begin{aligned} \omega_V &= t_V + J_F t_V J_F^* = u^* t_U u + u^* du + J_F (u^* t_U u + u^* du) J_F^* \\ &= u^* J_F u^* J_F^* t_U u J_F u J_F^* + J_F (u^* J_F u^* J_F^* t_U u J_F u J_F^*) J_F^* \\ &\quad + u^* J_F u^* J_F^* (du) J_F u J_F^* + u^* J_F u^* J_F^* J_F (du) J_F^* \\ &= g^{-1} (t_U + J_F t_U J_F^*) g + g^{-1} dg = g^{-1} \omega_U g + g^{-1} dg. \end{aligned}$$

Thus, $U \mapsto \omega_U$ indeed defines a G_F -connection.

Proposition 6.4. Let $(\mathcal{B}, \mathcal{E}, J)$ be a gauge module. A connection on E is of the form $\bigoplus_{(i,j)} \left(\nabla^i \otimes \mathbb{I} + \mathbb{I} \otimes \overline{\nabla^j} \right)$ if and only if it induces a connection on the principal $\mathcal{U}(A_F)$ -bundle Q from Proposition 5.2.

Proof. Consider a local trivialisation (U, h_U) of P , and let $\omega_U \in \Omega^1(U, \mathfrak{u}(A_F))$ be a local connection form on Q , yielding a connection ∇ on $E = Q \times_{\mathcal{U}(A_F)} \mathcal{H}_F$. Since the decomposition $\mathfrak{u}(A_F) = \bigoplus_{i \in I} \mathfrak{u}(N_i)$ is preserved by the action of $\mathcal{U}(A_F)$, we can write $\omega_U = \bigoplus_{i \in I} \omega_i$, where each $\omega_i \in \Omega^1(U, \mathfrak{u}(N_i))$ yields a connection ∇^i on E_i . For $x \in U$, the connection form ω_U acts on $(E_i \otimes \overline{E}_j)|_x \simeq \mathbb{C}^{N_i} \otimes \mathbb{C}^{N_j}$ as

$$\omega(v_i \otimes \overline{w}_j) = \omega_i v_i \otimes \overline{w}_j + v_i \otimes \overline{w}_j \omega_j^*,$$

from which it follows that $\nabla = \bigoplus_{(i,j)} \left(\nabla^i \otimes \mathbb{I} + \mathbb{I} \otimes \overline{\nabla^j} \right)$.

For the converse, consider a connection on E of the form $\nabla = \bigoplus_{(i,j)} (\nabla^i \otimes \mathbb{I} + \mathbb{I} \otimes \overline{\nabla^j})$. On a local trivialisation $(U, h_U)_i$ of E_i , each connection ∇^i yields a local connection form $\omega_i \in \Omega^1(U, \mathfrak{u}(N_i))$. Then $\omega_U := \bigoplus_{i \in I} \omega_i \in \Omega^1(U, \mathfrak{u}(A_F))$ is a connection form on Q that induces ∇ . \square

Definition 6.5. Let $I_p^\infty = (\mathcal{B}, \mathcal{E}, D_I, J_I)$ be a massive principal module of KO-dimension k over M , where M now has dimension 4. Let ∇ be a G_F -compatible connection on \mathcal{E} . We construct the real almost-commutative manifold $I_p^\infty \times_\nabla M$ as in Definition III.4. Since I_p^∞ is now a massive principal module (instead of a more general internal space), we will refer to $I_p^\infty \times_\nabla M$ as a *principal almost-commutative manifold*.

If I_p^∞ is even with grading γ_I , we obtain a real even almost-commutative manifold $I_p^\infty \times_\nabla M$. Since the connection ∇ is G_F -compatible, it automatically commutes with J_I and γ_I (see Lemma 6.3). Moreover, the same condition implies that the induced connection $[\nabla, \cdot]$ on $\text{End } E$ restricts to B . It then follows from Proposition 3.5 that $I_p^\infty \times_\nabla M$ is a real even spectral triple of KO-dimension $4 + k \pmod{8}$.

We continue in the remainder of this section, as in the usual approach for globally trivial almost-commutative manifolds (see Refs. 2 and 22 or the review Ref. 4), by generating the gauge fields and Higgs fields via inner fluctuations, and subsequently calculating the spectral action.

B. Inner fluctuations

Let $(\mathcal{B}, \mathcal{H}, D)$ be a spectral triple. Consider the *generalised one-forms* given by

$$\Omega_D^1(\mathcal{B}) := \left\{ \sum_j a_j [D, b_j] \mid a_j, b_j \in \mathcal{B} \right\}.$$

For the canonical triple $(\mathcal{A}, L^2(S), \not{D})$ of a spin manifold M , the generalised one-forms $\Omega_D^1(\mathcal{A})$ are simply given by the Clifford multiplication c of the usual one-forms $\Omega^1(M)$. To be precise, for smooth functions $f_1, f_2 \in \mathcal{A}$, we obtain $f_1 [\not{D}, f_2] = -if_1 c(df_2)$.

Definition 6.6. Let $(\mathcal{B}, \mathcal{H}, D, J)$ be a real spectral triple. An *inner fluctuation* of the operator D is a self-adjoint element $A = A^* \in \Omega_D^1(\mathcal{B})$. Such an inner fluctuation yields the *fluctuated operator*

$$D_A := D + A + \varepsilon' J A J^*,$$

where the sign $\varepsilon' = \pm 1$ is determined by the KO-dimension of the spectral triple (see Definition II.23).

For the remainder of this paper, we again assume that the dimension of M is equal to 4. We would like to show that, for a principal almost-commutative manifold, these inner fluctuations yield gauge fields and scalar fields (the latter are interpreted as Higgs fields in the noncommutative Standard Model). The inner fluctuations of the twisted Dirac operator $\not{D}_E := \mathbb{I} \otimes_\nabla \not{D}$ are (finite sums of) elements of the form

$$a[\not{D}_E, b] = -i(\mathbb{I} \otimes c) \circ (a[\nabla, b] \otimes \mathbb{I}),$$

for $a, b \in \mathcal{B}$, where c denotes Clifford multiplication. The fact that ∇ is a G_F -compatible connection ensures that $a[\nabla, b] \in \mathcal{B} \otimes_{\mathcal{A}} \Omega^1(M) \simeq \Omega^1(M, B)$. Requiring that $a[\not{D}_E, b]$ is self-adjoint then implies that $a[\nabla, b] \in \Omega^1(M, \mathfrak{u}(B))$, where $\mathfrak{u}(B)$ contains the anti-hermitian elements of B . An arbitrary inner fluctuation of \not{D}_E is thus given by

$$\alpha := \sum_j a_j [\nabla, b_j] \in \Omega^1(M, \mathfrak{u}(B)).$$

We can then write $J a[\not{D}_E, b] J^* = -i(\mathbb{I} \otimes c) \circ (J_I \alpha J_I^* \otimes \mathbb{I})$, and consequently we have

$$a[\not{D}_E, b] + J a[\not{D}_E, b] J^* = -i(\mathbb{I} \otimes c) \circ ((\alpha + J_I \alpha J_I^*) \otimes \mathbb{I}).$$

The inner fluctuations of the operator $D_I \otimes \gamma_5$ are of the form $\phi \otimes \gamma_5$, where

$$\phi = \phi^* := \sum_j a_j [D_I, b_j] \in \Gamma^\infty(\text{End}(\mathbb{E})).$$

Proposition 6.7. The fluctuated Dirac operator $D_A := D + A + JAJ^*$ for a real even almost-commutative manifold is of the form

$$D_A = 1 \otimes_{\nabla'} \not{D} + \Phi \otimes \gamma_5,$$

where $\nabla' := \nabla + \beta$ for some $\beta \in \Omega^1(M, \text{ad } \mathbb{P})$, and $\Phi = \Phi^* := D_I + \phi + J_I \phi J_I^* \in \Gamma^\infty(\text{End}(\mathbb{E}))$ for some $\phi = \phi^* := \sum_j a_j [D_I, b_j]$.

Proof. The expression $\beta = \alpha + J_I \alpha J_I^*$ is an $\text{ad } \mathbb{P}$ -valued 1-form on M (see Lemma 6.2). Noting that $\varepsilon' = 1$ by assumption, the statement follows straightforwardly. \square

The construction of $I_{\mathbb{P}}^\infty \times_{\nabla} M$ explicitly uses the choice of a connection ∇ . However, we now show that this choice is irrelevant once we take the inner fluctuations into account. We need the following lemma.

Lemma 6.8. Let $\mathcal{B} \rightarrow M$ be a unital $*$ -algebra bundle, and let $\tilde{\nabla}$ be a connection on $\mathcal{B} = \Gamma^\infty(\mathcal{B})$ such that $\tilde{\nabla}(1) = 0$, where 1 denotes the identity section. Write $\mathcal{A} = C^\infty(M)$. Then

$$\left\{ \sum_j a_j \tilde{\nabla}(b_j) \mid a_j, b_j \in \mathcal{B} \right\} = \mathcal{B} \otimes_{\mathcal{A}} \Omega^1(M) \simeq \Omega^1(M, \mathcal{B}). \tag{8}$$

Consequently, $\Omega^1(M, \mathfrak{u}(\mathcal{B}))$ is given by the anti-hermitian elements in $\left\{ \sum_j a_j \tilde{\nabla}(b_j) \mid a_j, b_j \in \mathcal{B} \right\}$.

Proof. Since $\tilde{\nabla}(b) \in \mathcal{B} \otimes_{\mathcal{A}} \Omega^1(M)$, the left-hand side of Eq. (8) is clearly contained in the right hand side of Eq. (8). For the converse inclusion, first suppose that both a_j and b_j are in $\mathcal{A} \subset \mathcal{Z}(\mathcal{B})$. In that case,

$$\left\{ \sum_j f_j \tilde{\nabla}(g_j \text{Id}_{\mathcal{B}}) \mid f_j, g_j \in \mathcal{A} \right\} \simeq \left\{ \sum_j f_j dg_j \mid f_j, g_j \in \mathcal{A} \right\} = \Omega^1(M).$$

It follows from this that

$$\left\{ \sum_j a_j \tilde{\nabla}(g_j 1) \mid a_j \in \mathcal{B}, g_j \in \mathcal{A} \right\} = \mathcal{B} \otimes_{\mathcal{A}} \Omega^1(M).$$

Of course, the left-hand side of the previous equation is contained in $\left\{ \sum_j a_j \tilde{\nabla}(b_j) \mid a_j, b_j \in \mathcal{B} \right\}$, which proves the other inclusion. \square

Proposition 6.9. Let $\mathbb{P} \times_{G_F} F = (\mathcal{B}, \mathcal{E}, J_I)$ be a principal module over M (for simplicity we consider here the massless case $D_I = 0$) with two (G_F -compatible) connections ∇ and ∇' . Then $\mathbb{I} \otimes_{\nabla'} \not{D}$ is obtained as an inner fluctuation of $\mathbb{I} \otimes_{\nabla} \not{D}$.

Proof. The difference between the two connections $\beta := \nabla' - \nabla$ is an element in $\Omega^1(M, \text{ad } \mathbb{P})$. By Lemma 6.2 there exists a (unique) element $\alpha \in \Omega^1(M, \mathfrak{u}) \subset \Omega^1(M, \mathfrak{u}(\mathcal{B}))$ such that $\beta = \alpha + J_I \alpha J_I^*$. The connection $\tilde{\nabla} = [\nabla, \cdot]$ on $\text{End}(\mathcal{E})$ restricts to a connection on \mathcal{B} , and satisfies $\tilde{\nabla}(1) = 0$. Lemma 6.8 now implies that β is obtained as an inner fluctuation. \square

Remark 6.10. We have seen that considering inner fluctuations of the Dirac operator essentially replaces the G_F -connection ∇ (chosen in the construction of the almost-commutative manifold $I_{\mathbb{P}}^\infty \times_{\nabla} M$) by a different (arbitrary) G_F -connection ∇' . Therefore, after taking into account the inner fluctuations, our construction of principal almost-commutative manifolds is essentially independent of the initial choice of the connection ∇ .

However, we also note that the endomorphisms Φ obtained through inner fluctuations in general remain dependent on the initial choice of D_I .

C. The spectral action

As mentioned immediately below Definition II.13, the dynamics of a gauge theory can be obtained from a gauge-invariant action functional. In the case of almost-commutative manifolds, such an action functional can be formulated in terms of the spectral triple.

Let us first recall the definitions of the bosonic and fermionic action functionals for an arbitrary spectral triple $T = (\mathcal{A}, \mathcal{H}, D)$. The bosonic part of the action functional is given by the *spectral action*,⁶ defined as

$$S_b(T) := \text{Tr} \left(f \left(\frac{D_A}{\Lambda} \right) \right).$$

Here Tr denotes the operator trace on $B(\mathcal{H})$, D_A is the fluctuated Dirac operator, $f : \mathbb{R} \rightarrow \mathbb{R}$ is some positive even function, and $\Lambda \in \mathbb{R}$ is a (large) cut-off parameter. The function f is assumed to decay sufficiently rapidly at infinity so that the trace of $f(D_A/\Lambda)$ exists. In particular, f could be considered as a smooth approximation to a cut-off function (and as such it counts the number of eigenvalues of D_A whose absolute values are smaller than Λ), but this viewpoint is not necessary for the following.

If the spectral triple is even (with grading γ) and has a real structure J of KO-dimension 2, the *fermionic action*³⁰ is defined as

$$S_f(T) := \frac{1}{2} \langle J\tilde{\xi}, D_A\tilde{\xi} \rangle,$$

where $\tilde{\xi}$ is the Grassmann variable corresponding to a vector $\xi \in \mathcal{H}^+$ (i.e. $\gamma\xi = \xi$).

We quote the following well-known result:

Proposition 6.11 (see, e.g., Sec. 2.6.1 of Ref. 4). *For a real spectral triple $T = (\mathcal{A}, \mathcal{H}, D, J, \gamma)$ of KO-dimension 2, the action functionals $S_b(T)$ and $S_f(T)$ are invariant under the action of the gauge group $\mathcal{G}(T)$.*

We now provide explicit formulas for the spectral action of principal almost-commutative manifolds (formulas for the fermionic action will only be given for the example of electrodynamics in Sec. VIIB). The spectral action was calculated in Refs. 2 and 30 for the product triple $M \times F$, where F was chosen in order to describe the full Standard Model of elementary particle physics. In the remainder of this section, we largely follow the notation of Ref. 4, where also detailed derivations of the formulas provided here can be found.

For the canonical triple $(C^\infty(M), L^2(S), \mathcal{D})$ of a smooth compact 4-dimensional Riemannian spin manifold M , the spectral action yields the asymptotic formula

$$S_b(M) \sim_{\Lambda \rightarrow \infty} \int_M \mathcal{L}_M(g_{\mu\nu}) \sqrt{|g|} d^4x + \mathcal{O}(\Lambda^{-1}),$$

where g is the Riemannian metric on M . The Lagrangian \mathcal{L}_M is given by

$$\mathcal{L}_M(g_{\mu\nu}) := \frac{f_4 \Lambda^4}{2\pi^2} - \frac{f_2 \Lambda^2}{24\pi^2} s + \frac{f(0)}{16\pi^2} \left(\frac{1}{30} \Delta s - \frac{1}{20} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{11}{360} R^* R^* \right). \quad (9)$$

Here s denotes the scalar curvature of M , Δ is the scalar Laplacian, C is the Weyl curvature, and $R^* R^*$ is a topological term, which integrates to (a multiple of) the Euler class. The coefficients f_k (for $k > 0$) are the moments of f , defined as

$$f_k := \int_0^\infty f(t) t^{k-1} dt.$$

We now provide the spectral action for a principal almost-commutative manifold. As all calculations are local, the result is exactly the same as for the spectral action of a product triple $M \times F$, and we refer to Ref. 4 for the detailed calculations.

In Proposition 6.7, we saw that the fluctuated Dirac operator is determined by a connection $\nabla' = \nabla + \beta$ and an endomorphism Φ on E . From here on we shall work on a local trivialisation (U, h_U) , where we can write $\nabla|_U = h_U^{-1} \circ (d + \omega_U) \circ h_U$, and define the local \mathfrak{g}_F -valued 1-form $B := \omega_U + h_U \circ \beta|_U \circ h_U^{-1} \in \Omega^1(U, \mathfrak{g}_F)$ (for ease of notation we do not make the dependence of B

on the local chart U explicit). Thus B is the local connection form for ∇' . Using a local coordinate basis ∂_μ , we define $B_\mu := B(\partial_\mu) \in C^\infty(U, \mathfrak{g}_F)$. We omit the local trivialisation h_ν from our notation, so we write, e.g., $\nabla'_\mu = \partial_\mu + B_\mu$. Furthermore, we introduce the notation

$$D_\mu \Phi := [\nabla'_\mu, \Phi] = \partial_\mu \Phi + [B_\mu, \Phi], \quad F_{\mu\nu} := \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu].$$

Proposition 6.12. *The spectral action for a principal almost-commutative manifold $I_P^\infty \times_\nabla M$ is asymptotically given by the local formula*

$$S_b(I_P^\infty \times_\nabla M) \sim_{\Lambda \rightarrow \infty} \int_M \mathcal{L}(g_{\mu\nu}, B_\mu, \Phi) \sqrt{|g|} d^4x + \mathcal{O}(\Lambda^{-1}),$$

for

$$\mathcal{L}(g_{\mu\nu}, B_\mu, \Phi) := N \mathcal{L}_M(g_{\mu\nu}) + \mathcal{L}_B(g_{\mu\nu}, B_\mu) + \mathcal{L}_H(g_{\mu\nu}, B_\mu, \Phi).$$

Here $\mathcal{L}_M(g_{\mu\nu})$ is given in Eq. (9), and N is the rank of E . \mathcal{L}_B gives the kinetic term of the gauge field and equals

$$\mathcal{L}_B(g_{\mu\nu}, B_\mu) := \frac{f(0)}{24\pi^2} \text{tr}(F_{\mu\nu} F^{\mu\nu}),$$

where tr denotes the fibre-wise trace for endomorphisms on the bundle $E \otimes S$. \mathcal{L}_H gives the Higgs Lagrangian given by

$$\begin{aligned} \mathcal{L}_H(g_{\mu\nu}, B_\mu, \Phi) := & -\frac{2f_2\Lambda^2}{4\pi^2} \text{tr}(\Phi^2) + \frac{f(0)}{8\pi^2} \text{tr}(\Phi^4) + \frac{f(0)}{24\pi^2} \Delta(\text{tr}(\Phi^2)) \\ & + \frac{f(0)}{48\pi^2} s \text{tr}(\Phi^2) + \frac{f(0)}{8\pi^2} \text{tr}((D_\mu \Phi)(D^\mu \Phi)), \end{aligned}$$

where the first two terms form the Higgs potential, the third is a boundary term, the fourth couples the Higgs field to the scalar curvature, and finally we have the kinetic term including interactions with the gauge field.

Remark 6.13. Although the above explicit formulas for the spectral action are exactly the same as for a product triple $M \times F$, there can nonetheless be a significant difference, because the constant matrix D_F is replaced by a global endomorphism D_I . For a product triple $M \times F$, the inner fluctuations of $\gamma_5 \otimes D_F$ also lead to global endomorphisms of the form $\gamma_5 \otimes \Phi$, where $\Phi \in \Gamma^\infty(\text{End}(E))$ (though this Φ would be more restricted than in our construction). However, there may be components of D_F that are not affected by inner fluctuations, and hence remain constant (this occurs for instance for the Majorana masses of right-handed neutrinos in the case of the noncommutative Standard Model²). In the case of a principal almost-commutative manifold, these components could be non-constant from the start. Hence, compared to the case of product triples, derivatives of the field Φ might contain additional terms. This difference is not yet visible in the general formulas above, but it may have consequences once we look at concrete examples (see Remark 7.4).

D. Gauge theory

The results of this section can be summarised as follows, which is the main result of our paper:

Theorem 6.14. *Let M be a smooth compact 4-dimensional Riemannian spin manifold. Consider a massive even principal module $I_P^\infty = (\mathcal{B}, \mathcal{E}, D_I, \gamma_I, J_I)$ of KO-dimension k over M . Let ∇ be a G_F -compatible connection on \mathcal{E} . If M is simply connected, then the principal almost-commutative manifold $I_P^\infty \times_\nabla M$ of KO-dimension $4 + k \pmod{8}$ describes a classical gauge theory over M with gauge group $\mathcal{G}(I_P^\infty \times_\nabla M)$.*

Proof. The principal module I_P^∞ is constructed from a principal G_F -bundle P over M , such that \mathcal{B} and \mathcal{E} are given by smooth sections of bundles associated to P . By assumption M is simply connected,

so it follows from Theorem 4.12 that we have the isomorphism $\mathcal{G}(I_{\mathbb{P}}^{\infty} \times_{\nabla} M) \simeq \mathcal{G}(P)$. We have seen in Sec. VI B that the inner fluctuations transform a G_F -compatible connection on \mathcal{E} to another G_F -compatible connection, which hence corresponds to a connection on P (and by Proposition 6.9 any connection on P can be obtained in this way). Finally, the spectral action and the fermionic action provide a gauge-invariant action functional (see Proposition 6.11). Thus the principal almost-commutative manifold $I_{\mathbb{P}}^{\infty} \times_{\nabla} M$ provides all the necessary ingredients for a classical gauge theory over M , as described in Definition 2.13. \square

VII. EXAMPLES

In this section, we adapt two simple examples of (globally trivial) gauge theories in the context of noncommutative geometry to the globally non-trivial case. In each example, we assume (as before) that the underlying manifold M is a smooth compact 4-dimensional Riemannian spin manifold.

In Sec. VII A, we describe the Yang-Mills case that was studied in Ref. 3, and provided the motivation for this work. In particular, we show that the Yang-Mills case provides examples of principal modules that cannot be described by gauge modules. In Sec. VII B, we discuss the abelian gauge theory of electrodynamics, based on the (globally trivial) description in Ref. 31. We will describe the resulting (globally non-trivial) gauge theory, and provide explicit formulas for both the spectral action and the fermionic action.

A. Yang-Mills

Globally trivial Yang-Mills theory was already studied in the setting of spectral triples by Chamseddine and Connes.⁶ It is described by the (real, even) finite spectral triple

$$F_{\text{YM}} := (M_N(\mathbb{C}), M_N(\mathbb{C}), D_F = 0, J_F = (\cdot)^*, \gamma_F = \text{id}),$$

where the algebra $M_N(\mathbb{C})$ acts on the Hilbert space $M_N(\mathbb{C})$ by left-multiplication. The KO-dimension of this spectral triple is 0 and the structure group G_F is equal to $PSU(N)$.

This has been generalised to the globally non-trivial case in Ref. 3. Let $B \rightarrow M$ be an arbitrary $*$ -algebra bundle with fibre $M_N(\mathbb{C})$, and let $\mathcal{B} = \Gamma^{\infty}(B)$ be its unital, involutive $C^{\infty}(M)$ -module algebra of sections. We consider the real even internal space

$$I_{\text{YM}}^{\infty} := (\mathcal{B}, \mathcal{B}, D_I = 0, J_I = (\cdot)^*, \gamma_I = \text{id}).$$

For a general principal module $P \times_{G_F} F$, we do not know how to reconstruct the principal bundle P from the module. However, in the Yang-Mills case we do.

Lemma 7.1. *There exists a principal $PSU(N)$ -bundle $P \rightarrow M$ (unique up to isomorphism) such that $I_{\text{YM}}^{\infty} \simeq P \times_{PSU(N)} F_{\text{YM}}$.*

Proof. The transition functions of the $*$ -algebra bundle B take values in $\text{Aut}(M_N(\mathbb{C})) \simeq PSU(N)$ (where $PSU(N)$ acts on $M_N(\mathbb{C})$ by conjugation). Hence by Theorem 2.7 we can reconstruct a principal $PSU(N)$ -bundle P such that $B \simeq P \times_{PSU(N)} M_N(\mathbb{C})$. Since $PSU(N)$ is the full automorphism group of the fibre, the bundle P is uniquely defined. \square

Remark 7.2. Note that I_{YM}^{∞} will in general *not* be a gauge module. If this were the case, the structure group $PSU(N)$ of B could be lifted to $U(N)$ by Proposition 5.2. This is only possible if the Dixmier-Douady class $\delta(B) \in \check{H}^3(M, \mathbb{Z})$ is identically zero (see, e.g., Chap. 5 of Ref. 32 or Ref. 33 for more details on Dixmier-Douady classes), which is equivalent to saying that B is an endomorphism bundle (note that this is consistent with the condition $B_i = \text{End}(E_i)$ in Definition V.1). Since not every $*$ -algebra bundle with fibre $M_N(\mathbb{C})$ has zero Dixmier-Douady class (see, e.g., Ref. 33), this example shows that there exist principal modules that are not gauge modules. However, in our description of gauge theories in Sec. VI, we have restricted our attention to simply connected, 4-dimensional manifolds, and it turns out that in this case the Dixmier-Douady class always vanishes

(as we will prove below). It is unclear if there exist other examples of principal modules that are not gauge modules.

Proposition 7.3. *Let \mathcal{B} be a $*$ -algebra bundle with fibre $M_N(\mathbb{C})$ over a simply connected, 4-dimensional, oriented, compact manifold M . Then the Dixmier-Douady class of \mathcal{B} is identically zero.*

Proof. Since M is simply connected, its fundamental group is trivial, and hence (see, e.g., Theorem 2.A.1 of Ref. 11) the first singular homology group $H_1(M, \mathbb{Z})$ is trivial. By Poincaré duality (see, e.g., Proposition 3.25 and Theorem 3.30 of Ref. 11) it then follows that the third cohomology group $H^3(M, \mathbb{Z})$ is also trivial. The Dixmier-Douady class by definition takes values in the third Čech cohomology group $\check{H}^3(M, \mathbb{Z})$. Since for compact manifolds these cohomology groups are equal, it follows that $\check{H}^3(M, \mathbb{Z})$ is trivial and hence that the Dixmier-Douady class of \mathcal{B} must vanish. \square

A connection $\nabla : \mathcal{B} \rightarrow \mathcal{B} \otimes_{\mathcal{A}} \Omega^1(M)$ is $PSU(N)$ -compatible (cf. Sec. VIA) if and only if it satisfies the algebraic identities (see Sec. 3.2 of Ref. 3)

$$\nabla(ab) = \nabla(a)b + a\nabla(b), \quad (\nabla a)^* = \nabla(a^*), \quad \forall a, b \in \mathcal{B}.$$

Such a connection thus corresponds to a connection form ω on P . If we pick any such connection, we can then consider the (principal) almost-commutative manifold

$$I_{\text{YM}}^\infty \times_\nabla M := (\Gamma^\infty(\mathcal{B}), L^2(\mathcal{B} \otimes \mathcal{S}), \not{D}_{\mathcal{B}}, J_I \otimes J_M, \gamma_I \otimes \gamma_5).$$

If M is simply connected, the group $\mathcal{G}(I_{\text{YM}}^\infty \times_\nabla M)$ is isomorphic to $\mathcal{G}(P)$, and $I_{\text{YM}}^\infty \times_\nabla M$ describes a $PSU(N)$ gauge theory (P, ω) over M . We denote the local connection form of ∇ by B_μ , and its curvature tensor by $F_{\mu\nu}$. From Proposition 6.12 we find that the spectral action yields the Lagrangian

$$\mathcal{L}(g_{\mu\nu}, B_\mu) = N^2 \mathcal{L}_M(g_{\mu\nu}) + \mathcal{L}_{\text{YM}}(g_{\mu\nu}, B_\mu),$$

where the Yang-Mills Lagrangian is given (up to a normalisation constant) by the usual expression:

$$\mathcal{L}_{\text{YM}}(g_{\mu\nu}, B_\mu) := \frac{f(0)}{24\pi^2} \text{tr}(F_{\mu\nu} F^{\mu\nu}).$$

B. Electrodynamics

The example of (globally trivial) Electrodynamics in the context of noncommutative geometry appeared in Ref. 31. Here we describe its generalisation to the globally non-trivial case. The finite spectral triple for electrodynamics is given by³¹

$$F_{\text{ED}} := (\mathbb{C}^2, \mathbb{C}^4, D_F, \gamma_F, J_F).$$

We shall generalise this finite triple to a massive even gauge module I_{ED}^∞ over M . First, we set the algebra \mathcal{B} to be of the form

$$\mathcal{B} := \mathcal{A} \oplus \mathcal{A} = C^\infty(M) \oplus C^\infty(M).$$

Let L be a complex line bundle over M , with a given hermitian structure, so that its structure group is $U(1)$. We shall take two identical copies of this line bundle, which we denote by E_L and E_R , with smooth sections $\mathcal{E}_L = \Gamma^\infty(E_L)$ and $\mathcal{E}_R = \Gamma^\infty(E_R)$. Then the Hilbert $\mathcal{B} - \mathcal{A}$ -bimodule \mathcal{E} is defined as

$$\mathcal{E} := (\mathcal{E}_L \oplus \mathcal{E}_R) \oplus (\overline{\mathcal{E}_L} \oplus \overline{\mathcal{E}_R}),$$

where the first component of \mathcal{B} acts on $\mathcal{E}_L \oplus \mathcal{E}_R$, and the second component acts on its conjugate. On this decomposition, the grading is defined as $\gamma_I := 1 \oplus (-1) \oplus (-1) \oplus 1$. The real structure J_I is the anti-linear map $\mathcal{E}_{L,R} \mapsto \overline{\mathcal{E}_{L,R}}$ and $\overline{\mathcal{E}_{L,R}} \mapsto \mathcal{E}_{L,R}$ of KO-dimension 6 (see Definition II.23). We then have the subalgebra $\mathcal{B}_J \simeq \mathcal{A} \subset \mathcal{B}$, where the injection is given by $a \mapsto a \oplus a$. Imposing all conditions

in Definition VI.1, the “mass matrix” D_I is restricted to be of the form

$$D_I := \begin{pmatrix} 0 & d & 0 & 0 \\ \bar{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{d} \\ 0 & 0 & d & 0 \end{pmatrix},$$

where $d \in C^\infty(M)$ (see Sec. 4.1.1 of Ref. 31).

Remark 7.4. In order to interpret d as a mass parameter, it would have to be given by a single real-valued parameter. For this reason we restrict ourselves to the case $d = -im$ (see Remark 4.4 of Ref. 31). We stress here that in general (as mentioned in Remark 3.2) the mass m is not a fixed parameter, but a function on M (although it can be chosen to be constant). In other words, our framework allows the mass of a particle to vary from point to point in M , so essentially the Yukawa mass parameter is replaced by a Yukawa field. This could of course have significant physical implications, which we intend to study in future work.

The module $I_{\text{ED}}^\infty = (\mathcal{B}, \mathcal{E}, D_I, \gamma_I, J_I)$ defined in this way is in fact a massive even gauge module. To be precise, if we write $\mathcal{E}_1 := \Gamma^\infty(\mathbb{L}) = \mathcal{E}_L = \mathcal{E}_R$ and $\mathcal{E}_2 := \mathcal{A}$, then we have $\mathcal{B}_1 = \text{End}_{\mathcal{A}}(\mathcal{E}_1) = \Gamma^\infty(\mathbb{L} \otimes \mathbb{L}^*) \simeq \mathcal{A}$ and also $\mathcal{B}_2 \simeq \mathcal{A}$. Furthermore, the module \mathcal{E} can be written as

$$\mathcal{E} \simeq \bigoplus_{(i,j) \in K} \mathcal{E}_i \otimes_{\mathcal{A}} \overline{\mathcal{E}}_j, \quad K := \{(1, 2), (1, 2), (2, 1), (2, 1)\}.$$

The hermitian structure on \mathbb{L} determines a class of transition functions of \mathbb{L} taking values in $U(1)$, so using Theorem 2.7 we can uniquely reconstruct a principal $U(1)$ -bundle \mathbb{P} , and we have $I_{\text{ED}}^\infty \simeq \mathbb{P} \times_{U(1)} F_{\text{ED}}$ as massless modules (i.e., ignoring the mass matrices D_F and D_I).

Proposition 7.5. The gauge group is given by

$$\mathcal{G}(I_{\text{ED}}^\infty) \simeq \mathcal{U}(\mathcal{B})/\mathcal{U}(\mathcal{B}_j) \simeq \Gamma^\infty(\text{Ad } \mathbb{P}) \simeq C^\infty(M, U(1)).$$

Proof. Note that the group bundle $\text{Ad } \mathbb{P} \simeq M \times U(1)$ is globally trivial, because the structure group $U(1)$ is abelian.

As in Sec. IV B 1, the main thing to prove is the surjectivity of the map $\phi_* : \mathcal{U}(\mathcal{B}) \rightarrow \Gamma^\infty(\text{Ad } \mathbb{P})$, which is given by $\phi_*(u) = uJuJ^*$. But for $u = (u_1, u_2) \in \mathcal{U}(\mathcal{B})$, this map is given by

$$(u_1, u_2) \mapsto \begin{pmatrix} u_1 u_2^* & 0 \\ 0 & u_2 u_1^* \end{pmatrix},$$

so $\phi_*(u_1, u_2)$ can be identified with $u_1 u_2^*$. Hence each $v \in \Gamma^\infty(\text{Ad } \mathbb{P}) \simeq C^\infty(M, U(1))$ is the image of $(v, 1) \in \mathcal{U}(\mathcal{B})$. \square

Remark 7.6. Note that in this particular example it is not necessary to require that M is simply connected, as we did in the general case (see Theorem 4.12).

An element $\lambda \in \mathcal{G}(I_{\text{ED}}^\infty)$ acts on $\mathcal{E}_L \oplus \mathcal{E}_R$ as multiplication by λ , and acts on $\overline{\mathcal{E}}_L \oplus \overline{\mathcal{E}}_R$ as multiplication by $\bar{\lambda}$.

Pick a connection $\nabla^{\mathbb{L}}$ on \mathbb{L} , and let the connection ∇ on \mathcal{E} be given by

$$\nabla := \nabla^{\mathbb{L}} \oplus \nabla^{\mathbb{L}} \oplus \overline{\nabla^{\mathbb{L}}} \oplus \overline{\nabla^{\mathbb{L}}}.$$

On a local trivialisation (say on a neighbourhood U), the connection $\nabla^{\mathbb{L}}$ is determined by a local connection form $\omega_U^{\mathbb{L}} \in \Omega^1(U, i\mathbb{R})$, where $i\mathbb{R}$ is the Lie algebra of $U(1)$. For the connection ∇ on \mathcal{E} this yields the connection form

$$\omega_U = \omega_U^{\mathbb{L}} \oplus \omega_U^{\mathbb{L}} \oplus \overline{\omega_U^{\mathbb{L}}} \oplus \overline{\omega_U^{\mathbb{L}}} = \omega_U^{\mathbb{L}} (1 \oplus 1 \oplus (-1) \oplus (-1)),$$

where the last equality follows because the action of $\overline{\omega_U^{\mathbb{L}}}$ is given by (right) multiplication with $\omega_U^{\mathbb{L}*} = -\omega_U^{\mathbb{L}}$.

Now consider the almost-commutative manifold $I_{\text{ED}}^\infty \times_{\nabla} M$ of KO-dimension 2, which (by Theorem 6.14) describes a $U(1)$ -gauge theory over M . Taking inner fluctuations simply amounts to choosing a different connection ∇^L (see Proposition 6.9), while there is no Higgs field (because D_I commutes with \mathcal{B}). Hence we ignore these inner fluctuations, and simply consider the local gauge field $A_\mu := \omega_v^L(\partial_\mu)$, on some coordinate basis ∂_μ . Its curvature is defined as $\mathcal{F}_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$. From Proposition 6.12 (see also Proposition 4.2 of Ref. 31), we find that the spectral action for $I_{\text{ED}}^\infty \times_{\nabla} M$ is asymptotically given by the local formula

$$S_b(I_{\text{ED}}^\infty \times_{\nabla} M) \sim_{\Lambda \rightarrow \infty} \int_M \mathcal{L}(g_{\mu\nu}, A_\mu, m) \sqrt{|g|} d^4x + \mathcal{O}(\Lambda^{-1}),$$

for

$$\mathcal{L}(g_{\mu\nu}, A_\mu, m) := 4\mathcal{L}_M(g_{\mu\nu}) + \mathcal{L}_A(g_{\mu\nu}, A_\mu) + \mathcal{L}_H(g_{\mu\nu}, m).$$

Here $\mathcal{L}_M(g_{\mu\nu})$ is the Lagrangian Eq. (9), and $\mathcal{L}_H(g_{\mu\nu}, m)$ yields additional terms depending on the mass m and the scalar curvature s :

$$\mathcal{L}_H(g_{\mu\nu}, m) := -\frac{2f_2\Lambda^2 m^2}{\pi^2} + \frac{f(0)m^4}{2\pi^2} + \frac{f(0)m^2 s}{12\pi^2}.$$

The Lagrangian for the gauge field is given by

$$\mathcal{L}_A(g_{\mu\nu}, A_\mu) := \frac{f(0)}{6\pi^2} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}.$$

The interaction of the $U(1)$ gauge field with the fermions is described by the fermionic action, and is given by (see Proposition 4.3 and Theorem 4.5 of Ref. 31)

$$S_f(I_{\text{ED}}^\infty \times_{\nabla} M) = \int_M \mathcal{L}_f(g_{\mu\nu}, A_\mu, m) \sqrt{|g|} d^4x,$$

for the Lagrangian

$$\mathcal{L}_f(g_{\mu\nu}, A_\mu, m) := -i (J_M \tilde{\chi}, (\gamma^\mu (\nabla_\mu^S - A_\mu) - m) \tilde{\psi}),$$

where χ and ψ are two Dirac spinors in $L^2(S)$. We summarise this as follows:

Proposition 7.7. The total Lagrangian for $I_{\text{ED}}^\infty \times_{\nabla} M$ is given by a gravitational part

$$\mathcal{L}_{\text{grav}}(g_{\mu\nu}, m) := 4\mathcal{L}_M(g_{\mu\nu}) + \mathcal{L}_H(g_{\mu\nu}, m),$$

and a part for electrodynamics

$$\mathcal{L}_{\text{ED}}(g_{\mu\nu}, A_\mu, m) := -i (J_M \tilde{\chi}, (\gamma^\mu (\nabla_\mu^S - A_\mu) - m) \tilde{\psi}) + \frac{f(0)}{6\pi^2} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}.$$

VIII. OUTLOOK

One of the main ideas in the development of noncommutative geometry has been the translation of geometric data into (operator-)algebraic data. In this light, it is somewhat unsatisfactory that our definition of principal modules relies entirely on the geometric notion of a principal bundle. Our discussion of gauge modules is an attempt to provide a purely algebraic approach, but as we have shown, these gauge modules only yield a proper subclass of principal modules. It is still an open question how arbitrary principal modules should be described algebraically, that is, what algebraic structure on a triplet $(\mathcal{B}, \mathcal{E}, J)$ would completely characterise the properties of a principal module. The decompositions $\mathcal{E} = \oplus_{i,j \in I} \mathcal{E}_{ij}$ and $\mathcal{B} = \oplus \mathcal{B}_i$ (as described in Sec. IV B 1) are not yet enough to ensure that $(\mathcal{B}, \mathcal{E}, J)$ is a principal module. On the other hand, the condition that $\mathcal{E}_{ij} = \mathcal{E}_i \otimes_A \bar{\mathcal{E}}_j$ (modulo multiplicities) along with $\mathcal{B}_i = \text{End}(\mathcal{E}_i)$, as for gauge modules, is in fact too strong.

As mentioned in Remark 4.8, the principal bundle P can only be reconstructed from the associated vector bundle $E = P \times_{G_F} \mathcal{H}_F$ if we also know the corresponding equivalence class of G_F -atlases. It is not clear if there exists a geometric structure on E , for which this equivalence class

corresponds precisely to those transition functions that preserve the geometric structure. If one has such a geometric structure on E , this might provide the possibility of finding an algebraic equivalent structure on the module \mathcal{E} . We intend to return to these questions in the future.

In Sec. VII, we described two basic examples, namely, Yang-Mills theory and electrodynamics. It would of course be more interesting to also put the description of the noncommutative Standard Model² into our globally non-trivial framework. This should certainly be possible, though it would require some small modifications to accommodate real algebras (in this paper, we have always assumed that our algebras are complex). In particular, for real algebras, the resulting gauge group would not automatically be unimodular (see also Remark 4.13), and one would have to impose unimodularity by hand (as in Sec. 2.5 of Ref. 2). More importantly, as we also mentioned in Remarks 3.2 and 7.4, in our framework the mass parameters (i.e., the Yukawa couplings and the Majorana terms) of the theory are not restricted to be constant, but they are allowed to vary on spacetime. Such variation of the Majorana mass then naturally leads to a new scalar field σ , which was used in Ref. 34 to restore the consistency of the noncommutative Standard Model with the experimental value of the Higgs mass. In addition, however, the variation of the Yukawa couplings will also have its effect on the physical theory. We hope to provide a more detailed study of these physical implications in a future work.

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